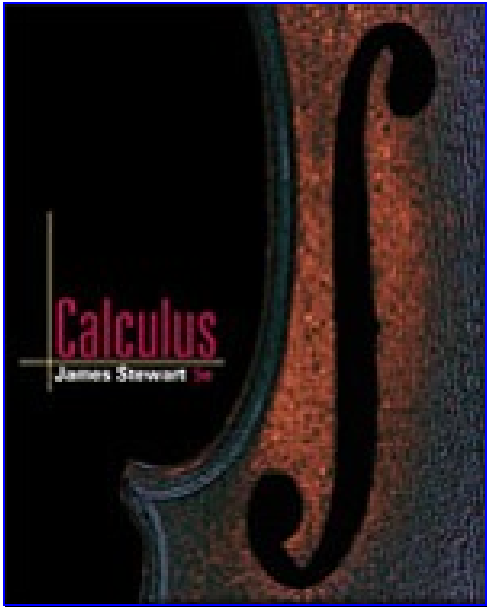


Stewart’s Complete Solutions Manual

Calculus 5e

“Classic Version”



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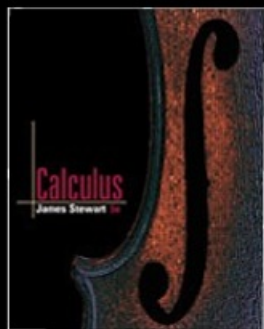
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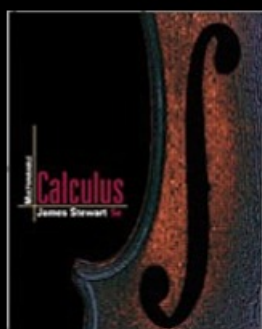
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Chapter 1

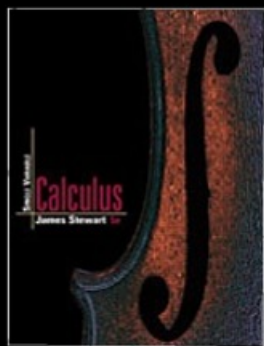
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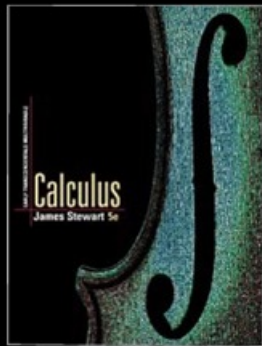
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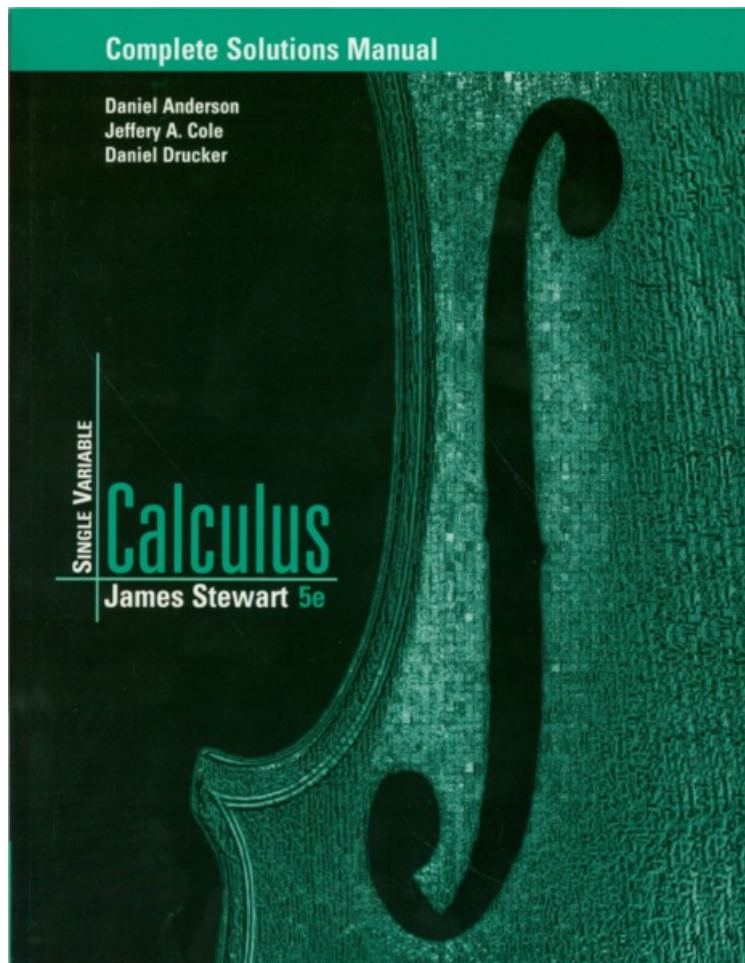
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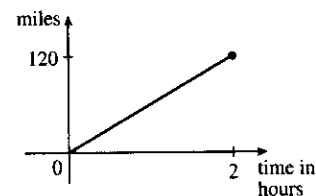


1 □ FUNCTIONS AND MODELS

1.1 Four Ways to Represent a Function

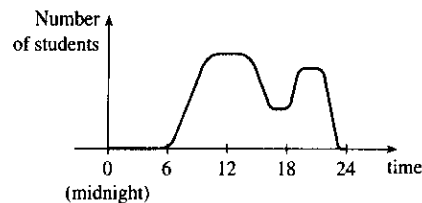
In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. (a) The point $(-1, -2)$ is on the graph of f , so $f(-1) = -2$.
(b) When $x = 2$, y is about 2.8, so $f(2) \approx 2.8$.
(c) $f(x) = 2$ is equivalent to $y = 2$. When $y = 2$, we have $x = -3$ and $x = 1$.
(d) Reasonable estimates for x when $y = 0$ are $x = -2.5$ and $x = 0.3$.
(e) The domain of f consists of all x -values on the graph of f . For this function, the domain is $-3 \leq x \leq 3$, or $[-3, 3]$. The range of f consists of all y -values on the graph of f . For this function, the range is $-2 \leq y \leq 3$, or $[-2, 3]$.
(f) As x increases from -1 to 3 , y increases from -2 to 3 . Thus, f is increasing on the interval $[-1, 3]$.
2. (a) The point $(-4, -2)$ is on the graph of f , so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g , so $g(3) = 4$.
(b) We are looking for the values of x for which the y -values are equal. The y -values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2 .
(c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.
(d) As x increases from 0 to 4 , y decreases from 3 to -1 . Thus, f is decreasing on the interval $[0, 4]$.
(e) The domain of f consists of all x -values on the graph of f . For this function, the domain is $-4 \leq x \leq 4$, or $[-4, 4]$. The range of f consists of all y -values on the graph of f . For this function, the range is $-2 \leq y \leq 3$, or $[-2, 3]$.
(f) The domain of g is $[-4, 3]$ and the range is $[0.5, 4]$.
3. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \leq a \leq 115$. In Figure 11, the range of the north-south acceleration is approximately $-325 \leq a \leq 485$. In Figure 12, the range of the east-west acceleration is approximately $-210 \leq a \leq 200$.
4. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t . The domain of the function is $\{t \mid 0 \leq t \leq 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \leq d \leq 120\}$, where d is measured in miles.

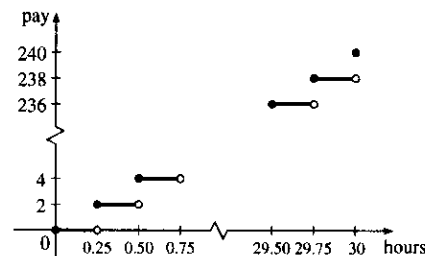


2 □ CHAPTER 1 FUNCTIONS AND MODELS

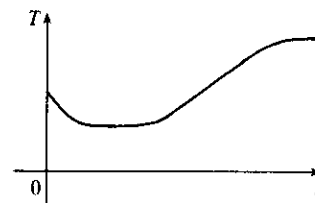
Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \leq t \leq 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \leq N \leq k\}$, where N is an integer and k is the largest number of students on campus at once.



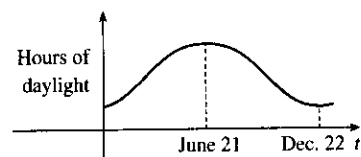
Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h . The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \dots, 238.00, 240.00\}$.



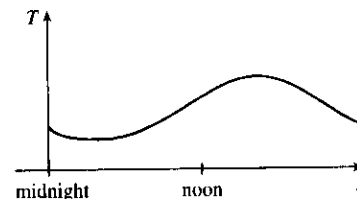
5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.
6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.
7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.
8. No, the curve is not the graph of a function since for $x = 0, \pm 1$, and ± 2 , there are infinitely many points on the curve.
9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.
10. The salesman travels away from home from 8 to 9 A.M. and is then stationary until 10:00. The salesman travels farther away from 10 until noon. There is no change in his distance from home until 1:00, at which time the distance from home decreases until 3:00. Then the distance starts increasing again, reaching the maximum distance away from home at 5:00. There is no change from 5 until 6, and then the distance decreases rapidly until 7:00 P.M., at which time the salesman reaches home.
11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



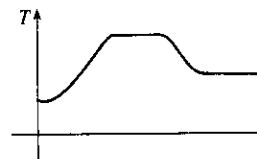
12. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22.



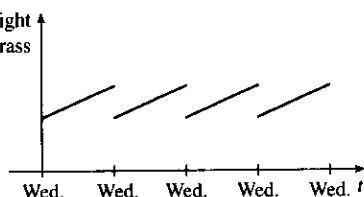
13. Of course, this graph depends strongly on the geographical location!



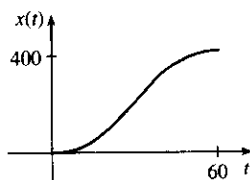
14. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



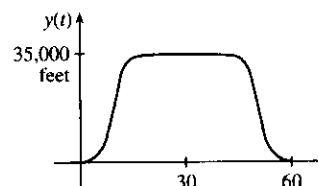
15. Height of grass



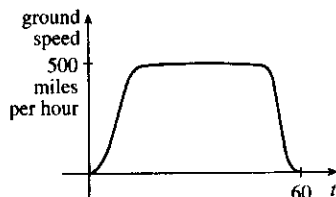
16. (a)



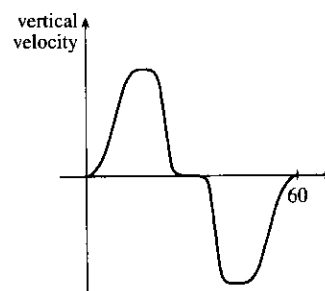
- (b)



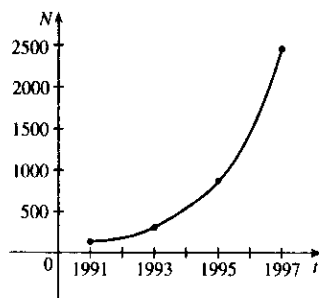
- (c)



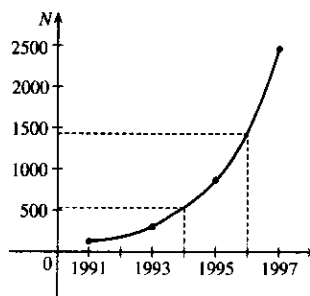
- (d)



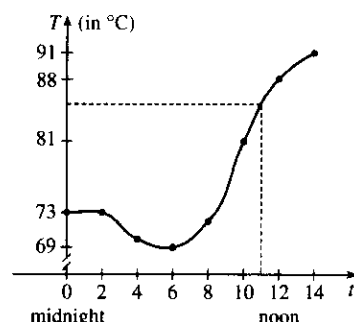
17. (a)



(b) From the graph, we estimate the number of cell-phone subscribers in Malaysia to be about 540 in 1994 and 1450 in 1996.



18. (a)



(b) From the graph in part (a), we estimate the temperature at 11:00 A.M. to be about 84.5°C .

19. $f(x) = 3x^2 - x + 2$.

$$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12.$$

$$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16.$$

$$f(a) = 3a^2 - a + 2.$$

$$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2.$$

$$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a - 1 + 2 = 3a^2 + 5a + 4.$$

$$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4.$$

$$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2.$$

$$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$$

$$[f(a)]^2 = [3a^2 - a + 2]^2 = (3a^2 - a + 2)(3a^2 - a + 2)$$

$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$

$$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$$

20. A spherical balloon with radius $r+1$ has volume $V(r+1) = \frac{4}{3}\pi(r+1)^3 = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1) - V(r) = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2 + 3r + 1)$.

21. $f(x) = x - x^2$, so

$$f(2+h) = 2+h - (2+h)^2 = 2+h - (4+4h+h^2) = 2+h-4-4h-h^2 = -(h^2+3h+2),$$

$$f(x+h) = x+h - (x+h)^2 = x+h - (x^2+2xh+h^2) = x+h-x^2-2xh-h^2, \text{ and}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{x+h-x^2-2xh-h^2-x+x^2}{h} = \frac{h-2xh-h^2}{h} = \frac{h(1-2x-h)}{h} = 1-2x-h.$$

22. $f(x) = \frac{x}{x+1}$, so $f(2+h) = \frac{2+h}{2+h+1} = \frac{2+h}{3+h}$, $f(x+h) = \frac{x+h}{x+h+1}$, and

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} = \frac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)} = \frac{1}{(x+h+1)(x+1)}.$$

23. $f(x) = x/(3x-1)$ is defined for all x except when $0 = 3x-1 \Leftrightarrow x = \frac{1}{3}$, so the domain is $\{x \in \mathbb{R} \mid x \neq \frac{1}{3}\} = (-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$.

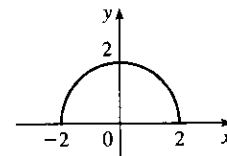
24. $f(x) = (5x+4)/(x^2+3x+2)$ is defined for all x except when $0 = x^2+3x+2 \Leftrightarrow 0 = (x+2)(x+1) \Leftrightarrow x = -2$ or -1 , so the domain is $\{x \in \mathbb{R} \mid x \neq -2, -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

25. $f(t) = \sqrt{t} + \sqrt[3]{t}$ is defined when $t \geq 0$. These values of t give real number results for $\sqrt{t}$, whereas any value of t gives a real number result for $\sqrt[3]{t}$. The domain is $[0, \infty)$.

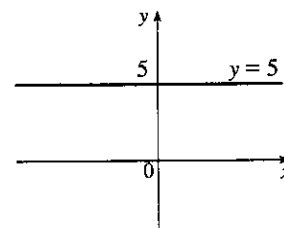
26. $g(u) = \sqrt{u} + \sqrt{4-u}$ is defined when $u \geq 0$ and $4-u \geq 0 \Leftrightarrow u \leq 4$. Thus, the domain is $0 \leq u \leq 4 = [0, 4]$.

27. $h(x) = 1/\sqrt[4]{x^2-5x}$ is defined when $x^2-5x > 0 \Leftrightarrow x(x-5) > 0$. Note that $x^2-5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

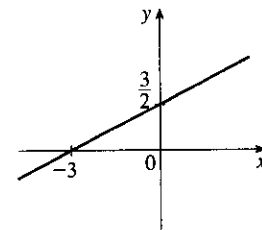
28. $h(x) = \sqrt{4-x^2}$. Now $y = \sqrt{4-x^2} \Rightarrow y^2 = 4-x^2 \Leftrightarrow x^2+y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\{x \mid 4-x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.



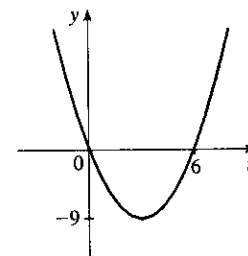
29. $f(x) = 5$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a horizontal line with y -intercept 5.



30. $F(x) = \frac{1}{2}(x+3)$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of F is a line with x -intercept -3 and y -intercept $\frac{3}{2}$.

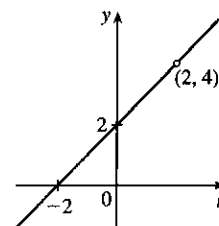


31. $f(t) = t^2 - 6t$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t -intercepts, let $y = 0$ and solve for t . $0 = t^2 - 6t = t(t-6) \Rightarrow t = 0$ and $t = 6$. The t -coordinate of the vertex is halfway between the t -intercepts, that is, at $t = 3$. Since $f(3) = 3^2 - 6 \cdot 3 = -9$, the vertex is $(3, -9)$.

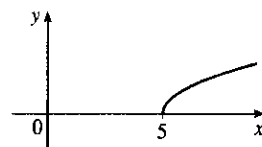


6 □ CHAPTER 1 FUNCTIONS AND MODELS

32. $H(t) = \frac{4-t^2}{2-t} = \frac{(2+t)(2-t)}{2-t}$, so for $t \neq 2$, $H(t) = 2+t$. The domain is $\{t \mid t \neq 2\}$. So the graph of H is the same as the graph of the function $f(t) = t+2$ (a line) except for the hole at $(2, 4)$.



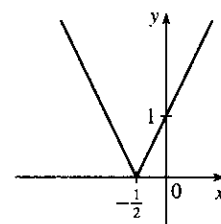
33. $g(x) = \sqrt{x-5}$ is defined when $x-5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x-5} \Rightarrow y^2 = x-5 \Rightarrow x = y^2+5$, we see that g is the top half of a parabola.



$$34. F(x) = |2x+1| = \begin{cases} 2x+1 & \text{if } 2x+1 \geq 0 \\ -(2x+1) & \text{if } 2x+1 < 0 \end{cases}$$

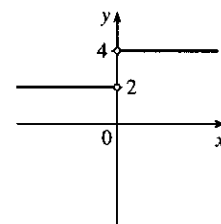
$$= \begin{cases} 2x+1 & \text{if } x \geq -\frac{1}{2} \\ -2x-1 & \text{if } x < -\frac{1}{2} \end{cases}$$

The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



35. $G(x) = \frac{3x+|x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

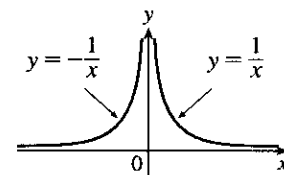
$$G(x) = \begin{cases} \frac{3x+x}{x} & \text{if } x > 0 \\ \frac{3x-x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \frac{4x}{x} & \text{if } x > 0 \\ \frac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$



Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.

36. $g(x) = \frac{|x|}{x^2}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

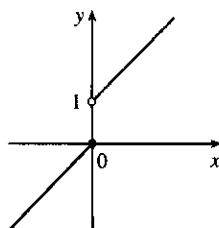
$$g(x) = \begin{cases} \frac{x}{x^2} & \text{if } x > 0 \\ \frac{-x}{x^2} & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x} & \text{if } x < 0 \end{cases}$$



Note that g is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.

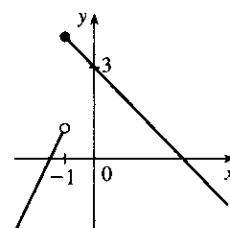
37. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$

Domain is $\mathbb{R}$, or $(-\infty, \infty)$.



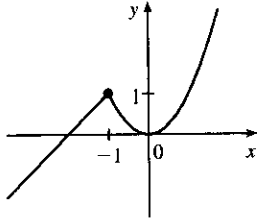
38. $f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ 3-x & \text{if } x \geq -1 \end{cases}$

Domain is $\mathbb{R}$, or $(-\infty, \infty)$.



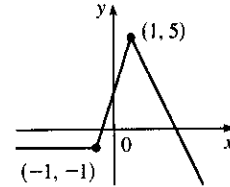
$$39. f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Note that for $x = -1$, both $x+2$ and x^2 are equal to 1. Domain is $\mathbb{R}$.



$$40. f(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ 3x+2 & \text{if } -1 < x < 1 \\ 7-2x & \text{if } x \geq 1 \end{cases}$$

Domain is $\mathbb{R}$.

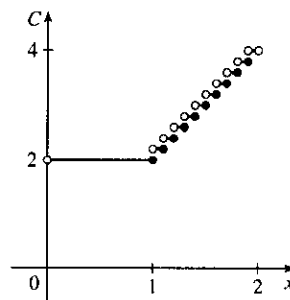


41. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of this line segment is $\frac{-6 - 1}{4 - (-2)} = -\frac{7}{6}$, so an equation is $y - 1 = -\frac{7}{6}(x + 2)$. The function is $f(x) = -\frac{7}{6}x - \frac{4}{3}$, $-2 \leq x \leq 4$.
42. The slope of this line segment is $\frac{3 - (-2)}{6 - (-3)} = \frac{5}{9}$, so an equation is $y + 2 = \frac{5}{9}(x + 3)$. The function is $f(x) = \frac{5}{9}x - \frac{1}{3}$, $-3 \leq x \leq 6$.
43. We need to solve the given equation for y . $x + (y - 1)^2 = 0 \Leftrightarrow (y - 1)^2 = -x \Leftrightarrow y - 1 = \pm\sqrt{-x} \Leftrightarrow y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \leq 0$.
44. $(x - 1)^2 + y^2 = 1 \Leftrightarrow y = \pm\sqrt{1 - (x - 1)^2} = \pm\sqrt{2x - x^2}$. The top half is given by the function $f(x) = \sqrt{2x - x^2}$, $0 \leq x \leq 2$.
45. For $-1 \leq x \leq 2$, the graph is the line with slope 1 and y -intercept 1, that is, the line $y = x + 1$. For $2 < x \leq 4$, the graph is the line with slope $-\frac{3}{2}$ and x -intercept 4 [which corresponds to the point $(4, 0)$], so $y - 0 = -\frac{3}{2}(x - 4) = -\frac{3}{2}x + 6$. So the function is $f(x) = \begin{cases} x+1 & \text{if } -1 \leq x \leq 2 \\ -\frac{3}{2}x+6 & \text{if } 2 < x \leq 4 \end{cases}$
46. For $x \leq 0$, the graph is the line $y = 2$. For $0 < x \leq 1$, the graph is the line with slope -2 and y -intercept 2, that is, the line $y = -2x + 2$. For $x > 1$, the graph is the line with slope 1 and x -intercept 1, that is, the line $y = 1(x - 1) = x - 1$. So the function is $f(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ -2x+2 & \text{if } 0 < x \leq 1 \\ x-1 & \text{if } 1 < x \end{cases}$
47. Let the length and width of the rectangle be L and W . Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \frac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W , then $5 < L < 10$ would be the domain.
48. Let the length and width of the rectangle be L and W . Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W , then $L > 4$ would be the domain.

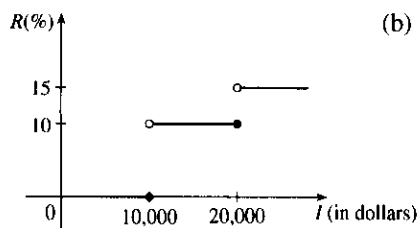
49. Let the length of a side of the equilateral triangle be x . Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + (\frac{1}{2}x)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.
50. Let the volume of the cube be V and the length of an edge be L . Then $V = L^3$ so $L = \sqrt[3]{V}$, and the surface area is $S(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, with domain $V > 0$.
51. Let each side of the base of the box have length x , and let the height of the box be h . Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.
52. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \frac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \Leftrightarrow 2h = 30 - x - \frac{1}{2}\pi x \Leftrightarrow h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,
- $$A(x) = x \frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2 = 15x - \frac{4}{8}x^2 - \frac{\pi}{8}x^2 = 15x - x^2\left(\frac{\pi+4}{8}\right)$$
- Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \Leftrightarrow 30 - x - \frac{1}{2}\pi x > 0 \Leftrightarrow 60 > 2x + \pi x \Leftrightarrow x < \frac{60}{2+\pi}$. Hence, the domain of A is $0 < x < \frac{60}{2+\pi}$.
53. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so $V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$. The sides L , W , and x must be positive. Thus, $L > 0 \Leftrightarrow 20 - 2x > 0 \Leftrightarrow x < 10$; $W > 0 \Leftrightarrow 12 - 2x > 0 \Leftrightarrow x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

54.

$$C(x) = \begin{cases} \$2.00 & \text{if } 0.0 < x \leq 1.0 \\ 2.20 & \text{if } 1.0 < x \leq 1.1 \\ 2.40 & \text{if } 1.1 < x \leq 1.2 \\ 2.60 & \text{if } 1.2 < x \leq 1.3 \\ 2.80 & \text{if } 1.3 < x \leq 1.4 \\ 3.00 & \text{if } 1.4 < x \leq 1.5 \\ 3.20 & \text{if } 1.5 < x \leq 1.6 \\ 3.40 & \text{if } 1.6 < x \leq 1.7 \\ 3.60 & \text{if } 1.7 < x \leq 1.8 \\ 3.80 & \text{if } 1.8 < x \leq 1.9 \\ 4.00 & \text{if } 1.9 < x < 2.0 \end{cases}$$



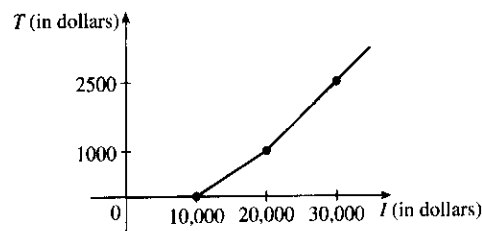
55. (a)

(b) On \$14,000, tax is assessed on \$4000, and $10\%(\$4000) = \400 .

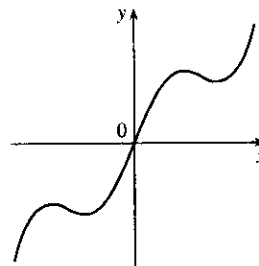
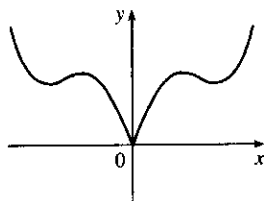
On \$26,000, tax is assessed on \$16,000, and

$$10\%(\$10,000) + 15\%(\$6000) = \$1000 + \$900 = \$1900.$$

- (c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10,000, 0)$ to $(20,000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20,000$ is the ray with initial point $(20,000, 1000)$ that passes through $(30,000, 2500)$.



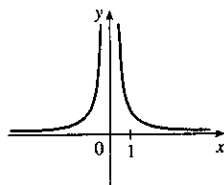
56. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.
57. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y -axis.
58. f is not an even function since it is not symmetric with respect to the y -axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y -axis.
59. (a) Because an even function is symmetric with respect to the y -axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.
 (b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.
60. (a) If f is even, we get the rest of the graph by reflecting about the y -axis. (b) If f is odd, we get the rest of the graph by rotating 180° about the origin.



61. $f(x) = x^{-2}$.

$$f(-x) = (-x)^{-2} = \frac{1}{(-x)^2} = \frac{1}{x^2} = x^{-2} = f(x)$$

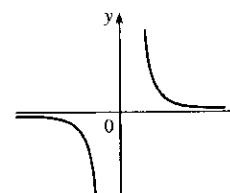
So f is an even function.



62. $f(x) = x^{-3}$.

$$f(-x) = (-x)^{-3} = \frac{1}{(-x)^3} = \frac{1}{-x^3} = -\frac{1}{x^3} = -(x^{-3}) = -f(x)$$

So f is odd.

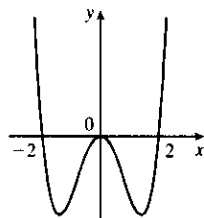


63. $f(x) = x^2 + x$, so $f(-x) = (-x)^2 + (-x) = x^2 - x$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

64. $f(x) = x^4 - 4x^2$.

$$\begin{aligned} f(-x) &= (-x)^4 - 4(-x)^2 \\ &= x^4 - 4x^2 = f(x) \end{aligned}$$

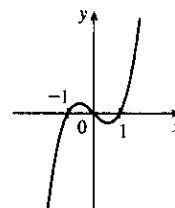
So f is even.



65. $f(x) = x^3 - x$.

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) = -x^3 + x \\ &= -(x^3 - x) = -f(x) \end{aligned}$$

So f is odd.

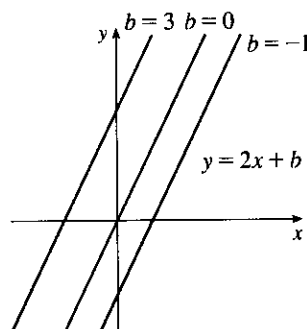


66. $f(x) = 3x^3 + 2x^2 + 1$, so $f(-x) = 3(-x)^3 + 2(-x)^2 + 1 = -3x^3 + 2x^2 + 1$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

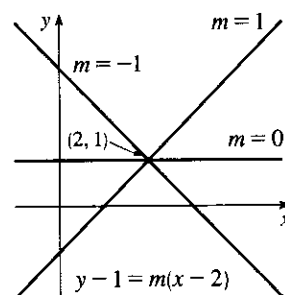
1.2 Mathematical Models: A Catalog of Essential Functions

- $f(x) = \sqrt[5]{x}$ is a root function with $n = 5$.
 - $g(x) = \sqrt{1-x^2}$ is an algebraic function because it is a root of a polynomial.
 - $h(x) = x^9 + x^4$ is a polynomial of degree 9.
 - $r(x) = \frac{x^2+1}{x^3+x}$ is a rational function because it is a ratio of polynomials.
 - $s(x) = \tan 2x$ is a trigonometric function.
 - $t(x) = \log_{10} x$ is a logarithmic function.
- $y = (x-6)/(x+6)$ is a rational function because it is a ratio of polynomials.
 - $y = x + x^2/\sqrt{x-1}$ is an algebraic function because it involves polynomials and roots of polynomials.
 - $y = 10^x$ is an exponential function (notice that x is the *exponent*).
 - $y = x^{10}$ is a power function (notice that x is the *base*).
 - $y = 2t^6 + t^4 - \pi$ is a polynomial of degree 6.
 - $y = \cos \theta + \sin \theta$ is a trigonometric function.
- We notice from the figure that g and h are even functions (symmetric with respect to the y -axis) and that f is an odd function (symmetric with respect to the origin). So (b) $[y = x^5]$ must be f . Since g is flatter than h near the origin, we must have (c) $[y = x^8]$ matched with g and (a) $[y = x^2]$ matched with h .
- The graph of $y = 3x$ is a line (choice G).
 - $y = 3^x$ is an exponential function (choice f).
 - $y = x^3$ is an odd polynomial function or power function (choice F).
 - $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

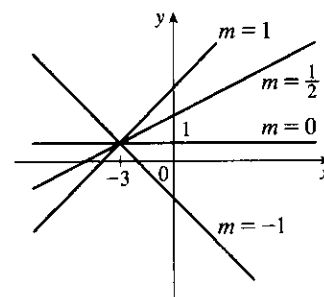
5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y -intercept.



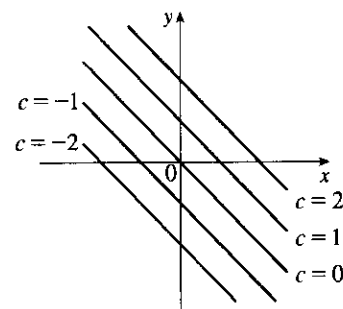
- (b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f . We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.



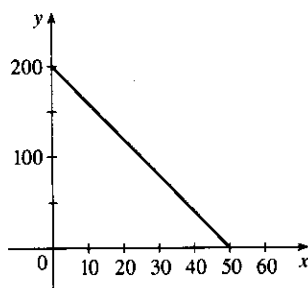
- (c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.
6. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1 . The y -intercept is c .

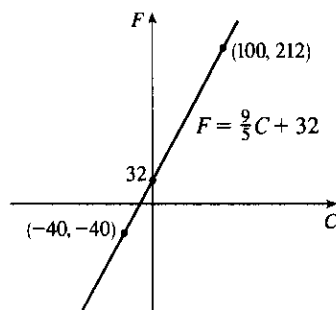


8. (a)



- (b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y -intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x -intercept of 50 is the smallest rental fee that results in no spaces rented.

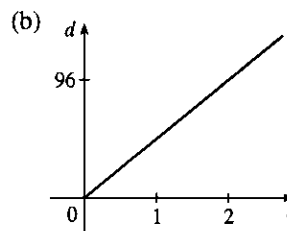
9. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1°C . (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F -intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

10. (a) Let d = distance traveled (in miles) and t = time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50 \text{ minutes} = 50 \cdot \frac{1}{60} = \frac{5}{6} \text{ h}$, $d = 40$. Thus we have two points: $(0, 0)$ and $(\frac{5}{6}, 40)$, so

$$m = \frac{40 - 0}{\frac{5}{6} - 0} = 48 \text{ and so } d = 48t.$$

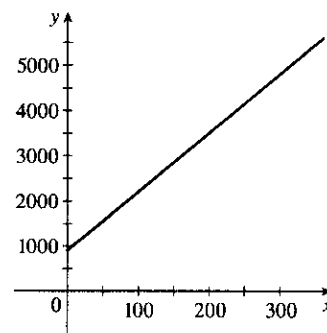


(c) The slope is 48 and represents the car's speed in mi/h.

11. (a) Using N in place of x and T in place of y , we find the slope to be $\frac{T_2 - T_1}{N_2 - N_1} = \frac{80 - 70}{173 - 113} = \frac{10}{60} = \frac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \Leftrightarrow T - 80 = \frac{1}{6}N - \frac{173}{6} \Leftrightarrow T = \frac{1}{6}N + \frac{307}{6} \left[\frac{307}{6} = 51.1\bar{6} \right]$.
- (b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1°F .
- (c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\bar{6}^\circ\text{F} \approx 76^\circ\text{F}$.

12. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$ we get the slope $\frac{4800 - 2200}{300 - 100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \Leftrightarrow y = 13x + 900$.

- (b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.
- (c) The y -intercept is 900 and it represents the fixed daily costs of operating the factory.



13. (a) We are given $\frac{\text{change in pressure}}{10 \text{ feet change in depth}} = \frac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.
- (b) When $P = 100$, then $100 = 0.434d + 15 \Leftrightarrow 0.434d = 85 \Leftrightarrow d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is 100 lb/in^2 at a depth of approximately 196 feet.

14. (a) Using d in place of x and C in place of y , we find the slope to be

$$\frac{C_2 - C_1}{d_2 - d_1} = \frac{460 - 380}{800 - 480} = \frac{80}{320} = \frac{1}{4}$$

So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \Leftrightarrow$

$$C - 460 = \frac{1}{4}d - 200 \Leftrightarrow C = \frac{1}{4}d + 260.$$

- (b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.

The cost of driving 1500 miles is \$635.

- (d) The y -intercept represents the fixed cost, \$260.

- (e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

15. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a \cos(bx) + c$ seems appropriate.

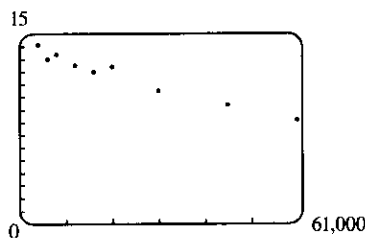
- (b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

16. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

- (b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Some values are given to many decimal places. These are the results given by several computer algebra systems—rounding is left to the reader.

17. (a)

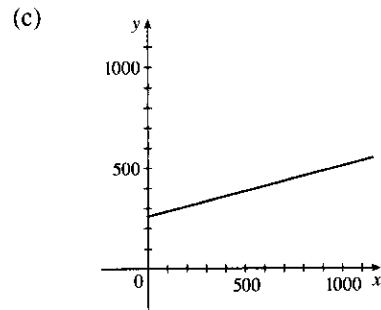
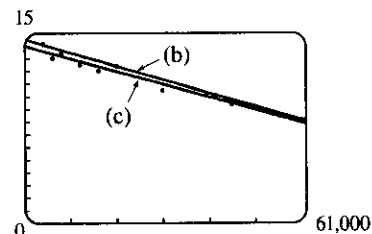


A linear model does seem appropriate.

- (b) Using the points (4000, 14.1) and (60,000, 8.2), we obtain

$$y - 14.1 = \frac{8.2 - 14.1}{60,000 - 4000}(x - 4000) \text{ or, equivalently,}$$

$$y \approx -0.000105357x + 14.521429.$$



The slope of the line represents the cost per mile, \$0.25.

- (c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$. The following commands and screens illustrate how to find the least squares regression line on a TI-83 Plus. Enter the data into list one (L1) and list two (L2). Press **STAT** **1** to enter the editor.

L1	L2	L3	1
4000	14.1		
6000	13		
8000	13.4		
12000	12.5		
16000	12		
20000	12.4		
30000	10.5		
L1 = {4000, 6000, 8...			

L1	L2	L3	2
12000	12.5		
16000	12		
20000	12.4		
30000	10.5		
45000	8.4		
60000	8.2		
L2(10) =			

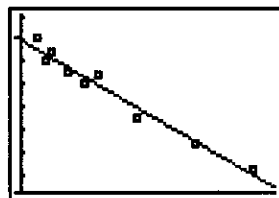
Find the regression line and store it in Y_1 . Press **2nd** **QUIT** **STAT** **▸** **4** **VARS** **▸** **1** **1** **ENTER**.

LinReg(ax+b) Y_1	LinReg $y=ax+b$ $a=-9.978546E-5$ $b=13.95076408$	Plot1 Plot2 Plot3 Y_1 -9.978545618 $7893E-5X+13.9507$ 64077085 $Y_2=$ $Y_3=$ $Y_4=$ $Y_5=$
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Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing **ENTER** or by pressing **2nd** **STAT PLOT** **1** **ENTER**.

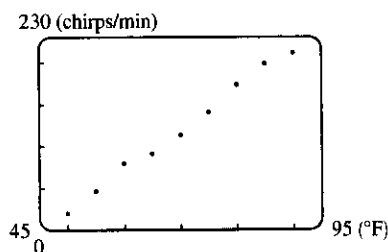
2nd STAT PLOT 1 Plot1...On 2 Plot2...Off 3 Plot3...Off 4 PlotsOff	Plot2 Plot3 Off Off Type: $\square$ $\square$ $\square$ Xlist: L1 Ylist: L2 Mark: $\square$ +
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Now press **ZOOM** **9** to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

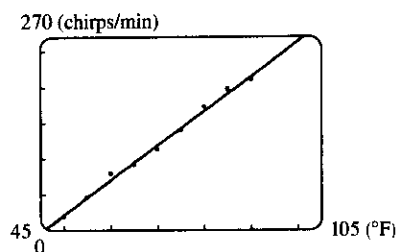


- (d) When $x = 25,000$, $y \approx 11.456$; or about 11.5 per 100 population.
 (e) When $x = 80,000$, $y \approx 5.968$; or about a 6% chance.
 (f) When $x = 200,000$, y is negative, so the model does not apply.

18. (a)



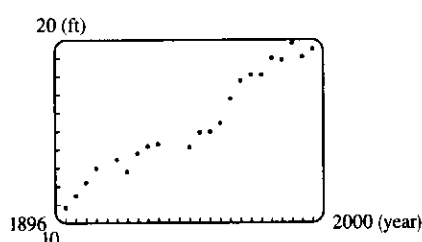
(b)



Using a computing device, we obtain the least squares regression line $y = 4.856x - 220.96$.

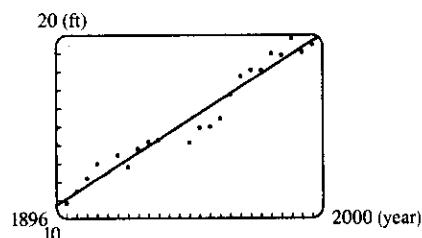
(c) When $x = 100^\circ \text{F}$, $y = 264.7 \approx 265$ chirps/min.

19. (a)



A linear model does seem appropriate.

(b)

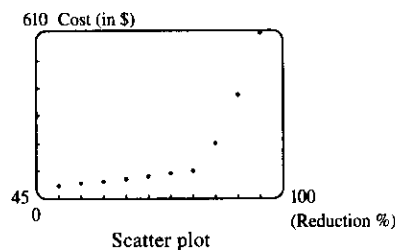


Using a computing device, we obtain the least squares regression line $y = 0.089119747x - 158.2403249$, where x is the year and y is the height in feet.

(c) When $x = 2000$, the model gives $y \approx 20.00$ ft. Note that the actual winning height for the 2000 Olympics is *less than* the winning height for 1996—so much for that prediction.

(d) When $x = 2100$, $y \approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

20. By looking at the scatter plot of the data, we rule out the linear and logarithmic models.



We try various models:

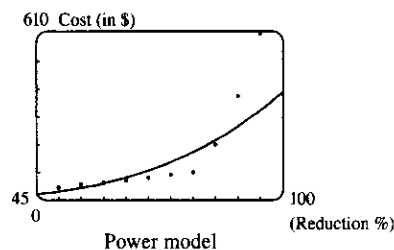
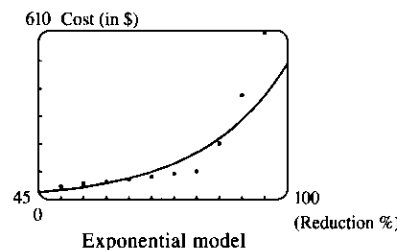
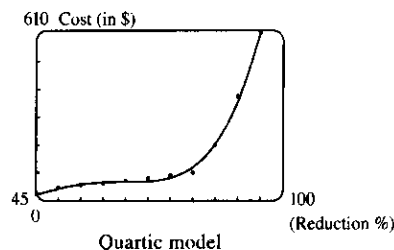
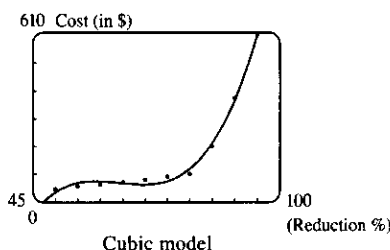
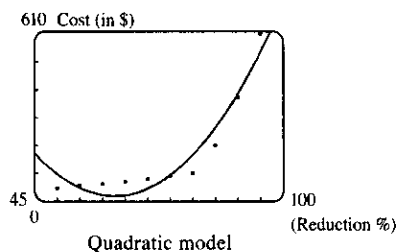
Quadratic: $y = 0.496x^2 - 62.2893x + 1970.639$

Cubic: $y = 0.0201243201x^3 - 3.88037296x^2 + 247.6754468x - 5163.935198$

Quartic: $y = 0.0002951049x^4 - 0.0654560995x^3 + 5.27525641x^2 - 180.2266511x + 2203.210956$

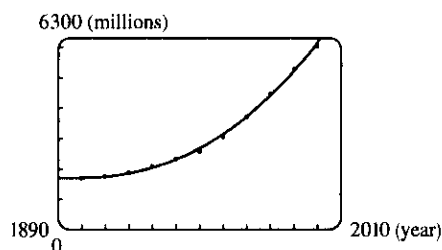
Exponential: $y = 2.41422994(1.054516914)^x$

Power: $y = 0.000022854971x^{3.616078251}$



After examining the graphs of these models, we see that the cubic and quartic models are clearly the best.

21.



Using a computing device, we obtain the cubic function $y = ax^3 + bx^2 + cx + d$ with $a = 0.0012937$, $b = -7.06142$, $c = 12,823$, and $d = -7,743,770$. When $x = 1925$, $y \approx 1914$ (million).

22. (a) $T = 1.000396048d^{1.499661718}$

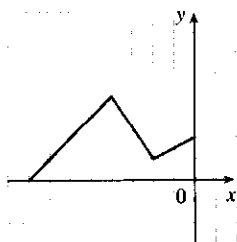
(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

1.3 New Functions from Old Functions

- If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.
 - If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.
 - If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.
 - If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.
 - If the graph of f is reflected about the x -axis, its equation becomes $y = -f(x)$.
 - If the graph of f is reflected about the y -axis, its equation becomes $y = f(-x)$.
 - If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.
 - If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.
- To obtain the graph of $y = 5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5.
 - To obtain the graph of $y = f(x - 5)$ from the graph of $y = f(x)$, shift the graph 5 units to the right.
 - To obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$, reflect the graph about the x -axis.

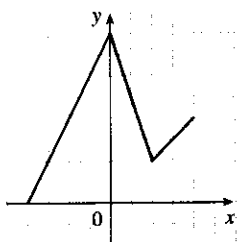
- (d) To obtain the graph of $y = -5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and reflect it about the x -axis.
- (e) To obtain the graph of $y = f(5x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 5.
- (f) To obtain the graph of $y = 5f(x) - 3$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and shift it 3 units downward.
3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.
- (b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.
- (c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.
- (d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x -axis. Its equation is $y = -f(x + 4)$.
- (e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

4. (a) To graph $y = f(x + 4)$ we shift the graph of f , 4 units to the left.



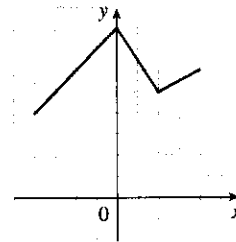
The point $(2, 1)$ on the graph of f corresponds to the point $(2 - 4, 1) = (-2, 1)$.

- (c) To graph $y = 2f(x)$ we stretch the graph of f vertically by a factor of 2.



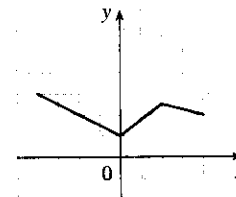
The point $(2, 1)$ on the graph of f corresponds to the point $(2, 2 \cdot 1) = (2, 2)$.

- (b) To graph $y = f(x) + 4$ we shift the graph of f , 4 units upward.



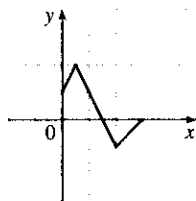
The point $(2, 1)$ on the graph of f corresponds to the point $(2, 1 + 4) = (2, 5)$.

- (d) To graph $y = -\frac{1}{2}f(x) + 3$, we shrink the graph of f vertically by a factor of 2, then reflect the resulting graph about the x -axis, then shift the resulting graph 3 units upward.



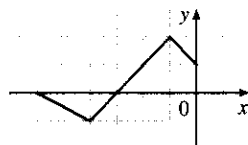
The point $(2, 1)$ on the graph of f corresponds to the point $(2, -\frac{1}{2} \cdot 1 + 3) = (2, 2.5)$.

5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



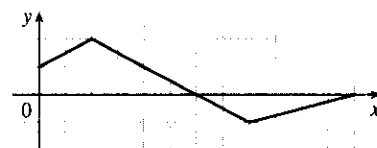
The point $(4, -1)$ on the graph of f corresponds to the point $(\frac{1}{2} \cdot 4, -1) = (2, -1)$.

- (c) To graph $y = f(-x)$ we reflect the graph of f about the y -axis.



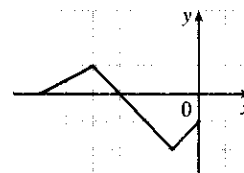
The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

- (b) To graph $y = f(\frac{1}{2}x)$ we stretch the graph of f horizontally by a factor of 2.



The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

- (d) To graph $y = -f(-x)$ we reflect the graph of f about the y -axis, then about the x -axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$\begin{aligned} y &= 2f(x - 2) = 2\sqrt{3(x - 2) - (x - 2)^2} \\ &= 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10} \end{aligned}$$

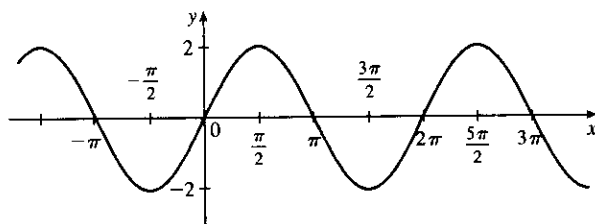
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x -axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\text{reflect about } x\text{-axis}} \underbrace{f(x + 4)}_{\text{shift 4 units left}} \underbrace{- 1}_{\text{shift 1 unit down}}$$

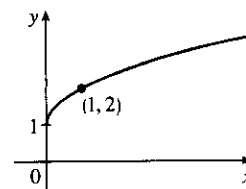
This function can be written as

$$\begin{aligned} y &= -f(x + 4) - 1 = -\sqrt{3(x + 4) - (x + 4)^2} - 1 = -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 \\ &= -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

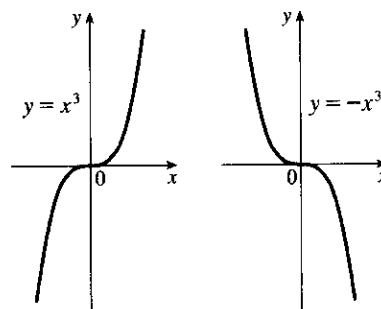
8. (a) The graph of $y = 2 \sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.



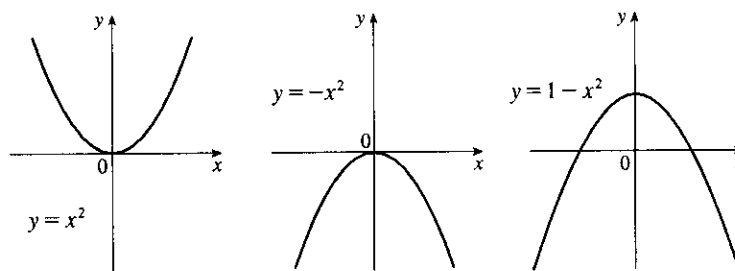
- (b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



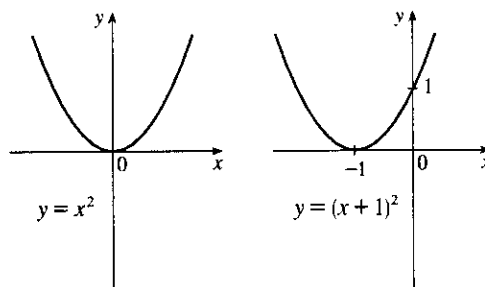
9. $y = -x^3$: Start with the graph of $y = x^3$ and reflect about the x -axis. Note: Reflecting about the y -axis gives the same result since substituting $-x$ for x gives us $y = (-x)^3 = -x^3$.



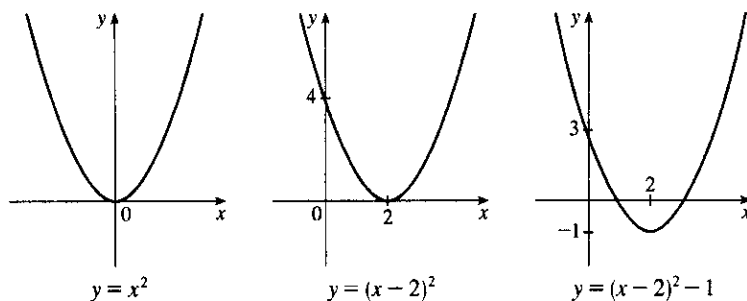
10. $y = 1 - x^2 = -x^2 + 1$: Start with the graph of $y = x^2$, reflect about the x -axis, and then shift 1 unit upward.



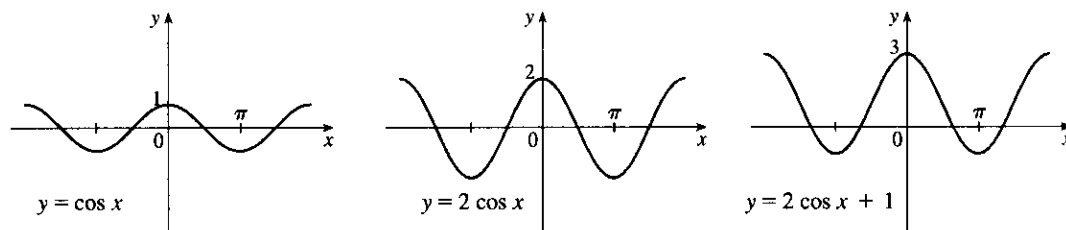
11. $y = (x + 1)^2$: Start with the graph of $y = x^2$ and shift 1 unit to the left.



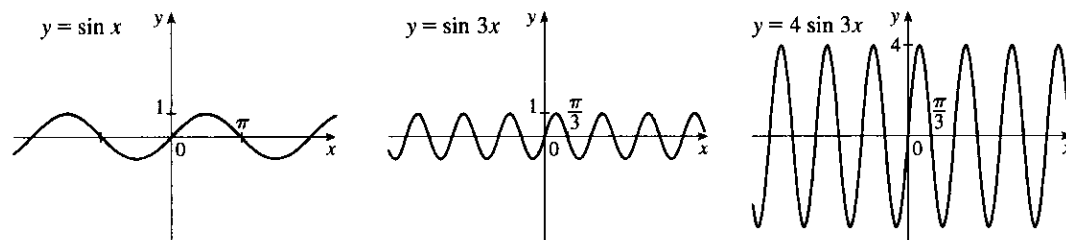
12. $y = x^2 - 4x + 3 = (x^2 - 4x + 4) - 1 = (x - 2)^2 - 1$: Start with the graph of $y = x^2$, shift 2 units to the right, and then shift 1 unit downward.



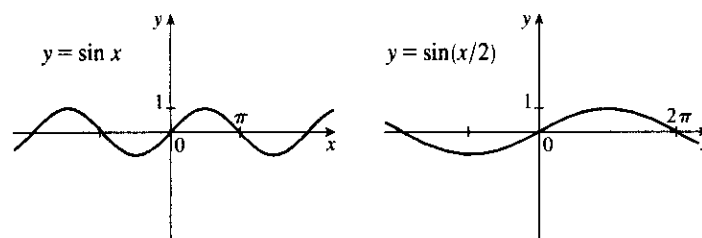
13. $y = 1 + 2 \cos x$: Start with the graph of $y = \cos x$, stretch vertically by a factor of 2, and then shift 1 unit upward.



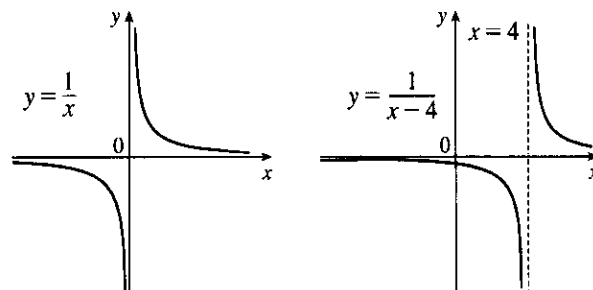
14. $y = 4 \sin 3x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 4.



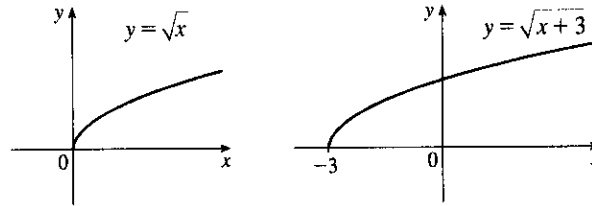
15. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



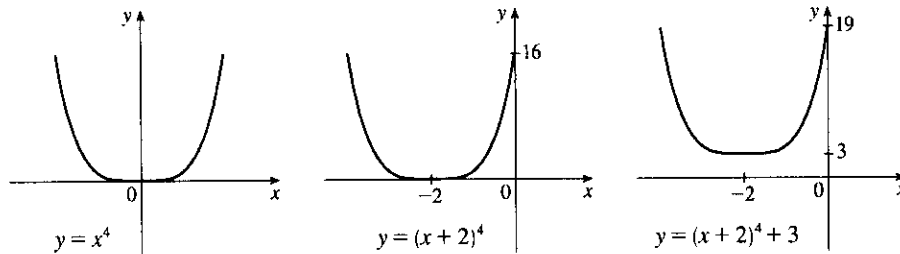
16. $y = 1/(x - 4)$: Start with the graph of $y = 1/x$ and shift 4 units to the right.



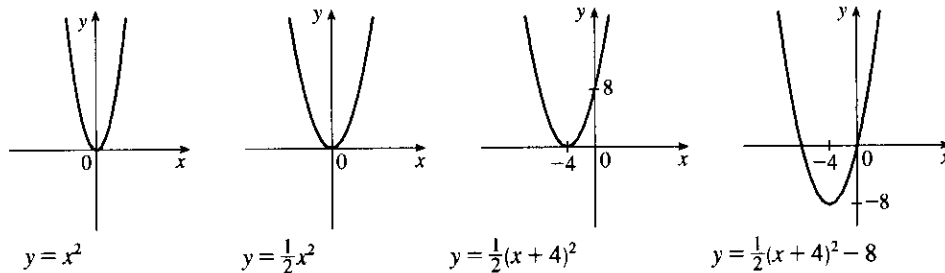
17. $y = \sqrt{x+3}$: Start with the graph of $y = \sqrt{x}$ and shift 3 units to the left.



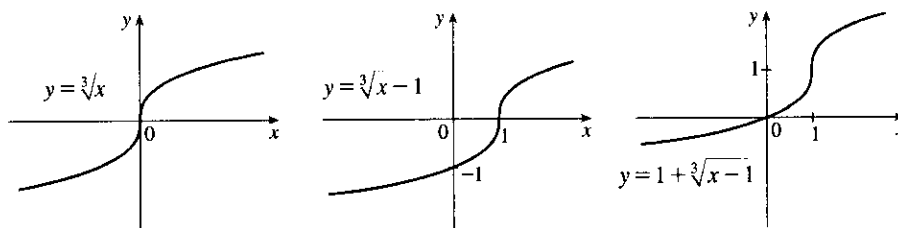
18. $y = (x+2)^4 + 3$: Start with the graph of $y = x^4$, shift 2 units to the left, and then shift 3 units upward.



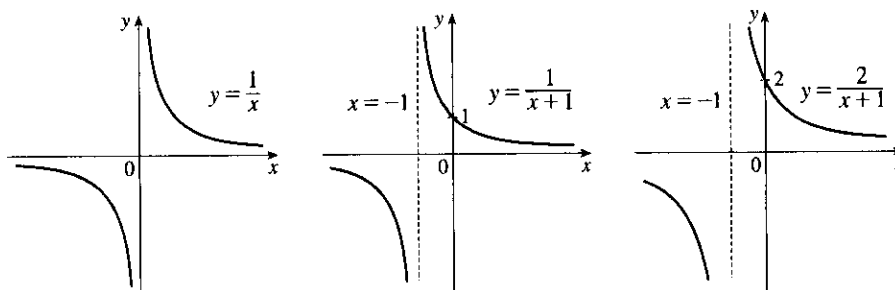
19. $y = \frac{1}{2}(x^2 + 8x) = \frac{1}{2}(x^2 + 8x + 16) - 8 = \frac{1}{2}(x+4)^2 - 8$: Start with the graph of $y = x^2$, compress vertically by a factor of 2, shift 4 units to the left, and then shift 8 units downward.



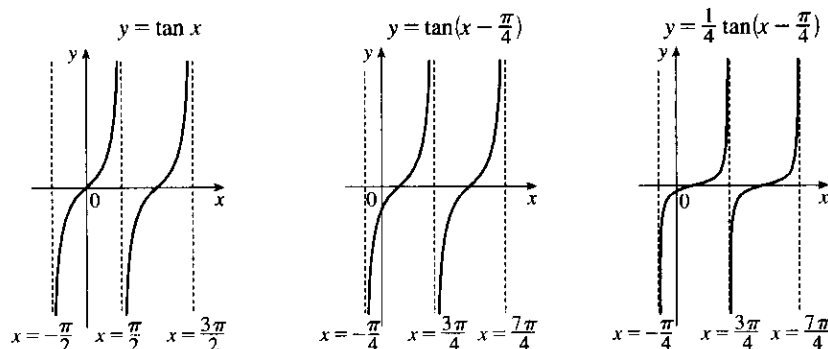
20. $y = 1 + \sqrt[3]{x-1}$: Start with the graph of $y = \sqrt[3]{x}$, shift 1 unit to the right, and then shift 1 unit upward.



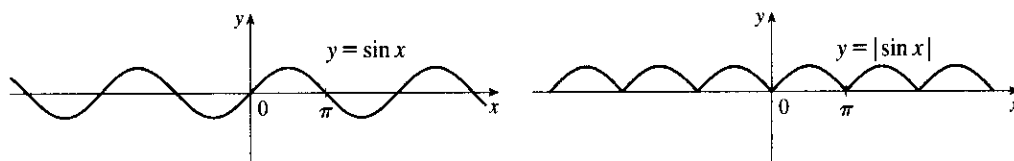
21. $y = 2/(x+1)$: Start with the graph of $y = 1/x$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



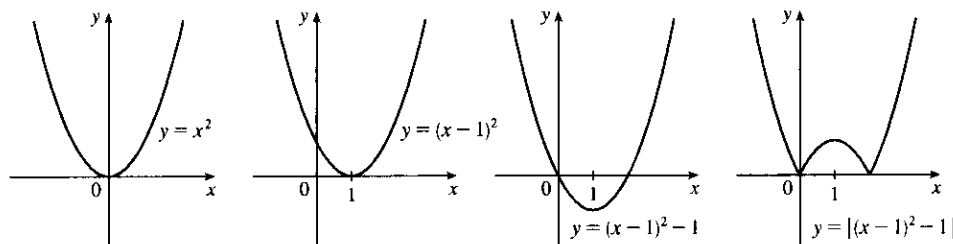
22. $y = \frac{1}{4} \tan(x - \frac{\pi}{4})$: Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



23. $y = |\sin x|$: Start with the graph of $y = \sin x$ and reflect all the parts of the graph below the x -axis about the x -axis.

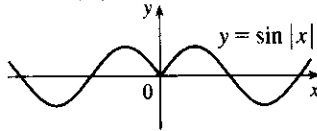


24. $y = |x^2 - 2x| = |x^2 - 2x + 1 - 1| = |(x - 1)^2 - 1|$: Start with the graph of $y = x^2$, shift 1 unit right, shift 1 unit downward, and reflect the portion of the graph below the x -axis about the x -axis.

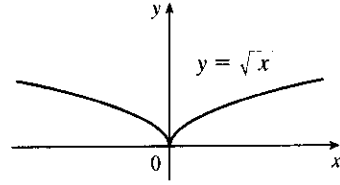
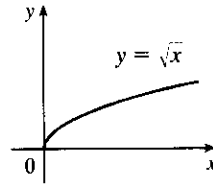


25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2 \sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 A.M. to 6:18 P.M.) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.
26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35 \sin\left(\frac{2\pi}{5.4}t\right)$.
27. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y -axis is reflected about the y -axis.

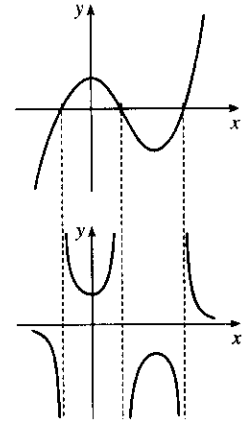
(b) $y = \sin |x|$



(c) $y = \sqrt{|x|}$

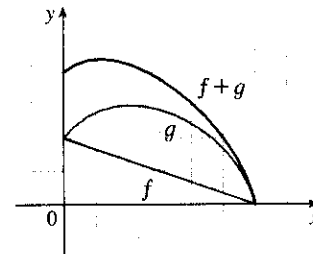


28. The most important features of the given graph are the x -intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x -values where there are x -intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



29. Assuming that successive horizontal and vertical gridlines are a unit apart, we can make a table of approximate values as follows.

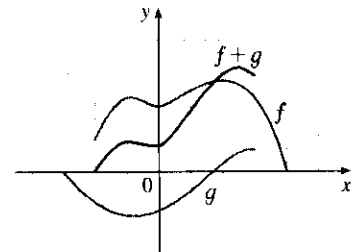
x	0	1	2	3	4	5	6
$f(x)$	2	1.7	1.3	1.0	0.7	0.3	0
$g(x)$	2	2.7	3	2.8	2.4	1.7	0
$f(x) + g(x)$	4	4.4	4.3	3.8	3.1	2.0	0



Connecting the points $(x, f(x) + g(x))$ with a smooth curve gives an approximation to the graph of $f + g$. Extra points can be plotted between those listed above if necessary.

30. First note that the domain of $f + g$ is the intersection of the domains of f and g ; that is, $f + g$ is only defined where both f and g are defined. Taking the horizontal and vertical units of length to be the distances between successive vertical and horizontal gridlines, we can make a table of approximate values as follows:

x	-2	-1	0	1	2	2.5	3
$f(x)$	-1	2.2	2.0	2.4	2.7	2.7	2.3
$g(x)$	1	-1.3	-1.2	-0.6	0.3	0.5	0.7
$f(x) + g(x)$	0	0.9	0.8	1.8	3.0	3.2	3.0



Extra values of x (like the value 2.5 in the table above) can be added as needed.

31. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g .

$$(f + g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1, \quad D = \mathbb{R}.$$

$$(f - g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1, \quad D = \mathbb{R}.$$

$$(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2, \quad D = \mathbb{R}.$$

$$\left(\frac{f}{g}\right)(x) = \frac{x^3 + 2x^2}{3x^2 - 1}, \quad D = \left\{x \mid x \neq \pm \frac{1}{\sqrt{3}}\right\} \text{ since } 3x^2 - 1 \neq 0.$$

32. $f(x) = \sqrt{1+x}$, $D = [-1, \infty)$; $g(x) = \sqrt{1-x}$, $D = (-\infty, 1]$.

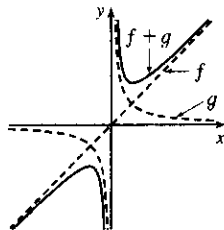
$$(f + g)(x) = \sqrt{1+x} + \sqrt{1-x}, \quad D = (-\infty, 1] \cap [-1, \infty) = [-1, 1].$$

$$(f - g)(x) = \sqrt{1+x} - \sqrt{1-x}, \quad D = [-1, 1].$$

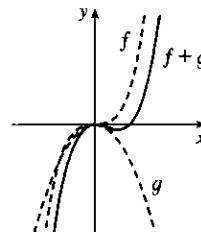
$$(fg)(x) = \sqrt{1+x} \cdot \sqrt{1-x} = \sqrt{1-x^2}, \quad D = [-1, 1].$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{1+x}}{\sqrt{1-x}}, \quad D = [-1, 1]. \text{ We must exclude } x = 1 \text{ since it would make } \frac{f}{g} \text{ undefined.}$$

33. $f(x) = x$, $g(x) = 1/x$



34. $f(x) = x^3$, $g(x) = -x^2$



35. $f(x) = 2x^2 - x$; $g(x) = 3x + 2$. $D = \mathbb{R}$ for both f and g , and hence for their composites.

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2)^2 - (3x + 2) = 2(9x^2 + 12x + 4) - 3x - 2 = 18x^2 + 21x + 6.$$

$$(g \circ f)(x) = g(f(x)) = g(2x^2 - x) = 3(2x^2 - x) + 2 = 6x^2 - 3x + 2.$$

$$(f \circ f)(x) = f(f(x)) = f(2x^2 - x) = 2(2x^2 - x)^2 - (2x^2 - x) = 2(4x^4 - 4x^3 + x^2) - 2x^2 + x = 8x^4 - 8x^3 + x.$$

$$(g \circ g)(x) = g(g(x)) = g(3x + 2) = 3(3x + 2) + 2 = 9x + 6 + 2 = 9x + 8.$$

36. $f(x) = 1 - x^3$, $D = \mathbb{R}$; $g(x) = 1/x$, $D = \{x \mid x \neq 0\}$.

$$(f \circ g)(x) = f(g(x)) = f(1/x) = 1 - (1/x)^3 = 1 - 1/x^3, \quad D = \{x \mid x \neq 0\}.$$

$$(g \circ f)(x) = g(f(x)) = g(1 - x^3) = 1/(1 - x^3), \quad D = \{x \mid 1 - x^3 \neq 0\} = \{x \mid x \neq 1\}.$$

$$(f \circ f)(x) = f(f(x)) = f(1 - x^3) = 1 - (1 - x^3)^3 = 1 - (1 - 3x^3 + 3x^6 - x^9) = x^9 - 3x^6 + 3x^3, \quad D = \mathbb{R}.$$

$$(g \circ g)(x) = g(g(x)) = g(1/x) = 1/(1/x) = x, \quad D = \{x \mid x \neq 0\} \text{ because } 0 \text{ is not in the domain of } g.$$

37. $f(x) = \sin x$, $D = \mathbb{R}$; $g(x) = 1 - \sqrt{x}$, $D = [0, \infty)$.

$$(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}) = \sin(1 - \sqrt{x}), \quad D = [0, \infty).$$

$$(g \circ f)(x) = g(f(x)) = g(\sin x) = 1 - \sqrt{\sin x}. \text{ For } \sqrt{\sin x} \text{ to be defined, we must have}$$

$$\sin x \geq 0 \Leftrightarrow x \in [0, \pi] \cup [2\pi, 3\pi] \cup [-2\pi, -\pi] \cup [4\pi, 5\pi] \cup [-4\pi, -3\pi] \cup \dots, \text{ so}$$

$$D = \{x \mid x \in [2n\pi, \pi + 2n\pi], \text{ where } n \text{ is an integer}\}.$$

$$(f \circ f)(x) = f(f(x)) = f(\sin x) = \sin(\sin x), \quad D = \mathbb{R}.$$

$$(g \circ g)(x) = g(g(x)) = g(1 - \sqrt{x}) = 1 - \sqrt{1 - \sqrt{x}},$$

$$D = \{x \geq 0 \mid 1 - \sqrt{x} \geq 0\} = \{x \geq 0 \mid 1 \geq \sqrt{x}\} = \{x \geq 0 \mid \sqrt{x} \leq 1\} = [0, 1].$$

38. $f(x) = 1 - 3x, \quad D = \mathbb{R}; \quad g(x) = 5x^2 + 3x + 2, \quad D = \mathbb{R}.$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(5x^2 + 3x + 2) = 1 - 3(5x^2 + 3x + 2) \\ &= 1 - 15x^2 - 9x - 6 = -15x^2 - 9x - 5, \quad D = \mathbb{R}.\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(1 - 3x) = 5(1 - 3x)^2 + 3(1 - 3x) + 2 = 5(1 - 6x + 9x^2) + 3 - 9x + 2 \\ &= 5 - 30x + 45x^2 - 9x + 5 = 45x^2 - 39x + 10, \quad D = \mathbb{R}.\end{aligned}$$

$$(f \circ f)(x) = f(f(x)) = f(1 - 3x) = 1 - 3(1 - 3x) = 1 - 3 + 9x = 9x - 2, \quad D = \mathbb{R}.$$

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) = g(5x^2 + 3x + 2) = 5(5x^2 + 3x + 2)^2 + 3(5x^2 + 3x + 2) + 2 \\ &= 5(25x^4 + 30x^3 + 29x^2 + 12x + 4) + 15x^2 + 9x + 6 + 2 \\ &= 125x^4 + 150x^3 + 145x^2 + 60x + 20 + 15x^2 + 9x + 8 \\ &= 125x^4 + 150x^3 + 160x^2 + 69x + 28, \quad D = \mathbb{R}.\end{aligned}$$

39. $f(x) = x + \frac{1}{x}, \quad D = \{x \mid x \neq 0\}; \quad g(x) = \frac{x+1}{x+2}, \quad D = \{x \mid x \neq -2\}.$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f\left(\frac{x+1}{x+2}\right) = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{x+1}{x+2} + \frac{x+2}{x+1} \\ &= \frac{(x+1)(x+1) + (x+2)(x+2)}{(x+2)(x+1)} = \frac{(x^2 + 2x + 1) + (x^2 + 4x + 4)}{(x+2)(x+1)} = \frac{2x^2 + 6x + 5}{(x+2)(x+1)}\end{aligned}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$, the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}.$

$$(g \circ f)(x) = g(f(x)) = g\left(x + \frac{1}{x}\right) = \frac{\left(x + \frac{1}{x}\right) + 1}{\left(x + \frac{1}{x}\right) + 2} = \frac{\frac{x^2 + 1 + x}{x}}{\frac{x^2 + 1 + 2x}{x}} = \frac{x^2 + x + 1}{x^2 + 2x + 1} = \frac{x^2 + x + 1}{(x+1)^2}.$$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$, the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}.$

$$\begin{aligned}(f \circ f)(x) &= f(f(x)) = f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right) + \frac{1}{x + \frac{1}{x}} = x + \frac{1}{x} + \frac{1}{\frac{x^2 + 1}{x}} = x + \frac{1}{x} + \frac{x}{x^2 + 1} \\ &= \frac{x(x)(x^2 + 1) + 1(x^2 + 1) + x(x)}{x(x^2 + 1)} = \frac{x^4 + x^2 + x^2 + 1 + x^2}{x(x^2 + 1)} \\ &= \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}, \quad D = \{x \mid x \neq 0\}.\end{aligned}$$

$$(g \circ g)(x) = g(g(x)) = g\left(\frac{x+1}{x+2}\right) = \frac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2} = \frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}} = \frac{x+1+x+2}{x+1+2x+4} = \frac{2x+3}{3x+5}.$$

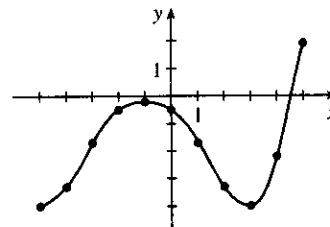
Since $g(x)$ is not defined for $x = -2$ and $g(g(x))$ is not defined for $x = -\frac{5}{3}$, the domain of $(g \circ g)(x)$ is $D = \{x \mid x \neq -2, -\frac{5}{3}\}.$

40. $f(x) = \sqrt{2x+3}$, $D = \{x \mid x \geq -\frac{3}{2}\}$; $g(x) = x^2 + 1$, $D = \mathbb{R}$.
 $(f \circ g)(x) = f(x^2 + 1) = \sqrt{2(x^2 + 1) + 3} = \sqrt{2x^2 + 5}$, $D = \mathbb{R}$.
 $(g \circ f)(x) = g(\sqrt{2x+3}) = (\sqrt{2x+3})^2 + 1 = (2x+3) + 1 = 2x+4$, $D = \{x \mid x \geq -\frac{3}{2}\}$.
 $(f \circ f)(x) = f(\sqrt{2x+3}) = \sqrt{2(\sqrt{2x+3}) + 3} = \sqrt{2\sqrt{2x+3} + 3}$, $D = \{x \mid x \geq -\frac{3}{2}\}$.
 $(g \circ g)(x) = g(x^2 + 1) = (x^2 + 1)^2 + 1 = (x^4 + 2x^2 + 1) + 1 = x^4 + 2x^2 + 2$, $D = \mathbb{R}$.
41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-1)) = f(2(x-1))$
 $= 2(x-1) + 1 = 2x-1$
42. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(1-x)) = f((1-x)^2)$
 $= 2(1-x)^2 - 1 = 2x^2 - 4x + 1$
43. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^2 + 2)$
 $= f(x^2 + 6x + 11) = \sqrt{(x^2 + 6x + 11) - 1} = \sqrt{x^2 + 6x + 10}$
44. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x+3})) = f(\cos \sqrt{x+3}) = \frac{2}{\cos \sqrt{x+3} + 1}$
45. Let $g(x) = x^2 + 1$ and $f(x) = x^{10}$. Then $(f \circ g)(x) = f(g(x)) = (x^2 + 1)^{10} = F(x)$.
46. Let $g(x) = \sqrt{x}$ and $f(x) = \sin x$. Then $(f \circ g)(x) = f(g(x)) = \sin(\sqrt{x}) = F(x)$.
47. Let $g(x) = x^2$ and $f(x) = \frac{x}{x+4}$. Then $(f \circ g)(x) = f(g(x)) = \frac{x^2}{x^2+4} = G(x)$.
48. Let $g(x) = x+3$ and $f(x) = 1/x$. Then $(f \circ g)(x) = f(g(x)) = 1/(x+3) = G(x)$.
49. Let $g(t) = \cos t$ and $f(t) = \sqrt{t}$. Then $(f \circ g)(t) = f(g(t)) = \sqrt{\cos t} = u(t)$.
50. Let $g(t) = \tan t$ and $f(t) = \frac{t}{1+t}$. Then $(f \circ g)(t) = f(g(t)) = \frac{\tan t}{1+\tan t} = u(t)$.
51. Let $h(x) = x^2$, $g(x) = 3^x$, and $f(x) = 1-x$. Then $(f \circ g \circ h)(x) = 1-3^{x^2} = H(x)$.
52. Let $h(x) = \sqrt{x}$, $g(x) = x-1$, and $f(x) = \sqrt[3]{x}$. Then $(f \circ g \circ h)(x) = \sqrt[3]{\sqrt{x}-1} = H(x)$.
53. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then $(f \circ g \circ h)(x) = (\sec \sqrt{x})^4 = \sec^4(\sqrt{x}) = H(x)$.
54. (a) $f(g(1)) = f(6) = 5$ (b) $g(f(1)) = g(3) = 2$
(c) $f(f(1)) = f(3) = 4$ (d) $g(g(1)) = g(6) = 3$
(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$ (f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$
55. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g . Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f .
(b) $g(f(0)) = g(0) = 3$
(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$
(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x -coordinate 6.
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$
(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

56. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

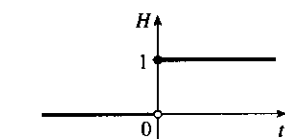
x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



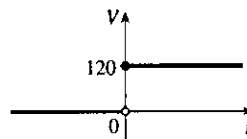
57. (a) Using the relationship $\text{distance} = \text{rate} \cdot \text{time}$ with the radius r as the distance, we have $r(t) = 60t$.
 (b) $A = \pi r^2 \Rightarrow (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t .
58. (a) $d = rt \Rightarrow d(t) = 350t$
 (b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s : $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.
 (c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

59. (a)



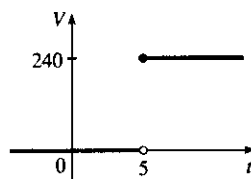
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

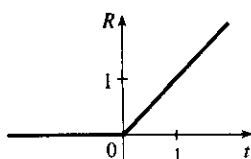
(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

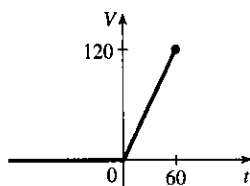
60. (a) $R(t) = tH(t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



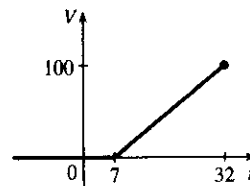
$$(b) \quad V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$$

so $V(t) = 2tH(t), t \leq 60$.



$$(c) \quad V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4(t - 7) & \text{if } 7 \leq t \leq 32 \end{cases}$$

so $V(t) = 4(t - 7)H(t - 7), t \leq 32$.



61. (a) By examining the variable terms in g and h , we deduce that we must square g to get the terms $4x^2$ and $4x$ in h .
 If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 + c = 4x^2 + 4x + (1 + c)$.
 Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But

$$h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5, \text{ so we see that } g(x) = x^2 + x - 1.$$

62. We need a function g so that $g(f(x)) = g(x + 4) = h(x) = 4x - 1 = 4(x + 4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

63. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad [\text{because } g \text{ is even}] = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

64. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

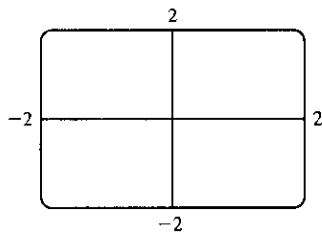
Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

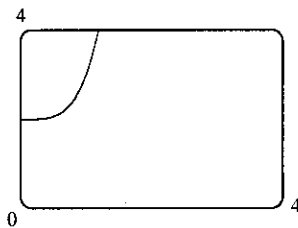
1.4 Graphing Calculators and Computers

1. $f(x) = x^4 + 2$

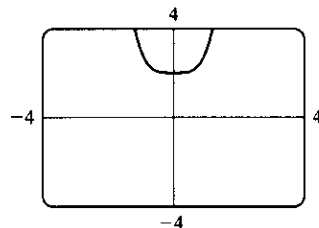
(a) $[-2, 2]$ by $[-2, 2]$



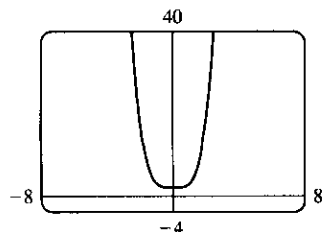
(b) $[0, 4]$ by $[0, 4]$



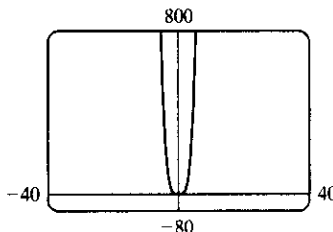
(c) $[-4, 4]$ by $[-4, 4]$



(d) $[-8, 8]$ by $[-4, 40]$



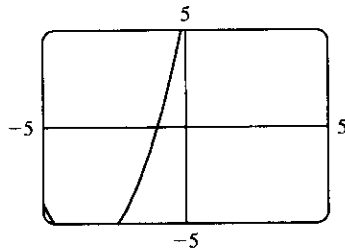
(e) $[-40, 40]$ by $[-80, 800]$



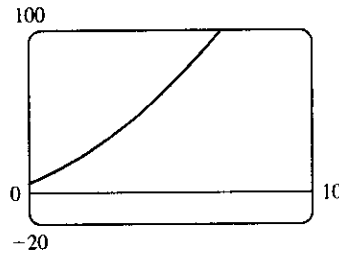
The most appropriate graph is produced in viewing rectangle (d).

2. $f(x) = x^2 + 7x + 6$

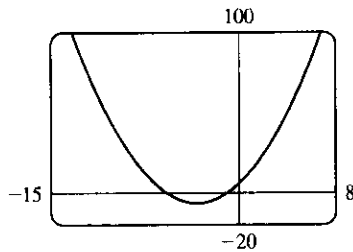
(a) $[-5, 5]$ by $[-5, 5]$



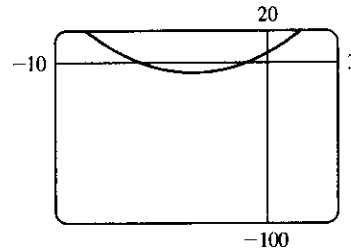
(b) $[0, 10]$ by $[-20, 100]$



(c) $[-15, 8]$ by $[-20, 100]$



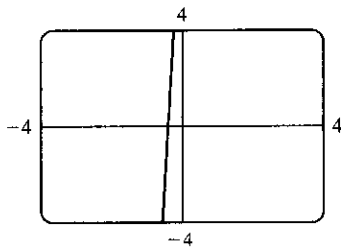
(d) $[-10, 3]$ by $[-100, 20]$



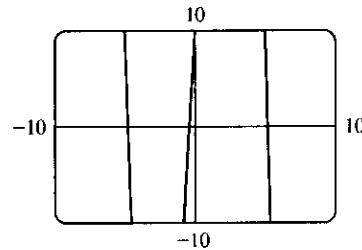
The most appropriate graph is produced in viewing rectangle (c).

3. $f(x) = 10 + 25x - x^3$

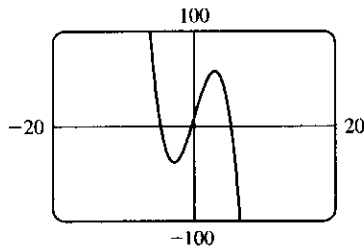
(a) $[-4, 4]$ by $[-4, 4]$



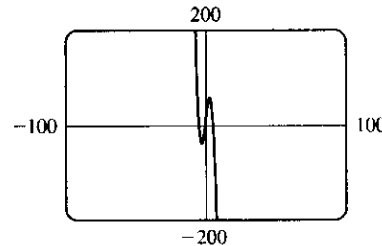
(b) $[-10, 10]$ by $[-10, 10]$



(c) $[-20, 20]$ by $[-100, 100]$



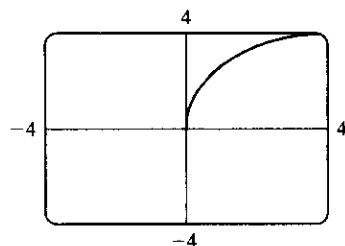
(d) $[-100, 100]$ by $[-200, 200]$



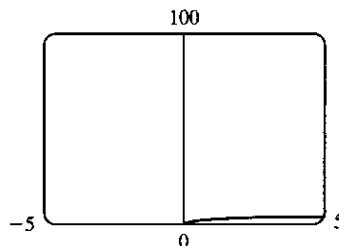
The most appropriate graph is produced in viewing rectangle (c) because the maximum and minimum points are fairly easy to see and estimate.

4. $f(x) = \sqrt{8x - x^2}$

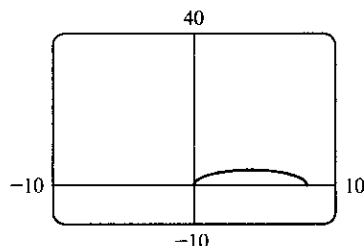
(a) $[-4, 4]$ by $[-4, 4]$



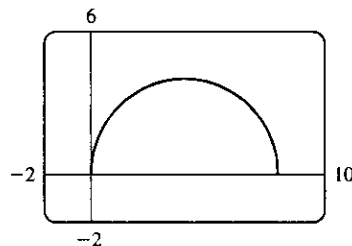
(b) $[-5, 5]$ by $[0, 100]$



(c) $[-10, 10]$ by $[-10, 40]$

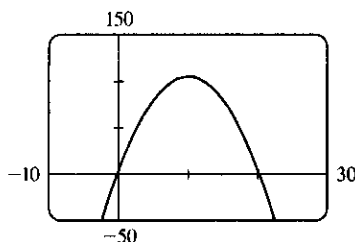


(d) $[-2, 10]$ by $[-2, 6]$

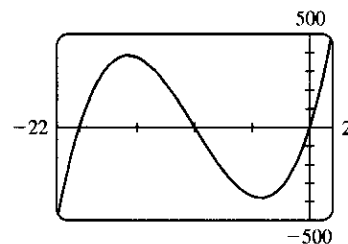


The most appropriate graph is produced in viewing rectangle (d).

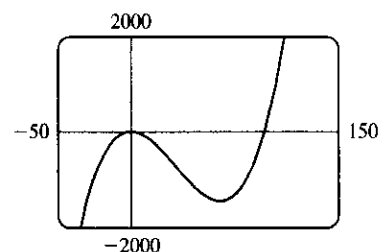
5. Since the graph of $f(x) = 5 + 20x - x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



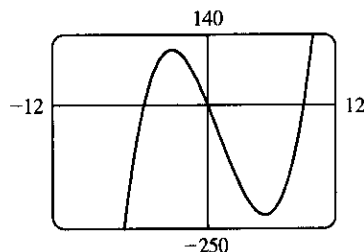
6. An appropriate viewing rectangle for $f(x) = x^3 + 30x^2 + 200x$ should include the high and low points.



7. $f(x) = 0.01x^3 - x^2 + 5$. Graphing f in a standard viewing rectangle, $[-10, 10]$ by $[-10, 10]$, shows us what appears to be a parabola. But since this is a cubic polynomial, we know that a larger viewing rectangle will reveal a minimum point as well as the maximum point. After some trial and error, we choose the viewing rectangle $[-50, 150]$ by $[-2000, 2000]$.



8. $f(x) = x(x+6)(x-9)$

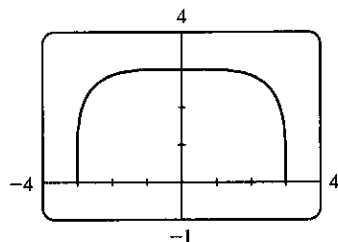


9. $f(x) = \sqrt[4]{81 - x^4}$ is defined when

$$81 - x^4 \geq 0 \Leftrightarrow x^4 \leq 81 \Leftrightarrow |x| \leq 3, \text{ so}$$

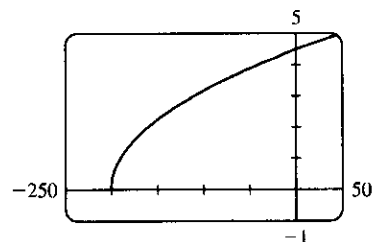
the domain of f is $[-3, 3]$. Also

$$0 \leq \sqrt[4]{81 - x^4} \leq \sqrt[4]{81} = 3, \text{ so the range is } [0, 3].$$

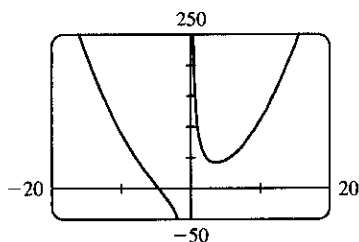


10. $f(x) = \sqrt{0.1x + 20}$ is defined when

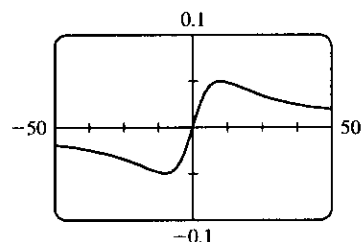
$$0.1x + 20 \geq 0 \Leftrightarrow x \geq -200, \text{ so the domain of } f \text{ is } [-200, \infty).$$



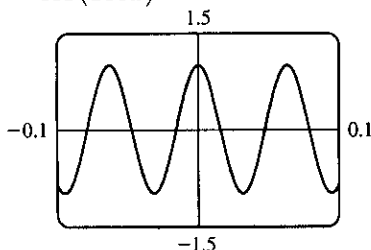
11. The graph of $f(x) = x^2 + (100/x)$ has a vertical asymptote of $x = 0$. As you zoom out, the graph of f looks more and more like that of $y = x^2$.



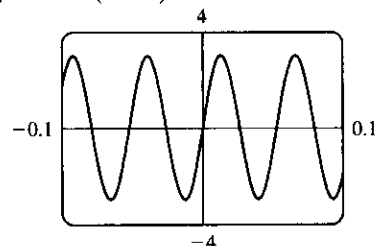
12. The graph of $f(x) = x/(x^2 + 100)$ is symmetric with respect to the origin.



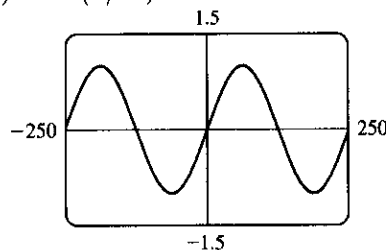
13. $f(x) = \cos(100x)$



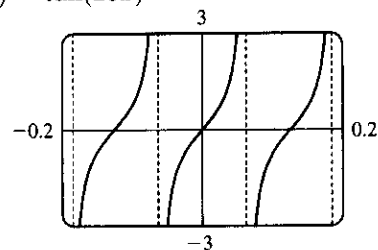
14. $f(x) = 3 \sin(120x)$



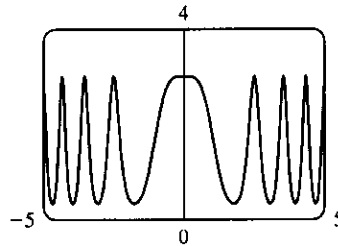
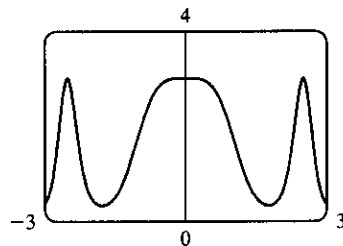
15. $f(x) = \sin(x/40)$



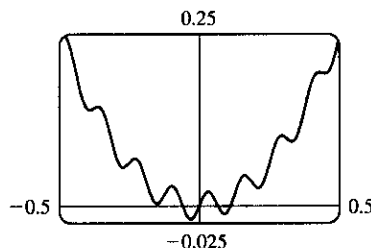
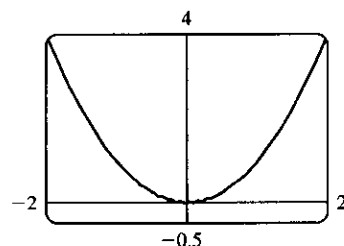
16. $f(x) = \tan(25x)$



17. $y = 3^{\cos(x^2)}$



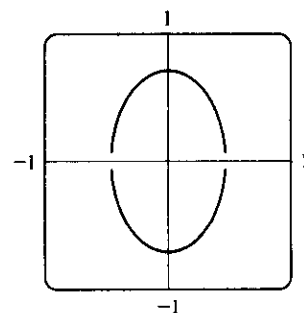
18. $y = x^2 + 0.02 \sin(50x)$



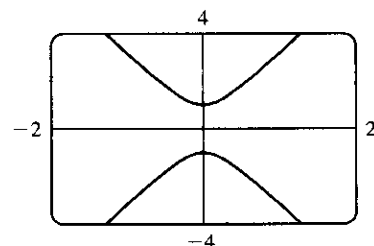
19. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

$$4x^2 + 2y^2 = 1 \Leftrightarrow 2y^2 = 1 - 4x^2 \Leftrightarrow y^2 = \frac{1 - 4x^2}{2}$$

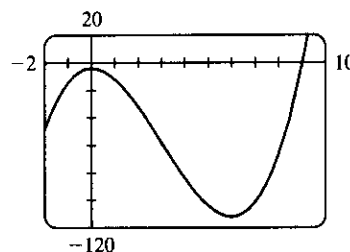
$$\Leftrightarrow y = \pm \sqrt{\frac{1 - 4x^2}{2}}$$



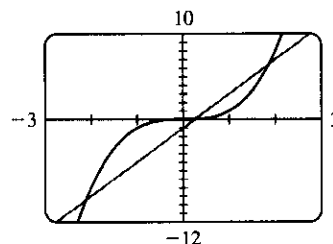
20. $y^2 - 9x^2 = 1 \Leftrightarrow y^2 = 1 + 9x^2 \Leftrightarrow y = \pm \sqrt{1 + 9x^2}$



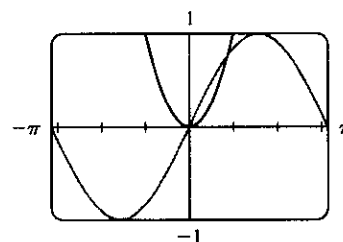
21. From the graph of $f(x) = x^3 - 9x^2 - 4$, we see that there is one solution of the equation $f(x) = 0$ and it is slightly larger than 9. By zooming in or using a root or zero feature, we obtain $x \approx 9.05$.



22. We see that the graphs of $f(x) = x^3$ and $g(x) = 4x - 1$ intersect three times. The x -coordinates of these points (which are the solutions of the equation) are approximately -2.11 , 0.25 , and 1.86 . Alternatively, we could find these values by finding the zeros of $h(x) = x^3 - 4x + 1$.

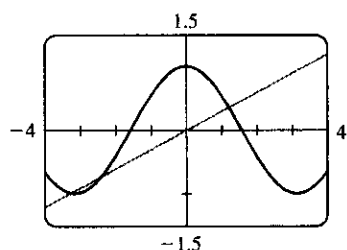


23. We see that the graphs of $f(x) = x^2$ and $g(x) = \sin x$ intersect twice. One solution is $x = 0$. The other solution of $f = g$ is the x -coordinate of the point of intersection in the first quadrant. Using an intersect feature or zooming in, we find this value to be approximately 0.88. Alternatively, we could find that value by finding the positive zero of $h(x) = x^2 - \sin x$.



Note: After producing the graph on a TI-83 Plus, we can find the approximate value 0.88 by using the following keystrokes: **2nd** **CALC** **5** **ENTER** **ENTER** **1** **ENTER**. The “1” is just a guess for 0.88.

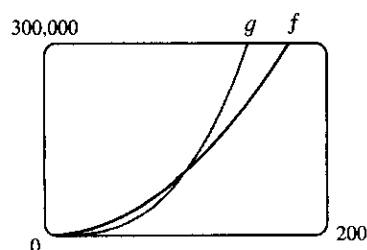
24. (a)



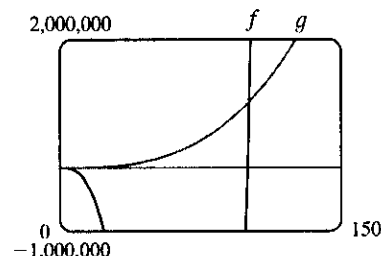
The x -coordinates of the three points of intersection are $x \approx -3.29, -2.36$ and 1.20 .

(b) Using trial and error, we find that $m \approx 0.3365$. Note that m could also be negative.

25. $g(x) = x^3/10$ is larger than $f(x) = 10x^2$ whenever $x > 100$.

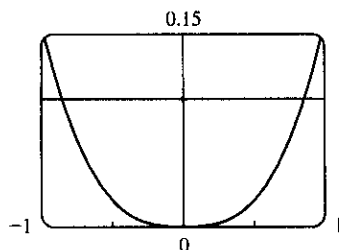


26. $f(x) = x^4 - 100x^3$ is larger than $g(x) = x^3$ whenever $x > 101$.



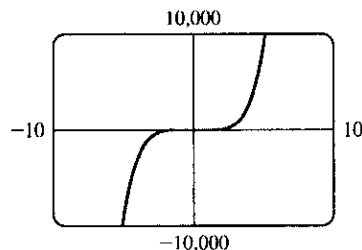
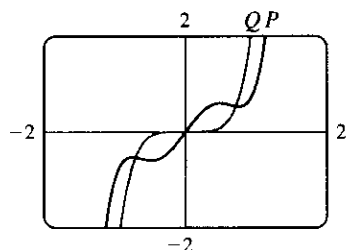
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27.



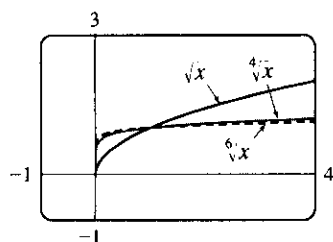
We see from the graphs of $y = |\sin x - x|$ and $y = 0.1$ that there are two solutions to the equation $|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values.

28.

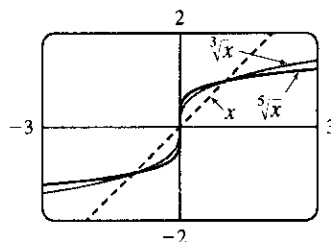


$P(x) = 3x^5 - 5x^3 + 2x$,
 $Q(x) = 3x^5$. These graphs are significantly different only in the region close to the origin. The larger a viewing rectangle one chooses, the more similar the two graphs look.

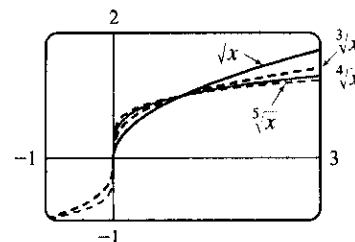
29. (a) The root functions $y = \sqrt{x}$,
 $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$,
 $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$

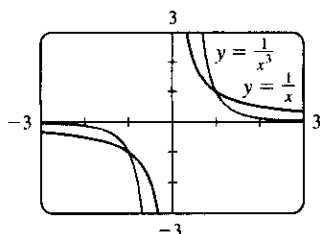


(c) The root functions $y = \sqrt{x}$,
 $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$

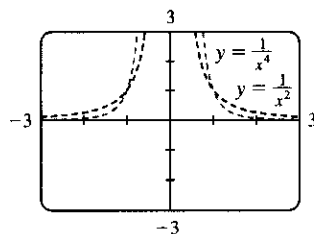


- (d) • For any n , the n th root of 0 is 0 and the n th root of 1 is 1; that is, all n th root functions pass through the points $(0, 0)$ and $(1, 1)$.
- For odd n , the domain of the n th root function is $\mathbb{R}$, while for even n , it is $\{x \in \mathbb{R} \mid x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.
- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

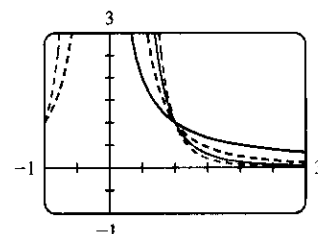
30. (a) The functions $y = 1/x$ and
 $y = 1/x^3$



(b) The functions $y = 1/x^2$
and $y = 1/x^4$

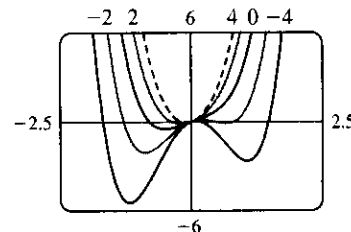


(c) The functions $y = 1/x$, $y = 1/x^2$,
 $y = 1/x^3$ and $y = 1/x^4$

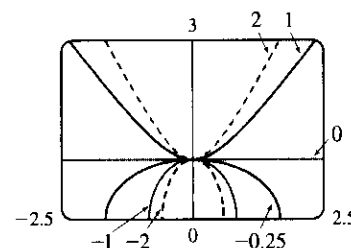


- (d) • The graphs of all functions of the form $y = 1/x^n$ pass through the point $(1, 1)$.
- If n is even, the graph of the function is entirely above the x -axis. The graphs of $1/x^n$ for n even are similar to one another.
 - If n is odd, the function is positive for positive x and negative for negative x . The graphs of $1/x^n$ for n odd are similar to one another.
 - As n increases, the graphs of $1/x^n$ approach 0 faster as $x \rightarrow \infty$.

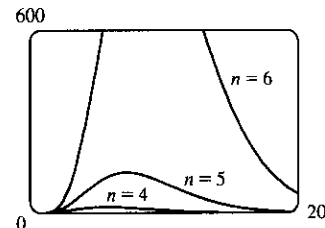
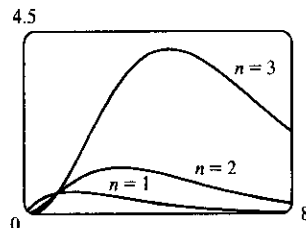
31. $f(x) = x^4 + cx^2 + x$. If $c < 0$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c = 0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.



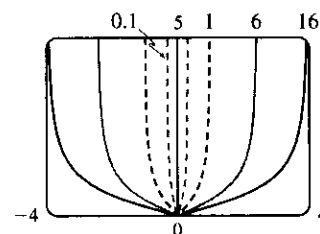
32. $f(x) = \sqrt{1 + cx^2}$. If $c < 0$, the function is only defined on $[-1/\sqrt{-c}, 1/\sqrt{-c}]$, and its graph is the top half of an ellipse. If $c = 0$, the graph is the line $y = 1$. If $c > 0$, the graph is the top half of a hyperbola. As c approaches 0, these curves become flatter and approach the line $y = 1$.



33. $y = x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n , the function approaches 0 as $x \rightarrow \infty$.

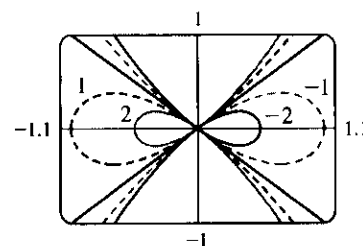


34. $y = \frac{|x|}{\sqrt{c - x^2}}$. The “bullet” becomes broader as c increases.



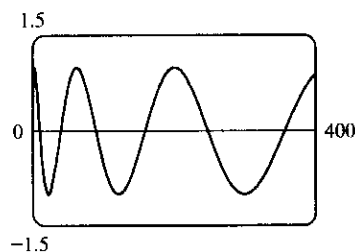
35. $y^2 = cx^3 + x^2$

If $c < 0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0, the larger the loop is. (In the limiting case, $c = 0$, the loop is “infinite”, that is, it doesn’t close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



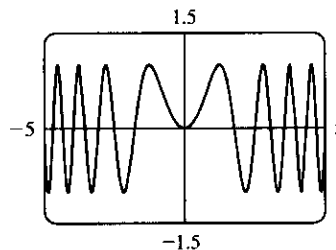
36. (a) $y = \sin(\sqrt{x})$

This function is not periodic; it oscillates less frequently as x increases.



(b) $y = \sin(x^2)$

This function oscillates more frequently as $|x|$ increases. Note also that this function is even, whereas $\sin x$ is odd.



37. The graphing window is 95 pixels wide and we want to start with $x = 0$ and end with $x = 2\pi$. Since there are 94 “gaps” between pixels, the distance between pixels is $\frac{2\pi-0}{94}$. Thus, the x -values that the calculator actually plots are $x = 0 + \frac{2\pi}{94} \cdot n$, where $n = 0, 1, 2, \dots, 93, 94$. For $y = \sin 2x$, the actual points plotted by the calculator are $(\frac{2\pi}{94} \cdot n, \sin(2 \cdot \frac{2\pi}{94} \cdot n))$ for $n = 0, 1, \dots, 94$. For $y = \sin 96x$, the points plotted are $(\frac{2\pi}{94} \cdot n, \sin(96 \cdot \frac{2\pi}{94} \cdot n))$ for $n = 0, 1, \dots, 94$. But

$$\begin{aligned}\sin(96 \cdot \frac{2\pi}{94} \cdot n) &= \sin(94 \cdot \frac{2\pi}{94} \cdot n + 2 \cdot \frac{2\pi}{94} \cdot n) = \sin(2\pi n + 2 \cdot \frac{2\pi}{94} \cdot n) \\ &= \sin(2 \cdot \frac{2\pi}{94} \cdot n) \quad [\text{by periodicity of sine}], \quad n = 0, 1, \dots, 94\end{aligned}$$

So the y -values, and hence the points, plotted for $y = \sin 96x$ are identical to those plotted for $y = \sin 2x$.

Note: Try graphing $y = \sin 94x$. Can you see why all the y -values are zero?

38. As in Exercise 37, we know that the points being plotted for $y = \sin 45x$ are $(\frac{2\pi}{94} \cdot n, \sin(45 \cdot \frac{2\pi}{94} \cdot n))$ for $n = 0, 1, \dots, 94$. But

$$\begin{aligned}\sin(45 \cdot \frac{2\pi}{94} \cdot n) &= \sin(47 \cdot \frac{2\pi}{94} \cdot n - 2 \cdot \frac{2\pi}{94} \cdot n) = \sin(n\pi - 2 \cdot \frac{2\pi}{94} \cdot n) \\ &= \sin(n\pi) \cos(2 \cdot \frac{2\pi}{94} \cdot n) - \cos(n\pi) \sin(2 \cdot \frac{2\pi}{94} \cdot n) \quad [\text{Subtraction formula for the sine}] \\ &= 0 \cdot \cos(2 \cdot \frac{2\pi}{94} \cdot n) - (\pm 1) \sin(2 \cdot \frac{2\pi}{94} \cdot n) = \pm \sin(2 \cdot \frac{2\pi}{94} \cdot n), \quad n = 0, 1, \dots, 94\end{aligned}$$

So the y -values, and hence the points, plotted for $y = \sin 45x$ lie on either $y = \sin 2x$ or $y = -\sin 2x$.

1 Review

CONCEPT CHECK

- (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B . The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
(b) If f is a function with domain A , then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.
(c) Use the Vertical Line Test on page 17.
- The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

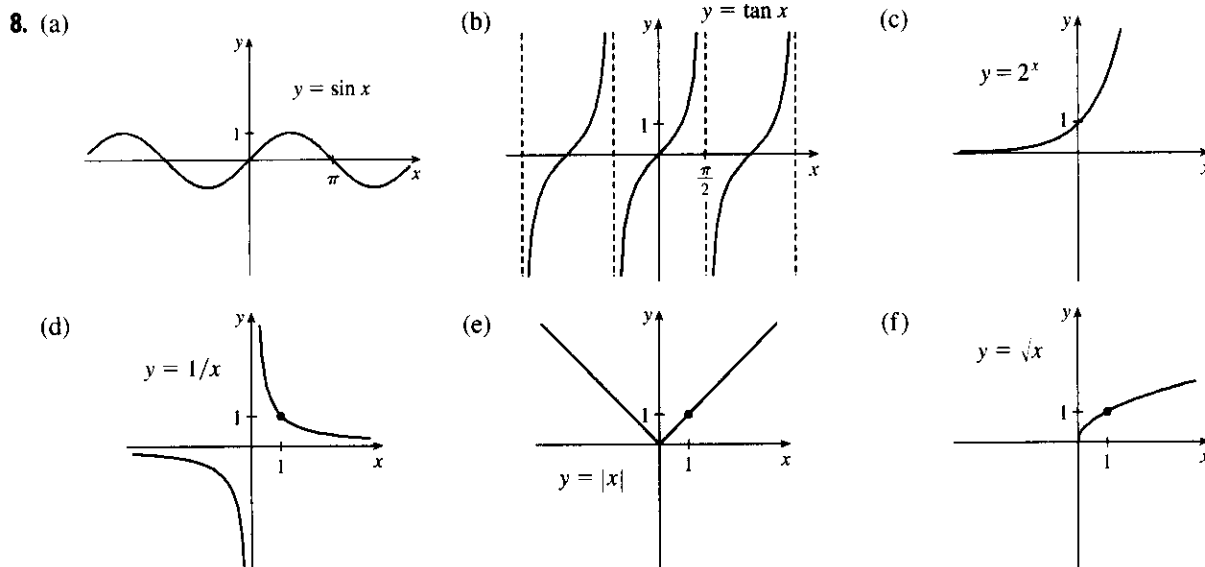
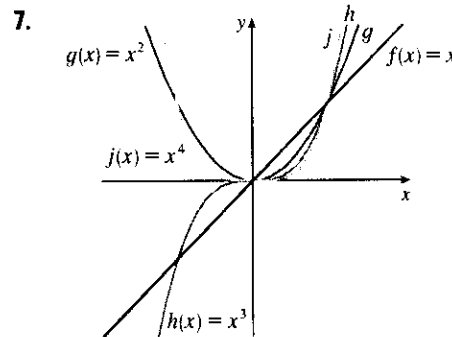
Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) An **even function** f satisfies $f(-x) = f(x)$ for every number x in its domain. It is symmetric with respect to the y -axis.
- (b) An **odd function** g satisfies $g(-x) = -g(x)$ for every number x in its domain. It is symmetric with respect to the origin.
4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I .
5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.
6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$
- (b) Power function: $f(x) = x^2$, $f(x) = x^a$
- (c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$
- (d) Quadratic function: $f(x) = x^2 + x + 1$,
 $f(x) = ax^2 + bx + c$
- (e) Polynomial of degree 5: $f(x) = x^5 + 2$
- (f) Rational function: $f(x) = \frac{x}{x+2}$, $f(x) = \frac{P(x)}{Q(x)}$ where
 $P(x)$ and $Q(x)$ are polynomials



9. (a) The domain of $f + g$ is the intersection of the domain of f and the domain of g ; that is, $A \cap B$.
- (b) The domain of fg is also $A \cap B$.
- (c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.
10. Given two functions f and g , the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f .
11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.
- (b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.
- (c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.
- (d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

- (e) If the graph of f is reflected about the x -axis, its equation becomes $y = -f(x)$.
 (f) If the graph of f is reflected about the y -axis, its equation becomes $y = f(-x)$.
 (g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.
 (h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.
 (i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f(\frac{1}{2}x)$.
 (j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

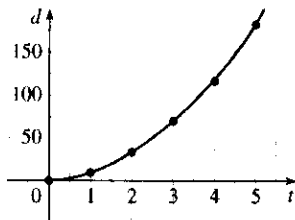
TRUE-FALSE QUIZ

- False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s + t) = (-1 + 1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s + t)$.
- False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.
- False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.
- True. If $x_1 < x_2$ and f is a decreasing function, then the y -values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.
- True. See the Vertical Line Test.
- False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

EXERCISES

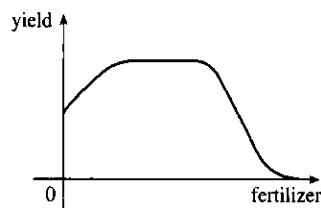
- When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.
 - $f(x) = 3 \Rightarrow x \approx 2.3, 5.6$
 - The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.
 - The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.
 - f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.
 - f is odd since its graph is symmetric about the origin.
- This curve is *not* the graph of a function of x since it *fails* the Vertical Line Test.
 - This curve *is* the graph of a function of x since it *passes* the Vertical Line Test. The domain is $[-3, 3]$ and the range is $[-2, 3]$.

3. (a)



- (b) From the graph, we see that the distance traveled after 4.5 seconds is slightly less than 150 feet.

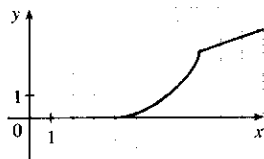
4.



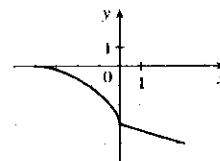
There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.

5. $f(x) = \sqrt{4 - 3x^2}$. Domain: $4 - 3x^2 \geq 0 \Rightarrow 3x^2 \leq 4 \Rightarrow x^2 \leq \frac{4}{3} \Rightarrow |x| \leq \frac{2}{\sqrt{3}}$. Range: $y \geq 0$ and $y \leq \sqrt{4} \Rightarrow 0 \leq y \leq 2$.
6. $g(x) = \frac{1}{x+1}$. Domain: $x+1 \neq 0 \Rightarrow x \neq -1$. Range: all reals except 0 ($y = 0$ is the horizontal asymptote for g .)
7. $y = 1 + \sin x$. Domain: $\mathbb{R}$. Range: $-1 \leq \sin x \leq 1 \Rightarrow 0 \leq 1 + \sin x \leq 2 \Rightarrow 0 \leq y \leq 2$.
8. $y = \tan 2x$. Domain: $2x \neq \frac{\pi}{2} + \pi n \Rightarrow x \neq \frac{\pi}{4} + \frac{\pi}{2}n$. Range: the tangent function takes on all real values, so the range is $\mathbb{R}$.
9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.
 (b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.
 (c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.
 (d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the “-2” inside the parentheses), and then shift the resulting graph 2 units downward.
 (e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x -axis.
 (f) To obtain the graph of $y = 3 - f(x)$, we reflect the graph of $y = f(x)$ about the x -axis, and then shift the resulting graph 3 units upward.

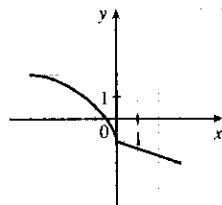
10. (a) To obtain the graph of $y = f(x - 8)$, we shift the graph of $y = f(x)$ right 8 units.



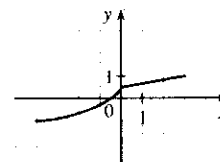
- (b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x -axis.



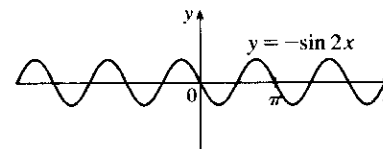
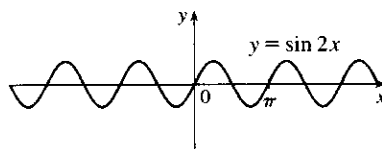
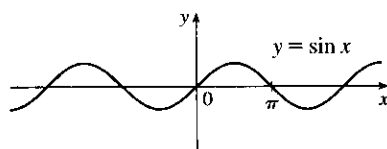
- (c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x -axis, and then shift the resulting graph 2 units upward.



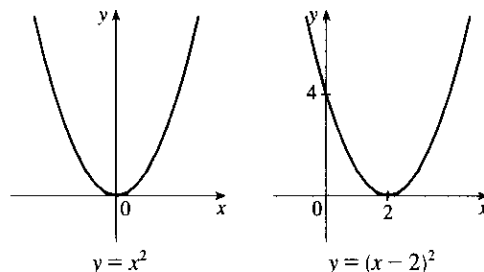
- (d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



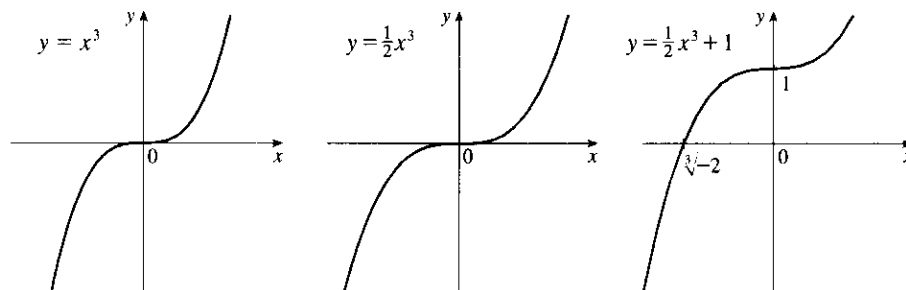
11. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x -axis.



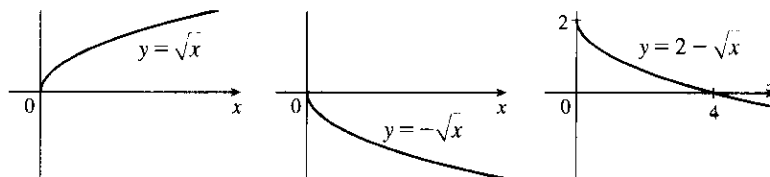
12. $y = (x - 2)^2$: Start with the graph of $y = x^2$ and shift 2 units to the right.



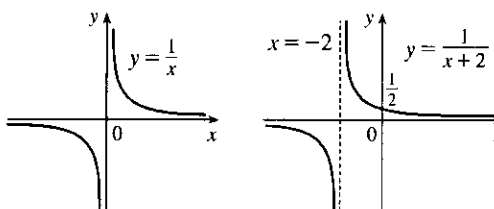
13. $y = 1 + \frac{1}{2}x^3$: Start with the graph of $y = x^3$, compress vertically by a factor of 2, and shift 1 unit upward.



14. $y = 2 - \sqrt{x}$: Start with the graph of $y = \sqrt{x}$, reflect about the x -axis, and shift 2 units upward.



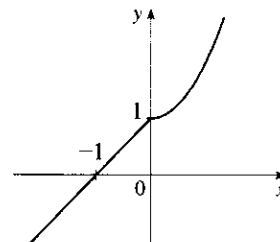
15. $f(x) = \frac{1}{x+2}$: Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



$$16. f(x) = \begin{cases} 1+x & \text{if } x < 0 \\ 1+x^2 & \text{if } x \geq 0 \end{cases}$$

On $(-\infty, 0)$, graph $y = 1 + x$ (the line with slope 1 and y -intercept 1) with open endpoint $(0, 1)$.

On $[0, \infty)$, graph $y = 1 + x^2$ (the rightmost half of the parabola $y = x^2$ shifted 1 unit upward) with closed endpoint $(0, 1)$.



17. (a) The terms of f are a mixture of odd and even powers of x , so f is neither even nor odd.
 (b) The terms of f are all odd powers of x , so f is odd.
 (c) $f(-x) = \cos((-x)^2) = \cos(x^2) = f(x)$, so f is even.
 (d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

18. For the line segment from $(-2, 2)$ to $(-1, 0)$, the slope is $\frac{0-2}{-1+2} = -2$, and an equation is $y - 0 = -2(x + 1)$ or, equivalently, $y = -2x - 2$. The circle has equation $x^2 + y^2 = 1$; the top half has equation $y = \sqrt{1 - x^2}$ (we have solved for positive y .) Thus, $f(x) = \begin{cases} -2x - 2 & \text{if } -2 \leq x \leq -1 \\ \sqrt{1 - x^2} & \text{if } -1 < x \leq 1 \end{cases}$

19. $f(x) = \sqrt{x}$, $D = [0, \infty)$; $g(x) = \sin x$, $D = \mathbb{R}$.

$(f \circ g)(x) = f(g(x)) = f(\sin x) = \sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have $\sin x \geq 0 \Leftrightarrow x \in [0, \pi], [2\pi, 3\pi], [-2\pi, -\pi], [4\pi, 5\pi], [-4\pi, -3\pi], \dots$, so $D = \{x \mid x \in [2n\pi, \pi + 2n\pi], \text{ where } n \text{ is an integer}\}$.

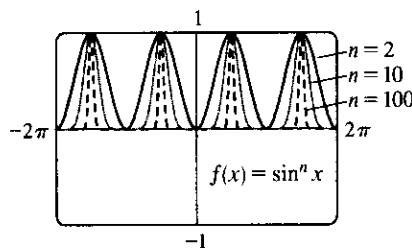
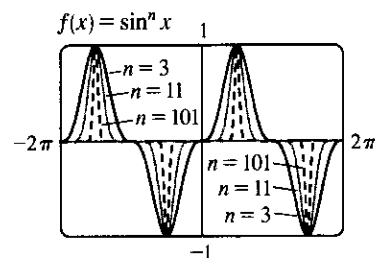
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sin \sqrt{x}$. x must be greater than or equal to 0 for $\sqrt{x}$ to be defined, so $D = [0, \infty)$.

$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. $D = [0, \infty)$.

$(g \circ g)(x) = g(g(x)) = g(\sin x) = \sin(\sin x)$. $D = \mathbb{R}$.

20. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = F(x)$.

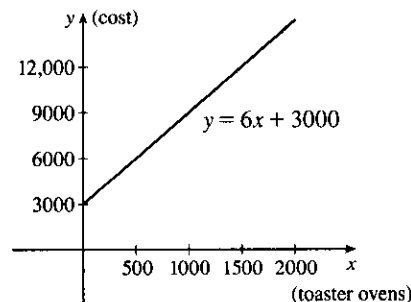
21. The graphs of $f(x) = \sin^n x$, where n is a positive integer, all have domain $\mathbb{R}$. For odd n , the range is $[-1, 1]$ and for even n , the range is $[0, 1]$. For odd n , the functions are odd and symmetric with respect to the origin. For even n , the functions are even and symmetric with respect to the y -axis. As n becomes large, the graphs become less rounded and more "spiky."



22. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points $(1000, 9000)$ and $(1500, 12,000)$, we get an equation of a

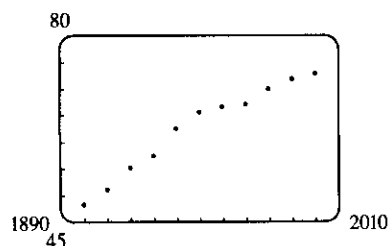
$$\text{line: } y - 9000 = \frac{12,000 - 9000}{1500 - 1000} (x - 1000) \Rightarrow$$

$$y = 6(x - 1000) + 9000 \Rightarrow y = 6x + 3000.$$



- (b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.
(c) The y -intercept of 3000 represents the overhead cost—the cost incurred without producing anything.

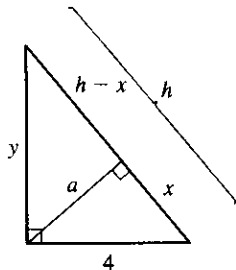
- 23.



Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$ gives us an estimate of 77.6 years for the year 2010.

□ PRINCIPLES OF PROBLEM SOLVING

1.

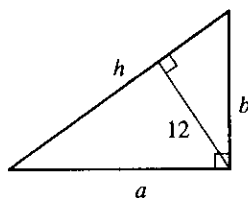


By using the area formula for a triangle, $\frac{1}{2}(\text{base})(\text{height})$, in two ways,

we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \frac{4y}{h}$. Since $4^2 + y^2 = h^2$,

$$y = \sqrt{h^2 - 16}, \text{ and } a = \frac{4\sqrt{h^2 - 16}}{h}.$$

2.



Refer to Example 1, where we obtained $h = \frac{P^2 - 100}{2P}$. The 100 came from 4 times the area of the triangle.

In this case, the area of the triangle is $\frac{1}{2}(h)(12) = 6h$. Thus, $h = \frac{P^2 - 4(6h)}{2P} \Rightarrow 2Ph = P^2 - 24h \Rightarrow$

$$2Ph + 24h = P^2 \Rightarrow h(2P + 24) = P^2 \Rightarrow h = \frac{P^2}{2P + 24}.$$

$$3. |2x - 1| = \begin{cases} 2x - 1 & \text{if } x \geq \frac{1}{2} \\ 1 - 2x & \text{if } x < \frac{1}{2} \end{cases} \quad \text{and} \quad |x + 5| = \begin{cases} x + 5 & \text{if } x \geq -5 \\ -x - 5 & \text{if } x < -5 \end{cases}$$

Therefore, we consider the three cases $x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x - 5) = 3 \Leftrightarrow x = 3$, which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1 - 2x - (x + 5) = 3 \Leftrightarrow x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x - 1 - (x + 5) = 3 \Leftrightarrow x = 9$.

So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

$$4. |x - 1| = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases} \quad \text{and} \quad |x - 3| = \begin{cases} x - 3 & \text{if } x \geq 3 \\ 3 - x & \text{if } x < 3 \end{cases}$$

Therefore, we consider the three cases $x < 1$, $1 \leq x < 3$, and $x \geq 3$.

If $x < 1$, we must have $1 - x - (3 - x) \geq 5 \Leftrightarrow 0 \geq 7$, which is false.

If $1 \leq x < 3$, we must have $x - 1 - (3 - x) \geq 5 \Leftrightarrow x \geq \frac{9}{2}$, which is false because $x < 3$.

If $x \geq 3$, we must have $x - 1 - (x - 3) \geq 5 \Leftrightarrow 2 \geq 5$, which is false.

All three cases lead to falsehoods, so the inequality has no solution.

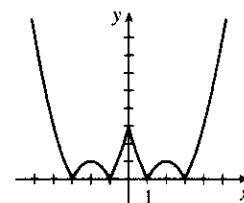
5. $f(x) = |x^2 - 4|x| + 3|$. If $x \geq 0$, then $f(x) = |x^2 - 4x + 3| = |(x-1)(x-3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

Case (ii): If $1 < x \leq 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.

This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y -axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.



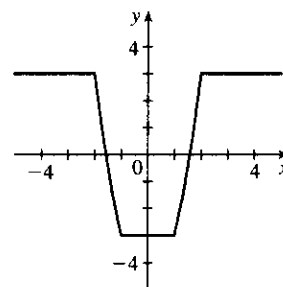
6. $g(x) = |x^2 - 1| - |x^2 - 4|$.

$$|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } |x| \geq 1 \\ 1 - x^2 & \text{if } |x| < 1 \end{cases} \quad \text{and} \quad |x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } |x| \geq 2 \\ 4 - x^2 & \text{if } |x| < 2 \end{cases}$$

So for $0 \leq |x| < 1$, $g(x) = 1 - x^2 - (4 - x^2) = -3$, for

$1 \leq |x| < 2$, $g(x) = x^2 - 1 - (4 - x^2) = 2x^2 - 5$, and for

$|x| \geq 2$, $g(x) = x^2 - 1 - (x^2 - 4) = 3$.



7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

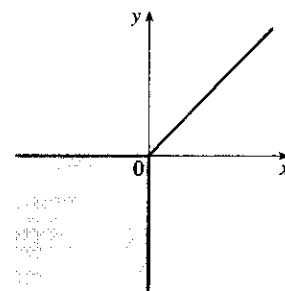
(1) $\frac{x \geq 0, y \geq 0}{2x = 2y}$	(2) $\frac{x \geq 0, y < 0}{2x = 0}$	(3) $\frac{x < 0, y \geq 0}{0 = 2y}$	(4) $\frac{x < 0, y < 0}{0 = 0}$
$x = y$	$x = 0$	$0 = y$	

Case 1 gives us the line $y = x$ with nonnegative x and y .

Case 2 gives us the portion of the y -axis with y negative.

Case 3 gives us the portion of the x -axis with x negative.

Case 4 gives us the entire third quadrant.

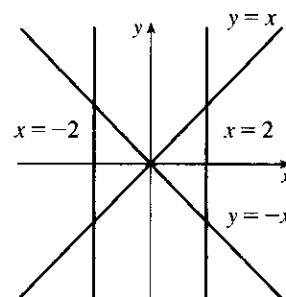


8. $x^4 - 4x^2 - x^2y^2 + 4y^2 = 0 \Leftrightarrow x^2(x^2 - 4) - y^2(x^2 - 4) = 0 \Leftrightarrow$

$$(x^2 - y^2)(x^2 - 4) = 0 \Leftrightarrow (x + y)(x - y)(x + 2)(x - 2) = 0.$$

So the graph of the equation consists of the graphs of the four lines

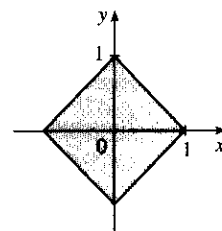
$y = -x$, $y = x$, $x = -2$, and $x = 2$.



9. $|x| + |y| \leq 1$. The boundary of the region has equation $|x| + |y| = 1$.

In quadrants I, II, III, and IV, this becomes the lines $x + y = 1$,

$-x + y = 1$, $-x - y = 1$, and $x - y = 1$ respectively.



10. $|x - y| + |x| - |y| \leq 2$

$$\text{Case (i): } x > y > 0 \quad \Leftrightarrow \quad x - y + x - y \leq 2 \quad \Leftrightarrow \quad x - y \leq 1 \quad \Leftrightarrow \quad y \geq x - 1$$

$$\text{Case (ii): } y > x > 0 \quad \Leftrightarrow \quad y - x + x - y \leq 2 \quad \Leftrightarrow \quad 0 \leq 2 \text{ (true)}$$

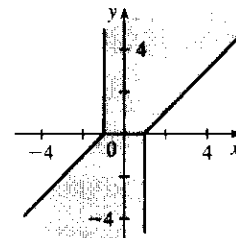
$$\text{Case (iii): } x > 0 \text{ and } y < 0 \quad \Leftrightarrow \quad x - y + x + y \leq 2 \quad \Leftrightarrow \quad 2x \leq 2 \quad \Leftrightarrow \quad x \leq 1$$

$$\text{Case (iv): } x < 0 \text{ and } y > 0 \quad \Leftrightarrow \quad y - x - x - y \leq 2 \quad \Leftrightarrow \quad -2x \leq 2 \quad \Leftrightarrow \quad x \geq -1$$

$$\text{Case (v): } y < x < 0 \quad \Leftrightarrow \quad x - y - x + y \leq 2 \quad \Leftrightarrow \quad 0 \leq 2 \text{ (true)}$$

$$\text{Case (vi): } x < y < 0 \quad \Leftrightarrow \quad y - x - x + y \leq 2 \quad \Leftrightarrow \quad y - x \leq 1 \quad \Leftrightarrow \quad y \leq x + 1$$

Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through 180° about the origin.



11. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip.

For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed

$$\text{for the entire trip is } \frac{\text{total distance}}{\text{total time}} = \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{30} + \frac{d}{60}} = \frac{2d}{\frac{2d}{60}} = \frac{120d}{2d + d} = \frac{120d}{3d} = 40. \text{ The average speed for}$$

the entire trip is 40 mi/h.

12. Let $f = \sin$, $g = x$, and $h = x$. Then the left-hand side of the equation is

$$f \circ (g + h) = \sin(x + x) = \sin 2x = 2 \sin x \cos x; \text{ and the right-hand side is}$$

$$f \circ g + f \circ h = \sin x + \sin x = 2 \sin x. \text{ The two sides are not equal, so the given statement is false.}$$

13. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m . Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n .

14. Let S_n be the statement that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

- S_1 is true because $[2(1) - 1] = 1 = 1^2$.
- Assume S_k is true, that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then

$$1 + 3 + 5 + \cdots + (2k - 1) + [(2k + 1) - 1] = 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\ = k^2 + (2k + 1) = (k + 1)^2$$

which shows that S_{k+1} is true.

- Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n .

15. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \dots$

$$f_1(x) = f_0(f_0(x)) = f_0(x^2) = (x^2)^2 = x^4, f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8,$$

$$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \dots \text{Thus, a general formula is } f_n(x) = x^{2^{n+1}}.$$

16. (a) $f_0(x) = 1/(2 - x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \dots$

$$f_1(x) = f_0\left(\frac{1}{2-x}\right) = \frac{1}{2 - \frac{1}{2-x}} = \frac{2-x}{2(2-x)-1} = \frac{2-x}{3-2x},$$

$$f_2(x) = f_0\left(\frac{2-x}{3-2x}\right) = \frac{1}{2 - \frac{2-x}{3-2x}} = \frac{3-2x}{2(3-2x)-(2-x)} = \frac{3-2x}{4-3x},$$

$$f_3(x) = f_0\left(\frac{3-2x}{4-3x}\right) = \frac{1}{2 - \frac{3-2x}{4-3x}} = \frac{4-3x}{2(4-3x)-(3-2x)} = \frac{4-3x}{5-4x}, \dots$$

$$\text{Thus, we conjecture that the general formula is } f_n(x) = \frac{n+1-nx}{n+2-(n+1)x}.$$

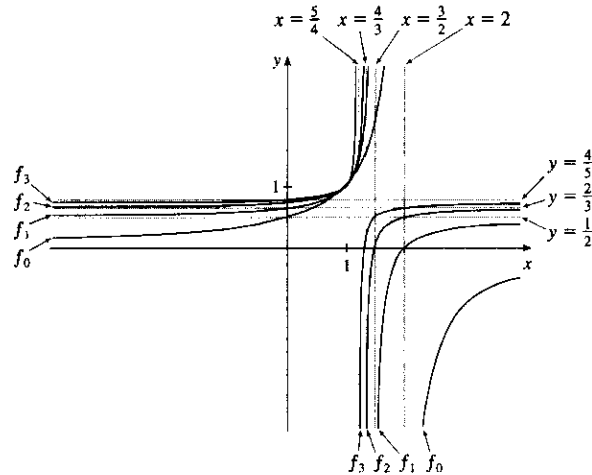
To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for

$$n = 1. \text{ Assume that the formula is true for } n = k; \text{ that is, } f_k(x) = \frac{k+1-kx}{k+2-(k+1)x}. \text{ Then}$$

$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k+1-kx}{k+2-(k+1)x}\right) = \frac{1}{2 - \frac{k+1-kx}{k+2-(k+1)x}} \\ = \frac{k+2-(k+1)x}{2[k+2-(k+1)x] - (k+1-kx)} = \frac{k+2-(k+1)x}{k+3-(k+2)x}$$

This shows that the formula for f_n is true for $n = k + 1$. Therefore, by mathematical induction, the formula is true for all positive integers n .

(b)

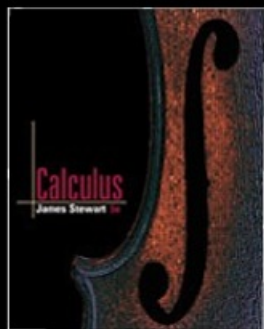


From the graph, we can make several observations:

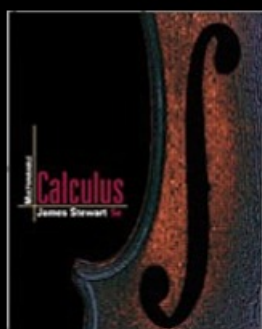
- The values at $x = a$ keep increasing as k increases.
- The vertical asymptote gets closer to $x = 1$ as k increases.
- The horizontal asymptote gets closer to $y = 1$ as k increases.
- The x -intercept for f_{k+1} is the value of the vertical asymptote for f_k .
- The y -intercept for f_k is the value of the horizontal asymptote for f_{k+1} .

Chapter 2

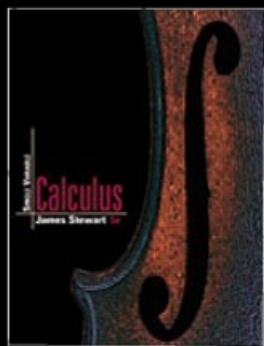
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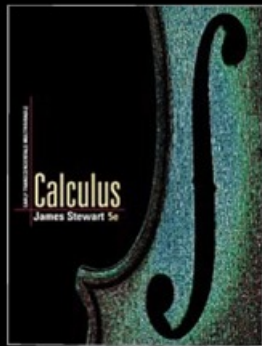
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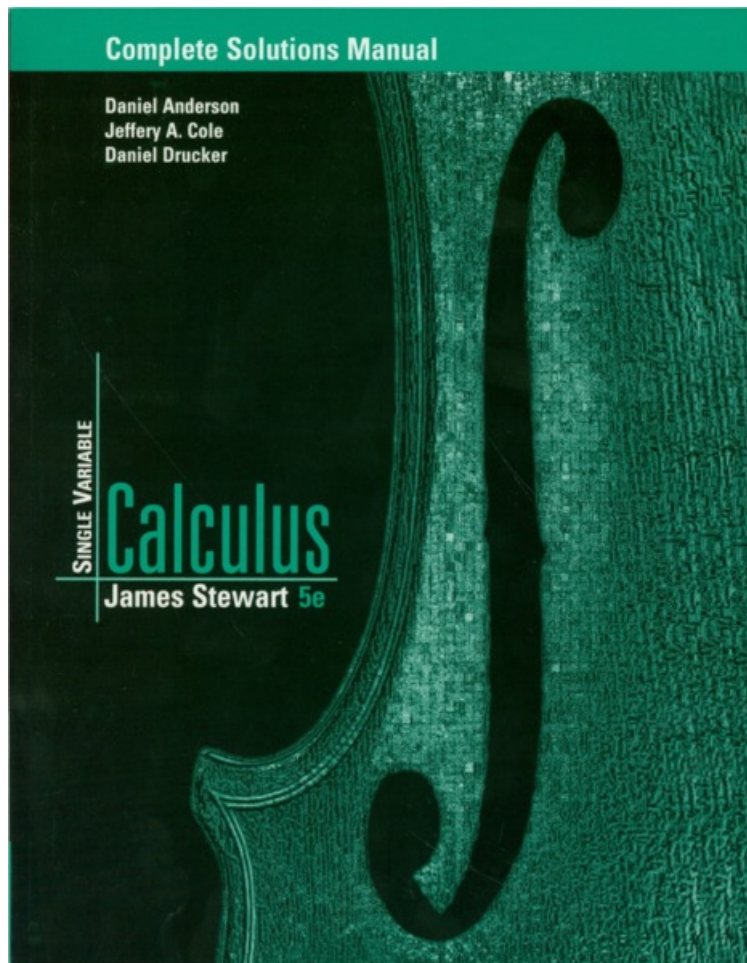
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2 □ LIMITS AND RATES OF CHANGE

2.1 The Tangent and Velocity Problems

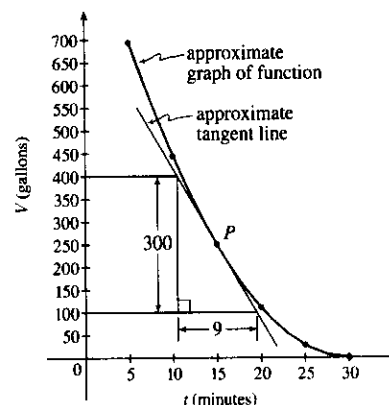
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope = m_{PQ}
5	(5, 694)	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	(10, 444)	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	(20, 111)	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	(25, 28)	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	(30, 0)	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\bar{6}$

- (b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

- (c) From the graph, we can estimate the slope of the tangent line at P to be $-\frac{300}{9} = -33.\bar{3}$.



2. (a) Slope = $\frac{2948 - 2530}{42 - 36} = \frac{418}{6} \approx 69.67$ (b) Slope = $\frac{2948 - 2661}{42 - 38} = \frac{287}{4} = 71.75$
(c) Slope = $\frac{2948 - 2806}{42 - 40} = \frac{142}{2} = 71$ (d) Slope = $\frac{3080 - 2948}{44 - 42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. For the curve $y = x/(1+x)$ and the point $P(1, \frac{1}{2})$:

(a)

	x	Q	m_{PQ}
(i)	0.5	(0.5, 0.333333)	0.333333
(ii)	0.9	(0.9, 0.473684)	0.263158
(iii)	0.99	(0.99, 0.497487)	0.251256
(iv)	0.999	(0.999, 0.499750)	0.250125
(v)	1.5	(1.5, 0.6)	0.2
(vi)	1.1	(1.1, 0.523810)	0.238095
(vii)	1.01	(1.01, 0.502488)	0.248756
(viii)	1.001	(1.001, 0.500250)	0.249875

- (b) The slope appears to be $\frac{1}{4}$.

(c) $y - \frac{1}{2} = \frac{1}{4}(x - 1)$ or
 $y = \frac{1}{4}x + \frac{1}{4}$.

4. For the curve
- $y = \sqrt{x+4}$
- and the point
- $P(5, 3)$
- :

(a)

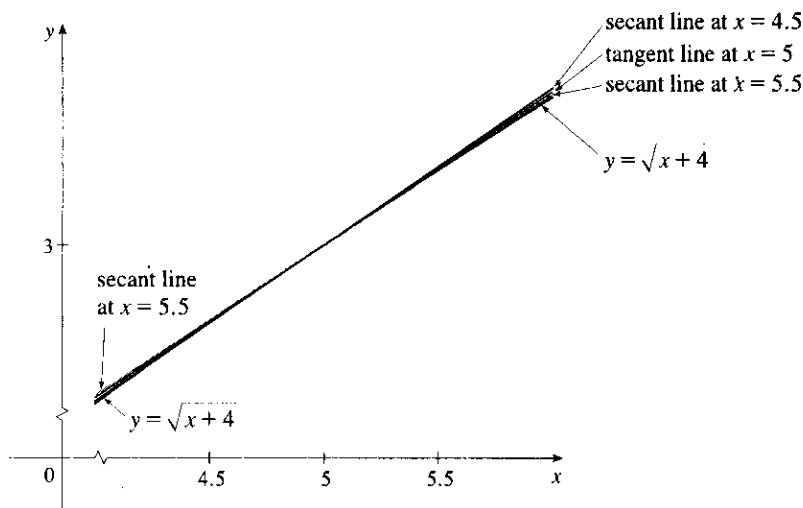
	x	Q	m_{PQ}
(i)	4.5	(4.5, 2.915476)	0.169048
(ii)	4.9	(4.9, 2.983287)	0.167132
(iii)	4.99	(4.99, 2.998333)	0.166713
(iv)	4.999	(4.999, 2.999833)	0.166671
(v)	5.5	(5.5, 3.082207)	0.164414
(vi)	5.1	(5.1, 3.016621)	0.166206
(vii)	5.01	(5.01, 3.001666)	0.166620
(viii)	5.001	(5.001, 3.000167)	0.166662

(b) The slope appears to be $\frac{1}{6}$.

(c) $y - 3 = \frac{1}{6}(x - 5)$ or

$$y = \frac{1}{6}x + \frac{13}{6}$$

(d)



5. (a)
- $y = y(t) = 40t - 16t^2$
- . At
- $t = 2$
- ,
- $y = 40(2) - 16(2)^2 = 16$
- . The average velocity between times 2 and
- $2 + h$

$$\text{is } v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{[40(2+h) - 16(2+h)^2] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $[2, 2.5]: h = 0.5, v_{\text{ave}} = -32 \text{ ft/s}$

(ii) $[2, 2.1]: h = 0.1, v_{\text{ave}} = -25.6 \text{ ft/s}$

(iii) $[2, 2.05]: h = 0.05, v_{\text{ave}} = -24.8 \text{ ft/s}$

(iv) $[2, 2.01]: h = 0.01, v_{\text{ave}} = -24.16 \text{ ft/s}$

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s .

6. The average velocity between
- t
- and
- $t + h$
- seconds is

$$\frac{58(t+h) - 0.83(t+h)^2 - (58t - 0.83t^2)}{h} = \frac{58h - 1.66th - 0.83h^2}{h} = 58 - 1.66t - 0.83h \text{ if } h \neq 0.$$

(a) Here $t = 1$, so the average velocity is $58 - 1.66 - 0.83h = 56.34 - 0.83h$.

(i) $[1, 2]: h = 1, 55.51 \text{ m/s}$

(ii) $[1, 1.5]: h = 0.5, 55.925 \text{ m/s}$

(iii) $[1, 1.1]: h = 0.1, 56.257 \text{ m/s}$

(iv) $[1, 1.01]: h = 0.01, 56.3317 \text{ m/s}$

(v) $[1, 1.001]: h = 0.001, 56.33917 \text{ m/s}$

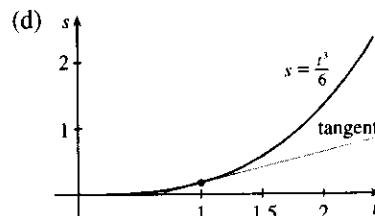
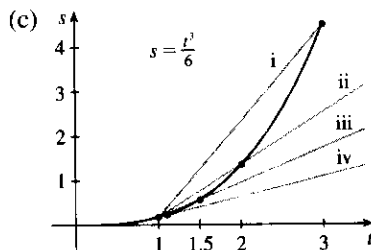
(b) The instantaneous velocity after 1 second is 56.34 m/s .

7. $s = s(t) = t^3/6$. Average velocity between times 1 and $1 + h$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{(1+h)^3/6 - 1/6}{h} = \frac{h^3 + 3h^2 + 3h}{6h} = \frac{h^2 + 3h + 3}{6} \text{ if } h \neq 0.$$

- (a) (i) $[1, 3]: h = 2, v_{\text{ave}} = \frac{13}{6} \text{ ft/s}$ (ii) $[1, 2]: h = 1, v_{\text{ave}} = \frac{7}{6} \text{ ft/s}$
 (iii) $[1, 1.5]: h = 0.5, v_{\text{ave}} = \frac{19}{24} \text{ ft/s}$ (iv) $[1, 1.1]: h = 0.1, v_{\text{ave}} = \frac{331}{600} \text{ ft/s}$

- (b) As h approaches 0, the velocity approaches $\frac{3}{6} = \frac{1}{2} \text{ ft/s}$.



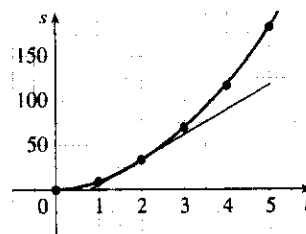
8. Average velocity between times $t = 2$ and $t = 2 + h$ is given by $\frac{s(2+h) - s(2)}{h}$.

(a) (i) $h = 3 \Rightarrow v_{\text{av}} = \frac{s(5) - s(2)}{5 - 2} = \frac{178 - 32}{3} = \frac{146}{3} \approx 48.7 \text{ ft/s}$

(ii) $h = 2 \Rightarrow v_{\text{av}} = \frac{s(4) - s(2)}{4 - 2} = \frac{119 - 32}{2} = \frac{87}{2} = 43.5 \text{ ft/s}$

(iii) $h = 1 \Rightarrow v_{\text{av}} = \frac{s(3) - s(2)}{3 - 2} = \frac{70 - 32}{1} = 38 \text{ ft/s}$

- (b) Using the points $(0.8, 0)$ and $(5, 118)$ from the approximate tangent line, the instantaneous velocity at $t = 2$ is about $\frac{118 - 0}{5 - 0.8} \approx 28 \text{ ft/s}$.



9. For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

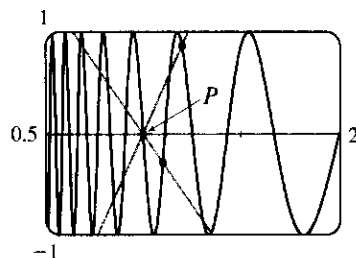
(a)

x	Q	m_{PQ}
2	(2, 0)	0
1.5	(1.5, 0.8660)	1.7321
1.4	(1.4, -0.4339)	-1.0847
1.3	(1.3, -0.8230)	-2.7433
1.2	(1.2, 0.8660)	4.3301
1.1	(1.1, -0.2817)	-2.8173

x	Q	m_{PQ}
0.5	(0.5, 0)	0
0.6	(0.6, 0.8660)	-2.1651
0.7	(0.7, 0.7818)	-2.6061
0.8	(0.8, 1)	-5
0.9	(0.9, -0.3420)	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x -values much closer to 1 in order to get accurate estimates of its slope.

- (c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108 . So we estimate that the slope of the tangent line at P is about -31.4 .

2.2 The Limit of a Function

- As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.
- As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.
- $\lim_{x \rightarrow -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
 - $\lim_{x \rightarrow 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.
- $\lim_{x \rightarrow 0} f(x) = 3$
 - $\lim_{x \rightarrow 3^-} f(x) = 4$
 - $\lim_{x \rightarrow 3^+} f(x) = 2$
 - $\lim_{x \rightarrow 3} f(x)$ does not exist because the limits in part (b) and part (c) are not equal.
 - $f(3) = 3$
- $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim_{x \rightarrow 1^-} f(x) = 2$.
 - $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim_{x \rightarrow 1^+} f(x) = 3$.
 - $\lim_{x \rightarrow 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.
 - $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim_{x \rightarrow 5} f(x) = 4$.
 - $f(5)$ is not defined, so it doesn't exist.
- $\lim_{x \rightarrow -2^-} g(x) = -1$
 - $\lim_{x \rightarrow -2^+} g(x) = 1$
 - $\lim_{x \rightarrow -2} g(x)$ doesn't exist
 - $g(-2) = 1$
 - $\lim_{x \rightarrow 2^-} g(x) = 1$
 - $\lim_{x \rightarrow 2^+} g(x) = 2$
 - $\lim_{x \rightarrow 2} g(x)$ doesn't exist
 - $\lim_{x \rightarrow 2} g(x)$ doesn't exist
 - $\lim_{x \rightarrow 4^+} g(x)$ doesn't exist
 - $\lim_{x \rightarrow 4^-} g(x) = 2$
 - $g(0)$ doesn't exist
 - $\lim_{x \rightarrow 0} g(x) = 0$

7. (a) $\lim_{t \rightarrow 0^-} g(t) = -1$

(b) $\lim_{t \rightarrow 0^+} g(t) = -2$

 (c) $\lim_{t \rightarrow 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t \rightarrow 2^-} g(t) = 2$

(e) $\lim_{t \rightarrow 2^+} g(t) = 0$

 (f) $\lim_{t \rightarrow 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$

(h) $\lim_{t \rightarrow 4} g(t) = 3$

8. (a) $\lim_{x \rightarrow 2} R(x) = -\infty$

(b) $\lim_{x \rightarrow 5} R(x) = \infty$

(c) $\lim_{x \rightarrow -3^-} R(x) = -\infty$

(d) $\lim_{x \rightarrow -3^+} R(x) = \infty$

 (e) The equations of the vertical asymptotes are $x = -3$, $x = 2$, and $x = 5$.

9. (a) $\lim_{x \rightarrow -7} f(x) = -\infty$

(b) $\lim_{x \rightarrow -3} f(x) = \infty$

(c) $\lim_{x \rightarrow 0} f(x) = \infty$

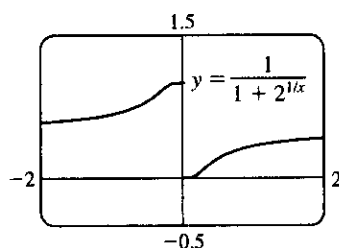
(d) $\lim_{x \rightarrow 6^-} f(x) = -\infty$

(e) $\lim_{x \rightarrow 6^+} f(x) = \infty$

 (f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

 10. $\lim_{t \rightarrow 12^-} f(t) = 150$ mg and $\lim_{t \rightarrow 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

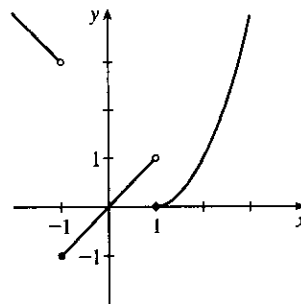
11.



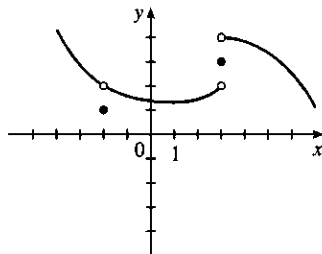
(a) $\lim_{x \rightarrow 0^-} f(x) = 1$

(b) $\lim_{x \rightarrow 0^+} f(x) = 0$

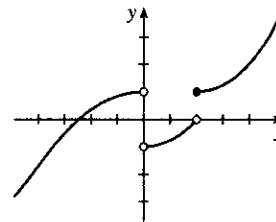
 (c) $\lim_{x \rightarrow 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

 12. $\lim_{x \rightarrow a} f(x)$ exists for all a except $a = \pm 1$.


13. $\lim_{x \rightarrow 3^+} f(x) = 4$, $\lim_{x \rightarrow 3^-} f(x) = 2$,
 $\lim_{x \rightarrow -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$



14. $\lim_{x \rightarrow 0^-} f(x) = 1$, $\lim_{x \rightarrow 0^+} f(x) = -1$,
 $\lim_{x \rightarrow 2^-} f(x) = 0$, $\lim_{x \rightarrow 2^+} f(x) = 1$, $f(2) = 1$,
 $f(0)$ is undefined



15. For $f(x) = \frac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$	x	$f(x)$
2.5	0.714286	1.9	0.655172
2.1	0.677419	1.95	0.661017
2.05	0.672131	1.99	0.665552
2.01	0.667774	1.995	0.666110
2.005	0.667221	1.999	0.666556
2.001	0.666778		

It appears that $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x^2 - x - 2} = 0.6 = \frac{2}{3}$.

16. For $f(x) = \frac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$	x	$f(x)$
0	0	-2	2
-0.5	-1	-1.5	3
-0.9	-9	-1.1	11
-0.95	-19	-1.01	101
-0.99	-99	-1.001	1001
-0.999	-999		

It appears that $\lim_{x \rightarrow -1} \frac{x^2 - 2x}{x^2 - x - 2}$ does not exist

since $f(x) \rightarrow -\infty$ as $x \rightarrow -1^-$ and $f(x) \rightarrow \infty$ as $x \rightarrow -1^+$.

17. For $f(x) = \frac{\sin x}{x + \tan x}$:

x	$f(x)$
± 1	0.329033
± 0.5	0.458209
± 0.2	0.493331
± 0.1	0.498333
± 0.05	0.499583
± 0.01	0.499983

It appears that $\lim_{x \rightarrow 0} \frac{\sin x}{x + \tan x} = 0.5 = \frac{1}{2}$.

18. For $f(x) = \frac{\sqrt{x} - 4}{x - 16}$:

x	$f(x)$	x	$f(x)$
17	0.123106	15	0.127017
16.5	0.124038	15.5	0.125992
16.1	0.124805	15.9	0.125196
16.05	0.124902	15.95	0.125098
16.01	0.124980	15.99	0.125020

It appears that $\lim_{x \rightarrow 16} \frac{\sqrt{x} - 4}{x - 16} = 0.125 = \frac{1}{8}$.

19. For $f(x) = \frac{\sqrt{x+4} - 2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
-1	0.267949
-0.5	0.258343
-0.1	0.251582
-0.05	0.250786
-0.01	0.250156

It appears that $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4}$.

20. For $f(x) = \frac{\tan 3x}{\tan 5x}$:

x	$f(x)$
± 0.2	0.439279
± 0.1	0.566236
± 0.05	0.591893
± 0.01	0.599680
± 0.001	0.599997

It appears that $\lim_{x \rightarrow 0} \frac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5}$.

21. For $f(x) = \frac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5}$.

22. For $f(x) = \frac{9^x - 5^x}{x}$:

x	$f(x)$
0.5	1.527864
0.1	0.711120
0.05	0.646496
0.01	0.599082
0.001	0.588906

x	$f(x)$
-0.5	0.227761
-0.1	0.485984
-0.05	0.534447
-0.01	0.576706
-0.001	0.586669

It appears that $\lim_{x \rightarrow 0} \frac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is $\ln(9/5)$.

23. $\lim_{x \rightarrow 5^+} \frac{6}{x-5} = \infty$ since $(x-5) \rightarrow 0$ as $x \rightarrow 5^+$ and $\frac{6}{x-5} > 0$ for $x > 5$.

24. $\lim_{x \rightarrow 5^-} \frac{6}{x-5} = -\infty$ since $(x-5) \rightarrow 0$ as $x \rightarrow 5^-$ and $\frac{6}{x-5} < 0$ for $x < 5$.

25. $\lim_{x \rightarrow 1} \frac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \rightarrow 1$.

26. $\lim_{x \rightarrow 0} \frac{x-1}{x^2(x+2)} = -\infty$ since $x^2 \rightarrow 0$ as $x \rightarrow 0$ and $\frac{x-1}{x^2(x+2)} < 0$ for $0 < x < 1$ and for $-2 < x < 0$.

27. $\lim_{x \rightarrow -2^+} \frac{x-1}{x^2(x+2)} = -\infty$ since $(x+2) \rightarrow 0$ as $x \rightarrow -2^+$ and $\frac{x-1}{x^2(x+2)} < 0$ for $-2 < x < 0$.

28. $\lim_{x \rightarrow \pi^-} \csc x = \lim_{x \rightarrow \pi^-} (1/\sin x) = \infty$ since $\sin x \rightarrow 0$ as $x \rightarrow \pi^-$ and $\sin x > 0$ for $0 < x < \pi$.

29. $\lim_{x \rightarrow (-\pi/2)^-} \sec x = \lim_{x \rightarrow (-\pi/2)^-} (1/\cos x) = -\infty$ since $\cos x \rightarrow 0$ as $x \rightarrow (-\pi/2)^-$ and $\cos x < 0$ for $-\pi < x < -\pi/2$.

30. $\lim_{x \rightarrow 1^+} \frac{x+1}{x \sin \pi x} = -\infty$ since $\frac{x+1}{x} \rightarrow 2$ as $x \rightarrow 1^+$ and $\sin \pi x \rightarrow 0$ through negative values as $x \rightarrow 1^+$.

31. (a) $f(x) = 1/(x^3 - 1)$

x	$f(x)$
0.5	-1.14
0.9	-3.69
0.99	-33.7
0.999	-333.7
0.9999	-3333.7
0.99999	-33,333.7

x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

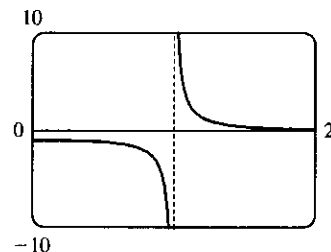
From these calculations, it seems that $\lim_{x \rightarrow 1^-} f(x) = -\infty$ and $\lim_{x \rightarrow 1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x \rightarrow 1^-} f(x) = -\infty$.

If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x \rightarrow 1^+} f(x) = \infty$.

(c) It appears from the graph of f that $\lim_{x \rightarrow 1^-} f(x) = -\infty$ and

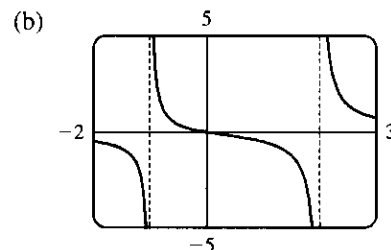
$$\lim_{x \rightarrow 1^+} f(x) = \infty.$$



32. (a) $y = \frac{x}{x^2 - x - 2} = \frac{x}{(x-2)(x+1)}$. Therefore, as $x \rightarrow -1^+$ or

$x \rightarrow 2^+$, the denominator approaches 0, and $y > 0$ for $x < -1$ and for $x > 2$, so $\lim_{x \rightarrow -1^+} y = \lim_{x \rightarrow 2^+} y = \infty$. Also, as $x \rightarrow -1^-$ or

$x \rightarrow 2^-$, the denominator approaches 0 and $y < 0$ for $-1 < x < 2$, so $\lim_{x \rightarrow -1^-} y = \lim_{x \rightarrow 2^-} y = -\infty$.

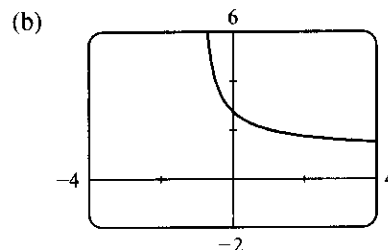


33. (a) Let
- $h(x) = (1+x)^{1/x}$
- .

x	$h(x)$
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830
-0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

It appears that $\lim_{x \rightarrow 0} (1+x)^{1/x} \approx 2.71828$, which is approximately e .

In Section 7.4 we will see that the value of the limit is exactly e .



34. For the curve
- $y = 2^x$
- and the points
- $P(0, 1)$
- and
- $Q(x, 2^x)$
- :

x	Q	m_{PQ}
0.1	(0.1, 1.0717735)	0.71773
0.01	(0.01, 1.0069556)	0.69556
0.001	(0.001, 1.0006934)	0.69339
0.0001	(0.0001, 1.0000693)	0.69317

The slope appears to be about 0.693.

35. For
- $f(x) = x^2 - (2^x/1000)$
- :

(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x \rightarrow 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000572
0.02	-0.000614
0.01	-0.000907
0.005	-0.000978
0.003	-0.000993
0.001	-0.001000

It appears that $\lim_{x \rightarrow 0} f(x) = -0.001$.

36. $h(x) = \frac{\tan x - x}{x^3}$

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

(c)

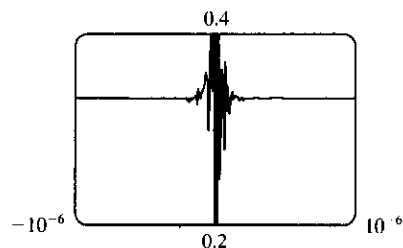
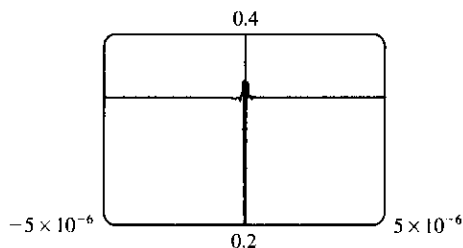
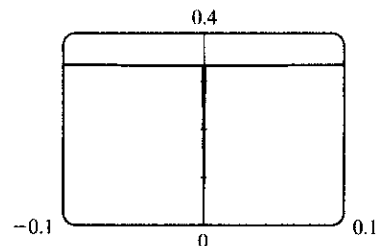
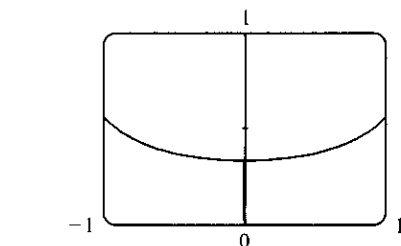
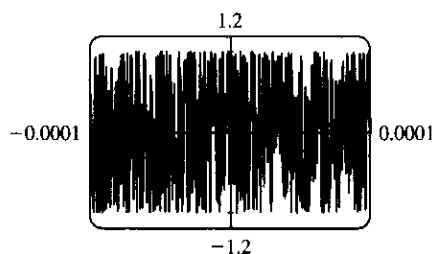
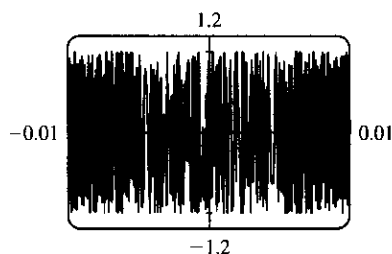
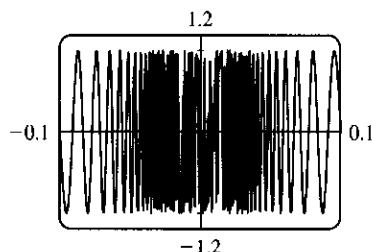
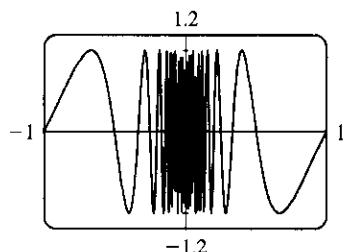
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

(b) It seems that $\lim_{x \rightarrow 0} h(x) = \frac{1}{3}$.

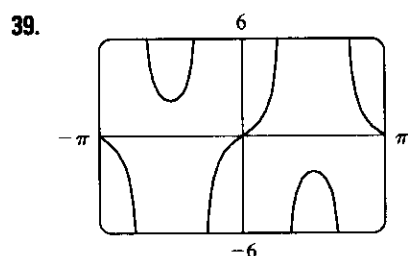
Here the values will vary from one calculator to another.

Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

37. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \rightarrow 0$.

38. $\lim_{v \rightarrow c^-} m = \lim_{v \rightarrow c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}}$. As $v \rightarrow c^-$, $\sqrt{1 - v^2/c^2} \rightarrow 0^+$, and $m \rightarrow \infty$.

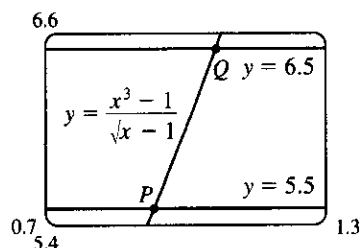


There appear to be vertical asymptotes of the curve $y = \tan(2 \sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2 \sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2} n$. Since $-1 \leq \sin x \leq 1$, we must have $\sin x = \pm \frac{\pi}{4}$ and so $x = \pm \sin^{-1} \frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$).

Just as 150° is the reference angle for 30° , $\pi - \sin^{-1} \frac{\pi}{4}$ is the reference angle for $\sin^{-1} \frac{\pi}{4}$. So $x = \pm(\pi - \sin^{-1} \frac{\pi}{4})$ are also equations of the vertical asymptotes (corresponding to $x \approx \pm 2.24$).

40. (a) Let $y = (x^3 - 1)/(\sqrt{x} - 1)$.

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



From the table and the graph, we guess that the limit of y as x approaches 1 is 6.

(b) We need to have $5.5 < \frac{x^3 - 1}{\sqrt{x} - 1} < 6.5$. From the graph we obtain the approximate points of intersection

$P(0.9313853, 5.5)$ and $Q(1.0649004, 6.5)$. Now $1 - 0.9313853 \approx 0.0686$ and $1.0649004 - 1 \approx 0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim_{x \rightarrow a} [f(x) + h(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} h(x)$
 $= -3 + 8 = 5$

(b) $\lim_{x \rightarrow a} [f(x)]^2 = \left[\lim_{x \rightarrow a} f(x) \right]^2 = (-3)^2 = 9$

(c) $\lim_{x \rightarrow a} \sqrt[3]{h(x)} = \sqrt[3]{\lim_{x \rightarrow a} h(x)} = \sqrt[3]{8} = 2$

(d) $\lim_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{\lim_{x \rightarrow a} f(x)} = \frac{1}{-3} = -\frac{1}{3}$

(e) $\lim_{x \rightarrow a} \frac{f(x)}{h(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} h(x)} = \frac{-3}{8} = -\frac{3}{8}$

(f) $\lim_{x \rightarrow a} \frac{g(x)}{f(x)} = \frac{\lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} f(x)} = \frac{0}{-3} = 0$

(g) The limit does not exist, since $\lim_{x \rightarrow a} g(x) = 0$ but $\lim_{x \rightarrow a} f(x) \neq 0$.

(h) $\lim_{x \rightarrow a} \frac{2f(x)}{h(x) - f(x)} = \frac{2 \lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} h(x) - \lim_{x \rightarrow a} f(x)} = \frac{2(-3)}{8 - (-3)} = -\frac{6}{11}$

$$2. (a) \lim_{x \rightarrow 2} [f(x) + g(x)] = \lim_{x \rightarrow 2} f(x) + \lim_{x \rightarrow 2} g(x) = 2 + 0 = 2$$

(b) $\lim_{x \rightarrow 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

$$(c) \lim_{x \rightarrow 0} [f(x)g(x)] = \lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x) = 0 \cdot 1.3 = 0$$

(d) Since $\lim_{x \rightarrow -1} g(x) = 0$ and g is in the denominator, but $\lim_{x \rightarrow -1} f(x) = -1 \neq 0$, the given limit does not exist.

$$(e) \lim_{x \rightarrow 2} x^3 f(x) = \left[\lim_{x \rightarrow 2} x^3 \right] \left[\lim_{x \rightarrow 2} f(x) \right] = 2^3 \cdot 2 = 16$$

$$(f) \lim_{x \rightarrow 1} \sqrt{3 + f(x)} = \sqrt{3 + \lim_{x \rightarrow 1} f(x)} = \sqrt{3 + 1} = 2$$

$$\begin{aligned} 3. \lim_{x \rightarrow -2} (3x^4 + 2x^2 - x + 1) &= \lim_{x \rightarrow -2} 3x^4 + \lim_{x \rightarrow -2} 2x^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 && \text{[Limit Laws 1 and 2]} \\ &= 3 \lim_{x \rightarrow -2} x^4 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 && [3] \\ &= 3(-2)^4 + 2(-2)^2 - (-2) + (1) && [9, 8, \text{ and } 7] \\ &= 48 + 8 + 2 + 1 = 59 \end{aligned}$$

$$\begin{aligned} 4. \lim_{x \rightarrow 2} \frac{2x^2 + 1}{x^2 + 6x - 4} &= \frac{\lim_{x \rightarrow 2} (2x^2 + 1)}{\lim_{x \rightarrow 2} (x^2 + 6x - 4)} && \text{[Limit Law 5]} \\ &= \frac{2 \lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} x^2 + 6 \lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 4} && [2, 1, \text{ and } 3] \\ &= \frac{2(2)^2 + 1}{(2)^2 + 6(2) - 4} = \frac{9}{12} = \frac{3}{4} && [9, 7, \text{ and } 8] \end{aligned}$$

$$\begin{aligned} 5. \lim_{x \rightarrow 3} (x^2 - 4)(x^3 + 5x - 1) &= \lim_{x \rightarrow 3} (x^2 - 4) \cdot \lim_{x \rightarrow 3} (x^3 + 5x - 1) && \text{[Limit Law 4]} \\ &= \left(\lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 4 \right) \cdot \left(\lim_{x \rightarrow 3} x^3 + 5 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 1 \right) && [2, 1, \text{ and } 3] \\ &= (3^2 - 4) \cdot (3^3 + 5 \cdot 3 - 1) && [7, 8, \text{ and } 9] \\ &= 5 \cdot 41 = 205 \end{aligned}$$

$$\begin{aligned} 6. \lim_{t \rightarrow -1} (t^2 + 1)^3 (t + 3)^5 &= \lim_{t \rightarrow -1} (t^2 + 1)^3 \cdot \lim_{t \rightarrow -1} (t + 3)^5 && \text{[Limit Law 4]} \\ &= \left[\lim_{t \rightarrow -1} (t^2 + 1) \right]^3 \cdot \left[\lim_{t \rightarrow -1} (t + 3) \right]^5 && [6] \\ &= \left[\lim_{t \rightarrow -1} t^2 + \lim_{t \rightarrow -1} 1 \right]^3 \cdot \left[\lim_{t \rightarrow -1} t + \lim_{t \rightarrow -1} 3 \right]^5 && [1] \\ &= [(-1)^2 + 1]^3 \cdot [-1 + 3]^5 = 8 \cdot 32 = 256 && [9, 7, \text{ and } 8] \end{aligned}$$

$$7. \lim_{x \rightarrow 1} \left(\frac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3 = \left(\lim_{x \rightarrow 1} \frac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3 \quad [6]$$

$$= \left[\frac{\lim_{x \rightarrow 1} (1 + 3x)}{\lim_{x \rightarrow 1} (1 + 4x^2 + 3x^4)} \right]^3 \quad [5]$$

$$= \left[\frac{\lim_{x \rightarrow 1} 1 + 3 \lim_{x \rightarrow 1} x}{\lim_{x \rightarrow 1} 1 + 4 \lim_{x \rightarrow 1} x^2 + 3 \lim_{x \rightarrow 1} x^4} \right]^3 \quad [2, 1, \text{ and } 3]$$

$$= \left[\frac{1 + 3(1)}{1 + 4(1)^2 + 3(1)^4} \right]^3 = \left[\frac{4}{8} \right]^3 = \left(\frac{1}{2} \right)^3 = \frac{1}{8} \quad [7, 8, \text{ and } 9]$$

$$8. \lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6} = \sqrt{\lim_{u \rightarrow -2} (u^4 + 3u + 6)} \quad [11]$$

$$= \sqrt{\lim_{u \rightarrow -2} u^4 + 3 \lim_{u \rightarrow -2} u + \lim_{u \rightarrow -2} 6} \quad [1, 2, \text{ and } 3]$$

$$= \sqrt{(-2)^4 + 3(-2) + 6} \quad [9, 8, \text{ and } 7]$$

$$= \sqrt{16 - 6 + 6} = \sqrt{16} = 4$$

$$9. \lim_{x \rightarrow 4^-} \sqrt{16 - x^2} = \sqrt{\lim_{x \rightarrow 4^-} (16 - x^2)} \quad [11]$$

$$= \sqrt{\lim_{x \rightarrow 4^-} 16 - \lim_{x \rightarrow 4^-} x^2} \quad [2]$$

$$= \sqrt{16 - (4)^2} = 0 \quad [7 \text{ and } 9]$$

10. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \rightarrow 2$, just as in Example 3. Remember that in finding $\lim_{x \rightarrow a} f(x)$, we never consider $x = a$.

$$11. \lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x + 3)(x - 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 3) = 2 + 3 = 5$$

$$12. \lim_{x \rightarrow -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \lim_{x \rightarrow -4} \frac{(x + 4)(x + 1)}{(x + 4)(x - 1)} = \lim_{x \rightarrow -4} \frac{x + 1}{x - 1} = \frac{-4 + 1}{-4 - 1} = \frac{-3}{-5} = \frac{3}{5}$$

$$13. \lim_{x \rightarrow 2} \frac{x^2 - x + 6}{x - 2} \text{ does not exist since } x - 2 \rightarrow 0 \text{ but } x^2 - x + 6 \rightarrow 8 \text{ as } x \rightarrow 2.$$

$$14. \lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \rightarrow 4} \frac{x(x - 4)}{(x - 4)(x + 1)} = \lim_{x \rightarrow 4} \frac{x}{x + 1} = \frac{4}{4 + 1} = \frac{4}{5}$$

$$15. \lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3} = \lim_{t \rightarrow -3} \frac{(t + 3)(t - 3)}{(2t + 1)(t + 3)} = \lim_{t \rightarrow -3} \frac{t - 3}{2t + 1} = \frac{-3 - 3}{2(-3) + 1} = \frac{-6}{-5} = \frac{6}{5}$$

$$16. \lim_{x \rightarrow -1} \frac{x^2 - 4x}{x^2 - 3x - 4} \text{ does not exist since } x^2 - 3x - 4 \rightarrow 0 \text{ but } x^2 - 4x \rightarrow 5 \text{ as } x \rightarrow -1.$$

$$17. \lim_{h \rightarrow 0} \frac{(4 + h)^2 - 16}{h} = \lim_{h \rightarrow 0} \frac{(16 + 8h + h^2) - 16}{h} = \lim_{h \rightarrow 0} \frac{8h + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(8 + h)}{h} = \lim_{h \rightarrow 0} (8 + h) = 8 + 0 = 8$$

$$18. \lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \lim_{x \rightarrow 1} \frac{x^2 + x + 1}{x + 1} = \frac{1^2 + 1 + 1}{1 + 1} = \frac{3}{2}$$

$$\begin{aligned}
 19. \lim_{h \rightarrow 0} \frac{(1+h)^4 - 1}{h} &= \lim_{h \rightarrow 0} \frac{(1 + 4h + 6h^2 + 4h^3 + h^4) - 1}{h} = \lim_{h \rightarrow 0} \frac{4h + 6h^2 + 4h^3 + h^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(4 + 6h + 4h^2 + h^3)}{h} = \lim_{h \rightarrow 0} (4 + 6h + 4h^2 + h^3) = 4 + 0 + 0 + 0 = 4
 \end{aligned}$$

$$\begin{aligned}
 20. \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} &= \lim_{h \rightarrow 0} \frac{(8 + 12h + 6h^2 + h^3) - 8}{h} = \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} (12 + 6h + h^2) = 12 + 0 + 0 = 12
 \end{aligned}$$

$$21. \lim_{t \rightarrow 9} \frac{9-t}{3-\sqrt{t}} = \lim_{t \rightarrow 9} \frac{(3+\sqrt{t})(3-\sqrt{t})}{3-\sqrt{t}} = \lim_{t \rightarrow 9} (3+\sqrt{t}) = 3 + \sqrt{9} = 6$$

$$\begin{aligned}
 22. \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+h} + 1)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 23. \lim_{x \rightarrow 7} \frac{\sqrt{x+2} - 3}{x-7} &= \lim_{x \rightarrow 7} \frac{\sqrt{x+2} - 3}{x-7} \cdot \frac{\sqrt{x+2} + 3}{\sqrt{x+2} + 3} = \lim_{x \rightarrow 7} \frac{(x+2) - 9}{(x-7)(\sqrt{x+2} + 3)} \\
 &= \lim_{x \rightarrow 7} \frac{x-7}{(x-7)(\sqrt{x+2} + 3)} = \lim_{x \rightarrow 7} \frac{1}{\sqrt{x+2} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 24. \lim_{x \rightarrow 2} \frac{x^4 - 16}{x-2} &= \lim_{x \rightarrow 2} \frac{(x+2)(x-2)(x^2+4)}{x-2} = \lim_{x \rightarrow 2} (x+2)(x^2+4) = \lim_{x \rightarrow 2} (x+2) \lim_{x \rightarrow 2} (x^2+4) \\
 &= (2+2)(2^2+4) = 32
 \end{aligned}$$

$$25. \lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x} = \lim_{x \rightarrow -4} \frac{\frac{x+4}{4x}}{4+x} = \lim_{x \rightarrow -4} \frac{x+4}{4x(4+x)} = \lim_{x \rightarrow -4} \frac{1}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}$$

$$26. \lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2+t} \right) = \lim_{t \rightarrow 0} \frac{(t^2+t) - t}{t(t^2+t)} = \lim_{t \rightarrow 0} \frac{t^2}{t \cdot t(t+1)} = \lim_{t \rightarrow 0} \frac{1}{t+1} = \frac{1}{0+1} = 1$$

$$\begin{aligned}
 27. \lim_{x \rightarrow 9} \frac{x^2 - 81}{\sqrt{x} - 3} &= \lim_{x \rightarrow 9} \frac{(x-9)(x+9)}{\sqrt{x} - 3} = \lim_{x \rightarrow 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)}{\sqrt{x} - 3} \quad \left[\begin{array}{l} \text{factor } x-9 \text{ as a} \\ \text{difference of squares} \end{array} \right] \\
 &= \lim_{x \rightarrow 9} [(\sqrt{x}+3)(x+9)] = (\sqrt{9}+3)(9+9) = 6 \cdot 18 = 108
 \end{aligned}$$

$$\begin{aligned}
 28. \lim_{h \rightarrow 0} \frac{(3+h)^{-1} - 3^{-1}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \rightarrow 0} \frac{3 - (3+h)}{h(3+h)3} = \lim_{h \rightarrow 0} \frac{-h}{h(3+h)3} \\
 &= \lim_{h \rightarrow 0} \left[-\frac{1}{3(3+h)} \right] = -\frac{1}{\lim_{h \rightarrow 0} [3(3+h)]} = -\frac{1}{3(3+0)} = -\frac{1}{9}
 \end{aligned}$$

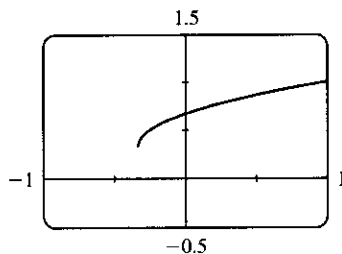
$$\begin{aligned}
 29. \lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) &= \lim_{t \rightarrow 0} \frac{1 - \sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t \rightarrow 0} \frac{(1 - \sqrt{1+t})(1 + \sqrt{1+t})}{t\sqrt{1+t}(1 + \sqrt{1+t})} = \lim_{t \rightarrow 0} \frac{-t}{t\sqrt{1+t}(1 + \sqrt{1+t})} \\
 &= \lim_{t \rightarrow 0} \frac{-1}{\sqrt{1+t}(1 + \sqrt{1+t})} = \frac{-1}{\sqrt{1+0}(1 + \sqrt{1+0})} = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 30. \lim_{x \rightarrow 1} \frac{\sqrt{x} - x^2}{1 - \sqrt{x}} &= \lim_{x \rightarrow 1} \frac{\sqrt{x}(1 - x^{3/2})}{1 - \sqrt{x}} = \lim_{x \rightarrow 1} \frac{\sqrt{x}(1 - \sqrt{x})(1 + \sqrt{x} + x)}{1 - \sqrt{x}} \quad [\text{difference of cubes}] \\
 &= \lim_{x \rightarrow 1} [\sqrt{x}(1 + \sqrt{x} + x)] = \lim_{x \rightarrow 1} [1(1 + 1 + 1)] = 3
 \end{aligned}$$

Another method: We “add and subtract” 1 in the numerator, and then split up the fraction:

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{\sqrt{x} - x^2}{1 - \sqrt{x}} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1) + (1 - x^2)}{1 - \sqrt{x}} = \lim_{x \rightarrow 1} \left[-1 + \frac{(1 - x)(1 + x)}{1 - \sqrt{x}} \right] \\
 &= \lim_{x \rightarrow 1} \left[-1 + \frac{(1 - \sqrt{x})(1 + \sqrt{x})(1 + x)}{1 - \sqrt{x}} \right] = -1 + (1 + \sqrt{1})(1 + 1) = 3
 \end{aligned}$$

31. (a)



$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+3x}-1} \approx \frac{2}{3}$$

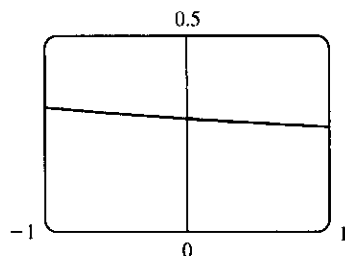
(b)

x	$f(x)$
-0.001	0.6661663
-0.0001	0.6666167
-0.00001	0.6666617
-0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

$$\begin{aligned}
 (c) \lim_{x \rightarrow 0} \left(\frac{x}{\sqrt{1+3x}-1} \cdot \frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1} \right) &= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{(1+3x)-1} = \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{3x} \\
 &= \frac{1}{3} \lim_{x \rightarrow 0} (\sqrt{1+3x}+1) \quad [\text{Limit Law 3}] \\
 &= \frac{1}{3} \left[\lim_{x \rightarrow 0} (\sqrt{1+3x}) + \lim_{x \rightarrow 0} 1 \right] \quad [1 \text{ and } 11] \\
 &= \frac{1}{3} \left(\sqrt{\lim_{x \rightarrow 0} 1 + 3 \lim_{x \rightarrow 0} x} + 1 \right) \quad [1, 3, \text{ and } 7] \\
 &= \frac{1}{3} (\sqrt{1+3 \cdot 0} + 1) \quad [7 \text{ and } 8] \\
 &= \frac{1}{3} (1 + 1) = \frac{2}{3}
 \end{aligned}$$

32. (a)



$$\lim_{x \rightarrow 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \approx 0.29$$

(b)

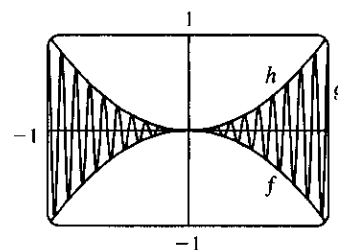
x	$f(x)$
-0.001	0.2886992
-0.0001	0.2886775
-0.00001	0.2886754
-0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887.

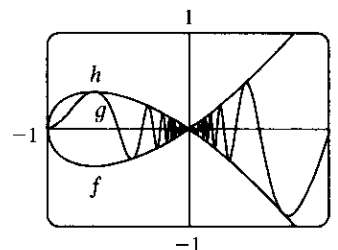
$$\begin{aligned}
 \text{(c)} \quad \lim_{x \rightarrow 0} \left(\frac{\sqrt{3+x} - \sqrt{3}}{x} \cdot \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} \right) &= \lim_{x \rightarrow 0} \frac{(3+x) - 3}{x(\sqrt{3+x} + \sqrt{3})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} \\
 &= \frac{\lim_{x \rightarrow 0} 1}{\lim_{x \rightarrow 0} \sqrt{3+x} + \lim_{x \rightarrow 0} \sqrt{3}} && \text{[Limit Laws 5 and 1]} \\
 &= \frac{1}{\sqrt{\lim_{x \rightarrow 0} (3+x)} + \sqrt{3}} && \text{[7 and 11]} \\
 &= \frac{1}{\sqrt{3+0} + \sqrt{3}} && \text{[1, 7, and 8]} \\
 &= \frac{1}{2\sqrt{3}}
 \end{aligned}$$

33. Let $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$ and $h(x) = x^2$. Then

$$\begin{aligned}
 -1 \leq \cos 20\pi x \leq 1 &\Rightarrow -x^2 \leq x^2 \cos 20\pi x \leq x^2 \Rightarrow \\
 f(x) \leq g(x) \leq h(x). &\text{ So since } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0, \text{ by the} \\
 \text{Squeeze Theorem we have } \lim_{x \rightarrow 0} g(x) &= 0.
 \end{aligned}$$

34. Let $f(x) = -\sqrt{x^3 + x^2}$, $g(x) = \sqrt{x^3 + x^2} \sin(\pi/x)$, and

$$\begin{aligned}
 h(x) &= \sqrt{x^3 + x^2}. \text{ Then } -1 \leq \sin(\pi/x) \leq 1 \Rightarrow \\
 -\sqrt{x^3 + x^2} &\leq \sqrt{x^3 + x^2} \sin(\pi/x) \leq \sqrt{x^3 + x^2} \Rightarrow \\
 f(x) \leq g(x) \leq h(x). &\text{ So since } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0, \text{ by the} \\
 \text{Squeeze Theorem we have } \lim_{x \rightarrow 0} g(x) &= 0.
 \end{aligned}$$

35. $1 \leq f(x) \leq x^2 + 2x + 2$ for all x . Now $\lim_{x \rightarrow -1} 1 = 1$ and

$$\begin{aligned}
 \lim_{x \rightarrow -1} (x^2 + 2x + 2) &= \lim_{x \rightarrow -1} x^2 + 2 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 2 = (-1)^2 + 2(-1) + 2 = 1. \text{ Therefore, by the Squeeze} \\
 \text{Theorem, } \lim_{x \rightarrow -1} f(x) &= 1.
 \end{aligned}$$

36. $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$. Now $\lim_{x \rightarrow 1} 3x = 3$ and $\lim_{x \rightarrow 1} (x^3 + 2) = \lim_{x \rightarrow 1} x^3 + \lim_{x \rightarrow 1} 2 = 1^3 + 2 = 3$.

Therefore, by the Squeeze Theorem, $\lim_{x \rightarrow 1} f(x) = 3$.

37. $-1 \leq \cos(2/x) \leq 1 \Rightarrow -x^4 \leq x^4 \cos(2/x) \leq x^4$. Since $\lim_{x \rightarrow 0} (-x^4) = 0$ and $\lim_{x \rightarrow 0} x^4 = 0$, we have

$\lim_{x \rightarrow 0} [x^4 \cos(2/x)] = 0$ by the Squeeze Theorem.

38. $-1 \leq \sin(2\pi/x) \leq 1 \Rightarrow 0 \leq \sin^2(2\pi/x) \leq 1 \Rightarrow 1 \leq 1 + \sin^2(2\pi/x) \leq 2 \Rightarrow$

$\sqrt{x} \leq \sqrt{x} [1 + \sin^2(2\pi/x)] \leq 2\sqrt{x}$. Since $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ and $\lim_{x \rightarrow 0^+} 2\sqrt{x} = 0$, we have

$\lim_{x \rightarrow 0^+} [\sqrt{x} (1 + \sin^2(2\pi/x))] = 0$ by the Squeeze Theorem.

39. If $x > -4$, then $|x + 4| = x + 4$, so $\lim_{x \rightarrow -4^+} |x + 4| = \lim_{x \rightarrow -4^+} (x + 4) = -4 + 4 = 0$.

If $x < -4$, then $|x + 4| = -(x + 4)$, so $\lim_{x \rightarrow -4^-} |x + 4| = \lim_{x \rightarrow -4^-} -(x + 4) = -(-4 + 4) = 0$.

Since the right and left limits are equal, $\lim_{x \rightarrow -4} |x + 4| = 0$.

40. If $x < -4$, then $|x + 4| = -(x + 4)$, so $\lim_{x \rightarrow -4^-} \frac{|x + 4|}{x + 4} = \lim_{x \rightarrow -4^-} \frac{-(x + 4)}{x + 4} = \lim_{x \rightarrow -4^-} (-1) = -1$.

41. If $x > 2$, then $|x - 2| = x - 2$, so $\lim_{x \rightarrow 2^+} \frac{|x - 2|}{x - 2} = \lim_{x \rightarrow 2^+} \frac{x - 2}{x - 2} = \lim_{x \rightarrow 2^+} 1 = 1$. If $x < 2$, then

$|x - 2| = -(x - 2)$, so $\lim_{x \rightarrow 2^-} \frac{|x - 2|}{x - 2} = \lim_{x \rightarrow 2^-} \frac{-(x - 2)}{x - 2} = \lim_{x \rightarrow 2^-} -1 = -1$. The right and left limits are

different, so $\lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2}$ does not exist.

42. If $x > \frac{3}{2}$, then $|2x - 3| = 2x - 3$, so

$\lim_{x \rightarrow 1.5^+} \frac{2x^2 - 3x}{|2x - 3|} = \lim_{x \rightarrow 1.5^+} \frac{2x^2 - 3x}{2x - 3} = \lim_{x \rightarrow 1.5^+} \frac{x(2x - 3)}{2x - 3} = \lim_{x \rightarrow 1.5^+} x = 1.5$. If $x < \frac{3}{2}$, then

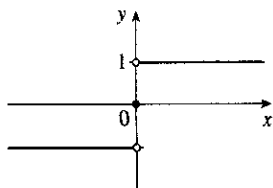
$|2x - 3| = 3 - 2x$, so $\lim_{x \rightarrow 1.5^-} \frac{2x^2 - 3x}{|2x - 3|} = \lim_{x \rightarrow 1.5^-} \frac{2x^2 - 3x}{-(2x - 3)} = \lim_{x \rightarrow 1.5^-} \frac{x(2x - 3)}{-(2x - 3)} = \lim_{x \rightarrow 1.5^-} -x = -1.5$.

The right and left limits are different, so $\lim_{x \rightarrow 1.5} \frac{2x^2 - 3x}{|2x - 3|}$ does not exist.

43. Since $|x| = -x$ for $x < 0$, we have $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{-x} \right) = \lim_{x \rightarrow 0^-} \frac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

44. Since $|x| = x$ for $x > 0$, we have $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0^+} 0 = 0$.

45. (a)



- (b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim_{x \rightarrow 0^+} \operatorname{sgn} x = \lim_{x \rightarrow 0^+} 1 = 1$.

- (ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim_{x \rightarrow 0^-} \operatorname{sgn} x = \lim_{x \rightarrow 0^-} -1 = -1$.

- (iii) Since $\lim_{x \rightarrow 0^-} \operatorname{sgn} x \neq \lim_{x \rightarrow 0^+} \operatorname{sgn} x$, $\lim_{x \rightarrow 0} \operatorname{sgn} x$ does not exist.

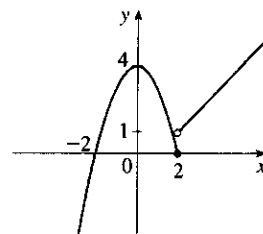
- (iv) Since $|\operatorname{sgn} x| = 1$ for $x \neq 0$, $\lim_{x \rightarrow 0} |\operatorname{sgn} x| = \lim_{x \rightarrow 0} 1 = 1$.

$$46. (a) \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (4 - x^2) = \lim_{x \rightarrow 2^-} 4 - \lim_{x \rightarrow 2^-} x^2 = 4 - 4 = 0$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x - 1) = \lim_{x \rightarrow 2^+} x - \lim_{x \rightarrow 2^+} 1 = 2 - 1 = 1$$

(b) No, $\lim_{x \rightarrow 2} f(x)$ does not exist since $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$.

(c)

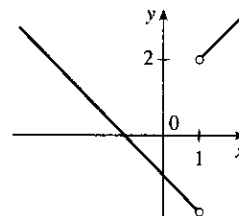


$$47. (a) (i) \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2$$

$$(ii) \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{-(x - 1)} = \lim_{x \rightarrow 1^-} -(x + 1) = -2$$

(b) No, $\lim_{x \rightarrow 1} F(x)$ does not exist since $\lim_{x \rightarrow 1^+} F(x) \neq \lim_{x \rightarrow 1^-} F(x)$.

(c)



$$48. (a) (i) \lim_{x \rightarrow 0^+} h(x) = \lim_{x \rightarrow 0^+} x^2 = 0^2 = 0$$

$$(ii) \lim_{x \rightarrow 0^-} h(x) = \lim_{x \rightarrow 0^-} x = 0, \text{ so } \lim_{x \rightarrow 0} h(x) = 0.$$

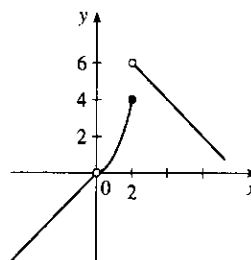
$$(iii) \lim_{x \rightarrow 1} h(x) = \lim_{x \rightarrow 1} x^2 = 1^2 = 1$$

$$(iv) \lim_{x \rightarrow 2^-} h(x) = \lim_{x \rightarrow 2^-} x^2 = 2^2 = 4$$

$$(v) \lim_{x \rightarrow 2^+} h(x) = \lim_{x \rightarrow 2^+} (8 - x) = 8 - 2 = 6$$

(vi) Since $\lim_{x \rightarrow 2^-} h(x) \neq \lim_{x \rightarrow 2^+} h(x)$, $\lim_{x \rightarrow 2} h(x)$ does not exist.

(b)



$$49. (a) (i) \lfloor x \rfloor = -2 \text{ for } -2 \leq x < -1, \text{ so } \lim_{x \rightarrow -2^+} \lfloor x \rfloor = \lim_{x \rightarrow -2^+} (-2) = -2$$

(ii) $\lfloor x \rfloor = -3$ for $-3 \leq x < -2$, so $\lim_{x \rightarrow -2^-} \lfloor x \rfloor = \lim_{x \rightarrow -2^-} (-3) = -3$. The right and left limits are different, so $\lim_{x \rightarrow -2} \lfloor x \rfloor$ does not exist.

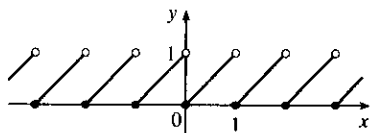
$$(iii) \lfloor x \rfloor = -3 \text{ for } -3 \leq x < -2, \text{ so } \lim_{x \rightarrow -2.4} \lfloor x \rfloor = \lim_{x \rightarrow -2.4} (-3) = -3.$$

$$(b) (i) \lfloor x \rfloor = n - 1 \text{ for } n - 1 \leq x < n, \text{ so } \lim_{x \rightarrow n^-} \lfloor x \rfloor = \lim_{x \rightarrow n^-} (n - 1) = n - 1.$$

$$(ii) \lfloor x \rfloor = n \text{ for } n \leq x < n + 1, \text{ so } \lim_{x \rightarrow n^+} \lfloor x \rfloor = \lim_{x \rightarrow n^+} n = n.$$

(c) $\lim_{x \rightarrow a} \lfloor x \rfloor$ exists $\Leftrightarrow a$ is not an integer.

50. (a)



$$(b) (i) \lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} (x - \lfloor x \rfloor) = \lim_{x \rightarrow n^-} [x - (n - 1)] = n - (n - 1) = 1$$

$$(ii) \lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} (x - \lfloor x \rfloor) = \lim_{x \rightarrow n^+} (x - n) = n - n = 0$$

(c) $\lim_{x \rightarrow a} f(x)$ exists $\Leftrightarrow a$ is not an integer.

51. The graph of $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a . Thus, $\lim_{x \rightarrow 2^-} f(x) = -1$ and $\lim_{x \rightarrow 2^+} f(x) = -1$, so $\lim_{x \rightarrow 2} f(x) = -1$. However, $f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = 2 + (-2) = 0$, so $\lim_{x \rightarrow 2} f(x) \neq f(2)$.

52. $\lim_{v \rightarrow c^-} \left(L_0 \sqrt{1 - \frac{v^2}{c^2}} \right) = L_0 \sqrt{1 - 1} = 0$. As the velocity approaches the speed of light, the length approaches 0. A left-hand limit is necessary since L is not defined for $v > c$.

53. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned} \lim_{x \rightarrow a} p(x) &= \lim_{x \rightarrow a} (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) \\ &= a_0 + a_1 \lim_{x \rightarrow a} x + a_2 \lim_{x \rightarrow a} x^2 + \cdots + a_n \lim_{x \rightarrow a} x^n \\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a) \end{aligned}$$

Thus, for any polynomial p , $\lim_{x \rightarrow a} p(x) = p(a)$.

54. Let $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Thus,

$$\lim_{x \rightarrow a} r(x) = \lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{\lim_{x \rightarrow a} p(x)}{\lim_{x \rightarrow a} q(x)} \quad [\text{Limit Law 5}] = \frac{p(a)}{q(a)} \quad [\text{Exercise 53}] = r(a).$$

55. Observe that $0 \leq f(x) \leq x^2$ for all x , and $\lim_{x \rightarrow 0} 0 = 0 = \lim_{x \rightarrow 0} x^2$. So, by the Squeeze Theorem, $\lim_{x \rightarrow 0} f(x) = 0$.

56. Let $f(x) = \lfloor x \rfloor$ and $g(x) = -\lfloor x \rfloor$. Then $\lim_{x \rightarrow 3} f(x)$ and $\lim_{x \rightarrow 3} g(x)$ do not exist (Example 10) but

$$\lim_{x \rightarrow 3} [f(x) + g(x)] = \lim_{x \rightarrow 3} (\lfloor x \rfloor - \lfloor x \rfloor) = \lim_{x \rightarrow 3} 0 = 0.$$

57. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59.

Thus, either f or g is 0 for any value of x . Then $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist, but

$$\lim_{x \rightarrow 0} [f(x)g(x)] = \lim_{x \rightarrow 0} 0 = 0.$$

58.
$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x \rightarrow 2} \left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} \right) \\ &= \lim_{x \rightarrow 2} \left[\frac{(\sqrt{6-x})^2 - 2^2}{(\sqrt{3-x})^2 - 1^2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} \right] = \lim_{x \rightarrow 2} \left(\frac{6-x-4}{3-x-1} \cdot \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} \right) \\ &= \lim_{x \rightarrow 2} \frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)} = \lim_{x \rightarrow 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2} \end{aligned}$$

59. Since the denominator approaches 0 as $x \rightarrow -2$, the limit will exist only if the numerator also approaches 0 as $x \rightarrow -2$. In order for this to happen, we need $\lim_{x \rightarrow -2} (3x^2 + ax + a + 3) = 0 \Leftrightarrow$

$$3(-2)^2 + a(-2) + a + 3 = 0 \Leftrightarrow 12 - 2a + a + 3 = 0 \Leftrightarrow a = 15. \text{ With } a = 15, \text{ the limit becomes}$$

$$\lim_{x \rightarrow -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \rightarrow -2} \frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x \rightarrow -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

60. Solution 1: First, we find the coordinates of P and Q as functions of r . Then we can find the equation of the line determined by these two points, and thus find the x -intercept (the point R), and take the limit as $r \rightarrow 0$.

The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x - 1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x - 1)^2 \Leftrightarrow r^2 = 1 + 2x - 1 \Leftrightarrow x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y -coordinate, we get

$(\frac{1}{2}r^2)^2 + y^2 = r^2 \Leftrightarrow y^2 = r^2(1 - \frac{1}{4}r^2) \Leftrightarrow y = r\sqrt{1 - \frac{1}{4}r^2}$ (the positive y -value). So the coordinates of Q are $(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2})$. The equation of the line joining P and Q is thus

$y - r = \frac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x -intercept, and get

$$x = -r \frac{\frac{1}{2}r^2}{r(\sqrt{1 - \frac{1}{4}r^2} - 1)} = \frac{-\frac{1}{2}r^2(\sqrt{1 - \frac{1}{4}r^2} + 1)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right).$$

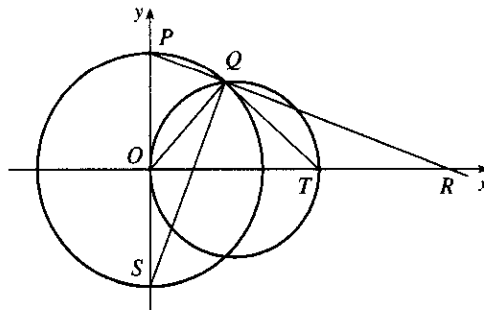
Now we take the limit as $r \rightarrow 0^+$: $\lim_{r \rightarrow 0^+} x = \lim_{r \rightarrow 0^+} 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right) = \lim_{r \rightarrow 0^+} 2(\sqrt{1} + 1) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS).

So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also

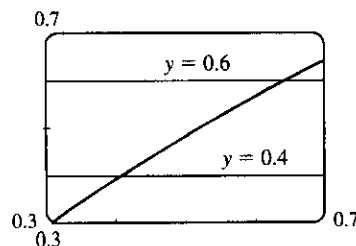
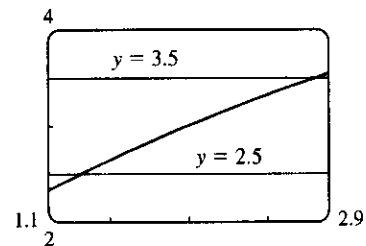
$\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T , that is, the point $(4, 0)$, as above.



2.4 The Precise Definition of a Limit

1. (a) To have $5x + 3$ within a distance of 0.1 of 13, we must have $12.9 \leq 5x + 3 \leq 13.1 \Rightarrow 9.9 \leq 5x \leq 10.1 \Rightarrow 1.98 \leq x \leq 2.02$. Thus, x must be within 0.02 units of 2 so that $5x + 3$ is within 0.1 of 13.
 (b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.
2. (a) To have $6x - 1$ within a distance of 0.01 of 29, we must have $28.99 \leq 6x - 1 \leq 29.01 \Rightarrow 29.99 \leq 6x \leq 30.01 \Rightarrow 4.998\bar{3} \leq x \leq 5.001\bar{6}$. Thus, x must be within $0.001\bar{6}$ units of 5 so that $6x - 1$ is within 0.01 of 29.
 (b) As in part (a) with 0.001 in place of 0.01, we obtain $0.0001\bar{6}$.
 (c) As in part (a) with 0.0001 in place of 0.01, we obtain $0.00001\bar{6}$.

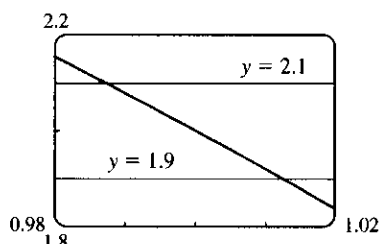
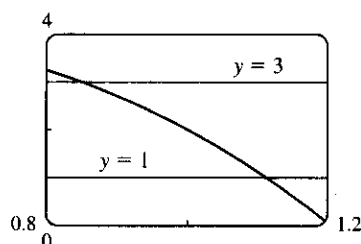
3. On the left side of $x = 2$, we need $|x - 2| < \left| \frac{10}{7} - 2 \right| = \frac{4}{7}$. On the right side, we need $|x - 2| < \left| \frac{10}{3} - 2 \right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x - 2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.
4. On the left side, we need $|x - 5| < |4 - 5| = 1$. On the right side, we need $|x - 5| < |5.7 - 5| = 0.7$. For both conditions to be satisfied at once, we need the more restrictive condition to hold; that is, $|x - 5| < 0.7$. So we can choose $\delta = 0.7$, or any smaller positive number.
5. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold — namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.
6. The left-hand question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the right-hand question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left| \frac{1}{\sqrt{2}} - 1 \right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left| \sqrt{\frac{3}{2}} - 1 \right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).
7. $|\sqrt{4x+1} - 3| < 0.5 \Leftrightarrow 2.5 < \sqrt{4x+1} < 3.5$. We plot the three parts of this inequality on the same screen and identify the x -coordinates of the points of intersection using the cursor. It appears that the inequality holds for $1.3125 \leq x \leq 2.8125$. Since $|2 - 1.3125| = 0.6875$ and $|2 - 2.8125| = 0.8125$, we choose $0 < \delta < \min \{0.6875, 0.8125\} = 0.6875$.
8. $|\sin x - \frac{1}{2}| < 0.1 \Leftrightarrow 0.4 < \sin x < 0.6$. From the graph, we see that for this inequality to hold, we need $0.42 \leq x \leq 0.64$. So since $|0.5 - 0.42| = 0.08$ and $|0.5 - 0.64| = 0.14$, we choose $0 < \delta \leq \min \{0.08, 0.14\} = 0.08$.



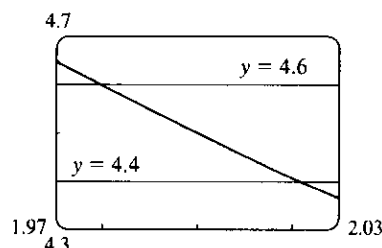
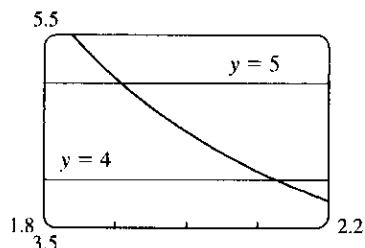
9. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $|(4 + x - 3x^3) - 2| < 1 \Leftrightarrow$

$1 < 4 + x - 3x^3 < 3$ whenever $0 < |x - 1| < \delta$. If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \leq x \leq 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that

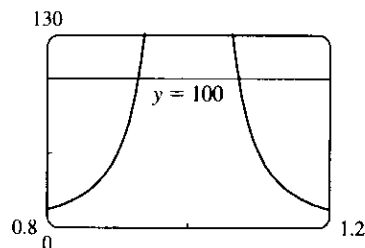
$|(4 + x - 3x^3) - 2| < 0.1 \Leftrightarrow 1.9 < 4 + x - 3x^3 < 2.1$ whenever $0 < |x - 1| < \delta$. From the graph, we see that we need $0.988 \leq x \leq 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



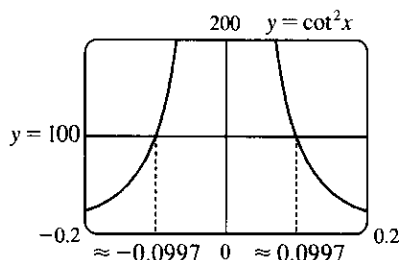
10. For $\varepsilon = 0.5$, we need $1.91 \leq x \leq 2.125$. So since $|2 - 1.91| = 0.09$ and $|2 - 2.125| = 0.125$, we can take $0 < \delta \leq 0.09$. For $\varepsilon = 0.1$, we need $1.980 \leq x \leq 2.021$. So since $|2 - 1.980| = 0.02$ and $|2 - 2.021| = 0.021$, we can take $\delta = 0.02$ (or any smaller positive number).



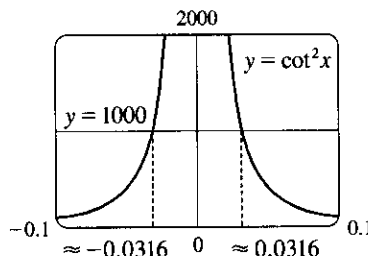
11. From the graph, we see that $\frac{x}{(x^2 + 1)(x - 1)^2} > 100$ whenever $0.93 \leq x \leq 1.07$. So since $|1 - 0.93| = 0.07$ and $|1 - 1.07| = 0.07$, we can take $\delta = 0.07$ (or any smaller positive number).



12. For $M = 100$, we need $-0.0997 < x < 0$ or $0 < x < 0.0997$. Thus, we choose $\delta = 0.0997$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 100$.



For $M = 1000$, we need $-0.0316 < x < 0$ or $0 < x < 0.0316$. Thus, we choose $\delta = 0.0316$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 1000$.



13. (a) $A = \pi r^2$ and $A = 1000 \text{ cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow$

$$r = \sqrt{\frac{1000}{\pi}} \quad [r > 0] \approx 17.8412 \text{ cm.}$$

(b) $|A - 1000| \leq 5 \Rightarrow -5 \leq \pi r^2 - 1000 \leq 5 \Rightarrow 1000 - 5 \leq \pi r^2 \leq 1000 + 5 \Rightarrow$

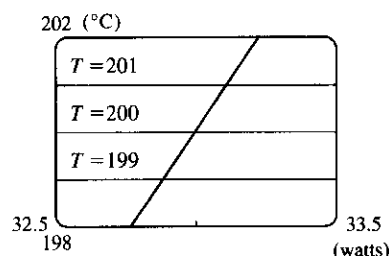
$$\sqrt{\frac{995}{\pi}} \leq r \leq \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \leq r \leq 17.8858. \quad \sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466 \text{ and}$$

$$\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455. \text{ So if the machinist gets the radius within } 0.0445 \text{ cm of } 17.8412, \text{ the area will be within } 5 \text{ cm}^2 \text{ of } 1000.$$

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

14. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \Rightarrow$

$$0.1w^2 + 2.155w + 20 = 200 \Rightarrow [\text{by the quadratic formula or from the graph}] \quad w \approx 33.0 \text{ watts } (w > 0)$$



(b) From the graph, $199 \leq T \leq 201 \Rightarrow 32.89 < w < 33.11$.

(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

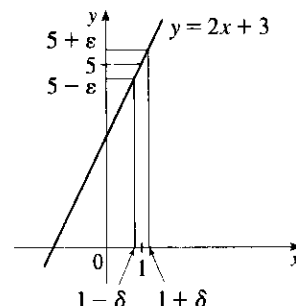
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then

$$|(2x + 3) - 5| < \varepsilon. \text{ But } |(2x + 3) - 5| < \varepsilon \Leftrightarrow |2x - 2| < \varepsilon$$

$$\Leftrightarrow 2|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \varepsilon/2. \text{ So if we choose } \delta = \varepsilon/2,$$

$$\text{then } 0 < |x - 1| < \delta \Rightarrow |(2x + 3) - 5| < \varepsilon. \text{ Thus,}$$

$$\lim_{x \rightarrow 1} (2x + 3) = 5 \text{ by the definition of a limit.}$$



16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then

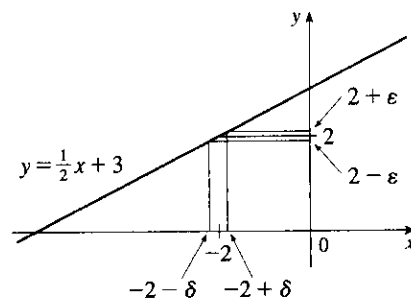
$$\left| \left(\frac{1}{2}x + 3 \right) - 2 \right| < \varepsilon. \text{ But } \left| \left(\frac{1}{2}x + 3 \right) - 2 \right| < \varepsilon \Leftrightarrow$$

$$\left| \frac{1}{2}x + 1 \right| < \varepsilon \Leftrightarrow \frac{1}{2}|x + 2| < \varepsilon \Leftrightarrow |x - (-2)| < 2\varepsilon. \text{ So if}$$

we choose $\delta = 2\varepsilon$, then $0 < |x - (-2)| < \delta \Rightarrow$

$$\left| \left(\frac{1}{2}x + 3 \right) - 2 \right| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow -2} \left(\frac{1}{2}x + 3 \right) = 2 \text{ by the definition of a}$$

limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then

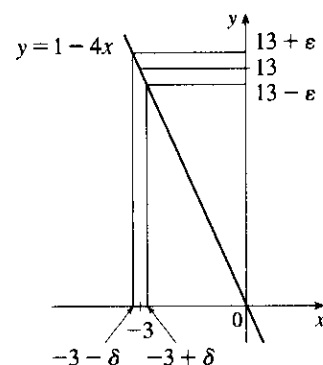
$$|(1 - 4x) - 13| < \varepsilon. \text{ But } |(1 - 4x) - 13| < \varepsilon \Leftrightarrow$$

$$|-4x - 12| < \varepsilon \Leftrightarrow |-4||x + 3| < \varepsilon \Leftrightarrow |x - (-3)| < \varepsilon/4.$$

So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \Rightarrow$

$$|(1 - 4x) - 13| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow -3} (1 - 4x) = 13 \text{ by the definition of}$$

a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then

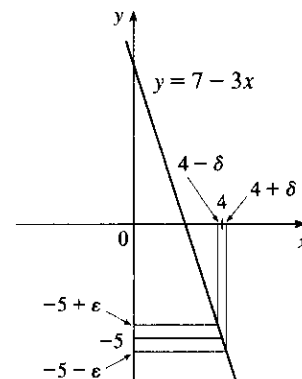
$$|(7 - 3x) - (-5)| < \varepsilon. \text{ But } |(7 - 3x) - (-5)| < \varepsilon \Leftrightarrow$$

$$|-3x + 12| < \varepsilon \Leftrightarrow |-3||x - 4| < \varepsilon \Leftrightarrow |x - 4| < \varepsilon/3. \text{ So}$$

if we choose $\delta = \varepsilon/3$, then $0 < |x - 4| < \delta \Rightarrow$

$$|(7 - 3x) - (-5)| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow 4} (7 - 3x) = -5 \text{ by the definition}$$

of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon \Leftrightarrow \frac{1}{5}|x - 3| < \varepsilon \Leftrightarrow$

$$|x - 3| < 5\varepsilon. \text{ So choose } \delta = 5\varepsilon. \text{ Then } 0 < |x - 3| < \delta \Rightarrow |x - 3| < 5\varepsilon \Rightarrow \frac{|x - 3|}{5} < \varepsilon \Rightarrow$$

$$\left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon. \text{ By the definition of a limit, } \lim_{x \rightarrow 3} \frac{x}{5} = \frac{3}{5}.$$

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 6| < \delta$, then $\left| \left(\frac{x}{4} + 3 \right) - \frac{9}{2} \right| < \varepsilon \Leftrightarrow \left| \frac{x}{4} - \frac{3}{2} \right| < \varepsilon \Leftrightarrow$

$$\frac{1}{4}|x - 6| < \varepsilon \Leftrightarrow |x - 6| < 4\varepsilon. \text{ So choose } \delta = 4\varepsilon. \text{ Then } 0 < |x - 6| < \delta \Rightarrow |x - 6| < 4\varepsilon \Rightarrow$$

$$\frac{|x - 6|}{4} < \varepsilon \Rightarrow \left| \frac{x}{4} - \frac{6}{4} \right| < \varepsilon \Rightarrow \left| \left(\frac{x}{4} + 3 \right) - \frac{9}{2} \right| < \varepsilon. \text{ By the definition of a limit, } \lim_{x \rightarrow 6} \left(\frac{x}{4} + 3 \right) = \frac{9}{2}.$$

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-5)| < \delta$, then $\left| \left(4 - \frac{3}{5}x \right) - 7 \right| < \varepsilon \Leftrightarrow$

$$\left| -\frac{3}{5}x - 3 \right| < \varepsilon \Leftrightarrow \frac{3}{5}|x + 5| < \varepsilon \Leftrightarrow |x - (-5)| < \frac{5}{3}\varepsilon. \text{ So choose } \delta = \frac{5}{3}\varepsilon. \text{ Then } |x - (-5)| < \delta \Rightarrow$$

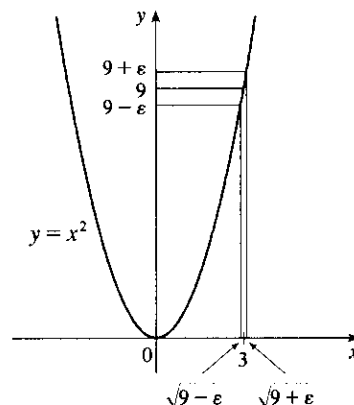
$$\left| \left(4 - \frac{3}{5}x \right) - 7 \right| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow -5} \left(4 - \frac{3}{5}x \right) = 7 \text{ by the definition of a limit.}$$

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left| \frac{x^2 + x - 12}{x - 3} - 7 \right| < \varepsilon$. Notice that if $0 < |x - 3|$, then $x \neq 3$, so $\frac{x^2 + x - 12}{x - 3} = \frac{(x + 4)(x - 3)}{x - 3} = x + 4$. Thus, when $0 < |x - 3|$, we have
- $$\left| \frac{x^2 + x - 12}{x - 3} - 7 \right| < \varepsilon \Leftrightarrow |(x + 4) - 7| < \varepsilon \Leftrightarrow |x - 3| < \varepsilon.$$
- We take $\delta = \varepsilon$ and see that $0 < |x - 3| < \delta$
- $$\Rightarrow \left| \frac{x^2 + x - 12}{x - 3} - 7 \right| < \varepsilon.$$
- By the definition of a limit, $\lim_{x \rightarrow 3} \frac{x^2 + x - 12}{x - 3} = 7$.
23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.
24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$. But $|c - c| = 0$, so this will be true no matter what δ we pick.
25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $|x^2 - 0| < \varepsilon \Leftrightarrow x^2 < \varepsilon \Leftrightarrow |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 0| < \delta \Rightarrow |x^2 - 0| < \varepsilon$. Thus, $\lim_{x \rightarrow 0} x^2 = 0$ by the definition of a limit.
26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $|x^3 - 0| < \varepsilon \Leftrightarrow |x|^3 < \varepsilon \Leftrightarrow |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x - 0| < \delta \Rightarrow |x^3 - 0| < \varepsilon$. Thus, $\lim_{x \rightarrow 0} x^3 = 0$ by the definition of a limit.
27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $||x| - 0| < \varepsilon$. But $||x|| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim_{x \rightarrow 0} |x| = 0$ by the definition of a limit.
28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $9 - \delta < x < 9$, then $|\sqrt[4]{9 - x} - 0| < \varepsilon \Leftrightarrow \sqrt[4]{9 - x} < \varepsilon \Leftrightarrow 9 - x < \varepsilon^4 \Leftrightarrow 9 - \varepsilon^4 < x < 9$. So take $\delta = \varepsilon^4$. Then $9 - \delta < x < 9 \Rightarrow |\sqrt[4]{9 - x} - 0| < \varepsilon$. Thus, $\lim_{x \rightarrow 9^-} \sqrt[4]{9 - x} = 0$ by the definition of a limit.
29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|(x^2 - 4x + 5) - 1| < \varepsilon \Leftrightarrow |x^2 - 4x + 4| < \varepsilon \Leftrightarrow |(x - 2)^2| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 2| < \delta \Leftrightarrow |x - 2| < \sqrt{\varepsilon} \Leftrightarrow |(x - 2)^2| < \varepsilon$. Thus, $\lim_{x \rightarrow 2} (x^2 - 4x + 5) = 1$ by the definition of a limit.
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|(x^2 + x - 4) - 8| < \varepsilon \Leftrightarrow |x^2 + x - 12| < \varepsilon \Leftrightarrow |(x - 3)(x + 4)| < \varepsilon$. Notice that if $|x - 3| < 1$, then $-1 < x - 3 < 1 \Rightarrow 6 < x + 4 < 8 \Rightarrow |x + 4| < 8$. So take $\delta = \min\{1, \varepsilon/8\}$. Then $0 < |x - 3| < \delta \Leftrightarrow |(x - 3)(x + 4)| \leq |8(x - 3)| = 8 \cdot |x - 3| < 8\delta \leq \varepsilon$. Thus, $\lim_{x \rightarrow 3} (x^2 + x - 4) = 8$ by the definition of a limit.
31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(x^2 - 1) - 3| < \varepsilon$ or upon simplifying we need $|x^2 - 4| < \varepsilon$ whenever $0 < |x + 2| < \delta$. Notice that if $|x + 2| < 1$, then $-1 < x + 2 < 1 \Rightarrow -5 < x - 2 < -3 \Rightarrow |x - 2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x + 2| < \delta \Rightarrow |x - 2| < 5$ and $|x + 2| < \varepsilon/5$, so $|(x^2 - 1) - 3| = |(x + 2)(x - 2)| = |x + 2||x - 2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim_{x \rightarrow -2} (x^2 - 1) = 3$.
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|x^3 - 8| < \varepsilon$. Now $|x^3 - 8| = |(x - 2)(x^2 + 2x + 4)|$. If $|x - 2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 < 3^2 + 2(3) + 4 = 19$ and so $|x^3 - 8| = |x - 2|(x^2 + 2x + 4) < 19|x - 2|$. So if we take $\delta = \min\{1, \frac{\varepsilon}{19}\}$, then

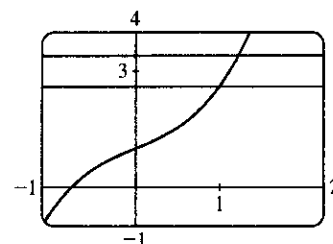
$0 < |x - 2| < \delta \Rightarrow |x^3 - 8| = |x - 2|(x^2 + 2x + 4) < \frac{\epsilon}{19} \cdot 19 = \epsilon$. Thus, by the definition of a limit,
 $\lim_{x \rightarrow 2} x^3 = 8$.

33. Given $\epsilon > 0$, we let $\delta = \min\{2, \frac{\epsilon}{8}\}$. If $0 < |x - 3| < \delta$, then $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow 4 < x + 3 < 8 \Rightarrow |x + 3| < 8$. Also $|x - 3| < \frac{\epsilon}{8}$, so $|x^2 - 9| = |x + 3||x - 3| < 8 \cdot \frac{\epsilon}{8} = \epsilon$. Thus,
 $\lim_{x \rightarrow 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9 - \epsilon}$ and $\delta_2 = \sqrt{9 + \epsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9 + \epsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \epsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\epsilon) = \frac{(216 + 108\epsilon + 12\sqrt{336 + 324\epsilon + 81\epsilon^2})^{2/3} - 12}{6(216 + 108\epsilon + 12\sqrt{336 + 324\epsilon + 81\epsilon^2})^{1/3}}. \text{ Thus, } \delta = x(\epsilon) - 1.$$

(c) If $\epsilon = 0.4$, then $x(\epsilon) \approx 1.093272342$ and $\delta = x(\epsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. 1. *Guessing a value for δ* Let $\epsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \epsilon$ whenever

$$0 < |x - 2| < \delta. \text{ But } \left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2 - x}{2x}\right| = \frac{|x - 2|}{|2x|} < \epsilon. \text{ We find a positive constant } C \text{ such that } \frac{1}{|2x|} < C \Rightarrow \frac{|x - 2|}{|2x|} < C|x - 2| \text{ and we can make } C|x - 2| < \epsilon \text{ by taking } |x - 2| < \frac{\epsilon}{C} = \delta. \text{ We restrict } x \text{ to lie in the}$$

interval $|x - 2| < 1 \Rightarrow 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\epsilon\}$.

2. *Showing that δ works* Given $\epsilon > 0$ we let $\delta = \min\{1, 2\epsilon\}$. If $0 < |x - 2| < \delta$, then $|x - 2| < 1 \Rightarrow$

$$1 < x < 3 \Rightarrow \frac{1}{|2x|} < \frac{1}{2} \text{ (as in part 1). Also } |x - 2| < 2\epsilon, \text{ so } \left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x - 2|}{|2x|} < \frac{1}{2} \cdot 2\epsilon = \epsilon. \text{ This shows that } \lim_{x \rightarrow 2} (1/x) = \frac{1}{2}.$$

37. 1. Guessing a value for δ Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever

$0 < |x - a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{|x - a|}{C} < \varepsilon$, and we take $|x - a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a . If $|x - a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x - a| < \delta$, then $|x - a| < \frac{1}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so $|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon}{\left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)} = \varepsilon$. Therefore, $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim_{t \rightarrow 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \Rightarrow |H(t) - L| < \frac{1}{2} \Leftrightarrow L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \Rightarrow L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \Rightarrow L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim_{t \rightarrow 0} H(t)$ does not exist.

39. Suppose that $\lim_{x \rightarrow 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \leq |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim_{x \rightarrow 0} f(x)$ does not exist.

40. First suppose that $\lim_{x \rightarrow a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Then $a - \delta < x < a \Rightarrow 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus, $\lim_{x \rightarrow a^-} f(x) = L$. Also $a < x < a + \delta \Rightarrow 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim_{x \rightarrow a^+} f(x) = L$. Now suppose $\lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim_{x \rightarrow a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \Rightarrow |f(x) - L| < \varepsilon$. Since $\lim_{x \rightarrow a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2 \Rightarrow |f(x) - L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2 . Then $0 < |x - a| < \delta \Rightarrow a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim_{x \rightarrow a} f(x) = L$. So we have proved that $\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$.

$$41. \frac{1}{(x+3)^4} > 10,000 \Leftrightarrow (x+3)^4 < \frac{1}{10,000} \Leftrightarrow |x+3| < \sqrt[4]{\frac{1}{10,000}} \Leftrightarrow |x - (-3)| < \frac{1}{10}$$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x + 3| < \delta \Rightarrow 1/(x + 3)^4 > M$. Now $\frac{1}{(x + 3)^4} > M \Leftrightarrow$

$$(x + 3)^4 < \frac{1}{M} \Leftrightarrow |x + 3| < \sqrt[4]{\frac{1}{M}}. \text{ So take } \delta = \sqrt[4]{\frac{1}{M}}. \text{ Then } 0 < |x + 3| < \delta = \sqrt[4]{\frac{1}{M}} \Rightarrow$$

$$\frac{1}{(x + 3)^4} > M, \text{ so } \lim_{x \rightarrow -3} \frac{1}{(x + 3)^4} = \infty.$$

43. Let $N < 0$ be given. Then, for $x < -1$, we have $\frac{5}{(x + 1)^3} < N \Leftrightarrow \frac{5}{N} < (x + 1)^3 \Leftrightarrow \sqrt[3]{\frac{5}{N}} < x + 1$. Let

$$\delta = -\sqrt[3]{\frac{5}{N}}. \text{ Then } -1 - \delta < x < -1 \Rightarrow \sqrt[3]{\frac{5}{N}} < x + 1 < 0 \Rightarrow \frac{5}{(x + 1)^3} < N, \text{ so}$$

$$\lim_{x \rightarrow -1^-} \frac{5}{(x + 1)^3} = -\infty.$$

44. (a) Let M be given. Since $\lim_{x \rightarrow a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow$

$$f(x) > M + 1 - c. \text{ Since } \lim_{x \rightarrow a} g(x) = c, \text{ there exists } \delta_2 > 0 \text{ such that } 0 < |x - a| < \delta_2 \Rightarrow |g(x) - c| < 1$$

$$\Rightarrow g(x) > c - 1. \text{ Let } \delta \text{ be the smaller of } \delta_1 \text{ and } \delta_2. \text{ Then } 0 < |x - a| < \delta \Rightarrow$$

$$f(x) + g(x) > (M + 1 - c) + (c - 1) = M. \text{ Thus, } \lim_{x \rightarrow a} [f(x) + g(x)] = \infty.$$

(b) Let $M > 0$ be given. Since $\lim_{x \rightarrow a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow$

$$|g(x) - c| < c/2 \Rightarrow g(x) > c/2. \text{ Since } \lim_{x \rightarrow a} f(x) = \infty, \text{ there exists } \delta_2 > 0 \text{ such that } 0 < |x - a| < \delta_2$$

$$\Rightarrow f(x) > 2M/c. \text{ Let } \delta = \min\{\delta_1, \delta_2\}. \text{ Then } 0 < |x - a| < \delta \Rightarrow f(x)g(x) > \frac{2M}{c} \cdot \frac{c}{2} = M, \text{ so}$$

$$\lim_{x \rightarrow a} f(x)g(x) = \infty.$$

(c) Let $N < 0$ be given. Since $\lim_{x \rightarrow a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow$

$$|g(x) - c| < -c/2 \Rightarrow g(x) < c/2. \text{ Since } \lim_{x \rightarrow a} f(x) = \infty, \text{ there exists } \delta_2 > 0 \text{ such that } 0 < |x - a| < \delta_2$$

$$\Rightarrow f(x) > 2N/c. \text{ (Note that } c < 0 \text{ and } N < 0 \Rightarrow 2N/c > 0.) \text{ Let } \delta = \min\{\delta_1, \delta_2\}. \text{ Then}$$

$$0 < |x - a| < \delta \Rightarrow f(x) > 2N/c \Rightarrow f(x)g(x) < \frac{2N}{c} \cdot \frac{c}{2} = N, \text{ so } \lim_{x \rightarrow a} f(x)g(x) = -\infty.$$

2.5 Continuity

1. From Definition 1, $\lim_{x \rightarrow 4} f(x) = f(4)$.

2. The graph of f has no hole, jump, or vertical asymptote.

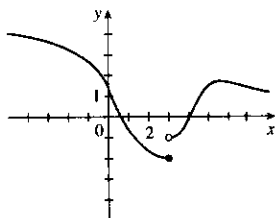
3. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number:
 -4 (removable), -2 (jump), 2 (jump), 4 (infinite).

(b) f is continuous from the left at -2 since $\lim_{x \rightarrow -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since

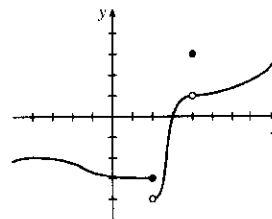
$$\lim_{x \rightarrow 2^+} f(x) = f(2) \text{ and } \lim_{x \rightarrow 4^+} f(x) = f(4). \text{ It is continuous from neither side at } -4 \text{ since } f(-4) \text{ is undefined.}$$

4. g is continuous on $[-4, -2)$, $(-2, 2)$, $[2, 4)$, $(4, 6)$, and $(6, 8)$.

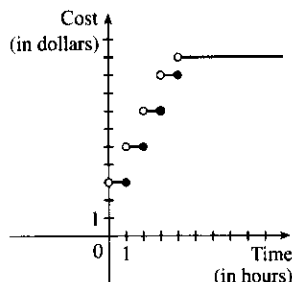
5. The graph of $y = f(x)$ must have a discontinuity at $x = 3$ and must show that $\lim_{x \rightarrow 3^-} f(x) = f(3)$.



6.



7. (a)



- (b) There are discontinuities at times $t = 1, 2, 3$, and 4 . A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.
 (b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.
 (c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values—at a cliff, for example.
 (d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.
 (e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.
9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x \rightarrow 3} [2f(x) - g(x)] &= 2 \lim_{x \rightarrow 3} f(x) - \lim_{x \rightarrow 3} g(x) && \text{[by Limit Laws 2 and 3]} \\ &= 2f(3) - g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x = 3\text{]} \\ &= 2 \cdot 5 - g(3) = 10 - g(3)\end{aligned}$$

Since it is given that $\lim_{x \rightarrow 3} [2f(x) - g(x)] = 4$, we have $10 - g(3) = 4$, so $g(3) = 6$.

10. $\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (x^2 + \sqrt{7-x}) = \lim_{x \rightarrow 4} x^2 + \sqrt{\lim_{x \rightarrow 4} 7 - \lim_{x \rightarrow 4} x} = 4^2 + \sqrt{7-4} = 16 + \sqrt{3} = f(4).$

By the definition of continuity, f is continuous at $a = 4$.

11. $\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x + 2x^3)^4 = \left(\lim_{x \rightarrow -1} x + 2 \lim_{x \rightarrow -1} x^3 \right)^4 = [-1 + 2(-1)^3]^4 = (-3)^4 = 81 = f(-1).$

By the definition of continuity, f is continuous at $a = -1$.

12. $\lim_{x \rightarrow 4} g(x) = \lim_{x \rightarrow 4} \frac{x+1}{2x^2-1} = \frac{\lim_{x \rightarrow 4} x + \lim_{x \rightarrow 4} 1}{2 \lim_{x \rightarrow 4} x^2 - \lim_{x \rightarrow 4} 1} = \frac{4+1}{2(4)^2-1} = \frac{5}{31} = g(4).$ So g is continuous at 4.

$$13. \text{ For } a > 2, \text{ we have } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{2x+3}{x-2} = \frac{\lim_{x \rightarrow a} (2x+3)}{\lim_{x \rightarrow a} (x-2)} \quad [\text{Limit Law 5}] = \frac{2 \lim_{x \rightarrow a} x + \lim_{x \rightarrow a} 3}{\lim_{x \rightarrow a} x - \lim_{x \rightarrow a} 2}$$

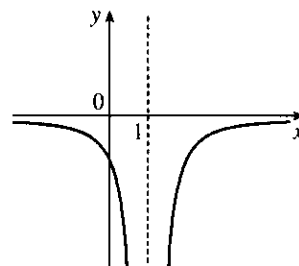
$[1, 2, \text{ and } 3] = \frac{2a+3}{a-2}$ [7 and 8] $= f(a)$. Thus, f is continuous at $x = a$ for every a in $(2, \infty)$; that is, f is continuous on $(2, \infty)$.

$$14. \text{ For } a < 3, \text{ we have } \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} 2\sqrt{3-x} = 2 \lim_{x \rightarrow a} \sqrt{3-x} \quad [\text{Limit Law 3}] = 2\sqrt{\lim_{x \rightarrow a} (3-x)} \quad [11]$$

$$= 2\sqrt{\lim_{x \rightarrow a} 3 - \lim_{x \rightarrow a} x} \quad [2] = 2\sqrt{3-a} \quad [7 \text{ and } 8] = g(a), \text{ so } g \text{ is continuous at } x = a \text{ for every } a \text{ in } (-\infty, 3).$$

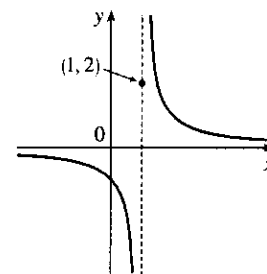
Also, $\lim_{x \rightarrow 3^-} g(x) = 0 = g(3)$, so g is continuous from the left at 3. Thus, g is continuous on $(-\infty, 3]$.

$$15. f(x) = -\frac{1}{(x-1)^2} \text{ is discontinuous at 1 since } f(1) \text{ is not defined.}$$



$$16. f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \text{ is discontinuous at 1 because}$$

$\lim_{x \rightarrow 1} f(x)$ does not exist.



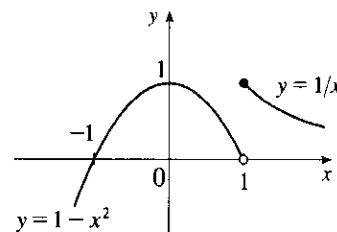
$$17. f(x) = \begin{cases} 1-x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$$

The left-hand limit of f at $a = 1$ is

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1-x^2) = 0. \text{ The right-hand limit of } f \text{ at } a = 1$$

is $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1/x) = 1$. Since these limits are not equal,

$\lim_{x \rightarrow 1} f(x)$ does not exist and f is discontinuous at 1.

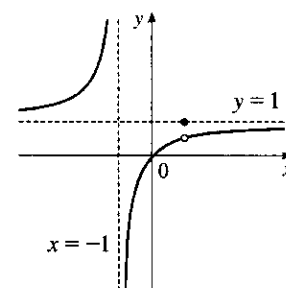


$$18. f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2-x}{x^2-1} = \lim_{x \rightarrow 1} \frac{x(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{x}{x+1} = \frac{1}{2},$$

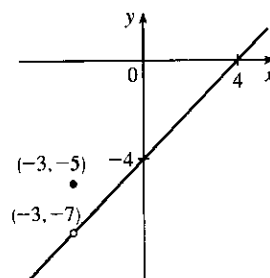
but $f(1) = 1$, so f is discontinuous at 1.



$$19. f(x) = \begin{cases} \frac{x^2 - x - 12}{x + 3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases} = \begin{cases} x - 4 & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases}$$

So $\lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} (x - 4) = -7$ and $f(-3) = -5$.

Since $\lim_{x \rightarrow -3} f(x) \neq f(-3)$, f is discontinuous at -3 .

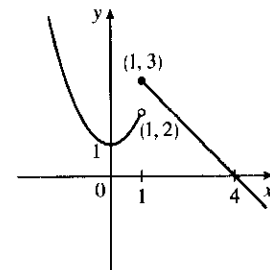


$$20. f(x) = \begin{cases} 1 + x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1 + x^2) = 1 + 1^2 = 2 \text{ and}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4 - x) = 4 - 1 = 3.$$

Thus, f is discontinuous at 1 because $\lim_{x \rightarrow 1} f(x)$ does not exist.



21. $F(x) = \frac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\{x \mid x^2 + 5x + 6 \neq 0\} = \{x \mid (x + 3)(x + 2) \neq 0\} = \{x \mid x \neq -3, -2\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1 + x^3$ are continuous on $\mathbb{R}$. By part 4 of Theorem 4, the product $G(x) = \sqrt[3]{x}(1 + x^3)$ is continuous on its domain, $\mathbb{R}$.

23. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x - 1}$ is continuous on its domain, $[\frac{1}{2}, \infty)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x - 1}$ is continuous on $[\frac{1}{2}, \infty)$.

24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By part 5 of Theorem 4, $h(x) = \frac{\sin x}{x + 1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

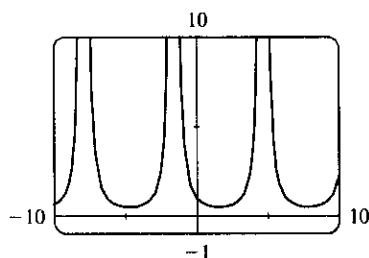
25. By Theorem 5, the polynomial $1 - x^2$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\cos$ is continuous on its domain, $\mathbb{R}$. By Theorem 9, $\cos(1 - x^2)$ is continuous on its domain, which is $\mathbb{R}$.

26. By Theorem 5, the polynomial $2x$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\tan$ is continuous at every number in its domain; that is, $\{x \mid x \neq \frac{\pi}{2} + \pi n\}$. By Theorem 9, $\tan 2x$ is continuous on its domain, which is $\{x \mid 2x \neq \frac{\pi}{2} + \pi n\} = \{x \mid x \neq \frac{\pi}{4} + \frac{\pi}{2}n\}$ (the odd multiples of $\frac{\pi}{4}$).

27. By Theorem 7, the root function $\sqrt{x}$ and the trigonometric function $\sin x$ are continuous on their domains, $[0, \infty)$ and $(-\infty, \infty)$, respectively. Thus, the product $F(x) = \sqrt{x} \sin x$ is continuous on the intersection of those domains, $[0, \infty)$, by part 4 of Theorem 4.

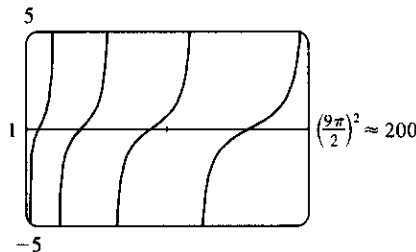
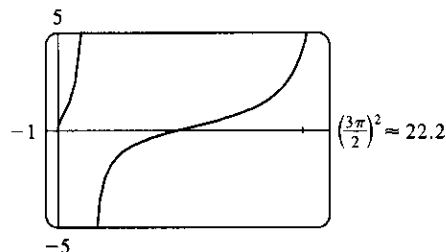
28. The sine and cosine functions are continuous everywhere by Theorem 7, so $F(x) = \sin(\cos(\sin x))$, which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

29.



$y = \frac{1}{1 + \sin x}$ is undefined and hence discontinuous when $1 + \sin x = 0 \Leftrightarrow \sin x = -1 \Leftrightarrow x = -\frac{\pi}{2} + 2\pi n$, n an integer. The figure shows discontinuities for $n = -1, 0$, and 1 ; that is, $-\frac{5\pi}{2} \approx -7.85$, $-\frac{\pi}{2} \approx -1.57$, and $\frac{3\pi}{2} \approx 4.71$.

30.



The function $y = f(x) = \tan \sqrt{x}$ is continuous throughout its domain because it is the composite of a trigonometric function and a root function. The square root function has domain $[0, \infty)$ and the tangent function has domain $\{x \mid x \neq \frac{\pi}{2} + \pi n\}$. So f is discontinuous when $x < 0$ and when $\sqrt{x} = \frac{\pi}{2} + \pi n \Rightarrow x = (\frac{\pi}{2} + \pi n)^2$, where n is a nonnegative integer. Note that as x increases, the distance between discontinuities increases.

31. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x+5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \frac{5 + \sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim_{x \rightarrow 4} f(x) = f(4) = \frac{7}{3}$.

32. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim_{x \rightarrow \pi} f(x) = f(\pi) = \sin(\pi + \sin \pi) = \sin \pi = 0$.

$$33. f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1 \text{ and } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x} = 1. \text{ Thus, } \lim_{x \rightarrow 1} f(x) \text{ exists and equals } 1. \text{ Also,}$$

$$f(1) = \sqrt{1} = 1. \text{ Thus, } f \text{ is continuous at } x = 1. \text{ We conclude that } f \text{ is continuous on } (-\infty, \infty).$$

$$34. f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$$

By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim_{x \rightarrow (\pi/4)^-} f(x) = \lim_{x \rightarrow (\pi/4)^-} \sin x = \sin \frac{\pi}{4} = 1/\sqrt{2}$ since

the sine function is continuous at $\pi/4$. Similarly, $\lim_{x \rightarrow (\pi/4)^+} f(x) = \lim_{x \rightarrow (\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the

cosine function at $\pi/4$. Thus, $\lim_{x \rightarrow (\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$.

Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

$$35. f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$$

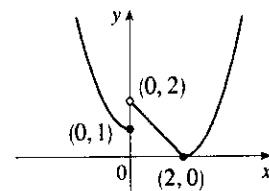
f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on

each of these intervals. Now $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (1 + x^2) = 1$ and

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (2 - x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0.

Also, $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2 - x) = 0$, $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x - 2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2.

The only number at which f is discontinuous is 0.



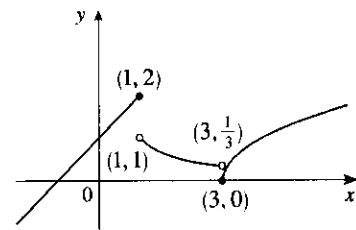
$$36. f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x - 3} & \text{if } x \geq 3 \end{cases}$$

f is continuous on $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 1) = 2$ and

$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1/x) = 1$, so f is discontinuous at 1.

Since $f(1) = 2$, f is continuous from the left at 1. Also, $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (1/x) = 1/3$, and

$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \sqrt{x - 3} = 0 = f(3)$, so f is discontinuous at 3, but it is continuous from the right at 3.



$$37. f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

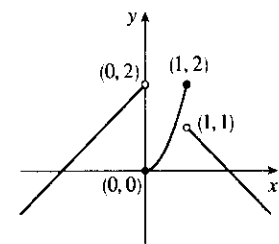
f is continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ since on each of these intervals it is a polynomial. Now $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x + 2) = 2$ and

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2x^2 = 0$, so f is discontinuous at 0. Since $f(0) = 0$, f

is continuous from the right at 0. Also $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2x^2 = 2$ and

$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since

$f(1) = 2$, f is continuous from the left at 1.



38. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$.

$\lim_{r \rightarrow R^-} F(r) = \lim_{r \rightarrow R^-} \frac{GMr}{R^3} = \frac{GM}{R^2}$ and $\lim_{r \rightarrow R^+} F(r) = \lim_{r \rightarrow R^+} \frac{GM}{r^2} = \frac{GM}{R^2}$, so $\lim_{r \rightarrow R} F(r) = \frac{GM}{R^2}$. Since $F(R) = \frac{GM}{R^2}$, F is continuous at R . Therefore, F is a continuous function of r .

39. f is continuous on $(-\infty, 3)$ and $(3, \infty)$. Now $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (cx + 1) = 3c + 1$ and

$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (cx^2 - 1) = 9c - 1$. So f is continuous $\Leftrightarrow 3c + 1 = 9c - 1 \Leftrightarrow 6c = 2 \Leftrightarrow c = \frac{1}{3}$.

Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{1}{3}$.

40. The functions $x^2 - c^2$ and $cx + 20$, considered on the intervals $(-\infty, 4)$ and $[4, \infty)$ respectively, are continuous for any value of c . So the only possible discontinuity is at $x = 4$. For the function to be continuous at $x = 4$, the left-hand and right-hand limits must be the same. Now $\lim_{x \rightarrow 4^-} g(x) = \lim_{x \rightarrow 4^-} (x^2 - c^2) = 16 - c^2$ and

$$\lim_{x \rightarrow 4^+} g(x) = \lim_{x \rightarrow 4^+} (cx + 20) = 4c + 20 = g(4). \text{ Thus, } 16 - c^2 = 4c + 20 \Leftrightarrow c^2 + 4c + 4 = 0 \Leftrightarrow c = -2.$$

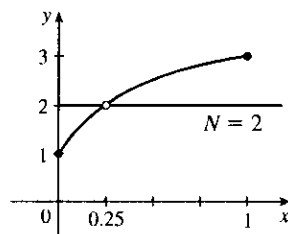
41. (a) $f(x) = \frac{x^2 - 2x - 8}{x + 2} = \frac{(x - 4)(x + 2)}{x + 2}$ has a removable discontinuity at -2 because $g(x) = x - 4$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -2$. [The discontinuity is removed by defining $f(-2) = -6$.]

(b) $f(x) = \frac{x - 7}{|x - 7|} \Rightarrow \lim_{x \rightarrow 7^-} f(x) = -1$ and $\lim_{x \rightarrow 7^+} f(x) = 1$. Thus, $\lim_{x \rightarrow 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

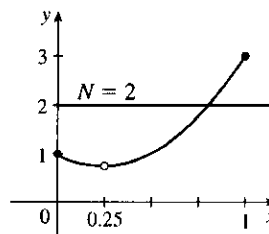
(c) $f(x) = \frac{x^3 + 64}{x + 4} = \frac{(x + 4)(x^2 - 4x + 16)}{x + 4}$ has a removable discontinuity at -4 because $g(x) = x^2 - 4x + 16$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -4$. [The discontinuity is removed by defining $f(-4) = 48$.]

(d) $f(x) = \frac{3 - \sqrt{x}}{9 - x} = \frac{3 - \sqrt{x}}{(3 - \sqrt{x})(3 + \sqrt{x})}$ has a removable discontinuity at 9 because $g(x) = \frac{1}{3 + \sqrt{x}}$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq 9$. [The discontinuity is removed by defining $f(9) = \frac{1}{6}$.]

42.



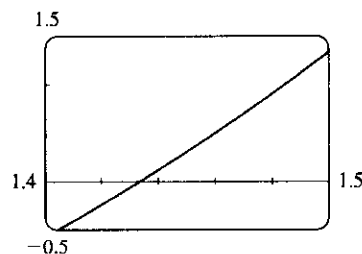
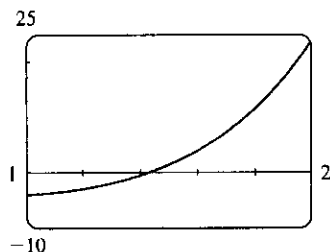
f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

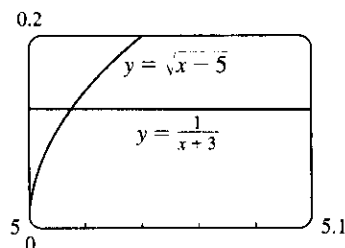
43. $f(x) = x^3 - x^2 + x$ is continuous on the interval $[2, 3]$, $f(2) = 6$, and $f(3) = 21$. Since $6 < 10 < 21$, there is a number c in $(2, 3)$ such that $f(c) = 10$ by the Intermediate Value Theorem.
44. $f(x) = x^2$ is continuous on the interval $[1, 2]$, $f(1) = 1$, and $f(2) = 4$. Since $1 < 2 < 4$, there is a number c in $(1, 2)$ such that $f(c) = c^2 = 2$ by the Intermediate Value Theorem.
45. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.
46. $f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$.
47. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

48. $f(x) = \tan x - 2x$ is continuous on the interval $[0, 1.4]$, $f(1) = \tan 1 - 2 \approx -0.44$, and $f(1.4) = \tan 1.4 - 2.8 \approx 3.00$. Since $-0.44 < 0 < 3.00$, there is a number c in $(0, 1.4)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\tan x - 2x = 0$, or $\tan x = 2x$, in the interval $(0, 1.4)$.
49. (a) $f(x) = \sin x - 2 + x$ is continuous on $[0, 2]$, $f(0) = -2$, and $f(2) = \sin 2 \approx 0.91$. Since $-2 < 0 < 0.91$, there is a number c in $(0, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - 2 + x = 0$, or $\sin x = 2 - x$, in the interval $(0, 2)$.
- (b) $f(1.10) \approx -0.009 < 0$ and $f(1.11) \approx 0.006 > 0$, so there is a root between 1.10 and 1.11.
50. (a) $f(x) = x^5 - x^2 + 2x + 3$ is continuous on $[-1, 0]$, $f(-1) = -1 < 0$, and $f(0) = 3 > 0$. Since $-1 < 0 < 3$, there is a number c in $(-1, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^2 + 2x + 3 = 0$ in the interval $(-1, 0)$.
- (b) $f(-0.88) \approx -0.062 < 0$ and $f(-0.87) \approx 0.0047 > 0$, so there is a root between -0.88 and -0.87 .
51. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number c in $(1, 2)$ such that $f(c) = c^5 - c^2 - 4 = 0$.
- (b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.



52. (a) Let $f(x) = \sqrt{x-5} - \frac{1}{x+3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\frac{1}{c+3} = \sqrt{c-5}$.

- (b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



53. $(\Rightarrow)$ If f is continuous at a , then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h \rightarrow 0} f(a+h) = f\left(\lim_{h \rightarrow 0} (a+h)\right) = f(a).$$

- $(\Leftarrow)$ Let $\varepsilon > 0$. Since $\lim_{h \rightarrow 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \Rightarrow$

$$|f(a+h) - f(a)| < \varepsilon. \text{ So if } 0 < |x - a| < \delta, \text{ then } |f(x) - f(a)| = |f(a + (x - a)) - f(a)| < \varepsilon.$$

Thus, $\lim_{x \rightarrow a} f(x) = f(a)$ and so f is continuous at a .

$$\begin{aligned}
 54. \lim_{h \rightarrow 0} \sin(a+h) &= \lim_{h \rightarrow 0} (\sin a \cos h + \cos a \sin h) = \lim_{h \rightarrow 0} (\sin a \cos h) + \lim_{h \rightarrow 0} (\cos a \sin h) \\
 &= \left(\lim_{h \rightarrow 0} \sin a \right) \left(\lim_{h \rightarrow 0} \cos h \right) + \left(\lim_{h \rightarrow 0} \cos a \right) \left(\lim_{h \rightarrow 0} \sin h \right) \\
 &= (\sin a)(1) + (\cos a)(0) = \sin a
 \end{aligned}$$

55. As in the previous exercise, we must show that $\lim_{h \rightarrow 0} \cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}
 \lim_{h \rightarrow 0} \cos(a+h) &= \lim_{h \rightarrow 0} (\cos a \cos h - \sin a \sin h) \\
 &= \lim_{h \rightarrow 0} (\cos a \cos h) - \lim_{h \rightarrow 0} (\sin a \sin h) \\
 &= \left(\lim_{h \rightarrow 0} \cos a \right) \left(\lim_{h \rightarrow 0} \cos h \right) - \left(\lim_{h \rightarrow 0} \sin a \right) \left(\lim_{h \rightarrow 0} \sin h \right) \\
 &= (\cos a)(1) - (\sin a)(0) = \cos a
 \end{aligned}$$

56. (a) Since f is continuous at a , $\lim_{x \rightarrow a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have

$$\lim_{x \rightarrow a} (cf)(x) = \lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x) = cf(a) = (cf)(a). \text{ Therefore, } cf \text{ is continuous at } a.$$

- (b) Since f and g are continuous at a , $\lim_{x \rightarrow a} f(x) = f(a)$ and $\lim_{x \rightarrow a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the

$$\text{Quotient Law of Limits: } \lim_{x \rightarrow a} \left(\frac{f}{g} \right)(x) = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g} \right)(a). \text{ Thus, } \frac{f}{g} \text{ is continuous}$$

at a .

57. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval

$(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1 , there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x \rightarrow a} f(x) \neq f(a)$.

[In fact, $\lim_{x \rightarrow a} f(x)$ does not even exist.]

58. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is continuous at 0 . To see why, note that $-|x| \leq g(x) \leq |x|$, so by the Squeeze

Theorem $\lim_{x \rightarrow 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval

$(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a , there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x \rightarrow a} g(x) \neq g(a)$.

59. If there is such a number, it satisfies the equation $x^3 + 1 = x \Leftrightarrow x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

60. (a) $\lim_{x \rightarrow 0^+} F(x) = 0$ and $\lim_{x \rightarrow 0^-} F(x) = 0$, so $\lim_{x \rightarrow 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x \rightarrow a} F(x) = \lim_{x \rightarrow a} x = a = F(a)$. For $a < 0$, $\lim_{x \rightarrow a} F(x) = \lim_{x \rightarrow a} (-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I . Then for $a \in I$, $\lim_{x \rightarrow a} |f(x)| = \left| \lim_{x \rightarrow a} f(x) \right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I , use the appropriate one-sided limit.) So $|f|$ is continuous on I .

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

61. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \Leftrightarrow u(t_0) = d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

2.6 Tangents, Velocities, and Other Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(3)}{x - 3}$.

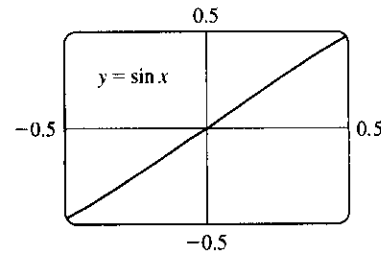
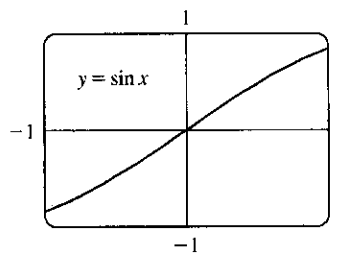
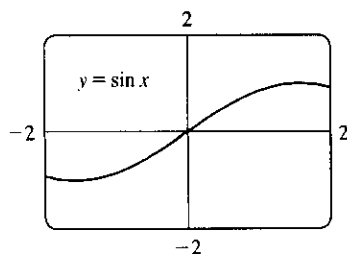
(b) This is the limit of the slope of the secant line PQ as Q approaches P : $m = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$.

2. (a) Average velocity = $\frac{\Delta s}{\Delta t} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$

(b) Instantaneous velocity = $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

3. The slope at D is the largest positive slope, followed by the positive slope at E . The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D, E, C, A , and B .

4. The curve looks more like a line as the viewing rectangle gets smaller.



5. (a) (i) Using Definition 1,

$$\begin{aligned}
 m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow -3} \frac{f(x) - f(-3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x^2 + 2x) - (3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x+3)(x-1)}{x+3} \\
 &= \lim_{x \rightarrow -3} (x-1) = -4
 \end{aligned}$$

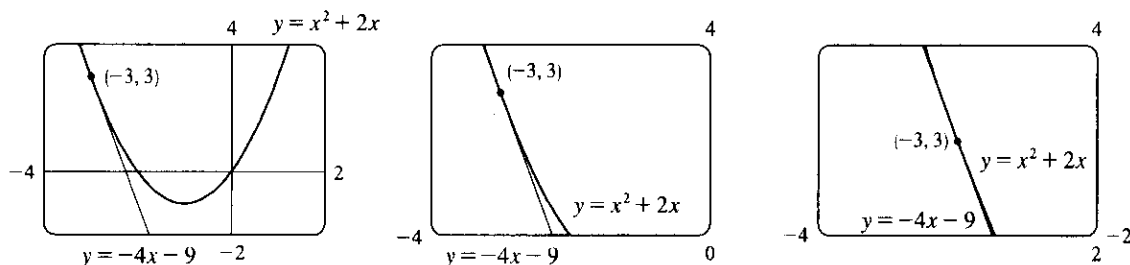
- (ii) Using Equation 2,

$$\begin{aligned}
 m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h \rightarrow 0} \frac{[(-3+h)^2 + 2(-3+h)] - (3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{9 - 6h + h^2 - 6 + 2h - 3}{h} = \lim_{h \rightarrow 0} \frac{h(h-4)}{h} = \lim_{h \rightarrow 0} (h-4) = -4
 \end{aligned}$$

- (b) Using the point-slope form of the equation of a line, an equation of the tangent line is
- $y - 3 = -4(x + 3)$
- .

Solving for y gives us $y = -4x - 9$, which is the slope-intercept form of the equation of the tangent line.

- (c)

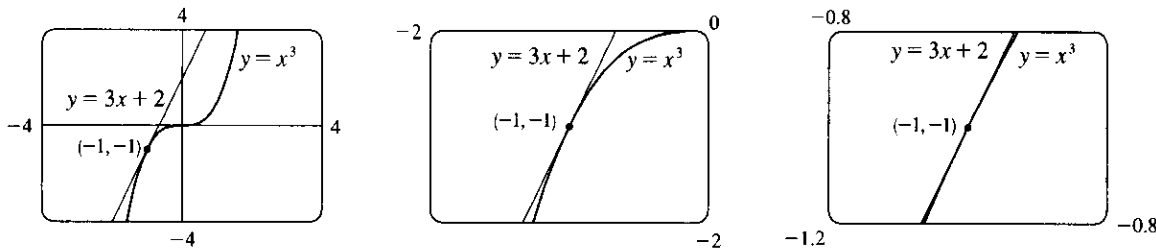


$$\begin{aligned}
 6. (a) (i) m &= \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{x^3 - (-1)}{x + 1} = \lim_{x \rightarrow -1} \frac{(x+1)(x^2 - x + 1)}{x + 1} \\
 &= \lim_{x \rightarrow -1} (x^2 - x + 1) = 3
 \end{aligned}$$

$$\begin{aligned}
 (ii) m &= \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{(-1+h)^3 - (-1)}{h} = \lim_{h \rightarrow 0} \frac{h^3 - 3h^2 + 3h - 1 + 1}{h} \\
 &= \lim_{h \rightarrow 0} (h^2 - 3h + 3) = 3
 \end{aligned}$$

$$(b) y - (-1) = 3[x - (-1)] \Leftrightarrow y + 1 = 3x + 3 \Leftrightarrow y = 3x + 2$$

- (c)



7. Using (2) with
- $f(x) = 1 + 2x - x^3$
- and
- $P(1, 2)$
- ,

$$\begin{aligned}
 m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[1 + 2(1+h) - (1+h)^3] - 2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1 + 2 + 2h - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \rightarrow 0} \frac{-h^3 - 3h^2 - h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(-h^2 - 3h - 1)}{h} = \lim_{h \rightarrow 0} (-h^2 - 3h - 1) = -1
 \end{aligned}$$

$$\text{Tangent line: } y - 2 = -1(x - 1) \Leftrightarrow y - 2 = -x + 1 \Leftrightarrow y = -x + 3$$

8. Using (1),

$$\begin{aligned}
 m &= \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - \sqrt{2(4)+1}}{x-4} = \lim_{x \rightarrow 4} \frac{\sqrt{2x+1}-3}{x-4} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} \\
 &= \lim_{x \rightarrow 4} \frac{(2x+1)-3^2}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x \rightarrow 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} \\
 &= \lim_{x \rightarrow 4} \frac{2}{(\sqrt{2x+1}+3)} = \frac{2}{3+3} = \frac{1}{3}.
 \end{aligned}$$

$$\text{Tangent line: } y-3 = \frac{1}{3}(x-4) \Leftrightarrow y-3 = \frac{1}{3}x - \frac{4}{3} \Leftrightarrow y = \frac{1}{3}x + \frac{5}{3}$$

9. Using (1) with $f(x) = \frac{x-1}{x-2}$ and $P(3, 2)$,

$$\begin{aligned}
 m &= \lim_{x \rightarrow 3} \frac{f(x) - f(a)}{x-a} = \lim_{x \rightarrow 3} \frac{\frac{x-1}{x-2} - 2}{x-3} = \lim_{x \rightarrow 3} \frac{\frac{x-1-2(x-2)}{x-2}}{x-3} = \lim_{x \rightarrow 3} \frac{3-x}{(x-2)(x-3)} \\
 &= \lim_{x \rightarrow 3} \frac{-1}{x-2} = \frac{-1}{1} = -1.
 \end{aligned}$$

$$\text{Tangent line: } y-2 = -1(x-3) \Leftrightarrow y-2 = -x+3 \Leftrightarrow y = -x+5$$

$$10. \text{ Using (1), } m = \lim_{x \rightarrow 0} \frac{\frac{2x}{(x+1)^2} - 0}{x-0} = \lim_{x \rightarrow 0} \frac{2x}{x(x+1)^2} = \lim_{x \rightarrow 0} \frac{2}{(x+1)^2} = \frac{2}{1^2} = 2.$$

$$\text{Tangent line: } y-0 = 2(x-0) \Leftrightarrow y = 2x$$

$$\begin{aligned}
 11. (a) \quad m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a} = \lim_{x \rightarrow a} \frac{2/(x+3) - 2/(a+3)}{x-a} = \lim_{x \rightarrow a} \frac{2(a+3) - 2(x+3)}{(x-a)(x+3)(a+3)} \\
 &= \lim_{x \rightarrow a} \frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x \rightarrow a} \frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}
 \end{aligned}$$

$$(b) \quad (i) \ a = -1 \Rightarrow m = \frac{-2}{(-1+3)^2} = -\frac{1}{2} \qquad (ii) \ a = 0 \Rightarrow m = \frac{-2}{(0+3)^2} = -\frac{2}{9}$$

$$(iii) \ a = 1 \Rightarrow m = \frac{-2}{(1+3)^2} = -\frac{1}{8}$$

12. (a) Using (1),

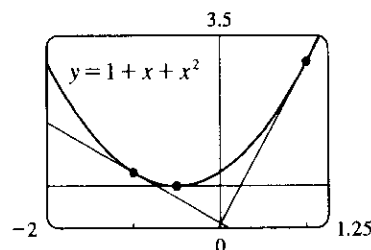
$$\begin{aligned}
 m &= \lim_{x \rightarrow a} \frac{(1+x+x^2) - (1+a+a^2)}{x-a} = \lim_{x \rightarrow a} \frac{x+x^2-a-a^2}{x-a} = \lim_{x \rightarrow a} \frac{x-a+(x-a)(x+a)}{x-a} \\
 &= \lim_{x \rightarrow a} \frac{(x-a)(1+x+a)}{x-a} = \lim_{x \rightarrow a} (1+x+a) = 1+2a
 \end{aligned}$$

$$(b) \quad (i) \ x = -1 \Rightarrow m = 1+2(-1) = -1$$

$$(ii) \ x = -\frac{1}{2} \Rightarrow m = 1+2\left(-\frac{1}{2}\right) = 0$$

$$(iii) \ x = 1 \Rightarrow m = 1+2(1) = 3$$

(c)



13. (a) Using (1),

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{(x^3 - 4x + 1) - (a^3 - 4a + 1)}{x - a} = \lim_{x \rightarrow a} \frac{(x^3 - a^3) - 4(x - a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x - a)(x^2 + ax + a^2) - 4(x - a)}{x - a} = \lim_{x \rightarrow a} (x^2 + ax + a^2 - 4) = 3a^2 - 4 \end{aligned}$$

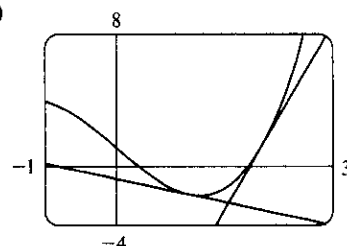
(b) At $(1, -2)$: $m = 3(1)^2 - 4 = -1$, so an equation of the tangent line

$$\text{is } y - (-2) = -1(x - 1) \Leftrightarrow y = -x - 1. \text{ At } (2, 1):$$

$$m = 3(2)^2 - 4 = 8, \text{ so an equation of the tangent line is}$$

$$y - 1 = 8(x - 2) \Leftrightarrow y = 8x - 15.$$

(c)



14. (a) Using (1),

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a} = \lim_{x \rightarrow a} \frac{\frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x \rightarrow a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}(x - a)(\sqrt{a} + \sqrt{x})} \\ &= \lim_{x \rightarrow a} \frac{a - x}{\sqrt{ax}(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \rightarrow a} \frac{-1}{\sqrt{ax}(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2} \end{aligned}$$

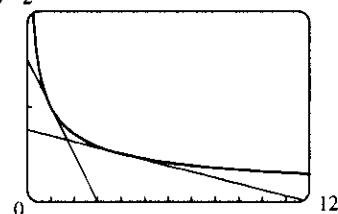
(b) At $(1, 1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}.$$

At $(4, \frac{1}{2})$: $m = -\frac{1}{16}$, so an equation of the tangent line is

$$y - \frac{1}{2} = -\frac{1}{16}(x - 4) \Leftrightarrow y = -\frac{1}{16}x + \frac{3}{4}.$$

(c) 2



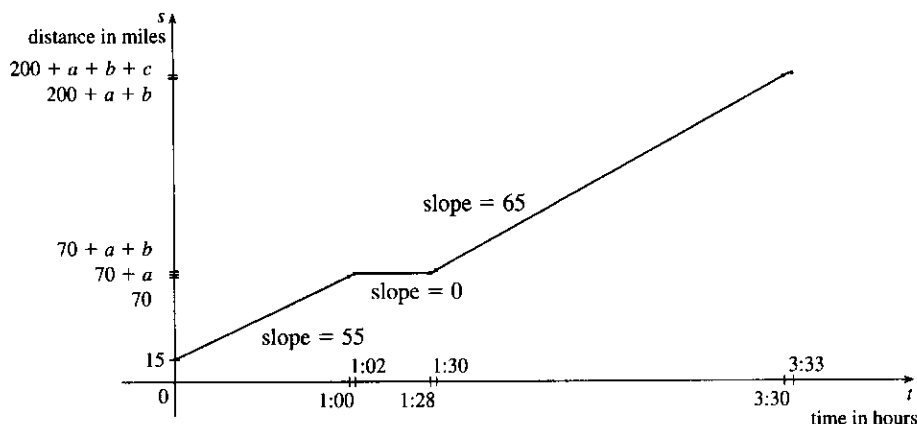
15. (a) Since the slope of the tangent at $t = 0$ is 0, the car's initial velocity was 0.

(b) The slope of the tangent is greater at C than at B , so the car was going faster at C .

(c) Near A , the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B , the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C , so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E , the slope of the tangent is 0, so the car did not move during that time.

16. Let a denote the distance traveled from 1:00 to 1:02, b from 1:28 to 1:30, and c from 3:30 to 3:33, where all the times are relative to $t = 0$ at the beginning of the trip.



17. Let $s(t) = 40t - 16t^2$.

$$\begin{aligned} v(2) &= \lim_{t \rightarrow 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \rightarrow 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t \rightarrow 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t \rightarrow 2} \frac{-8(2t^2 - 5t + 2)}{t - 2} \\ &= \lim_{t \rightarrow 2} \frac{-8(t - 2)(2t - 1)}{t - 2} = -8 \lim_{t \rightarrow 2} (2t - 1) = -8(3) = -24 \end{aligned}$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

$$\begin{aligned} 18. (a) \quad v(1) &= \lim_{h \rightarrow 0} \frac{H(1+h) - H(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(58 + 58h - 0.83 - 1.66h - 0.83h^2) - 57.17}{h} = \lim_{h \rightarrow 0} (56.34 - 0.83h) = 56.34 \text{ m/s} \end{aligned}$$

$$\begin{aligned} (b) \quad v(a) &= \lim_{h \rightarrow 0} \frac{H(a+h) - H(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2) - (58a - 0.83a^2)}{h} \\ &= \lim_{h \rightarrow 0} (58 - 1.66a - 0.83h) = 58 - 1.66a \text{ m/s} \end{aligned}$$

(c) The arrow strikes the moon when the height is 0, that is, $58t - 0.83t^2 = 0 \Leftrightarrow t(58 - 0.83t) = 0 \Leftrightarrow t = \frac{58}{0.83} \approx 69.9$ s (since t can't be 0).

(d) Using the time from part (c), $v\left(\frac{58}{0.83}\right) = 58 - 1.66\left(\frac{58}{0.83}\right) = -58$ m/s. Thus, the arrow will have a velocity of -58 m/s.

$$\begin{aligned} 19. \quad v(a) &= \lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \rightarrow 0} \frac{4(a+h)^3 + 6(a+h) + 2 - (4a^3 + 6a + 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4a^3 + 12a^2h + 12ah^2 + 4h^3 + 6a + 6h + 2 - 4a^3 - 6a - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{12a^2h + 12ah^2 + 4h^3 + 6h}{h} = \lim_{h \rightarrow 0} (12a^2 + 12ah + 4h^2 + 6) = (12a^2 + 6) \text{ m/s} \end{aligned}$$

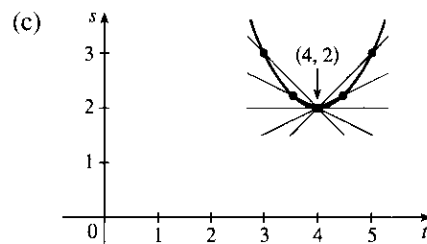
So $v(1) = 12(1)^2 + 6 = 18$ m/s, $v(2) = 12(2)^2 + 6 = 54$ m/s, and $v(3) = 12(3)^2 + 6 = 114$ m/s.

20. (a) The average velocity between times
- t
- and
- $t + h$
- is

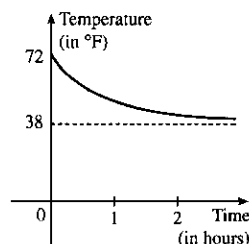
$$\begin{aligned}\frac{s(t+h) - s(t)}{(t+h) - t} &= \frac{(t+h)^2 - 8(t+h) + 18 - (t^2 - 8t + 18)}{h} \\ &= \frac{t^2 + 2th + h^2 - 8t - 8h + 18 - t^2 + 8t - 18}{h} = \frac{2th + h^2 - 8h}{h} \\ &= (2t + h - 8) \text{ m/s}\end{aligned}$$

(i) $[3, 4]$: $t = 3$, $h = 4 - 3 = 1$, so the average velocity is $2(3) + 1 - 8 = -1$ m/s.(ii) $[3.5, 4]$: $t = 3.5$, $h = 0.5$, so the average velocity is $2(3.5) + 0.5 - 8 = -0.5$ m/s.(iii) $[4, 5]$: $t = 4$, $h = 1$, so the average velocity is $2(4) + 1 - 8 = 1$ m/s.(iv) $[4, 4.5]$: $t = 4$, $h = 0.5$, so the average velocity is $2(4) + 0.5 - 8 = 0.5$ m/s.

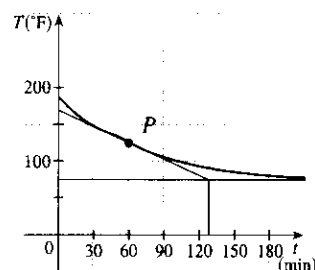
$$(b) v(t) = \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \rightarrow 0} (2t + h - 8) = 2t - 8, \text{ so } v(4) = 0.$$



21. The sketch shows the graph for a room temperature of
- 72°
- and a refrigerator temperature of
- 38°
- . The initial rate of change is greater in magnitude than the rate of change after an hour.



22. The slope of the tangent (that is, the rate of change of temperature with respect to time) at
- $t = 1$
- h seems to be about
- $\frac{75 - 168}{132 - 0} \approx -0.7^\circ\text{F/min}$
- .



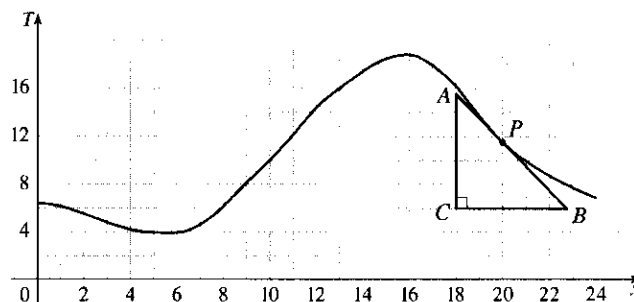
23. (a) (i) $[20, 23]$: $\frac{7.9 - 11.5}{23 - 20} = -1.2^\circ\text{C/h}$

(ii) $[20, 22]$: $\frac{9.0 - 11.5}{22 - 20} = -1.25^\circ\text{C/h}$

(iii) $[20, 21]$: $\frac{10.2 - 11.5}{21 - 20} = -1.3^\circ\text{C/h}$

- (b) In the figure, we estimate
- A
- to be
- $(18, 15.5)$
- and
- B
- as
- $(23, 6)$
- . So the slope is

$$\frac{6 - 15.5}{23 - 18} = -1.9^\circ\text{C/h at 8:00 P.M.}$$



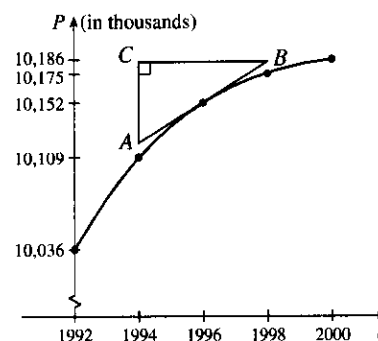
24. (a) (i) [1992, 1996]: $\frac{P(1996) - P(1992)}{1996 - 1992} = \frac{10,152 - 10,036}{4} = \frac{116}{4} = 29$ thousand/year
- (ii) [1994, 1996]: $\frac{P(1996) - P(1994)}{1996 - 1994} = \frac{10,152 - 10,109}{2} = \frac{43}{2} = 21.5$ thousand/year
- (iii) [1996, 1998]: $\frac{P(1998) - P(1996)}{1998 - 1996} = \frac{10,175 - 10,152}{2} = \frac{23}{2} = 11.5$ thousand/year

(b) Using the values from (ii) and (iii), we have $\frac{21.5 + 11.5}{2} = 16.5$ thousand/year.

(c) Estimating A as (1994, 10,125) and B as

(1998, 10,182), the slope at 1996 is

$$\frac{10,182 - 10,125}{1998 - 1994} = \frac{57}{4} = 14.25 \text{ thousand/year.}$$

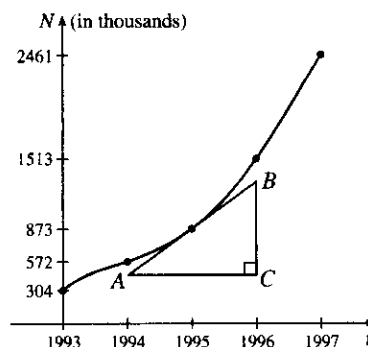


25. (a) (i) [1995, 1997]: $\frac{N(1997) - N(1995)}{1997 - 1995} = \frac{2461 - 873}{2} = \frac{1588}{2} = 794$ thousand/year
- (ii) [1995, 1996]: $\frac{N(1996) - N(1995)}{1996 - 1995} = \frac{1513 - 873}{1} = 640$ thousand/year
- (iii) [1994, 1995]: $\frac{N(1995) - N(1994)}{1995 - 1994} = \frac{873 - 572}{1} = 301$ thousand/year

(b) Using the values from (ii) and (iii), we have $\frac{640 + 301}{2} = \frac{941}{2} = 470.5$ thousand/year.

(c) Estimating A as (1994, 420) and B as (1996, 1275), the slope

$$\text{at 1995 is } \frac{1275 - 420}{1996 - 1994} = \frac{855}{2} = 427.5 \text{ thousand/year}$$

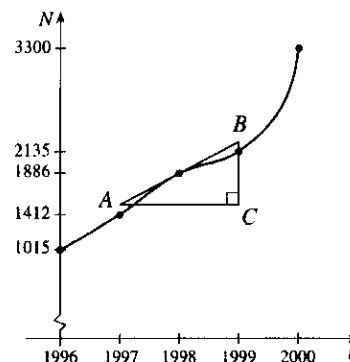


26. (a) (i) [1996, 1998]: $\frac{N(1998) - N(1996)}{1998 - 1996} = \frac{1886 - 1015}{2} = \frac{871}{2} = 435.5$ locations/year
- (ii) [1997, 1998]: $\frac{N(1998) - N(1997)}{1998 - 1997} = \frac{1886 - 1412}{1} = 474$ locations/year
- (iii) [1998, 1999]: $\frac{N(1999) - N(1998)}{1999 - 1998} = \frac{2135 - 1886}{1} = 249$ locations/year

(b) Using the values from (ii) and (iii), we have $\frac{474 + 249}{2} = \frac{723}{2} = 361.5 \approx 362$ locations/year.

(c) Estimating A as (1997, 1525) and B as (1999, 2250), the slope

$$\text{at 1998 is } \frac{2250 - 1525}{1999 - 1997} = \frac{725}{2} = 362.5 \text{ locations/year.}$$



$$27. (a) (i) \frac{\Delta C}{\Delta x} = \frac{C(105) - C(100)}{105 - 100} = \frac{6601.25 - 6500}{5} = \$20.25/\text{unit.}$$

$$(ii) \frac{\Delta C}{\Delta x} = \frac{C(101) - C(100)}{101 - 100} = \frac{6520.05 - 6500}{1} = \$20.05/\text{unit.}$$

$$(b) \frac{C(100 + h) - C(100)}{h} = \frac{[5000 + 10(100 + h) + 0.05(100 + h)^2] - 6500}{h} = \frac{20h + 0.05h^2}{h} \\ = 20 + 0.05h, h \neq 0$$

$$\text{So the instantaneous rate of change is } \lim_{h \rightarrow 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \rightarrow 0} (20 + 0.05h) = \$20/\text{unit.}$$

$$28. \Delta V = V(t + h) - V(t) = 100,000 \left(1 - \frac{t + h}{60}\right)^2 - 100,000 \left(1 - \frac{t}{60}\right)^2 \\ = 100,000 \left[\left(1 - \frac{t + h}{60} + \frac{(t + h)^2}{3600}\right) - \left(1 - \frac{t}{60} + \frac{t^2}{3600}\right) \right] = 100,000 \left(-\frac{h}{30} + \frac{2th}{3600} + \frac{h^2}{3600}\right) \\ = \frac{100,000}{3600} h (-120 + 2t + h) = \frac{250}{9} h (-120 + 2t + h)$$

Dividing ΔV by h and then letting $h \rightarrow 0$, we see that the instantaneous rate of change is $\frac{500}{9} (t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\bar{3}$	100,000
10	$-2777.\bar{7}$	69,444. $\bar{4}$
20	$-2222.\bar{2}$	44,444. $\bar{4}$
30	$-1666.\bar{6}$	25,000
40	$-1111.\bar{1}$	11,111. $\bar{1}$
50	$-555.\bar{5}$	2,777. $\bar{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

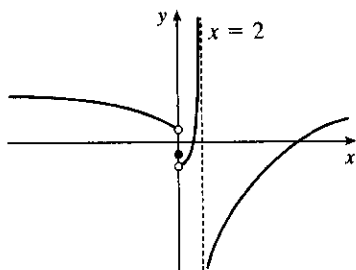
44 □ CHAPTER 2 LIMITS AND RATES OF CHANGE

4. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)
5. False. Consider $\lim_{x \rightarrow 5} \frac{x(x-5)}{x-5}$ or $\lim_{x \rightarrow 5} \frac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).
6. False. Consider $\lim_{x \rightarrow 6} [f(x)g(x)] = \lim_{x \rightarrow 6} \left[(x-6) \frac{1}{x-6} \right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.
7. True. A polynomial is continuous everywhere, so $\lim_{x \rightarrow b} p(x)$ exists and is equal to $p(b)$.
8. False. Consider $\lim_{x \rightarrow 0} [f(x) - g(x)] = \lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{x^4} \right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞ .
9. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$
10. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let
$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$$
 There is no number $c \in [0, 3]$ with $f(c) = 0$.
11. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.
12. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.
13. True, by the definition of a limit with $\varepsilon = 1$.
14. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$
Then $f(x) > 1$ for all x , but $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x^2 + 1) = 1$.

EXERCISES

1. (a) (i) $\lim_{x \rightarrow 2^+} f(x) = 3$ (ii) $\lim_{x \rightarrow -3^+} f(x) = 0$
(iii) $\lim_{x \rightarrow -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2 .) (iv) $\lim_{x \rightarrow 4} f(x) = 2$
(v) $\lim_{x \rightarrow 0} f(x) = \infty$ (vi) $\lim_{x \rightarrow 2^-} f(x) = -\infty$
- (b) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.
- (c) f is discontinuous at $x = -3, 0, 2$, and 4 . The discontinuities are jump, infinite, infinite, and removable, respectively.

2.



$$3. \lim_{x \rightarrow 0} \cos(x + \sin x) = \cos \left[\lim_{x \rightarrow 0} (x + \sin x) \right] \quad [\text{by Theorem 2.5.8}] = \cos 0 = 1$$

$$4. \text{ Since rational functions are continuous, } \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{3^2 - 9}{3^2 + 2(3) - 3} = \frac{0}{12} = 0.$$

$$5. \lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \rightarrow -3} \frac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x \rightarrow -3} \frac{x-3}{x-1} = \frac{-3-3}{-3-1} = \frac{-6}{-4} = \frac{3}{2}$$

$$6. \lim_{x \rightarrow 1^+} \frac{x^2 - 9}{x^2 + 2x - 3} = -\infty \text{ since } x^2 + 2x - 3 \rightarrow 0 \text{ as } x \rightarrow 1^+ \text{ and } \frac{x^2 - 9}{x^2 + 2x - 3} < 0 \text{ for } 1 < x < 3.$$

$$7. \lim_{h \rightarrow 0} \frac{(h-1)^3 + 1}{h} = \lim_{h \rightarrow 0} \frac{(h^3 - 3h^2 + 3h - 1) + 1}{h} = \lim_{h \rightarrow 0} \frac{h^3 - 3h^2 + 3h}{h} = \lim_{h \rightarrow 0} (h^2 - 3h + 3) = 3$$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(h-1)^3 + 1}{h} &= \lim_{h \rightarrow 0} \frac{(h-1)^3 + 1^3}{h} = \lim_{h \rightarrow 0} \frac{[(h-1) + 1][(h-1)^2 - 1(h-1) + 1^2]}{h} \\ &= \lim_{h \rightarrow 0} [(h-1)^2 - h + 2] = 1 - 0 + 2 = 3 \end{aligned}$$

$$8. \lim_{t \rightarrow 2} \frac{t^2 - 4}{t^3 - 8} = \lim_{t \rightarrow 2} \frac{(t+2)(t-2)}{(t-2)(t^2 + 2t + 4)} = \lim_{t \rightarrow 2} \frac{t+2}{t^2 + 2t + 4} = \frac{2+2}{4+4+4} = \frac{4}{12} = \frac{1}{3}$$

$$9. \lim_{r \rightarrow 9} \frac{\sqrt{r}}{(r-9)^4} = \infty \text{ since } (r-9)^4 \rightarrow 0 \text{ as } r \rightarrow 9 \text{ and } \frac{\sqrt{r}}{(r-9)^4} > 0 \text{ for } r \neq 9.$$

$$10. \lim_{v \rightarrow 4^+} \frac{4-v}{|4-v|} = \lim_{v \rightarrow 4^+} \frac{4-v}{-(4-v)} = \lim_{v \rightarrow 4^+} \frac{1}{-1} = -1$$

$$11. \lim_{s \rightarrow 16} \frac{4-\sqrt{s}}{s-16} = \lim_{s \rightarrow 16} \frac{4-\sqrt{s}}{(\sqrt{s}+4)(\sqrt{s}-4)} = \lim_{s \rightarrow 16} \frac{-1}{\sqrt{s}+4} = \frac{-1}{\sqrt{16}+4} = -\frac{1}{8}$$

$$12. \lim_{v \rightarrow 2} \frac{v^2 + 2v - 8}{v^4 - 16} = \lim_{v \rightarrow 2} \frac{(v+4)(v-2)}{(v+2)(v-2)(v^2+4)} = \lim_{v \rightarrow 2} \frac{v+4}{(v+2)(v^2+4)} = \frac{2+4}{(2+2)(2^2+4)} = \frac{3}{16}$$

$$13. \frac{|x-8|}{x-8} = \begin{cases} \frac{x-8}{x-8} & \text{if } x-8 > 0 \\ \frac{-(x-8)}{x-8} & \text{if } x-8 < 0 \end{cases} = \begin{cases} 1 & \text{if } x > 8 \\ -1 & \text{if } x < 8 \end{cases}$$

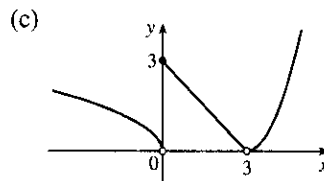
$$\text{Thus, } \lim_{x \rightarrow 8^-} \frac{|x-8|}{x-8} = \lim_{x \rightarrow 8^-} (-1) = -1.$$

$$14. \lim_{x \rightarrow 9^+} (\sqrt{x-9} + \lfloor x+1 \rfloor) = \lim_{x \rightarrow 9^+} \sqrt{x-9} + \lim_{x \rightarrow 9^+} \lfloor x+1 \rfloor = \sqrt{9-9} + 10 = 10$$

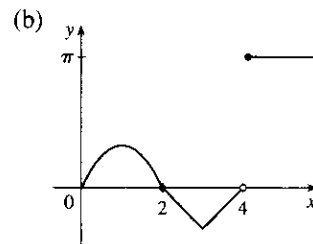
$$15. \lim_{x \rightarrow 0} \frac{1 - \sqrt{1-x^2}}{x} \cdot \frac{1 + \sqrt{1-x^2}}{1 + \sqrt{1-x^2}} = \lim_{x \rightarrow 0} \frac{1 - (1-x^2)}{x(1 + \sqrt{1-x^2})} = \lim_{x \rightarrow 0} \frac{x^2}{x(1 + \sqrt{1-x^2})} = \lim_{x \rightarrow 0} \frac{x}{1 + \sqrt{1-x^2}} = 0$$

$$16. \lim_{x \rightarrow 2} \frac{\sqrt{x+2} - \sqrt{2x}}{x(x-2)} \cdot \frac{\sqrt{x+2} + \sqrt{2x}}{\sqrt{x+2} + \sqrt{2x}} = \lim_{x \rightarrow 2} \frac{-(x-2)}{x(x-2)(\sqrt{x+2} + \sqrt{2x})} = \lim_{x \rightarrow 2} \frac{-1}{x(\sqrt{x+2} + \sqrt{2x})} = -\frac{1}{8}$$

17. Since $2x - 1 \leq f(x) \leq x^2$ for $0 < x < 3$ and $\lim_{x \rightarrow 1} (2x - 1) = 1 = \lim_{x \rightarrow 1} x^2$, we have $\lim_{x \rightarrow 1} f(x) = 1$ by the Squeeze Theorem.
18. Let $f(x) = -x^2$, $g(x) = x^2 \cos(1/x^2)$ and $h(x) = x^2$. Then since $|\cos(1/x^2)| \leq 1$ for $x \neq 0$, we have $f(x) \leq g(x) \leq h(x)$ for $x \neq 0$, and so $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0 \Rightarrow \lim_{x \rightarrow 0} g(x) = 0$ by the Squeeze Theorem.
19. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 5| < \delta$, then $|(7x - 27) - 8| < \varepsilon \Leftrightarrow |7x - 35| < \varepsilon \Leftrightarrow |x - 5| < \varepsilon/7$. So take $\delta = \varepsilon/7$. Then $0 < |x - 5| < \delta \Rightarrow |(7x - 27) - 8| < \varepsilon$. Thus, $\lim_{x \rightarrow 5} (7x - 27) = 8$ by the definition of a limit.
20. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0 < |x - 0| < \delta$, then $|\sqrt[3]{x} - 0| < \varepsilon$. Now $|\sqrt[3]{x} - 0| = |\sqrt[3]{x}| < \varepsilon \Rightarrow |x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0 < |x - 0| = |x| < \varepsilon^3 \Rightarrow |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim_{x \rightarrow 0} \sqrt[3]{x} = 0$.
21. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $|x^2 - 3x - (-2)| < \varepsilon$. First, note that if $|x - 2| < 1$, then $-1 < x - 2 < 1$, so $0 < x - 1 < 2 \Rightarrow |x - 1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x - 2| < \delta \Rightarrow |x^2 - 3x - (-2)| = |(x - 2)(x - 1)| = |x - 2||x - 1| < (\varepsilon/2)(2) = \varepsilon$. Thus, $\lim_{x \rightarrow 2} (x^2 - 3x) = -2$ by the definition of a limit.
22. Given $M > 0$, we need $\delta > 0$ such that if $0 < x - 4 < \delta$, then $2/\sqrt{x - 4} > M$. This is true $\Leftrightarrow \sqrt{x - 4} < 2/M \Leftrightarrow x - 4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x - 4 < \delta \Rightarrow 2/\sqrt{x - 4} > M$. So by the definition of a limit, $\lim_{x \rightarrow 4^+} (2/\sqrt{x - 4}) = \infty$.
23. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3 - x$ if $0 \leq x < 3$, $f(x) = (x - 3)^2$ if $x > 3$.
- (i) $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (3 - x) = 3$ (ii) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sqrt{-x} = 0$
- (iii) Because of (i) and (ii), $\lim_{x \rightarrow 0} f(x)$ does not exist. (iv) $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (3 - x) = 0$
- (v) $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x - 3)^2 = 0$ (vi) Because of (iv) and (v), $\lim_{x \rightarrow 3} f(x) = 0$.
- (b) f is discontinuous at 0 since $\lim_{x \rightarrow 0} f(x)$ does not exist.
- f is discontinuous at 3 since $f(3)$ does not exist.



24. (a) $g(x) = 2x - x^2$ if $0 \leq x \leq 2$, $g(x) = 2 - x$ if $2 < x \leq 3$, $g(x) = x - 4$ if $3 < x < 4$, $g(x) = \pi$ if $x \geq 4$. Therefore,
- $\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} (2x - x^2) = 0$ and $\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} (2 - x) = 0$. Thus, $\lim_{x \rightarrow 2} g(x) = 0 = g(2)$, so g is continuous at 2.
- $\lim_{x \rightarrow 3^-} g(x) = \lim_{x \rightarrow 3^-} (2 - x) = -1$ and $\lim_{x \rightarrow 3^+} g(x) = \lim_{x \rightarrow 3^+} (x - 4) = -1$. Thus, $\lim_{x \rightarrow 3} g(x) = -1 = g(3)$, so g is continuous at 3.
- $\lim_{x \rightarrow 4^-} g(x) = \lim_{x \rightarrow 4^-} (x - 4) = 0$ and $\lim_{x \rightarrow 4^+} g(x) = \lim_{x \rightarrow 4^+} \pi = \pi$. Thus, $\lim_{x \rightarrow 4} g(x)$ does not exist, so g is discontinuous at 4. But $\lim_{x \rightarrow 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.
- (b)



25. x^3 is continuous on $\mathbb{R}$ since it is a polynomial and $\cos x$ is also continuous on $\mathbb{R}$, so the product $x^3 \cos x$ is continuous on $\mathbb{R}$. The root function $\sqrt[4]{x}$ is continuous on its domain, $[0, \infty)$, and so the sum, $h(x) = \sqrt[4]{x} + x^3 \cos x$, is continuous on its domain, $[0, \infty)$.
26. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$, so the composition $\sqrt{x^2 - 9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \frac{\sqrt{x^2 - 9}}{x^2 - 2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$.
27. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.
28. Let $f(x) = 2 \sin x - 3 + 2x$. Now f is continuous on $[0, 1]$ and $f(0) = -3 < 0$ and $f(1) = 2 \sin 1 - 1 \approx 0.68 > 0$. So by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $f(c) = 0$, that is, the equation $2 \sin x = 3 - 2x$ has a root in $(0, 1)$.

29. (a) The slope of the tangent line at $(2, 1)$ is

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x \rightarrow 2} \frac{8 - 2x^2}{x - 2} = \lim_{x \rightarrow 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x \rightarrow 2} \frac{-2(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} [-2(x + 2)] = -2 \cdot 4 = -8 \end{aligned}$$

(b) An equation of this tangent line is $y - 1 = -8(x - 2)$ or $y = -8x + 17$.

30. For a general point with x -coordinate a , we have

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{2/(1 - 3x) - 2/(1 - 3a)}{x - a} = \lim_{x \rightarrow a} \frac{2(1 - 3a) - 2(1 - 3x)}{(1 - 3a)(1 - 3x)(x - a)} \\ &= \lim_{x \rightarrow a} \frac{6(x - a)}{(1 - 3a)(1 - 3x)(x - a)} = \lim_{x \rightarrow a} \frac{6}{(1 - 3a)(1 - 3x)} = \frac{6}{(1 - 3a)^2} \end{aligned}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x - 0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x + 1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

31. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1 + h]$ is

$$v_{\text{ave}} = \frac{s(1 + h) - s(1)}{(1 + h) - 1} = \frac{1 + 2(1 + h) + (1 + h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}.$$

So for the following intervals the average velocities are:

- (i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s
- (ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s
- (iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s
- (iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\lim_{h \rightarrow 0} \frac{s(1 + h) - s(1)}{h} = \lim_{h \rightarrow 0} \frac{10 + h}{4} = \frac{10}{4} = 2.5$ m/s.

32. (a) When V increases from 200 in^3 to 250 in^3 , we have $\Delta V = 250 - 200 = 50 \text{ in}^3$, and since $P = 800/V$,

$$\Delta P = P(250) - P(200) = \frac{800}{250} - \frac{800}{200} = 3.2 - 4 = -0.8 \text{ lb/in}^2. \text{ So the average rate of change is}$$

$$\frac{\Delta P}{\Delta V} = \frac{-0.8}{50} = -0.016 \frac{\text{lb/in}^2}{\text{in}^3}.$$

- (b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\Delta V}{\Delta P} &= \lim_{h \rightarrow 0} \frac{V(P+h) - V(P)}{h} = \lim_{h \rightarrow 0} \frac{800/(P+h) - 800/P}{h} \\ &= \lim_{h \rightarrow 0} \frac{800[P - (P+h)]}{h(P+h)P} = \lim_{h \rightarrow 0} \frac{-800}{(P+h)P} = -\frac{800}{P^2} \end{aligned}$$

which is inversely proportional to the square of P .

33. $\left| \frac{x+1}{x-1} - 3 \right| < 0.2 \Rightarrow -0.2 < \frac{x+1}{x-1} - 3 < 0.2 \Rightarrow$

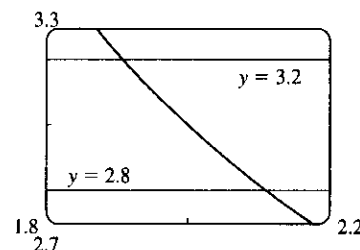
$$2.8 < \frac{x+1}{x-1} < 3.2. \text{ Graphing the functions } y = 2.8,$$

$y = (x+1)/(x-1)$, and $y = 3.2$ on the interval $[1.8, 2.2]$, we see

that the inequality holds whenever $1.91 < x < 2.11$ (approximately).

So since $|2 - 1.91| = 0.09$ and $|2 - 2.11| = 0.11$, any positive

$\delta \leq 0.09$ will do.



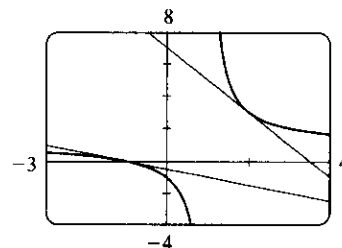
34. The slope of the tangent to $y = \frac{x+1}{x-1}$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\frac{(x+h)+1}{(x+h)-1} - \frac{x+1}{x-1}}{h} &= \lim_{h \rightarrow 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{h(x-1)(x+h-1)} = -\frac{2}{(x-1)^2} \end{aligned}$$

$$\text{So at } (2, 3), m = -\frac{2}{(2-1)^2} = -2 \Rightarrow y - 3 = -2(x - 2) \Rightarrow$$

$$y = -2x + 7. \text{ At } (-1, 0), m = -\frac{2}{(-1-1)^2} = -\frac{1}{2} \Rightarrow$$

$$y = -\frac{1}{2}(x+1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}.$$



35. $|f(x)| \leq g(x) \Leftrightarrow -g(x) \leq f(x) \leq g(x)$ and $\lim_{x \rightarrow a} g(x) = 0 = \lim_{x \rightarrow a} -g(x)$. Thus, by the Squeeze Theorem,

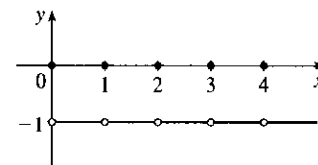
$$\lim_{x \rightarrow a} f(x) = 0.$$

36. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n , $\llbracket n \rrbracket + \llbracket -n \rrbracket = n - n = 0$, and for any real number k which is not an integer,

$$\llbracket k \rrbracket + \llbracket -k \rrbracket = \llbracket k \rrbracket + (-\llbracket k \rrbracket - 1) = -1. \text{ So } \lim_{x \rightarrow a} f(x) \text{ exists}$$

(and is equal to -1) for all values of a .

- (b) f is discontinuous at all integers.



PROBLEMS PLUS

1. Let $t = \sqrt[3]{x}$, so $x = t^3$. Then $t \rightarrow 1$ as $x \rightarrow 1$, so

$$\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} = \lim_{t \rightarrow 1} \frac{t^3 - 1}{t^3 - 1} = \lim_{t \rightarrow 1} \frac{(t-1)(t^2+t+1)}{(t-1)(t^2+t+1)} = \lim_{t \rightarrow 1} \frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x} + 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)$.

2. First rationalize the numerator: $\lim_{x \rightarrow 0} \frac{\sqrt{ax+b}-2}{x} \cdot \frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x \rightarrow 0} \frac{ax+b-4}{x(\sqrt{ax+b}+2)}$. Now since the

denominator approaches 0 as $x \rightarrow 0$, the limit will exist only if the numerator also approaches 0 as $x \rightarrow 0$. So we require that $a(0) + b - 4 = 0 \Rightarrow b = 4$. So the equation becomes

$$\lim_{x \rightarrow 0} \frac{a}{\sqrt{ax+4}+2} = 1 \Rightarrow \frac{a}{\sqrt{4}+2} = 1 \Rightarrow a = 4. \text{ Therefore, } a = b = 4.$$

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x - 1 < 0$ and $2x + 1 > 0$, so $|2x - 1| = -(2x - 1)$ and $|2x + 1| = 2x + 1$.

$$\text{Therefore, } \lim_{x \rightarrow 0} \frac{|2x-1| - |2x+1|}{x} = \lim_{x \rightarrow 0} \frac{-(2x-1) - (2x+1)}{x} = \lim_{x \rightarrow 0} \frac{-4x}{x} = \lim_{x \rightarrow 0} (-4) = -4.$$

4. Let R be the midpoint of OP , so the coordinates of R are $(\frac{1}{2}x, \frac{1}{2}x^2)$ since the coordinates of P are (x, x^2) . Let

$Q = (0, a)$. Since the slope $m_{OP} = \frac{x^2}{x} = x$, $m_{QR} = -\frac{1}{x}$ (negative reciprocal). But

$$m_{QR} = \frac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \frac{x^2 - 2a}{x}, \text{ so we conclude that } -1 = \frac{x^2 - 2a}{x} \Rightarrow 2a = x^2 + 1 \Rightarrow a = \frac{1}{2}x^2 + \frac{1}{2}.$$

As $x \rightarrow 0$, $a \rightarrow \frac{1}{2}$, and the limiting position of Q is $(0, \frac{1}{2})$.

5. Since $[x] \leq x < [x] + 1$, we have $\frac{[x]}{[x]} \leq \frac{x}{[x]} < \frac{[x]+1}{[x]} \Rightarrow 1 \leq \frac{x}{[x]} < 1 + \frac{1}{[x]}$ for $x \geq 1$. As $x \rightarrow \infty$, $[x] \rightarrow \infty$, so $\frac{1}{[x]} \rightarrow 0$ and $1 + \frac{1}{[x]} \rightarrow 1$. Thus, $\lim_{x \rightarrow \infty} \frac{x}{[x]} = 1$ by the Squeeze Theorem.

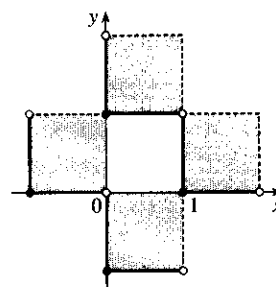
6. (a) $[x]^2 + [y]^2 = 1$. Since $[x]^2$ and $[y]^2$ are positive integers or 0, there are only 4 cases:

$$\text{Case (i): } [x] = 1, [y] = 0 \Rightarrow 1 \leq x < 2 \text{ and } 0 \leq y < 1$$

$$\text{Case (ii): } [x] = -1, [y] = 0 \Rightarrow -1 \leq x < 0 \text{ and } 0 \leq y < 1$$

$$\text{Case (iii): } [x] = 0, [y] = 1 \Rightarrow 0 \leq x < 1 \text{ and } 1 \leq y < 2$$

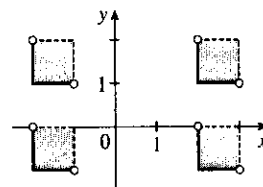
$$\text{Case (iv): } [x] = 0, [y] = -1 \Rightarrow 0 \leq x < 1 \text{ and } -1 \leq y < 0$$



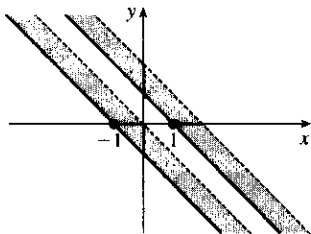
- (b) $[x]^2 - [y]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

$$\{(x, y) \mid [x] = \pm 2, [y] = \pm 1\} =$$

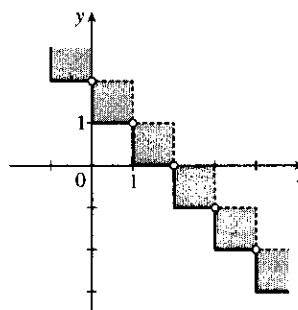
$$\left\{ (x, y) \mid \begin{array}{l} 2 \leq x < 3 \text{ or } -2 \leq x < -1, \\ 1 \leq y < 2 \text{ or } -1 \leq y < 0 \end{array} \right\}.$$



$$\begin{aligned} (c) \quad [x+y]^2 = 1 &\Rightarrow [x+y] = \pm 1 \\ &\Rightarrow 1 \leq x+y < 2 \text{ or} \\ &\quad -1 \leq x+y < 0 \end{aligned}$$



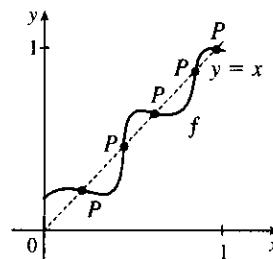
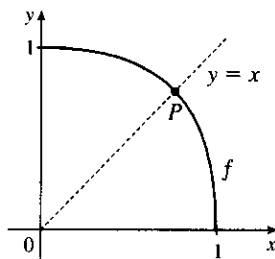
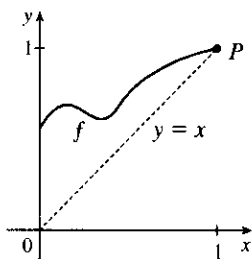
$$\begin{aligned} (d) \quad \text{For } n \leq x < n+1, [x] = n. \text{ Then } [x] + [y] = 1 &\Rightarrow \\ [y] = 1-n &\Rightarrow 1-n \leq y < 2-n. \text{ Choosing integer} \\ \text{values for } n &\text{ produces the graph.} \end{aligned}$$



7. f is continuous on $(-\infty, a)$ and (a, ∞) . To make f continuous on $\mathbb{R}$, we must have continuity at a . Thus,

$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) &\Rightarrow \lim_{x \rightarrow a^+} x^2 = \lim_{x \rightarrow a^-} (x+1) \Rightarrow a^2 = a+1 \Rightarrow a^2 - a - 1 = 0 \Rightarrow \\ \text{[by the quadratic formula]} \quad a &= (1 \pm \sqrt{5})/2 \approx 1.618 \text{ or } -0.618. \end{aligned}$$

8. (a) Here are a few possibilities:



- (b) The “obstacle” is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

- (c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point.

Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove.

So assume $f(0) \neq 0$. For the same reason we can assume that $f(1) \neq 1$. Then $F(0) = f(0) > 0$ and

$F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

$$9. \quad \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \left(\frac{1}{2} [f(x) + g(x)] + \frac{1}{2} [f(x) - g(x)] \right)$$

$$= \frac{1}{2} \lim_{x \rightarrow a} [f(x) + g(x)] + \frac{1}{2} \lim_{x \rightarrow a} [f(x) - g(x)]$$

$$= \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}, \text{ and}$$

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} \left([f(x) + g(x)] - f(x) \right) = \lim_{x \rightarrow a} [f(x) + g(x)] - \lim_{x \rightarrow a} f(x) = 2 - \frac{3}{2} = \frac{1}{2}.$$

$$\text{So } \lim_{x \rightarrow a} [f(x)g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right] = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

Another solution: Since $\lim_{x \rightarrow a} [f(x) + g(x)]$ and $\lim_{x \rightarrow a} [f(x) - g(x)]$ exist, we must have

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) + g(x)]^2 &= \left(\lim_{x \rightarrow a} [f(x) + g(x)] \right)^2 \text{ and } \lim_{x \rightarrow a} [f(x) - g(x)]^2 = \left(\lim_{x \rightarrow a} [f(x) - g(x)] \right)^2, \text{ so} \\ \lim_{x \rightarrow a} [f(x)g(x)] &= \lim_{x \rightarrow a} \frac{1}{4} ([f(x) + g(x)]^2 - [f(x) - g(x)]^2) \quad [\text{because all of the } f^2 \text{ and } g^2 \text{ cancel}] \\ &= \frac{1}{4} \left(\lim_{x \rightarrow a} [f(x) + g(x)]^2 - \lim_{x \rightarrow a} [f(x) - g(x)]^2 \right) = \frac{1}{4} (2^2 - 1^2) = \frac{3}{4}. \end{aligned}$$

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular

from P , as shown. We see from $\angle NCP$ that $\tan 2\theta = \frac{y}{1-x}$, and

from $\angle NBP$ that $\tan \theta = y/x$. Using the double-angle formula for

tangents, we get $\frac{y}{1-x} = \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2(y/x)}{1 - (y/x)^2}$.

After a bit of simplification, this becomes $\frac{1}{1-x} = \frac{2x}{x^2 - y^2} \Leftrightarrow$

$y^2 = x(3x - 2)$. As the altitude AM decreases in length, the point P will approach the x -axis, that is, $y \rightarrow 0$, so the limiting location of P must be one of the roots of the equation $x(3x - 2) = 0$. Obviously it is not $x = 0$ (the point P can never be to the left of the altitude AM , which it would have to be in order to approach 0) so it must be $3x - 2 = 0$, that is, $x = \frac{2}{3}$.

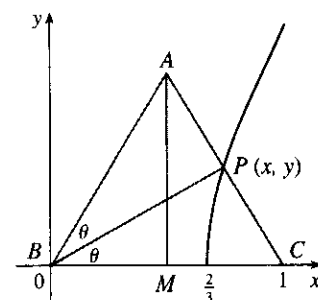
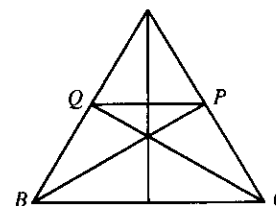
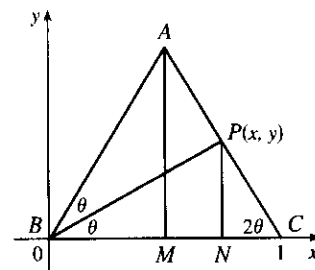
Solution 2: We add a few lines to the original diagram, as shown. Now

note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by

symmetry) and similarly $\angle CQP = \angle QCB$. So $\triangle BPQ$ and $\triangle CQP$

are isosceles, and the line segments BQ , QP and PC are all of equal length. As $|AM| \rightarrow 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C , as above.

- (b) The equation $y^2 = x(3x - 2)$ calculated in part (a) is the equation of the curve traced out by P . Now as $|AM| \rightarrow \infty$, $2\theta \rightarrow \frac{\pi}{2}$, $\theta \rightarrow \frac{\pi}{4}$, $x \rightarrow 1$, and since $\tan \theta = y/x$, $y \rightarrow 1$. Thus, P only traces out the part of the curve with $0 \leq y < 1$.



11. (a) Consider $G(x) = T(x + 180^\circ) - T(x)$. Fix any number a . If $G(a) = 0$, we are done:

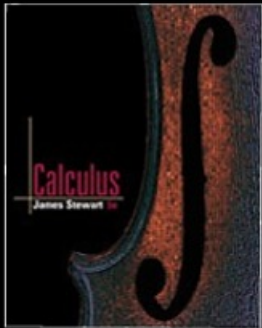
Temperature at a = Temperature at $a + 180^\circ$. If $G(a) > 0$, then $G(a + 180^\circ) = T(a + 360^\circ) - T(a + 180^\circ) = T(a) - T(a + 180^\circ) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a + 180^\circ]$. If $G(a) < 0$, then a similar argument applies.

- (b) Yes. The same argument applies.

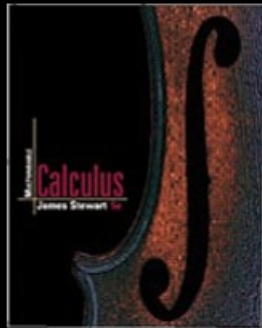
- (c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

Chapter 3

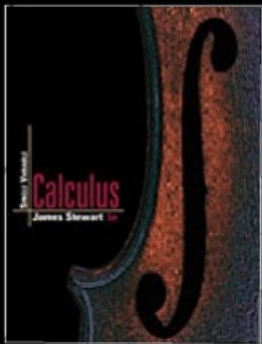
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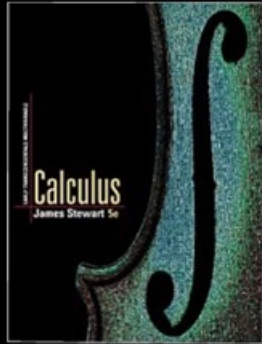
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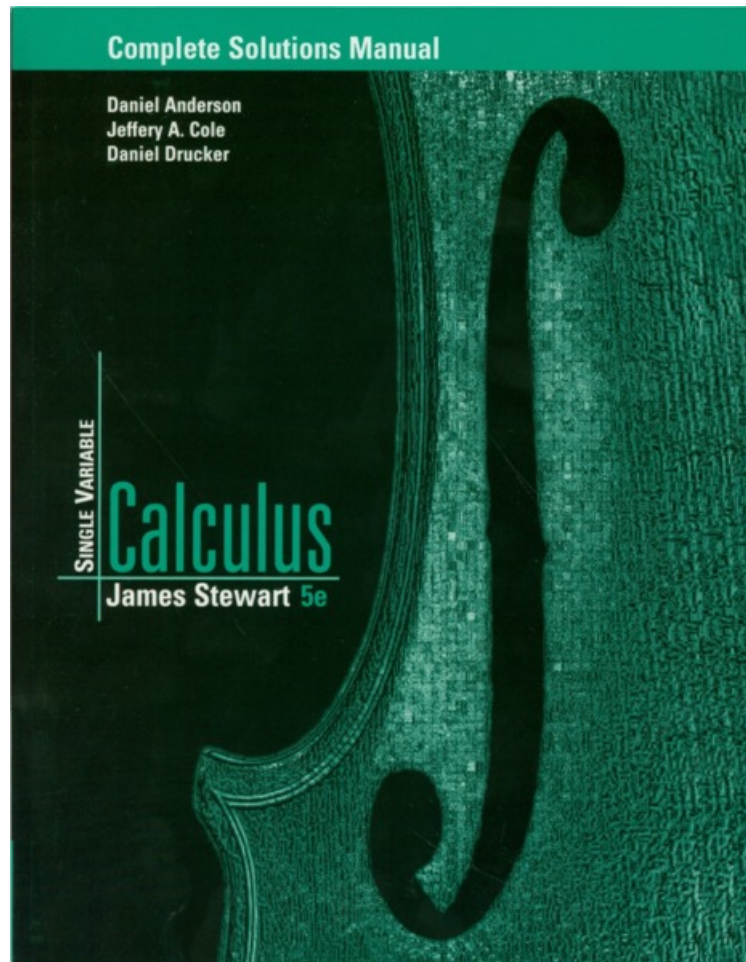
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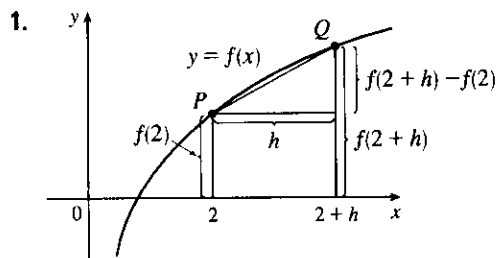


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3 □ DERIVATIVES

3.1 Derivatives



The line from $P(2, f(2))$ to $Q(2+h, f(2+h))$ is the line that has slope $\frac{f(2+h) - f(2)}{h}$.

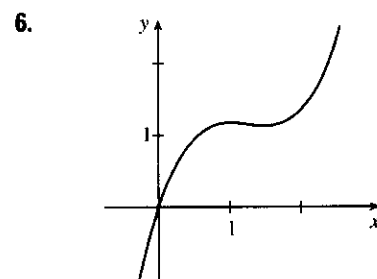
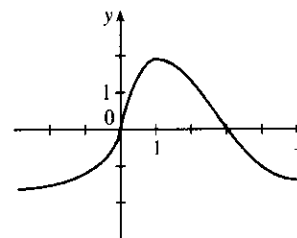
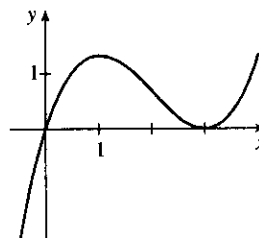
2. As h decreases, the line PQ becomes steeper, so its slope increases. So

$$0 < \frac{f(4) - f(2)}{4 - 2} < \frac{f(3) - f(2)}{3 - 2} < \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}. \text{ Thus, } 0 < \frac{1}{2} [f(4) - f(2)] < f(3) - f(2) < f'(2).$$

3. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

4. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

5. We begin by drawing a curve through the origin at a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Lastly, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.



7. Using Definition 2 with $f(x) = 3x^2 - 5x$ and the point $(2, 2)$, we have

$$\begin{aligned} f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{[3(2+h)^2 - 5(2+h)] - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(12 + 12h + 3h^2 - 10 - 5h) - 2}{h} = \lim_{h \rightarrow 0} \frac{3h^2 + 7h}{h} = \lim_{h \rightarrow 0} (3h + 7) = 7. \end{aligned}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = 7(x - 2)$ or $y = 7x - 12$.

8. Using Definition 2 with $g(x) = 1 - x^3$ and the point $(0, 1)$, we have

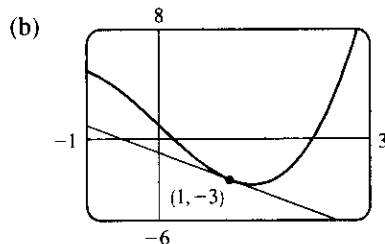
$$g'(0) = \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{[1 - (0+h)^3] - 1}{h} = \lim_{h \rightarrow 0} \frac{(1 - h^3) - 1}{h} = \lim_{h \rightarrow 0} (-h^2) = 0.$$

So an equation of the tangent line is $y - 1 = 0(x - 0)$ or $y = 1$.

9. (a) Using Definition 2 with $F(x) = x^3 - 5x + 1$ and the point $(1, -3)$, we have

$$\begin{aligned} F'(1) &= \lim_{h \rightarrow 0} \frac{F(1+h) - F(1)}{h} = \lim_{h \rightarrow 0} \frac{[(1+h)^3 - 5(1+h) + 1] - (-3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1 + 3h + 3h^2 + h^3 - 5 - 5h + 1) + 3}{h} = \lim_{h \rightarrow 0} \frac{h^3 + 3h^2 - 2h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h^2 + 3h - 2)}{h} = \lim_{h \rightarrow 0} (h^2 + 3h - 2) = -2 \end{aligned}$$

So an equation of the tangent line at $(1, -3)$ is $y - (-3) = -2(x - 1) \Leftrightarrow y = -2x - 1$.



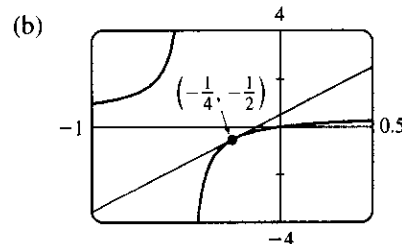
10. (a) $G'(a) = \lim_{h \rightarrow 0} \frac{G(a+h) - G(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{a+h}{1+2(a+h)} - \frac{a}{1+2a}}{h}$

$$= \lim_{h \rightarrow 0} \frac{a + 2a^2 + h + 2ah - a - 2a^2 - 2ah}{h(1+2a+2h)(1+2a)} = \lim_{h \rightarrow 0} \frac{1}{(1+2a+2h)(1+2a)} = (1+2a)^{-2}$$

So the slope of the tangent at the point $(-\frac{1}{4}, -\frac{1}{2})$ is

$$m = [1 + 2(-\frac{1}{4})]^{-2} = 4, \text{ and thus an equation is}$$

$$y + \frac{1}{2} = 4(x + \frac{1}{4}) \text{ or } y = 4x + \frac{1}{2}.$$

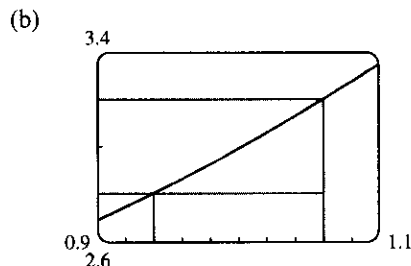


11. (a) $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{3^{1+h} - 3^1}{h}.$

So let $F(h) = \frac{3^{1+h} - 3}{h}$. We calculate:

h	$F(h)$	h	$F(h)$
0.1	3.484	-0.1	3.121
0.01	3.314	-0.01	3.278
0.001	3.298	-0.001	3.294
0.0001	3.296	-0.0001	3.296

We estimate that $f'(1) \approx 3.296$.



From the graph, we estimate that the slope of the tangent is about

$$\frac{3.2 - 2.8}{1.06 - 0.94} = \frac{0.4}{0.12} \approx 3.3.$$

$$12. (a) g'\left(\frac{\pi}{4}\right) = \lim_{h \rightarrow 0} \frac{g\left(\frac{\pi}{4} + h\right) - g\left(\frac{\pi}{4}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - \tan\left(\frac{\pi}{4}\right)}{h}.$$

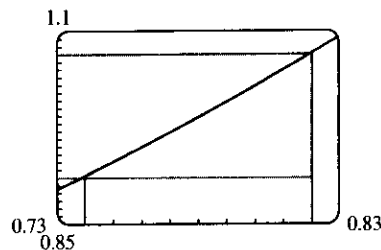
$$\text{So let } G(h) = \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h}.$$

We calculate:

h	$G(h)$	h	$G(h)$
0.1	2.2305	-0.1	1.8237
0.01	2.0203	-0.01	1.9803
0.001	2.0020	-0.001	1.9980
0.0001	2.0002	-0.0001	1.9998

We estimate that $g'\left(\frac{\pi}{4}\right) = 2$.

(b)



From the graph, we estimate that the slope of the

$$\text{tangent is about } \frac{1.07 - 0.91}{0.82 - 0.74} = \frac{0.16}{0.08} = 2.$$

13. Use Definition 2 with $f(x) = 3 - 2x + 4x^2$.

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{[3 - 2(a+h) + 4(a+h)^2] - (3 - 2a + 4a^2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(3 - 2a - 2h + 4a^2 + 8ah + 4h^2) - (3 - 2a + 4a^2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2h + 8ah + 4h^2}{h} = \lim_{h \rightarrow 0} \frac{h(-2 + 8a + 4h)}{h} = \lim_{h \rightarrow 0} (-2 + 8a + 4h) = -2 + 8a$$

$$14. f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{[(a+h)^4 - 5(a+h)] - (a^4 - 5a)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5a - 5h) - (a^4 - 5a)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5h}{h} = \lim_{h \rightarrow 0} \frac{h(4a^3 + 6a^2h + 4ah^2 + h^3 - 5)}{h}$$

$$= \lim_{h \rightarrow 0} (4a^3 + 6a^2h + 4ah^2 + h^3 - 5) = 4a^3 - 5$$

15. Use Definition 2 with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2(a+h)+1}{(a+h)+3} - \frac{2a+1}{a+3}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h \rightarrow 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h \rightarrow 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h \rightarrow 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

$$\begin{aligned}
 16. f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{(a+h)^2 + 1}{(a+h) - 2} - \frac{a^2 + 1}{a - 2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(a^2 + 2ah + h^2 + 1)(a - 2) - (a^2 + 1)(a + h - 2)}{h(a + h - 2)(a - 2)} \\
 &= \lim_{h \rightarrow 0} \frac{(a^3 - 2a^2 + 2a^2h - 4ah + ah^2 - 2h^2 + a - 2) - (a^3 + a^2h - 2a^2 + a + h - 2)}{h(a + h - 2)(a - 2)} \\
 &= \lim_{h \rightarrow 0} \frac{a^2h - 4ah + ah^2 - 2h^2 - h}{h(a + h - 2)(a - 2)} = \lim_{h \rightarrow 0} \frac{h(a^2 - 4a + ah - 2h - 1)}{h(a + h - 2)(a - 2)} \\
 &= \lim_{h \rightarrow 0} \frac{a^2 - 4a + ah - 2h - 1}{(a + h - 2)(a - 2)} = \frac{a^2 - 4a - 1}{(a - 2)^2}
 \end{aligned}$$

17. Use Definition 2 with $f(x) = 1/\sqrt{x+2}$.

$$\begin{aligned}
 f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{(a+h)+2}} - \frac{1}{\sqrt{a+2}}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{a+2} - \sqrt{a+h+2}}{\sqrt{a+h+2}\sqrt{a+2}}}{h} = \lim_{h \rightarrow 0} \left[\frac{\sqrt{a+2} - \sqrt{a+h+2}}{h\sqrt{a+h+2}\sqrt{a+2}} \cdot \frac{\sqrt{a+2} + \sqrt{a+h+2}}{\sqrt{a+2} + \sqrt{a+h+2}} \right] \\
 &= \lim_{h \rightarrow 0} \frac{(a+2) - (a+h+2)}{h\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2} + \sqrt{a+h+2})} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2} + \sqrt{a+h+2})} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2} + \sqrt{a+h+2})} \\
 &= \frac{-1}{(\sqrt{a+2})^2(2\sqrt{a+2})} = -\frac{1}{2(a+2)^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
 18. f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{3(a+h)+1} - \sqrt{3a+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{3a+3h+1} - \sqrt{3a+1})(\sqrt{3a+3h+1} + \sqrt{3a+1})}{h(\sqrt{3a+3h+1} + \sqrt{3a+1})} \\
 &= \lim_{h \rightarrow 0} \frac{(3a+3h+1) - (3a+1)}{h(\sqrt{3a+3h+1} + \sqrt{3a+1})} = \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{3a+3h+1} + \sqrt{3a+1})} \\
 &= \lim_{h \rightarrow 0} \frac{3}{\sqrt{3a+3h+1} + \sqrt{3a+1}} = \frac{3}{2\sqrt{3a+1}}
 \end{aligned}$$

Note that the answers to Exercises 19–24 are not unique.

19. By Definition 2, $\lim_{h \rightarrow 0} \frac{(1+h)^{10} - 1}{h} = f'(1)$, where $f(x) = x^{10}$ and $a = 1$.

Or: By Definition 2, $\lim_{h \rightarrow 0} \frac{(1+h)^{10} - 1}{h} = f'(0)$, where $f(x) = (1+x)^{10}$ and $a = 0$.

20. By Definition 2, $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16+h} - 2}{h} = f'(16)$, where $f(x) = \sqrt[4]{x}$ and $a = 16$.

Or: By Definition 2, $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16+h} - 2}{h} = f'(0)$, where $f(x) = \sqrt[4]{16+x}$ and $a = 0$.

21. By Equation 3, $\lim_{x \rightarrow 5} \frac{2^x - 32}{x - 5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

22. By Equation 3, $\lim_{x \rightarrow \pi/4} \frac{\tan x - 1}{x - \pi/4} = f'(\pi/4)$, where $f(x) = \tan x$ and $a = \pi/4$.

23. By Definition 2, $\lim_{h \rightarrow 0} \frac{\cos(\pi + h) + 1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By Definition 2, $\lim_{h \rightarrow 0} \frac{\cos(\pi + h) + 1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

24. By Equation 3, $\lim_{t \rightarrow 1} \frac{t^4 + t - 2}{t - 1} = f'(1)$, where $f(t) = t^4 + t$ and $a = 1$.

25. $v(2) = f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{[(2+h)^2 - 6(2+h) - 5] - [2^2 - 6(2) - 5]}{h}$
 $= \lim_{h \rightarrow 0} \frac{(4 + 4h + h^2 - 12 - 6h - 5) - (-13)}{h} = \lim_{h \rightarrow 0} \frac{h^2 - 2h}{h} = \lim_{h \rightarrow 0} (h - 2) = -2 \text{ m/s}$

26. $v(2) = f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$
 $= \lim_{h \rightarrow 0} \frac{[2(2+h)^3 - (2+h) + 1] - [2(2)^3 - 2 + 1]}{h}$
 $= \lim_{h \rightarrow 0} \frac{(2h^3 + 12h^2 + 24h + 16 - 2 - h + 1) - 15}{h}$
 $= \lim_{h \rightarrow 0} \frac{2h^3 + 12h^2 + 23h}{h} = \lim_{h \rightarrow 0} (2h^2 + 12h + 23) = 23 \text{ m/s}$

27. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

28. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

29. (a) $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are (gal/h)/(mi/h).

(b) The fuel consumption is decreasing by 0.05 (gal/h)/(mi/h) as the car's speed reaches 20 mi/h. So if you increase your speed to 21 mi/h, you could expect to decrease your fuel consumption by about 0.05 (gal/h)/(mi/h).

30. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).
 (b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t = 8$ and $t = 12$. Let

$$A = \frac{T(8) - T(10)}{8 - 10} = \frac{72 - 81}{-2} = 4.5 \text{ and } B = \frac{T(12) - T(10)}{12 - 10} = \frac{88 - 81}{2} = 3.5. \text{ Then}$$

$$T'(10) = \lim_{t \rightarrow 10} \frac{T(t) - T(10)}{t - 10} \approx \frac{A + B}{2} = \frac{4.5 + 3.5}{2} = 4^\circ\text{F/h.}$$

32. **For 1910:** We will average the difference quotients obtained using the years 1900 and 1920.

$$\text{Let } A = \frac{E(1900) - E(1910)}{1900 - 1910} = \frac{48.3 - 51.1}{-10} = 0.28 \text{ and}$$

$$B = \frac{E(1920) - E(1910)}{1920 - 1910} = \frac{55.2 - 51.1}{10} = 0.41. \text{ Then}$$

$$E'(1910) = \lim_{t \rightarrow 1910} \frac{E(t) - E(1910)}{t - 1910} \approx \frac{A + B}{2} = 0.345. \text{ This means that life expectancy at birth was increasing at about } 0.345 \text{ year/year in } 1910.$$

For 1950: Using data for 1940 and 1960 in a similar fashion, we obtain $E'(1950) \approx [0.31 + 0.10]/2 = 0.205$. So life expectancy at birth was increasing at about 0.205 year/year in 1950.

33. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are (mg/L)/°C.
 (b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points (0, 14) and (32, 6). So $S'(16) \approx \frac{6 - 14}{32 - 0} = -\frac{8}{32} = -0.25 \text{ (mg/L)/}^\circ\text{C}$. This means that as the temperature increases past 16°C , the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/°C.
34. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are (cm/s)/°C.
 (b) For $T = 15^\circ\text{C}$, it appears the tangent line to the curve goes through the points (10, 25) and (20, 32). So $S'(15) \approx \frac{32 - 25}{20 - 10} = 0.7 \text{ (cm/s)/}^\circ\text{C}$. This tells us that at $T = 15^\circ\text{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 (cm/s)/°C. In a similar fashion for $T = 25^\circ\text{C}$, we can use the points (20, 35) and (25, 25) to obtain $S'(25) \approx \frac{25 - 35}{25 - 20} = -2 \text{ (cm/s)/}^\circ\text{C}$. As it gets warmer than 20°C , the maximum sustainable speed decreases rapidly.

35. Since $f(x) = x \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \rightarrow 0} \sin(1/h). \text{ This limit does not exist since } \sin(1/h)$$

takes the values -1 and 1 on any interval containing 0 . (Compare with Example 4 in Section 2.2.)

36. Since $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

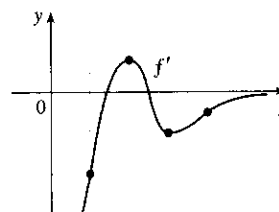
$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \rightarrow 0} h \sin(1/h).$$

Since $-1 \leq \sin \frac{1}{h} \leq 1$, we have $-|h| \leq |h| \sin \frac{1}{h} \leq |h| \Rightarrow -|h| \leq h \sin \frac{1}{h} \leq |h|$. Because $\lim_{h \rightarrow 0} (-|h|) = 0$ and $\lim_{h \rightarrow 0} |h| = 0$, we know that $\lim_{h \rightarrow 0} \left(h \sin \frac{1}{h} \right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

3.2 The Derivative as a Function

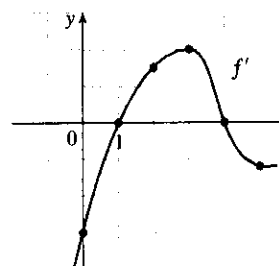
1. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f , it appears that

- | | |
|------------------------|--------------------------|
| (a) $f'(1) \approx -2$ | (b) $f'(2) \approx 0.8$ |
| (c) $f'(3) \approx -1$ | (d) $f'(4) \approx -0.5$ |



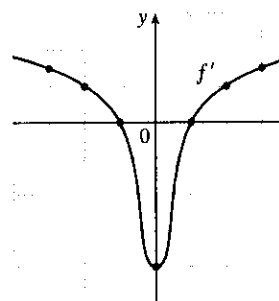
2. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f , it appears that

- | | |
|-------------------------|--------------------------|
| (a) $f'(0) \approx -3$ | (b) $f'(1) \approx 0$ |
| (c) $f'(2) \approx 1.5$ | (d) $f'(3) \approx 2$ |
| (e) $f'(4) \approx 0$ | (f) $f'(5) \approx -1.2$ |



3. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

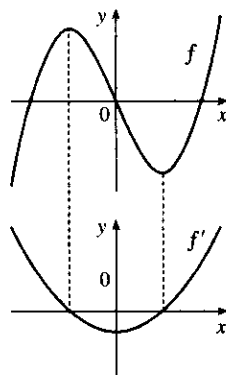
- | | |
|--------------------------|------------------------|
| (a) $f'(-3) \approx 1.5$ | (b) $f'(-2) \approx 1$ |
| (c) $f'(-1) \approx 0$ | (d) $f'(0) \approx -4$ |
| (e) $f'(1) \approx 0$ | (f) $f'(2) \approx 1$ |
| (g) $f'(3) \approx 1.5$ | |



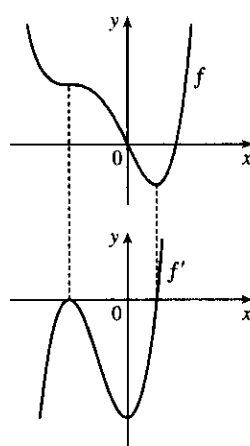
4. (a)' = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.
- (b)' = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.
- (c)' = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.
- (d)' = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 5–13: First plot x -intercepts on the graph of f' for any horizontal tangents on the graph of f . Look for any corners on the graph of f — there will be a discontinuity on the graph of f' . On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

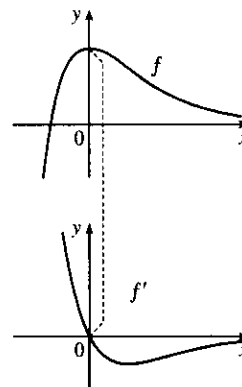
5.



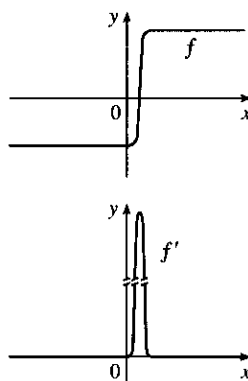
6.



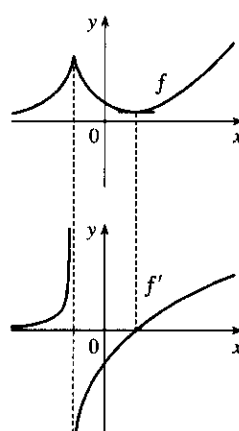
7.



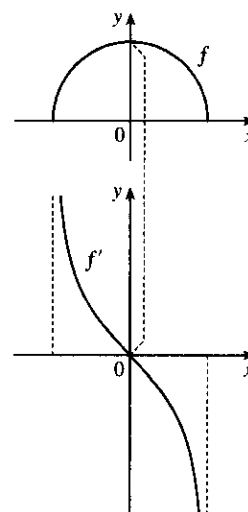
8.



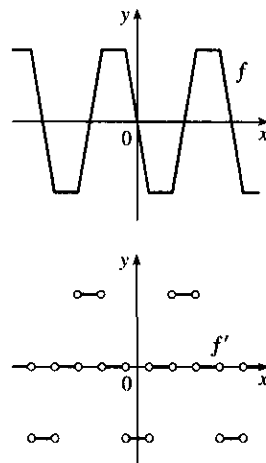
9.



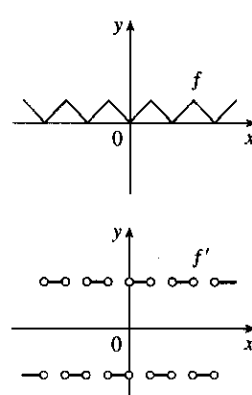
10.



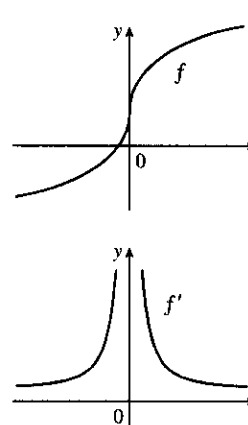
11.



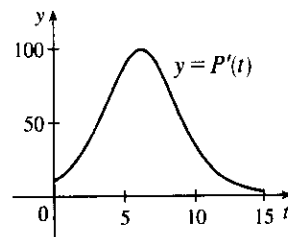
12.



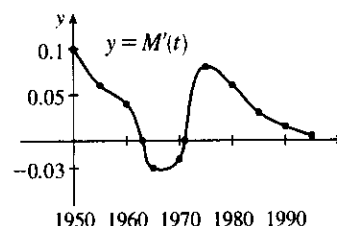
13.



14. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y -values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y -values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M' . The derivative is negative for the years 1963 to 1971.



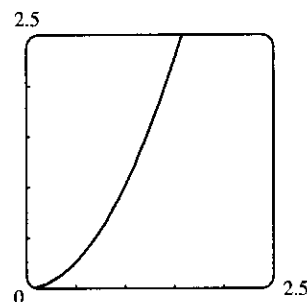
16. See Figure 1 in Section 3.5.

17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'(\frac{1}{2}) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

- (b) By symmetry, $f'(-x) = -f'(x)$. So $f'(-\frac{1}{2}) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

- (c) It appears that $f'(x)$ is twice the value of x , so we guess that $f'(x) = 2x$.

$$\begin{aligned} \text{(d) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} = \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x \end{aligned}$$

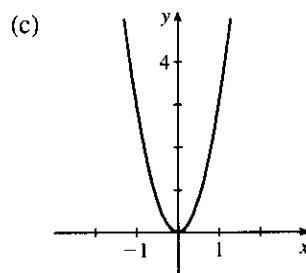


18. (a) By zooming in, we estimate that $f'(0) = 0$, $f'(\frac{1}{2}) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

- (b) By symmetry, $f'(-x) = f'(x)$. So $f'(-\frac{1}{2}) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

- (d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

$$\begin{aligned} \text{(e) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$



19. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{37 - 37}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0$
Domain of $f = \text{domain of } f' = \mathbb{R}$.

$$\begin{aligned}
 20. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[12 + 7(x+h)] - (12 + 7x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12 + 7x + 7h - 12 - 7x}{h} = \lim_{h \rightarrow 0} \frac{7h}{h} = \lim_{h \rightarrow 0} 7 = 7
 \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned}
 21. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[1 - 3(x+h)^2] - (1 - 3x^2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[1 - 3(x^2 + 2xh + h^2)] - (1 - 3x^2)}{h} = \lim_{h \rightarrow 0} \frac{1 - 3x^2 - 6xh - 3h^2 - 1 + 3x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-6xh - 3h^2}{h} = \lim_{h \rightarrow 0} \frac{h(-6x - 3h)}{h} = \lim_{h \rightarrow 0} (-6x - 3h) = -6x
 \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned}
 22. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[5(x+h)^2 + 3(x+h) - 2] - (5x^2 + 3x - 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5x^2 + 10xh + 5h^2 + 3x + 3h - 2 - 5x^2 - 3x + 2}{h} = \lim_{h \rightarrow 0} \frac{10xh + 5h^2 + 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(10x + 5h + 3)}{h} = \lim_{h \rightarrow 0} (10x + 5h + 3) = 10x + 3
 \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned}
 23. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h) + 5] - (x^3 - 3x + 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 5) - (x^3 - 3x + 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3
 \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned}
 24. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h+\sqrt{x+h}) - (x+\sqrt{x})}{h} \\
 &= \lim_{h \rightarrow 0} \left(\frac{h}{h} + \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h \rightarrow 0} \left[1 + \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \right] \\
 &= \lim_{h \rightarrow 0} \left(1 + \frac{1}{\sqrt{x+h} + \sqrt{x}} \right) = 1 + \frac{1}{\sqrt{x} + \sqrt{x}} = 1 + \frac{1}{2\sqrt{x}}
 \end{aligned}$$

Domain of $f = [0, \infty)$, domain of $f' = (0, \infty)$.

$$\begin{aligned}
 25. g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} \left[\frac{\sqrt{1+2(x+h)} + \sqrt{1+2x}}{\sqrt{1+2(x+h)} + \sqrt{1+2x}} \right] \\
 &= \lim_{h \rightarrow 0} \frac{(1+2x+2h) - (1+2x)}{h[\sqrt{1+2(x+h)} + \sqrt{1+2x}]} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{1+2x+2h} + \sqrt{1+2x}} = \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}
 \end{aligned}$$

Domain of $g = [-\frac{1}{2}, \infty)$, domain of $g' = (-\frac{1}{2}, \infty)$.

$$\begin{aligned}
 26. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3+(x+h)}{1-3(x+h)} - \frac{3+x}{1-3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3+x+h)(1-3x) - (3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)} \\
 &= \lim_{h \rightarrow 0} \frac{(3-9x+x-3x^2+h-3hx) - (3-9x-9h+x-3x^2-3hx)}{h(1-3x-3h)(1-3x)} \\
 &= \lim_{h \rightarrow 0} \frac{10h}{h(1-3x-3h)(1-3x)} = \lim_{h \rightarrow 0} \frac{10}{(1-3x-3h)(1-3x)} = \frac{10}{(1-3x)^2}
 \end{aligned}$$

Domain of f = domain of $f' = (-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$.

$$\begin{aligned}
 27. G'(t) &= \lim_{h \rightarrow 0} \frac{G(t+h) - G(t)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4(t+h)}{(t+h)+1} - \frac{4t}{t+1}}{h} = \lim_{h \rightarrow 0} \frac{\frac{4(t+h)(t+1) - 4t(t+h+1)}{(t+h+1)(t+1)}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(4t^2 + 4ht + 4t + 4h) - (4t^2 + 4ht + 4t)}{h(t+h+1)(t+1)} \\
 &= \lim_{h \rightarrow 0} \frac{4h}{h(t+h+1)(t+1)} = \lim_{h \rightarrow 0} \frac{4}{(t+h+1)(t+1)} = \frac{4}{(t+1)^2}
 \end{aligned}$$

Domain of G = domain of $G' = (-\infty, -1) \cup (-1, \infty)$.

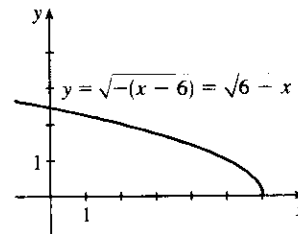
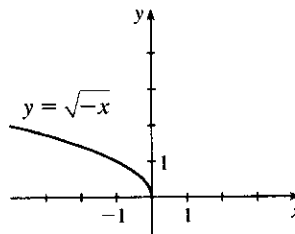
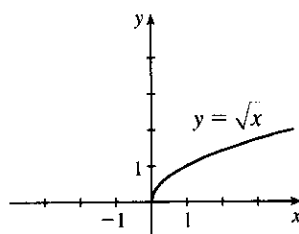
$$\begin{aligned}
 28. g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{x^2 - (x+h)^2}{h(x+h)^2 x^2} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 - (x^2 + 2xh + h^2)}{h} = \lim_{h \rightarrow 0} \frac{-2xh - h^2}{h(x+h)^2 x^2} = \lim_{h \rightarrow 0} \frac{-2x - h}{(x+h)^2 x^2} = \frac{-2x}{x^4} \\
 &= -2x^{-3}
 \end{aligned}$$

Domain of g = domain of $g' = \{x \mid x \neq 0\}$.

$$\begin{aligned}
 29. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \rightarrow 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3
 \end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

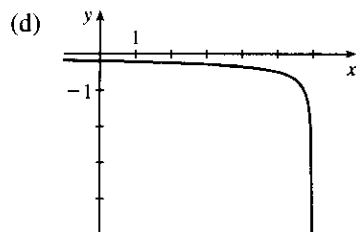
30. (a)



(b) Note that the third graph in part (a) has small negative values for its slope, f' ; but as $x \rightarrow 6^-$, $f' \rightarrow -\infty$.
See the graph in part (d).

$$\begin{aligned}
 \text{(c) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{6 - (x+h)} - \sqrt{6-x}}{h} \left[\frac{\sqrt{6 - (x+h)} + \sqrt{6-x}}{\sqrt{6 - (x+h)} + \sqrt{6-x}} \right] \\
 &= \lim_{h \rightarrow 0} \frac{[6 - (x+h)] - (6-x)}{h [\sqrt{6 - (x+h)} + \sqrt{6-x}]} = \lim_{h \rightarrow 0} \frac{-h}{h(\sqrt{6-x-h} + \sqrt{6-x})} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{6-x-h} + \sqrt{6-x}} = \frac{-1}{2\sqrt{6-x}}
 \end{aligned}$$

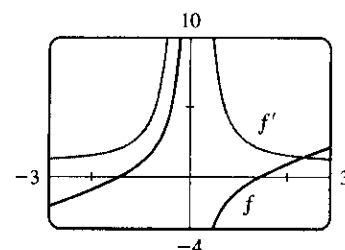
Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.



$$\begin{aligned}
 \text{31. (a) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\left[x+h - \left(\frac{2}{x+h} \right) \right] - \left[x - \left(\frac{2}{x} \right) \right]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left[h - \frac{2}{(x+h)} + \frac{2}{x} \right]}{h} = \lim_{h \rightarrow 0} \left[1 + \frac{-2x + 2(x+h)}{hx(x+h)} \right] = \lim_{h \rightarrow 0} \left[1 + \frac{2h}{hx(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \left[1 + \frac{2}{x(x+h)} \right] = 1 + \frac{2}{x^2}
 \end{aligned}$$

(b) Notice that when f has steep tangent lines, $f'(x)$ is very large.

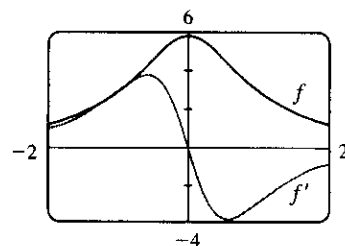
When f is flatter, $f'(x)$ is smaller.



$$\begin{aligned}
 \text{32. (a) } f'(t) &= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0} \frac{\frac{6}{1 + (t+h)^2} - \frac{6}{1+t^2}}{h} = \lim_{h \rightarrow 0} \frac{6 + 6t^2 - 6 - 6(t+h)^2}{h[1 + (t+h)^2](1+t^2)} \\
 &= \lim_{h \rightarrow 0} \frac{-12th - 6h^2}{h[1 + (t+h)^2](1+t^2)} = \lim_{h \rightarrow 0} \frac{-12t - 6h}{[1 + (t+h)^2](1+t^2)} = \frac{-12t}{(1+t^2)^2}
 \end{aligned}$$

(b) Notice that f has a horizontal tangent when $t = 0$. This corresponds

to $f'(0) = 0$. f' is positive when f is increasing and negative when f is decreasing.



33. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim_{h \rightarrow 0} \frac{U(t+h) - U(t)}{h} \approx \frac{U(t+h) - U(t)}{h}$ for small values of h .

For 1991: $U'(1991) = \frac{U(1992) - U(1991)}{1992 - 1991} = \frac{7.5 - 6.8}{1} = 0.70$

For 1992: We estimate $U'(1992)$ by using $h = -1$ and $h = 1$, and then average the two results to obtain a final estimate.

$h = -1 \Rightarrow U'(1992) \approx \frac{U(1991) - U(1992)}{1991 - 1992} = \frac{6.8 - 7.5}{-1} = 0.70;$

$h = 1 \Rightarrow U'(1992) \approx \frac{U(1993) - U(1992)}{1993 - 1992} = \frac{6.9 - 7.5}{1} = -0.60.$

So we estimate that $U'(1992) \approx \frac{1}{2}[0.70 + (-0.60)] = 0.05$.

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$U'(t)$	0.70	0.05	-0.70	-0.65	-0.35	-0.35	-0.45	-0.35	-0.25	-0.20

34. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h} \approx \frac{P(t+h) - P(t)}{h}$ for small values of h .

For 1950: $P'(1950) = \frac{P(1960) - P(1950)}{1960 - 1950} = \frac{35.7 - 31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h = -10$ and $h = 10$, and then average the two results to obtain a final estimate.

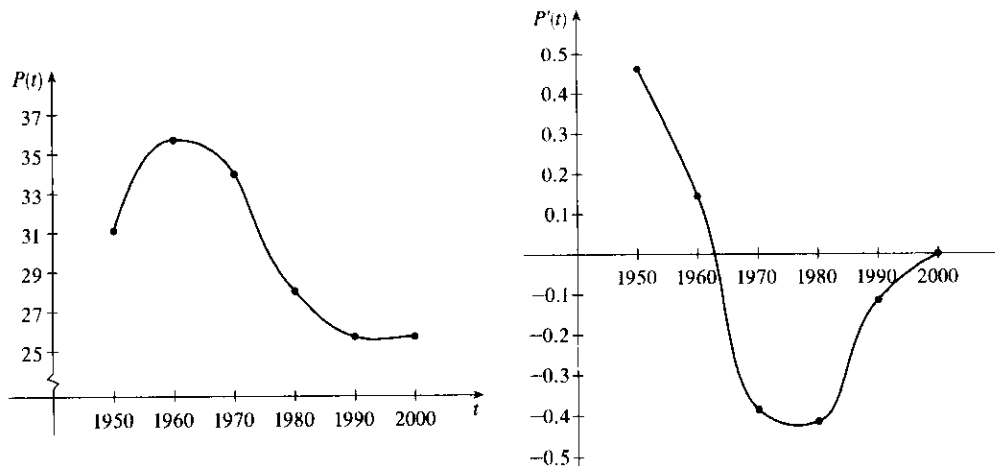
$h = -10 \Rightarrow P'(1960) \approx \frac{P(1950) - P(1960)}{1950 - 1960} = \frac{31.1 - 35.7}{-10} = 0.46$

$h = 10 \Rightarrow P'(1960) \approx \frac{P(1970) - P(1960)}{1970 - 1960} = \frac{34.0 - 35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46 + (-0.17)] = 0.145$.

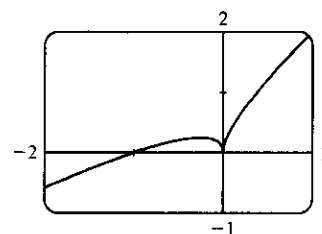
t	1950	1960	1970	1980	1990	2000
$P'(t)$	0.460	0.145	-0.385	-0.415	-0.115	0.000

(c)

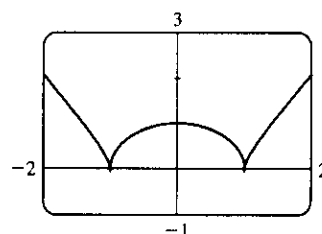


- (d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, and 1995.
35. f is not differentiable at $x = -1$ or at $x = 11$ because the graph has vertical tangents at those points; at $x = 4$, because there is a discontinuity there; and at $x = 8$, because the graph has a corner there.
36. (a) g is discontinuous at $x = -2$ (a removable discontinuity), at $x = 0$ (g is not defined there), and at $x = 5$ (a jump discontinuity).
- (b) g is not differentiable at the points mentioned in part (a) (by Theorem 4), nor is it differentiable at $x = -1$ (corner), $x = 2$ (vertical tangent), or $x = 4$ (vertical tangent).

37. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



38. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so f is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = \pm 1$.



39. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})} \\ &= \lim_{x \rightarrow a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \frac{1}{3}a^{-2/3} \end{aligned}$$

- (b) $f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{h} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

- (c) $\lim_{x \rightarrow 0} |f'(x)| = \lim_{x \rightarrow 0} \frac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

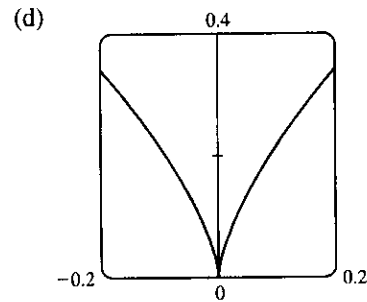
40. (a) $g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^{2/3} - 0}{x} = \lim_{x \rightarrow 0} \frac{1}{x^{1/3}}$, which does not exist.

$$\begin{aligned} \text{(b) } g'(a) &= \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \rightarrow a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})} \\ &= \lim_{x \rightarrow a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \frac{2}{3}a^{-1/3} \end{aligned}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and

$$\lim_{x \rightarrow 0} |g'(x)| = \lim_{x \rightarrow 0} \frac{2}{3|x|^{1/3}} = \infty. \text{ This shows that}$$

g has a vertical tangent line at $x = 0$.

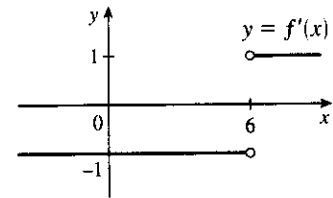


$$41. f(x) = |x - 6| = \begin{cases} -(x - 6) & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases} = \begin{cases} 6 - x & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases}$$

$$\lim_{x \rightarrow 6^+} \frac{f(x) - f(6)}{x - 6} = \lim_{x \rightarrow 6^+} \frac{|x - 6| - 0}{x - 6} = \lim_{x \rightarrow 6^+} \frac{x - 6}{x - 6} = \lim_{x \rightarrow 6^+} 1 = 1.$$

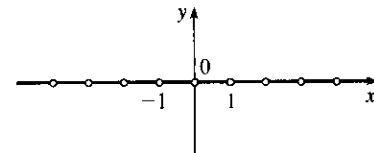
$$\begin{aligned} \text{But } \lim_{x \rightarrow 6^-} \frac{f(x) - f(6)}{x - 6} &= \lim_{x \rightarrow 6^-} \frac{|x - 6| - 0}{x - 6} = \lim_{x \rightarrow 6^-} \frac{6 - x}{x - 6} \\ &= \lim_{x \rightarrow 6^-} (-1) = -1 \end{aligned}$$

$$\text{So } f'(6) = \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6} \text{ does not exist. However, } f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$$

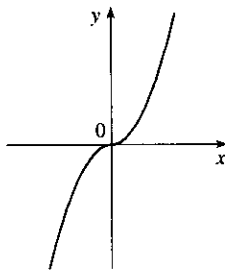


Another way of writing the answer is $f'(x) = \frac{x - 6}{|x - 6|}$.

42. $f(x) = \llbracket x \rrbracket$ is not continuous at any integer n , so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a , so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



$$43. (a) f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$.

[See Exercise 3.2.17(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

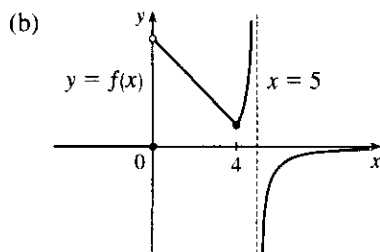
$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x|x|}{x} = \lim_{x \rightarrow 0} |x| = 0.$$

So f is differentiable at 0. Thus, f is differentiable for all x .

$$(c) \text{ From part (b), we have } f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} = 2|x|.$$

44. (a) $f'_-(4) = \lim_{h \rightarrow 0^-} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0^-} \frac{5 - (4+h) - 1}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$ and

$$f'_+(4) = \lim_{h \rightarrow 0^+} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{1}{5 - (4+h)} - 1}{h} = \lim_{h \rightarrow 0^+} \frac{1 - (1-h)}{h(1-h)} = \lim_{h \rightarrow 0^+} \frac{1}{1-h} = 1.$$



(c)
$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ 1/(5 - x) & \text{if } x \geq 4 \end{cases}$$

These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (5 - x) = 5 \neq 0 = \lim_{x \rightarrow 0^-} f(x), \quad \lim_{x \rightarrow 0^-} f(x)$$

does not exist, so f is discontinuous (and therefore not differentiable) at 0.

At 4 we have $\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} (5 - x) = 1$ and $\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \frac{1}{5 - x} = 1$, so $\lim_{x \rightarrow 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5.

(d) From (a), f is not differentiable at 4 since $f'_-(4) \neq f'_+(4)$, and from (c), f is not differentiable at 0 or 5.

45. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f[-(x-h)] - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} \\ &= - \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] = - \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

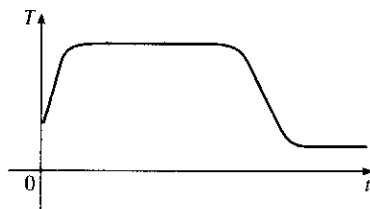
Therefore, f' is odd.

(b) If f is odd, then

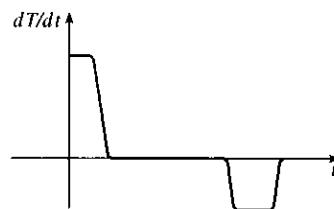
$$\begin{aligned} f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f[-(x-h)] - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

46. (a)

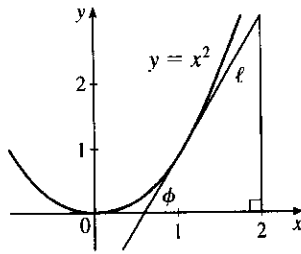


(c)



(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

47.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ . Then the slope of the tangent line ℓ is $m = \Delta y / \Delta x = \tan \phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63^\circ$.

3.3 Differentiation Formulas

1. $f(x) = 186.5$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.
2. $f(x) = \sqrt{30}$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.
3. $f(x) = 5x - 1 \Rightarrow f'(x) = 5 - 0 = 5$
4. $F(x) = -4x^{10} \Rightarrow F'(x) = -4(10x^{10-1}) = -40x^9$
5. $f(x) = x^2 + 3x - 4 \Rightarrow f'(x) = 2x^{2-1} + 3 - 0 = 2x + 3$
6. $g(x) = 5x^8 - 2x^5 + 6 \Rightarrow g'(x) = 5(8x^{8-1}) - 2(5x^{5-1}) + 0 = 40x^7 - 10x^4$
7. $f(t) = \frac{1}{4}(t^4 + 8) \Rightarrow f'(t) = \frac{1}{4}(t^4 + 8)' = \frac{1}{4}(4t^{4-1} + 0) = t^3$
8. $f(t) = \frac{1}{2}t^6 - 3t^4 + t \Rightarrow f'(t) = \frac{1}{2}(6t^5) - 3(4t^3) + 1 = 3t^5 - 12t^3 + 1$
9. $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = \frac{4}{3}\pi(3r^2) = 4\pi r^2$
10. $R(t) = 5t^{-3/5} \Rightarrow R'(t) = 5\left[-\frac{3}{5}t^{(-3/5)-1}\right] = -3t^{-8/5}$
11. $Y(t) = 6t^{-9} \Rightarrow Y'(t) = 6(-9)t^{-10} = -54t^{-10}$
12. $R(x) = \frac{\sqrt{10}}{x^7} = \sqrt{10}x^{-7} \Rightarrow R'(x) = -7\sqrt{10}x^{-8} = -\frac{7\sqrt{10}}{x^8}$
13. $F(x) = \left(\frac{1}{2}x\right)^5 = \left(\frac{1}{2}\right)^5 x^5 = \frac{1}{32}x^5 \Rightarrow F'(x) = \frac{1}{32}(5x^4) = \frac{5}{32}x^4$
14. $f(t) = \sqrt{t} - \frac{1}{\sqrt{t}} = t^{1/2} - t^{-1/2} \Rightarrow f'(t) = \frac{1}{2}t^{-1/2} - \left(-\frac{1}{2}t^{-3/2}\right) = \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}$
15. $y = x^{-2/5} \Rightarrow y' = -\frac{2}{5}x^{(-2/5)-1} = -\frac{2}{5}x^{-7/5} = -\frac{2}{5x^{7/5}}$
16. $y = \sqrt[3]{x} = x^{1/3} \Rightarrow y' = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
17. $y = 4\pi^2 \Rightarrow y' = 0$ since $4\pi^2$ is a constant.
18. $g(u) = \sqrt{2}u + \sqrt{3}u = \sqrt{2}u + \sqrt{3}\sqrt{u} \Rightarrow g'(u) = \sqrt{2}(1) + \sqrt{3}\left(\frac{1}{2}u^{-1/2}\right) = \sqrt{2} + \sqrt{3}/(2\sqrt{u})$
19. $v = t^2 - \frac{1}{\sqrt[4]{t^3}} = t^2 - t^{-3/4} \Rightarrow v' = 2t - \left(-\frac{3}{4}\right)t^{-7/4} = 2t + \frac{3}{4t^{7/4}} = 2t + \frac{3}{4t\sqrt[4]{t^3}}$
20. $u = \sqrt[3]{t^2} + 2\sqrt{t^3} = t^{2/3} + 2t^{3/2} \Rightarrow u' = \frac{2}{3}t^{-1/3} + 2\left(\frac{3}{2}\right)t^{1/2} = \frac{2}{3\sqrt[3]{t}} + 3\sqrt{t}$
21. Product Rule: $y = (x^2 + 1)(x^3 + 1) \Rightarrow$
 $y' = (x^2 + 1)(3x^2) + (x^3 + 1)(2x) = 3x^4 + 3x^2 + 2x^4 + 2x = 5x^4 + 3x^2 + 2x$.
 Multiplying first: $y = (x^2 + 1)(x^3 + 1) = x^5 + x^3 + x^2 + 1 \Rightarrow y' = 5x^4 + 3x^2 + 2x$ (equivalent).

22. Quotient Rule: $F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}} = \frac{x - 3x^{3/2}}{x^{1/2}} \Rightarrow$

$$F'(x) = \frac{x^{1/2} \left(1 - \frac{9}{2}x^{1/2}\right) - (x - 3x^{3/2}) \left(\frac{1}{2}x^{-1/2}\right)}{(x^{1/2})^2}$$

$$= \frac{x^{1/2} - \frac{9}{2}x - \frac{1}{2}x^{1/2} + \frac{3}{2}x}{x} = \frac{\frac{1}{2}x^{1/2} - 3x}{x} = \frac{1}{2}x^{-1/2} - 3$$

Simplifying first: $F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}} = \sqrt{x} - 3x = x^{1/2} - 3x \Rightarrow F'(x) = \frac{1}{2}x^{-1/2} - 3$ (equivalent).

For this problem, simplifying first seems to be the better method.

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate use of the Product and Quotient Rules, respectively.

23. $V(x) = (2x^3 + 3)(x^4 - 2x) \overset{\text{PR}}{\Rightarrow}$

$$V'(x) = (2x^3 + 3)(4x^3 - 2) + (x^4 - 2x)(6x^2) = (8x^6 + 8x^3 - 6) + (6x^6 - 12x^3) = 14x^6 - 4x^3 - 6$$

24. $Y(u) = (u^{-2} + u^{-3})(u^5 - 2u^2) \overset{\text{PR}}{\Rightarrow}$

$$Y'(u) = (u^{-2} + u^{-3})(5u^4 - 4u) + (u^5 - 2u^2)(-2u^{-3} - 3u^{-4})$$

$$= (5u^2 - 4u^{-1} + 5u - 4u^{-2}) + (-2u^2 - 3u + 4u^{-1} + 6u^{-2}) = 3u^2 + 2u + 2u^{-2}$$

25. $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \overset{\text{PR}}{\Rightarrow}$

$$F'(y) = (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5})$$

$$= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2})$$

$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4$$

26. $y = \sqrt{x}(x - 1) = x^{3/2} - x^{1/2} \Rightarrow y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{-1/2}(3x - 1)$ [factor out $\frac{1}{2}x^{-1/2}$]

or $y' = \frac{3x - 1}{2\sqrt{x}}$.

27. $g(x) = \frac{3x - 1}{2x + 1} \overset{\text{QR}}{\Rightarrow} g'(x) = \frac{(2x + 1)(3) - (3x - 1)(2)}{(2x + 1)^2} = \frac{6x + 3 - 6x + 2}{(2x + 1)^2} = \frac{5}{(2x + 1)^2}$

28. $f(t) = \frac{2t}{4 + t^2} \overset{\text{QR}}{\Rightarrow} f'(t) = \frac{(4 + t^2)(2) - (2t)(2t)}{(4 + t^2)^2} = \frac{8 + 2t^2 - 4t^2}{(4 + t^2)^2} = \frac{8 - 2t^2}{(4 + t^2)^2}$

29. $y = \frac{t^2}{3t^2 - 2t + 1} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(3t^2 - 2t + 1)(2t) - t^2(6t - 2)}{(3t^2 - 2t + 1)^2} = \frac{2t[3t^2 - 2t + 1 - t(3t - 1)]}{(3t^2 - 2t + 1)^2}$$

$$= \frac{2t(3t^2 - 2t + 1 - 3t^2 + t)}{(3t^2 - 2t + 1)^2} = \frac{2t(1 - t)}{(3t^2 - 2t + 1)^2}$$

30. $y = \frac{t^3 + t}{t^4 - 2} \overset{\text{QR}}{\Rightarrow} y' = \frac{(t^4 - 2)(3t^2 + 1) - (t^3 + t)(4t^3)}{(t^4 - 2)^2} = \frac{(3t^6 + t^4 - 6t^2 - 2) - (4t^6 + 4t^4)}{(t^4 - 2)^2}$

$$= \frac{-t^6 - 3t^4 - 6t^2 - 2}{(t^4 - 2)^2} = -\frac{t^6 + 3t^4 + 6t^2 + 2}{(t^4 - 2)^2}$$

$$31. y = \frac{v^3 - 2v\sqrt{v}}{v} = v^2 - 2\sqrt{v} = v^2 - 2v^{1/2} \Rightarrow y' = 2v - 2\left(\frac{1}{2}\right)v^{-1/2} = 2v - v^{-1/2}.$$

We can change the form of the answer as follows: $2v - v^{-1/2} = 2v - \frac{1}{\sqrt{v}} = \frac{2v\sqrt{v} - 1}{\sqrt{v}} = \frac{2v^{3/2} - 1}{\sqrt{v}}$

$$32. y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \Rightarrow$$

$$y' = \frac{(\sqrt{x} + 1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x} - 1)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x} + 1)^2} = \frac{\frac{1}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{2} + \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2} = \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2}$$

$$33. y = \frac{1}{x^4 + x^2 + 1} \Rightarrow y' = \frac{(x^4 + x^2 + 1)(0) - 1(4x^3 + 2x)}{(x^4 + x^2 + 1)^2} = -\frac{2x(2x^2 + 1)}{(x^4 + x^2 + 1)^2}$$

$$34. y = x^2 + x + x^{-1} + x^{-2} \Rightarrow y' = 2x + 1 - x^{-2} - 2x^{-3}$$

$$35. y = ax^2 + bx + c \Rightarrow y' = 2ax + b$$

$$36. y = A + \frac{B}{x} + \frac{C}{x^2} = A + Bx^{-1} + Cx^{-2} \Rightarrow y' = -Bx^{-2} - 2Cx^{-3} = -\frac{B}{x^2} - 2\frac{C}{x^3}$$

$$37. y = \frac{r^2}{1 + \sqrt{r}} \Rightarrow$$

$$y' = \frac{(1 + \sqrt{r})(2r) - r^2\left(\frac{1}{2}r^{-1/2}\right)}{(1 + \sqrt{r})^2} = \frac{2r + 2r^{3/2} - \frac{1}{2}r^{3/2}}{(1 + \sqrt{r})^2} = \frac{2r + \frac{3}{2}r^{3/2}}{(1 + \sqrt{r})^2} = \frac{\frac{1}{2}r(4 + 3r^{1/2})}{(1 + \sqrt{r})^2} = \frac{r(4 + 3\sqrt{r})}{2(1 + \sqrt{r})^2}$$

$$38. y = \frac{cx}{1 + cx} \Rightarrow y' = \frac{(1 + cx)(c) - (cx)(c)}{(1 + cx)^2} = \frac{c + c^2x - c^2x}{(1 + cx)^2} = \frac{c}{(1 + cx)^2}$$

$$39. y = \sqrt[3]{t}(t^2 + t + t^{-1}) = t^{1/3}(t^2 + t + t^{-1}) = t^{7/3} + t^{4/3} + t^{-2/3} \Rightarrow$$

$$y' = \frac{7}{3}t^{4/3} + \frac{4}{3}t^{1/3} - \frac{2}{3}t^{-5/3} = \frac{1}{3}t^{-5/3}(7t^{9/3} + 4t^{6/3} - 2) = (7t^3 + 4t^2 - 2)/(3t^{5/3})$$

$$40. y = \frac{u^6 - 2u^3 + 5}{u^2} = u^4 - 2u + 5u^{-2} \Rightarrow$$

$$y' = 4u^3 - 2 - 10u^{-3} = 2u^{-3}(2u^6 - u^3 - 5) = 2(2u^6 - u^3 - 5)/u^3$$

$$41. f(x) = \frac{x}{x + c/x} \Rightarrow$$

$$f'(x) = \frac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \frac{c}{x}\right)^2} = \frac{x + c/x - x + c/x}{\left(\frac{x^2 + c}{x}\right)^2} = \frac{2c/x}{\frac{(x^2 + c)^2}{x^2}} = \frac{2cx}{(x^2 + c)^2}$$

$$42. f(x) = \frac{ax + b}{cx + d} \Rightarrow f'(x) = \frac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2} = \frac{acx + ad - acx - bc}{(cx + d)^2} = \frac{ad - bc}{(cx + d)^2}$$

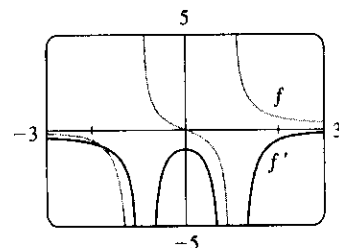
$$43. P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \Rightarrow$$

$$P'(x) = na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1$$

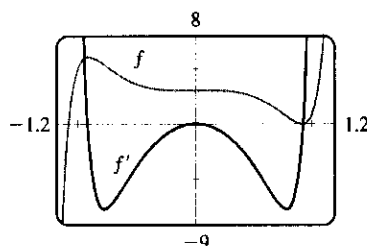
$$44. f(x) = \frac{x}{x^2 - 1} \Rightarrow$$

$$f'(x) = \frac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} = \frac{-x^2 - 1}{(x^2 - 1)^2} = -\frac{x^2 + 1}{(x^2 - 1)^2}$$

Notice that the slopes of all tangents to f are negative and $f'(x) < 0$ always.

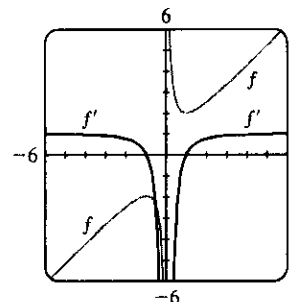


$$45. f(x) = 3x^{15} - 5x^3 + 3 \Rightarrow f'(x) = 45x^{14} - 15x^2.$$



Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

$$46. f(x) = x + 1/x = x + x^{-1} \Rightarrow f'(x) = 1 - x^{-2} = 1 - 1/x^2.$$



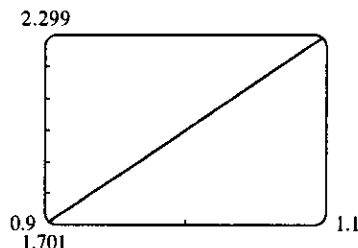
Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

47. To graphically estimate the value of $f'(1)$ for $f(x) = 3x^2 - x^3$, we'll graph f in the viewing rectangle $[1 - 0.1, 1 + 0.1]$ by $[f(0.9), f(1.1)]$, as shown in the figure. [When assigning values to the window variables, it is convenient to use $Y_1(0.9)$ for $Y_{\min}$ and $Y_1(1.1)$ for $Y_{\max}$.] If we have sufficiently zoomed in on the graph of f , we should obtain a graph that looks like a diagonal line; if not, graph again with $1 - 0.01$ and $1 + 0.01$, etc.

Estimated value:

$$f'(1) \approx \frac{2.299 - 1.701}{1.1 - 0.9} = \frac{0.598}{0.2} = 2.99.$$

Exact value: $f(x) = 3x^2 - x^3 \Rightarrow f'(x) = 6x - 3x^2$,
so $f'(1) = 6 - 3 = 3$.

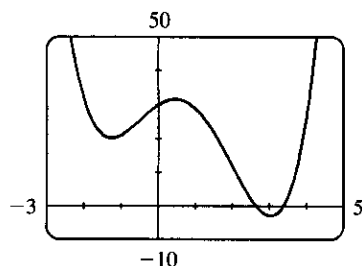


48. See the previous exercise. Since f is a decreasing function, assign $Y_1(3.9)$ to $Y_{\max}$ and $Y_1(4.1)$ to $Y_{\min}$.

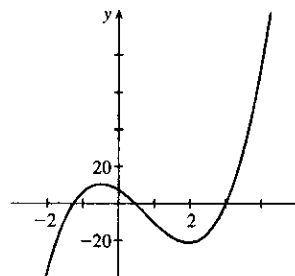
Estimated value: $f'(4) \approx \frac{0.49386 - 0.50637}{4.1 - 3.9} = \frac{-0.01251}{0.2} = -0.06255$.

Exact value: $f(x) = x^{-1/2} \Rightarrow f'(x) = -\frac{1}{2}x^{-3/2}$, so $f'(4) = -\frac{1}{2}(4^{-3/2}) = -\frac{1}{2}(\frac{1}{8}) = -\frac{1}{16} = -0.0625$.

49. (a)

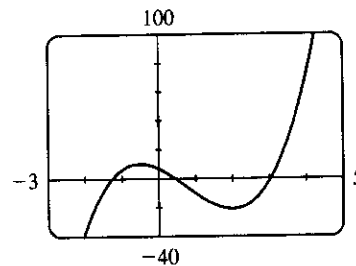


(b)

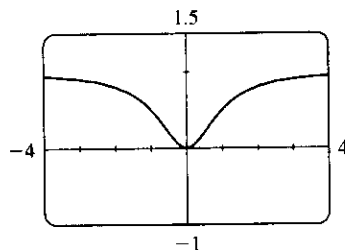


From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3) . The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞) .

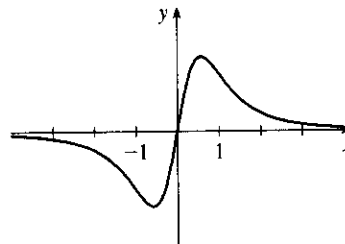
(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \Rightarrow$
 $f'(x) = 4x^3 - 9x^2 - 12x + 7$



50. (a)

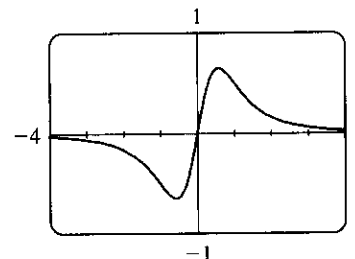


(b)



From the graph in part (a), it appears that g' is zero at $x = 0$. The slopes are negative (so g' is negative) on $(-\infty, 0)$. The slopes are positive (so g' is positive) on $(0, \infty)$.

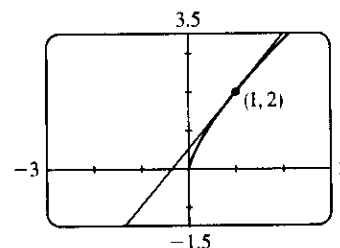
(c) $g(x) = \frac{x^2}{x^2 + 1} \Rightarrow$
 $g'(x) = \frac{(x^2 + 1)(2x) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$



51. $y = \frac{2x}{x+1} \Rightarrow y' = \frac{(x+1)(2) - (2x)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. At $(1, 1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

52. $y = \frac{\sqrt{x}}{x+1} \Rightarrow y' = \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1)}{(x+1)^2} = \frac{(x+1) - (2x)}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2}$. At $(4, 0.4)$, $y' = \frac{-3}{100} = -0.03$, and an equation of the tangent line is $y - 0.4 = -0.03(x - 4)$, or $y = -0.03x + 0.52$.

53. $y = f(x) = x + \sqrt{x} \Rightarrow f'(x) = 1 + \frac{1}{2}x^{-1/2}$. So the slope of the tangent line at $(1, 2)$ is $f'(1) = 1 + \frac{1}{2}(1) = \frac{3}{2}$ and its equation is $y - 2 = \frac{3}{2}(x - 1)$ or $y = \frac{3}{2}x + \frac{1}{2}$.



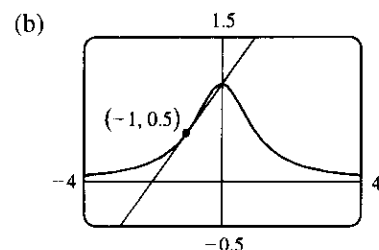
54. $y = (1 + 2x)^2 = 1 + 4x + 4x^2 \Rightarrow y' = 4 + 8x$. At $(1, 9)$, $y' = 12$ and an equation of the tangent line is $y - 9 = 12(x - 1)$ or $y = 12x - 3$.

55. (a) $y = f(x) = \frac{1}{1+x^2} \Rightarrow$

$$f'(x) = \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}. \text{ So the slope of the}$$

tangent line at the point $(-1, \frac{1}{2})$ is $f'(-1) = \frac{2}{2^2} = \frac{1}{2}$ and its

equation is $y - \frac{1}{2} = \frac{1}{2}(x + 1)$ or $y = \frac{1}{2}x + 1$.

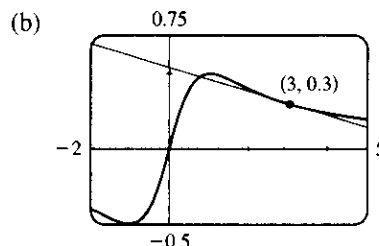


56. (a) $y = f(x) = \frac{x}{1+x^2} \Rightarrow$

$$f'(x) = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}. \text{ So the slope of the}$$

tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is

$y - 0.3 = -0.08(x - 3)$ or $y = -0.08x + 0.54$.



57. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\frac{f}{g}\right)'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \frac{(-3)(6) - (1)(2)}{(-3)^2} = -\frac{20}{9}$

(c) $\left(\frac{g}{f}\right)'(5) = \frac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \frac{(1)(2) - (-3)(6)}{(1)^2} = 20$

58. We are given that $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$.

(a) $(f+g)'(3) = f'(3) + g'(3) = -6 + 5 = -1$

(b) $(fg)'(3) = f(3)g'(3) + g(3)f'(3) = (4)(5) + (2)(-6) = 20 - 12 = 8$

(c) $\left(\frac{f}{g}\right)'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{[g(3)]^2} = \frac{(2)(-6) - (4)(5)}{(2)^2} = \frac{-32}{4} = -8$

(d) $\left(\frac{f}{f-g}\right)'(3) = \frac{[f(3) - g(3)]f'(3) - f(3)[f'(3) - g'(3)]}{[f(3) - g(3)]^2}$
 $= \frac{(4-2)(-6) - 4(-6-5)}{(4-2)^2} = \frac{-12+44}{2^2} = 8$

59. $f(x) = \sqrt{x}g(x) \Rightarrow f'(x) = \sqrt{x}g'(x) + g(x) \cdot \frac{1}{2}x^{-1/2}$, so

$$f'(4) = \sqrt{4}g'(4) + g(4) \cdot \frac{1}{2\sqrt{4}} = 2 \cdot 7 + 8 \cdot \frac{1}{4} = 16.$$

60. $\frac{d}{dx} \left[\frac{h(x)}{x} \right] = \frac{xh'(x) - h(x) \cdot 1}{x^2} \Rightarrow \frac{d}{dx} \left[\frac{h(x)}{x} \right]_{x=2} = \frac{2h'(2) - h(2)}{2^2} = \frac{2(-3) - (4)}{4} = \frac{-10}{4} = -2.5$

61. (a) From the graphs of f and g , we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f ; $g(1) = 1$ since the point $(1, 1)$ is on the graph of g ; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\frac{4-0}{2-0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\frac{0-4}{2-(-2)} = -1$. Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)f'(1) = 2 \cdot (-1) + 1 \cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \frac{2(-\frac{1}{3}) - 3 \cdot \frac{2}{3}}{2^2} = \frac{-\frac{8}{3}}{4} = -\frac{2}{3}$

62. (a) $P(x) = F(x)G(x)$, so $P'(2) = F(2)G'(2) + G(2)F'(2) = 3 \cdot \frac{2}{4} + 2 \cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \frac{G(7)F'(7) - F(7)G'(7)}{[G(7)]^2} = \frac{1 \cdot \frac{1}{4} - 5 \cdot (-\frac{2}{3})}{1^2} = \frac{1}{4} + \frac{10}{3} = \frac{43}{12}$

63. (a) $y = xg(x) \Rightarrow y' = xg'(x) + g(x) \cdot 1 = xg'(x) + g(x)$

(b) $y = \frac{x}{g(x)} \Rightarrow y' = \frac{g(x) \cdot 1 - xg'(x)}{[g(x)]^2} = \frac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \frac{g(x)}{x} \Rightarrow y' = \frac{xg'(x) - g(x) \cdot 1}{(x)^2} = \frac{xg'(x) - g(x)}{x^2}$

64. (a) $y = x^2 f(x) \Rightarrow y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \frac{f(x)}{x^2} \Rightarrow y' = \frac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \frac{xf'(x) - 2f(x)}{x^3}$

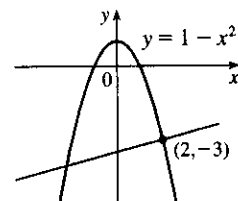
(c) $y = \frac{x^2}{f(x)} \Rightarrow y' = \frac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \frac{1 + xf(x)}{\sqrt{x}} \Rightarrow$

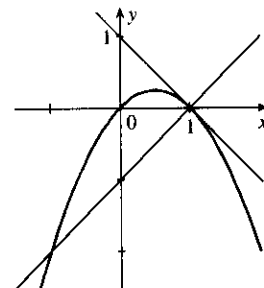
$$y' = \frac{\sqrt{x}[xf'(x) + f(x)] - [1 + xf(x)] \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

65. $y = f(x) = 1 - x^2 \Rightarrow f'(x) = -2x$, so the tangent line at $(2, -3)$ has slope $f'(2) = -4$. The normal line has slope $-\frac{1}{-4} = \frac{1}{4}$ and equation $y + 3 = \frac{1}{4}(x - 2)$ or $y = \frac{1}{4}x - \frac{7}{2}$.



66. $y = f(x) = x - x^2 \Rightarrow f'(x) = 1 - 2x$. So $f'(1) = -1$, and the slope of the normal line is the negative reciprocal of that of the tangent line, that is, $-1/(-1) = 1$. So the equation of the normal line at $(1, 0)$ is $y - 0 = 1(x - 1)$
 $\Leftrightarrow y = x - 1$. Substituting this into the equation of the parabola, we obtain $x - 1 = x - x^2 \Leftrightarrow x = \pm 1$. The solution $x = -1$ is the one we require. Substituting $x = -1$ into the equation of the parabola to find the y -coordinate, we have $y = -2$. So the point of intersection is $(-1, -2)$, as shown in the sketch.



67. $y = x^3 - x^2 - x + 1$ has a horizontal tangent when $y' = 3x^2 - 2x - 1 = 0$. $(3x + 1)(x - 1) = 0 \Leftrightarrow x = 1$ or $-\frac{1}{3}$. Therefore, the points are $(1, 0)$ and $(-\frac{1}{3}, \frac{32}{27})$.

68. $y = \frac{x-1}{x+1} \Rightarrow y' = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. If the tangent intersects the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to $x - 2y = 2$,

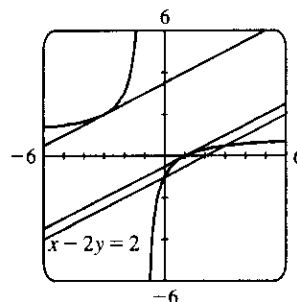
that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\frac{2}{(a+1)^2} = \frac{1}{2} \Rightarrow$

$(a+1)^2 = 4 \Rightarrow a+1 = \pm 2 \Rightarrow a = 1$ or -3 . When $a = 1$, $y = 0$

and the equation of the tangent is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$.

When $a = -3$, $y = 2$ and the equation of the tangent is $y - 2 = \frac{1}{2}(x + 3)$

or $y = \frac{1}{2}x + \frac{7}{2}$.



69. If $y = f(x) = \frac{x}{x+1}$, then $f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent

line is $y - \frac{a}{a+1} = \frac{1}{(a+1)^2}(x - a)$. This line passes through $(1, 2)$ when $2 - \frac{a}{a+1} = \frac{1}{(a+1)^2}(1 - a) \Leftrightarrow$

$2(a+1)^2 - a(a+1) = 1 - a \Leftrightarrow 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \Leftrightarrow a^2 + 4a + 1 = 0$.

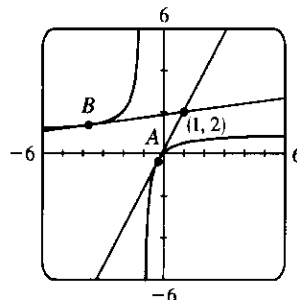
The quadratic formula gives the roots of this equation as $a = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

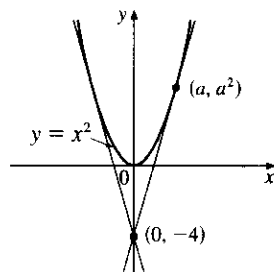
$$\begin{aligned} f(-2 \pm \sqrt{3}) &= \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}} \\ &= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2}, \end{aligned}$$

the lines touch the curve at $A(-2 + \sqrt{3}, \frac{1 - \sqrt{3}}{2}) \approx (-0.27, -0.37)$ and

$B(-2 - \sqrt{3}, \frac{1 + \sqrt{3}}{2}) \approx (-3.73, 1.37)$.



70.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation

$y - (-4) = 2a(x - 0) \Leftrightarrow y = 2ax - 4$. Since (a, a^2) also lies on

the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

71. $y = 6x^3 + 5x - 3 \Rightarrow m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x , so $m \geq 5$ for all x .

72. If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\frac{\Delta y}{\Delta x} = \frac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \frac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \Leftrightarrow$

$a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \Leftrightarrow a = 5$ or -1 . If $a = -1$, the point is $(-1, 0)$ and the slope is -1 , so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11 , so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

73. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) $y = \sqrt{x}(x^4 + x + 1)(2x - 3)$. Using part (a), we have

$$\begin{aligned} y' &= \frac{1}{2\sqrt{x}}(x^4 + x + 1)(2x - 3) + \sqrt{x}(4x^3 + 1)(2x - 3) + \sqrt{x}(x^4 + x + 1)(2) \\ &= (x^4 + x + 1)\frac{2x - 3}{2\sqrt{x}} + \sqrt{x}[(4x^3 + 1)(2x - 3) + 2(x^4 + x + 1)] \end{aligned}$$

74. (a) Putting $f = g = h$ in part (a), we have

$$\frac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x).$$

(b) $y = (x^4 + 3x^3 + 17x + 82)^3 \Rightarrow y' = 3(x^4 + 3x^3 + 17x + 82)^2(4x^3 + 9x^2 + 17)$

75. $y = f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f , so $f(-2) = 6 \Rightarrow -8a + 4b - 2c + d = 6$ (1). The point $(2, 0)$ is on f , so $f(2) = 0 \Rightarrow 8a + 4b + 2c + d = 0$ (2). Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \Rightarrow 12a - 4b + c = 0$ (3) and $f'(2) = 0 \Rightarrow 12a + 4b + c = 0$ (4). Subtracting equation (3) from (4) gives $8b = 0 \Rightarrow b = 0$. Adding (1) and (2) gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From (3) we have $c = -12a$, so (2) becomes $8a + 4(0) + 2(-12a) + 3 = 0 \Rightarrow 3 = 16a \Rightarrow a = \frac{3}{16}$. Now $c = -12a = -12(\frac{3}{16}) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

76. (a) $s(0) = 100,000$ subscribers and $n(0) = 1.2$ phone lines per subscriber. $s'(0) = 1000$ subscribers/month and $n'(0) = 0.01$ phone line per subscriber/month.

(b) The total number of lines is given by $L(t) = s(t)n(t)$. To find $L'(0)$, we first find $L'(t)$ using the Product Rule. $L'(t) = s(t)n'(t) + n(t)s'(t) \Rightarrow L'(0) = s(0)n'(0) + n(0)s'(0) = 100,000(0.01) + 1.2(1000) = 2200$ phone lines/month.

77. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \Rightarrow T'(1999) = P(1999)A'(1999) + A(1999)P'(1999) = (961,400)(\$1400/\text{yr}) + (\$30,593)(9200/\text{yr}) = \$1,345,960,000/\text{yr} + \$281,455,600/\text{yr} = \$1,627,415,600/\text{yr}$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

78. (a) $f(20) = 10,000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold. $f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \Rightarrow R'(p) = pf'(p) + f(p) \cdot 1 \Rightarrow R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10,000 = 3000$. This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but that that loss is more than made up for by the additional revenue due to the increase in price.

79. $f(x) = 2 - x$ if $x \leq 1$ and $f(x) = x^2 - 2x + 2$ if $x > 1$. Now we compute the right- and left-hand derivatives defined in Exercise 3.2.44:

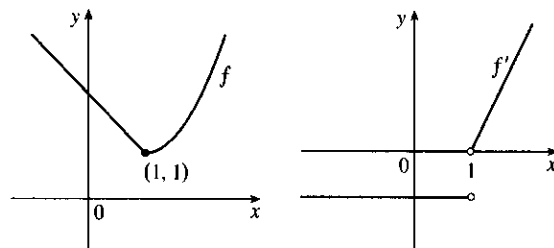
$$f'_-(1) = \lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{2 - (1+h) - 1}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = \lim_{h \rightarrow 0^-} -1 = -1 \text{ and}$$

$$f'_+(1) = \lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{(1+h)^2 - 2(1+h) + 2 - 1}{h} = \lim_{h \rightarrow 0^+} \frac{h^2}{h} = \lim_{h \rightarrow 0^+} h = 0.$$

Thus, $f'(1)$ does not exist since $f'_-(1) \neq f'_+(1)$,

so f is not differentiable at 1. But $f'(x) = -1$

for $x < 1$ and $f'(x) = 2x - 2$ if $x > 1$.



$$80. g(x) = \begin{cases} -1 - 2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

$$\lim_{h \rightarrow 0^-} \frac{g(-1+h) - g(-1)}{h} = \lim_{h \rightarrow 0^-} \frac{[-1 - 2(-1+h)] - 1}{h} = \lim_{h \rightarrow 0^-} \frac{-2h}{h} = \lim_{h \rightarrow 0^-} (-2) = -2 \text{ and}$$

$$\lim_{h \rightarrow 0^+} \frac{g(-1+h) - g(-1)}{h} = \lim_{h \rightarrow 0^+} \frac{(-1+h)^2 - 1}{h} = \lim_{h \rightarrow 0^+} \frac{-2h + h^2}{h} = \lim_{h \rightarrow 0^+} (-2 + h) = -2,$$

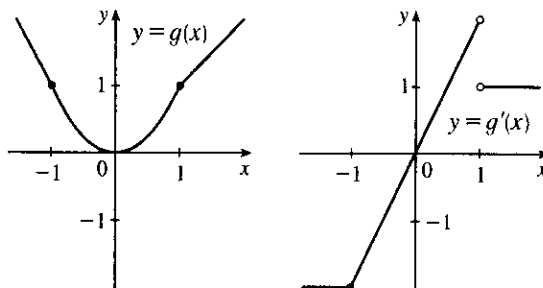
so g is differentiable at -1 and $g'(-1) = -2$.

$$\lim_{h \rightarrow 0^-} \frac{g(1) - g(1)}{h} = \lim_{h \rightarrow 0^-} \frac{(1)^2 - 1}{h} = \lim_{h \rightarrow 0^-} \frac{2h + h^2}{h} = \lim_{h \rightarrow 0^-} (2 + h) = 2 \text{ and}$$

$$\lim_{h \rightarrow 0^+} \frac{g(1) - g(1)}{h} = \lim_{h \rightarrow 0^+} \frac{(1) - 1}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1, \text{ so } g'(1) \text{ does not exist.}$$

Thus, g is differentiable except when $x = 1$, and

$$g'(x) = \begin{cases} -2 & \text{if } x < -1 \\ 2x & \text{if } -1 \leq x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$



81. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \Leftrightarrow |x| < 3 \Leftrightarrow -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \leq -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \geq 3 \end{cases} \Rightarrow$$

$$f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

To show that $f'(3)$ does not exist we investigate $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 3.2.44.

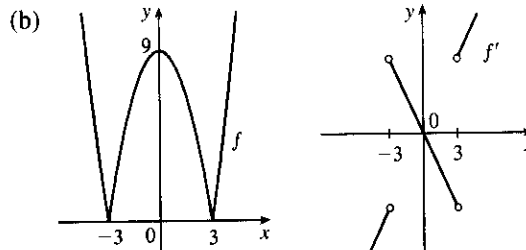
$$f'_-(3) = \lim_{h \rightarrow 0^-} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0^-} \frac{[-(3+h)^2 + 9] - 0}{h} = \lim_{h \rightarrow 0^-} (-6 - h) = -6 \quad \text{and}$$

$$f'_+(3) = \lim_{h \rightarrow 0^+} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0^+} \frac{[(3+h)^2 - 9] - 0}{h} = \lim_{h \rightarrow 0^+} \frac{6h + h^2}{h} = \lim_{h \rightarrow 0^+} (6 + h) = 6.$$

Since the left and right limits are different,

$$\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \text{ does not exist, that is,}$$

$f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3 .



- 82.** If $x > 1$, then $h(x) = |x - 1| + |x + 2| = x - 1 + x + 2 = 2x + 1$.

If $-2 < x < 1$, then $h(x) = -(x - 1) + x + 2 = 3$.

If $x < -2$, then $h(x) = -(x - 1) - (x + 2) = -2x - 1$. Therefore,

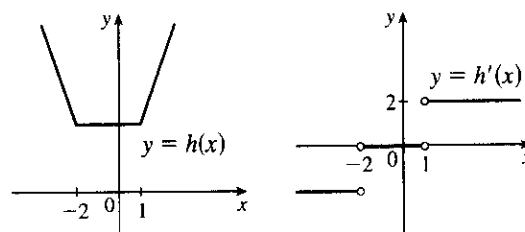
$$h(x) = \begin{cases} -2x - 1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x + 1 & \text{if } x > 1 \end{cases} \Rightarrow h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

To see that $h'(1) = \lim_{x \rightarrow 1} \frac{h(x) - h(1)}{x - 1}$ does not exist,

observe that $\lim_{x \rightarrow 1^-} \frac{h(x) - h(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{3 - 3}{3 - 1} = 0$ but

$$\lim_{x \rightarrow 1^+} \frac{h(x) - h(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{2x - 2}{x - 1} = 2. \text{ Similarly,}$$

$h'(-2)$ does not exist.



- 83.** $y = f(x) = ax^2 \Rightarrow f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \Leftrightarrow y = -2x + b$, is seen to be -2 , so we must have $4a = -2 \Leftrightarrow a = -\frac{1}{2}$. So when $x = 2$, the point in question has y -coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \Leftrightarrow b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.
- 84.** f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.
- 85.** $F = f/g \Rightarrow f = Fg \Rightarrow f' = F'g + Fg' \Rightarrow F' = \frac{f' - Fg'}{g} = \frac{f' - (f/g)g'}{g} = \frac{f'g - fg'}{g^2}$
- 86.** (a) $xy = c \Rightarrow y = \frac{c}{x}$. Let $P = \left(a, \frac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\frac{c}{a^2}$. Its equation is $y - \frac{c}{a} = -\frac{c}{a^2}(x - a)$ or $y = -\frac{c}{a^2}x + \frac{2c}{a}$, so its y -intercept is $\frac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x -intercept is $2a$. The midpoint of the line segment joining $\left(0, \frac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \frac{c}{a}\right) = P$.

(b) We know the x - and y -intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

87. Solution 1: Let $f(x) = x^{1000}$. Then, by the definition of a derivative,

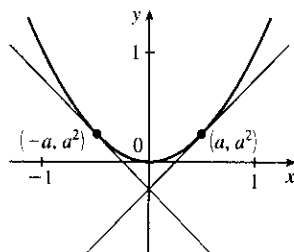
$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1}. \text{ But this is just the limit we want to find, and we know (from the}$$

Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1) = \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} \\ &= 1000, \text{ as above.} \end{aligned}$$

88.



In order for the two tangents to intersect on the y -axis, the points of tangency must be at equal distances from the y -axis, since the parabola $y = x^2$ is symmetric about the y -axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation

$y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1 , so $(-2a)(2a) = -1 \Leftrightarrow a^2 = \frac{1}{4} \Leftrightarrow a = \frac{1}{2}$. So the lines intersect at $(0, -\frac{1}{4})$.

3.4 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^2 - 10t + 12 \Rightarrow v(t) = f'(t) = 2t - 10$

(b) $v(3) = 2(3) - 10 = -4$ ft/s

(c) The particle is at rest when $v(t) = 0 \Leftrightarrow 2t - 10 = 0 \Leftrightarrow t = 5$ s.

(d) The particle is moving in the positive direction when $v(t) > 0 \Leftrightarrow 2t - 10 > 0 \Leftrightarrow 2t > 10 \Leftrightarrow t > 5$.

(e) Since the particle is moving in the positive direction (f)

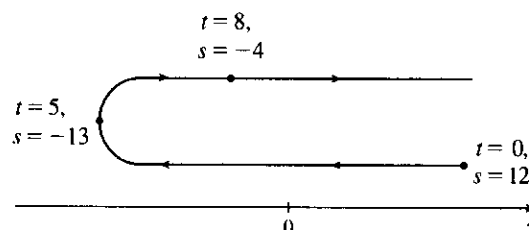
and in the negative direction, we need to calculate the

distance traveled in the intervals $[0, 5]$ and $[5, 8]$

separately. $|f(5) - f(0)| = |-13 - 12| = 25$ ft and

$|f(8) - f(5)| = |-4 - (-13)| = 9$ ft. The total

distance traveled during the first 8 s is $25 + 9 = 34$ ft.



$$2. (a) s = f(t) = t^3 - 9t^2 + 15t + 10 \Rightarrow v(t) = f'(t) = 3t^2 - 18t + 15 = 3(t-1)(t-5)$$

$$(b) v(3) = 3(2)(-2) = -12 \text{ ft/s}$$

$$(c) v(t) = 0 \Leftrightarrow t = 1 \text{ s or } 5 \text{ s}$$

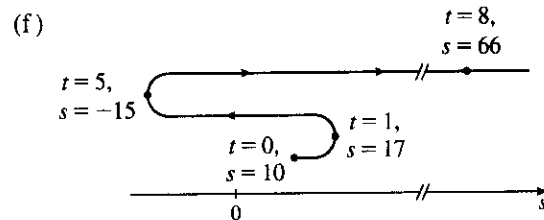
$$(d) v(t) > 0 \Leftrightarrow 0 \leq t < 1 \text{ or } t > 5$$

$$(e) |f(1) - f(0)| = |17 - 10| = 7,$$

$$|f(5) - f(1)| = |-15 - 17| = 32, \text{ and}$$

$$|f(8) - f(5)| = |66 - (-15)| = 81.$$

$$\text{Total distance} = 7 + 32 + 81 = 120 \text{ ft.}$$



$$3. (a) s = f(t) = t^3 - 12t^2 + 36t \Rightarrow v(t) = f'(t) = 3t^2 - 24t + 36$$

$$(b) v(3) = 27 - 72 + 36 = -9 \text{ ft/s}$$

$$(c) \text{The particle is at rest when } v(t) = 0. 3t^2 - 24t + 36 = 0 \Rightarrow 3(t-2)(t-6) = 0 \Rightarrow t = 2 \text{ s or } 6 \text{ s.}$$

$$(d) \text{The particle is moving in the positive direction when } v(t) > 0. 3(t-2)(t-6) > 0 \Leftrightarrow 0 \leq t < 2 \text{ or } t > 6.$$

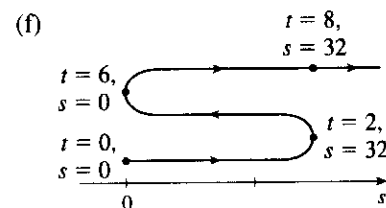
(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 2]$, $[2, 6]$, and $[6, 8]$ separately.

$$|f(2) - f(0)| = |32 - 0| = 32.$$

$$|f(6) - f(2)| = |0 - 32| = 32.$$

$$|f(8) - f(6)| = |32 - 0| = 32.$$

$$\text{The total distance is } 32 + 32 + 32 = 96 \text{ ft.}$$



$$4. (a) s = f(t) = t^4 - 4t + 1 \Rightarrow v(t) = f'(t) = 4t^3 - 4$$

$$(b) v(3) = 4(3)^3 - 4 = 104 \text{ ft/s}$$

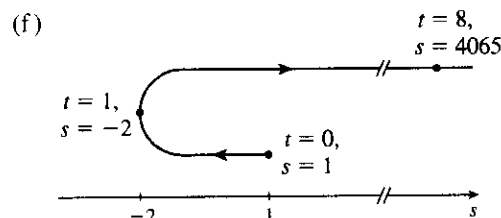
$$(c) \text{It is at rest when } v(t) = 4(t^3 - 1) = 4(t-1)(t^2 + t + 1) = 0 \Leftrightarrow t = 1 \text{ s.}$$

$$(d) \text{It moves in the positive direction when } 4(t^3 - 1) > 0 \Leftrightarrow t > 1.$$

$$(e) \text{Distance in positive direction} = |f(8) - f(1)| = |4065 - (-2)| = 4067 \text{ ft}$$

$$\text{Distance in negative direction} = |f(1) - f(0)| = |-2 - 1| = 3 \text{ ft}$$

$$\text{Total distance traveled} = 4067 + 3 = 4070 \text{ ft}$$



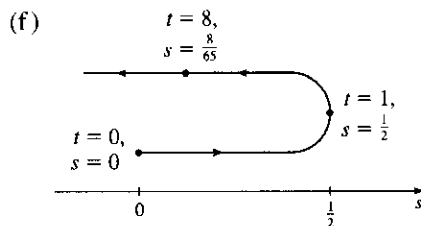
$$5. (a) s = \frac{t}{t^2 + 1} \Rightarrow v(t) = s'(t) = \frac{(t^2 + 1)(1) - t(2t)}{(t^2 + 1)^2} = \frac{1 - t^2}{(t^2 + 1)^2}$$

$$(b) v(3) = \frac{1 - (3)^2}{(3^2 + 1)^2} = \frac{1 - 9}{10^2} = \frac{-8}{100} = -\frac{2}{25} \text{ ft/s}$$

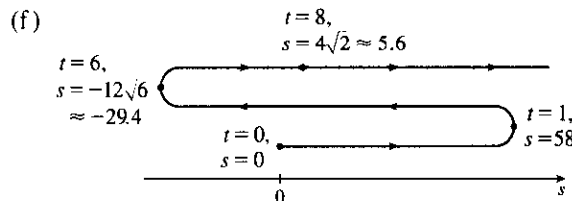
$$(c) \text{It is at rest when } v = 0 \Leftrightarrow 1 - t^2 = 0 \Leftrightarrow t = 1 \text{ s } [t \neq -1 \text{ since } t \geq 0].$$

$$(d) \text{It moves in the positive direction when } v > 0 \Leftrightarrow 1 - t^2 > 0 \Leftrightarrow t^2 < 1 \Leftrightarrow 0 \leq t < 1.$$

- (e) Distance in positive direction $= |s(1) - s(0)| = \left| \frac{1}{2} - 0 \right| = \frac{1}{2}$ ft
 Distance in negative direction $= |s(8) - s(1)| = \left| \frac{8}{65} - \frac{1}{2} \right| = \frac{49}{130}$ ft
 Total distance traveled $= \frac{1}{2} + \frac{49}{130} = \frac{57}{65}$ ft



6. (a) $s = \sqrt{t}(3t^2 - 35t + 90) = 3t^{5/2} - 35t^{3/2} + 90t^{1/2} \Rightarrow$
 $v(t) = s'(t) = \frac{15}{2}t^{3/2} - \frac{105}{2}t^{1/2} + 45t^{-1/2} = \frac{15}{2}t^{-1/2}(t^2 - 7t + 6) = \frac{15}{2\sqrt{t}}(t-1)(t-6)$
 (b) $v(3) = \frac{15}{2\sqrt{3}}(2)(-3) = -15\sqrt{3}$ ft/s
 (c) It is at rest when $v = 0 \Leftrightarrow t = 1$ s or 6 s.
 (d) It moves in the positive direction when $v > 0 \Leftrightarrow (t-1)(t-6) > 0 \Leftrightarrow 0 \leq t < 1$ or $t > 6$.
 (e) Distance in positive direction $= |s(1) - s(0)| + |s(8) - s(6)| = |58 - 0| + |4\sqrt{2} - (-12\sqrt{6})|$
 $= 58 + 4\sqrt{2} + 12\sqrt{6} \approx 93.05$ ft
 Distance in negative direction $= |s(6) - s(1)| = |-12\sqrt{6} - 58| = 58 + 12\sqrt{6} \approx 87.39$ ft
 Total distance traveled $= 58 + 4\sqrt{2} + 12\sqrt{6} + 58 + 12\sqrt{6} = 116 + 4\sqrt{2} + 24\sqrt{6} \approx 180.44$ ft



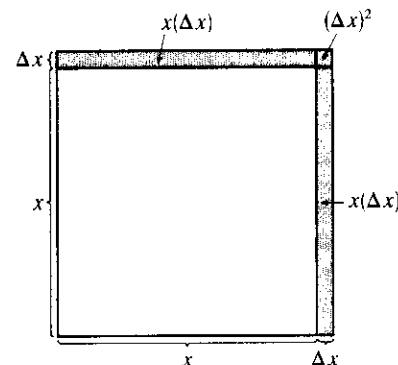
7. $s(t) = t^3 - 4.5t^2 - 7t \Rightarrow v(t) = s'(t) = 3t^2 - 9t - 7 = 5 \Leftrightarrow 3t^2 - 9t - 12 = 0 \Leftrightarrow$
 $3(t-4)(t+1) = 0 \Leftrightarrow t = 4$ or -1 . Since $t \geq 0$, the particle reaches a velocity of 5 m/s at $t = 4$ s.
8. (a) $s = 5t + 3t^2 \Rightarrow v(t) = \frac{ds}{dt} = 5 + 6t$, so $v(2) = 5 + 6(2) = 17$ m/s.
 (b) $v(t) = 35 \Rightarrow 5 + 6t = 35 \Rightarrow 6t = 30 \Rightarrow t = 5$ s.
9. (a) $h = 10t - 0.83t^2 \Rightarrow v(t) = \frac{dh}{dt} = 10 - 1.66t$, so $v(3) = 10 - 1.66(3) = 5.02$ m/s.
 (b) $h = 25 \Rightarrow 10t - 0.83t^2 = 25 \Rightarrow 0.83t^2 - 10t + 25 = 0 \Rightarrow t = \frac{10 \pm \sqrt{17}}{1.66} \approx 3.54$ or 8.51 .
 The value $t_1 = (10 - \sqrt{17})/1.66$ corresponds to the time it takes for the stone to rise 25 m and
 $t_2 = (10 + \sqrt{17})/1.66$ corresponds to the time when the stone is 25 m high on the way down. Thus,
 $v(t_1) = 10 - 1.66[(10 - \sqrt{17})/1.66] = \sqrt{17} \approx 4.12$ m/s.

10. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \Leftrightarrow 32t = 80 \Leftrightarrow t = \frac{5}{2}$. So the maximum height is $s(\frac{5}{2}) = 80(\frac{5}{2}) - 16(\frac{5}{2})^2 = 200 - 100 = 100$ ft.
- (b) $s(t) = 80t - 16t^2 = 96 \Leftrightarrow 16t^2 - 80t + 96 = 0 \Leftrightarrow 16(t^2 - 5t + 6) = 0 \Leftrightarrow 16(t-3)(t-2) = 0$. So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

11. (a) $A(x) = x^2 \Rightarrow A'(x) = 2x$. $A'(15) = 30$ mm²/mm is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

- (b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$.

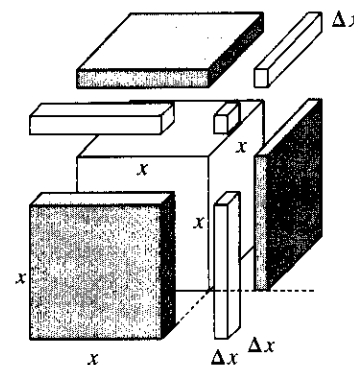
The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx . From the figure, $\Delta A = 2x(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.



12. (a) $V(x) = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$. $\frac{dV}{dx} \Big|_{x=3} = 3(3)^2 = 27$ mm³/mm is the rate at which the volume is increasing as x increases past 3 mm.

- (b) The surface area is $S(x) = 6x^2$, so

$V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx . From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.



13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

$$(i) \frac{A(3) - A(2)}{3 - 2} = \frac{9\pi - 4\pi}{1} = 5\pi$$

$$(ii) \frac{A(2.5) - A(2)}{2.5 - 2} = \frac{6.25\pi - 4\pi}{0.5} = 4.5\pi$$

$$(iii) \frac{A(2.1) - A(2)}{2.1 - 2} = \frac{4.41\pi - 4\pi}{0.1} = 4.1\pi$$

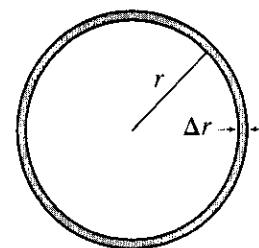
- (b) $A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

- (c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr . Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr , so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically,

$$\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2.$$

So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore,

$$\Delta A/\Delta r \approx 2\pi r.$$



14. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \Rightarrow A'(t) = 7200\pi t \Rightarrow$

(a) $A'(1) = 7200\pi \text{ cm}^2/\text{s}$ (b) $A'(3) = 21,600\pi \text{ cm}^2/\text{s}$ (c) $A'(5) = 36,000\pi \text{ cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

15. $S(r) = 4\pi r^2 \Rightarrow S'(r) = 8\pi r \Rightarrow$

(a) $S'(1) = 8\pi \text{ ft}^2/\text{ft}$ (b) $S'(2) = 16\pi \text{ ft}^2/\text{ft}$ (c) $S'(3) = 24\pi \text{ ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

16. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\frac{V(8) - V(5)}{8 - 5} = \frac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi \mu\text{m}^3/\mu\text{m}$

(ii) $\frac{V(6) - V(5)}{6 - 5} = \frac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\bar{3}\pi \mu\text{m}^3/\mu\text{m}$

(iii) $\frac{V(5.1) - V(5)}{5.1 - 5} = \frac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\bar{3}\pi \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi \mu\text{m}^3/\mu\text{m}$.

- (c) $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 13(c), we can say that the change in the volume of the spherical shell, ΔV , is approximately equal to its thickness, Δr , times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6 \text{ kg/m}$ (b) $\rho(2) = 12 \text{ kg/m}$ (c) $\rho(3) = 18 \text{ kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

18. $V(t) = 5000(1 - \frac{1}{40}t)^2 = 5000(1 - \frac{1}{20}t + \frac{1}{1600}t^2) \Rightarrow V'(t) = 5000(-\frac{1}{20} + \frac{1}{800}t) = -250(1 - \frac{1}{40}t)$

(a) $V'(5) = -250(1 - \frac{5}{40}) = -218.75 \text{ gal/min}$ (b) $V'(10) = -250(1 - \frac{10}{40}) = -187.5 \text{ gal/min}$

(c) $V'(20) = -250(1 - \frac{20}{40}) = -125 \text{ gal/min}$ (d) $V'(40) = -250(1 - \frac{40}{40}) = 0 \text{ gal/min}$

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250 \text{ gal/min}$. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75 \text{ A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5 \text{ A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3} \text{ s}$.

20. (a) $F = \frac{GmM}{r^2} = (GmM)r^{-2} \Rightarrow \frac{dF}{dr} = -2(GmM)r^{-3} = -\frac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20,000) = -2$, find $F'(10,000)$. $-2 = -\frac{2GmM}{20,000^3} \Rightarrow GmM = 20,000^3$.

$F'(10,000) = -\frac{2(20,000^3)}{10,000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$

21. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P .

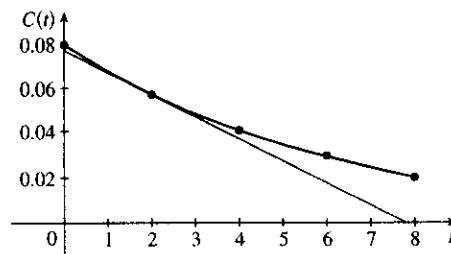
$$PV = C \Rightarrow V = \frac{C}{P} \Rightarrow \frac{dV}{dP} = -\frac{C}{P^2}.$$

- (b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.

$$(c) \beta = -\frac{1}{V} \frac{dV}{dP} = -\frac{1}{V} \left(-\frac{C}{P^2} \right) = \frac{C}{(PV)P} = \frac{C}{CP} = \frac{1}{P}$$

$$\begin{aligned} 22. (a) \quad (i) \quad \frac{C(6) - C(2)}{6 - 2} &= \frac{0.0295 - 0.0570}{4} \\ &= -0.006875 \text{ (moles/L)/min} \\ (ii) \quad \frac{C(4) - C(2)}{4 - 2} &= \frac{0.0408 - 0.0570}{2} \\ &= -0.008 \text{ (moles/L)/min} \\ (iii) \quad \frac{C(2) - C(0)}{2 - 0} &= \frac{0.0570 - 0.0800}{2} \\ &= -0.0115 \text{ (moles/L)/min} \end{aligned}$$

$$(b) \text{ Slope} = \frac{\Delta C}{\Delta t} \approx -\frac{0.077}{7.8} \approx -0.01 \text{ (moles/L)/min}$$



$$\begin{aligned} 23. (a) \quad 1920: m_1 &= \frac{1860 - 1750}{1920 - 1910} = \frac{110}{10} = 11, m_2 = \frac{2070 - 1860}{1930 - 1920} = \frac{210}{10} = 21, \\ (m_1 + m_2)/2 &= (11 + 21)/2 = 16 \text{ million/year} \\ 1980: m_1 &= \frac{4450 - 3710}{1980 - 1970} = \frac{740}{10} = 74, m_2 = \frac{5280 - 4450}{1990 - 1980} = \frac{830}{10} = 83, \\ (m_1 + m_2)/2 &= (74 + 83)/2 = 78.5 \text{ million/year} \end{aligned}$$

- (b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx 0.0012937063$, $b \approx -7.061421911$, $c \approx 12,822.97902$, and $d \approx -7,743,770.396$.

$$(c) P(t) = at^3 + bt^2 + ct + d \Rightarrow P'(t) = 3at^2 + 2bt + c \text{ (in millions of people per year)}$$

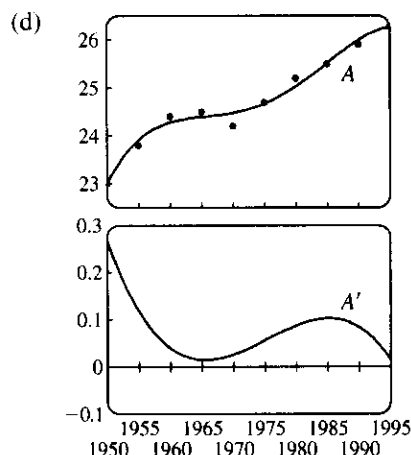
$$\begin{aligned} (d) P'(1920) &= 3(0.0012937063)(1920)^2 + 2(-7.061421911)(1920) + 12,822.97902 \\ &\approx 14.48 \text{ million/year [smaller than the answer in part (a), but close to it]} \\ P'(1980) &\approx 75.29 \text{ million/year (smaller, but close)} \end{aligned}$$

- (e) $P'(1985) \approx 81.62$ million/year, so the rate of growth in 1985 was about 81.62 million/year.

$$24. (a) A(t) = at^4 + bt^3 + ct^2 + dt + e, \text{ where } a = -5.8275058275396 \times 10^{-6}, b = 0.0460458430461, \\ c = -136.43277039706, d = 179,661.02676871, \text{ and } e = -88,717,597.060767.$$

$$(b) A(t) = at^4 + bt^3 + ct^2 + dt + e \Rightarrow A'(t) = 4at^3 + 3bt^2 + 2ct + d$$

$$(c) A'(1990) \approx 0.0833 \text{ years of age per year}$$



25. (a) $[C] = \frac{a^2 kt}{akt + 1} \Rightarrow$

$$\text{rate of reaction} = \frac{d[C]}{dt} = \frac{(akt + 1)(a^2 k) - (a^2 kt)(ak)}{(akt + 1)^2} = \frac{a^2 k(akt + 1 - akt)}{(akt + 1)^2} = \frac{a^2 k}{(akt + 1)^2}$$

(b) If $x = [C]$, then $a - x = a - \frac{a^2 kt}{akt + 1} = \frac{a^2 kt + a - a^2 kt}{akt + 1} = \frac{a}{akt + 1}$.

So $k(a - x)^2 = k\left(\frac{a}{akt + 1}\right)^2 = \frac{a^2 k}{(akt + 1)^2} = \frac{d[C]}{dt}$ [from part (a)] $= \frac{dx}{dt}$.

26. $\frac{1}{p} = \frac{1}{f} - \frac{1}{q} \Leftrightarrow \frac{1}{p} = \frac{q - f}{fq} \Leftrightarrow p = \frac{fq}{q - f}$. So $\frac{dp}{dq} = \frac{f(q - f) - fq}{(q - f)^2} = -\frac{f^2}{(q - f)^2}$.

27. (a) Using $v = \frac{P}{4\eta l} (R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r :

$$v(r) = \frac{3000}{4(0.027)3} (0.01^2 - r^2). \quad v(0) = 0.925 \text{ cm/s}, \quad v(0.005) = 0.694 \text{ cm/s}, \quad v(0.01) = 0.$$

(b) $v(r) = \frac{P}{4\eta l} (R^2 - r^2) \Rightarrow v'(r) = \frac{P}{4\eta l} (-2r) = -\frac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$$v'(r) = -\frac{3000r}{2(0.027)3}. \quad v'(0) = 0, \quad v'(0.005) = -92.592 \text{ (cm/s)/cm}, \quad \text{and } v'(0.01) = -185.185 \text{ (cm/s)/cm}.$$

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

28. (a) (i) $f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left(\frac{1}{2} \sqrt{\frac{T}{\rho}}\right) L^{-1} \Rightarrow \frac{df}{dL} = -\left(\frac{1}{2} \sqrt{\frac{T}{\rho}}\right) L^{-2} = -\frac{1}{2L^2} \sqrt{\frac{T}{\rho}}$

(ii) $f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left(\frac{1}{2L\sqrt{\rho}}\right) T^{1/2} \Rightarrow \frac{df}{dT} = \frac{1}{2} \left(\frac{1}{2L\sqrt{\rho}}\right) T^{-1/2} = \frac{1}{4L\sqrt{T\rho}}$

(iii) $f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left(\frac{\sqrt{T}}{2L}\right) \rho^{-1/2} \Rightarrow \frac{df}{d\rho} = -\frac{1}{2} \left(\frac{\sqrt{T}}{2L}\right) \rho^{-3/2} = -\frac{\sqrt{T}}{4L\rho^{3/2}}$

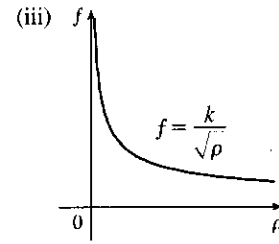
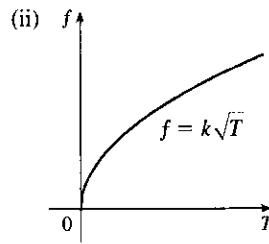
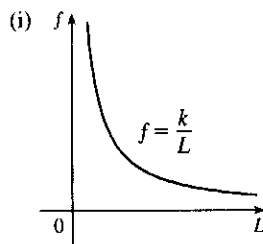
(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\frac{df}{dL} < 0$ and L is decreasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

(ii) $\frac{df}{dT} > 0$ and T is increasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

SECTION 3.4 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES

- (iii) $\frac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow f$ is decreasing $\Rightarrow$ lower note



29. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \Rightarrow C'(x) = 3 + 0.02x + 0.0006x^2$
 (b) $C'(100) = 3 + 0.02(100) + 0.0006(10,000) = 3 + 2 + 6 = \$11/\text{pair}$. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the cost of the 101st pair.
 (c) The cost of manufacturing the 101st pair of jeans is

$$C(101) - C(100) = (2000 + 303 + 102.01 + 206.0602) - (2000 + 300 + 100 + 200) = 11.0702 \approx \$11.07$$

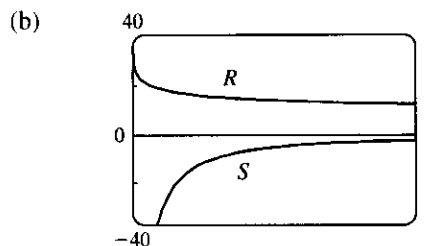
30. (a) $C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3 \Rightarrow C'(x) = 0.16 - 0.0012x + 0.000009x^2 \Rightarrow C'(100) = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.
 (b) $C(101) - C(100) = 97.13030299 - 97 \approx \0.13 .

31. (a) $A(x) = \frac{p(x)}{x} \Rightarrow A'(x) = \frac{xp'(x) - p(x) \cdot 1}{x^2} = \frac{xp'(x) - p(x)}{x^2}$. $A'(x) > 0 \Rightarrow A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

- (b) $p'(x)$ is greater than the average productivity $\Rightarrow p'(x) > A(x) \Rightarrow p'(x) > \frac{p(x)}{x} \Rightarrow xp'(x) > p(x) \Rightarrow xp'(x) - p(x) > 0 \Rightarrow \frac{xp'(x) - p(x)}{x^2} > 0 \Rightarrow A'(x) > 0$.

32. (a) $S = \frac{dR}{dx} = \frac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$

$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$



At low levels of brightness, R is quite large [$R(0) = 40$] and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

33. $PV = nRT \Rightarrow T = \frac{PV}{nR} = \frac{PV}{(10)(0.0821)} = \frac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821} [P(t)V'(t) + V(t)P'(t)] = \frac{1}{0.821} [(8)(-0.15) + (10)(0.10)] \approx -0.2436 \text{ K/min.}$$

34. (a) If $dP/dt = 0$, the population is stable (it is constant).

$$(b) \frac{dP}{dt} = 0 \Rightarrow \beta P = r_0 \left(1 - \frac{P}{P_c}\right) P \Rightarrow \frac{\beta}{r_0} = 1 - \frac{P}{P_c} \Rightarrow \frac{P}{P_c} = 1 - \frac{\beta}{r_0} \Rightarrow P = P_c \left(1 - \frac{\beta}{r_0}\right).$$

If $P_c = 10,000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10,000 \left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10,000 \left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is,

$$\frac{dC}{dt} = 0 \text{ and } \frac{dW}{dt} = 0.$$

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\frac{dC}{dt} = aC - bCW$ and $\frac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$,

$b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ (1) and

$-0.05W + 0.0001CW = 0$ (2). Adding 10 times (2) to (1) eliminates the CW -terms and gives us

$0.05C - 0.5W = 0 \Rightarrow C = 10W$. Substituting $C = 10W$ into (1) results in

$$0.05(10W) - 0.001(10W)W = 0 \Leftrightarrow 0.5W - 0.01W^2 = 0 \Leftrightarrow 50W - W^2 = 0 \Leftrightarrow$$

$$W(50 - W) = 0 \Leftrightarrow W = 0 \text{ or } 50. \text{ Since } C = 10W, C = 0 \text{ or } 500. \text{ Thus, the population pairs } (C, W)$$

that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.5 Derivatives of Trigonometric Functions

$$1. f(x) = x - 3 \sin x \Rightarrow f'(x) = 1 - 3 \cos x$$

$$2. f(x) = x \sin x \Rightarrow f'(x) = x \cdot \cos x + (\sin x) \cdot 1 = x \cos x + \sin x$$

$$3. y = \sin x + 10 \tan x \Rightarrow y' = \cos x + 10 \sec^2 x$$

$$4. y = 2 \csc x + 5 \cos x \Rightarrow y' = -2 \csc x \cot x - 5 \sin x$$

$$5. g(t) = t^3 \cos t \Rightarrow g'(t) = t^3(-\sin t) + (\cos t) \cdot 3t^2 = 3t^2 \cos t - t^3 \sin t \text{ or } t^2(3 \cos t - t \sin t)$$

$$6. g(t) = 4 \sec t + \tan t \Rightarrow g'(t) = 4 \sec t \tan t + \sec^2 t$$

$$7. h(\theta) = \theta \csc \theta - \cot \theta \Rightarrow h'(\theta) = \theta(-\csc \theta \cot \theta) + (\csc \theta) \cdot 1 - (-\csc^2 \theta) = \csc \theta - \theta \csc \theta \cot \theta + \csc^2 \theta$$

$$8. y = u(a \cos u + b \cot u) \Rightarrow$$

$$y' = u(-a \sin u - b \csc^2 u) + (a \cos u + b \cot u) \cdot 1 = a \cos u + b \cot u - au \sin u - bu \csc^2 u$$

$$9. y = \frac{x}{\cos x} \Rightarrow y' = \frac{(\cos x)(1) - (x)(-\sin x)}{(\cos x)^2} = \frac{\cos x + x \sin x}{\cos^2 x}$$

$$10. y = \frac{1 + \sin x}{x + \cos x} \Rightarrow$$

$$y' = \frac{(x + \cos x)(\cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} = \frac{x \cos x + \cos^2 x - (1 - \sin^2 x)}{(x + \cos x)^2}$$

$$= \frac{x \cos x + \cos^2 x - (\cos^2 x)}{(x + \cos x)^2} = \frac{x \cos x}{(x + \cos x)^2}$$

$$11. f(\theta) = \frac{\sec \theta}{1 + \sec \theta} \Rightarrow$$

$$f'(\theta) = \frac{(1 + \sec \theta)(\sec \theta \tan \theta) - (\sec \theta)(\sec \theta \tan \theta)}{(1 + \sec \theta)^2} = \frac{(\sec \theta \tan \theta)[(1 + \sec \theta) - \sec \theta]}{(1 + \sec \theta)^2} = \frac{\sec \theta \tan \theta}{(1 + \sec \theta)^2}$$

$$12. y = \frac{\tan x - 1}{\sec x} \Rightarrow$$

$$\frac{dy}{dx} = \frac{\sec x \sec^2 x - (\tan x - 1) \sec x \tan x}{\sec^2 x} = \frac{\sec x (\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

Another method: Simplify y first: $y = \sin x - \cos x \Rightarrow y' = \cos x + \sin x$.

$$13. y = \frac{\sin x}{x^2} \Rightarrow y' = \frac{x^2 \cos x - (\sin x)(2x)}{(x^2)^2} = \frac{x(x \cos x - 2 \sin x)}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}$$

$$14. y = \csc \theta (\theta + \cot \theta) \Rightarrow$$

$$y' = \csc \theta (1 - \csc^2 \theta) + (\theta + \cot \theta)(-\csc \theta \cot \theta) = \csc \theta (1 - \csc^2 \theta - \theta \cot \theta - \cot^2 \theta)$$

$$= \csc \theta (-\cot^2 \theta - \theta \cot \theta - \cot^2 \theta) \quad [1 + \cot^2 \theta = \csc^2 \theta]$$

$$= \csc \theta (-\theta \cot \theta - 2 \cot^2 \theta) = -\csc \theta \cot \theta (\theta + 2 \cot \theta)$$

$$15. y = \sec \theta \tan \theta \Rightarrow y' = \sec \theta (\sec^2 \theta) + \tan \theta (\sec \theta \tan \theta) = \sec \theta (\sec^2 \theta + \tan^2 \theta)$$

Using the identity $1 + \tan^2 \theta = \sec^2 \theta$, we can write alternative forms of the answer as

$$\sec \theta (1 + 2 \tan^2 \theta) \quad \text{or} \quad \sec \theta (2 \sec^2 \theta - 1)$$

$$16. \text{Recall that if } y = fgh, \text{ then } y' = f'gh + fg'h + fgh'. y = x \sin x \cos x \Rightarrow$$

$$\frac{dy}{dx} = \sin x \cos x + x \cos x \cos x + x \sin x (-\sin x) = \sin x \cos x + x \cos^2 x - x \sin^2 x$$

$$17. \frac{d}{dx} (\csc x) = \frac{d}{dx} \left(\frac{1}{\sin x} \right) = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$18. \frac{d}{dx} (\sec x) = \frac{d}{dx} \left(\frac{1}{\cos x} \right) = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$19. \frac{d}{dx} (\cot x) = \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$20. f(x) = \cos x \Rightarrow$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \rightarrow 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) = \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h}$$

$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

$$21. y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow \text{the slope of the tangent line at } \left(\frac{\pi}{4}, 1 \right) \text{ is } \sec^2 \frac{\pi}{4} = (\sqrt{2})^2 = 2 \text{ and an equation of the tangent line is } y - 1 = 2(x - \frac{\pi}{4}) \text{ or } y = 2x + 1 - \frac{\pi}{2}.$$

$$22. y = (1+x) \cos x \Rightarrow y' = (1+x)(-\sin x) + \cos x \cdot 1. \text{ At } (0, 1), y' = 1, \text{ and an equation of the tangent line is } y - 1 = 1(x - 0), \text{ or } y = x + 1.$$

$$23. y = x + \cos x \Rightarrow y' = 1 - \sin x. \text{ At } (0, 1), y' = 1, \text{ and an equation of the tangent line is } y - 1 = 1(x - 0), \text{ or } y = x + 1.$$

24. $y = \frac{1}{\sin x + \cos x} \Rightarrow y' = -\frac{\cos x - \sin x}{(\sin x + \cos x)^2}$ [Reciprocal Rule]. At $(0, 1)$, $y' = -\frac{1-0}{(0+1)^2} = -1$, and an equation of the tangent line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

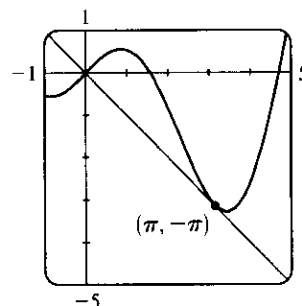
25. (a) $y = x \cos x \Rightarrow y' = x(-\sin x) + \cos x(1) = \cos x - x \sin x$.

So the slope of the tangent at the point $(\pi, -\pi)$ is

$\cos \pi - \pi \sin \pi = -1 - \pi(0) = -1$, and an equation is

$y + \pi = -(x - \pi)$ or $y = -x$.

(b)



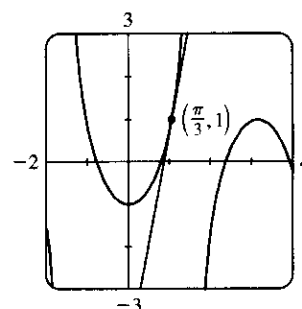
26. (a) $y = \sec x - 2 \cos x \Rightarrow y' = \sec x \tan x + 2 \sin x \Rightarrow$

the slope of the tangent line at $(\frac{\pi}{3}, 1)$ is

$\sec \frac{\pi}{3} \tan \frac{\pi}{3} + 2 \sin \frac{\pi}{3} = 2 \cdot \sqrt{3} + 2 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$ and an equation is

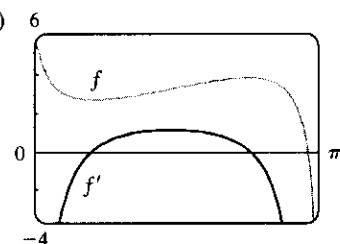
$y - 1 = 3\sqrt{3}(x - \frac{\pi}{3})$ or $y = 3\sqrt{3}x + 1 - \pi\sqrt{3}$.

(b)



27. (a) $f(x) = 2x + \cot x \Rightarrow f'(x) = 2 - \csc^2 x$

(b)

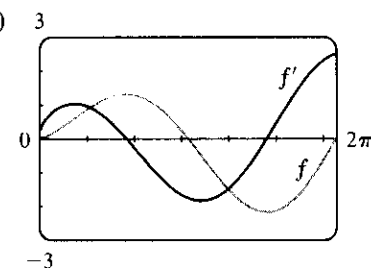


Notice that $f'(x) = 0$ when f has a horizontal tangent.

f' is positive when f is increasing and f' is negative when f is decreasing. Also, $f'(x)$ is large negative when the graph of f is steep.

28. (a) $f(x) = \sqrt{x} \sin x \Rightarrow f'(x) = \sqrt{x} \cos x + (\sin x)\left(\frac{1}{2}x^{-1/2}\right) = \sqrt{x} \cos x + \frac{\sin x}{2\sqrt{x}}$

(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent.

f' is positive when f is increasing and f' is negative when f is decreasing.

29. $f(x) = x + 2 \sin x$ has a horizontal tangent when $f'(x) = 0 \Leftrightarrow 1 + 2 \cos x = 0 \Leftrightarrow \cos x = -\frac{1}{2} \Leftrightarrow x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm \frac{\pi}{3}$ units from π . This allows us to write the solutions in the more compact equivalent form $(2n + 1)\pi \pm \frac{\pi}{3}$, n an integer.

$$30. y = \frac{\cos x}{2 + \sin x} \Rightarrow$$

$$y' = \frac{(2 + \sin x)(-\sin x) - \cos x \cos x}{(2 + \sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} = \frac{-2\sin x - 1}{(2 + \sin x)^2} = 0 \text{ when}$$

$$-2\sin x - 1 = 0 \Leftrightarrow \sin x = -\frac{1}{2} \Leftrightarrow x = \frac{11\pi}{6} + 2\pi n \text{ or } x = \frac{7\pi}{6} + 2\pi n, n \text{ an integer. So } y = \frac{1}{\sqrt{3}} \text{ or}$$

$y = -\frac{1}{\sqrt{3}}$ and the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}\right), \left(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

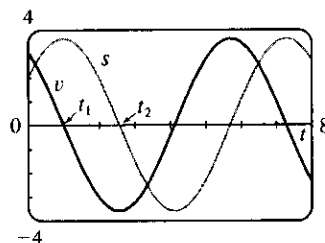
$$31. (a) x(t) = 8 \sin t \Rightarrow v(t) = x'(t) = 8 \cos t$$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8 \sin \frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$ and velocity

$$v\left(\frac{2\pi}{3}\right) = 8 \cos \frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4. \text{ Since } v\left(\frac{2\pi}{3}\right) < 0, \text{ the particle is moving to the left.}$$

$$32. (a) s(t) = 2 \cos t + 3 \sin t \Rightarrow v(t) = -2 \sin t + 3 \cos t$$

(b)

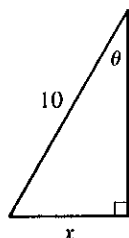


(c) $s = 0 \Rightarrow t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \Rightarrow t_1 \approx 0.98, s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$; that is, when $t = t_2 + n\pi, n$ a positive integer.

33.



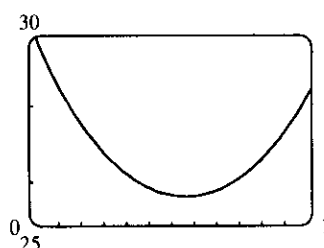
From the diagram we can see that $\sin \theta = x/10 \Leftrightarrow x = 10 \sin \theta$. We want to find the rate of change of x with respect to θ ; that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta = 10(\cos \theta)$. So when $\theta = \frac{\pi}{3}$,

$$dx/d\theta = 10 \cos \frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5 \text{ ft/rad}$$

$$34. (a) F = \frac{\mu W}{\mu \sin \theta + \cos \theta} \Rightarrow \frac{dF}{d\theta} = \frac{(\mu \sin \theta + \cos \theta)(0) - \mu W(\mu \cos \theta - \sin \theta)}{(\mu \sin \theta + \cos \theta)^2} = \frac{\mu W(\sin \theta - \mu \cos \theta)}{(\mu \sin \theta + \cos \theta)^2}$$

$$(b) \frac{dF}{d\theta} = 0 \Rightarrow \mu W(\sin \theta - \mu \cos \theta) = 0 \Rightarrow \sin \theta = \mu \cos \theta \Rightarrow \tan \theta = \mu \Rightarrow \theta = \tan^{-1} \mu$$

(c)



From the graph of $F = \frac{0.6(50)}{0.6 \sin \theta + \cos \theta}$ for $0 \leq \theta \leq 1$, we see that

$$\frac{dF}{d\theta} = 0 \Rightarrow \theta \approx 0.54. \text{ Checking this with part (b) and } \mu = 0.6,$$

we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

$$\begin{aligned}
 35. \lim_{x \rightarrow 0} \frac{\sin 3x}{x} &= \lim_{x \rightarrow 0} \frac{3 \sin 3x}{3x} \quad [\text{multiply numerator and denominator by 3}] \\
 &= 3 \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} \quad [\text{as } x \rightarrow 0, 3x \rightarrow 0] \\
 &= 3 \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \quad [\text{let } \theta = 3x] \\
 &= 3(1) \quad [\text{Equation 2}] \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 36. \lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x} &= \lim_{x \rightarrow 0} \left(\frac{\sin 4x}{x} \cdot \frac{x}{\sin 6x} \right) = \lim_{x \rightarrow 0} \frac{4 \sin 4x}{4x} \cdot \lim_{x \rightarrow 0} \frac{6x}{6 \sin 6x} \\
 &= 4 \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{1}{6} \lim_{x \rightarrow 0} \frac{6x}{\sin 6x} = 4(1) \cdot \frac{1}{6}(1) = \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 37. \lim_{t \rightarrow 0} \frac{\tan 6t}{\sin 2t} &= \lim_{t \rightarrow 0} \left(\frac{\sin 6t}{t} \cdot \frac{1}{\cos 6t} \cdot \frac{t}{\sin 2t} \right) = \lim_{t \rightarrow 0} \frac{6 \sin 6t}{6t} \cdot \lim_{t \rightarrow 0} \frac{1}{\cos 6t} \cdot \lim_{t \rightarrow 0} \frac{2t}{2 \sin 2t} \\
 &= 6 \lim_{t \rightarrow 0} \frac{\sin 6t}{6t} \cdot \lim_{t \rightarrow 0} \frac{1}{\cos 6t} \cdot \frac{1}{2} \lim_{t \rightarrow 0} \frac{2t}{\sin 2t} = 6(1) \cdot \frac{1}{1} \cdot \frac{1}{2}(1) = 3
 \end{aligned}$$

$$38. \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\cos \theta - 1}{\theta}}{\frac{\sin \theta}{\theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} = \frac{0}{1} = 0$$

$$39. \lim_{\theta \rightarrow 0} \frac{\sin(\cos \theta)}{\sec \theta} = \frac{\sin\left(\lim_{\theta \rightarrow 0} \cos \theta\right)}{\lim_{\theta \rightarrow 0} \sec \theta} = \frac{\sin 1}{1} = \sin 1$$

$$\begin{aligned}
 40. \lim_{t \rightarrow 0} \frac{\sin^2 3t}{t^2} &= \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{t} \cdot \frac{\sin 3t}{t} \right) = \lim_{t \rightarrow 0} \frac{\sin 3t}{t} \cdot \lim_{t \rightarrow 0} \frac{\sin 3t}{t} \\
 &= \left(\lim_{t \rightarrow 0} \frac{\sin 3t}{t} \right)^2 = \left(3 \lim_{t \rightarrow 0} \frac{\sin 3t}{3t} \right)^2 = (3 \cdot 1)^2 = 9
 \end{aligned}$$

$$\begin{aligned}
 41. \lim_{x \rightarrow 0} \frac{\cot 2x}{\csc x} &= \lim_{x \rightarrow 0} \frac{\cos 2x \sin x}{\sin 2x} = \lim_{x \rightarrow 0} \cos 2x \left[\frac{(\sin x)/x}{(\sin 2x)/x} \right] = \lim_{x \rightarrow 0} \cos 2x \left[\frac{\lim_{x \rightarrow 0} [(\sin x)/x]}{2 \lim_{x \rightarrow 0} [(\sin 2x)/2x]} \right] \\
 &= 1 \cdot \frac{1}{2 \cdot 1} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 42. \lim_{x \rightarrow \pi/4} \frac{\sin x - \cos x}{\cos 2x} &= \lim_{x \rightarrow \pi/4} \frac{\sin x - \cos x}{\cos^2 x - \sin^2 x} = \lim_{x \rightarrow \pi/4} \frac{\sin x - \cos x}{(\cos x + \sin x)(\cos x - \sin x)} \\
 &= \lim_{x \rightarrow \pi/4} \frac{-1}{\cos x + \sin x} = \frac{-1}{\cos \frac{\pi}{4} + \sin \frac{\pi}{4}} = \frac{-1}{\sqrt{2}}
 \end{aligned}$$

43. Divide numerator and denominator by θ . ($\sin \theta$ also works.)

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\theta}}{1 + \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}}{1 + \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta}} = \frac{1}{1 + 1 \cdot 1} = \frac{1}{2}$$

$$44. \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2} = \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x \rightarrow 1} \frac{1}{x+2} \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x-1} = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

45. (a) $\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} \Rightarrow \sec^2 x = \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \frac{1}{\cos^2 x}$.
- (b) $\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} \Rightarrow \sec x \tan x = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x \tan x = \frac{\sin x}{\cos^2 x}$.
- (c) $\frac{d}{dx} (\sin x + \cos x) = \frac{d}{dx} \frac{1 + \cot x}{\csc x} \Rightarrow$

$$\cos x - \sin x = \frac{\csc x (-\csc^2 x) - (1 + \cot x)(-\csc x \cot x)}{\csc^2 x} = \frac{\csc x [-\csc^2 x + (1 + \cot x) \cot x]}{\csc^2 x}$$

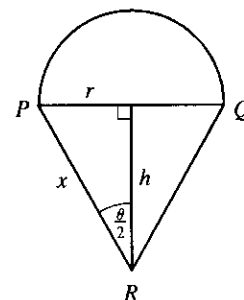
$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x}$$
 So $\cos x - \sin x = \frac{\cot x - 1}{\csc x}$.

46. Let $|PR| = x$. Then we get the following formulas for r and h in terms of θ and x :

$$\sin \frac{\theta}{2} = \frac{r}{x} \Rightarrow r = x \sin \frac{\theta}{2} \text{ and } \cos \frac{\theta}{2} = \frac{h}{x} \Rightarrow h = x \cos \frac{\theta}{2}.$$

Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)} &= \lim_{\theta \rightarrow 0^+} \frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \frac{r}{h} = \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \frac{x \sin(\theta/2)}{x \cos(\theta/2)} \\ &= \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \tan(\theta/2) = 0. \end{aligned}$$



47. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ , we can see that $\sin \frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r \sin \frac{\theta}{2}$.

So $\lim_{\theta \rightarrow 0^+} \frac{s}{d} = \lim_{\theta \rightarrow 0^+} \frac{r\theta}{2r \sin(\theta/2)} = \lim_{\theta \rightarrow 0^+} \frac{2 \cdot (\theta/2)}{2 \sin(\theta/2)} = \lim_{\theta \rightarrow 0} \frac{\theta/2}{\sin(\theta/2)} = 1$. [This is just the reciprocal of the limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ combined with the fact that as $\theta \rightarrow 0$, $\frac{\theta}{2} \rightarrow 0$ also.]

3.6 The Chain Rule

- Let $u = g(x) = 4x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\cos u)(4) = 4 \cos 4x$.
- Let $u = g(x) = 4 + 3x$ and $y = f(u) = \sqrt{u} = u^{1/2}$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-1/2}(3) = \frac{3}{2\sqrt{u}} = \frac{3}{2\sqrt{4+3x}}$.
- Let $u = g(x) = 1 - x^2$ and $y = f(u) = u^{10}$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (10u^9)(-2x) = -20x(1 - x^2)^9$.
- Let $u = g(x) = \sin x$ and $y = f(u) = \tan u$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\sec^2 u)(\cos x) = \sec^2(\sin x) \cdot \cos x$, or equivalently, $[\sec(\sin x)]^2 \cos x$.
- Let $u = g(x) = \sin x$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-1/2} \cos x = \frac{\cos x}{2\sqrt{u}} = \frac{\cos x}{2\sqrt{\sin x}}$.
- Let $u = g(x) = \sqrt{x}$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\cos u)\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos u}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{2\sqrt{x}}$.
- $F(x) = (x^3 + 4x)^7 \Rightarrow F'(x) = 7(x^3 + 4x)^6(3x^2 + 4)$ [or $7x^6(x^2 + 4)^6(3x^2 + 4)$]
- $F(x) = (x^2 - x + 1)^3 \Rightarrow F'(x) = 3(x^2 - x + 1)^2(2x - 1)$

9. $F(x) = \sqrt[4]{1+2x+x^3} = (1+2x+x^3)^{1/4} \Rightarrow$
 $F'(x) = \frac{1}{4}(1+2x+x^3)^{-3/4} \cdot \frac{d}{dx}(1+2x+x^3) = \frac{1}{4(1+2x+x^3)^{3/4}} \cdot (2+3x^2)$
 $= \frac{2+3x^2}{4(1+2x+x^3)^{3/4}} = \frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$
10. $f(x) = (1+x^4)^{2/3} \Rightarrow f'(x) = \frac{2}{3}(1+x^4)^{-1/3}(4x^3) = \frac{8x^3}{3\sqrt[3]{1+x^4}}$
11. $g(t) = \frac{1}{(t^4+1)^3} = (t^4+1)^{-3} \Rightarrow g'(t) = -3(t^4+1)^{-4}(4t^3) = -12t^3(t^4+1)^{-4} = \frac{-12t^3}{(t^4+1)^4}$
12. $f(t) = \sqrt[3]{1+\tan t} = (1+\tan t)^{1/3} \Rightarrow f'(t) = \frac{1}{3}(1+\tan t)^{-2/3} \sec^2 t = \frac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}}$
13. $y = \cos(a^3+x^3) \Rightarrow y' = -\sin(a^3+x^3) \cdot 3x^2$ [a^3 is just a constant] $= -3x^2 \sin(a^3+x^3)$
14. $y = a^3 + \cos^3 x \Rightarrow y' = 3(\cos x)^2(-\sin x)$ [a^3 is just a constant] $= -3 \sin x \cos^2 x$
15. $y = \cot(x/2) \Rightarrow y' = -\csc^2(x/2) \cdot \frac{1}{2} = -\frac{1}{2} \csc^2(x/2)$
16. $y = 4 \sec 5x \Rightarrow y' = 4 \sec 5x \tan 5x(5) = 20 \sec 5x \tan 5x$
17. $g(x) = (1+4x)^5(3+x-x^2)^8 \Rightarrow$
 $g'(x) = (1+4x)^5 \cdot 8(3+x-x^2)^7(1-2x) + (3+x-x^2)^8 \cdot 5(1+4x)^4 \cdot 4$
 $= 4(1+4x)^4(3+x-x^2)^7 [2(1+4x)(1-2x) + 5(3+x-x^2)]$
 $= 4(1+4x)^4(3+x-x^2)^7 [(2+4x-16x^2) + (15+5x-5x^2)]$
 $= 4(1+4x)^4(3+x-x^2)^7 (17+9x-21x^2)$
18. $h(t) = (t^4-1)^3(t^3+1)^4 \Rightarrow$
 $h'(t) = (t^4-1)^3 \cdot 4(t^3+1)^3(3t^2) + (t^3+1)^4 \cdot 3(t^4-1)^2(4t^3)$
 $= 12t^2(t^4-1)^2(t^3+1)^3 [(t^4-1) + t(t^3+1)] = 12t^2(t^4-1)^2(t^3+1)^3 (2t^4+t-1)$
19. $y = (2x-5)^4(8x^2-5)^{-3} \Rightarrow$
 $y' = 4(2x-5)^3(2)(8x^2-5)^{-3} + (2x-5)^4(-3)(8x^2-5)^{-4}(16x)$
 $= 8(2x-5)^3(8x^2-5)^{-3} - 48x(2x-5)^4(8x^2-5)^{-4}$
 [This simplifies to $8(2x-5)^3(8x^2-5)^{-4}(-4x^2+30x-5)$.]
20. $y = (x^2+1)(x^2+2)^{1/3} \Rightarrow$
 $y' = 2x(x^2+2)^{1/3} + (x^2+1)\left(\frac{1}{3}\right)(x^2+2)^{-2/3}(2x) = 2x(x^2+2)^{1/3} \left[1 + \frac{x^2+1}{3(x^2+2)}\right]$
21. $y = x^3 \cos nx \Rightarrow y' = x^3(-\sin nx)(n) + \cos nx (3x^2) = x^2(3 \cos nx - nx \sin nx)$
22. $y = x \sin \sqrt{x} \Rightarrow y' = x \cos \sqrt{x} \cdot \frac{1}{2}x^{-1/2} + \sin \sqrt{x} \cdot 1 = \frac{1}{2}\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x}$
23. $y = \sin(x \cos x) \Rightarrow y' = \cos(x \cos x) \cdot [x(-\sin x) + \cos x \cdot 1] = (\cos x - x \sin x) \cos(x \cos x)$

$$24. f(x) = \frac{x}{\sqrt{7-3x}} \Rightarrow$$

$$f'(x) = \frac{\sqrt{7-3x}(1) - x \cdot \frac{1}{2}(7-3x)^{-1/2} \cdot (-3)}{(\sqrt{7-3x})^2} = \frac{\sqrt{7-3x} + \frac{3x}{2\sqrt{7-3x}}}{(7-3x)^1}$$

$$= \frac{2(7-3x) + 3x}{2(7-3x)^{3/2}} = \frac{14-3x}{2(7-3x)^{3/2}}$$

$$25. F(z) = \sqrt{\frac{z-1}{z+1}} = \left(\frac{z-1}{z+1}\right)^{1/2} \Rightarrow$$

$$F'(z) = \frac{1}{2} \left(\frac{z-1}{z+1}\right)^{-1/2} \cdot \frac{d}{dz} \left(\frac{z-1}{z+1}\right) = \frac{1}{2} \left(\frac{z+1}{z-1}\right)^{1/2} \cdot \frac{(z+1)(1) - (z-1)(1)}{(z+1)^2}$$

$$= \frac{1}{2} \frac{(z+1)^{1/2}}{(z-1)^{1/2}} \cdot \frac{z+1-z+1}{(z+1)^2} = \frac{1}{2} \frac{(z+1)^{1/2}}{(z-1)^{1/2}} \cdot \frac{2}{(z+1)^2} = \frac{1}{(z-1)^{1/2}(z+1)^{3/2}}$$

$$26. G(y) = \frac{(y-1)^4}{(y^2+2y)^5} \Rightarrow$$

$$G'(y) = \frac{(y^2+2y)^5 \cdot 4(y-1)^3 \cdot 1 - (y-1)^4 \cdot 5(y^2+2y)^4(2y+2)}{[(y^2+2y)^5]^2}$$

$$= \frac{2(y^2+2y)^4(y-1)^3[2(y^2+2y) - 5(y-1)(y+1)]}{(y^2+2y)^{10}}$$

$$= \frac{2(y-1)^3[(2y^2+4y) + (-5y^2+5)]}{(y^2+2y)^6} = \frac{2(y-1)^3(-3y^2+4y+5)}{(y^2+2y)^6}$$

$$27. y = \frac{r}{\sqrt{r^2+1}} \Rightarrow$$

$$y' = \frac{\sqrt{r^2+1}(1) - r \cdot \frac{1}{2}(r^2+1)^{-1/2}(2r)}{(\sqrt{r^2+1})^2} = \frac{\sqrt{r^2+1} - \frac{r^2}{\sqrt{r^2+1}}}{(\sqrt{r^2+1})^2} = \frac{\frac{\sqrt{r^2+1}\sqrt{r^2+1} - r^2}{\sqrt{r^2+1}}}{(\sqrt{r^2+1})^2}$$

$$= \frac{(r^2+1) - r^2}{(\sqrt{r^2+1})^3} = \frac{1}{(r^2+1)^{3/2}} \text{ or } (r^2+1)^{-3/2}$$

Another solution: Write y as a product and make use of the Product Rule. $y = r(r^2+1)^{-1/2} \Rightarrow$

$$y' = r \cdot -\frac{1}{2}(r^2+1)^{-3/2}(2r) + (r^2+1)^{-1/2} \cdot 1$$

$$= (r^2+1)^{-3/2}[-r^2 + (r^2+1)^1] = (r^2+1)^{-3/2}(1) = (r^2+1)^{-3/2}$$

The step that students usually have trouble with is factoring out $(r^2+1)^{-3/2}$. But this is no different than factoring out x^2 from $x^2 + x^5$; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

$$28. y = \frac{\cos \pi x}{\sin \pi x + \cos \pi x} \Rightarrow$$

$$y' = \frac{(\sin \pi x + \cos \pi x)(-\pi \sin \pi x) - (\cos \pi x)(\pi \cos \pi x - \pi \sin \pi x)}{(\sin \pi x + \cos \pi x)^2}$$

$$= \frac{-\pi \sin^2 \pi x - \pi \sin \pi x \cos \pi x - \pi \cos^2 \pi x + \pi \sin \pi x \cos \pi x}{(\sin \pi x + \cos \pi x)^2}$$

$$= \frac{-\pi(\sin^2 \pi x + \cos^2 \pi x)}{(\sin \pi x + \cos \pi x)^2} = \frac{-\pi}{(\sin \pi x + \cos \pi x)^2} \text{ or } \frac{-\pi}{1 + 2 \sin \pi x \cos \pi x}$$

$$29. y = \tan(\cos x) \Rightarrow y' = \sec^2(\cos x) \cdot (-\sin x) = -\sin x \sec^2(\cos x)$$

$$30. y = \frac{\sin^2 x}{\cos x} \Rightarrow$$

$$y' = \frac{\cos x (2 \sin x \cos x) - \sin^2 x (-\sin x)}{\cos^2 x} = \frac{\sin x (2 \cos^2 x + \sin^2 x)}{\cos^2 x} = \frac{\sin x (1 + \cos^2 x)}{\cos^2 x}$$

$$= \sin x (1 + \sec^2 x)$$

$$\text{Another method: } y = \tan x \sin x \Rightarrow y' = \sec^2 x \sin x + \tan x \cos x = \sec^2 x \sin x + \sin x$$

$$31. y = \sin \sqrt{1+x^2} \Rightarrow y' = \cos \sqrt{1+x^2} \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = (x \cos \sqrt{1+x^2}) / \sqrt{1+x^2}$$

$$32. y = \tan^2(3\theta) = (\tan 3\theta)^2 \Rightarrow y' = 2(\tan 3\theta) \cdot \frac{d}{d\theta}(\tan 3\theta) = 2 \tan 3\theta \cdot \sec^2 3\theta \cdot 3 = 6 \tan 3\theta \sec^2 3\theta$$

$$33. y = (1 + \cos^2 x)^6 \Rightarrow y' = 6(1 + \cos^2 x)^5 \cdot 2 \cos x (-\sin x) = -12 \cos x \sin x (1 + \cos^2 x)^5$$

$$34. y = x \sin \frac{1}{x} \Rightarrow y' = \sin \frac{1}{x} + x \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

$$35. y = \sec^2 x + \tan^2 x = (\sec x)^2 + (\tan x)^2 \Rightarrow$$

$$y' = 2(\sec x)(\sec x \tan x) + 2(\tan x)(\sec^2 x) = 2 \sec^2 x \tan x + 2 \sec^2 x \tan x = 4 \sec^2 x \tan x$$

$$36. y = \cot(x^2) + \cot^2 x = \cot(x^2) + (\cot x)^2 \Rightarrow$$

$$y' = -\csc^2(x^2) \cdot 2x + 2(\cot x)^1 (-\csc^2 x) = -2x \csc^2(x^2) - 2 \cot x \csc^2 x$$

$$37. y = \cot^2(\sin \theta) = [\cot(\sin \theta)]^2 \Rightarrow$$

$$y' = 2[\cot(\sin \theta)] \cdot \frac{d}{d\theta}[\cot(\sin \theta)] = 2 \cot(\sin \theta) \cdot [-\csc^2(\sin \theta) \cdot \cos \theta] = -2 \cos \theta \cot(\sin \theta) \csc^2(\sin \theta)$$

$$38. y = \sin(\sin(\sin x)) \Rightarrow y' = \cos(\sin(\sin x)) \frac{d}{dx}(\sin(\sin x)) = \cos(\sin(\sin x)) \cos(\sin x) \cos x$$

$$39. y = \sqrt{x + \sqrt{x}} \Rightarrow y' = \frac{1}{2}(x + \sqrt{x})^{-1/2} \left(1 + \frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x + \sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right)$$

$$40. y = \sqrt{x + \sqrt{x + \sqrt{x}}} \Rightarrow y' = \frac{1}{2}(x + \sqrt{x + \sqrt{x}})^{-1/2} \left[1 + \frac{1}{2}(x + \sqrt{x})^{-1/2} \left(1 + \frac{1}{2}x^{-1/2}\right)\right]$$

$$41. y = \sin(\tan \sqrt{\sin x}) \Rightarrow$$

$$y' = \cos(\tan \sqrt{\sin x}) \cdot \frac{d}{dx}(\tan \sqrt{\sin x}) = \cos(\tan \sqrt{\sin x}) \sec^2 \sqrt{\sin x} \cdot \frac{d}{dx}(\sin x)^{1/2}$$

$$= \cos(\tan \sqrt{\sin x}) \sec^2 \sqrt{\sin x} \cdot \frac{1}{2}(\sin x)^{-1/2} \cdot \cos x$$

$$= \cos(\tan \sqrt{\sin x}) \left(\sec^2 \sqrt{\sin x}\right) \left(\frac{1}{2\sqrt{\sin x}}\right) (\cos x)$$

$$42. y = \sqrt{\cos(\sin^2 x)} \Rightarrow y' = \frac{1}{2}(\cos(\sin^2 x))^{-1/2} [-\sin(\sin^2 x)](2 \sin x \cos x) = -\frac{\sin(\sin^2 x) \sin x \cos x}{\sqrt{\cos(\sin^2 x)}}$$

$$43. y = (1 + 2x)^{10} \Rightarrow y' = 10(1 + 2x)^9 \cdot 2 = 20(1 + 2x)^9. \text{ At } (0, 1), y' = 20(1 + 0)^9 = 20, \text{ and an equation of the tangent line is } y - 1 = 20(x - 0), \text{ or } y = 20x + 1.$$

$$44. y = \sin x + \sin^2 x \Rightarrow y' = \cos x + 2 \sin x \cos x. \text{ At } (0, 0), y' = 1, \text{ and an equation of the tangent line is } y - 0 = 1(x - 0), \text{ or } y = x.$$

$$45. y = \sin(\sin x) \Rightarrow y' = \cos(\sin x) \cdot \cos x. \text{ At } (\pi, 0), y' = \cos(\sin \pi) \cdot \cos \pi = \cos(0) \cdot (-1) = 1(-1) = -1, \text{ and an equation of the tangent line is } y - 0 = -1(x - \pi), \text{ or } y = -x + \pi.$$

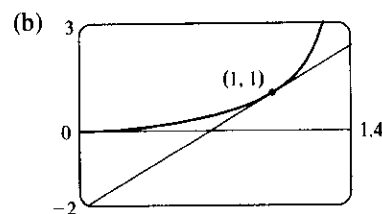
46. $y = \sqrt{5+x^2} \Rightarrow y' = \frac{1}{2}(5+x^2)^{-1/2}(2x) = x/\sqrt{5+x^2}$. At $(2, 3)$, $y' = \frac{2}{3}$, and an equation of the tangent line is $y - 3 = \frac{2}{3}(x - 2)$, or $y = \frac{2}{3}x + \frac{5}{3}$.

47. (a) $y = f(x) = \tan\left(\frac{\pi}{4}x\right) \Rightarrow f'(x) = \sec^2\left(\frac{\pi}{4}x\right)\left(2 \cdot \frac{\pi}{4}x\right)$.

The slope of the tangent at $(1, 1)$ is thus

$$f'(1) = \sec^2\left(\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right) = 2 \cdot \frac{\pi}{2} = \pi, \text{ and its equation is}$$

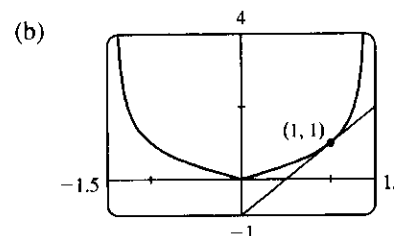
$$y - 1 = \pi(x - 1) \text{ or } y = \pi x - \pi + 1.$$



48. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \frac{x}{\sqrt{2-x^2}} \Rightarrow$

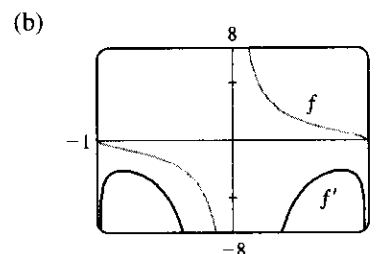
$$\begin{aligned} f'(x) &= \frac{\sqrt{2-x^2}(1) - x(\frac{1}{2})(2-x^2)^{-1/2}(-2x)}{(\sqrt{2-x^2})^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}} \\ &= \frac{(2-x^2) + x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}} \end{aligned}$$

So at $(1, 1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x - 1)$ or $y = 2x - 1$.



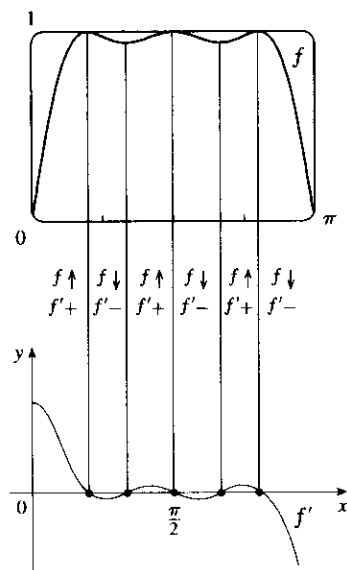
49. (a) $f(x) = \frac{\sqrt{1-x^2}}{x} \Rightarrow$

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) - \sqrt{1-x^2}(1)}{x^2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} \\ &= \frac{-x^2 - (1-x^2)}{x^2 \sqrt{1-x^2}} = \frac{-1}{x^2 \sqrt{1-x^2}} \end{aligned}$$



Notice that all tangents to the graph of f have negative slopes and $f'(x) < 0$ always.

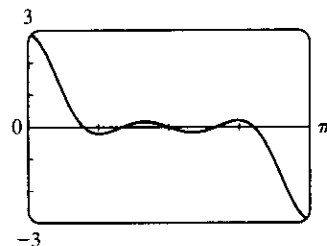
50. (a)



From the graph of f , we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f' . From the symmetry of the graph of f , we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

$$(b) f(x) = \sin(x + \sin 2x) \Rightarrow$$

$$\begin{aligned} f'(x) &= \cos(x + \sin 2x) \cdot \frac{d}{dx}(x + \sin 2x) \\ &= \cos(x + \sin 2x)(1 + 2 \cos 2x) \end{aligned}$$



51. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2 \sin x + \sin^2 x \Rightarrow$
 $f'(x) = 2 \cos x + 2 \sin x \cos x = 0 \Leftrightarrow 2 \cos x (1 + \sin x) = 0 \Leftrightarrow \cos x = 0$ or $\sin x = -1$, so
 $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f(\frac{\pi}{2}) = 3$ and $f(\frac{3\pi}{2}) = -1$, so the points on the curve
 with a horizontal tangent are $(\frac{\pi}{2} + 2n\pi, 3)$ and $(\frac{3\pi}{2} + 2n\pi, -1)$, where n is any integer.
52. $f(x) = \sin 2x - 2 \sin x \Rightarrow f'(x) = 2 \cos 2x - 2 \cos x = 4 \cos^2 x - 2 \cos x - 2$, and
 $4 \cos^2 x - 2 \cos x - 2 = 0 \Leftrightarrow (\cos x - 1)(4 \cos x + 2) = 0 \Leftrightarrow \cos x = 1$ or $\cos x = -\frac{1}{2}$. So $x = 2n\pi$ or
 $(2n + 1)\pi \pm \frac{\pi}{3}$, n any integer.
53. $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) \cdot g'(x)$,
 so $F'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot g'(3) = 7 \cdot 4 = 28$. Notice that we did not use $f'(3) = 2$.
54. $w = u \circ v \Rightarrow w(x) = u(v(x)) \Rightarrow w'(x) = u'(v(x)) \cdot v'(x)$, so
 $w'(0) = u'(v(0)) \cdot v'(0) = u'(2) \cdot v'(0) = 4 \cdot 5 = 20$. The other pieces of information, $u(0) = 1$, $u'(0) = 3$, and
 $v'(2) = 6$, were not needed.
55. (a) $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.
 (b) $H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.
56. (a) $F(x) = f(f(x)) \Rightarrow F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.
 (b) $G(x) = g(g(x)) \Rightarrow G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.
57. (a) $u(x) = f(g(x)) \Rightarrow u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note
 that f is linear from $(2, 4)$ to $(6, 3)$, so its slope is $\frac{3-4}{6-2} = -\frac{1}{4}$. To find $g'(1)$, note that g is linear from $(0, 6)$
 to $(2, 0)$, so its slope is $\frac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = (-\frac{1}{4})(-3) = \frac{3}{4}$.
 (b) $v(x) = g(f(x)) \Rightarrow v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist
 since $g'(2)$ does not exist.
 (c) $w(x) = g(g(x)) \Rightarrow w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note
 that g is linear from $(2, 0)$ to $(5, 2)$, so its slope is $\frac{2-0}{5-2} = \frac{2}{3}$. Thus, $g'(3)g'(1) = (\frac{2}{3})(-3) = -2$.
58. (a) $h(x) = f(f(x)) \Rightarrow h'(x) = f'(f(x))f'(x)$.
 So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.
 (b) $g(x) = f(x^2) \Rightarrow g'(x) = f'(x^2) \cdot \frac{d}{dx}(x^2) = f'(x^2)(2x)$.
 So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(1.5) = 6$.

59. $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x)$. So $h'(0.5) = f'(g(0.5))g'(0.5) = f'(0.1)g'(0.5)$. We can estimate the derivatives by taking the average of two secant slopes.

$$\text{For } f'(0.1): m_1 = \frac{14.8 - 12.6}{0.1 - 0} = 22, m_2 = \frac{18.4 - 14.8}{0.2 - 0.1} = 36. \text{ So } f'(0.1) \approx \frac{m_1 + m_2}{2} = \frac{22 + 36}{2} = 29.$$

$$\text{For } g'(0.5): m_1 = \frac{0.10 - 0.17}{0.5 - 0.4} = -0.7, m_2 = \frac{0.05 - 0.10}{0.6 - 0.5} = -0.5. \text{ So } g'(0.5) \approx \frac{m_1 + m_2}{2} = -0.6.$$

$$\text{Hence, } h'(0.5) = f'(0.1)g'(0.5) \approx (29)(-0.6) = -17.4.$$

60. $g(x) = f(f(x)) \Rightarrow g'(x) = f'(f(x))f'(x)$. So $g'(1) = f'(f(1))f'(1) = f'(2)f'(1)$.

$$\text{For } f'(2): m_1 = \frac{3.1 - 2.4}{2.0 - 1.5} = 1.4, m_2 = \frac{4.4 - 3.1}{2.5 - 2.0} = 2.6. \text{ So } f'(2) \approx \frac{m_1 + m_2}{2} = 2.$$

$$\text{For } f'(1): m_1 = \frac{2.0 - 1.8}{1.0 - 0.5} = 0.4, m_2 = \frac{2.4 - 2.0}{1.5 - 1.0} = 0.8. \text{ So } f'(1) \approx \frac{m_1 + m_2}{2} = 0.6.$$

$$\text{Hence, } g'(1) = f'(2)f'(1) \approx (2)(0.6) = 1.2.$$

61. (a) $F(x) = f(\cos x) \Rightarrow F'(x) = f'(\cos x) \frac{d}{dx}(\cos x) = -\sin x f'(\cos x)$

$$(b) G(x) = \cos(f(x)) \Rightarrow G'(x) = -\sin(f(x)) f'(x)$$

62. (a) $F(x) = f(x^\alpha) \Rightarrow F'(x) = f'(x^\alpha) \frac{d}{dx}(x^\alpha) = f'(x^\alpha) \alpha x^{\alpha-1}$

$$(b) G(x) = [f(x)]^\alpha \Rightarrow G'(x) = \alpha [f(x)]^{\alpha-1} f'(x)$$

63. (a) $f(x) = L(x^4) \Rightarrow f'(x) = L'(x^4) \cdot 4x^3 = (1/x^4) \cdot 4x^3 = 4/x$ for $x > 0$.

$$(b) g(x) = L(4x) \Rightarrow g'(x) = L'(4x) \cdot 4 = (1/(4x)) \cdot 4 = 1/x$$
 for $x > 0$.

$$(c) F(x) = [L(x)]^4 \Rightarrow F'(x) = 4[L(x)]^3 \cdot L'(x) = 4[L(x)]^3 \cdot (1/x) = 4[L(x)]^3/x$$

$$(d) G(x) = L(1/x) \Rightarrow G'(x) = L'(1/x) \cdot (-1/x^2) = (1/(1/x)) \cdot (-1/x^2) = x \cdot (-1/x^2) = -1/x$$
 for $x > 0$.

64. $r(x) = f(g(h(x))) \Rightarrow r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so

$$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$$

65. $s(t) = 10 + \frac{1}{4} \sin(10\pi t) \Rightarrow$ the velocity after t seconds is

$$v(t) = s'(t) = \frac{1}{4} \cos(10\pi t)(10\pi) = \frac{5\pi}{2} \cos(10\pi t) \text{ cm/s.}$$

66. (a) $s = A \cos(\omega t + \delta) \Rightarrow$ velocity $= s' = -\omega A \sin(\omega t + \delta)$.

$$(b) \text{ If } A \neq 0 \text{ and } \omega \neq 0, \text{ then } s' = 0 \Leftrightarrow \sin(\omega t + \delta) = 0 \Leftrightarrow \omega t + \delta = n\pi \Leftrightarrow t = \frac{n\pi - \delta}{\omega},$$

n an integer.

67. (a) $B(t) = 4.0 + 0.35 \sin \frac{2\pi t}{5.4} \Rightarrow \frac{dB}{dt} = \left(0.35 \cos \frac{2\pi t}{5.4}\right) \left(\frac{2\pi}{5.4}\right) = \frac{0.7\pi}{5.4} \cos \frac{2\pi t}{5.4} = \frac{7\pi}{54} \cos \frac{2\pi t}{5.4}$

$$(b) \text{ At } t = 1, \frac{dB}{dt} = \frac{7\pi}{54} \cos \frac{2\pi}{5.4} \approx 0.16.$$

68. $L(t) = 12 + 2.8 \sin\left(\frac{2\pi}{365}(t - 80)\right) \Rightarrow L'(t) = 2.8 \cos\left(\frac{2\pi}{365}(t - 80)\right) \left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

69. (a) Derive gives $g'(t) = \frac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get

$$g'(t) = 9 \frac{(t-2)^8}{(2t+1)^9} - 18 \frac{(t-2)^9}{(2t+1)^{10}}, \text{ and the simplification command results in the above expression.}$$

- (b) Derive gives $y' = 2(x^3 - x + 1)^3(2x + 1)^4(17x^3 + 6x^2 - 9x + 3)$ without simplifying.

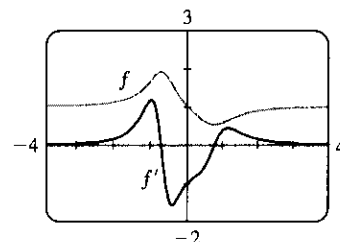
With either Maple or Mathematica, we first get

$y' = 10(2x + 1)^4(x^3 - x + 1)^4 + 4(2x + 1)^5(x^3 - x + 1)^3(3x^2 - 1)$. If we use Mathematica's Factor or Simplify, or Maple's factor, we get the above expression, but Maple's simplify gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

70. (a) $f(x) = \left(\frac{x^4 - x + 1}{x^4 + x + 1}\right)^{1/2}$. Derive gives $f'(x) = \frac{(3x^4 - 1)\sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}}{(x^4 + x + 1)(x^4 - x + 1)}$ whereas either Maple or Mathematica give $f'(x) = \frac{3x^4 - 1}{\sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}(x^4 + x + 1)^2}$ after simplification.

(b) $f'(x) = 0 \Leftrightarrow 3x^4 - 1 = 0 \Leftrightarrow x = \pm\sqrt[4]{\frac{1}{3}} \approx \pm 0.7598$.

- (c) $f'(x) = 0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.



71. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get

$$f'(x) = f'(-x) \frac{d}{dx}(-x) = -f'(-x). \text{ Thus, } f'(-x) = -f'(x), \text{ so } f' \text{ is odd.}$$

- (b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

72. $\left[\frac{f(x)}{g(x)}\right]' = \{f(x)[g(x)]^{-1}\}' = f'(x)[g(x)]^{-1} + (-1)[g(x)]^{-2}g'(x)f(x)$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

73. (a) $\frac{d}{dx}(\sin^n x \cos nx) = n \sin^{n-1} x \cos x \cos nx + \sin^n x (-n \sin nx)$ [Product Rule]

$$= n \sin^{n-1} x (\cos nx \cos x - \sin nx \sin x)$$
 [factor out $n \sin^{n-1} x$]

$$= n \sin^{n-1} x \cos(nx + x)$$
 [Addition Formula for cosine]

$$= n \sin^{n-1} x \cos[(n+1)x]$$
 [factor out x]

(b) $\frac{d}{dx}(\cos^n x \cos nx) = n \cos^{n-1} x (-\sin x) \cos nx + \cos^n x (-n \sin nx)$ [Product Rule]

$$= -n \cos^{n-1} x (\cos nx \sin x + \sin nx \cos x)$$
 [factor out $-n \cos^{n-1} x$]

$$= -n \cos^{n-1} x \sin(nx + x)$$
 [Addition Formula for sine]

$$= -n \cos^{n-1} x \sin[(n+1)x]$$
 [factor out x]

74. "The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x " $\Leftrightarrow$

$$\frac{d}{dx} y^5 = 80 \frac{dy}{dx} \Leftrightarrow 5y^4 \frac{dy}{dx} = 80 \frac{dy}{dx} \Leftrightarrow 5y^4 = 80 \quad (\text{Note that } dy/dx \neq 0 \text{ since the curve never has a horizontal tangent})$$

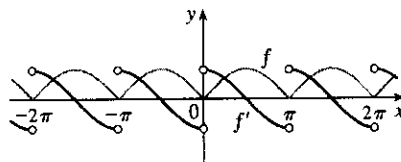
$$\Leftrightarrow y^4 = 16 \Leftrightarrow y = 2 \quad (\text{since } y > 0 \text{ for all } x)$$

75. Since $\theta^\circ = (\frac{\pi}{180})\theta$ rad, we have $\frac{d}{d\theta}(\sin \theta^\circ) = \frac{d}{d\theta}(\sin \frac{\pi}{180}\theta) = \frac{\pi}{180} \cos \frac{\pi}{180}\theta = \frac{\pi}{180} \cos \theta^\circ$.

76. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.
 f is not differentiable at $x = 0$.

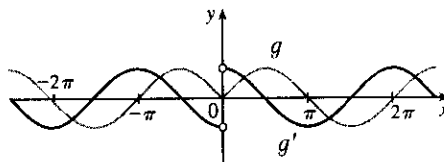
(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \Rightarrow$

$$f'(x) = \frac{1}{2}(\sin^2 x)^{-1/2} 2 \sin x \cos x = \frac{\sin x}{|\sin x|} \cos x = \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin |x| = \sin \sqrt{x^2} \Rightarrow g'(x) = \cos |x| \cdot \frac{x}{|x|} = \frac{x}{|x|} \cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$



g is not differentiable at 0.

77. First note that products and differences of polynomials are polynomials and that the derivative of a polynomial is

also a polynomial. When $n = 1$, $f^{(1)}(x) = \left(\frac{P(x)}{Q(x)}\right)' = \frac{Q(x)P'(x) - P(x)Q'(x)}{[Q(x)]^2} = \frac{A_1(x)}{[Q(x)]^{1+1}}$, where

$A_1(x) = Q(x)P'(x) - P(x)Q'(x)$. Suppose the result is true for $n = k$, where $k \geq 1$. Then

$$f^{(k)}(x) = \frac{A_k(x)}{[Q(x)]^{k+1}}, \text{ so}$$

$$\begin{aligned} f^{(k+1)}(x) &= \left(\frac{A_k(x)}{[Q(x)]^{k+1}}\right)' = \frac{[Q(x)]^{k+1}A'_k(x) - A_k(x) \cdot (k+1)[Q(x)]^k \cdot Q'(x)}{\{[Q(x)]^{k+1}\}^2} \\ &= \frac{[Q(x)]^{k+1}A'_k(x) - (k+1)A_k(x)[Q(x)]^k Q'(x)}{[Q(x)]^{2k+2}} \\ &= \frac{[Q(x)]^k \{[Q(x)]^1 A'_k(x) - (k+1)A_k(x)Q'(x)\}}{[Q(x)]^k [Q(x)]^{k+2}} = \frac{Q(x)A'_k(x) - (k+1)A_k(x)Q'(x)}{[Q(x)]^{k+2}} \\ &= A_{k+1}(x)/[Q(x)]^{k+2}, \text{ where } A_{k+1}(x) = Q(x)A'_k(x) - (k+1)A_k(x)Q'(x). \end{aligned}$$

We have shown that the formula holds for $n = 1$, and that when it holds for $n = k$ it also holds for $n = k + 1$.

Thus, by mathematical induction, the formula holds for all positive integers n .

3.7 Implicit Differentiation

$$1. (a) \frac{d}{dx}(xy + 2x + 3x^2) = \frac{d}{dx}(4) \Rightarrow (x \cdot y' + y \cdot 1) + 2 + 6x = 0 \Rightarrow xy' = -y - 2 - 6x \Rightarrow y' = \frac{-y - 2 - 6x}{x} \text{ or } y' = -6 - \frac{y + 2}{x}.$$

$$(b) xy + 2x + 3x^2 = 4 \Rightarrow xy = 4 - 2x - 3x^2 \Rightarrow y = \frac{4 - 2x - 3x^2}{x} = \frac{4}{x} - 2 - 3x, \text{ so } y' = -\frac{4}{x^2} - 3.$$

$$(c) \text{ From part (a), } y' = \frac{-y - 2 - 6x}{x} = \frac{-(4/x - 2 - 3x) - 2 - 6x}{x} = \frac{-4/x - 3x}{x} = -\frac{4}{x^2} - 3.$$

$$2. (a) \frac{d}{dx}(4x^2 + 9y^2) = \frac{d}{dx}(36) \Rightarrow 8x + 18y \cdot y' = 0 \Rightarrow y' = -\frac{8x}{18y} = -\frac{4x}{9y}$$

$$(b) 4x^2 + 9y^2 = 36 \Rightarrow 9y^2 = 36 - 4x^2 \Rightarrow y^2 = \frac{4}{9}(9 - x^2) \Rightarrow y = \pm \frac{2}{3}\sqrt{9 - x^2}, \text{ so } y' = \pm \frac{2}{3} \cdot \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = \mp \frac{2x}{3\sqrt{9 - x^2}}$$

$$(c) \text{ From part (a), } y' = -\frac{4x}{9y} = -\frac{4x}{9(\pm \frac{2}{3}\sqrt{9 - x^2})} = \mp \frac{2x}{3\sqrt{9 - x^2}}.$$

$$3. (a) \frac{d}{dx}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{d}{dx}(1) \Rightarrow -\frac{1}{x^2} - \frac{1}{y^2}y' = 0 \Rightarrow -\frac{1}{y^2}y' = \frac{1}{x^2} \Rightarrow y' = -\frac{y^2}{x^2}$$

$$(b) \frac{1}{x} + \frac{1}{y} = 1 \Rightarrow \frac{1}{y} = 1 - \frac{1}{x} = \frac{x-1}{x} \Rightarrow y = \frac{x}{x-1}, \text{ so } y' = \frac{(x-1)(1) - (x)(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}.$$

$$(c) y' = -\frac{y^2}{x^2} = -\frac{[x/(x-1)]^2}{x^2} = -\frac{x^2}{x^2(x-1)^2} = -\frac{1}{(x-1)^2}$$

$$4. (a) \frac{d}{dx}(\sqrt{x} + \sqrt{y}) = \frac{d}{dx}(4) \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$(b) \sqrt{y} = 4 - \sqrt{x} \Rightarrow y = (4 - \sqrt{x})^2 = 16 - 8\sqrt{x} + x \Rightarrow y' = -\frac{4}{\sqrt{x}} + 1$$

$$(c) y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{4 - \sqrt{x}}{\sqrt{x}} = -\frac{4}{\sqrt{x}} + 1$$

$$5. \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1) \Rightarrow 2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow y' = -\frac{x}{y}$$

$$6. \frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(1) \Rightarrow 2x - 2yy' = 0 \Rightarrow 2x = 2yy' \Rightarrow y' = \frac{x}{y}$$

$$7. \frac{d}{dx}(x^3 + x^2y + 4y^2) = \frac{d}{dx}(6) \Rightarrow 3x^2 + (x^2y' + y \cdot 2x) + 8yy' = 0 \Rightarrow x^2y' + 8yy' = -3x^2 - 2xy \Rightarrow (x^2 + 8y)y' = -3x^2 - 2xy \Rightarrow y' = -\frac{3x^2 + 2xy}{x^2 + 8y} = -\frac{x(3x + 2y)}{x^2 + 8y}$$

$$8. \frac{d}{dx}(x^2 - 2xy + y^3) = \frac{d}{dx}(c) \Rightarrow 2x - 2(xy' + y \cdot 1) + 3y^2y' = 0 \Rightarrow 2x - 2y = 2xy' - 3y^2y' \Rightarrow 2x - 2y = y'(2x - 3y^2) \Rightarrow y' = \frac{2x - 2y}{2x - 3y^2}$$

$$9. \frac{d}{dx}(x^2y + xy^2) = \frac{d}{dx}(3x) \Rightarrow (x^2y' + y \cdot 2x) + (x \cdot 2yy' + y^2 \cdot 1) = 3 \Rightarrow$$

$$x^2y' + 2xyy' = 3 - 2xy - y^2 \Rightarrow y'(x^2 + 2xy) = 3 - 2xy - y^2 \Rightarrow y' = \frac{3 - 2xy - y^2}{x^2 + 2xy}$$

$$10. \frac{d}{dx}(y^5 + x^2y^3) = \frac{d}{dx}(1 + x^4y) \Rightarrow 5y^4y' + x^2 \cdot 3y^2y' + y^3 \cdot 2x = 0 + x^4y' + y \cdot 4x^3 \Rightarrow$$

$$y'(5y^4 + 3x^2y^2 - x^4) = 4x^3y - 2xy^3 \Rightarrow y' = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}$$

$$11. \frac{d}{dx}(x^2y^2 + x \sin y) = \frac{d}{dx}(4) \Rightarrow x^2 \cdot 2yy' + y^2 \cdot 2x + x \cos y \cdot y' + \sin y \cdot 1 = 0 \Rightarrow$$

$$2x^2yy' + x \cos y \cdot y' = -2xy^2 - \sin y \Rightarrow (2x^2y + x \cos y)y' = -2xy^2 - \sin y \Rightarrow y' = \frac{-2xy^2 - \sin y}{2x^2y + x \cos y}$$

$$12. \frac{d}{dx}(1 + x) = \frac{d}{dx}[\sin(xy^2)] \Rightarrow 1 = [\cos(xy^2)](x \cdot 2yy' + y^2 \cdot 1) \Rightarrow 1 = 2xy \cos(xy^2)y' + y^2 \cos(xy^2)$$

$$\Rightarrow 1 - y^2 \cos(xy^2) = 2xy \cos(xy^2)y' \Rightarrow y' = \frac{1 - y^2 \cos(xy^2)}{2xy \cos(xy^2)}$$

$$13. \frac{d}{dx}(4 \cos x \sin y) = \frac{d}{dx}(1) \Rightarrow 4[\cos x \cdot \cos y \cdot y' + \sin y \cdot (-\sin x)] = 0 \Rightarrow$$

$$y'(4 \cos x \cos y) = 4 \sin x \sin y \Rightarrow y' = \frac{4 \sin x \sin y}{4 \cos x \cos y} = \tan x \tan y$$

$$14. \frac{d}{dx}[y \sin(x^2)] = \frac{d}{dx}[x \sin(y^2)] \Rightarrow y \cos(x^2) \cdot 2x + \sin(x^2) \cdot y' = x \cos(y^2) \cdot 2yy' + \sin(y^2) \cdot 1 \Rightarrow$$

$$y'[\sin(x^2) - 2xy \cos(y^2)] = \sin(y^2) - 2xy \cos(x^2) \Rightarrow y' = \frac{\sin(y^2) - 2xy \cos(x^2)}{\sin(x^2) - 2xy \cos(y^2)}$$

$$15. \frac{d}{dx}[\tan(x/y)] = \frac{d}{dx}(x + y) \Rightarrow \sec^2(x/y) \cdot \frac{y \cdot 1 - x \cdot y'}{y^2} = 1 + y' \Rightarrow$$

$$y \sec^2(x/y) - x \sec^2(x/y) \cdot y' = y^2 + y^2 y' \Rightarrow y \sec^2(x/y) - y^2 = y^2 y' + x \sec^2(x/y) \Rightarrow$$

$$y \sec^2(x/y) - y^2 = [y^2 + x \sec^2(x/y)] \cdot y' \Rightarrow y' = \frac{y \sec^2(x/y) - y^2}{y^2 + x \sec^2(x/y)}$$

$$16. \frac{d}{dx}(\sqrt{x+y}) = \frac{d}{dx}(1 + x^2y^2) \Rightarrow \frac{1}{2}(x+y)^{-1/2}(1+y') = x^2 \cdot 2yy' + y^2 \cdot 2x \Rightarrow$$

$$\frac{1}{2\sqrt{x+y}} + \frac{y'}{2\sqrt{x+y}} = 2x^2yy' + 2xy^2 \Rightarrow 1 + y' = 4x^2y\sqrt{x+y}y' + 4xy^2\sqrt{x+y} \Rightarrow$$

$$y' - 4x^2y\sqrt{x+y}y' = 4xy^2\sqrt{x+y} - 1 \Rightarrow y'(1 - 4x^2y\sqrt{x+y}) = 4xy^2\sqrt{x+y} - 1 \Rightarrow$$

$$y' = \frac{4xy^2\sqrt{x+y} - 1}{1 - 4x^2y\sqrt{x+y}}$$

$$17. \sqrt{xy} = 1 + x^2y \Rightarrow \frac{1}{2}(xy)^{-1/2}(xy' + y \cdot 1) = 0 + x^2y' + y \cdot 2x \Rightarrow \frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} = x^2y' + 2xy$$

$$\Rightarrow y'\left(\frac{x}{2\sqrt{xy}} - x^2\right) = 2xy - \frac{y}{2\sqrt{xy}} \Rightarrow y'\left(\frac{x - 2x^2\sqrt{xy}}{2\sqrt{xy}}\right) = \frac{4xy\sqrt{xy} - y}{2\sqrt{xy}} \Rightarrow y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$$

18. $\tan(x - y) = \frac{y}{1 + x^2} \Rightarrow (1 + x^2) \tan(x - y) = y \Rightarrow$
 $(1 + x^2) \sec^2(x - y) \cdot (1 - y') + \tan(x - y) \cdot 2x = y' \Rightarrow$
 $(1 + x^2) \sec^2(x - y) - (1 + x^2) \sec^2(x - y) \cdot y' + 2x \tan(x - y) = y' \Rightarrow$
 $(1 + x^2) \sec^2(x - y) + 2x \tan(x - y) = [1 + (1 + x^2) \sec^2(x - y)] \cdot y' \Rightarrow$
 $y' = \frac{(1 + x^2) \sec^2(x - y) + 2x \tan(x - y)}{1 + (1 + x^2) \sec^2(x - y)}$
19. $xy = \cot(xy) \Rightarrow y + xy' = -\csc^2(xy)(y + xy') \Rightarrow (y + xy')[1 + \csc^2(xy)] = 0 \Rightarrow$
 $y + xy' = 0$ [since $1 + \csc^2(xy) > 0$] $\Rightarrow y' = -y/x$
20. $\sin x + \cos y = \sin x \cos y \Rightarrow \cos x - \sin y \cdot y' = \sin x (-\sin y \cdot y') + \cos y \cos x \Rightarrow$
 $(\sin x \sin y - \sin y) y' = \cos x \cos y - \cos x \Rightarrow y' = \frac{\cos x (\cos y - 1)}{\sin y (\sin x - 1)}$
21. $\frac{d}{dx} \{1 + f(x) + x^2[f(x)]^3\} = \frac{d}{dx}(0) \Rightarrow f'(x) + x^2 \cdot 3[f(x)]^2 \cdot f'(x) + [f(x)]^3 \cdot 2x = 0$. If $x = 1$, we have
 $f'(1) + 1^2 \cdot 3[f(1)]^2 \cdot f'(1) + [f(1)]^3 \cdot 2(1) = 0 \Rightarrow f'(1) + 1 \cdot 3 \cdot 2^2 \cdot f'(1) + 2^3 \cdot 2 = 0 \Rightarrow$
 $f'(1) + 12f'(1) = -16 \Rightarrow 13f'(1) = -16 \Rightarrow f'(1) = -\frac{16}{13}$.
22. $\frac{d}{dx} [g(x) + x \sin g(x)] = \frac{d}{dx}(x^2) \Rightarrow g'(x) + x \cos g(x) \cdot g'(x) + \sin g(x) \cdot 1 = 2x$. If $x = 1$, we have
 $g'(1) + 1 \cos g(1) \cdot g'(1) + \sin g(1) = 2(1) \Rightarrow g'(1) + \cos 0 \cdot g'(1) + \sin 0 = 2 \Rightarrow g'(1) + g'(1) = 2 \Rightarrow$
 $2g'(1) = 2 \Rightarrow g'(1) = 1$.
23. $y^4 + x^2 y^2 + yx^4 = y + 1 \Rightarrow 4y^3 + \left(x^2 \cdot 2y + y^2 \cdot 2x \frac{dx}{dy}\right) + \left(y \cdot 4x^3 \frac{dx}{dy} + x^4 \cdot 1\right) = 1 \Rightarrow$
 $2xy^2 \frac{dx}{dy} + 4x^3 y \frac{dx}{dy} = 1 - 4y^3 - 2x^2 y - x^4 \Rightarrow \frac{dx}{dy} = \frac{1 - 4y^3 - 2x^2 y - x^4}{2xy^2 + 4x^3 y}$
24. $(x^2 + y^2)^2 = ax^2 y \Rightarrow 2(x^2 + y^2) \left(2x \frac{dx}{dy} + 2y\right) = 2axy \frac{dx}{dy} + ax^2 \Rightarrow \frac{dx}{dy} = \frac{ax^2 - 4y(x^2 + y^2)}{4x(x^2 + y^2) - 2axy}$
25. $x^2 + xy + y^2 = 3 \Rightarrow 2x + xy' + y \cdot 1 + 2yy' = 0 \Rightarrow xy' + 2yy' = -2x - y \Rightarrow$
 $y'(x + 2y) = -2x - y \Rightarrow y' = \frac{-2x - y}{x + 2y}$. When $x = 1$ and $y = 1$, we have $y' = \frac{-2 - 1}{1 + 2 \cdot 1} = \frac{-3}{3} = -1$, so
an equation of the tangent line is $y - 1 = -1(x - 1)$ or $y = -x + 2$.
26. $x^2 + 2xy - y^2 + x = 2 \Rightarrow 2x + 2(xy' + y \cdot 1) - 2yy' + 1 = 0 \Rightarrow 2xy' - 2yy' = -2x - 2y - 1 \Rightarrow$
 $y'(2x - 2y) = -2x - 2y - 1 \Rightarrow y' = \frac{-2x - 2y - 1}{2x - 2y}$. When $x = 1$ and $y = 2$, we have
 $y' = \frac{-2 - 4 - 1}{2 - 4} = \frac{-7}{-2} = \frac{7}{2}$, so an equation of the tangent line is $y - 2 = \frac{7}{2}(x - 1)$ or $y = \frac{7}{2}x - \frac{3}{2}$.

27. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \Rightarrow 2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y' - 1) \Rightarrow y' = 2y' - 1 \Rightarrow y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$ or $y = x + \frac{1}{2}$.

28. $x^{2/3} + y^{2/3} = 4 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \Rightarrow \frac{1}{\sqrt[3]{x}} + \frac{y'}{\sqrt[3]{y}} = 0 \Rightarrow y' = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\frac{1}{(-3\sqrt{3})^{1/3}} = -\frac{(-3\sqrt{3})^{2/3}}{-3\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$, so an equation of the tangent line is $y - 1 = \frac{1}{\sqrt{3}}(x + 3\sqrt{3})$ or $y = \frac{1}{\sqrt{3}}x + 4$.

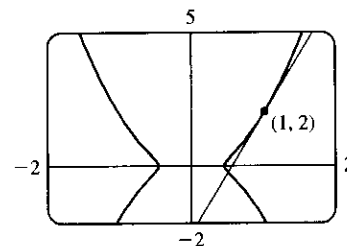
29. $2(x^2 + y^2)^2 = 25(x^2 - y^2) \Rightarrow 4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy') \Rightarrow 4(x + yy')(x^2 + y^2) = 25(x - yy') \Rightarrow 4yy'(x^2 + y^2) + 25yy' = 25x - 4x(x^2 + y^2) \Rightarrow y' = \frac{25x - 4x(x^2 + y^2)}{25y + 4y(x^2 + y^2)}$. When $x = 3$ and $y = 1$, we have $y' = \frac{75 - 120}{25 + 40} = -\frac{45}{65} = -\frac{9}{13}$, so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x - 3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

30. $y^2(y^2 - 4) = x^2(x^2 - 5) \Rightarrow y^4 - 4y^2 = x^4 - 5x^2 \Rightarrow 4y^3y' - 8yy' = 4x^3 - 10x$. When $x = 0$ and $y = -2$, we have $-32y' + 16y' = 0 \Rightarrow -16y' = 0 \Rightarrow y' = 0$, so an equation of the tangent line is $y + 2 = 0(x - 0)$ or $y = -2$.

31. (a) $y^2 = 5x^4 - x^2 \Rightarrow 2yy' = 5(4x^3) - 2x \Rightarrow y' = \frac{10x^3 - x}{y}$. (b)

So at the point $(1, 2)$ we have $y' = \frac{10(1)^3 - 1}{2} = \frac{9}{2}$, and an

equation of the tangent line is $y - 2 = \frac{9}{2}(x - 1)$ or $y = \frac{9}{2}x - \frac{5}{2}$.



32. (a) $y^2 = x^3 + 3x^2 \Rightarrow 2yy' = 3x^2 + 3(2x) \Rightarrow y' = \frac{3x^2 + 6x}{2y}$. So at the point $(1, -2)$ we have

$y' = \frac{3(1)^2 + 6(1)}{2(-2)} = -\frac{9}{4}$, and an equation of the tangent line is $y + 2 = -\frac{9}{4}(x - 1)$ or $y = -\frac{9}{4}x + \frac{1}{4}$.

(b) The curve has a horizontal tangent where $y' = 0 \Leftrightarrow$

$$3x^2 + 6x = 0 \Leftrightarrow 3x(x + 2) = 0 \Leftrightarrow x = 0 \text{ or } x = -2. \text{ But}$$

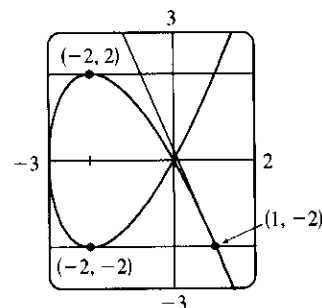
note that at $x = 0, y = 0$ also, so the derivative does not exist. At

$x = -2, y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$, so $y = \pm 2$. So the

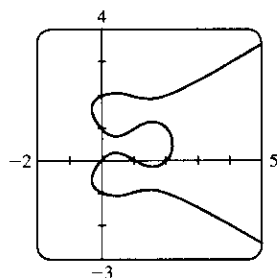
two points at which the curve has a horizontal tangent are $(-2, -2)$

and $(-2, 2)$.

(c)



33. (a)



There are eight points with horizontal tangents:
four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph.

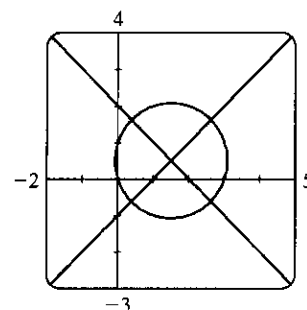
By modifying the equation in other ways, we can generate the other graphs.

$$(b) y' = \frac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \Rightarrow y' = -1 \text{ at}$$

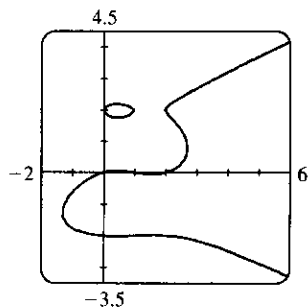
$$(0, 1) \text{ and } y' = \frac{1}{3} \text{ at } (0, 2).$$

Equations of the tangent lines are $y = -x + 1$
and $y = \frac{1}{3}x + 2$.

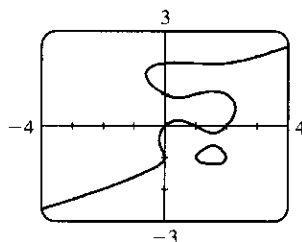
$$(c) y' = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{3}\sqrt{3}$$



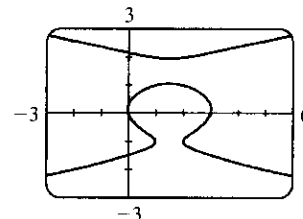
$$\begin{aligned} y(y^2 - 1)(y - 2) \\ = x(x - 1)(x - 2)(x - 3) \end{aligned}$$



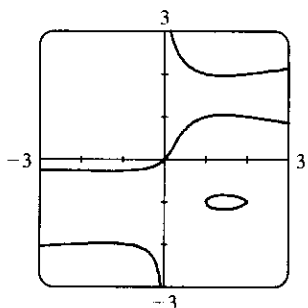
$$\begin{aligned} y(y^2 - 4)(y - 2) \\ = x(x - 1)(x - 2) \end{aligned}$$



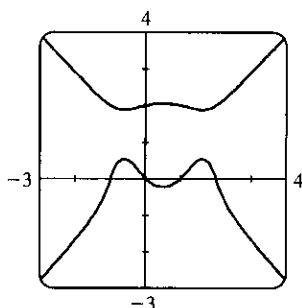
$$\begin{aligned} y(y + 1)(y^2 - 1)(y - 2) \\ = x(x - 1)(x - 2) \end{aligned}$$



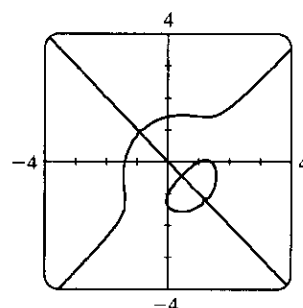
$$\begin{aligned} (y + 1)(y^2 - 1)(y - 2) \\ = (x - 1)(x - 2) \end{aligned}$$



$$\begin{aligned} x(y + 1)(y^2 - 1)(y - 2) \\ = y(x - 1)(x - 2) \end{aligned}$$

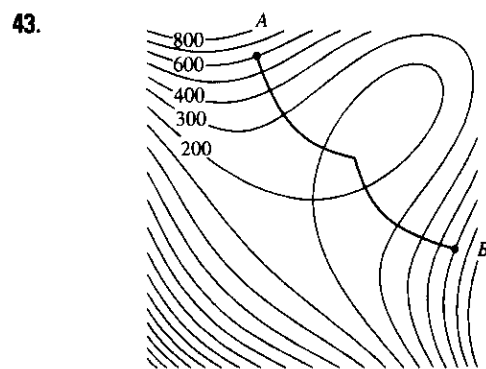


$$\begin{aligned} y(y^2 + 1)(y - 2) \\ = x(x^2 - 1)(x - 2) \end{aligned}$$

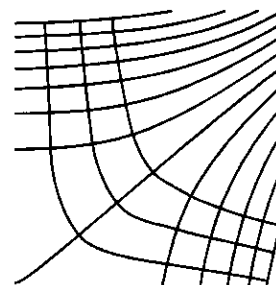


$$\begin{aligned} y(y + 1)(y^2 - 2) \\ = x(x - 1)(x^2 - 2) \end{aligned}$$

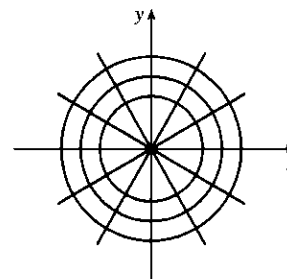
41. $2x^2 + y^2 = 3$ and $x = y^2$ intersect when $2x^2 + x - 3 = 0 \Leftrightarrow (2x + 3)(x - 1) = 0 \Leftrightarrow x = -\frac{3}{2}$ or 1, but $-\frac{3}{2}$ is extraneous since $x = y^2$ is nonnegative. When $x = 1$, $1 = y^2 \Rightarrow y = \pm 1$, so there are two points of intersection: $(1, \pm 1)$. $2x^2 + y^2 = 3 \Rightarrow 4x + 2yy' = 0 \Rightarrow y' = -2x/y$, and $x = y^2 \Rightarrow 1 = 2yy' \Rightarrow y' = 1/(2y)$. At $(1, 1)$, the slopes are $m_1 = -2(1)/1 = -2$ and $m_2 = 1/(2 \cdot 1) = \frac{1}{2}$, so the curves are orthogonal (since m_1 and m_2 are negative reciprocals of each other). By symmetry, the curves are also orthogonal at $(1, -1)$.
42. $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ intersect when $4x^2 + 9(x^2 - 5) = 72 \Leftrightarrow 13x^2 = 117 \Leftrightarrow x = \pm 3$, so there are four points of intersection: $(\pm 3, \pm 2)$. $x^2 - y^2 = 5 \Rightarrow 2x - 2yy' = 0 \Rightarrow y' = x/y$, and $4x^2 + 9y^2 = 72 \Rightarrow 8x + 18yy' = 0 \Leftrightarrow y' = -4x/9y$. At $(3, 2)$, the slopes are $m_1 = \frac{3}{2}$ and $m_2 = -\frac{2}{3}$, so the curves are orthogonal. By symmetry, the curves are also orthogonal at $(3, -2)$, $(-3, 2)$ and $(-3, -2)$.



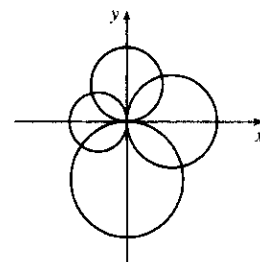
44. The orthogonal family represents the direction of the wind.



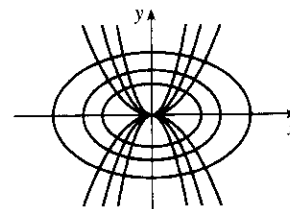
45. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O .
 $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -x/y$, so the slope of the tangent line at $P_0(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0 , which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.



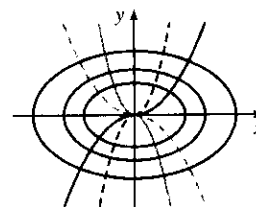
46. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ (1) and $x_0^2 + y_0^2 = by_0$ (2). Now $x^2 + y^2 = ax \Rightarrow 2x + 2yy' = a \Rightarrow y' = \frac{a - 2x}{2y}$ and $x^2 + y^2 = by \Rightarrow 2x + 2yy' = by' \Rightarrow y' = \frac{2x}{b - 2y}$. Thus, the curves are orthogonal at $(x_0, y_0) \Leftrightarrow \frac{a - 2x_0}{2y_0} = -\frac{b - 2y_0}{2x_0} \Leftrightarrow 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \Leftrightarrow ax_0 + by_0 = 2(x_0^2 + y_0^2)$, which is true by (1) and (2).



47. $y = cx^2 \Rightarrow y' = 2cx$ and $x^2 + 2y^2 = k \Rightarrow 2x + 4yy' = 0 \Rightarrow$
 $2yy' = -x \Rightarrow y' = -\frac{x}{2(y)} = -\frac{x}{2(cx^2)} = -\frac{1}{2cx}$, so the curves are
 orthogonal.



48. $y = ax^3 \Rightarrow y' = 3ax^2$ and $x^2 + 3y^2 = b \Rightarrow 2x + 6yy' = 0 \Rightarrow$
 $3yy' = -x \Rightarrow y' = -\frac{x}{3(y)} = -\frac{x}{3(ax^3)} = -\frac{1}{3ax^2}$, so the curves are
 orthogonal.



49. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x -axis, let $y = 0$ and solve for x .

$y = 0 \Rightarrow x^2 - x(0) + 0^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x -axis at the points $(\pm\sqrt{3}, 0)$. Using implicit differentiation to find y' , we get $2x - xy' - y + 2yy' = 0 \Rightarrow y'(2y - x) = y - 2x$
 $\Leftrightarrow y' = \frac{y - 2x}{2y - x}$. So y' at $(\sqrt{3}, 0)$ is $\frac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $(-\sqrt{3}, 0)$ is $\frac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent
 lines at these points are parallel.

50. (a) We use implicit differentiation to find $y' = \frac{y - 2x}{2y - x}$ as in Exercise 49.

The slope of the tangent line at $(-1, 1)$ is $m = \frac{1 - 2(-1)}{2(1) - (-1)} = \frac{3}{3} = 1$,

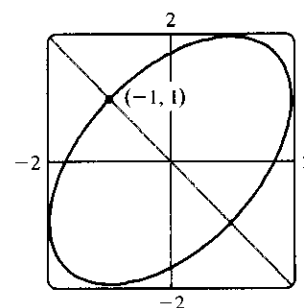
so the slope of the normal line is $-\frac{1}{m} = -1$, and its equation is

$y - 1 = -1(x + 1) \Leftrightarrow y = -x$. Substituting $-x$ for y in the

equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \Rightarrow$

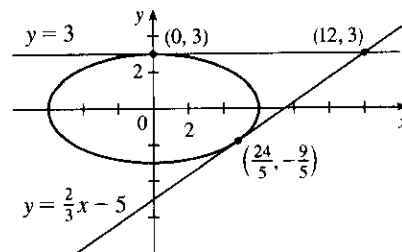
$3x^2 = 3 \Leftrightarrow x = \pm 1$. So the normal line must intersect the ellipse
 again at $x = 1$, and since the equation of the line is $y = -x$, the other
 point of intersection must be $(1, -1)$.

(b)



51. $x^2y^2 + xy = 2 \Rightarrow x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \Leftrightarrow y'(2x^2y + x) = -2xy^2 - y \Leftrightarrow$
 $y' = -\frac{2xy^2 + y}{2x^2y + x}$. So $-\frac{2xy^2 + y}{2x^2y + x} = -1 \Leftrightarrow 2xy^2 + y = 2x^2y + x \Leftrightarrow y(2xy + 1) = x(2xy + 1) \Leftrightarrow$
 $y(2xy + 1) - x(2xy + 1) = 0 \Leftrightarrow (2xy + 1)(y - x) = 0 \Leftrightarrow xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \Rightarrow$
 $x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \Rightarrow x^4 + x^2 = 2 \Leftrightarrow$
 $x^4 + x^2 - 2 = 0 \Leftrightarrow (x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \Leftrightarrow x = \pm 1$.
 Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

52. $x^2 + 4y^2 = 36 \Rightarrow 2x + 8yy' = 0 \Rightarrow y' = -\frac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\frac{a}{4b}(x - 12)$, so $b - 3 = -\frac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \Rightarrow a + b = 3 \Rightarrow b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36 \Leftrightarrow a^2 + 36 - 24a + 4a^2 = 36 \Leftrightarrow 5a^2 - 24a = 0 \Leftrightarrow a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$. So the two points on the ellipse are $(0, 3)$ and $(\frac{24}{5}, -\frac{9}{5})$. Using $y - 3 = -\frac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line $y - 3 = 0$ or $y = 3$. With $(a, b) = (\frac{24}{5}, -\frac{9}{5})$, we have $y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \Leftrightarrow y - 3 = \frac{2}{3}(x - 12) \Leftrightarrow y = \frac{2}{3}x - 5$. A graph of the ellipse and the tangent lines confirms our results.



53. $x^2 + 4y^2 = 5 \Rightarrow 2x + 4(2yy') = 0 \Rightarrow y' = -\frac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/(3 - (-5)) = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\frac{a}{4b}$, or as $\frac{b - 0}{a - (-5)} = \frac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\frac{a}{4b} = \frac{b}{a + 5} \Leftrightarrow 4b^2 = -a^2 - 5a \Leftrightarrow a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \Leftrightarrow a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \Rightarrow b = 1$, since the point is on the top half of the ellipse. So $\frac{h}{8} = \frac{b}{a + 5} = \frac{1}{-1 + 5} = \frac{1}{4} \Rightarrow h = 2$. So the lamp is located 2 units above the x -axis.

3.8 Higher Derivatives

1. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f' , since at the points where c has a horizontal tangent, neither a nor b is equal to 0.
2. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a , so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.
3. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.
4. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

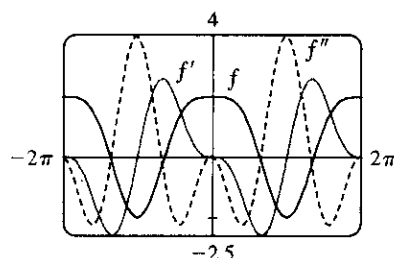
5. $f(x) = x^5 + 6x^2 - 7x \Rightarrow f'(x) = 5x^4 + 12x - 7 \Rightarrow f''(x) = 20x^3 + 12$
6. $f(t) = t^8 - 7t^6 + 2t^4 \Rightarrow f'(t) = 8t^7 - 42t^5 + 8t^3 \Rightarrow f''(t) = 56t^6 - 210t^4 + 24t^2$
7. $y = \cos 2\theta \Rightarrow y' = -2 \sin 2\theta \Rightarrow y'' = -4 \cos 2\theta$
8. $y = \theta \sin \theta \Rightarrow y' = \theta \cos \theta + \sin \theta \Rightarrow y'' = \theta(-\sin \theta) + \cos \theta \cdot 1 + \cos \theta = 2 \cos \theta - \theta \sin \theta$
9. $F(t) = (1 - 7t)^6 \Rightarrow F'(t) = 6(1 - 7t)^5(-7) = -42(1 - 7t)^5 \Rightarrow$
 $F''(t) = -42 \cdot 5(1 - 7t)^4(-7) = 1470(1 - 7t)^4$
10. $g(x) = \frac{2x+1}{x-1} \Rightarrow g'(x) = \frac{(x-1)(2) - (2x+1)(1)}{(x-1)^2} = \frac{2x-2-2x-1}{(x-1)^2} = \frac{-3}{(x-1)^2} \text{ or } -3(x-1)^{-2}$
 $\Rightarrow g''(x) = -3(-2)(x-1)^{-3} = 6(x-1)^{-3} \text{ or } \frac{6}{(x-1)^3}$
11. $h(u) = \frac{1-4u}{1+3u} \Rightarrow h'(u) = \frac{(1+3u)(-4) - (1-4u)(3)}{(1+3u)^2} = \frac{-4-12u-3+12u}{(1+3u)^2} = \frac{-7}{(1+3u)^2} \text{ or}$
 $-7(1+3u)^{-2} \Rightarrow h''(u) = -7(-2)(1+3u)^{-3}(3) = 42(1+3u)^{-3} \text{ or } \frac{42}{(1+3u)^3}$
12. $H(s) = a\sqrt{s} + \frac{b}{\sqrt{s}} = as^{1/2} + bs^{-1/2} \Rightarrow$
 $H'(s) = a \cdot \frac{1}{2}s^{-1/2} + b \left(-\frac{1}{2}s^{-3/2}\right) = \frac{1}{2}as^{-1/2} - \frac{1}{2}bs^{-3/2} \Rightarrow$
 $H''(s) = \frac{1}{2}a \left(-\frac{1}{2}s^{-3/2}\right) - \frac{1}{2}b \left(-\frac{3}{2}s^{-5/2}\right) = -\frac{1}{4}as^{-3/2} + \frac{3}{4}bs^{-5/2}$
13. $h(x) = \sqrt{x^2+1} \Rightarrow h'(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}} \Rightarrow$
 $h''(x) = \frac{\sqrt{x^2+1} \cdot 1 - x \left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right]}{(\sqrt{x^2+1})^2} = \frac{(x^2+1)^{-1/2}[(x^2+1) - x^2]}{(x^2+1)^1} = \frac{1}{(x^2+1)^{3/2}}$
14. $y = x^n \Rightarrow y' = nx^{n-1} \Rightarrow y'' = n(n-1)x^{n-2}$
15. $y = (x^3+1)^{2/3} \Rightarrow y' = \frac{2}{3}(x^3+1)^{-1/3}(3x^2) = 2x^2(x^3+1)^{-1/3} \Rightarrow$
 $y'' = 2x^2 \left(-\frac{1}{3}\right)(x^3+1)^{-4/3}(3x^2) + (x^3+1)^{-1/3}(4x) = 4x(x^3+1)^{-1/3} - 2x^4(x^3+1)^{-4/3}$
16. $y = \frac{4x}{\sqrt{x+1}} \Rightarrow$
 $y' = \frac{\sqrt{x+1} \cdot 4 - 4x \cdot \frac{1}{2}(x+1)^{-1/2}}{(\sqrt{x+1})^2} = \frac{4\sqrt{x+1} - 2x/\sqrt{x+1}}{x+1} = \frac{4(x+1) - 2x}{(x+1)^{3/2}} = \frac{2x+4}{(x+1)^{3/2}} \Rightarrow$
 $y'' = \frac{(x+1)^{3/2} \cdot 2 - (2x+4) \cdot \frac{3}{2}(x+1)^{1/2}}{[(x+1)^{3/2}]^2} = \frac{(x+1)^{1/2}[2(x+1) - 3(2x+2)]}{(x+1)^3}$
 $= \frac{2x+2-3x-6}{(x+1)^{5/2}} = \frac{-x-4}{(x+1)^{5/2}}$
17. $H(t) = \tan 3t \Rightarrow H'(t) = 3 \sec^2 3t \Rightarrow$
 $H''(t) = 2 \cdot 3 \sec 3t \frac{d}{dt}(\sec 3t) = 6 \sec 3t (3 \sec 3t \tan 3t) = 18 \sec^2 3t \tan 3t$
18. $g(s) = s^2 \cos s \Rightarrow g'(s) = 2s \cos s - s^2 \sin s \Rightarrow$
 $g''(s) = 2 \cos s - 2s \sin s - 2s \sin s - s^2 \cos s = (2 - s^2) \cos s - 4s \sin s$

$$\begin{aligned}
 19. \quad g(\theta) &= \theta \csc \theta \Rightarrow g'(\theta) = -\theta \csc \theta \cot \theta + \csc \theta \Rightarrow \\
 g''(\theta) &= (-1) \csc \theta \cot \theta + (-\theta)(-\csc \theta \cot \theta) \cot \theta + (-\theta \csc \theta)(-\csc^2 \theta) - \csc \theta \cot \theta \\
 &= -\csc \theta \cot \theta + \theta \csc \theta \cot^2 \theta + \theta \csc^3 \theta - \csc \theta \cot \theta \\
 &= \csc \theta (\theta \csc^2 \theta + \theta \cot^2 \theta - 2 \cot \theta)
 \end{aligned}$$

$$\begin{aligned}
 20. \quad h(x) &= \frac{x+3}{x^2+2x} \Rightarrow \\
 h'(x) &= \frac{(x^2+2x)(1) - (x+3)(2x+2)}{(x^2+2x)^2} = \frac{(x^2+2x) - (2x^2+8x+6)}{(x^2+2x)^2} = -\frac{x^2+6x+6}{(x^2+2x)^2} \Rightarrow \\
 h''(x) &= -\frac{(x^2+2x)^2(2x+6) - (x^2+6x+6)2(x^2+2x)(2x+2)}{[(x^2+2x)^2]^2} \\
 &= -\frac{2(x^2+2x)[(x^2+2x)(x+3) - (x^2+6x+6)(2x+2)]}{(x^2+2x)^4} \\
 &= -\frac{2[(x^3+5x^2+6x) - (2x^3+14x^2+24x+12)]}{(x^2+2x)^3} = \frac{2(x^3+9x^2+18x+12)}{(x^2+2x)^3}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad (a) \quad f(x) &= 2 \cos x + \sin^2 x \Rightarrow f'(x) = 2(-\sin x) + 2 \sin x (\cos x) = \sin 2x - 2 \sin x \Rightarrow \\
 f''(x) &= 2 \cos 2x - 2 \cos x = 2(\cos 2x - \cos x)
 \end{aligned}$$

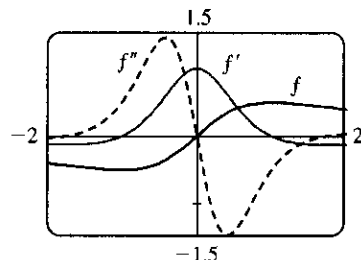
(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

$$\begin{aligned}
 22. \quad (a) \quad f(x) &= \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \Rightarrow \\
 f''(x) &= \frac{(x^2+1)^2(-2x) - (1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2x(2x^2-2-x^2-1)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}
 \end{aligned}$$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

$$\begin{aligned}
 23. \quad y &= \sqrt{2x+3} = (2x+3)^{1/2} \Rightarrow y' = \frac{1}{2}(2x+3)^{-1/2} \cdot 2 = (2x+3)^{-1/2} \Rightarrow \\
 y'' &= -\frac{1}{2}(2x+3)^{-3/2} \cdot 2 = -(2x+3)^{-3/2} \Rightarrow y''' = \frac{3}{2}(2x+3)^{-5/2} \cdot 2 = 3(2x+3)^{-5/2}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad y &= \frac{x}{2x-1} \Rightarrow y' = \frac{(2x-1)(1) - x(2)}{(2x-1)^2} = \frac{-1}{(2x-1)^2} \text{ or } -1(2x-1)^{-2} \Rightarrow \\
 y'' &= -1(-2)(2x-1)^{-3}(2) = 4(2x-1)^{-3} \Rightarrow \\
 y''' &= 4(-3)(2x-1)^{-4}(2) = -24(2x-1)^{-4} \text{ or } -24/(2x-1)^4
 \end{aligned}$$

$$25. f(t) = t \cos t \Rightarrow f'(t) = t(-\sin t) + \cos t \cdot 1 \Rightarrow f''(t) = t(-\cos t) - \sin t \cdot 1 - \sin t \Rightarrow \\ f'''(t) = t \sin t - \cos t \cdot 1 - \cos t - \cos t = t \sin t - 3 \cos t, \text{ so } f'''(0) = 0 - 3 = -3.$$

$$26. g(x) = \sqrt{5-2x} \Rightarrow g'(x) = \frac{1}{2}(5-2x)^{-1/2}(-2) = -(5-2x)^{-1/2} \Rightarrow \\ g''(x) = \frac{1}{2}(5-2x)^{-3/2}(-2) = -(5-2x)^{-3/2} \Rightarrow g'''(x) = \frac{3}{2}(5-2x)^{-5/2}(-2) = -3(5-2x)^{-5/2}, \text{ so } \\ g'''(2) = -3(1)^{-5/2} = -3.$$

$$27. f(\theta) = \cot \theta \Rightarrow f'(\theta) = -\csc^2 \theta \Rightarrow f''(\theta) = -2 \csc \theta (-\csc \theta \cot \theta) = 2 \csc^2 \theta \cot \theta \Rightarrow \\ f'''(\theta) = 2(-2 \csc^2 \theta \cot \theta) \cot \theta + 2 \csc^2 \theta (-\csc^2 \theta) = -2 \csc^2 \theta (2 \cot^2 \theta + \csc^2 \theta) \Rightarrow \\ f'''(\pi/6) = -2(2)^2 [2(\sqrt{3})^2 + (2)^2] = -80$$

$$28. g(x) = \sec x \Rightarrow g'(x) = \sec x \tan x \Rightarrow \\ g''(x) = \sec x \sec^2 x + \tan x (\sec x \tan x) = \sec^3 x + \sec x \tan^2 x = \sec^3 x + \sec x (\sec^2 x - 1) \\ = 2 \sec^3 x - \sec x \Rightarrow \\ g'''(x) = 6 \sec^2 x (\sec x \tan x) - \sec x \tan x = \sec x \tan x (6 \sec^2 x - 1) \Rightarrow \\ g'''(\pi/4) = \sqrt{2}(1)(6 \cdot 2 - 1) = 11\sqrt{2}$$

$$29. 9x^2 + y^2 = 9 \Rightarrow 18x + 2yy' = 0 \Rightarrow 2yy' = -18x \Rightarrow y' = -9x/y \Rightarrow \\ y'' = -9 \left(\frac{y \cdot 1 - x \cdot y'}{y^2} \right) = -9 \left(\frac{y - x(-9x/y)}{y^2} \right) = -9 \cdot \frac{y^2 + 9x^2}{y^3} = -9 \cdot \frac{9}{y^3} \quad [\text{since } x \text{ and } y \text{ must satisfy} \\ \text{the original equation, } 9x^2 + y^2 = 9]. \text{ Thus, } y'' = -81/y^3.$$

$$30. \sqrt{x} + \sqrt{y} = 1 \Rightarrow \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\sqrt{y}/\sqrt{x} \Rightarrow \\ y'' = -\frac{\sqrt{x}[1/(2\sqrt{y})]y' - \sqrt{y}[1/(2\sqrt{x})]}{x} = -\frac{\sqrt{x}(1/\sqrt{y})(-\sqrt{y}/\sqrt{x}) - \sqrt{y}(1/\sqrt{x})}{2x} = \frac{1 + \sqrt{y}/\sqrt{x}}{2x} \\ = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} = \frac{1}{2x\sqrt{x}} \text{ since } x \text{ and } y \text{ must satisfy the original equation, } \sqrt{x} + \sqrt{y} = 1.$$

$$31. x^3 + y^3 = 1 \Rightarrow 3x^2 + 3y^2y' = 0 \Rightarrow y' = -\frac{x^2}{y^2} \Rightarrow \\ y'' = -\frac{y^2(2x) - x^2 \cdot 2yy'}{(y^2)^2} = -\frac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\frac{2xy^4 + 2x^4y}{y^6} = -\frac{2xy(y^3 + x^3)}{y^6} = -\frac{2x}{y^5}, \\ \text{since } x \text{ and } y \text{ must satisfy the original equation, } x^3 + y^3 = 1.$$

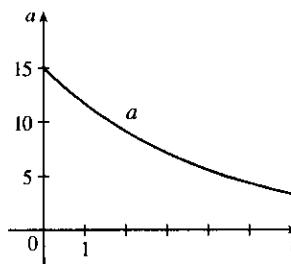
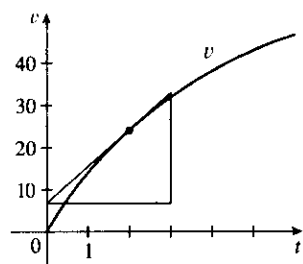
$$32. x^4 + y^4 = a^4 \Rightarrow 4x^3 + 4y^3y' = 0 \Rightarrow 4y^3y' = -4x^3 \Rightarrow y' = -x^3/y^3 \Rightarrow \\ y'' = -\left(\frac{y^3 \cdot 3x^2 - x^3 \cdot 3y^2y'}{(y^3)^2} \right) = -3x^2y^2 \cdot \frac{y - x(-x^3/y^3)}{y^6} = -3x^2 \cdot \frac{y^4 + x^4}{y^4y^3} = -3x^2 \cdot \frac{a^4}{y^7} = \frac{-3a^4x^2}{y^7}$$

$$33. f(x) = x^n \Rightarrow f'(x) = nx^{n-1} \Rightarrow f''(x) = n(n-1)x^{n-2} \Rightarrow \dots \Rightarrow \\ f^{(n)}(x) = n(n-1)(n-2)\dots 2 \cdot 1x^{n-n} = n!$$

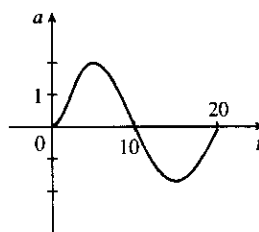
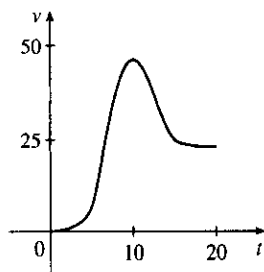
$$34. f(x) = \frac{1}{5x-1} = (5x-1)^{-1} \Rightarrow f'(x) = -1(5x-1)^{-2} \cdot 5 \Rightarrow f''(x) = (-1)(-2)(5x-1)^{-3} \cdot 5^2 \Rightarrow \\ f'''(x) = (-1)(-2)(-3)(5x-1)^{-4} \cdot 5^3 \Rightarrow \dots \Rightarrow f^{(n)}(x) = (-1)^n n! 5^n (5x-1)^{-(n+1)}$$

$$35. f(x) = (1+x)^{-1} \Rightarrow f'(x) = -1(1+x)^{-2}, f''(x) = 1 \cdot 2(1+x)^{-3}, f^{(3)}(x) = -1 \cdot 2 \cdot 3(1+x)^{-4}, \\ f^{(4)}(x) = 1 \cdot 2 \cdot 3 \cdot 4(1+x)^{-5}, \dots, f^{(n)}(x) = (-1)^n n! (1+x)^{-(n+1)}$$

36. $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow$
 $f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} \Rightarrow f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} \Rightarrow$
 $f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} = -\frac{1 \cdot 3 \cdot 5}{2^4}x^{-7/2} \Rightarrow$
 $f^{(5)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-9/2} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5}x^{-9/2} \Rightarrow \dots \Rightarrow$
 $f^{(n)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \dots \left(\frac{1}{2} - n + 1\right)x^{-(2n-1)/2} = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{2^n} x^{-(2n-1)/2}$
37. $f(x) = 1/(3x^3) = \frac{1}{3}x^{-3} \Rightarrow f'(x) = \frac{1}{3}(-3)x^{-4} \Rightarrow f''(x) = \frac{1}{3}(-3)(-4)x^{-5} \Rightarrow$
 $f'''(x) = \frac{1}{3}(-3)(-4)(-5)x^{-6} \Rightarrow \dots \Rightarrow$
 $f^{(n)}(x) = \frac{1}{3}(-3)(-4) \dots [-(n+2)]x^{-(n+3)} = \frac{(-1)^n \cdot 3 \cdot 4 \cdot 5 \cdot \dots \cdot (n+2)}{3x^{n+3}} \cdot \frac{2}{2} = \frac{(-1)^n (n+2)!}{6x^{n+3}}$
38. $D \sin x = \cos x \Rightarrow D^2 \sin x = -\sin x \Rightarrow D^3 \sin x = -\cos x \Rightarrow D^4 \sin x = \sin x$. The derivatives of $\sin x$ occur in a cycle of four. Since $74 = 4(18) + 2$, we have $D^{74} \sin x = D^2 \sin x = -\sin x$.
39. Let $f(x) = \cos x$. Then $Df(2x) = 2f'(2x)$, $D^2 f(2x) = 2^2 f''(2x)$, $D^3 f(2x) = 2^3 f'''(2x)$, $\dots$,
 $D^{(n)} f(2x) = 2^n f^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103 = 4(25) + 3$, we have $f^{(103)}(x) = f^{(3)}(x) = \sin x$ and $D^{103} \cos 2x = 2^{103} f^{(103)}(2x) = 2^{103} \sin 2x$.
40. Let $f(x) = x \sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then
 $f'(x) = h(x) + xh'(x)$, $f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x)$,
 $f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x)$, $\dots$, $f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x)$. Since
 $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = D^2 \sin x = -\sin x$ and $h^{(35)}(x) = -\cos x$. Thus,
 $D^{(35)} x \sin x = 35h^{(34)}(x) + xh^{(35)}(x) = -35 \sin x - x \cos x$.
41. By measuring the slope of the graph of $s = f(t)$ at $t = 0, 1, 2, 3, 4$, and 5 , and using the method of Example 1 in Section 3.2, we plot the graph of the velocity function $v = f'(t)$ in the first figure. The acceleration when $t = 2$ s is $a = f''(2)$, the slope of the tangent line to the graph of f' when $t = 2$. We estimate the slope of this tangent line to be $a(2) = f''(2) = v'(2) \approx \frac{27}{3} = 9 \text{ ft/s}^2$. Similar measurements enable us to graph the acceleration function in the second figure.



42. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.



(b) Drawing a tangent line at $t = 10$ on the graph of a , a appears to decrease by 10 ft/s^2 over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5 \text{ (ft/s}^2\text{)/s}$ or ft/s^3 .

43. (a) $s = 2t^3 - 15t^2 + 36t + 2 \Rightarrow v(t) = s'(t) = 6t^2 - 30t + 36 \Rightarrow a(t) = v'(t) = 12t - 30$

(b) $a(1) = 12 \cdot 1 - 30 = -18 \text{ m/s}^2$

(c) $v(t) = 6(t^2 - 5t + 6) = 6(t - 2)(t - 3) = 0$ when $t = 2$ or 3 and $a(2) = 24 - 30 = -6 \text{ m/s}^2$,
 $a(3) = 36 - 30 = 6 \text{ m/s}^2$.

44. (a) $s = 2t^3 - 3t^2 - 12t \Rightarrow v(t) = s'(t) = 6t^2 - 6t - 12 \Rightarrow a(t) = v'(t) = 12t - 6$

(b) $a(1) = 12 \cdot 1 - 6 = 6 \text{ m/s}^2$

(c) $v(t) = 6(t^2 - t - 2) = 6(t + 1)(t - 2) = 0$ when $t = -1$ or 2. Since $t \geq 0$, $t \neq -1$ and
 $a(2) = 24 - 6 = 18 \text{ m/s}^2$.

45. (a) $s = \sin(\frac{\pi}{6}t) + \cos(\frac{\pi}{6}t)$, $0 \leq t \leq 2$. $v(t) = s'(t) = \cos(\frac{\pi}{6}t) \cdot \frac{\pi}{6} - \sin(\frac{\pi}{6}t) \cdot \frac{\pi}{6} = \frac{\pi}{6} [\cos(\frac{\pi}{6}t) - \sin(\frac{\pi}{6}t)]$
 $\Rightarrow a(t) = v'(t) = \frac{\pi}{6} [-\sin(\frac{\pi}{6}t) \cdot \frac{\pi}{6} - \cos(\frac{\pi}{6}t) \cdot \frac{\pi}{6}] = -\frac{\pi^2}{36} [\sin(\frac{\pi}{6}t) + \cos(\frac{\pi}{6}t)]$

(b) $a(1) = -\frac{\pi^2}{36} [\sin(\frac{\pi}{6} \cdot 1) + \cos(\frac{\pi}{6} \cdot 1)] = -\frac{\pi^2}{36} [\frac{1}{2} + \frac{\sqrt{3}}{2}] = -\frac{\pi^2}{72} (1 + \sqrt{3}) \approx -0.3745 \text{ m/s}^2$

(c) $v(t) = 0$ for $0 \leq t \leq 2 \Rightarrow \cos(\frac{\pi}{6}t) = \sin(\frac{\pi}{6}t) \Rightarrow 1 = \frac{\sin(\frac{\pi}{6}t)}{\cos(\frac{\pi}{6}t)} \Rightarrow$

$\tan(\frac{\pi}{6}t) = 1 \Rightarrow \frac{\pi}{6}t = \tan^{-1} 1 \Rightarrow t = \frac{6}{\pi} \cdot \frac{\pi}{4} = \frac{3}{2} = 1.5 \text{ s. Thus,}$

$a(\frac{3}{2}) = -\frac{\pi^2}{36} [\sin(\frac{\pi}{6} \cdot \frac{3}{2}) + \cos(\frac{\pi}{6} \cdot \frac{3}{2})] = -\frac{\pi^2}{36} [\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}] = -\frac{\pi^2}{36} \sqrt{2} \approx -0.3877 \text{ m/s}^2$.

46. (a) $s = 2t^3 - 7t^2 + 4t + 1 \Rightarrow v(t) = s'(t) = 6t^2 - 14t + 4 \Rightarrow a(t) = v'(t) = 12t - 14$

(b) $a(1) = 12 - 14 = -2 \text{ m/s}^2$

(c) $v(t) = 2(3t^2 - 7t + 2) = 2(3t - 1)(t - 2) = 0$ when $t = \frac{1}{3}$ or 2 and $a(\frac{1}{3}) = 12(\frac{1}{3}) - 14 = -10 \text{ m/s}^2$,
 $a(2) = 12(2) - 14 = 10 \text{ m/s}^2$.

47. (a) $s(t) = t^4 - 4t^3 + 2 \Rightarrow v(t) = s'(t) = 4t^3 - 12t^2 \Rightarrow a(t) = v'(t) = 12t^2 - 24t = 12t(t - 2) = 0$
when $t = 0$ or 2.

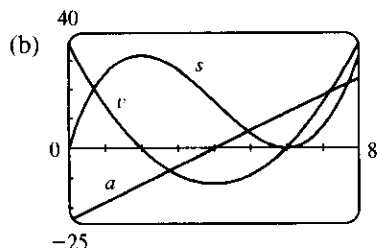
(b) $s(0) = 2 \text{ m}$, $v(0) = 0 \text{ m/s}$, $s(2) = -14 \text{ m}$, $v(2) = -16 \text{ m/s}$

48. (a) $s(t) = 2t^3 - 9t^2 \Rightarrow v(t) = s'(t) = 6t^2 - 18t \Rightarrow a(t) = v'(t) = 12t - 18 = 0$ when $t = 1.5$.

(b) $s(1.5) = -13.5 \text{ m}$, $v(1.5) = -13.5 \text{ m/s}$

49. (a) $s = f(t) = t^3 - 12t^2 + 36t$, $t \geq 0 \Rightarrow v(t) = f'(t) = 3t^2 - 24t + 36$.

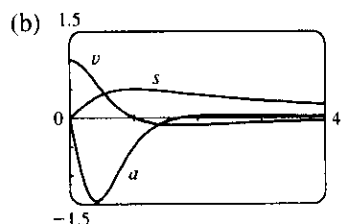
$a(t) = v'(t) = 6t - 24$. $a(3) = 6(3) - 24 = -6 \text{ (m/s)/s}$ or m/s^2 .



(c) The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 4$ and when $t > 6$. It is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and when $4 < t < 6$.

$$50. (a) x(t) = \frac{t}{1+t^2} \Rightarrow v(t) = x'(t) = \frac{(1+t^2)(1) - t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}.$$

$$a(t) = v'(t) = \frac{2t(t^2-3)}{(1+t^2)^3}. a(t) = 0 \Rightarrow 2t(t^2-3) = 0 \Rightarrow t = 0 \text{ or } \sqrt{3}$$



(c) v and a have the same sign and the particle is speeding up when $1 < t < \sqrt{3}$. The particle is slowing down and v and a have opposite signs when $0 < t < 1$ and when $t > \sqrt{3}$.

$$51. (a) y(t) = A \sin \omega t \Rightarrow v(t) = y'(t) = A\omega \cos \omega t \Rightarrow a(t) = v'(t) = -A\omega^2 \sin \omega t$$

$$(b) a(t) = -A\omega^2 \sin \omega t = -\omega^2(A \sin \omega t) = -\omega^2 y(t), \text{ so } a(t) \text{ is proportional to } y(t).$$

(c) speed $= |v(t)| = A\omega |\cos \omega t|$ is a maximum when $\cos \omega t = \pm 1$. But when $\cos \omega t = \pm 1$, we have $\sin \omega t = 0$, and $a(t) = -A\omega^2 \sin \omega t = -A\omega^2(0) = 0$.

52. By the Chain Rule, $a(t) = \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = \frac{dv}{ds} v(t) = v(t) \frac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

$$53. \text{ Let } P(x) = ax^2 + bx + c. \text{ Then } P'(x) = 2ax + b \text{ and } P''(x) = 2a. P''(2) = 2 \Rightarrow 2a = 2 \Rightarrow a = 1. \\ P'(2) = 3 \Rightarrow 2(1)(2) + b = 3 \Rightarrow 4 + b = 3 \Rightarrow b = -1.$$

$$P(2) = 5 \Rightarrow 1(2)^2 + (-1)(2) + c = 5 \Rightarrow 2 + c = 5 \Rightarrow c = 3. \text{ So } P(x) = x^2 - x + 3.$$

$$54. \text{ Let } Q(x) = ax^3 + bx^2 + cx + d. \text{ Then } Q'(x) = 3ax^2 + 2bx + c, Q''(x) = 6ax + 2b \text{ and } Q'''(x) = 6a. \text{ Thus,} \\ Q(1) = a + b + c + d = 1, Q'(1) = 3a + 2b + c = 3, Q''(1) = 6a + 2b = 6 \text{ and } Q'''(1) = 6a = 12. \text{ Solving} \\ \text{these four equations in four unknowns } a, b, c \text{ and } d \text{ we get } a = 2, b = -3, c = 3 \text{ and } d = -1, \text{ so} \\ Q(x) = 2x^3 - 3x^2 + 3x - 1.$$

$$55. y = A \sin x + B \cos x \Rightarrow y' = A \cos x - B \sin x \Rightarrow y'' = -A \sin x - B \cos x. \text{ Substituting into} \\ y'' + y' - 2y = \sin x \text{ gives us } (-3A - B) \sin x + (A - 3B) \cos x = 1 \sin x, \text{ so we must have } -3A - B = 1 \\ \text{and } A - 3B = 0. \text{ Solving for } A \text{ and } B, \text{ we add the first equation to three times the second to get } B = -\frac{1}{10} \\ \text{and } A = -\frac{3}{10}.$$

$$56. y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A. \text{ We substitute these expressions into the equation} \\ y'' + y' - 2y = x^2 \text{ to get}$$

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$

$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \Rightarrow A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \Rightarrow A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \Rightarrow -1 - \frac{1}{2} - 2C = 0 \Rightarrow C = -\frac{3}{4}$.

$$57. f(x) = xg(x^2) \Rightarrow f'(x) = x \cdot g'(x^2) \cdot 2x + g(x^2) \cdot 1 = g(x^2) + 2x^2 g'(x^2) \Rightarrow \\ f''(x) = g'(x^2) \cdot 2x + 2x^2 \cdot g''(x^2) \cdot 2x + g'(x^2) \cdot 4x = 6xg'(x^2) + 4x^3 g''(x^2)$$

$$58. f(x) = \frac{g(x)}{x} \Rightarrow f'(x) = \frac{xg'(x) - g(x)}{x^2} \Rightarrow$$

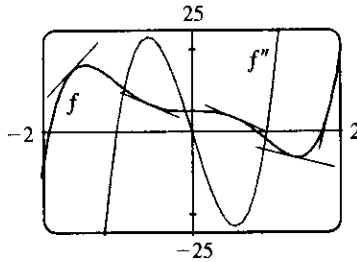
$$f''(x) = \frac{x^2[g'(x) + xg''(x) - g'(x)] - 2x[xg'(x) - g(x)]}{x^4} = \frac{x^2g''(x) - 2xg'(x) + 2g(x)}{x^3}$$

$$59. f(x) = g(\sqrt{x}) \Rightarrow f'(x) = g'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = \frac{g'(\sqrt{x})}{2\sqrt{x}} \Rightarrow$$

$$f''(x) = \frac{2\sqrt{x} \cdot g''(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} - g'(\sqrt{x}) \cdot 2 \cdot \frac{1}{2}x^{-1/2}}{(2\sqrt{x})^2} = \frac{x^{-1/2}[\sqrt{x}g''(\sqrt{x}) - g'(\sqrt{x})]}{4x}$$

$$= \frac{\sqrt{x}g''(\sqrt{x}) - g'(\sqrt{x})}{4x\sqrt{x}}$$

60.



$$f(x) = 3x^5 - 10x^3 + 5 \Rightarrow f'(x) = 15x^4 - 30x^2 \Rightarrow$$

$$f''(x) = 60x^3 - 60x = 60x(x^2 - 1) = 60x(x+1)(x-1)$$

So $f''(x) > 0$ when $-1 < x < 0$ or $x > 1$, and on these intervals the graph of f lies above its tangent lines; and $f''(x) < 0$ when $x < -1$ or $0 < x < 1$, and on these intervals the graph of f lies below its tangent lines.

$$61. (a) f(x) = \frac{1}{x^2 + x} \Rightarrow f'(x) = \frac{-(2x+1)}{(x^2+x)^2} \Rightarrow$$

$$f''(x) = \frac{(x^2+x)^2(-2) + (2x+1)(2)(x^2+x)(2x+1)}{(x^2+x)^4} = \frac{2(3x^2+3x+1)}{(x^2+x)^3} \Rightarrow$$

$$f'''(x) = \frac{(x^2+x)^3(2)(6x+3) - 2(3x^2+3x+1)(3)(x^2+x)^2(2x+1)}{(x^2+x)^6}$$

$$= \frac{-6(4x^3+6x^2+4x+1)}{(x^2+x)^4} \Rightarrow$$

$$f^{(4)}(x) = \frac{(x^2+x)^4(-6)(12x^2+12x+4) + 6(4x^3+6x^2+4x+1)(4)(x^2+x)^3(2x+1)}{(x^2+x)^8}$$

$$= \frac{24(5x^4+10x^3+10x^2+5x+1)}{(x^2+x)^5}$$

$$f^{(5)}(x) = ?$$

$$(b) f(x) = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \Rightarrow f'(x) = -x^{-2} + (x+1)^{-2} \Rightarrow f''(x) = 2x^{-3} - 2(x+1)^{-3} \Rightarrow$$

$$f'''(x) = (-3)(2)x^{-4} + (3)(2)(x+1)^{-4} \Rightarrow \dots \Rightarrow f^{(n)}(x) = (-1)^n n! [x^{-(n+1)} - (x+1)^{-(n+1)}]$$

$$62. (a) \text{ For } f(x) = \frac{7x+17}{2x^2-7x-4}, \text{ a CAS gives us } f'''(x) = \frac{-6(56x^4+544x^3-2184x^2+6184x-6139)}{(2x^2-7x-4)^4}$$

$$(b) \text{ Using a CAS we get } f(x) = \frac{7x+17}{2x^2-7x-4} = \frac{-3}{2x+1} + \frac{5}{x-4}. \text{ Now we differentiate three times to obtain}$$

$$f'''(x) = \frac{144}{(2x+1)^4} - \frac{30}{(x-4)^4}.$$

63. We will show that for each positive integer n , the n th derivative $f^{(n)}$ exists and equals one of $f, f', f'', f''', \dots, f^{(p-1)}$. Since $f^{(p)} = f$, the first p derivatives of f are $f', f'', f''', \dots, f^{(p-1)}$, and f . In particular, our statement is true for $n = 1$. Suppose that k is an integer, $k \geq 1$, for which f is k -times differentiable with $f^{(k)}$ in the set

$S = \{f, f', f'', \dots, f^{(p-1)}\}$. Since f is p -times differentiable, every member of S [including $f^{(k)}$] is differentiable, so $f^{(k+1)}$ exists and equals the derivative of some member of S . Thus, $f^{(k+1)}$ is in the set $\{f', f'', f''', \dots, f^{(p)}\}$, which equals S since $f^{(p)} = f$. We have shown that the statement is true for $n = 1$ and that its truth for $n = k$ implies its truth for $n = k + 1$. By mathematical induction, the statement is true for all positive integers n .

64. (a) Use the Product Rule repeatedly: $F = fg \Rightarrow F' = f'g + fg' \Rightarrow$

$$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''.$$

- (b) $F''' = f'''g + f''g' + 2(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \Rightarrow$

$$\begin{aligned} F^{(4)} &= f^{(4)}g + f'''g' + 3(f'''g' + f''g'') + 3(f''g'' + f'g''') + f'g''' + fg^{(4)} \\ &= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)} \end{aligned}$$

- (c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + n f^{(n-1)}g' + \binom{n}{2} f^{(n-2)}g'' + \dots + \binom{n}{k} f^{(n-k)}g^{(k)} + \dots + n f'g^{(n-1)} + fg^{(n)}$$

$$\text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

65. The Chain Rule says that $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$, so

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{du} \frac{du}{dx} \right) = \left[\frac{d}{dx} \left(\frac{dy}{du} \right) \right] \frac{du}{dx} + \frac{dy}{du} \frac{d}{dx} \left(\frac{du}{dx} \right) \quad [\text{Product Rule}] \\ &= \left[\frac{d}{du} \left(\frac{dy}{du} \right) \frac{du}{dx} \right] \frac{du}{dx} + \frac{dy}{du} \frac{d^2u}{dx^2} = \frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2} \end{aligned}$$

66. From Exercise 65, $\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2} \Rightarrow$

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{d}{dx} \frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 \right] + \frac{d}{dx} \left[\frac{dy}{du} \frac{d^2u}{dx^2} \right] \\ &= \left[\frac{d}{dx} \left(\frac{d^2y}{du^2} \right) \right] \left(\frac{du}{dx} \right)^2 + \left[\frac{d}{dx} \left(\frac{du}{dx} \right)^2 \right] \frac{d^2y}{du^2} + \left[\frac{d}{dx} \left(\frac{dy}{du} \right) \right] \frac{d^2u}{dx^2} + \left[\frac{d}{dx} \left(\frac{d^2u}{dx^2} \right) \right] \frac{dy}{du} \\ &= \left[\frac{d}{du} \left(\frac{d^2y}{du^2} \right) \frac{du}{dx} \right] \left(\frac{du}{dx} \right)^2 + 2 \frac{du}{dx} \frac{d^2u}{dx^2} \frac{d^2y}{du^2} + \left[\frac{d}{du} \left(\frac{dy}{du} \right) \frac{du}{dx} \right] \left(\frac{d^2u}{dx^2} \right) + \frac{d^3u}{dx^3} \frac{dy}{du} \\ &= \frac{d^3y}{du^3} \left(\frac{du}{dx} \right)^3 + 3 \frac{du}{dx} \frac{d^2u}{dx^2} \frac{d^2y}{du^2} + \frac{dy}{du} \frac{d^3u}{dx^3} \end{aligned}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

$$(\alpha) P(0) = 0$$

$$(\beta) P'(0) = 0 \text{ (for a smooth landing)}$$

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \Rightarrow P'(x) = 3ax^2 + 2bx + c$.

But $P'(0) = c = 0$ by condition β . So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell(3a\ell + 2b)$. Now by condition γ ,

$3a\ell + 2b = 0 \Rightarrow a = -\frac{2b}{3\ell}$. Therefore, $P(x) = -\frac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ , we get

$P(\ell) = -\frac{2b}{3\ell}\ell^3 + b\ell^2 = h \Rightarrow -\frac{2}{3}b\ell^2 + b\ell^2 = h \Rightarrow \frac{1}{3}b\ell^2 = h \Rightarrow b = \frac{3h}{\ell^2} \Rightarrow a = -\frac{2h}{\ell^3}$. So

$$y = P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2.$$

2. By condition (ii), $\frac{dx}{dt} = -v$ for all t , so $x(t) = \ell - vt$. Condition (iii) states that $\left| \frac{d^2y}{dt^2} \right| \leq k$. By the Chain Rule,

we have $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = -\frac{2h}{\ell^3}(3x^2) \frac{dx}{dt} + \frac{3h}{\ell^2}(2x) \frac{dx}{dt} = \frac{6hx^2v}{\ell^3} - \frac{6h xv}{\ell^2}$ (for $x \leq \ell$) $\Rightarrow$

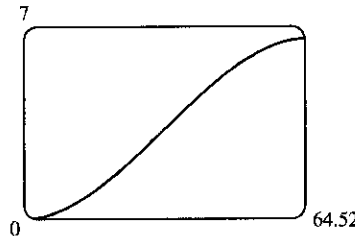
$\frac{d^2y}{dt^2} = \frac{6hv}{\ell^3}(2x) \frac{dx}{dt} - \frac{6hv}{\ell^2} \frac{dx}{dt} = -\frac{12hv^2}{\ell^3}x + \frac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so

$\left. \frac{d^2y}{dt^2} \right|_{t=0} = -\frac{12hv^2}{\ell^3}\ell + \frac{6hv^2}{\ell^2} = -\frac{6hv^2}{\ell^2}$. Thus, $\left| \frac{d^2y}{dt^2} \right|_{t=0} = \frac{6hv^2}{\ell^2} \leq k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 860 \text{ mi/h}^2$, $h = 35,000 \text{ ft} \times \frac{1 \text{ mi}}{5280 \text{ ft}}$, and $v = 300 \text{ mi/h}$ into the result of part (b):

$$\frac{6(35,000 \cdot \frac{1}{5280})(300)^2}{\ell^2} \leq 860 \Rightarrow \ell \geq 300 \sqrt{6 \cdot \frac{35,000}{5280 \cdot 860}} \approx 64.5 \text{ miles}.$$

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx 4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



APPLIED PROJECT Building a Better Roller Coaster

1. (a) $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$.

The origin is at P : $f(0) = 0 \Rightarrow c = 0$

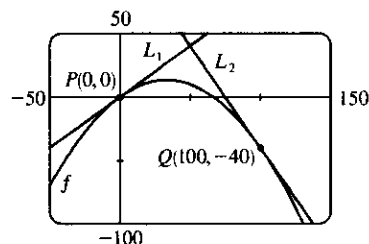
The slope of the ascent is 0.8: $f'(0) = 0.8 \Rightarrow b = 0.8$

The slope of the drop is -1.6: $f'(100) = -1.6 \Rightarrow 200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \Rightarrow 200a + 0.8 = -1.6 \Rightarrow 200a = -2.4 \Rightarrow a = -\frac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

- (c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$. The horizontal distance between P and Q is 100, so the y -coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6 , it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



- (d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L'_1(x) = 0.8$	$L''_1(x) = 0$
$[0, 10]$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L'_2(x) = -1.6$	$L''_2(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition—these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L'_1(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L''_1(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729,000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24,300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1,000,000p + 10,000q + 100r + s = -40$
	$h'(100) = L'_2(100)$	$-\frac{8}{5}$	$30,000p + 200q + r = -1.6$
	$h''(100) = L''_2(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
-100	-10	-1	1000	100	10	1	0	0	0	0	0
-20	-1	0	300	20	1	0	0	0	0	0	0
-2	0	0	60	2	0	0	0	0	0	0	0
-8100	-90	-1	0	0	0	0	729,000	8100	90	1	0
-180	-1	0	0	0	0	0	24,300	180	1	0	0
-2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	-40
0	0	0	0	0	0	0	30,000	200	1	0	-1.6
0	0	0	0	0	0	0	600	2	0	0	0

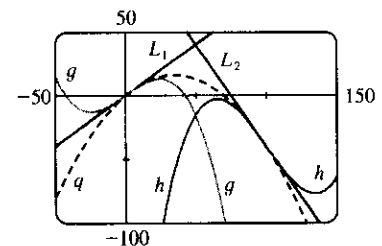
Solving the system gives us the formulas for q , g , and h .

$$\left. \begin{aligned} a &= -0.01\bar{3} = -\frac{1}{75} \\ b &= 0.9\bar{3} = \frac{14}{15} \\ c &= -0.4 = -\frac{4}{9} \end{aligned} \right\} q(x) = -\frac{1}{75}x^2 + \frac{14}{15}x - \frac{4}{9}$$

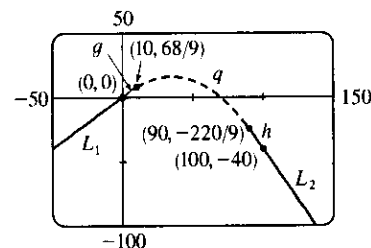
$$\left. \begin{aligned} k &= -0.000\bar{4} = -\frac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \frac{4}{5} \\ n &= 0 \end{aligned} \right\} g(x) = -\frac{1}{2250}x^3 + \frac{4}{5}x$$

$$\left. \begin{aligned} p &= 0.000\bar{4} = \frac{1}{2250} \\ q &= -0.1\bar{3} = -\frac{2}{15} \\ r &= 11.7\bar{3} = \frac{176}{15} \\ s &= -324.\bar{4} = -\frac{2920}{9} \end{aligned} \right\} h(x) = \frac{1}{2250}x^3 - \frac{2}{15}x^2 + \frac{176}{15}x - \frac{2920}{9}$$

(c) Graph of L_1 , q , g , h , and L_2 :



The graph of the five functions as a piecewise-defined function:

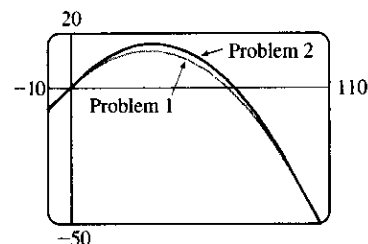


This is the piecewise-defined function assignment on a TI-83 Plus calculator, where $Y_2 = L_1$, $Y_6 = g$, $Y_5 = q$, $Y_7 = h$, and $Y_3 = L_2$.

```

Plot1 Plot2 Plot3
Y8=Y2*(X<0)+Y6*
(X≥0 and X<10)+Y
5*(X≥10 and X≤90
)+Y7*(X>90 and X
≤100)+Y3*(X>100)
Y9=
  
```

A comparison of the graphs in part 1(c) and part 2(c):



3.9 Related Rates

$$1. V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = 3x^2 \frac{dx}{dt}$$

$$2. (a) A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \quad (b) \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(30 \text{ m})(1 \text{ m/s}) = 60\pi \text{ m}^2/\text{s}$$

$$3. y = x^3 + 2x \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = (3x^2 + 2)(5) = 5(3x^2 + 2). \text{ When } x = 2, \frac{dy}{dt} = 5(14) = 70.$$

$$4. x^2 + y^2 = 25 \Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow x \frac{dx}{dt} = -y \frac{dy}{dt} \Rightarrow \frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

$$\text{When } y = 4, x^2 + 4^2 = 25 \Rightarrow x = \pm 3. \text{ For } \frac{dy}{dt} = 6, \frac{dx}{dt} = -\frac{4}{\pm 3}(6) = \mp 8.$$

$$5. z^2 = x^2 + y^2 \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right). \text{ When } x = 5 \text{ and } y = 12,$$

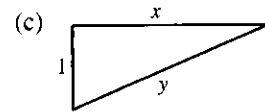
$$z^2 = 5^2 + 12^2 \Rightarrow z^2 = 169 \Rightarrow z = \pm 13. \text{ For } \frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = 3, \frac{dz}{dt} = \frac{1}{\pm 13} (5 \cdot 2 + 12 \cdot 3) = \pm \frac{46}{13}.$$

$$6. y = \sqrt{1+x^3} \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{1}{2}(1+x^3)^{-1/2} (3x^2) \frac{dx}{dt} = \frac{3x^2}{2\sqrt{1+x^3}} \frac{dx}{dt}. \text{ With } \frac{dy}{dt} = 4 \text{ when } x = 2 \text{ and}$$

$$y = 3, \text{ we have } 4 = \frac{3(4)}{2(3)} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2 \text{ cm/s.}$$

7. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

- (b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

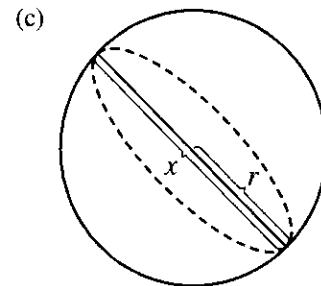


- (d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \Rightarrow 2y(dy/dt) = 2x(dx/dt)$.

- (e) $\frac{dy}{dt} = \frac{x}{y} \frac{dx}{dt} = \frac{x}{y} (500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\frac{dy}{dt} = \frac{\sqrt{3}}{2} (500) = 250\sqrt{3} \approx 433$ mi/h.

8. (a) Given: the rate of decrease of the surface area is 1 cm²/min. If we let t be time (in minutes) and S be the surface area (in cm²), then we are given that $dS/dt = -1$ cm²/s.

- (b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.



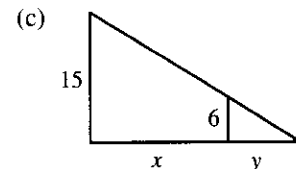
- (d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and $S = 4\pi r^2 = 4\pi(\frac{1}{2}x)^2 = \pi x^2 \Rightarrow$

$$\frac{dS}{dt} = \frac{dS}{dx} \frac{dx}{dt} = 2\pi x \frac{dx}{dt}.$$

- (e) $-1 = \frac{dS}{dt} = 2\pi x \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -\frac{1}{2\pi x}$. When $x = 10$, $\frac{dx}{dt} = -\frac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm/min.

9. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

- (b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\frac{d}{dt}(x + y)$ when $x = 40$ ft.

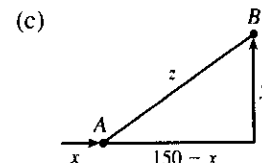


- (d) By similar triangles, $\frac{15}{6} = \frac{x+y}{y} \Rightarrow 15y = 6x + 6y \Rightarrow 9y = 6x \Rightarrow y = \frac{2}{3}x$.

- (e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x + y) = \frac{d}{dt}(x + \frac{2}{3}x) = \frac{5}{3} \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

10. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

- (b) Unknown: the rate at which the distance between the ships is changing at 4:00 P.M. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

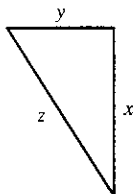


$$(d) z^2 = (150 - x)^2 + y^2 \Rightarrow 2z \frac{dz}{dt} = 2(150 - x) \left(-\frac{dx}{dt} \right) + 2y \frac{dy}{dt}$$

(e) At 4:00 P.M., $x = 4(35) = 140$ and $y = 4(25) = 100 \Rightarrow z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10,100}$. So

$$\frac{dz}{dt} = \frac{1}{z} \left[(x - 150) \frac{dx}{dt} + y \frac{dy}{dt} \right] = \frac{-10(35) + 100(25)}{\sqrt{10,100}} = \frac{215}{\sqrt{101}} \approx 21.4 \text{ km/h.}$$

11.



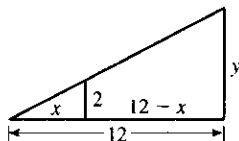
We are given that $\frac{dx}{dt} = 60$ mi/h and $\frac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \Rightarrow$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right).$$

After 2 hours, $x = 2(60) = 120$ and $y = 2(25) = 50 \Rightarrow z = \sqrt{120^2 + 50^2} = 130$,

$$\text{so } \frac{dz}{dt} = \frac{1}{z} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) = \frac{120(60) + 50(25)}{130} = 65 \text{ mi/h.}$$

12.

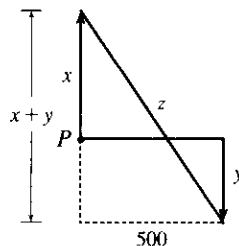


We are given that $\frac{dx}{dt} = 1.6$ m/s. By similar triangles, $\frac{y}{12} = \frac{2}{x} \Rightarrow y = \frac{24}{x}$

$$\Rightarrow \frac{dy}{dt} = -\frac{24}{x^2} \frac{dx}{dt} = -\frac{24}{x^2} (1.6). \text{ When } x = 8, \frac{dy}{dt} = -\frac{24(1.6)}{64} = -0.6 \text{ m/s,}$$

so the shadow is decreasing at a rate of 0.6 m/s.

13.



We are given that $\frac{dx}{dt} = 4$ ft/s and $\frac{dy}{dt} = 5$ ft/s. $z^2 = (x + y)^2 + 500^2 \Rightarrow$

$$2z \frac{dz}{dt} = 2(x + y) \left(\frac{dx}{dt} + \frac{dy}{dt} \right). \text{ 15 minutes after the woman starts, we have}$$

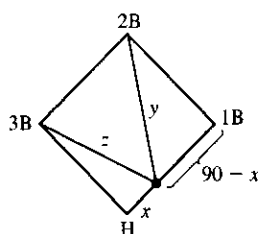
$$x = (4 \text{ ft/s})(20 \text{ min})(60 \text{ s/min}) = 4800 \text{ ft and } y = 5 \cdot 15 \cdot 60 = 4500 \Rightarrow$$

$$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86,740,000}, \text{ so}$$

$$\frac{dz}{dt} = \frac{x + y}{z} \left(\frac{dx}{dt} + \frac{dy}{dt} \right) = \frac{4800 + 4500}{\sqrt{86,740,000}} (4 + 5) = \frac{837}{\sqrt{8674}} \approx 8.99 \text{ ft/s.}$$

14. We are given that $\frac{dx}{dt} = 24$ ft/s.

(a)



$$y^2 = (90 - x)^2 + 90^2 \Rightarrow 2y \frac{dy}{dt} = 2(90 - x) \left(-\frac{dx}{dt} \right).$$

When $x = 45$, $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so

$$\frac{dy}{dt} = \frac{90 - x}{y} \left(-\frac{dx}{dt} \right) = \frac{45}{45\sqrt{5}} (-24) = -\frac{24}{\sqrt{5}},$$

so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer—and we do.

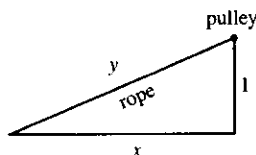
$$z^2 = x^2 + 90^2 \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt}. \text{ When } x = 45, z = 45\sqrt{5}, \text{ so } \frac{dz}{dt} = \frac{45}{45\sqrt{5}} (24) = \frac{24}{\sqrt{5}} \approx 10.7 \text{ ft/s.}$$

15. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\frac{dh}{dt} = 1$ cm/min and $\frac{dA}{dt} = 2$ cm²/min.

Using the Product Rule, we have $\frac{dA}{dt} = \frac{1}{2} \left(b \frac{dh}{dt} + h \frac{db}{dt} \right)$. When $h = 10$ and $A = 100$, we have

$$100 = \frac{1}{2}b(10) \Rightarrow \frac{1}{2}b = 10 \Rightarrow b = 20, \text{ so } 2 = \frac{1}{2} \left(20 \cdot 1 + 10 \frac{db}{dt} \right) \Rightarrow 4 = 20 + 10 \frac{db}{dt} \Rightarrow \frac{db}{dt} = \frac{4 - 20}{10} = -1.6 \text{ cm/min.}$$

16.

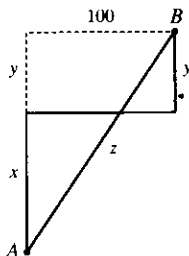


Given $\frac{dy}{dt} = -1$ m/s, find $\frac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \Rightarrow$

$$2y \frac{dy}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt} = -\frac{y}{x}. \text{ When } x = 8, y = \sqrt{65}, \text{ so}$$

$$\frac{dx}{dt} = -\frac{\sqrt{65}}{8}. \text{ Thus, the boat approaches the dock at } \frac{\sqrt{65}}{8} \approx 1.01 \text{ m/s.}$$

17.



We are given that $\frac{dx}{dt} = 35$ km/h and $\frac{dy}{dt} = 25$ km/h. $z^2 = (x + y)^2 + 100^2$

$$\Rightarrow 2z \frac{dz}{dt} = 2(x + y) \left(\frac{dx}{dt} + \frac{dy}{dt} \right). \text{ At 4:00 P.M., } x = 4(35) = 140 \text{ and}$$

$$y = 4(25) = 100 \Rightarrow z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67,600} = 260, \text{ so}$$

$$\frac{dz}{dt} = \frac{x + y}{z} \left(\frac{dx}{dt} + \frac{dy}{dt} \right) = \frac{140 + 100}{260} (35 + 25) = \frac{720}{13} \approx 55.4 \text{ km/h.}$$

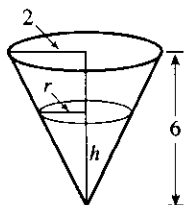
18. Let D denote the distance from the origin $(0, 0)$ to the point on the curve $y = \sqrt{x}$.

$$D = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + (\sqrt{x})^2} = \sqrt{x^2 + x} \Rightarrow$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + x)^{-1/2} (2x + 1) \frac{dx}{dt} = \frac{2x + 1}{2\sqrt{x^2 + x}} \frac{dx}{dt}. \text{ With } \frac{dx}{dt} = 3 \text{ when } x = 4,$$

$$\frac{dD}{dt} = \frac{9}{2\sqrt{20}} (3) = \frac{27}{4\sqrt{5}} \approx 3.02 \text{ cm/s.}$$

19.



If $C =$ the rate at which water is pumped in, then $\frac{dV}{dt} = C - 10,000$, where

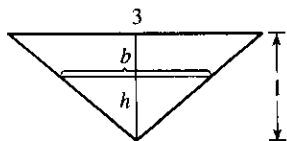
$$V = \frac{1}{3}\pi r^2 h \text{ is the volume at time } t. \text{ By similar triangles, } \frac{r}{2} = \frac{h}{6} \Rightarrow$$

$$r = \frac{1}{3}h \Rightarrow V = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{9}h^2 \frac{dh}{dt}.$$

$$\text{When } h = 200 \text{ cm, } \frac{dh}{dt} = 20 \text{ cm/min, so } C - 10,000 = \frac{\pi}{9}(200)^2(20) \Rightarrow$$

$$C = 10,000 + \frac{800,000}{9}\pi \approx 289,253 \text{ cm}^3/\text{min.}$$

20.

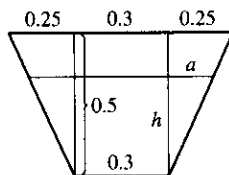


By similar triangles, $\frac{3}{1} = \frac{b}{h}$, so $b = 3h$. The trough has volume

$$V = \frac{1}{2}bh(10) = 5(3h)h = 15h^2 \Rightarrow 12 = \frac{dV}{dt} = 30h \frac{dh}{dt} \Rightarrow$$

$$\frac{dh}{dt} = \frac{2}{5h}. \text{ When } h = \frac{1}{2}, \frac{dh}{dt} = \frac{2}{5 \cdot \frac{1}{2}} = \frac{4}{5} \text{ ft/min.}$$

21.



The figure is labeled in meters. The area A of a trapezoid is

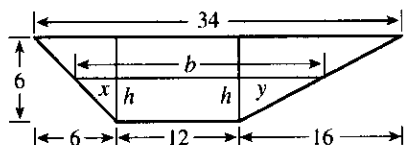
$\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is

$V = (10)\frac{1}{2}[0.3 + (0.3 + 2a)]h$. By similar triangles, $\frac{a}{h} = \frac{0.25}{0.5} = \frac{1}{2}$, so

$$2a = h \Rightarrow V = 5(0.6 + h)h = 3h + 5h^2. \text{ Now } \frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} \Rightarrow 0.2 = (3 + 10h) \frac{dh}{dt} \Rightarrow$$

$$\frac{dh}{dt} = \frac{0.2}{3 + 10h}. \text{ When } h = 0.3, \frac{dh}{dt} = \frac{0.2}{3 + 10(0.3)} = \frac{0.2}{6} \text{ m/min} = \frac{1}{30} \text{ m/min or } \frac{10}{3} \text{ cm/min.}$$

22.



The figure is drawn without the top 3 feet.

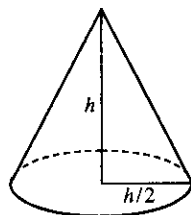
$V = \frac{1}{2}(b + 12)h(20) = 10(b + 12)h$ and, from similar triangles,

$$\frac{x}{h} = \frac{6}{6} \text{ and } \frac{y}{h} = \frac{16}{6} = \frac{8}{3}, \text{ so } b = x + 12 + y = h + 12 + \frac{8h}{3} = 12 + \frac{11h}{3}. \text{ Thus,}$$

$$V = 10\left(24 + \frac{11h}{3}\right)h = 240h + \frac{110h^2}{3} \text{ and so } 0.8 = \frac{dV}{dt} = \left(240 + \frac{220}{3}h\right) \frac{dh}{dt}. \text{ When } h = 5,$$

$$\frac{dh}{dt} = \frac{0.8}{240 + 5(220/3)} = \frac{3}{2275} \approx 0.00132 \text{ ft/min.}$$

23.

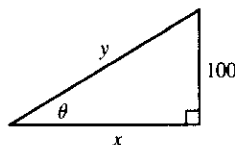


We are given that $\frac{dV}{dt} = 30 \text{ ft}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} \Rightarrow 30 = \frac{\pi h^2}{4} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{120}{\pi h^2}. \text{ When}$$

$$h = 10 \text{ ft, } \frac{dh}{dt} = \frac{120}{10^2 \pi} = \frac{6}{5\pi} \approx 0.38 \text{ ft/min.}$$

24.

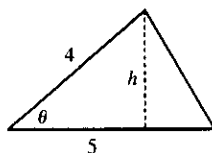


We are given $dx/dt = 8 \text{ ft/s}$. $\cot \theta = \frac{x}{100} \Rightarrow x = 100 \cot \theta \Rightarrow$

$$\frac{dx}{dt} = -100 \csc^2 \theta \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{\sin^2 \theta}{100} \cdot 8. \text{ When } y = 200,$$

$$\sin \theta = \frac{100}{200} = \frac{1}{2} \Rightarrow \frac{d\theta}{dt} = -\frac{(1/2)^2}{100} \cdot 8 = -\frac{1}{50} \text{ rad/s. The angle is decreasing at a rate of } \frac{1}{50} \text{ rad/s.}$$

25.



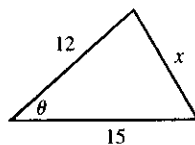
$A = \frac{1}{2}bh$, but $b = 5 \text{ m}$ and $\sin \theta = \frac{h}{4} \Rightarrow h = 4 \sin \theta$, so

$A = \frac{1}{2}(5)(4 \sin \theta) = 10 \sin \theta$. We are given $\frac{d\theta}{dt} = 0.06 \text{ rad/s}$, so

$$\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt} = (10 \cos \theta)(0.06) = 0.6 \cos \theta. \text{ When } \theta = \frac{\pi}{3},$$

$$\frac{dA}{dt} = 0.6\left(\cos \frac{\pi}{3}\right) = (0.6)\left(\frac{1}{2}\right) = 0.3 \text{ m}^2/\text{s}.$$

26.



We are given $d\theta/dt = 2^\circ/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

$$x^2 = 12^2 + 15^2 - 2(12)(15) \cos \theta = 369 - 360 \cos \theta \Rightarrow$$

$$2x \frac{dx}{dt} = 360 \sin \theta \frac{d\theta}{dt} \Rightarrow \frac{dx}{dt} = \frac{180 \sin \theta}{x} \frac{d\theta}{dt}. \text{ When } \theta = 60^\circ,$$

$$x = \sqrt{369 - 360 \cos 60^\circ} = \sqrt{189} = 3\sqrt{21}, \text{ so}$$

$$\frac{dx}{dt} = \frac{180 \sin 60^\circ}{3\sqrt{21}} \frac{\pi}{90} = \frac{\pi \sqrt{3}}{3\sqrt{21}} = \frac{\sqrt{7}\pi}{21} \approx 0.396 \text{ m/min.}$$

27. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P \frac{dV}{dt} + V \frac{dP}{dt} = 0$

$\Rightarrow \frac{dV}{dt} = -\frac{V}{P} \frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of $80 \text{ cm}^3/\text{min}$.

28. $PV^{1.4} = C \Rightarrow P \cdot 1.4V^{0.4} \frac{dV}{dt} + V^{1.4} \frac{dP}{dt} = 0 \Rightarrow \frac{dV}{dt} = -\frac{V^{1.4}}{P \cdot 1.4V^{0.4}} \frac{dP}{dt} = -\frac{V}{1.4P} \frac{dP}{dt}$. When

$V = 400$, $P = 80$ and $\frac{dP}{dt} = -10$, so we have $\frac{dV}{dt} = -\frac{400}{1.4(80)}(-10) = \frac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36 \text{ cm}^3/\text{min}$.

29. With $R_1 = 80$ and $R_2 = 100$, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400}$, so $R = \frac{400}{9}$. Differentiating

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \text{ with respect to } t, \text{ we have } -\frac{1}{R^2} \frac{dR}{dt} = -\frac{1}{R_1^2} \frac{dR_1}{dt} - \frac{1}{R_2^2} \frac{dR_2}{dt} \Rightarrow$$

$$\frac{dR}{dt} = R^2 \left(\frac{1}{R_1^2} \frac{dR_1}{dt} + \frac{1}{R_2^2} \frac{dR_2}{dt} \right). \text{ When } R_1 = 80 \text{ and } R_2 = 100,$$

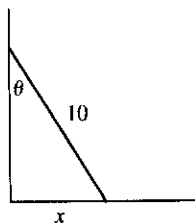
$$\frac{dR}{dt} = \frac{400^2}{9^2} \left[\frac{1}{80^2}(0.3) + \frac{1}{100^2}(0.2) \right] = \frac{107}{810} \approx 0.132 \Omega/\text{s}.$$

30. We want to find $\frac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\frac{dB}{dt} = \frac{dB}{dW} \frac{dW}{dL} \frac{dL}{dt} = \left(0.007 \cdot \frac{2}{3} W^{-1/3} \right) (0.12 \cdot 2.53 \cdot L^{1.53}) \left(\frac{20 - 15}{10,000,000} \right)$$

$$= \left[0.007 \cdot \frac{2}{3} (0.12 \cdot 18^{2.53})^{-1/3} \right] (0.12 \cdot 2.53 \cdot 18^{1.53}) \left(\frac{5}{10^7} \right) \approx 1.045 \times 10^{-8} \text{ g/yr}$$

31.

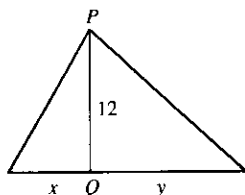


We are given that $\frac{dx}{dt} = 2 \text{ ft/s}$. $\sin \theta = \frac{x}{10} \Rightarrow x = 10 \sin \theta \Rightarrow$

$$\frac{dx}{dt} = 10 \cos \theta \frac{d\theta}{dt}. \text{ When } \theta = \frac{\pi}{4}, 2 = 10 \cos \frac{\pi}{4} \frac{d\theta}{dt} \Rightarrow$$

$$\frac{d\theta}{dt} = \frac{2}{10(1/\sqrt{2})} = \frac{\sqrt{2}}{5} \text{ rad/s.}$$

32.



Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$. Using the Pythagorean Theorem twice, we have

$\sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2} = 39$, the total length of the rope. Differentiating

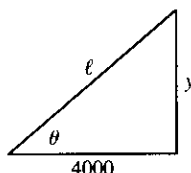
with respect to t , we get $\frac{x}{\sqrt{x^2 + 12^2}} \frac{dx}{dt} + \frac{y}{\sqrt{y^2 + 12^2}} \frac{dy}{dt} = 0$, so

$$\frac{dy}{dt} = -\frac{x \sqrt{y^2 + 12^2}}{y \sqrt{x^2 + 12^2}} \frac{dx}{dt}. \text{ Now when } x = -5, 39 = \sqrt{(-5)^2 + 12^2} + \sqrt{y^2 + 12^2} = 13 + \sqrt{y^2 + 12^2} \Leftrightarrow$$

$$\sqrt{y^2 + 12^2} = 26, \text{ and } y = \sqrt{26^2 - 12^2} = \sqrt{532}. \text{ So when } x = -5,$$

$$\frac{dy}{dt} = -\frac{(-5)(26)}{\sqrt{532}(13)}(-2) = -\frac{10}{\sqrt{133}} \approx -0.87 \text{ ft/s. So cart B is moving towards Q at about 0.87 ft/s.}$$

33. (a)



By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t , we obtain $2y \frac{dy}{dt} = 2\ell \frac{d\ell}{dt}$. We know that $\frac{dy}{dt} = 600$ ft/s, so when

$$y = 3000 \text{ ft}, \ell = \sqrt{4000^2 + 3000^2} = \sqrt{25,000,000} = 5000 \text{ ft and}$$

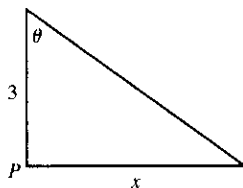
$$\frac{d\ell}{dt} = \frac{y}{\ell} \frac{dy}{dt} = \frac{3000}{5000}(600) = \frac{1800}{5} = 360 \text{ ft/s.}$$

$$(b) \text{ Here } \tan \theta = \frac{y}{4000} \Rightarrow \frac{d}{dt}(\tan \theta) = \frac{d}{dt}\left(\frac{y}{4000}\right) \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{4000} \frac{dy}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\cos^2 \theta}{4000} \frac{dy}{dt}.$$

$$\text{When } y = 3000 \text{ ft, } \frac{dy}{dt} = 600 \text{ ft/s, } \ell = 5000 \text{ and } \cos \theta = \frac{4000}{\ell} = \frac{4000}{5000} = \frac{4}{5}, \text{ so}$$

$$\frac{d\theta}{dt} = \frac{(4/5)^2}{4000}(600) = 0.096 \text{ rad/s.}$$

34.

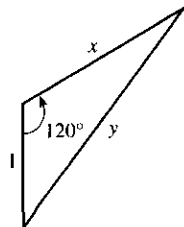


We are given that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3 \tan \theta \Rightarrow$

$$\frac{dx}{dt} = 3 \sec^2 \theta \frac{d\theta}{dt}. \text{ When } x = 1, \tan \theta = \frac{1}{3}, \text{ so } \sec^2 \theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9} \text{ and}$$

$$\frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80\pi}{3} \approx 83.8 \text{ km/min.}$$

35.



We are given that $\frac{dx}{dt} = 300$ km/h. By the Law of Cosines,

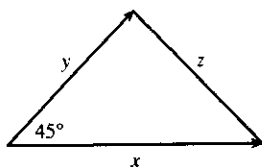
$$y^2 = x^2 + 1^2 - 2(1)(x) \cos 120^\circ = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1, \text{ so}$$

$$2y \frac{dy}{dt} = 2x \frac{dx}{dt} + \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{2x + 1}{2y} \frac{dx}{dt}. \text{ After 1 minute,}$$

$$x = \frac{300}{60} = 5 \text{ km} \Rightarrow y = \sqrt{5^2 + 5 + 1} = \sqrt{31} \text{ km} \Rightarrow$$

$$\frac{dy}{dt} = \frac{2(5) + 1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296 \text{ km/h.}$$

36.



We are given that $\frac{dx}{dt} = 3$ mi/h and $\frac{dy}{dt} = 2$ mi/h. By the Law of Cosines,

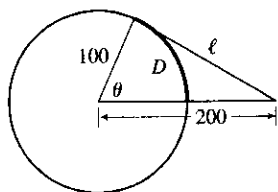
$$z^2 = x^2 + y^2 - 2xy \cos 45^\circ = x^2 + y^2 - \sqrt{2}xy \Rightarrow$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} - \sqrt{2}x \frac{dy}{dt} - \sqrt{2}y \frac{dx}{dt}. \text{ After 15 minutes } [= \frac{1}{4} \text{ h}],$$

$$\text{we have } x = \frac{3}{4} \text{ and } y = \frac{2}{4} = \frac{1}{2} \Rightarrow z^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{1}{2}\right) \Rightarrow z = \frac{\sqrt{13 - 6\sqrt{2}}}{4} \text{ and}$$

$$\frac{dz}{dt} = \frac{2}{\sqrt{13 - 6\sqrt{2}}} \left[2\left(\frac{3}{4}\right)3 + 2\left(\frac{1}{2}\right)2 - \sqrt{2}\left(\frac{3}{4}\right)2 - \sqrt{2}\left(\frac{1}{2}\right)3 \right] = \frac{2}{\sqrt{13 - 6\sqrt{2}}} \frac{13 - 6\sqrt{2}}{2} = \sqrt{13 - 6\sqrt{2}} \\ \approx 2.125 \text{ mi/h.}$$

37.



Let the distance between the runner and the friend be ℓ . Then by the Law of Cosines,

$$\ell^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cdot \cos \theta = 50,000 - 40,000 \cos \theta (*).$$

Differentiating implicitly with respect to t , we obtain

$$2\ell \frac{d\ell}{dt} = -40,000(-\sin \theta) \frac{d\theta}{dt}. \text{ Now if } D \text{ is the distance run when}$$

the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so

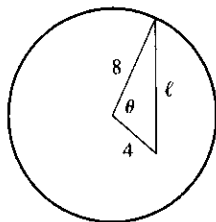
$$\theta = \frac{1}{100}D \Rightarrow \frac{d\theta}{dt} = \frac{1}{100} \frac{dD}{dt} = \frac{7}{100}. \text{ To substitute into the expression for } \frac{d\ell}{dt}, \text{ we must know } \sin \theta \text{ at the time}$$

$$\text{when } \ell = 200, \text{ which we find from } (*): 200^2 = 50,000 - 40,000 \cos \theta \Leftrightarrow \cos \theta = \frac{1}{4} \Rightarrow$$

$$\sin \theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}. \text{ Substituting, we get } 2(200) \frac{d\ell}{dt} = 40,000 \frac{\sqrt{15}}{4} \left(\frac{7}{100}\right) \Rightarrow$$

$$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78 \text{ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.}$$

38.



The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of

$$d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \text{ rad/h. Now, to relate } \theta \text{ to } \ell, \text{ we use the Law of}$$

$$\text{Cosines: } \ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos \theta = 80 - 64 \cos \theta (*).$$

Differentiating implicitly with respect to t , we get $2\ell \frac{d\ell}{dt} = -64(-\sin \theta) \frac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use (*) to find ℓ at 1:00:

$$\ell = \sqrt{80 - 64 \cos \frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}. \text{ Substituting, we get } 2\ell \frac{d\ell}{dt} = 64 \sin \frac{\pi}{6} \left(-\frac{11\pi}{6}\right) \Rightarrow$$

$$\frac{d\ell}{dt} = \frac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6. \text{ So at 1:00, the distance between the tips of the hands is}$$

$$\text{decreasing at a rate of } 18.6 \text{ mm/h} \approx 0.005 \text{ mm/s.}$$

3.10 Linear Approximations and Differentials

1. As in Example 1, $T(0) = 185$, $T(10) = 172$, $T(20) = 160$, and

$$T'(20) \approx \frac{T(10) - T(20)}{10 - 20} = \frac{172 - 160}{-10} = -1.2^\circ\text{F/min.}$$

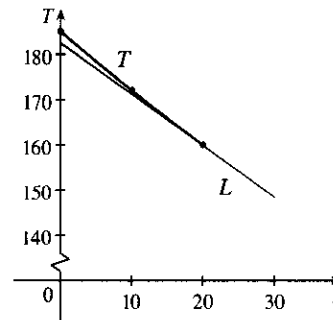
$$T(30) \approx T(20) + T'(20)(30 - 20) \approx 160 - 1.2(10) = 148^\circ\text{F.}$$

We would expect the temperature of the turkey to get closer to 75°F as time increases. Since the temperature decreased 13°F in the first 10 minutes and 12°F in the second 10 minutes, we can assume that the slopes of the tangent line are increasing through negative values:

$-1.3, -1.2, \dots$. Hence, the tangent lines are under the curve and 148°F

is an underestimate. From the figure, we estimate the slope of the tangent line at $t = 20$ to be $\frac{184 - 147}{0 - 30} = -\frac{37}{30}$.

Then the linear approximation becomes $T(30) \approx T(20) + T'(20) \cdot 10 \approx 160 - \frac{37}{30}(10) = 147\frac{2}{3} \approx 147.7$.



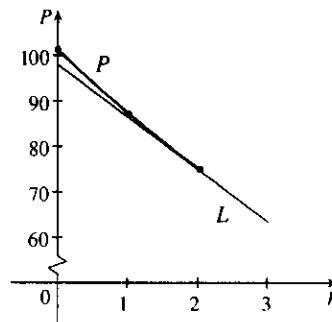
2. $P'(2) \approx \frac{P(1) - P(2)}{1 - 2} = \frac{87.1 - 74.9}{-1} = -12.2$ kilopascals/km.

$$P(3) \approx P(2) + P'(2)(3 - 2) \approx 74.9 - 12.2(1) = 62.7 \text{ kPa.}$$

From the figure, we estimate the slope of the tangent line at $h = 2$ to be

$$\frac{98 - 63}{0 - 3} = -\frac{35}{3}. \text{ Then the linear approximation becomes}$$

$$P(3) \approx P(2) + P'(2) \cdot 1 \approx 74.9 - \frac{35}{3} \approx 63.23 \text{ kPa.}$$



3. Extend the tangent line at the point $(2030, 21)$ to the t -axis.

Answers will vary based on this approximation—we'll use

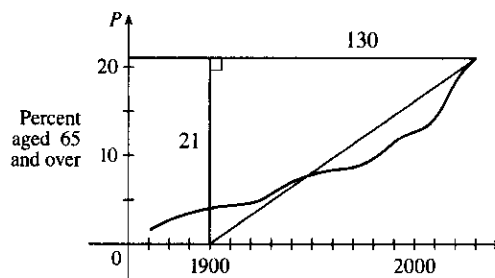
$t = 1900$ as our t -intercept. The linearization is then

$$\begin{aligned} P(t) &\approx P(2030) + P'(2030)(t - 2030) \\ &\approx 21 + \frac{21}{130}(t - 2030) \end{aligned}$$

$$P(2040) = 21 + \frac{21}{130}(2040 - 2030) \approx 22.6\%$$

$$P(2050) = 21 + \frac{21}{130}(2050 - 2030) \approx 24.2\%$$

These predictions are probably too high since the tangent line lies above the graph at $t = 2030$.



4. Let $A = \frac{N(1980) - N(1985)}{1980 - 1985} = \frac{15.0 - 17.0}{-5} = 0.4$ and $B = \frac{N(1990) - N(1985)}{1990 - 1985} = \frac{19.3 - 17.0}{5} = 0.46$.

$$\text{Then } N'(1985) = \lim_{t \rightarrow 1985} \frac{N(t) - N(1985)}{t - 1985} \approx \frac{A + B}{2} = 0.43 \text{ million/year. So}$$

$$N(1984) \approx N(1985) + N'(1985)(1984 - 1985) \approx 17.0 + 0.43(-1) = 16.57 \text{ million.}$$

$$N'(2000) \approx \frac{N(1995) - N(2000)}{1995 - 2000} = \frac{22.0 - 24.9}{-5} = 0.58 \text{ million/year.}$$

$$N(2006) \approx N(2000) + N'(2000)(2006 - 2000) \approx 24.9 + 0.58(6) = 28.38 \text{ million.}$$

5. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$, so $f(1) = 1$ and $f'(1) = 3$. With $a = 1$, $L(x) = f(a) + f'(a)(x - a)$ becomes $L(x) = f(1) + f'(1)(x - 1) = 1 + 3(x - 1) = 3x - 2$.

6. $f(x) = 1/\sqrt{2+x} = (2+x)^{-1/2} \Rightarrow f'(x) = -\frac{1}{2}(2+x)^{-3/2}$ so $f(0) = \frac{1}{\sqrt{2}}$ and $f'(0) = -1/(4\sqrt{2})$. So $L(x) = f(0) + f'(0)(x - 0) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x - 0) = \frac{1}{\sqrt{2}}(1 - \frac{1}{4}x)$.

7. $f(x) = \cos x \Rightarrow f'(x) = -\sin x$, so $f(\frac{\pi}{2}) = 0$ and $f'(\frac{\pi}{2}) = -1$. Thus, $L(x) = f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x - \frac{\pi}{2}) = 0 - 1(x - \frac{\pi}{2}) = -x + \frac{\pi}{2}$.

8. $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$, so $f(-8) = -2$ and $f'(-8) = \frac{1}{12}$. Thus, $L(x) = f(-8) + f'(-8)(x + 8) = -2 + \frac{1}{12}(x + 8) = \frac{1}{12}x - \frac{4}{3}$.

9. $f(x) = \sqrt{1-x} \Rightarrow f'(x) = \frac{-1}{2\sqrt{1-x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$. Therefore,

$$\begin{aligned}\sqrt{1-x} = f(x) &\approx f(0) + f'(0)(x - 0) \\ &= 1 + (-\frac{1}{2})(x - 0) = 1 - \frac{1}{2}x\end{aligned}$$

So $\sqrt{0.9} = \sqrt{1-0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$ and

$\sqrt{0.99} = \sqrt{1-0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

10. $g(x) = \sqrt[3]{1+x} = (1+x)^{1/3} \Rightarrow g'(x) = \frac{1}{3}(1+x)^{-2/3}$, so $g(0) = 1$ and $g'(0) = \frac{1}{3}$.

Therefore, $\sqrt[3]{1+x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$.

So $\sqrt[3]{0.95} = \sqrt[3]{1+(-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\bar{3}$, and

$\sqrt[3]{1.1} = \sqrt[3]{1+0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\bar{3}$.

11. $f(x) = \sqrt[3]{1-x} = (1-x)^{1/3} \Rightarrow f'(x) = -\frac{1}{3}(1-x)^{-2/3}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{3}$. Thus,

$f(x) \approx f(0) + f'(0)(x - 0) = 1 - \frac{1}{3}x$. We need

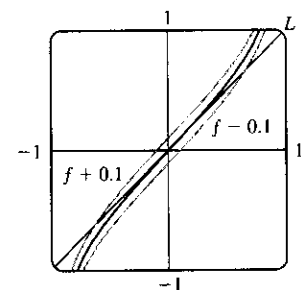
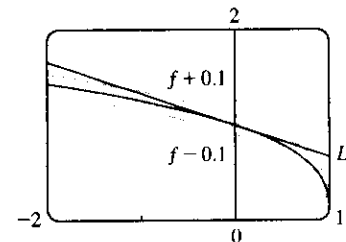
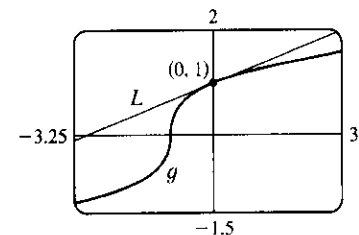
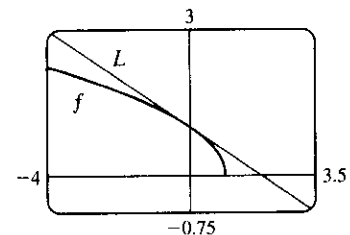
$\sqrt[3]{1-x} - 0.1 < 1 - \frac{1}{3}x < \sqrt[3]{1-x} + 0.1$, which is true when $-1.204 < x < 0.706$.

12. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$, so $f(0) = 0$ and $f'(0) = 1$.

Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1(x - 0) = x$.

We need $\tan x - 0.1 < x < \tan x + 0.1$, which is true when

$-0.63 < x < 0.63$.

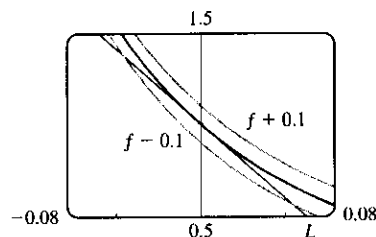


$$13. f(x) = \frac{1}{(1+2x)^4} = (1+2x)^{-4} \Rightarrow$$

$$f'(x) = -4(1+2x)^{-5}(2) = \frac{-8}{(1+2x)^5}, \text{ so } f(0) = 1 \text{ and } f'(0) = -8.$$

$$\text{Thus, } f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x.$$

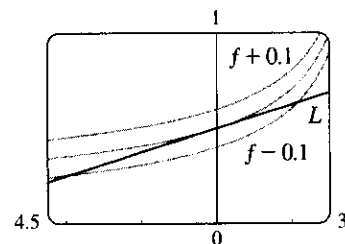
We need $1/(1+2x)^4 - 0.1 < 1 - 8x < 1/(1+2x)^4 + 0.1$, which is true when $-0.045 < x < 0.055$.



$$14. f(x) = \frac{1}{\sqrt{4-x}} \Rightarrow f'(x) = \frac{1}{2(4-x)^{3/2}} \text{ so } f(0) = \frac{1}{2} \text{ and}$$

$$f'(0) = \frac{1}{16}. \text{ So } f(x) \approx \frac{1}{2} + \frac{1}{16}(x-0) = \frac{1}{2} + \frac{1}{16}x. \text{ We need}$$

$$\frac{1}{\sqrt{4-x}} - 0.1 < \frac{1}{2} + \frac{1}{16}x < \frac{1}{\sqrt{4-x}} + 0.1, \text{ which is true when } -3.91 < x < 2.14.$$



$$15. \text{ If } y = f(x), \text{ then the differential } dy \text{ is equal to } f'(x) dx. y = x^4 + 5x \Rightarrow dy = (4x^3 + 5) dx.$$

$$16. y = \cos \pi x \Rightarrow dy = -\sin \pi x \cdot \pi dx = -\pi \sin \pi x dx$$

$$17. y = x^2 \tan x \Rightarrow dy = (x^2 \sec^2 x + \tan x \cdot 2x) dx = (x^2 \sec^2 x + 2x \tan x) dx$$

$$18. y = \sqrt{1+t^2} \Rightarrow dy = \frac{1}{2}(1+t^2)^{-1/2}(2t) dt = \frac{t}{\sqrt{1+t^2}} dt$$

$$19. y = \frac{u+1}{u-1} \Rightarrow dy = \frac{(u-1)(1) - (u+1)(1)}{(u-1)^2} du = \frac{-2}{(u-1)^2} du$$

$$20. y = (1+2r)^{-4} \Rightarrow dy = -4(1+2r)^{-5} \cdot 2 dr = -8(1+2r)^{-5} dr$$

$$21. (a) y = x^2 + 2x \Rightarrow dy = (2x+2) dx$$

$$(b) \text{ When } x = 3 \text{ and } dx = \frac{1}{2}, dy = [2(3) + 2] \left(\frac{1}{2}\right) = 4.$$

$$22. (a) y = x^3 - 6x^2 + 5x - 7 \Rightarrow dy = (3x^2 - 12x + 5) dx$$

$$(b) \text{ When } x = -2 \text{ and } dx = 0.1, dy = (12 + 24 + 5)(0.1) = 4.1.$$

$$23. (a) y = \sqrt{4+5x} \Rightarrow dy = \frac{1}{2}(4+5x)^{-1/2} \cdot 5 dx = \frac{5}{2\sqrt{4+5x}} dx$$

$$(b) \text{ When } x = 0 \text{ and } dx = 0.04, dy = \frac{5}{2\sqrt{4}}(0.04) = \frac{5}{4} \cdot \frac{1}{25} = \frac{1}{20} = 0.05.$$

$$24. (a) y = 1/(x+1) \Rightarrow dy = -\frac{1}{(x+1)^2} dx$$

$$(b) \text{ When } x = 1 \text{ and } dx = -0.01, dy = -\frac{1}{2^2}(-0.01) = \frac{1}{4} \cdot \frac{1}{100} = \frac{1}{400} = 0.0025.$$

$$25. (a) y = \tan x \Rightarrow dy = \sec^2 x dx$$

$$(b) \text{ When } x = \pi/4 \text{ and } dx = -0.1, dy = [\sec(\pi/4)]^2(-0.1) = (\sqrt{2})^2(-0.1) = -0.2.$$

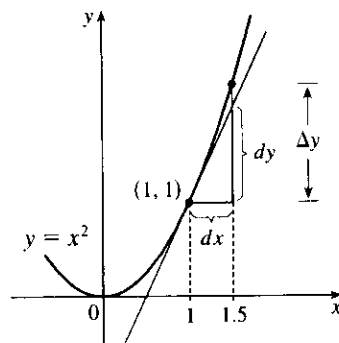
$$26. (a) y = \cos x \Rightarrow dy = -\sin x dx$$

$$(b) \text{ When } x = \pi/3 \text{ and } dx = 0.05, dy = -\sin(\pi/3)(0.05) = -0.5\sqrt{3}(0.05) = -0.025\sqrt{3} \approx -0.043.$$

$$27. y = x^2, x = 1, \Delta x = 0.5 \Rightarrow$$

$$\Delta y = (1.5)^2 - 1^2 = 1.25.$$

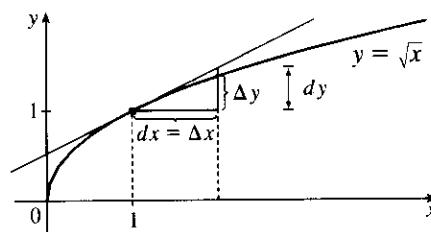
$$dy = 2x dx = 2(1)(0.5) = 1$$



$$28. y = \sqrt{x}, x = 1, \Delta x = 1 \Rightarrow$$

$$\Delta y = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1 \approx 0.414$$

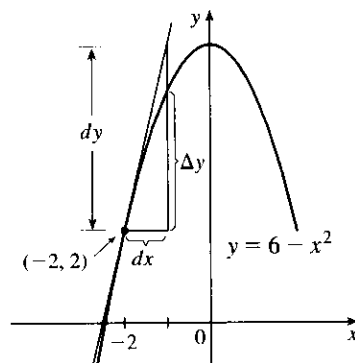
$$dy = \frac{1}{2\sqrt{x}} dx = \frac{1}{2}(1) = 0.5$$



$$29. y = 6 - x^2, x = -2, \Delta x = 0.4 \Rightarrow$$

$$\Delta y = (6 - (-1.6)^2) - (6 - (-2)^2) = 1.44$$

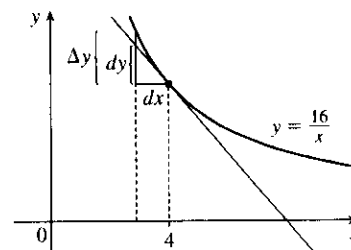
$$dy = -2x dx = -2(-2)(0.4) = 1.6$$



$$30. y = \frac{16}{x}, x = 4, \Delta x = -1 \Rightarrow$$

$$\Delta y = \frac{16}{3} - \frac{16}{4} = \frac{4}{3}.$$

$$dy = -\left(\frac{16}{x^2}\right) dx = -\left(\frac{16}{4^2}\right)(-1) = 1$$



$$31. y = f(x) = x^5 \Rightarrow dy = 5x^4 dx. \text{ When } x = 2 \text{ and } dx = 0.001, dy = 5(2)^4(0.001) = 0.08, \text{ so}$$

$$(2.001)^5 = f(2.001) \approx f(2) + dy = 32 + 0.08 = 32.08.$$

$$32. y = f(x) = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}} dx. \text{ When } x = 100 \text{ and } dx = -0.2, dy = \frac{1}{2\sqrt{100}}(-0.2) = -0.01, \text{ so}$$

$$\sqrt{99.8} = f(99.8) \approx f(100) + dy = 10 - 0.01 = 9.99.$$

$$33. y = f(x) = x^{2/3} \Rightarrow dy = \frac{2}{3\sqrt[3]{x}} dx. \text{ When } x = 8 \text{ and } dx = 0.06, dy = \frac{2}{3\sqrt[3]{8}}(0.06) = 0.02, \text{ so}$$

$$(8.06)^{2/3} = f(8.06) \approx f(8) + dy = 4 + 0.02 = 4.02.$$

$$34. y = f(x) = 1/x \Rightarrow dy = (-1/x^2) dx. \text{ When } x = 1000 \text{ and } dx = 2, dy = [-1/(1000)^2](2) = -0.000002, \text{ so}$$

$$1/1002 = f(1002) \approx f(1000) + dy = 1/1000 - 0.000002 = 0.000998$$

$$35. y = f(x) = \tan x \Rightarrow dy = \sec^2 x dx. \text{ When } x = 45^\circ \text{ and } dx = -1^\circ,$$

$$dy = \sec^2 45^\circ(-\pi/180) = (\sqrt{2})^2(-\pi/180) = -\pi/90, \text{ so}$$

$$\tan 44^\circ = f(44^\circ) \approx f(45^\circ) + dy = 1 - \pi/90 \approx 0.965.$$

$$36. y = f(x) = \cos x \Rightarrow dy = -\sin x dx. \text{ When } x = \frac{\pi}{6} \text{ and } dx = \frac{1.5\pi}{180},$$

$$dy = -\sin \frac{\pi}{6} \left(\frac{1.5\pi}{180}\right) = -\frac{1}{2} \left(\frac{\pi}{120}\right) = -\frac{\pi}{240}, \text{ so } \cos 31.5^\circ = f\left(\frac{31.5\pi}{180}\right) \approx f\left(\frac{\pi}{6}\right) + dy = \frac{\sqrt{3}}{2} - \frac{\pi}{240} \approx 0.853.$$

37. $y = f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.
38. If $y = x^6$, $y' = 6x^5$ and the tangent line approximation at $(1, 1)$ has slope 6. If the change in x is 0.01, the change in y on the tangent line is 0.06, and approximating $(1.01)^6$ with 1.06 is reasonable.
39. (a) If x is the edge length, then $V = x^3 \Rightarrow dV = 3x^2 dx$. When $x = 30$ and $dx = 0.1$,
 $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3 . The relative error is calculated by dividing the change in V , ΔV , by V . We approximate ΔV with dV .
 Relative error $= \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3x^2 dx}{x^3} = 3 \frac{dx}{x} = 3 \left(\frac{0.1}{30} \right) = 0.01$.
 Percentage error $= \text{relative error} \times 100\% = 0.01 \times 100\% = 1\%$.
- (b) $S = 6x^2 \Rightarrow dS = 12x dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2 .
 Relative error $= \frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{12x dx}{6x^2} = 2 \frac{dx}{x} = 2 \left(\frac{0.1}{30} \right) = 0.00\bar{6}$.
 Percentage error $= \text{relative error} \times 100\% = 0.00\bar{6} \times 100\% = 0.\bar{6}\%$.
40. (a) $A = \pi r^2 \Rightarrow dA = 2\pi r dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30 \text{ cm}^2$.
- (b) Relative error $= \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2 dr}{r} = \frac{2(0.2)}{24} = \frac{0.2}{12} = \frac{1}{60} = 0.01\bar{6}$.
 Percentage error $= \text{relative error} \times 100\% = 0.01\bar{6} \times 100\% = 1.\bar{6}\%$.
41. (a) For a sphere of radius r , the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so $r = C/(2\pi) \Rightarrow$
 $S = 4\pi(C/2\pi)^2 = C^2/\pi \Rightarrow dS = (2/\pi)C dC$. When $C = 84$ and $dC = 0.5$, $dS = \frac{2}{\pi}(84)(0.5) = \frac{84}{\pi}$,
 so the maximum error is about $\frac{84}{\pi} \approx 27 \text{ cm}^2$. Relative error $\approx \frac{dS}{S} = \frac{84/\pi}{84^2/\pi} = \frac{1}{84} \approx 0.012$
- (b) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\frac{C}{2\pi} \right)^3 = \frac{C^3}{6\pi^2} \Rightarrow dV = \frac{1}{2\pi^2} C^2 dC$. When $C = 84$ and $dC = 0.5$,
 $dV = \frac{1}{2\pi^2}(84)^2(0.5) = \frac{1764}{\pi^2}$, so the maximum error is about $\frac{1764}{\pi^2} \approx 179 \text{ cm}^3$. The relative error is
 approximately $\frac{dV}{V} = \frac{1764/\pi^2}{(84)^3/(6\pi^2)} = \frac{1}{56} \approx 0.018$.
42. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \Rightarrow dV = 2\pi r^2 dr$. When $r = \frac{1}{2}(50) = 25 \text{ m}$ and
 $dr = 0.05 \text{ cm} = 0.0005 \text{ m}$, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2 \text{ m}^3$.
43. (a) $V = \pi r^2 h \Rightarrow \Delta V \approx dV = 2\pi r h dr = 2\pi r h \Delta r$
- (b) The error is
 $\Delta V - dV = [\pi(r + \Delta r)^2 h - \pi r^2 h] - 2\pi r h \Delta r = \pi r^2 h + 2\pi r h \Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi r h \Delta r$
 $= \pi(\Delta r)^2 h$
44. $F = kR^4 \Rightarrow dF = 4kR^3 dR \Rightarrow \frac{dF}{F} = \frac{4kR^3 dR}{kR^4} = 4 \left(\frac{dR}{R} \right)$. Thus, the relative change in F is about
 4 times the relative change in R . So a 5% increase in the radius corresponds to a 20% increase in blood flow.

45. (a) $dc = \frac{dc}{dx} dx = 0 dx = 0$

(b) $d(cu) = \frac{d}{dx}(cu) dx = c \frac{du}{dx} dx = c du$

(c) $d(u+v) = \frac{d}{dx}(u+v) dx = \left(\frac{du}{dx} + \frac{dv}{dx} \right) dx = \frac{du}{dx} dx + \frac{dv}{dx} dx = du + dv$

(d) $d(uv) = \frac{d}{dx}(uv) dx = \left(u \frac{dv}{dx} + v \frac{du}{dx} \right) dx = u \frac{dv}{dx} dx + v \frac{du}{dx} dx = u dv + v du$

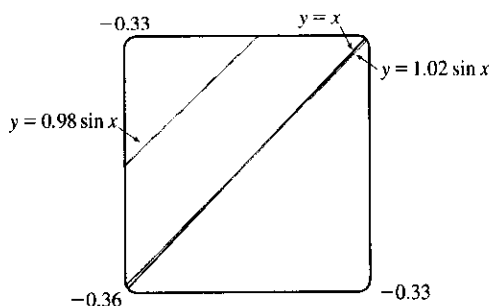
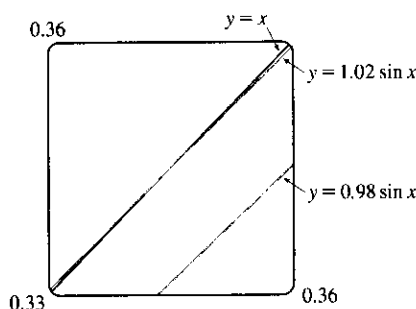
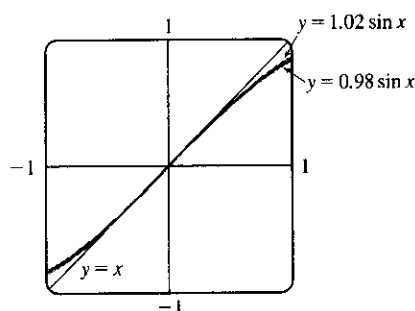
(e) $d\left(\frac{u}{v}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) dx = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} dx = \frac{v \frac{du}{dx} dx - u \frac{dv}{dx} dx}{v^2} = \frac{v du - u dv}{v^2}$

(f) $d(x^n) = \frac{d}{dx}(x^n) dx = nx^{n-1} dx$

46. (a) $f(x) = \sin x \Rightarrow f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

(b)



We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left| \frac{x - \sin x}{\sin x} \right| < 0.02 \Leftrightarrow -0.02 < \frac{x - \sin x}{\sin x} < 0.02 \Leftrightarrow$$

$$\begin{cases} -0.02 \sin x < x - \sin x < 0.02 \sin x & \text{if } \sin x > 0 \\ -0.02 \sin x > x - \sin x > 0.02 \sin x & \text{if } \sin x < 0 \end{cases} \Leftrightarrow \begin{cases} 0.98 \sin x < x < 1.02 \sin x & \text{if } \sin x > 0 \\ 1.02 \sin x < x < 0.98 \sin x & \text{if } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02 \sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure.). Converting 0.344 radians to degrees, we get $0.344 \left(\frac{180^\circ}{\pi} \right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

47. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x - 1) = 5 + 2(x - 1) = 2x + 3$.
 $f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.
- (b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.
48. (a) $g'(x) = \sqrt{x^2 + 5} \Rightarrow g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$.
 $g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85$.
- (b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g . Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

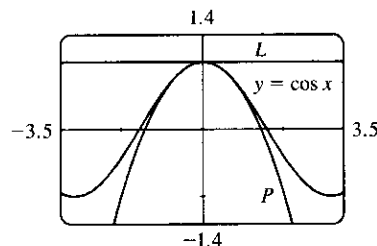
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & f(x) &= \cos x \\ P'(x) &= B + 2Cx & f'(x) &= -\sin x \\ P''(x) &= 2C & f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

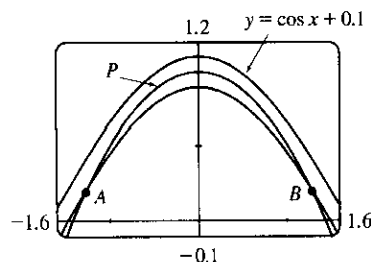
$$\begin{aligned} P(0) &= f(0): & A &= \cos 0 = 1 \\ P'(0) &= f'(0): & B &= -\sin 0 = 0 \\ P''(0) &= f''(0): & 2C &= -\cos 0 = -1 \Rightarrow C = -\frac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $|\cos x - (1 - \frac{1}{2}x^2)| < 0.1 \Leftrightarrow$
 $-0.1 < \cos x - (1 - \frac{1}{2}x^2) < 0.1 \Leftrightarrow 0.1 > (1 - \frac{1}{2}x^2) - \cos x > -0.1 \Leftrightarrow$
 $\cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \Leftrightarrow \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x -coordinates of B and A are about ± 1.26 . Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x - a) + C(x - a)^2$, then $P'(x) = B + 2C(x - a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$P(a) = f(a) : \quad A = f(a)$$

$$P'(a) = f'(a) : \quad B = f'(a)$$

$$P''(a) = f''(a) : \quad 2C = f''(a) \Rightarrow C = \frac{1}{2}f''(a)$$

Thus, $P(x) = A + B(x - a) + C(x - a)^2$ can be written in the form

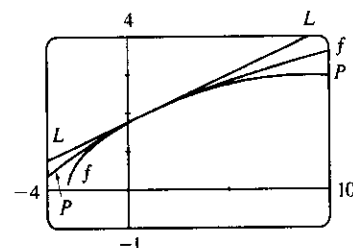
$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

4. From Example 2 in Section 3.10, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and

$$f'(x) = \frac{1}{2}(x + 3)^{-1/2}. \text{ So } f''(x) = -\frac{1}{4}(x + 3)^{-3/2} \Rightarrow$$

$f''(1) = -\frac{1}{32}$. From Problem 3, the quadratic approximation $P(x)$ is

$$\begin{aligned} \sqrt{x+3} &\approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \\ &= 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2 \end{aligned}$$



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain

$$T'_n(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots + nc_n(x - a)^{n-1}. \text{ Substituting } x = a \text{ gives}$$

$$T'_n(a) = c_1. \text{ Differentiating again, we have}$$

$$T''_n(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \cdots + (n-1)nc_n(x - a)^{n-2} \text{ and so } T''_n(a) = 2c_2. \text{ Continuing in this manner, we get}$$

$$T'''_n(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + \cdots + (n-2)(n-1)nc_n(x - a)^{n-3} \text{ and } T'''_n(a) = 2 \cdot 3c_3.$$

By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2 \cdot 3 \cdot 4c_4$ and in general, for any integer k between 1 and n ,

$$T_n^{(k)}(a) = 2 \cdot 3 \cdot 4 \cdot 5 \cdots k c_k = k! c_k \Rightarrow c_k = \frac{T_n^{(k)}(a)}{k!}$$

Because we want T_n and f to have the same derivatives at a , we require that $c_k = \frac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \dots, n$.

6. $T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$\begin{array}{ll} f(x) = \cos x & f(0) = \cos 0 = 1 \\ f'(x) = -\sin x & f'(0) = -\sin 0 = 0 \\ f''(x) = -\cos x & f''(0) = -1 \\ f'''(x) = \sin x & f'''(0) = 0 \\ f^{(4)}(x) = \cos x & f^{(4)}(0) = 1 \end{array}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$. From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

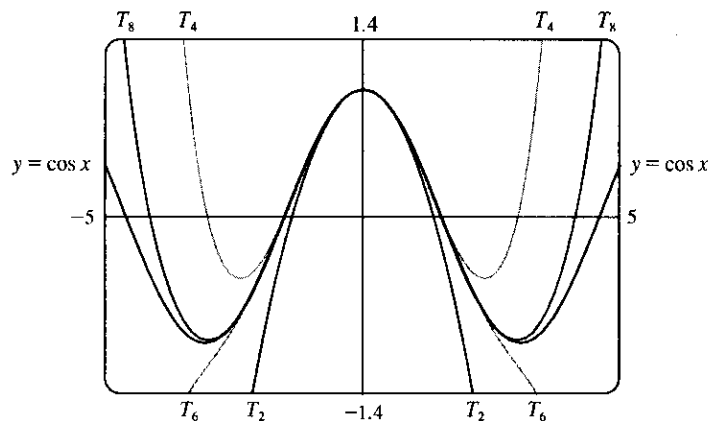
$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2 , T_4 , and T_6

consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$,

$$T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \text{ and } T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

We graph T_2 , T_4 , T_6 , T_8 , and f :



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

3 Review

CONCEPT CHECK

- See Definition 3.1.2 and the subsequent discussions on the interpretations of the derivative as the slope of a tangent and as a rate of change.
- A function f is differentiable at a number a if its derivative f' exists at $x = a$; that is, if $f'(a)$ exists.
 - See Theorem 3.2.4. This theorem also tells us that if f is *not* continuous at a , then f is *not* differentiable at a .
- The Power Rule: If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$. The derivative of a variable base raised to a constant power is the power times the base raised to the power minus one.
 - The Constant Multiple Rule: If c is a constant and f is a differentiable function, then $\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x)$.
The derivative of a constant times a function is the constant times the derivative of the function.
 - The Sum Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. The derivative of a sum of functions is the sum of the derivatives.
 - The Difference Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$. The derivative of a difference of functions is the difference of the derivatives.
 - The Product Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}g(x) + g(x) \frac{d}{dx}f(x)$.
The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.
 - The Quotient Rule: If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \frac{d}{dx}f(x) - f(x) \frac{d}{dx}g(x)}{[g(x)]^2}$.
The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.
 - The Chain Rule: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x))g'(x)$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.
- $y = x^n \Rightarrow y' = nx^{n-1}$
 - $y = \sin x \Rightarrow y' = \cos x$
 - $y = \cos x \Rightarrow y' = -\sin x$
 - $y = \tan x \Rightarrow y' = \sec^2 x$
 - $y = \csc x \Rightarrow y' = -\csc x \cot x$
 - $y = \sec x \Rightarrow y' = \sec x \tan x$
 - $y = \cot x \Rightarrow y' = -\csc^2 x$
- Implicit differentiation consists of differentiating both sides of an equation involving x and y with respect to x , and then solving the resulting equation for y' .
- The second derivative of a function f is the rate of change of the first derivative f' . The third derivative is the derivative (rate of change) of the second derivative. If f is the position function of an object, f' is its velocity function, f'' is its acceleration function, and f''' is its jerk function.
- The linearization L of f at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$.
 - If $y = f(x)$, then the differential dy is given by $dy = f'(x) dx$.
 - See Figure 6 in Section 3.10.

TRUE-FALSE QUIZ

1. False. See the warning after Theorem 3.2.4.
2. True. This is the Sum Rule.
3. False. See the warning before the Product Rule.
4. True. This is the Chain Rule.
5. True by the Chain Rule.
6. False. $\frac{d}{dx} f(\sqrt{x}) = \frac{f'(\sqrt{x})}{2\sqrt{x}}$ by the Chain Rule.
7. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$. So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.
8. True. $f'(r)$ exists $\Rightarrow f$ is differentiable at $r \Rightarrow f$ is continuous at $r \Rightarrow \lim_{x \rightarrow r} f(x) = f(r)$.
9. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative,

$$\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} = g'(2) = 80.$$
10. False. $\frac{d^2 y}{dx^2}$ is the second derivative while $\left(\frac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then

$$\frac{d^2 y}{dx^2} = 0, \text{ but } \left(\frac{dy}{dx}\right)^2 = 1.$$
11. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]
12. True. $\frac{d}{dx} (\tan^2 x) = 2 \tan x \sec^2 x$, and $\frac{d}{dx} (\sec^2 x) = 2 \sec x (\sec x \tan x) = 2 \tan x \sec^2 x$.

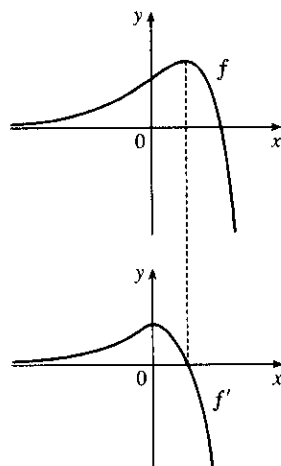
EXERCISES

1. Estimating the slopes of the tangent lines at $x = 2, 3$, and 5 , we obtain approximate values $0.4, 2$, and 0.1 . The slope of the tangent line at $x = 7$ is negative, so $f'(7) < 0$. Arranging the numbers in increasing order, we have:

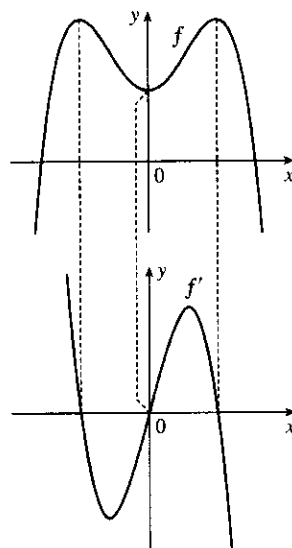
$$f'(7) < 0 < f'(5) < f'(2) < 1 < f'(3).$$
2. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.
3. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).
 (b) The total cost of paying off the loan is increasing by $\$1200/(\text{percent per year})$ as the interest rate reaches 10% . So if the interest rate goes up from 10% to 11% , the cost goes up approximately $\$1200$.
 (c) As r increases, C increases. So $f'(r)$ will always be positive.

For Exercises 4–6, see the hints before Exercise 5 in Section 3.2.

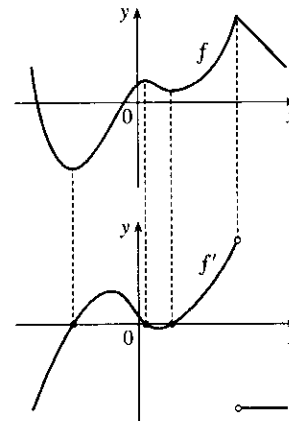
4.



5.



6.



7. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a . The graph of c has horizontal tangent lines to the left and right of the x -axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c . Therefore, a is the graph of f , c is the graph of f' , and b is the graph of f'' .

8. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$,

$$F'(1965) \approx \frac{-1.6}{10} = -0.16, \text{ and } F'(1987) \approx \frac{0.2}{10} = 0.02.$$

- (b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

- (c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

9. $B'(1990)$ is the rate at which the total value of U.S. banknotes in circulation is changing in billions of dollars per year. To estimate the value of $B'(1990)$, we will average the difference quotients obtained using the times $t = 1985$

$$\text{and } t = 1995. \text{ Let } A = \frac{B(1985) - B(1990)}{1985 - 1990} = \frac{182.0 - 268.2}{-5} = 17.24 \text{ and}$$

$$C = \frac{B(1995) - B(1990)}{1995 - 1990} = \frac{401.5 - 268.2}{5} = 26.66. \text{ Then}$$

$$B'(1990) = \lim_{t \rightarrow 1990} \frac{B(t) - B(1990)}{t - 1990} \approx \frac{A + C}{2} = \frac{17.24 + 26.66}{2} = 21.95 \text{ billions of dollars/year.}$$

10. $f(x) = \frac{4-x}{3+x} \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4-(x+h)}{3+(x+h)} - \frac{4-x}{3+x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(4-x-h)(3+x) - (4-x)(3+x+h)}{h(3+x+h)(3+x)} = \lim_{h \rightarrow 0} \frac{-7h}{h(3+x+h)(3+x)} \\ &= \lim_{h \rightarrow 0} \frac{-7}{(3+x+h)(3+x)} = -\frac{7}{(3+x)^2} \end{aligned}$$

11. $f(x) = x^3 + 5x + 4 \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 + 5(x+h) + 4 - (x^3 + 5x + 4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 + 5h}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 + 5) = 3x^2 + 5 \end{aligned}$$

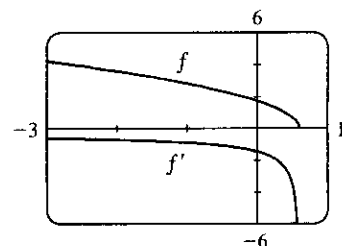
12. (a) $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h} \cdot \frac{\sqrt{3-5(x+h)} + \sqrt{3-5x}}{\sqrt{3-5(x+h)} + \sqrt{3-5x}}$

$$= \lim_{h \rightarrow 0} \frac{[3-5(x+h)] - (3-5x)}{h(\sqrt{3-5(x+h)} + \sqrt{3-5x})} = \lim_{h \rightarrow 0} \frac{-5}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f : (the radicand must be nonnegative) $3-5x \geq 0 \Rightarrow 5x \leq 3 \Rightarrow x \in (-\infty, \frac{3}{5}]$

Domain of f' : exclude $\frac{3}{5}$ because it makes the denominator zero; $x \in (-\infty, \frac{3}{5})$

(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.



13. $y = (x^4 - 3x^2 + 5)^3 \Rightarrow$

$$y' = 3(x^4 - 3x^2 + 5)^2 \cdot \frac{d}{dx}(x^4 - 3x^2 + 5) = 3(x^4 - 3x^2 + 5)^2(4x^3 - 6x) = 6x(x^4 - 3x^2 + 5)^2(2x^2 - 3)$$

14. $y = \cos(\tan x) \Rightarrow y' = -\sin(\tan x) \cdot \frac{d}{dx}(\tan x) = -\sin(\tan x)(\sec^2 x)$

15. $y = \sqrt{x} + \frac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \Rightarrow y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \frac{1}{2\sqrt{x}} - \frac{4}{3\sqrt[3]{x^7}}$

16. $y = \frac{3x-2}{\sqrt{2x+1}} \Rightarrow$

$$y' = \frac{\sqrt{2x+1}(3) - (3x-2)\frac{1}{2}(2x+1)^{-1/2}(2)}{(\sqrt{2x+1})^2} \cdot \frac{(2x+1)^{1/2}}{(2x+1)^{1/2}} = \frac{3(2x+1) - (3x-2)}{(2x+1)^{3/2}} = \frac{3x+5}{(2x+1)^{3/2}}$$

17. $y = 2x\sqrt{x^2+1} \Rightarrow$

$$y' = 2x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x) + \sqrt{x^2+1}(2) = \frac{2x^2}{\sqrt{x^2+1}} + 2\sqrt{x^2+1} = \frac{2x^2 + 2(x^2+1)}{\sqrt{x^2+1}} = \frac{2(2x^2+1)}{\sqrt{x^2+1}}$$

$$18. y = (x + 1/x^2)^{\sqrt{7}} \Rightarrow y' = \sqrt{7}(x + 1/x^2)^{\sqrt{7}-1}(1 - 2/x^3)$$

$$19. y = \frac{t}{1-t^2} \Rightarrow y' = \frac{(1-t^2)(1) - t(-2t)}{(1-t^2)^2} = \frac{1-t^2+2t^2}{(1-t^2)^2} = \frac{t^2+1}{(1-t^2)^2}$$

$$20. y = \sin(\cos x) \Rightarrow y' = \cos(\cos x)(-\sin x) = -\sin x \cos(\cos x)$$

$$21. y = \tan \sqrt{1-x} \Rightarrow y' = (\sec^2 \sqrt{1-x}) \left(\frac{1}{2\sqrt{1-x}} \right) (-1) = -\frac{\sec^2 \sqrt{1-x}}{2\sqrt{1-x}}$$

$$22. \text{Using the Reciprocal Rule, } g(x) = \frac{1}{f(x)} \Rightarrow g'(x) = -\frac{f'(x)}{[f(x)]^2}, \text{ we have } y = \frac{1}{\sin(x - \sin x)} \Rightarrow$$

$$y' = -\frac{\cos(x - \sin x)(1 - \cos x)}{\sin^2(x - \sin x)}.$$

$$23. \frac{d}{dx}(xy^4 + x^2y) = \frac{d}{dx}(x + 3y) \Rightarrow x \cdot 4y^3y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \Rightarrow$$

$$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \Rightarrow y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$$

$$24. y = \sec(1 + x^2) \Rightarrow y' = 2x \sec(1 + x^2) \tan(1 + x^2)$$

$$25. y = \frac{\sec 2\theta}{1 + \tan 2\theta} \Rightarrow$$

$$y' = \frac{(1 + \tan 2\theta)(\sec 2\theta \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta [(1 + \tan 2\theta) \tan 2\theta - \sec^2 2\theta]}{(1 + \tan 2\theta)^2}$$

$$= \frac{2 \sec 2\theta (\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta (\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \quad [1 + \tan^2 x = \sec^2 x]$$

$$26. \frac{d}{dx}(x^2 \cos y + \sin 2y) = \frac{d}{dx}(xy) \Rightarrow x^2(-\sin y \cdot y') + (\cos y)(2x) + \cos 2y \cdot 2y' = x \cdot y' + y \cdot 1 \Rightarrow$$

$$y'(-x^2 \sin y + 2 \cos 2y - x) = y - 2x \cos y \Rightarrow y' = \frac{y - 2x \cos y}{2 \cos 2y - x^2 \sin y - x}$$

$$27. y = (1 - x^{-1})^{-1} \Rightarrow$$

$$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$$

$$28. y = (x + \sqrt{x})^{-1/3} \Rightarrow y' = -\frac{1}{3}(x + \sqrt{x})^{-4/3} \left(1 + \frac{1}{2\sqrt{x}} \right)$$

$$29. \sin(xy) = x^2 - y \Rightarrow \cos(xy)(xy' + y \cdot 1) = 2x - y' \Rightarrow x \cos(xy)y' + y' = 2x - y \cos(xy) \Rightarrow$$

$$y'[x \cos(xy) + 1] = 2x - y \cos(xy) \Rightarrow y' = \frac{2x - y \cos(xy)}{x \cos(xy) + 1}$$

$$30. y = \sqrt{\sin \sqrt{x}} \Rightarrow y' = \frac{1}{2}(\sin \sqrt{x})^{-1/2}(\cos \sqrt{x}) \left(\frac{1}{2\sqrt{x}} \right) = \frac{\cos \sqrt{x}}{4\sqrt{x} \sin \sqrt{x}}$$

$$31. y = \cot(3x^2 + 5) \Rightarrow y' = -\csc^2(3x^2 + 5)(6x) = -6x \csc^2(3x^2 + 5)$$

$$32. y = \frac{(x + \lambda)^4}{x^4 + \lambda^4} \Rightarrow y' = \frac{(x^4 + \lambda^4)(4)(x + \lambda)^3 - (x + \lambda)^4(4x^3)}{(x^4 + \lambda^4)^2} = \frac{4(x + \lambda)^3(\lambda^4 - \lambda x^3)}{(x^4 + \lambda^4)^2}$$

$$33. y = \sin(\tan \sqrt{1 + x^3}) \Rightarrow y' = \cos(\tan \sqrt{1 + x^3})(\sec^2 \sqrt{1 + x^3})[3x^2/(2\sqrt{1 + x^3})]$$

$$34. y = (\sin mx)/x \Rightarrow y' = (mx \cos mx - \sin mx)/x^2$$

$$35. y = \tan^2(\sin \theta) = [\tan(\sin \theta)]^2 \Rightarrow y' = 2[\tan(\sin \theta)] \cdot \sec^2(\sin \theta) \cdot \cos \theta$$

$$36. x \tan y = y - 1 \Rightarrow \tan y + (x \sec^2 y) y' = y' \Rightarrow y' = \frac{\tan y}{1 - x \sec^2 y}$$

$$37. y = (x \tan x)^{1/5} \Rightarrow y' = \frac{1}{5}(x \tan x)^{-4/5}(\tan x + x \sec^2 x)$$

$$38. y = \frac{(x-1)(x-4)}{(x-2)(x-3)} = \frac{x^2 - 5x + 4}{x^2 - 5x + 6} \Rightarrow$$

$$y' = \frac{(x^2 - 5x + 6)(2x - 5) - (x^2 - 5x + 4)(2x - 5)}{(x^2 - 5x + 6)^2} = \frac{2(2x - 5)}{(x - 2)^2(x - 3)^2}$$

$$39. f(t) = \sqrt{4t+1} \Rightarrow f'(t) = \frac{1}{2}(4t+1)^{-1/2} \cdot 4 = 2(4t+1)^{-1/2} \Rightarrow$$

$$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2} \cdot 4 = -4/(4t+1)^{3/2}, \text{ so } f''(2) = -4/9^{3/2} = -\frac{4}{27}.$$

$$40. g(\theta) = \theta \sin \theta \Rightarrow g'(\theta) = \theta \cos \theta + \sin \theta \cdot 1 \Rightarrow g''(\theta) = \theta(-\sin \theta) + \cos \theta \cdot 1 + \cos \theta = 2 \cos \theta - \theta \sin \theta,$$

$$\text{so } g''(\pi/6) = 2 \cos(\pi/6) - (\pi/6) \sin(\pi/6) = 2(\sqrt{3}/2) - (\pi/6)(1/2) = \sqrt{3} - \pi/12.$$

$$41. x^6 + y^6 = 1 \Rightarrow 6x^5 + 6y^5 y' = 0 \Rightarrow y' = -x^5/y^5 \Rightarrow$$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4 y')}{(y^5)^2} = -\frac{5x^4 y^4 [y - x(-x^5/y^5)]}{y^{10}} = -\frac{5x^4 [(y^6 + x^6)/y^5]}{y^6} = -\frac{5x^4}{y^{11}}$$

$$42. f(x) = (2-x)^{-1} \Rightarrow f'(x) = (2-x)^{-2} \Rightarrow f''(x) = 2(2-x)^{-3} \Rightarrow f'''(x) = 2 \cdot 3(2-x)^{-4} \Rightarrow$$

$$f^{(4)}(x) = 2 \cdot 3 \cdot 4(2-x)^{-5}. \text{ In general, } f^{(n)}(x) = 2 \cdot 3 \cdot 4 \cdots n(2-x)^{-(n+1)} = \frac{n!}{(2-x)^{(n+1)}}.$$

$$43. \lim_{x \rightarrow 0} \frac{\sec x}{1 - \sin x} = \frac{\sec 0}{1 - \sin 0} = \frac{1}{1 - 0} = 1$$

$$44. \lim_{t \rightarrow 0} \frac{t^3}{\tan^3 2t} = \lim_{t \rightarrow 0} \frac{t^3 \cos^3 2t}{\sin^3 2t} = \lim_{t \rightarrow 0} \cos^3 2t \cdot \frac{1}{8 \frac{\sin^3 2t}{(2t)^3}} = \lim_{t \rightarrow 0} \frac{\cos^3 2t}{8 \left(\lim_{t \rightarrow 0} \frac{\sin 2t}{2t} \right)^3} = \frac{1}{8 \cdot 1^3} = \frac{1}{8}$$

$$45. y = 4 \sin^2 x \Rightarrow y' = 4 \cdot 2 \sin x \cos x. \text{ At } (\frac{\pi}{6}, 1), y' = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}, \text{ so an equation of the tangent line is}$$

$$y - 1 = 2\sqrt{3}(x - \frac{\pi}{6}), \text{ or } y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3.$$

$$46. y = \frac{x^2 - 1}{x^2 + 1} \Rightarrow y' = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}. \text{ At } (0, -1), y' = 0, \text{ so an equation of the}$$

$$\text{tangent line is } y + 1 = 0(x - 0), \text{ or } y = -1.$$

$$47. y = \sqrt{1 + 4 \sin x} \Rightarrow y' = \frac{1}{2}(1 + 4 \sin x)^{-1/2} \cdot 4 \cos x = \frac{2 \cos x}{\sqrt{1 + 4 \sin x}}. \text{ At } (0, 1), y' = \frac{2}{\sqrt{1}} = 2, \text{ so an}$$

$$\text{equation of the tangent line is } y - 1 = 2(x - 0), \text{ or } y = 2x + 1.$$

$$48. x^2 + 4xy + y^2 = 13 \Rightarrow 2x + 4(xy' + y \cdot 1) + 2yy' = 0 \Rightarrow x + 2xy' + 2y + yy' = 0 \Rightarrow$$

$$2xy' + yy' = -x - 2y \Rightarrow y'(2x + y) = -x - 2y \Rightarrow y' = \frac{-x - 2y}{2x + y}. \text{ At } (2, 1), y' = \frac{-2 - 2}{4 + 1} = -\frac{4}{5}, \text{ so}$$

$$\text{an equation of the tangent line is } y - 1 = -\frac{4}{5}(x - 2), \text{ or } y = -\frac{4}{5}x + \frac{13}{5}.$$

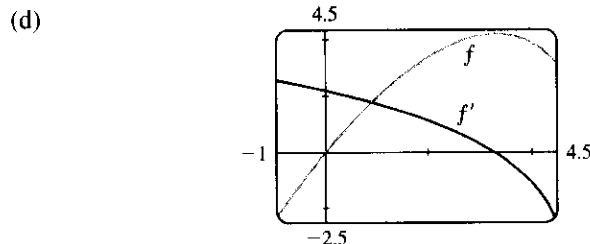
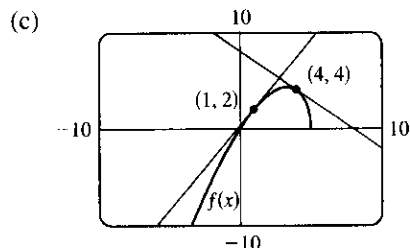
49. (a) $f(x) = x\sqrt{5-x} \Rightarrow$

$$f'(x) = x \left[\frac{1}{2}(5-x)^{-1/2}(-1) \right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x} \cdot \frac{2\sqrt{5-x}}{2\sqrt{5-x}}$$

$$= \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}} = \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}}$$

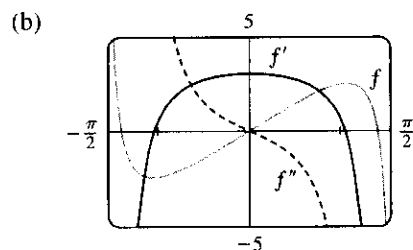
(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$. So an equation of the tangent line is $y - 2 = \frac{7}{4}(x - 1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$. So an equation of the tangent line is $y - 4 = -1(x - 4)$ or $y = -x + 8$.



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

50. (a) $f(x) = 4x - \tan x \Rightarrow f'(x) = 4 - \sec^2 x \Rightarrow f''(x) = -2 \sec x (\sec x \tan x) = -2 \sec^2 x \tan x$.



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f'' .

51. $y = \sin x + \cos x \Rightarrow y' = \cos x - \sin x = 0 \Leftrightarrow \cos x = \sin x$ and $0 \leq x \leq 2\pi \Leftrightarrow x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $(\frac{\pi}{4}, \sqrt{2})$ and $(\frac{5\pi}{4}, -\sqrt{2})$.

52. $x^2 + 2y^2 = 1 \Rightarrow 2x + 4yy' = 0 \Rightarrow y' = -x/(2y) = 1 \Leftrightarrow x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \Rightarrow 6y^2 = 1 \Rightarrow y = \pm \frac{1}{\sqrt{6}}$. The points are $(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ and $(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$.

53. $f(x) = (x-a)(x-b)(x-c) \Rightarrow f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$. So

$$\frac{f'(x)}{f(x)} = \frac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}.$$

54. (a) $\cos 2x = \cos^2 x - \sin^2 x \Rightarrow -2 \sin 2x = -2 \cos x \sin x - 2 \sin x \cos x \Leftrightarrow \sin 2x = 2 \sin x \cos x$

(b) $\sin(x+a) = \sin x \cos a + \cos x \sin a \Rightarrow \cos(x+a) = \cos x \cos a - \sin x \sin a$.

55. (a) $h(x) = f(x)g(x) \Rightarrow h'(x) = f(x)g'(x) + g(x)f'(x) \Rightarrow$

$$h'(2) = f(2)g'(2) + g(2)f'(2) = (3)(4) + (5)(-2) = 12 - 10 = 2$$

(b) $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x))g'(x) \Rightarrow F'(2) = f'(g(2))g'(2) = f'(5)(4) = 11 \cdot 4 = 44$

$$56. (a) P(x) = f(x)g(x) \Rightarrow P'(x) = f(x)g'(x) + g(x)f'(x) \Rightarrow$$

$$P'(2) = f(2)g'(2) + g(2)f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{9-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$$

$$(b) Q(x) = \frac{f(x)}{g(x)} \Rightarrow Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow$$

$$Q'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(4)(-1) - (1)(2)}{4^2} = \frac{-6}{16} = -\frac{3}{8}$$

$$(c) C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))g'(x) \Rightarrow$$

$$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$$

$$57. f(x) = x^2g(x) \Rightarrow f'(x) = x^2g'(x) + g(x)(2x) = x[xg'(x) + 2g(x)]$$

$$58. f(x) = g(x^2) \Rightarrow f'(x) = g'(x^2)(2x) = 2xg'(x^2)$$

$$59. f(x) = [g(x)]^2 \Rightarrow f'(x) = 2[g(x)]^1 \cdot g'(x) = 2g(x)g'(x)$$

$$60. f(x) = x^a g(x^b) \Rightarrow f'(x) = ax^{a-1}g(x^b) + x^a g'(x^b)(bx^{b-1}) = ax^{a-1}g(x^b) + bx^{a+b-1}g'(x^b)$$

$$61. f(x) = g(g(x)) \Rightarrow f'(x) = g'(g(x))g'(x)$$

$$62. f(x) = \sin(g(x)) \Rightarrow f'(x) = \cos(g(x)) \cdot g'(x)$$

$$63. f(x) = g(\sin x) \Rightarrow f'(x) = g'(\sin x) \cdot \cos x$$

$$64. f(x) = g(\tan \sqrt{x}) \Rightarrow$$

$$f'(x) = g'(\tan \sqrt{x}) \cdot \frac{d}{dx}(\tan \sqrt{x}) = g'(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x} \cdot \frac{d}{dx}(\sqrt{x}) = \frac{g'(\tan \sqrt{x}) \sec^2 \sqrt{x}}{2\sqrt{x}}$$

$$65. h(x) = \frac{f(x)g(x)}{f(x) + g(x)} \Rightarrow$$

$$h'(x) = \frac{[f(x) + g(x)][f(x)g'(x) + g(x)f'(x)] - f(x)g(x)[f'(x) + g'(x)]}{[f(x) + g(x)]^2}$$

$$= \frac{[f(x)]^2 g'(x) + f(x)g(x)f'(x) + f(x)g(x)g'(x) + [g(x)]^2 f'(x) - f(x)g(x)f'(x) - f(x)g(x)g'(x)}{[f(x) + g(x)]^2}$$

$$= \frac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x) + g(x)]^2}$$

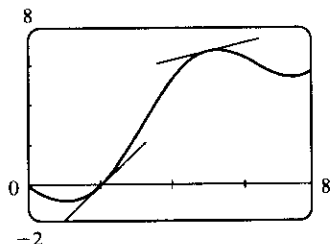
$$66. h(x) = \sqrt{\frac{f(x)}{g(x)}} \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{2\sqrt{f(x)/g(x)}[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$$

$$67. \text{ Using the Chain Rule repeatedly, } h(x) = f(g(\sin 4x)) \Rightarrow$$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}(\sin 4x)$$

$$= f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$$

68. (a)


 (b) The average rate of change is larger on $[2, 3]$.

 (c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

 (d) $f(x) = x - 2 \sin x \Rightarrow f'(x) = 1 - 2 \cos x$, so

$$f'(2) = 1 - 2 \cos 2 \approx 1.8323 \text{ and}$$

$$f'(5) = 1 - 2 \cos 5 \approx 0.4327.$$

 So $f'(2) > f'(5)$, as predicted in part (c).

 69. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

 70. (a) $x = \sqrt{b^2 + c^2 t^2} \Rightarrow v(t) = x' = [1/(2\sqrt{b^2 + c^2 t^2})] 2c^2 t = c^2 t / \sqrt{b^2 + c^2 t^2} \Rightarrow$

$$a(t) = v'(t) = \frac{c^2 \sqrt{b^2 + c^2 t^2} - c^2 t (c^2 t / \sqrt{b^2 + c^2 t^2})}{b^2 + c^2 t^2} = \frac{b^2 c^2}{(b^2 + c^2 t^2)^{3/2}}$$

 (b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

 71. (a) $y = t^3 - 12t + 3 \Rightarrow v(t) = y' = 3t^2 - 12 \Rightarrow a(t) = v'(t) = 6t$

 (b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \leq t < 2$.

 (c) Distance upward = $y(3) - y(2) = -6 - (-13) = 7$,

 Distance downward = $y(0) - y(2) = 3 - (-13) = 16$. Total distance = $7 + 16 = 23$.

 72. (a) $V = \frac{1}{3} \pi r^2 h \Rightarrow dV/dh = \frac{1}{3} \pi r^2$ [r constant]

 (b) $V = \frac{1}{3} \pi r^2 h \Rightarrow dV/dr = \frac{2}{3} \pi r h$ [h constant]

 73. The linear density ρ is the rate of change of mass m with respect to length x . $m = x(1 + \sqrt{x}) = x + x^{3/2} \Rightarrow \rho = dm/dx = 1 + \frac{3}{2} \sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2} \sqrt{4} = 4$ kg/m.

 74. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \Rightarrow C'(x) = 2 - 0.04x + 0.00021x^2$

 (b) $C'(100) = 2 - 4 + 2.1 = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

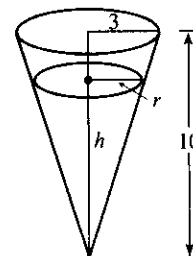
 (c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \0.10107 , slightly larger than $C'(100)$.

 75. If x = edge length, then $V = x^3 \Rightarrow dV/dt = 3x^2 dx/dt = 10 \Rightarrow dx/dt = 10/(3x^2)$ and $S = 6x^2 \Rightarrow dS/dt = (12x) dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm²/min.

 76. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3} \pi r^2 h$ and, from similar

 triangles, $\frac{r}{h} = \frac{3}{10} \Rightarrow V = \frac{\pi}{3} \left(\frac{3h}{10}\right)^2 h = \frac{3\pi}{100} h^3$, so

$$2 = \frac{dV}{dt} = \frac{9\pi}{100} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi(5)^2} = \frac{8}{9\pi} \text{ cm/s}$$

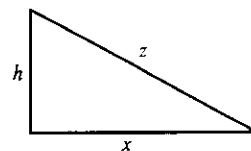
 when $h = 5$.


77. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt . $z^2 = x^2 + h^2 \Rightarrow$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2h \frac{dh}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}(15x + 5h). \text{ When } t = 3,$$

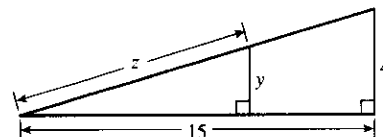
$$h = 45 + 3(5) = 60 \text{ and } x = 15(3) = 45 \Rightarrow z = \sqrt{45^2 + 60^2} = 75, \text{ so}$$

$$\frac{dz}{dt} = \frac{1}{75}[15(45) + 5(60)] = 13 \text{ ft/s.}$$



78. We are given $dz/dt = 30$ ft/s. By similar triangles, $\frac{y}{z} = \frac{4}{\sqrt{241}}$

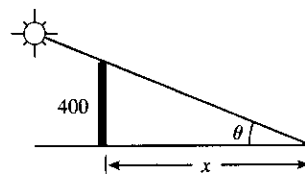
$$\Rightarrow y = \frac{4}{\sqrt{241}}z, \text{ so } \frac{dy}{dt} = \frac{4}{\sqrt{241}} \frac{dz}{dt} = \frac{120}{\sqrt{241}} \approx 7.7 \text{ ft/s.}$$



79. We are given $d\theta/dt = -0.25$ rad/h. $\tan \theta = 400/x \Rightarrow$

$$x = 400 \cot \theta \Rightarrow \frac{dx}{dt} = -400 \csc^2 \theta \frac{d\theta}{dt}. \text{ When } \theta = \frac{\pi}{6},$$

$$\frac{dx}{dt} = -400(2)^2(-0.25) = 400 \text{ ft/h.}$$

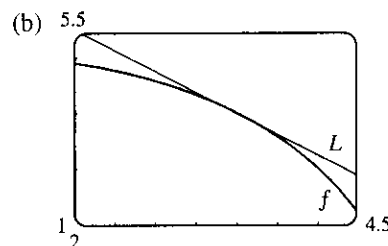


80. (a) $f(x) = \sqrt{25 - x^2} \Rightarrow$

$$f'(x) = \frac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}. \text{ So the linear}$$

approximation to $f(x)$ near 3 is

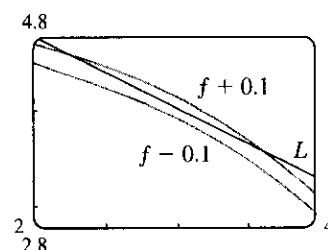
$$f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3).$$



- (c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$

and $4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that

these both hold for $2.24 < x < 3.66$.



81. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \Rightarrow f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is

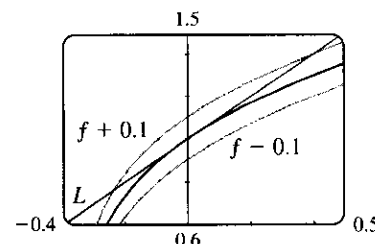
$$L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x. \text{ Thus, } \sqrt[3]{1 + 3x} \approx 1 + x \Rightarrow$$

$$\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01.$$

- (b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required

accuracy we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From

the graph, it appears that this is true when $-0.23 < x < 0.40$.

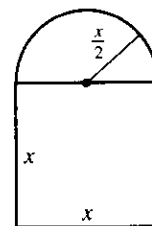


82. $y = x^3 - 2x^2 + 1 \Rightarrow dy = (3x^2 - 4x) dx$. When $x = 2$ and $dx = 0.2$, $dy = [3(2)^2 - 4(2)](0.2) = 0.8$.

83. $A = x^2 + \frac{1}{2}\pi(\frac{1}{2}x)^2 = (1 + \frac{\pi}{8})x^2 \Rightarrow dA = (2 + \frac{\pi}{4})x dx$.

When $x = 60$ and $dx = 0.1$, $dA = (2 + \frac{\pi}{4})60(0.1) = 12 + \frac{3\pi}{2}$, so

the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7 \text{ cm}^2$.



84. $\lim_{x \rightarrow 1} \frac{x^{17} - 1}{x - 1} = \left[\frac{d}{dx} x^{17} \right]_{x=1} = 17(1)^{16} = 17$

85. $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16+h} - 2}{h} = \left[\frac{d}{dx} \sqrt[4]{x} \right]_{x=16} = \frac{1}{4} x^{-3/4} \Big|_{x=16} = \frac{1}{4(\sqrt[4]{16})^3} = \frac{1}{32}$

86. $\lim_{\theta \rightarrow \pi/3} \frac{\cos \theta - 0.5}{\theta - \pi/3} = \left[\frac{d}{d\theta} \cos \theta \right]_{\theta=\pi/3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

87. $\lim_{x \rightarrow 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+\tan x} - \sqrt{1+\sin x})(\sqrt{1+\tan x} + \sqrt{1+\sin x})}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})}$
 $= \lim_{x \rightarrow 0} \frac{(1+\tan x) - (1+\sin x)}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})} = \lim_{x \rightarrow 0} \frac{\sin x(1/\cos x - 1)}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})} \cdot \frac{\cos x}{\cos x}$
 $= \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x}) \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}$
 $= \lim_{x \rightarrow 0} \frac{\sin x \cdot \sin^2 x}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x}) \cos x (1 + \cos x)}$
 $= \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^3 \lim_{x \rightarrow 0} \frac{1}{(\sqrt{1+\tan x} + \sqrt{1+\sin x}) \cos x (1 + \cos x)}$
 $= 1^3 \cdot \frac{1}{(\sqrt{1} + \sqrt{1}) \cdot 1 \cdot (1+1)} = \frac{1}{4}$

88. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain

$f(g(x)) = x \Rightarrow f'(g(x))g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x) = \frac{1}{1 + [f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \frac{1}{1 + x^2}$.

89. $\frac{d}{dx} [f(2x)] = x^2 \Rightarrow f'(2x) \cdot 2 = x^2 \Rightarrow f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}(\frac{1}{2}t)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

90. Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$,

so $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent

line is $y - c = -(c/b)^{1/3}(x - b)$ or $y = -(c/b)^{1/3}x + (c + b^{2/3}c^{1/3})$. Setting $y = 0$, we find that the x -intercept

is $b^{1/3}c^{2/3} + b = b^{1/3}(c^{2/3} + b^{2/3})$ and setting $x = 0$ we find that the y -intercept is

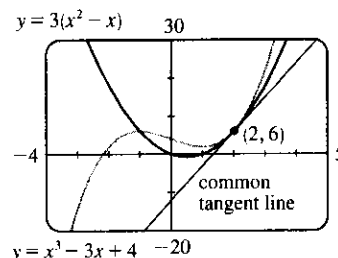
$c + b^{2/3}c^{1/3} = c^{1/3}(c^{2/3} + b^{2/3})$. So the length of the tangent line between these two points is

$$\begin{aligned}\sqrt{[b^{1/3}(c^{2/3} + b^{2/3})]^2 + [c^{1/3}(c^{2/3} + b^{2/3})]^2} &= \sqrt{b^{2/3}(a^{2/3})^2 + c^{2/3}(a^{2/3})^2} \\ &= \sqrt{(b^{2/3} + c^{2/3})a^{4/3}} = \sqrt{a^{2/3}a^{4/3}} \\ &= \sqrt{a^2} = a = \text{constant}\end{aligned}$$

□ PROBLEMS PLUS

1. Let a be the x -coordinate of Q . Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \Rightarrow a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4 \Rightarrow y' = 3x^2 - 3$, and $y = 3(x^2 - x) \Rightarrow y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2 . At $x = 0$, both tangents have slope -3 , but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \Rightarrow f(0) = 2f(0)$. Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

$$(b) f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{[f(0) + f(h) + 0^2h + 0h^2] - f(0)}{h} \\ = \lim_{h \rightarrow 0} \frac{f(h)}{h} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$$

$$(c) f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[f(x) + f(h) + x^2h + xh^2] - f(x)}{h} \\ = \lim_{h \rightarrow 0} \frac{f(h) + x^2h + xh^2}{h} = \lim_{h \rightarrow 0} \left[\frac{f(h)}{h} + x^2 + xh \right] = 1 + x^2$$

4. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2 \Rightarrow a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0) . We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates $(100, 50)$ into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0) \Rightarrow x_0^2 - 200x_0 + 5000 = 0 \Rightarrow x_0 = \frac{1}{2} \left[200 \pm \sqrt{200^2 - 4(5000)} \right] \Rightarrow x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $(100 - 50\sqrt{2}, 150 - 100\sqrt{2}) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

5. We use mathematical induction. Let S_n be the statement that $\frac{d^n}{dx^n} (\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned} \frac{d}{dx} (\sin^4 x + \cos^4 x) &= 4 \sin^3 x \cos x - 4 \cos^3 x \sin x = 4 \sin x \cos x (\sin^2 x - \cos^2 x) \\ &= -4 \sin x \cos x \cos 2x = -2 \sin 2x \cos 2x = -\sin 4x = \sin(-4x) \\ &= \cos\left(\frac{\pi}{2} - (-4x)\right) = \cos\left(\frac{\pi}{2} + 4x\right) = 4^{n-1} \cos(4x + n\frac{\pi}{2}) \quad \text{when } n = 1 \end{aligned}$$

Now assume S_k is true, that is, $\frac{d^k}{dx^k} (\sin^4 x + \cos^4 x) = 4^{k-1} \cos(4x + k\frac{\pi}{2})$. Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}} (\sin^4 x + \cos^4 x) &= \frac{d}{dx} \left[\frac{d^k}{dx^k} (\sin^4 x + \cos^4 x) \right] = \frac{d}{dx} \left[4^{k-1} \cos(4x + k\frac{\pi}{2}) \right] \\ &= -4^{k-1} \sin(4x + k\frac{\pi}{2}) \cdot \frac{d}{dx} (4x + k\frac{\pi}{2}) = -4^k \sin(4x + k\frac{\pi}{2}) \\ &= 4^k \sin(-4x - k\frac{\pi}{2}) = 4^k \cos\left(\frac{\pi}{2} - (-4x - k\frac{\pi}{2})\right) \\ &= 4^k \cos(4x + (k+1)\frac{\pi}{2}) \end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\frac{d^n}{dx^n} (\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\frac{\pi}{2})$ for every positive integer n , by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x = 1 - \frac{1}{2} \sin^2 2x = 1 - \frac{1}{4} (1 - \cos 4x) = \frac{3}{4} + \frac{1}{4} \cos 4x.$$

$$\text{Then we have } \frac{d^n}{dx^n} (\sin^4 x + \cos^4 x) = \frac{d^n}{dx^n} \left(\frac{3}{4} + \frac{1}{4} \cos 4x \right) = \frac{1}{4} \cdot 4^n \cos(4x + n\frac{\pi}{2}) = 4^{n-1} \cos(4x + n\frac{\pi}{2}).$$

6. If we divide $1 - x$ into x^n by long division, we find that $f(x) = \frac{x^n}{1-x} = -x^{n-1} - x^{n-2} - \dots - x - 1 + \frac{1}{1-x}$.

This can also be seen by multiplying the last expression by $1 - x$ and canceling terms on the right-hand side. So we

$$\text{let } g(x) = 1 + x + x^2 + \dots + x^{n-1}, \text{ so that } f(x) = \frac{1}{1-x} - g(x) \Rightarrow f^{(n)}(x) = \left(\frac{1}{1-x} \right)^{(n)} - g^{(n)}(x). \text{ But}$$

g is a polynomial of degree $(n-1)$, so its n th derivative is 0, and therefore $f^{(n)}(x) = \left(\frac{1}{1-x} \right)^{(n)}$. Now

$$\frac{d}{dx} (1-x)^{-1} = (-1)(1-x)^{-2}(-1) = (1-x)^{-2}, \quad \frac{d^2}{dx^2} (1-x)^{-1} = (-2)(1-x)^{-3}(-1) = 2(1-x)^{-3},$$

$$\frac{d^3}{dx^3} (1-x)^{-1} = (-3) \cdot 2(1-x)^{-4}(-1) = 3 \cdot 2(1-x)^{-4}, \quad \frac{d^4}{dx^4} (1-x)^{-1} = 4 \cdot 3 \cdot 2(1-x)^{-5}, \text{ and so on. So}$$

$$\text{after } n \text{ differentiations, we will have } f^{(n)}(x) = \left(\frac{1}{1-x} \right)^{(n)} = \frac{n!}{(1-x)^{n+1}}.$$

7. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit

from the points $(\pm x_0, x_0^2)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\frac{1}{\pm 2x_0}$ and pass through $(\pm x_0, x_0^2)$

respectively, so the normals have the equations $y - x_0^2 = -\frac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \frac{1}{2x_0}(x + x_0)$. The

common y -intercept is $x_0^2 + \frac{1}{2}$. We want to find the value of x_0 for which the distance from $(0, x_0^2 + \frac{1}{2})$ to (x_0, x_0^2) equals 1. The square of the distance is $(x_0 - 0)^2 + [x_0^2 - (x_0^2 + \frac{1}{2})]^2 = x_0^2 + \frac{1}{4} = 1 \Leftrightarrow x_0 = \pm \frac{\sqrt{3}}{2}$. For these values of x_0 , the y -intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $(0, \frac{5}{4})$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$.

Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \Leftrightarrow x^2 + x^4 - 2ax^2 + a^2 = 1 \Leftrightarrow x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \Leftrightarrow 1 - 4a + 4a^2 - 4a^2 + 4 = 0 \Leftrightarrow 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $(0, \frac{5}{4})$.

$$\begin{aligned} 8. \lim_{x \rightarrow a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} &= \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right] = \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (\sqrt{x} + \sqrt{a}) \right] \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (\sqrt{x} + \sqrt{a}) = f'(a) \cdot (\sqrt{a} + \sqrt{a}) = 2\sqrt{a} f'(a) \end{aligned}$$

9. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is $360 \text{ rpm} = 360 \cdot (2\pi \text{ rad}) / (60 \text{ s}) = 12\pi \text{ rad/s}$.] Then the position of A as a function of time is

$$A = (40 \cos \theta, 40 \sin \theta) = (40 \cos 12\pi t, 40 \sin 12\pi t), \text{ so } \sin \alpha = \frac{y}{1.2 \text{ m}} = \frac{40 \sin \theta}{120} = \frac{\sin \theta}{3} = \frac{1}{3} \sin 12\pi t.$$

- (a) Differentiating the expression for $\sin \alpha$, we get $\cos \alpha \cdot \frac{d\alpha}{dt} = \frac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi \cos \theta$.

When $\theta = \frac{\pi}{3}$, we have $\sin \alpha = \frac{1}{3} \sin \theta = \frac{\sqrt{3}}{6}$, so $\cos \alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}}$ and

$$\frac{d\alpha}{dt} = \frac{4\pi \cos \frac{\pi}{3}}{\cos \alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56 \text{ rad/s}.$$

- (b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2|OA||OP|\cos \theta \Rightarrow$

$$120^2 = 40^2 + |OP|^2 - 2 \cdot 40|OP|\cos \theta \Rightarrow |OP|^2 - (80 \cos \theta)|OP| - 12,800 = 0 \Rightarrow$$

$$\begin{aligned} |OP| &= \frac{1}{2} \left(80 \cos \theta \pm \sqrt{6400 \cos^2 \theta + 51,200} \right) = 40 \cos \theta \pm 40 \sqrt{\cos^2 \theta + 8} \\ &= 40 \left(\cos \theta + \sqrt{8 + \cos^2 \theta} \right) \text{ cm} \quad (\text{since } |OP| > 0) \end{aligned}$$

As a check, note that $|OP| = 160 \text{ cm}$ when $\theta = 0$ and $|OP| = 80\sqrt{2} \text{ cm}$ when $\theta = \frac{\pi}{2}$.

- (c) By part (b), the x -coordinate of P is given by $x = 40(\cos \theta + \sqrt{8 + \cos^2 \theta})$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta} \frac{d\theta}{dt} = 40 \left(-\sin \theta - \frac{2 \cos \theta \sin \theta}{2\sqrt{8 + \cos^2 \theta}} \right) \cdot 12\pi = -480\pi \sin \theta \left(1 + \frac{\cos \theta}{\sqrt{8 + \cos^2 \theta}} \right) \text{ cm/s}.$$

In particular, $dx/dt = 0 \text{ cm/s}$ when $\theta = 0$ and $dx/dt = -480\pi \text{ cm/s}$ when $\theta = \frac{\pi}{2}$.

10. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or

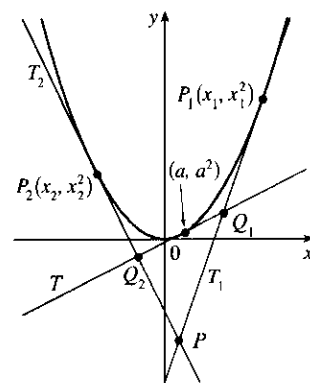
$y = 2x_1x - x_1^2$. The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \Rightarrow x = \frac{1}{2}(x_1 + x_2)$.

Therefore, the coordinates of P are $(\frac{1}{2}(x_1 + x_2), x_1x_2)$. So if the point of contact of T is (a, a^2) , then Q_1 is $(\frac{1}{2}(a + x_1), ax_1)$ and Q_2 is $(\frac{1}{2}(a + x_2), ax_2)$.

Therefore, $|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2(\frac{1}{4} + x_1^2)$ and

$|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2(\frac{1}{4} + x_1^2)$. So $\frac{|PQ_1|^2}{|PP_1|^2} = \frac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly

$\frac{|PQ_2|^2}{|PP_2|^2} = \frac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = \frac{a - x_2}{x_1 - x_2} + \frac{x_1 - a}{x_1 - x_2} = 1$.



11. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \rightarrow \infty$ and $y_T \rightarrow 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \rightarrow 3^+$ and $y_T \rightarrow \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N . We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation

of the tangent line: $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{2x}{9} + \frac{2y}{4}y' = 0$, so $y' = -\frac{4}{9}\frac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an

equation of the tangent line is $y - y_0 = -\frac{4}{9}\frac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as

$\frac{x_0x}{9} + \frac{y_0y}{4} = \frac{x_0^2}{9} + \frac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is

$$\frac{x_0x}{9} + \frac{y_0y}{4} = 1.$$

Therefore, the x -intercept x_T for the tangent line is given by $\frac{x_0x_T}{9} = 1 \Leftrightarrow x_T = \frac{9}{x_0}$, and the y -intercept y_T

is given by $\frac{y_0y_T}{4} = 1 \Leftrightarrow y_T = \frac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is

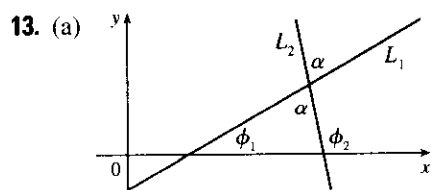
$-\frac{1}{y'(x_0, y_0)} = \frac{9}{4}\frac{y_0}{x_0}$, and its equation is $y - y_0 = \frac{9}{4}\frac{y_0}{x_0}(x - x_0)$. So the x -intercept x_N for the normal line is

given by $0 - y_0 = \frac{9}{4}\frac{y_0}{x_0}(x_N - x_0) \Rightarrow x_N = -\frac{4x_0}{9} + x_0 = \frac{5x_0}{9}$, and the y -intercept y_N is given by

$$y_N - y_0 = \frac{9}{4}\frac{y_0}{x_0}(0 - x_0) \Rightarrow y_N = -\frac{9y_0}{4} + y_0 = -\frac{5y_0}{4}.$$

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $(0, \frac{5}{3})$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $(-\frac{5}{2}, 0)$.

12. $\lim_{x \rightarrow 0} \frac{\sin(3+x)^2 - \sin 9}{x} = f'(3)$ where $f(x) = \sin x^2$. Now $f'(x) = (\cos x^2)(2x)$, so $f'(3) = 6 \cos 9$.



If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2 , then $m_1 = \tan \phi_1$ and $m_2 = \tan \phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180^\circ - \phi_2) = 180^\circ$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x - y)$, we have

$$\tan \alpha = \tan(\phi_2 - \phi_1) = \frac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_2 \tan \phi_1} \text{ and so}$$

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

- (b) (i) The parabolas intersect when $x^2 = (x-2)^2 \Rightarrow x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1, 1)$ is $m_1 = 2(1) = 2$. If $y = (x-2)^2$, then $y' = 2(x-2)$, so the slope of the tangent to $y = (x-2)^2$ at $(1, 1)$ is $m_2 = 2(1-2) = -2$. Therefore,

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-2 - 2}{1 + 2(-2)} = \frac{4}{3} \text{ and so } \alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ \text{ (or } 127^\circ\text{)}.$$

- (ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + (x^2 - 3) + 3 = 0 \Leftrightarrow 2x(x-2) = 0$

$$\Rightarrow x = 0 \text{ or } 2, \text{ but } 0 \text{ is extraneous. If } x = 2, \text{ then } y = \pm 1. \text{ If } x^2 - y^2 = 3 \text{ then } 2x - 2yy' = 0 \Rightarrow$$

$$y' = x/y \text{ and } x^2 - 4x + y^2 + 3 = 0 \Rightarrow 2x - 4 + 2yy' = 0 \Rightarrow y' = \frac{2-x}{y}. \text{ At } (2, 1) \text{ the slopes are}$$

$$m_1 = 2 \text{ and } m_2 = 0, \text{ so } \tan \alpha = \frac{0-2}{1+2 \cdot 0} = -2 \Rightarrow \alpha \approx 117^\circ. \text{ At } (2, -1) \text{ the slopes are } m_1 = -2 \text{ and}$$

$$m_2 = 0, \text{ so } \tan \alpha = \frac{0-(-2)}{1+(-2)(0)} = 2 \Rightarrow \alpha \approx 63^\circ \text{ (or } 117^\circ\text{)}.$$

14. $y^2 = 4px \Rightarrow 2yy' = 4p \Rightarrow y' = 2p/y \Rightarrow$ slope of tangent at $P(x_1, y_1)$ is $m_1 = 2p/y_1$. The slope of FP is $m_2 = \frac{y_1}{x_1 - p}$, so by the formula from Problem 13(a),

$$\begin{aligned} \tan \alpha &= \frac{y_1/(x_1 - p) - 2p/y_1}{1 + (2p/y_1)(y_1/(x_1 - p))} = \frac{y_1(x_1 - p) - 2p(x_1 - p)}{y_1(x_1 - p) + 2py_1} \\ &= \frac{4px_1 - 2px_1 + 2p^2}{x_1y_1 - py_1 + 2py_1} = \frac{2p(p + x_1)}{y_1(p + x_1)} = \frac{2p}{y_1} = \text{slope of tangent at } P = \tan \beta \end{aligned}$$

Since $0 \leq \alpha, \beta \leq \frac{\pi}{2}$, this proves that $\alpha = \beta$.

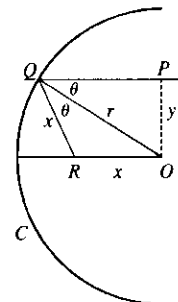
15. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so

$$|QR| = |RO| = x. \text{ By the Law of Cosines, } x^2 = x^2 + r^2 - 2rx \cos \theta. \text{ Hence,}$$

$$2rx \cos \theta = r^2, \text{ so } x = \frac{r^2}{2r \cos \theta} = \frac{r}{2 \cos \theta}. \text{ Note that as } y \rightarrow 0^+, \theta \rightarrow 0^+ \text{ (since}$$

$$\sin \theta = y/r), \text{ and hence } x \rightarrow \frac{r}{2 \cos 0} = \frac{r}{2}. \text{ Thus, as } P \text{ is taken closer and closer}$$

to the x -axis, the point R approaches the midpoint of the radius AO .



$$16. \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{g(x) - g(0)} = \lim_{x \rightarrow 0} \frac{\frac{f(x) - f(0)}{x - 0}}{\frac{g(x) - g(0)}{x - 0}} = \frac{\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}}{\lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0}} = \frac{f'(0)}{g'(0)}$$

$$\begin{aligned}
 17. \lim_{x \rightarrow 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2} &= \lim_{x \rightarrow 0} \frac{\sin a \cos 2x + \cos a \sin 2x - 2\sin a \cos x - 2\cos a \sin x + \sin a}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{\sin a (\cos 2x - 2\cos x + 1) + \cos a (\sin 2x - 2\sin x)}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{\sin a (2\cos^2 x - 1 - 2\cos x + 1) + \cos a (2\sin x \cos x - 2\sin x)}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{\sin a (2\cos x)(\cos x - 1) + \cos a (2\sin x)(\cos x - 1)}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{2(\cos x - 1)[\sin a \cos x + \cos a \sin x](\cos x + 1)}{x^2(\cos x + 1)} \\
 &= \lim_{x \rightarrow 0} \frac{-2\sin^2 x [\sin(a + x)]}{x^2(\cos x + 1)} = -2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \frac{\sin(a + x)}{\cos x + 1} = -2(1)^2 \frac{\sin(a + 0)}{\cos 0 + 1} = -\sin a
 \end{aligned}$$

18. Suppose that $y = mx + c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the

discriminant of the equation $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1 \Leftrightarrow (b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$ must

be 0; that is,

$$\begin{aligned}
 0 &= (2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) \\
 &= 4a^4c^2m^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4c^2m^2 + 4a^4b^2m^2 = 4a^2b^2(a^2m^2 + b^2 - c^2)
 \end{aligned}$$

Therefore, $a^2m^2 + b^2 - c^2 = 0$.

Now if a point (α, β) lies on the line $y = mx + c$, then $c = \beta - m\alpha$, so from above,

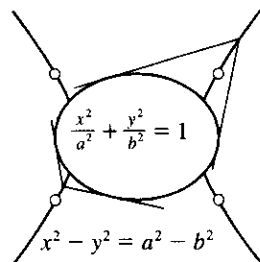
$$0 = a^2m^2 + b^2 - (\beta - m\alpha)^2 = (a^2 - \alpha^2)m^2 + 2\alpha\beta m + b^2 - \beta^2 \Leftrightarrow m^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}m + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0.$$

(a) Suppose that the two tangent lines from the point (α, β) to the ellipse have slopes m and $\frac{1}{m}$. Then m and $\frac{1}{m}$

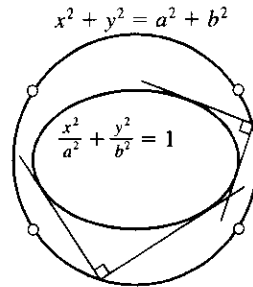
are roots of the equation $z^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}z + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0$. This implies that $(z - m)\left(z - \frac{1}{m}\right) = 0 \Leftrightarrow$

$z^2 - \left(m + \frac{1}{m}\right)z + m\left(\frac{1}{m}\right) = 0$, so equating the constant terms in the two quadratic equations, we get

$\frac{b^2 - \beta^2}{a^2 - \alpha^2} = m\left(\frac{1}{m}\right) = 1$, and hence $b^2 - \beta^2 = a^2 - \alpha^2$. So (α, β) lies on the hyperbola $x^2 - y^2 = a^2 - b^2$.



- (b) If the two tangent lines from the point (α, β) to the ellipse have slopes m and $-\frac{1}{m}$, then m and $-\frac{1}{m}$ are roots of the quadratic equation, and so $(z - m)\left(z + \frac{1}{m}\right) = 0$, and equating the constant terms as in part (a), we get $\frac{b^2 - \beta^2}{a^2 - \alpha^2} = -1$, and hence $b^2 - \beta^2 = \alpha^2 - a^2$. So the point (α, β) lies on the circle $x^2 + y^2 = a^2 + b^2$.



19. $y = x^4 - 2x^2 - x \Rightarrow y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \Rightarrow (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \Rightarrow 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \Rightarrow a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \frac{1}{3a}$. Hence, $a^2 + \frac{1}{9a^2} = \frac{2}{3}$, so $9a^4 + 1 = 6a^2 \Rightarrow 0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \Rightarrow a^2 = \frac{1}{3} \Rightarrow b^2 = \frac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

20. Suppose that the normal lines at the three points (a_1, a_1^2) , (a_2, a_2^2) , and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\frac{1}{2a_i}$ and its equation is $y - a_i^2 = -\frac{1}{2a_i}(x - a_i)$. We solve for the x -coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2) : $y = a_1^2 - \frac{1}{2a_1}(x - a_1) = a_2^2 - \frac{1}{2a_2}(x - a_2) \Rightarrow$

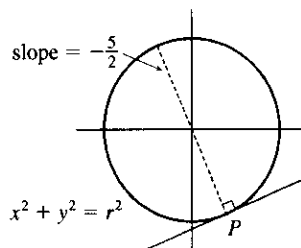
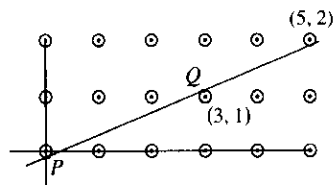
$$x \left(\frac{1}{2a_2} - \frac{1}{2a_1} \right) = a_2^2 - a_1^2 \Rightarrow x \left(\frac{a_1 - a_2}{2a_1 a_2} \right) = (-a_1 - a_2)(a_1 + a_2) \Leftrightarrow x = -2a_1 a_2 (a_1 + a_2) \quad (*).$$

Similarly, solving for the x -coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives

$$x = -2a_1 a_3 (a_1 + a_3) \quad (\dagger).$$

$$\begin{aligned} \text{Equating } (*) \text{ and } (\dagger) \text{ gives } a_2(a_1 + a_2) &= a_3(a_1 + a_3) \Leftrightarrow a_1(a_2 - a_3) = a_3^2 - a_2^2 = -(a_2 + a_3)(a_2 - a_3) \\ \Leftrightarrow a_1 &= -(a_2 + a_3) \Leftrightarrow a_1 + a_2 + a_3 = 0. \end{aligned}$$

21.



Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$. To find P , the point at which the line is tangent to the circle at $(0, 0)$,

$$\text{we simultaneously solve } x^2 + y^2 = r^2 \text{ and } y = -\frac{5}{2}x \Rightarrow x^2 + \frac{25}{4}x^2 = r^2 \Rightarrow x^2 = \frac{4}{29}r^2 \Rightarrow$$

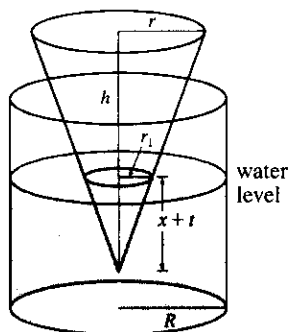
$$x = \frac{2}{\sqrt{29}}r, y = -\frac{5}{\sqrt{29}}r. \text{ To find } Q, \text{ we either use symmetry or solve } (x-3)^2 + (y-1)^2 = r^2 \text{ and}$$

$$y-1 = -\frac{5}{2}(x-3). \text{ As above, we get } x = 3 - \frac{2}{\sqrt{29}}r, y = 1 + \frac{5}{\sqrt{29}}r. \text{ Now the slope of the line } PQ \text{ is } \frac{2}{5}, \text{ so}$$

$$m_{PQ} = \frac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \frac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \frac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \frac{2}{5} \Rightarrow 5\sqrt{29} + 50r = 6\sqrt{29} - 8r \Leftrightarrow$$

$$58r = \sqrt{29} \Leftrightarrow r = \frac{\sqrt{29}}{58}. \text{ So the minimum value of } r \text{ for which any line with slope } \frac{2}{5} \text{ intersects circles with radius } r \text{ centered at the lattice points on the plane is } r = \frac{\sqrt{29}}{58} \approx 0.093.$$

22.



Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x meters, so the tip of the cone is $x + t$ meters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged). Using similar

$$\text{triangles, } \frac{r_1}{x+t} = \frac{r}{h} \Rightarrow r_1 = \frac{r}{h}(x+t).$$

volume of water and cone at time t	=	original volume of water	+	volume of submerged part of cone
$\pi R^2(H+x)$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi r_1^2(x+t)$
$\pi R^2 H + \pi R^2 x$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi \frac{r^2}{h^2}(x+t)^3$
$3h^2 R^2 x$	=	$r^2(x+t)^3$		

Differentiating implicitly with respect to t gives us

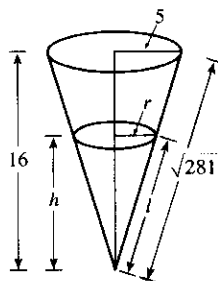
$$3h^2 R^2 \frac{dx}{dt} = r^2 \left[3(x+t)^2 \frac{dx}{dt} + 3(x+t)^2 \frac{dt}{dt} \right] \Rightarrow$$

$$\frac{dx}{dt} = \frac{r^2(x+t)^2}{h^2 R^2 - r^2(x+t)^2} \Rightarrow$$

$$\left. \frac{dx}{dt} \right|_{r+t=h} = \frac{r^2 h^2}{h^2 R^2 - r^2 h^2} = \frac{r^2}{R^2 - r^2}$$

Thus, the water level is rising at a rate of $\frac{r^2}{B^2 - r^2}$ cm/s at the instant the cone is completely submerged.

23.



By similar triangles, $\frac{r}{5} = \frac{h}{16} \Rightarrow r = \frac{5h}{16}$. The volume of the cone is

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{5h}{16}\right)^2 h = \frac{25\pi}{768}h^3$, so $\frac{dV}{dt} = \frac{25\pi}{256}h^2 \frac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2 \text{ cm}^3/\text{min}$) and what is oozing out ($k\pi r l$, where $\pi r l$ is the area of the cone and k is a proportionality constant). Thus, $\frac{dV}{dt} = 2 - k\pi r l$.

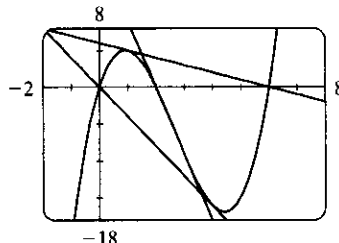
Equating the two expressions for $\frac{dV}{dt}$ and substituting $h = 10$, $\frac{dh}{dt} = -0.3$, $r = \frac{5(10)}{16} = \frac{25}{8}$, and $\frac{l}{\sqrt{281}} = \frac{10}{16}$

$$\Leftrightarrow l = \frac{5}{8}\sqrt{281}, \text{ we get } \frac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi \frac{25}{8} \cdot \frac{5}{8}\sqrt{281} \Leftrightarrow \frac{125k\pi\sqrt{281}}{64} = 2 + \frac{750\pi}{256}. \text{ Solving for } k$$

gives us $k = \frac{256 + 375\pi}{250\pi \sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi r l$, must equal the rate of the liquid

being poured in; that is, $\frac{dV}{dt} = 0$. $k\pi r l = \frac{256 + 375\pi}{250\pi\sqrt{281}} \cdot \pi \cdot \frac{25}{8} \cdot \frac{5\sqrt{281}}{8} = \frac{256 + 375\pi}{128} \approx 11.204 \text{ cm}^3/\text{min}$.

- 24.** (a) $f(x) = x(x-2)(x-6) = x^3 - 8x^2 + 12x \Rightarrow f'(x) = 3x^2 - 16x + 12$. The average of the first pair of zeros is $(0+2)/2 = 1$. At $x = 1$, the slope of the tangent line is $f'(1) = -1$, so an equation of the tangent line has the form $y = -1x + b$. Since $f(1) = 5$, we have $5 = -1 + b \Rightarrow b = 6$ and the tangent has equation $y = -x + 6$. Similarly, at $x = \frac{0+6}{2} = 3$, $y = -9x + 18$; at $x = \frac{2+6}{2} = 4$, $y = -4x$. From the graph, we see that each tangent line drawn at the average of two zeros intersects the graph of f at the third zero.



- (b) A CAS gives $f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$ or

$f'(x) = 3x^2 - 2(a + b + c)x + ab + ac + bc$. Using the Simplify command, we get

$f'\left(\frac{a+b}{2}\right) = -\frac{(a-b)^2}{4}$ and $f\left(\frac{a+b}{2}\right) = -\frac{(a-b)^2}{8}(a+b-2c)$, so an equation of the tangent line at $x = \frac{a+b}{2}$ is

$$y = -\frac{(a-b)^2}{4}\left(x - \frac{a+b}{2}\right) - \frac{(a-b)^2}{8}(a+b-2c)$$

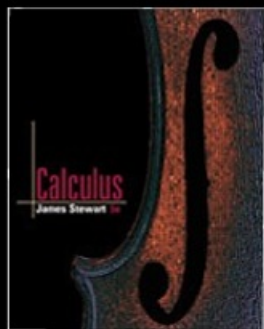
To find the x -intercept, let $y = 0$ and use the Solve command. The result is $x = c$.

Using Derive, we can begin by authoring the expression $(x-a)(x-b)(x-c)$. Now load the utility file Dif_apps. Next we author tangent (#1, $x, (a+b)/2$)—this is the command to find an equation of the tangent line of the function in #1 whose independent variable is x at the x -value $(a+b)/2$. We then simplify that expression and obtain the equation $y = \#3$. The form in expression #3 makes it easy to see that the x -intercept is the third zero, namely c . In a similar fashion we see that b is the x -intercept for the tangent line at $(a+c)/2$ and a is the x -intercept for the tangent line at $(b+c)/2$.

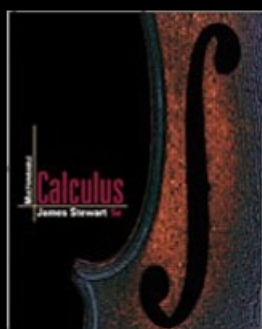
#1:	$(x - a) \cdot (x - b) \cdot (x - c)$	Author the function y=
#2:	$\text{TANGENT}\left[(x - a) \cdot (x - b) \cdot (x - c), x, \frac{a + b}{2}\right]$	Tangent(#1, x, (a+b)/2)
#3:	$\frac{(a^2 - 2 \cdot a \cdot b + b^2) \cdot (c - x)}{4}$	0.0s Simp(#2)

Chapter 4

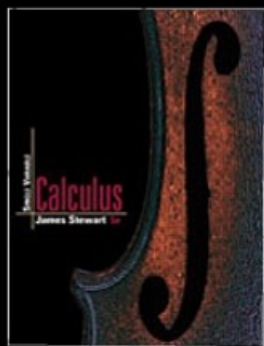
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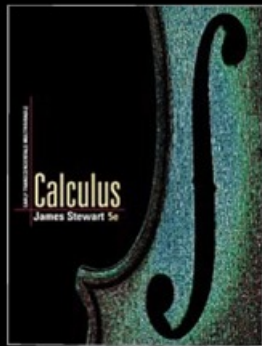
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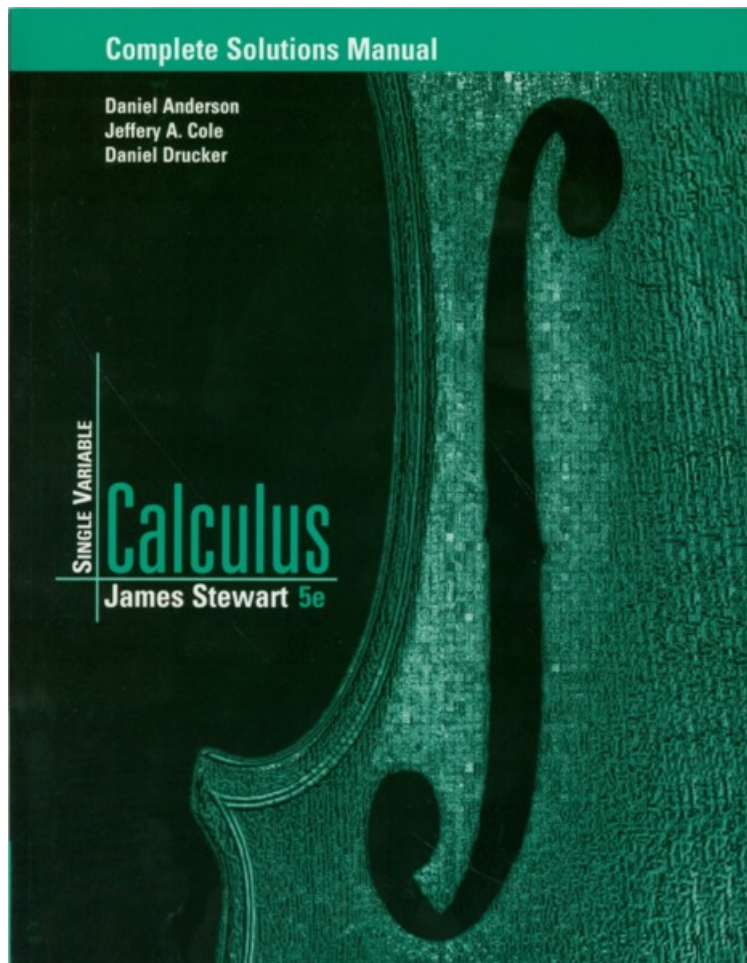
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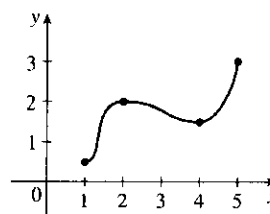
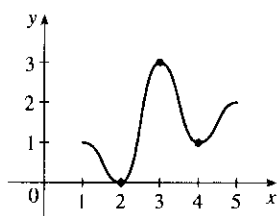
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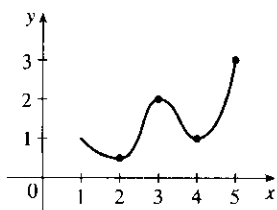
4 APPLICATIONS OF DIFFERENTIATION

4.1 Maximum and Minimum Values

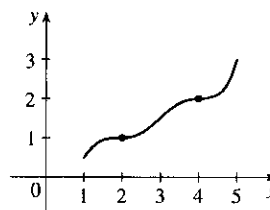
1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f , whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c .
2. (a) The Extreme Value Theorem
(b) See the Closed Interval Method.
3. Absolute maximum at b ; absolute minimum at d ; local maxima at b and e ; local minima at d and s ; neither a maximum nor a minimum at a , c , r , and t .
4. Absolute maximum at e ; absolute minimum at t ; local maxima at c , e , and s ; local minima at b , c , d , and r ; neither a maximum nor a minimum at a .
5. Absolute maximum value is $f(4) = 4$; absolute minimum value is $f(7) = 0$; local maximum values are $f(4) = 4$ and $f(6) = 3$; local minimum values are $f(2) = 1$ and $f(5) = 2$.
6. Absolute maximum value is $f(8) = 5$; absolute minimum value is $f(2) = 0$; local maximum values are $f(1) = 2$, $f(4) = 4$, and $f(6) = 3$; local minimum values are $f(2) = 0$, $f(5) = 2$, and $f(7) = 1$.
7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4
8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4



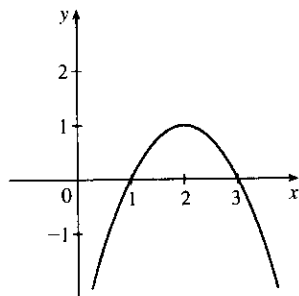
9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



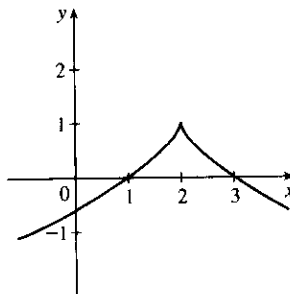
10. f has no local maximum or minimum, but 2 and 4 are critical numbers



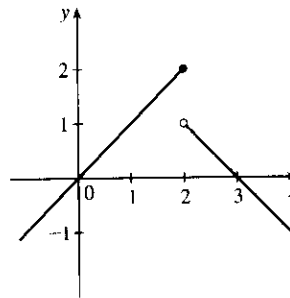
11. (a)



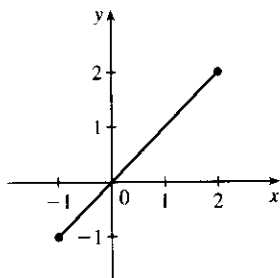
(b)



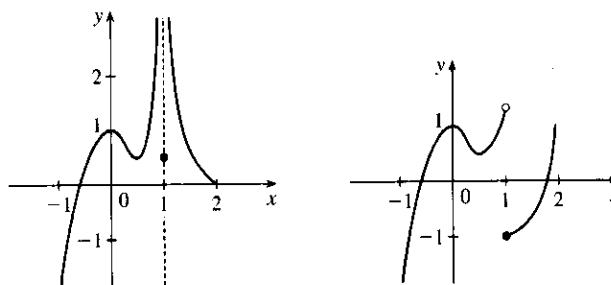
(c)



12. (a) Note that a local maximum cannot occur at an endpoint.

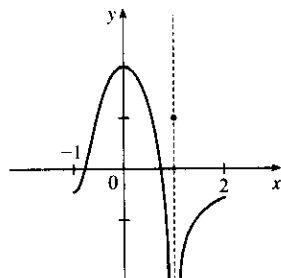


(b)

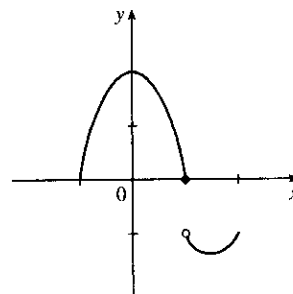


Note: By the Extreme Value Theorem, f must not be continuous.

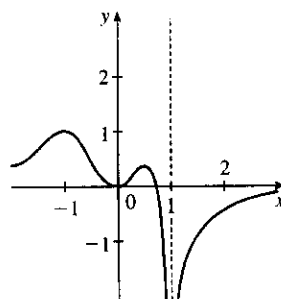
13. (a) Note: By the Extreme Value Theorem, f must not be continuous; because if it were, it would attain an absolute minimum.



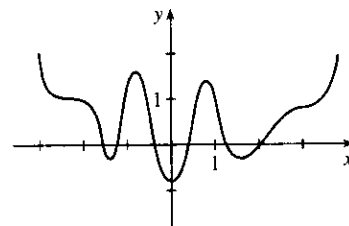
(b)



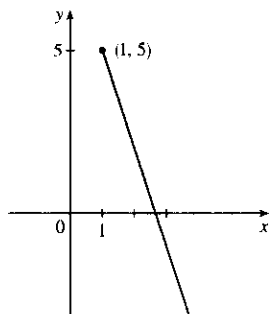
14. (a)



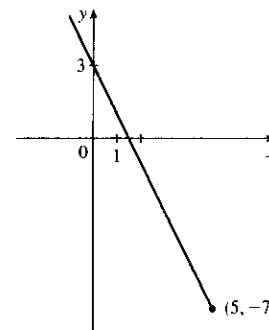
(b)



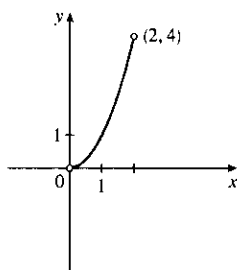
15. $f(x) = 8 - 3x$, $x \geq 1$. Absolute maximum $f(1) = 5$; no local maximum. No absolute or local minimum.



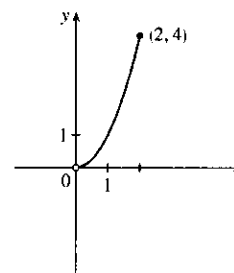
16. $f(x) = 3 - 2x$, $x \leq 5$. Absolute minimum $f(5) = -7$; no local minimum. No absolute or local maximum.



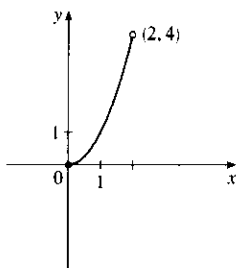
17. $f(x) = x^2$, $0 < x < 2$. No absolute or local maximum or minimum value.



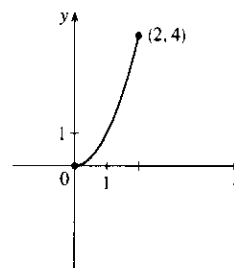
18. $f(x) = x^2$, $0 < x \leq 2$. Absolute maximum $f(2) = 4$; no local maximum. No absolute or local minimum.



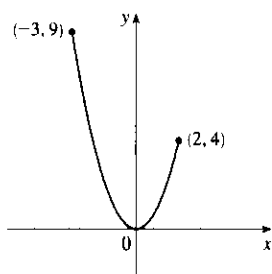
19. $f(x) = x^2$, $0 \leq x < 2$. Absolute minimum $f(0) = 0$; no local minimum. No absolute or local maximum.



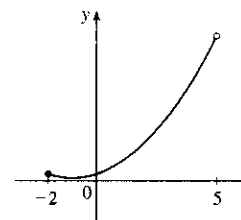
20. $f(x) = x^2$, $0 \leq x \leq 2$. Absolute maximum $f(2) = 4$. Absolute minimum $f(0) = 0$. No local maximum or minimum.



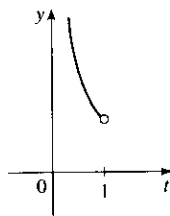
21. $f(x) = x^2$, $-3 \leq x \leq 2$. Absolute maximum $f(-3) = 9$. No local maximum. Absolute and local minimum $f(0) = 0$.



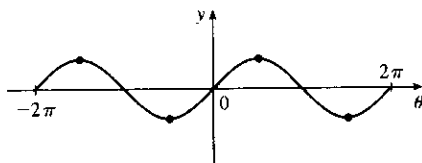
22. $f(x) = 1 + (x + 1)^2$, $-2 \leq x < 5$. No absolute or local maximum. Absolute and local minimum $f(-1) = 1$.



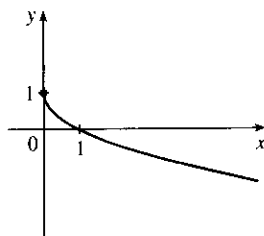
23. $f(t) = 1/t$, $0 < t < 1$. No maximum or minimum.



25. $f(\theta) = \sin \theta$, $-2\pi \leq \theta \leq 2\pi$. Absolute and local maxima $f(-\frac{3\pi}{2}) = f(\frac{\pi}{2}) = 1$. Absolute and local minima $f(-\frac{\pi}{2}) = f(\frac{3\pi}{2}) = -1$.

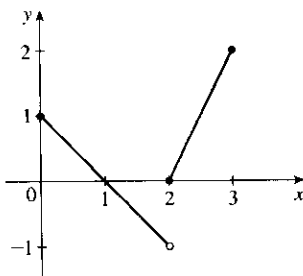


27. $f(x) = 1 - \sqrt{x}$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.

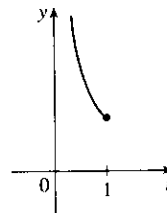


29. $f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 2 \\ 2x - 4 & \text{if } 2 \leq x \leq 3 \end{cases}$

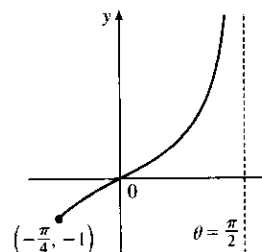
Absolute maximum $f(3) = 2$; no local maximum. No absolute or local minimum.



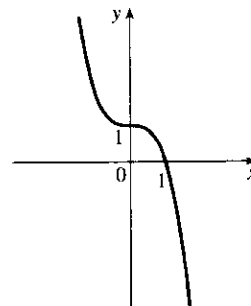
24. $f(t) = 1/t$, $0 < t \leq 1$. Absolute minimum $f(1) = 1$; no local minimum. No local or absolute maximum.



26. $f(\theta) = \tan \theta$, $-\frac{\pi}{4} \leq \theta < \frac{\pi}{2}$. Absolute minimum $f(-\frac{\pi}{4}) = -1$; no local minimum. No absolute or local maximum.

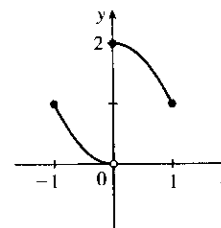


28. $f(x) = 1 - x^3$. No absolute or local extreme values.



30. $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x < 0 \\ 2 - x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$

Absolute and local maximum $f(0) = 2$. No absolute or local minimum.



31. $f(x) = 5x^2 + 4x \Rightarrow f'(x) = 10x + 4$. $f'(x) = 0 \Rightarrow x = -\frac{2}{5}$, so $-\frac{2}{5}$ is the only critical number.
32. $f(x) = x^3 + x^2 - x \Rightarrow f'(x) = 3x^2 + 2x - 1$. $f'(x) = 0 \Rightarrow (x+1)(3x-1) = 0 \Rightarrow x = -1, \frac{1}{3}$.
These are the only critical numbers.
33. $f(x) = x^3 + 3x^2 - 24x \Rightarrow f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8)$.
 $f'(x) = 0 \Rightarrow 3(x+4)(x-2) = 0 \Rightarrow x = -4, 2$. These are the only critical numbers.
34. $f(x) = x^3 + x^2 + x \Rightarrow f'(x) = 3x^2 + 2x + 1$. $f'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0 \Rightarrow$
 $x = \frac{-2 \pm \sqrt{4-12}}{6}$. Neither of these is a real number. Thus, there are no critical numbers.
35. $s(t) = 3t^4 + 4t^3 - 6t^2 \Rightarrow s'(t) = 12t^3 + 12t^2 - 12t$. $s'(t) = 0 \Rightarrow 12t(t^2 + t - 1) \Rightarrow t = 0$ or
 $t^2 + t - 1 = 0$. Using the quadratic formula to solve the latter equation gives us
 $t = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2} \approx 0.618, -1.618$. The three critical numbers are $0, \frac{-1 \pm \sqrt{5}}{2}$.
36. $f(z) = \frac{z+1}{z^2+z+1} \Rightarrow f'(z) = \frac{(z^2+z+1) \cdot 1 - (z+1)(2z+1)}{(z^2+z+1)^2} = \frac{-z^2-2z}{(z^2+z+1)^2} = 0 \Leftrightarrow$
 $z(z+2) = 0 \Rightarrow z = 0, -2$ are the critical numbers. (Note that $z^2+z+1 \neq 0$ since the discriminant < 0 .)
37. $g(x) = |2x+3| = \begin{cases} 2x+3 & \text{if } 2x+3 \geq 0 \\ -(2x+3) & \text{if } 2x+3 < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} 2 & \text{if } x > -\frac{3}{2} \\ -2 & \text{if } x < -\frac{3}{2} \end{cases}$
 $g'(x)$ is never 0, but $g'(x)$ does not exist for $x = -\frac{3}{2}$, so $-\frac{3}{2}$ is the only critical number.
38. $g(x) = x^{1/3} - x^{-2/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x+2) = \frac{x+2}{3x^{5/3}}$.
 $g'(-2) = 0$ and $g'(0)$ does not exist, but 0 is not in the domain of g , so the only critical number is -2 .
39. $g(t) = 5t^{2/3} + t^{5/3} \Rightarrow g'(t) = \frac{10}{3}t^{-1/3} + \frac{5}{3}t^{2/3}$. $g'(0)$ does not exist, so $t = 0$ is a critical number.
 $g'(t) = \frac{5}{3}t^{-1/3}(2+t) = 0 \Leftrightarrow t = -2$, so $t = -2$ is also a critical number.
40. $g(t) = \sqrt{t}(1-t) = t^{1/2} - t^{3/2} \Rightarrow g'(t) = \frac{1}{2\sqrt{t}} - \frac{3}{2}\sqrt{t}$. $g'(0)$ does not exist, so $t = 0$ is a critical number.
 $0 = g'(t) = \frac{1-3t}{2\sqrt{t}} \Rightarrow t = \frac{1}{3}$, so $t = \frac{1}{3}$ is also a critical number.
41. $F(x) = x^{4/5}(x-4)^2 \Rightarrow$
 $F'(x) = x^{4/5} \cdot 2(x-4) + (x-4)^2 \cdot \frac{4}{5}x^{-1/5} = \frac{1}{5}x^{-1/5}(x-4)[5 \cdot x \cdot 2 + (x-4) \cdot 4]$
 $= \frac{(x-4)(14x-16)}{5x^{1/5}} = \frac{2(x-4)(7x-8)}{5x^{1/5}} = 0$ when $x = 4, \frac{8}{7}$; and $F'(0)$ does not exist.
Critical numbers are $0, \frac{8}{7}, 4$.
42. $G(x) = \sqrt[3]{x^2-x} \Rightarrow G'(x) = \frac{1}{3}(x^2-x)^{-2/3}(2x-1)$. $G'(x)$ does not exist when $x^2-x = 0$, that is,
when $x = 0$ or 1 . $G'(x) = 0 \Leftrightarrow 2x-1 = 0 \Leftrightarrow x = \frac{1}{2}$. So the critical numbers are $x = 0, \frac{1}{2}, 1$.
43. $f(\theta) = 2\cos\theta + \sin^2\theta \Rightarrow f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$. $f'(\theta) = 0 \Rightarrow 2\sin\theta(\cos\theta-1) = 0 \Rightarrow$
 $\sin\theta = 0$ or $\cos\theta = 1 \Rightarrow \theta = n\pi$ (n an integer) or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions
 $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.
44. $g(\theta) = 4\theta - \tan\theta \Rightarrow g'(\theta) = 4 - \sec^2\theta$. $g'(\theta) = 0 \Rightarrow \sec^2\theta = 4 \Rightarrow \sec\theta = \pm 2 \Rightarrow$
 $\cos\theta = \pm \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, \text{ and } \frac{4\pi}{3} + 2n\pi$ are critical numbers.
Note: The values of θ that make $g'(\theta)$ undefined are not in the domain of g .

45. $f(x) = 3x^2 - 12x + 5$, $[0, 3]$. $f'(x) = 6x - 12 = 0 \Leftrightarrow x = 2$. Applying the Closed Interval Method, we find that $f(0) = 5$, $f(2) = -7$, and $f(3) = -4$. So $f(0) = 5$ is the absolute maximum value and $f(2) = -7$ is the absolute minimum value.
46. $f(x) = x^3 - 3x + 1$, $[0, 3]$. $f'(x) = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$, but -1 is not in $[0, 3]$. $f(0) = 1$, $f(1) = -1$, and $f(3) = 19$. So $f(3) = 19$ is the absolute maximum value and $f(1) = -1$ is the absolute minimum value.
47. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$. $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1) = 0 \Leftrightarrow x = 2, -1$. $f(-2) = -3$, $f(-1) = 8$, $f(2) = -19$, and $f(3) = -8$. So $f(-1) = 8$ is the absolute maximum value and $f(2) = -19$ is the absolute minimum value.
48. $f(x) = x^3 - 6x^2 + 9x + 2$, $[-1, 4]$. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3) = 0 \Leftrightarrow x = 1, 3$. $f(-1) = -14$, $f(1) = 6$, $f(3) = 2$, and $f(4) = 6$. So $f(1) = f(4) = 6$ is the absolute maximum value and $f(-1) = -14$ is the absolute minimum value.
49. $f(x) = x^4 - 2x^2 + 3$, $[-2, 3]$. $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1) = 0 \Leftrightarrow x = -1, 0, 1$. $f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, $f(3) = 66$. So $f(3) = 66$ is the absolute maximum value and $f(\pm 1) = 2$ is the absolute minimum value.
50. $f(x) = (x^2 - 1)^3$, $[-1, 2]$. $f'(x) = 3(x^2 - 1)^2(2x) = 6x(x + 1)^2(x - 1)^2 = 0 \Leftrightarrow x = -1, 0, 1$. $f(\pm 1) = 0$, $f(0) = -1$, and $f(2) = 27$. So $f(2) = 27$ is the absolute maximum value and $f(0) = -1$ is the absolute minimum value.
51. $f(x) = \frac{x}{x^2 + 1}$, $[0, 2]$. $f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = 0 \Leftrightarrow x = \pm 1$, but -1 is not in $[0, 2]$. $f(0) = 0$, $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$. So $f(1) = \frac{1}{2}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.
52. $f(x) = \frac{x^2 - 4}{x^2 + 4}$, $[-4, 4]$. $f'(x) = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2} = 0 \Leftrightarrow x = 0$. $f(\pm 4) = \frac{12}{20} = \frac{3}{5}$ and $f(0) = -1$. So $f(\pm 4) = \frac{3}{5}$ is the absolute maximum value and $f(0) = -1$ is the absolute minimum value.
53. $f(t) = t\sqrt{4 - t^2}$, $[-1, 2]$.
 $f'(t) = t \cdot \frac{1}{2}(4 - t^2)^{-1/2}(-2t) + (4 - t^2)^{1/2} \cdot 1 = \frac{-t^2}{\sqrt{4 - t^2}} + \sqrt{4 - t^2} = \frac{-t^2 + (4 - t^2)}{\sqrt{4 - t^2}} = \frac{4 - 2t^2}{\sqrt{4 - t^2}}$.
 $f'(t) = 0 \Rightarrow 4 - 2t^2 = 0 \Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$, but $t = -\sqrt{2}$ is not in the given interval, $[-1, 2]$.
 $f'(t)$ does not exist if $4 - t^2 = 0 \Rightarrow t = \pm 2$, but -2 is not in the given interval. $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. So $f(\sqrt{2}) = 2$ is the absolute maximum value and $f(-1) = -\sqrt{3}$ is the absolute minimum value.
54. $f(t) = \sqrt[3]{t}(8 - t)$, $[0, 8]$. $f(t) = 8t^{1/3} - t^{4/3} \Rightarrow f'(t) = \frac{8}{3}t^{-2/3} - \frac{4}{3}t^{1/3} = \frac{4}{3}t^{-2/3}(2 - t) = \frac{4(2 - t)}{3\sqrt[3]{t^2}}$.
 $f'(t) = 0 \Rightarrow t = 2$. $f'(t)$ does not exist if $t = 0$. $f(0) = 0$, $f(2) = 6\sqrt[3]{2} \approx 7.56$, and $f(8) = 0$.
 So $f(2) = 6\sqrt[3]{2}$ is the absolute maximum value and $f(0) = f(8) = 0$ is the absolute minimum value.
55. $f(x) = \sin x + \cos x$, $[0, \frac{\pi}{3}]$. $f'(x) = \cos x - \sin x = 0 \Leftrightarrow \sin x = \cos x \Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$. $f(0) = 1$, $f(\frac{\pi}{4}) = \sqrt{2} \approx 1.41$, $f(\frac{\pi}{3}) = \frac{\sqrt{3}+1}{2} \approx 1.37$. So $f(\frac{\pi}{4}) = \sqrt{2}$ is the absolute maximum value and $f(0) = 1$ is the absolute minimum value.

56. $f(x) = x - 2 \cos x, [-\pi, \pi]$. $f'(x) = 1 + 2 \sin x = 0 \Leftrightarrow \sin x = -\frac{1}{2} \Leftrightarrow x = -\frac{5\pi}{6}, -\frac{\pi}{6}$.

$$f(-\pi) = 2 - \pi \approx -1.14, f(-\frac{5\pi}{6}) = \sqrt{3} - \frac{5\pi}{6} \approx -0.886, f(-\frac{\pi}{6}) = -\frac{\pi}{6} - \sqrt{3} \approx -2.26,$$

$f(\pi) = \pi + 2 \approx 5.14$. So $f(\pi) = \pi + 2$ is the absolute maximum value and $f(-\frac{\pi}{6}) = -\frac{\pi}{6} - \sqrt{3}$ is the absolute minimum value.

57. $f(x) = x^a(1-x)^b, 0 \leq x \leq 1, a > 0, b > 0$.

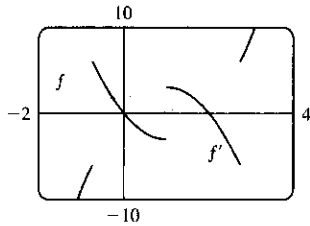
$$\begin{aligned} f'(x) &= x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1} [x \cdot b(-1) + (1-x) \cdot a] \\ &= x^{a-1}(1-x)^{b-1}(a - ax - bx) \end{aligned}$$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of f]. In the interval $(0, 1)$, $f'(x) = 0 \Leftrightarrow x = \frac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a} \left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

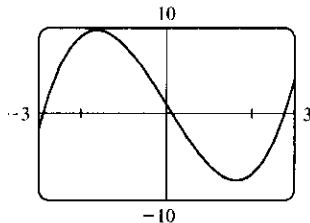
So $f\left(\frac{a}{a+b}\right) = \frac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

58.



We see that $f'(x) = 0$ at about $x = 0.0$ and 2.0 , and that $f'(x)$ does not exist at about $x = -0.7, 1.0$, and 2.7 , so the critical numbers of f are about $-0.7, 0.0, 1.0, 2.0$, and 2.7 .

59. (a)



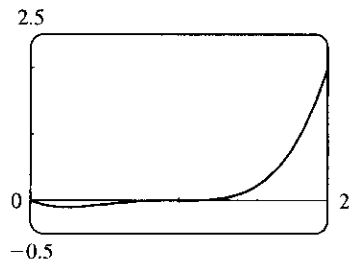
From the graph, it appears that the absolute maximum value is about $f(-1.63) = 9.71$, and the absolute minimum value is about $f(1.63) = -7.71$. These values make sense because the graph is symmetric about the point $(0, 1)$. ($y = x^3 - 8x$ is symmetric about the origin.)

(b) $f(x) = x^3 - 8x + 1 \Rightarrow f'(x) = 3x^2 - 8$. So $f'(x) = 0 \Rightarrow x = \pm\sqrt{\frac{8}{3}}$.

$$\begin{aligned} f\left(\pm\sqrt{\frac{8}{3}}\right) &= \left(\pm\sqrt{\frac{8}{3}}\right)^3 - 8\left(\pm\sqrt{\frac{8}{3}}\right) + 1 = \pm\frac{8}{3}\sqrt{\frac{8}{3}} \mp 8\sqrt{\frac{8}{3}} + 1 \\ &= -\frac{16}{3}\sqrt{\frac{8}{3}} + 1 = 1 - \frac{32\sqrt{6}}{9} \text{ [minimum]} \quad \text{or} \quad \frac{16}{3}\sqrt{\frac{8}{3}} + 1 = 1 + \frac{32\sqrt{6}}{9} \text{ [maximum]} \end{aligned}$$

(From the graph, we see that the extreme values do not occur at the endpoints.)

60. (a)



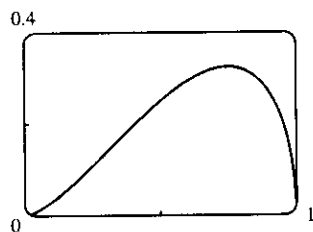
From the graph, it appears that the absolute maximum value is $f(2) = 2$, and that the absolute minimum value is about $f(0.25) = -0.11$.

(b) $f(x) = x^4 - 3x^3 + 3x^2 - x \Rightarrow f'(x) = 4x^3 - 9x^2 + 6x - 1 = (4x-1)(x-1)^2$.

So $f'(x) = 0 \Rightarrow x = \frac{1}{4}$ or $x = 1$. Now $f(1) = 1^4 - 3 \cdot 1^3 + 3 \cdot 1^2 - 1 = 0$ (not an extremum)

and $f(\frac{1}{4}) = (\frac{1}{4})^4 - 3(\frac{1}{4})^3 + 3(\frac{1}{4})^2 - \frac{1}{4} = -\frac{27}{256}$ (minimum). At the right endpoint we have $f(2) = 2^4 - 3 \cdot 2^3 + 3 \cdot 2^2 - 2 = 2$ (maximum).

61. (a)



From the graph, it appears that the absolute maximum value is about

$f(0.75) = 0.32$, and the absolute minimum value is

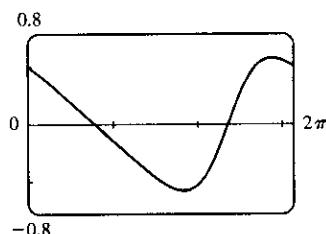
$f(0) = f(1) = 0$; that is, at both endpoints.

$$(b) f(x) = x\sqrt{x-x^2} \Rightarrow f'(x) = x \cdot \frac{1-2x}{2\sqrt{x-x^2}} + \sqrt{x-x^2} = \frac{(x-2x^2) + (2x-2x^2)}{2\sqrt{x-x^2}} = \frac{3x-4x^2}{2\sqrt{x-x^2}}.$$

$$\text{So } f'(x) = 0 \Rightarrow 3x - 4x^2 = 0 \Rightarrow x(3-4x) = 0 \Rightarrow x = 0 \text{ or } \frac{3}{4}. \quad f(0) = f(1) = 0 \text{ [minimum],}$$

$$\text{and } f(\frac{3}{4}) = \frac{3}{4}\sqrt{\frac{3}{4} - (\frac{3}{4})^2} = \frac{3\sqrt{3}}{16} \text{ [maximum].}$$

62. (a)



From the graph, it appears that the absolute maximum value is about

$f(5.76) = 0.58$, and the absolute minimum value is about

$f(3.67) = -0.58$.

$$(b) f(x) = \frac{\cos x}{2 + \sin x} \Rightarrow f'(x) = \frac{(2 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} = \frac{-1 - 2\sin x}{(2 + \sin x)^2}.$$

$$\text{So } f'(x) = 0 \Rightarrow \sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}. \text{ Now } f(\frac{7\pi}{6}) = \frac{-\sqrt{3}/2}{3/2} = -\frac{1}{\sqrt{3}} \text{ [minimum],}$$

$$\text{and } f(\frac{11\pi}{6}) = \frac{\sqrt{3}/2}{3/2} = \frac{1}{\sqrt{3}} \text{ [maximum].}$$

63. The density is defined as $\rho = \frac{\text{mass}}{\text{volume}} = \frac{1000}{V(T)}$ (in g/cm³). But a critical point of ρ will also be a critical point

of V [since $\frac{d\rho}{dT} = -1000V^{-2}\frac{dV}{dT}$ and V is never 0], and V is easier to differentiate than ρ .

$$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \Rightarrow$$

$V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$. Setting this equal to 0 and using the quadratic formula to

$$\text{find } T, \text{ we get } T = \frac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665^\circ\text{C or } 79.5318^\circ\text{C. Since}$$

we are only interested in the region $0^\circ\text{C} \leq T \leq 30^\circ\text{C}$, we check the density ρ at the endpoints and at 3.9665°C :

$$\rho(0) \approx \frac{1000}{999.87} \approx 1.00013; \rho(30) \approx \frac{1000}{1003.7628} \approx 0.99625; \rho(3.9665) \approx \frac{1000}{999.7447} \approx 1.000255. \text{ So water has its maximum density at about } 3.9665^\circ\text{C.}$$

$$64. F = \frac{\mu W}{\mu \sin \theta + \cos \theta} \Rightarrow \frac{dF}{d\theta} = \frac{(\mu \sin \theta + \cos \theta)(0) - \mu W(\mu \cos \theta - \sin \theta)}{(\mu \sin \theta + \cos \theta)^2} = \frac{-\mu W(\mu \cos \theta - \sin \theta)}{(\mu \sin \theta + \cos \theta)^2}.$$

So $\frac{dF}{d\theta} = 0 \Rightarrow \mu \cos \theta - \sin \theta = 0 \Rightarrow \mu = \frac{\sin \theta}{\cos \theta} = \tan \theta$. Substituting $\tan \theta$ for μ in F gives us

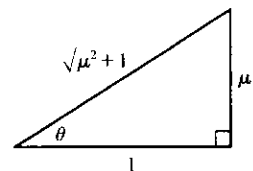
$$F = \frac{(\tan \theta)W}{(\tan \theta) \sin \theta + \cos \theta} = \frac{W \tan \theta}{\frac{\sin^2 \theta}{\cos \theta} + \cos \theta} = \frac{W \tan \theta \cos \theta}{\sin^2 \theta + \cos^2 \theta} = \frac{W \sin \theta}{1} = W \sin \theta.$$

If $\tan \theta = \mu$, then $\sin \theta = \frac{\mu}{\sqrt{\mu^2 + 1}}$ (see the figure), so $F = \frac{\mu}{\sqrt{\mu^2 + 1}}W$. We

compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F(\frac{\pi}{2}) = W$.

Now because $\frac{\mu}{\sqrt{\mu^2 + 1}} \leq 1$ and $\frac{\mu}{\sqrt{\mu^2 + 1}} \leq \mu$, we have that $\frac{\mu}{\sqrt{\mu^2 + 1}}W$

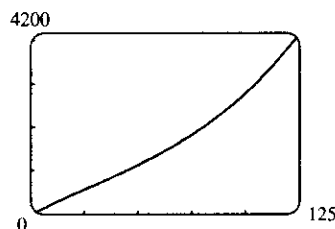
is less than or equal to each of $F(0)$ and $F(\frac{\pi}{2})$. Hence, $\frac{\mu}{\sqrt{\mu^2 + 1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan \theta = \mu$.



65. We apply the Closed Interval Method to the continuous function

$I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$ on $[0, 10]$. Its derivative is $I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683t^2 + 0.9196t - 0.6270$. Since I' exists for all t , the only critical numbers of I occur when $I'(t) = 0$. We use a root-finder on a computer algebra system (or a graphing device) to find that $I'(t) = 0$ when $t \approx -29.7186, 0.8231, 5.1309$, or 11.0459 , but only the second and third roots lie in the interval $[0, 10]$. The values of I at these critical numbers are $I(0.8231) \approx 99.09$ and $I(5.1309) \approx 100.67$. The values of I at the endpoints of the interval are $I(0) = 99.33$ and $I(10) \approx 96.86$. Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t = 10$ (midyear 1994).

66. (a)



The equation of the graph in the figure is

$$v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872.$$

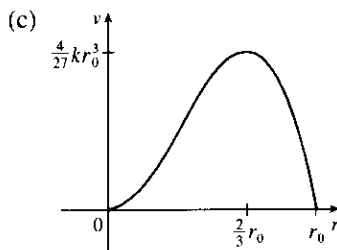
$$(b) a(t) = v'(t) = 0.00438t^2 - 0.23106t + 24.98169 \Rightarrow a'(t) = 0.00876t - 0.23106. a'(t) = 0 \Rightarrow$$

$$t_1 = \frac{0.23106}{0.00876} \approx 26.4. a(0) \approx 24.98, a(t_1) \approx 21.93, \text{ and } a(125) \approx 64.54. \text{ The maximum acceleration is about}$$

64.5 ft/s² and the minimum acceleration is about 21.93 ft/s².

67. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \Rightarrow v'(r) = 2kr_0r - 3kr^2. v'(r) = 0 \Rightarrow kr(2r_0 - 3r) = 0$
 $\Rightarrow r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0, \frac{2}{3}r_0$, and r_0 , we get $v(\frac{1}{2}r_0) = \frac{1}{8}kr_0^3$,
 $v(\frac{2}{3}r_0) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

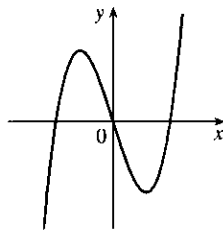
(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.



68. $g(x) = 2 + (x - 5)^3 \Rightarrow g'(x) = 3(x - 5)^2 \Rightarrow g'(5) = 0$, so 5 is a critical number. But $g(5) = 2$ and g takes on values > 2 and values < 2 in any open interval containing 5, so g does not have a local maximum or minimum at 5.
69. $f(x) = x^{101} + x^{51} + x + 1 \Rightarrow f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x , so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.
70. Suppose that f has a minimum value at c , so $f(x) \geq f(c)$ for all x near c . Then $g(x) = -f(x) \leq -f(c) = g(c)$ for all x near c , so $g(x)$ has a maximum value at c .
71. If f has a local minimum at c , then $g(x) = -f(x)$ has a local maximum at c , so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.
72. (a) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic and hence has either 2, 1, or 0 real roots, so $f(x)$ has either 2, 1 or 0 critical numbers.

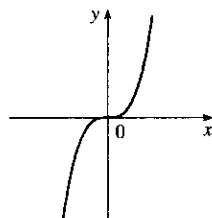
Case (i) (2 critical numbers):

$$f(x) = x^3 - 3x \Rightarrow f'(x) = 3x^2 - 3, \text{ so } x = -1, 1 \text{ are critical numbers.}$$



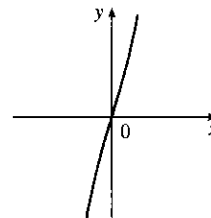
Case (ii) (1 critical number):

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2, \text{ so } x = 0 \text{ is the only critical number.}$$



Case (iii) (no critical number):

$$f(x) = x^3 + 3x \Rightarrow f'(x) = 3x^2 + 3, \text{ so there are no real roots.}$$



- (b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.

APPLIED PROJECT The Calculus of Rainbows

1. From Snell's Law, we have $\sin \alpha = \frac{4}{3} \sin \beta \Rightarrow \frac{d}{d\alpha}(\sin \alpha) = \frac{4}{3} \frac{d}{d\alpha}(\sin \beta) \Rightarrow \cos \alpha = \frac{4}{3} \cos \beta \frac{d\beta}{d\alpha} \Rightarrow \frac{d\beta}{d\alpha} = \frac{3 \cos \alpha}{4 \cos \beta}$. Now $D(\alpha) = \pi + 2\alpha - 4\beta \Rightarrow D'(\alpha) = 2 - 4 \frac{d\beta}{d\alpha} = 2 - 3 \frac{\cos \alpha}{\cos \beta}$. So $D'(\alpha) = 0 \Leftrightarrow 2 \cos \beta = 3 \cos \alpha$. Thus, $4 \cos^2 \beta = 9 \cos^2 \alpha \Rightarrow 4 - 4 \sin^2 \beta = 9 - 9 \sin^2 \alpha$. Since $3 \sin \alpha = 4 \sin \beta$, $\sin \beta = \frac{3}{4} \sin \alpha \Rightarrow 4 - 4 \left(\frac{3}{4} \sin \alpha\right)^2 = 9 - 9 \sin^2 \alpha \Rightarrow \left(9 - \frac{9}{4}\right) \sin^2 \alpha = 9 - 4 = 5 \Rightarrow \sin^2 \alpha = \frac{20}{27} \Rightarrow \sin \alpha = \sqrt{\frac{20}{27}}$. So $\alpha \approx 1.037$ radians or 59.4° . We show that this α does give the minimum on $[0, \frac{\pi}{2}]$: When $\alpha = 0$, $\sin \alpha = \frac{4}{3} \sin \beta \Rightarrow \beta = 0$, or $D(0) = \pi \approx 3.14$. When $\alpha = \frac{\pi}{2}$, $1 = \sin \frac{\pi}{2} = \frac{4}{3} \sin \beta \Rightarrow \sin \beta = \frac{3}{4} \Rightarrow \beta \approx 0.85$. So $D(\frac{\pi}{2}) \approx \pi + \pi - 4(0.85) \approx 2.88$. For $\alpha \approx 1.037$, $\sin \beta = \frac{3}{4} \sin \alpha = \frac{3}{4} \sqrt{\frac{20}{27}}$, so $\beta \approx 0.702 \Rightarrow D(\alpha) \approx \pi + 2(1.036) - 4(0.702) \approx 2.41$. So the minimum occurs when $\alpha \approx 1.04$ radians or 59.4° .

2. We repeat Problem 1 with k in place of $\frac{4}{3}$. So $\sin \alpha = k \sin \beta \Rightarrow \frac{d\beta}{d\alpha} = \frac{1}{k} \frac{\cos \alpha}{\cos \beta}$. Then

$$D'(\alpha) = 2 - 4 \frac{d\beta}{d\alpha} = 2 - \frac{4 \cos \alpha}{k \cos \beta} \text{ and } D'(\alpha) = 0 \Leftrightarrow k \cos \beta = 2 \cos \alpha. \text{ So } k^2 \cos^2 \beta = 4 \cos^2 \alpha \Rightarrow$$

$$k^2 - k^2 \sin^2 \beta = 4 - 4 \sin^2 \alpha \Rightarrow k^2 - \sin^2 \alpha = 4 - 4 \sin^2 \alpha \Rightarrow 3 \sin^2 \alpha = 4 - k^2 \Rightarrow$$

$$\sin \alpha = \sqrt{\frac{4 - k^2}{3}}. \text{ So for } k \approx 1.3318 \text{ (red light) the minimum occurs when } \sin \alpha_1 = \sqrt{\frac{4 - (1.3318)^2}{3}} \text{ or}$$

$\alpha_1 \approx 1.038$ radians, so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6^\circ$.

3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is

$$D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi, \text{ as required. Now } \sin \alpha = k \sin \beta \Rightarrow \frac{d\beta}{d\alpha} = \frac{1}{k} \frac{\cos \alpha}{\cos \beta}. \text{ So}$$

$$D'(\alpha) = 2 - 6 \frac{d\beta}{d\alpha} = 2 - \frac{6 \cos \alpha}{k \cos \beta} \text{ and } D'(\alpha) = 0 \Leftrightarrow k \cos \beta = 3 \cos \alpha. \text{ So } k^2 \cos^2 \beta = 9 \cos^2 \alpha \Rightarrow$$

$$k^2 - k^2 \sin^2 \beta = 9 - 9 \sin^2 \alpha \Rightarrow k^2 - \sin^2 \alpha = 9 - 9 \sin^2 \alpha \Rightarrow \sin^2 \alpha = \frac{9 - k^2}{8} \Rightarrow$$

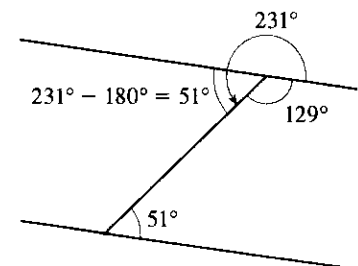
$$\sin \alpha = \sqrt{\frac{9 - k^2}{8}}. \text{ If } k = \frac{4}{3}, \text{ then the minimum occurs when } \sin \alpha_1 = \sqrt{\frac{9 - (4/3)^2}{8}} \text{ or } \alpha_1 \approx 1.254 \text{ radians.}$$

Thus, the minimum *counterclockwise* rotation is $D(\alpha_1) \approx 231^\circ$, which is equivalent to a *clockwise* rotation of $360^\circ - 231^\circ = 129^\circ$ (see the figure).

So the rainbow angle for the secondary rainbow is about

$180^\circ - 129^\circ = 51^\circ$, as required. In general, the rainbow angle for the

secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$.



4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k . But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow.

For $k \approx 1.3318$ (red light) the minimum occurs when $\sin \alpha_1 = \sqrt{\frac{9 - 1.3318^2}{8}}$ or $\alpha_1 \approx 1.254$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs when

$\sin \alpha_2 = \sqrt{\frac{9 - 1.3435^2}{8}}$ or $\alpha_2 \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6^\circ$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k , the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread-out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about 1.7° , whereas in the secondary rainbow it is about 3° .

4.2 The Mean Value Theorem

1. $f(x) = x^2 - 4x + 1$, $[0, 4]$. Since f is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also, $f(0) = 1 = f(4)$. $f'(c) = 0 \Leftrightarrow 2c - 4 = 0 \Leftrightarrow c = 2$, which is in the open interval $(0, 4)$, so $c = 2$ satisfies the conclusion of Rolle's Theorem.
2. $f(x) = x^3 - 3x^2 + 2x + 5$, $[0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Also, $f(0) = 5 = f(2)$.
 $f'(c) = 0 \Leftrightarrow 3c^2 - 6c + 2 = 0 \Leftrightarrow c = \frac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{1}{3}\sqrt{3}$, both in $(0, 2)$.
3. $f(x) = \sin 2\pi x$, $[-1, 1]$. f , being the composite of the sine function and the polynomial $2\pi x$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Also, $f(-1) = 0 = f(1)$.
 $f'(c) = 0 \Leftrightarrow 2\pi \cos 2\pi c = 0 \Leftrightarrow \cos 2\pi c = 0 \Leftrightarrow 2\pi c = \pm \frac{\pi}{2} + 2\pi n \Leftrightarrow c = \pm \frac{1}{4} + n$. If $n = 0$ or ± 1 , then $c = \pm \frac{1}{4}, \pm \frac{3}{4}$ is in $(-1, 1)$.
4. $f(x) = x\sqrt{x+6}$, $[-6, 0]$. f is continuous on its domain, $[-6, \infty)$, and differentiable on $(-6, \infty)$, so it is continuous on $[-6, 0]$ and differentiable on $(-6, 0)$. Also, $f(-6) = 0 = f(0)$. $f'(c) = 0 \Leftrightarrow \frac{3c + 12}{2\sqrt{c+6}} = 0$
 $\Leftrightarrow c = -4$, which is in $(-6, 0)$.
5. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.
6. $f(x) = (x-1)^{-2}$. $f(0) = (0-1)^{-2} = 1 = (2-1)^{-2} = f(2)$. $f'(x) = -2(x-1)^{-3} \Rightarrow f'(x)$ is never 0. This does not contradict Rolle's Theorem since $f'(1)$ does not exist.

13. $f(x) = \sqrt[3]{x}$, $[0, 1]$. f is continuous on $\mathbb{R}$ and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. $f'(c) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow \frac{1}{3c^{2/3}} = \frac{f(1) - f(0)}{1 - 0} \Leftrightarrow \frac{1}{3c^{2/3}} = \frac{1 - 0}{1} \Leftrightarrow 3c^{2/3} = 1 \Leftrightarrow c^{2/3} = \frac{1}{3} \Leftrightarrow c^2 = \left(\frac{1}{3}\right)^3 = \frac{1}{27} \Leftrightarrow c = \pm\sqrt{\frac{1}{27}} = \pm\frac{\sqrt{3}}{9}$, but only $\frac{\sqrt{3}}{9}$ is in $(0, 1)$.
14. $f(x) = \frac{x}{x+2}$, $[1, 4]$. f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $f'(c) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow \frac{2}{(c+2)^2} = \frac{\frac{2}{3} - \frac{1}{3}}{4 - 1} \Leftrightarrow (c+2)^2 = 18 \Leftrightarrow c = -2 \pm 3\sqrt{2}$. $-2 + 3\sqrt{2} \approx 2.24$ is in $(1, 4)$.
15. $f(x) = |x - 1|$. $f(3) - f(0) = |3 - 1| - |0 - 1| = 1$. Since $f'(c) = -1$ if $c < 1$ and $f'(c) = 1$ if $c > 1$, $f'(c)(3 - 0) = \pm 3$ and so is never equal to 1. This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.
16. $f(x) = \frac{x+1}{x-1}$. $f(2) - f(0) = 3 - (-1) = 4$. $f'(x) = \frac{1(x-1) - 1(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$. Since $f'(x) < 0$ for all x (except $x = 1$), $f'(c)(2 - 0)$ is always < 0 and hence cannot equal 4. This does not contradict the Mean Value Theorem since f is not continuous at $x = 1$.
17. Let $f(x) = 1 + 2x + x^3 + 4x^5$. Then $f(-1) = -6 < 0$ and $f(0) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c) = 0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a < b$. Then $f(a) = f(b) = 0$. Since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. By Rolle's Theorem, there is a number r in (a, b) such that $f'(r) = 0$. But $f'(x) = 2 + 3x^2 + 20x^4 \geq 2$ for all x , so $f'(x)$ can never be 0. This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.
18. Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. f is the sum of the polynomial $2x - 1$ and the scalar multiple $(-1) \cdot \sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x . By the Intermediate Value Theorem, there is a number c in $(0, \pi/2)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b) , Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \cos r > 0$ since $\cos r \leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.
19. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real roots a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b) , Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b) , which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation can't have two real roots in $[-2, 2]$. Hence, it has at most one real root in $[-2, 2]$.
20. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)(x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.

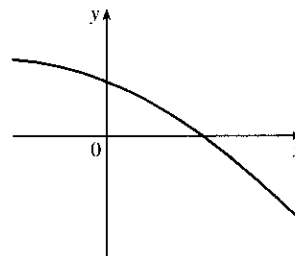
21. (a) Suppose that a cubic polynomial $P(x)$ has roots $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$ and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.
- (b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real roots, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \dots, c_{n+1}$ with $a_1 < c_1 < a_2, \dots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the n th degree polynomial $P'(x)$ has at least $n + 1$ roots. This contradiction shows that $P(x)$ has at most $n + 1$ real roots.
22. (a) Suppose that $f(a) = f(b) = 0$ where $a < b$. By Rolle's Theorem applied to f on $[a, b]$ there is a number c such that $a < c < b$ and $f'(c) = 0$.
- (b) Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$. By Rolle's Theorem applied to $f(x)$ on $[a, b]$ and $[b, c]$ there are numbers $a < d < b$ and $b < e < c$ with $f'(d) = 0$ and $f'(e) = 0$. By Rolle's Theorem applied to $f'(x)$ on $[d, e]$ there is a number g with $d < g < e$ such that $f''(g) = 0$.
- (c) Suppose that f is n times differentiable on $\mathbb{R}$ and has $n + 1$ distinct real roots. Then $f^{(n)}$ has at least one real root.
23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.
24. If $3 \leq f'(x) \leq 5$ for all x , then by the Mean Value Theorem, $f(8) - f(2) = f'(c) \cdot (8 - 2)$ for some c in $[2, 8]$. (f is differentiable for all x , so, in particular, f is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that $6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5 \Rightarrow 18 \leq f(8) - f(2) \leq 30$.
25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0 < c < 2$ with $f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{5}{2}$. But this is impossible since $f'(x) \leq 2 < \frac{5}{2}$ for all x , so no such function can exist.
26. Let $h = f - g$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b) , so is h , and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) - h(a) = h'(c)(b - a)$. Since $h'(c) < 0$, $h'(c)(b - a) < 0$, so $f(b) - g(b) = h(b) < 0$ and hence $f(b) < g(b)$.
27. We use Exercise 26 with $f(x) = \sqrt{1+x}$, $g(x) = 1 + \frac{1}{2}x$, and $a = 0$. Notice that $f(0) = 1 = g(0)$ and $f'(x) = \frac{1}{2\sqrt{1+x}} < \frac{1}{2} = g'(x)$ for $x > 0$. So by Exercise 26, $f(b) < g(b) \Rightarrow \sqrt{1+b} < 1 + \frac{1}{2}b$ for $b > 0$.
Another method: Apply the Mean Value Theorem directly to either $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$ or $g(x) = \sqrt{1+x}$ on $[0, b]$.
28. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $\frac{f(b) - f(-b)}{b - (-b)} = f'(c)$ for some $c \in (-b, b)$. But since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $\frac{f(b) + f(b)}{2b} = f'(c) \Rightarrow \frac{f(b)}{b} = f'(c)$.

29. Let $f(x) = \sin x$ and let $b < a$. Then $f(x)$ is continuous on $[b, a]$ and differentiable on (b, a) . By the Mean Value Theorem, there is a number $c \in (b, a)$ with $\sin a - \sin b = f(a) - f(b) = f'(c)(a - b) = (\cos c)(a - b)$. Thus, $|\sin a - \sin b| \leq |\cos c| |b - a| \leq |a - b|$. If $a < b$, then $|\sin a - \sin b| = |\sin b - \sin a| \leq |b - a| = |a - b|$. If $a = b$, both sides of the inequality are 0.
30. Suppose that $f'(x) = c$. Let $g(x) = cx$, so $g'(x) = c$. Then, by Corollary 7, $f(x) = g(x) + d$, where d is a constant, so $f(x) = cx + d$.
31. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$] so we cannot conclude that $f - g$ is constant (in fact it is not).
32. Let $v(t)$ be the velocity of the car t hours after 2:00 P.M. Then $\frac{v(1/6) - v(0)}{1/6 - 0} = \frac{50 - 30}{1/6} = 120$. By the Mean Value Theorem, there is a number c such that $0 < c < \frac{1}{6}$ with $v'(c) = 120$. Since $v'(t)$ is the acceleration at time t , the acceleration c hours after 2:00 P.M. is exactly 120 mi/h².
33. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t) = g(t) - h(t)$. By hypothesis, $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$, where b is the finishing time. Then by the Mean Value Theorem, there is a time c , with $0 < c < b$, such that $f'(c) = \frac{f(b) - f(0)}{b - 0}$. But $f(b) = f(0) = 0$, so $f'(c) = 0$. Since $f'(c) = g'(c) - h'(c) = 0$, we have $g'(c) = h'(c)$. So at time c , both runners have the same speed $g'(c) = h'(c)$.
34. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x) \neq 1$ for all x . Suppose f has more than one fixed point. Then there are numbers a and b such that $a < b$, $f(a) = a$, and $f(b) = b$. Applying the Mean Value Theorem to the function f on $[a, b]$, we find that there is a number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. But then $f'(c) = \frac{b - a}{b - a} = 1$, contradicting our assumption that $f'(x) \neq 1$ for every real number x . This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

4.3 How Derivatives Affect the Shape of a Graph

- f is increasing on $(0, 6)$ and $(8, 9)$.
 - f is decreasing on $(6, 8)$.
 - f is concave upward on $(2, 4)$ and $(7, 9)$.
 - f is concave downward on $(0, 2)$ and $(4, 7)$.
 - The points of inflection are $(2, 3)$, $(4, 4.5)$ and $(7, 4)$ (where the concavity changes).
- f is increasing on $(1, \approx 3.8)$ and $(5, \approx 6.5)$.
 - f is decreasing on $(0, 1)$, $(\approx 3.8, 5)$, $(\approx 6.5, 8)$, and $(8, 9)$.
 - f is concave upward on $(0, 3)$ and $(8, 9)$.
 - f is concave downward on $(3, 5)$ and $(5, 8)$.
 - The point of inflection is $(3, \approx 1.8)$ (where the concavity changes).

3. (a) Use the Increasing/Decreasing (I/D) Test.
 (b) Use the Concavity Test.
 (c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.
4. (a) See the First Derivative Test.
 (b) See the Second Derivative Test and the note that precedes Example 7.
5. (a) Since $f'(x) > 0$ on $(1, 5)$, f is increasing on this interval. Since $f'(x) < 0$ on $(0, 1)$ and $(5, 6)$, f is decreasing on these intervals.
 (b) Since $f'(x) = 0$ at $x = 1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 1$. Since $f'(x) = 0$ at $x = 5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 5$.
6. (a) $f'(x) > 0$ and f is increasing on $(0, 1)$ and $(3, 5)$. $f'(x) < 0$ and f is decreasing on $(1, 3)$ and $(5, 6)$.
 (b) Since $f'(x) = 0$ at $x = 1$ and $x = 5$ and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x = 1$ and $x = 5$. Since $f'(x) = 0$ at $x = 3$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 3$.
7. There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.
8. (a) f is increasing on the intervals where $f'(x) > 0$, namely, $(2, 4)$ and $(6, 9)$.
 (b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x = 2$ and at $x = 6$).
 (c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1, 3)$, $(5, 7)$, and $(8, 9)$. Similarly, f is concave downward when f' is decreasing—that is, on $(0, 1)$, $(3, 5)$, and $(7, 8)$.
 (d) f has inflection points at $x = 1, 3, 5, 7$, and 8 , since the direction of concavity changes at each of these values.
9. The function must be always decreasing and concave downward.



10. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.
 (b) The rate of increase has its maximum value at $t = 8$ hours.
 (c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.
 (d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

11. (a) $f(x) = x^3 - 12x + 1 \Rightarrow f'(x) = 3x^2 - 12 = 3(x+2)(x-2)$.

We don't need to include "3" in the chart to determine the sign of $f'(x)$.

Interval	$x+2$	$x-2$	$f'(x)$	f
$x < -2$	-	-	+	increasing on $(-\infty, -2)$
$-2 < x < 2$	+	-	-	decreasing on $(-2, 2)$
$x > 2$	+	+	+	increasing on $(2, \infty)$

So f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and f is decreasing on $(-2, 2)$.

(b) f changes from increasing to decreasing at $x = -2$ and from decreasing to increasing at $x = 2$. Thus, $f(-2) = 17$ is a local maximum value and $f(2) = -15$ is a local minimum value.

(c) $f''(x) = 6x$. $f''(x) > 0 \Leftrightarrow x > 0$ and $f''(x) < 0 \Leftrightarrow x < 0$. Thus, f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$. There is an inflection point where the concavity changes, at $(0, f(0)) = (0, 1)$.

12. (a) $f(x) = 5 - 3x^2 + x^3 \Rightarrow f'(x) = -6x + 3x^2 = 3x(x-2)$. Thus, $f'(x) > 0 \Leftrightarrow x < 0$ or $x > 2$ and $f'(x) < 0 \Leftrightarrow 0 < x < 2$. So f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and f is decreasing on $(0, 2)$.

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 2$. Thus, $f(0) = 5$ is a local maximum value and $f(2) = 1$ is a local minimum value.

(c) $f''(x) = -6 + 6x = 6(x-1)$. $f''(x) > 0 \Leftrightarrow x > 1$ and $f''(x) < 0 \Leftrightarrow x < 1$. Thus, f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$. There is an inflection point at $(1, 3)$.

13. (a) $f(x) = x^4 - 2x^2 + 3 \Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$.

Interval	$x+1$	x	$x-1$	$f'(x)$	f
$x < -1$	-	-	-	-	decreasing on $(-\infty, -1)$
$-1 < x < 0$	+	-	-	+	increasing on $(-1, 0)$
$0 < x < 1$	+	+	-	-	decreasing on $(0, 1)$
$x > 1$	+	+	+	+	increasing on $(1, \infty)$

So f is increasing on $(-1, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = -1$ and $x = 1$. Thus, $f(0) = 3$ is a local maximum value and $f(\pm 1) = 2$ are local minimum values.

(c) $f''(x) = 12x^2 - 4 = 12(x^2 - \frac{1}{3}) = 12(x + 1/\sqrt{3})(x - 1/\sqrt{3})$. $f''(x) > 0 \Leftrightarrow x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$ and $f''(x) < 0 \Leftrightarrow -1/\sqrt{3} < x < 1/\sqrt{3}$. Thus, f is concave upward on $(-\infty, -\sqrt{3}/3)$ and $(\sqrt{3}/3, \infty)$ and concave downward on $(-\sqrt{3}/3, \sqrt{3}/3)$. There are inflection points at $(\pm\sqrt{3}/3, \frac{22}{9})$.

14. (a) $f(x) = \frac{x^2}{x^2 + 3} \Rightarrow f'(x) = \frac{(x^2 + 3)(2x) - x^2(2x)}{(x^2 + 3)^2} = \frac{6x}{(x^2 + 3)^2}$. The denominator is positive so the sign

of $f'(x)$ is determined by the sign of x . Thus, $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x = 0$. Thus, $f(0) = 0$ is a local minimum value.

$$\begin{aligned} \text{(c) } f''(x) &= \frac{(x^2 + 3)^2(6) - 6x \cdot 2(x^2 + 3)(2x)}{[(x^2 + 3)^2]^2} = \frac{6(x^2 + 3)[x^2 + 3 - 4x^2]}{(x^2 + 3)^4} \\ &= \frac{6(3 - 3x^2)}{(x^2 + 3)^3} = \frac{-18(x + 1)(x - 1)}{(x^2 + 3)^3}. \end{aligned}$$

$f''(x) > 0 \Leftrightarrow -1 < x < 1$ and $f''(x) < 0 \Leftrightarrow x < -1$ or $x > 1$. Thus, f is concave upward on $(-1, 1)$ and concave downward on $(-\infty, -1)$ and $(1, \infty)$. There are inflection points at $(\pm 1, \frac{1}{4})$.

15. (a) $f(x) = x - 2 \sin x$ on $(0, 3\pi) \Rightarrow f'(x) = 1 - 2 \cos x$. $f'(x) > 0 \Leftrightarrow 1 - 2 \cos x > 0 \Leftrightarrow \cos x < \frac{1}{2}$
 $\Leftrightarrow \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$. $f'(x) < 0 \Leftrightarrow \cos x > \frac{1}{2} \Leftrightarrow 0 < x < \frac{\pi}{3}$ or $\frac{5\pi}{3} < x < \frac{7\pi}{3}$. So f is increasing on $(\frac{\pi}{3}, \frac{5\pi}{3})$ and $(\frac{7\pi}{3}, 3\pi)$, and f is decreasing on $(0, \frac{\pi}{3})$ and $(\frac{5\pi}{3}, \frac{7\pi}{3})$.

(b) f changes from increasing to decreasing at $x = \frac{5\pi}{3}$, and from decreasing to increasing at $x = \frac{\pi}{3}$ and at $x = \frac{7\pi}{3}$.
 Thus, $f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ is a local maximum value and $f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3} \approx -0.68$ and $f(\frac{7\pi}{3}) = \frac{7\pi}{3} - \sqrt{3} \approx 5.60$ are local minimum values.

(c) $f''(x) = 2 \sin x > 0 \Leftrightarrow 0 < x < \pi$ and $2\pi < x < 3\pi$, $f''(x) < 0 \Leftrightarrow \pi < x < 2\pi$. Thus, f is concave upward on $(0, \pi)$ and $(2\pi, 3\pi)$, and f is concave downward on $(\pi, 2\pi)$. There are inflection points at (π, π) and $(2\pi, 2\pi)$.

16. (a) $f(x) = \cos^2 x - 2 \sin x$, $0 \leq x \leq 2\pi$. $f'(x) = -2 \cos x \sin x - 2 \cos x = -2 \cos x (1 + \sin x)$. Note that $1 + \sin x \geq 0$ [since $\sin x \geq -1$], with equality $\Leftrightarrow \sin x = -1 \Leftrightarrow x = 3\pi/2$ [since $0 \leq x \leq 2\pi$]
 $\Rightarrow \cos x = 0$. Thus, $f'(x) > 0 \Leftrightarrow \cos x < 0 \Leftrightarrow \pi/2 < x < 3\pi/2$ and $f'(x) < 0 \Leftrightarrow \cos x > 0$
 $\Leftrightarrow 0 < x < \pi/2$ or $3\pi/2 < x < 2\pi$. Thus, f is increasing on $(\pi/2, 3\pi/2)$ and f is decreasing on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$.

(b) f changes from decreasing to increasing at $x = \pi/2$ and from increasing to decreasing at $x = 3\pi/2$. Thus, $f(\pi/2) = -2$ is a local minimum value and $f(3\pi/2) = 2$ is a local maximum value.

(c) $f''(x) = 2 \sin x (1 + \sin x) - 2 \cos^2 x = 2 \sin x + 2 \sin^2 x - 2(1 - \sin^2 x)$
 $= 4 \sin^2 x + 2 \sin x - 2 = 2(2 \sin x - 1)(\sin x + 1)$
 so $f''(x) > 0 \Leftrightarrow \sin x > \frac{1}{2} \Leftrightarrow \frac{\pi}{6} < x < \frac{5\pi}{6}$, and $f''(x) < 0 \Leftrightarrow \sin x < \frac{1}{2}$ and $\sin x \neq -1 \Leftrightarrow$
 $0 < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is concave upward on $(\frac{\pi}{6}, \frac{5\pi}{6})$ and concave downward on $(0, \frac{\pi}{6})$, $(\frac{5\pi}{6}, \frac{3\pi}{2})$, and $(\frac{3\pi}{2}, 2\pi)$. There are inflection points at $(\frac{\pi}{6}, -\frac{1}{4})$ and $(\frac{5\pi}{6}, -\frac{1}{4})$.

17. $f(x) = x^5 - 5x + 3 \Rightarrow f'(x) = 5x^4 - 5 = 5(x^2 + 1)(x + 1)(x - 1)$.

First Derivative Test: $f'(x) < 0 \Rightarrow -1 < x < 1$ and $f'(x) > 0 \Rightarrow x > 1$ or $x < -1$. Since f' changes from positive to negative at $x = -1$, $f(-1) = 7$ is a local maximum value; and since f' changes from negative to positive at $x = 1$, $f(1) = -1$ is a local minimum value.

Second Derivative Test: $f''(x) = 20x^3$. $f'(x) = 0 \Leftrightarrow x = \pm 1$. $f''(-1) = -20 < 0 \Rightarrow f(-1) = 7$ is a local maximum value. $f''(1) = 20 > 0 \Rightarrow f(1) = -1$ is a local minimum value.

Preference: For this function, the two tests are equally easy.

$$18. f(x) = \frac{x}{x^2 + 4} \Rightarrow f'(x) = \frac{(x^2 + 4) \cdot 1 - x(2x)}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2} = \frac{(2 + x)(2 - x)}{(x^2 + 4)^2}.$$

First Derivative Test: $f'(x) > 0 \Rightarrow -2 < x < 2$ and $f'(x) < 0 \Rightarrow x > 2$ or $x < -2$. Since f' changes from positive to negative at $x = 2$, $f(2) = \frac{1}{4}$ is a local maximum value; and since f' changes from negative to positive at $x = -2$, $f(-2) = -\frac{1}{4}$ is a local minimum value.

Second Derivative Test:

$$\begin{aligned} f''(x) &= \frac{(x^2 + 4)^2(-2x) - (4 - x^2) \cdot 2(x^2 + 4)(2x)}{[(x^2 + 4)^2]^2} \\ &= \frac{-2x(x^2 + 4)[(x^2 + 4) + 2(4 - x^2)]}{(x^2 + 4)^4} = \frac{-2x(12 - x^2)}{(x^2 + 4)^3} \end{aligned}$$

$f'(x) = 0 \Leftrightarrow x = \pm 2$. $f''(-2) = \frac{1}{16} > 0 \Rightarrow f(-2) = -\frac{1}{4}$ is a local minimum value.

$f''(2) = -\frac{1}{16} < 0 \Rightarrow f(2) = \frac{1}{4}$ is a local maximum value.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

19. $f(x) = x + \sqrt{1-x} \Rightarrow f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1) = 1 - \frac{1}{2\sqrt{1-x}}$. Note that f is defined for $1-x \geq 0$; that is, for $x \leq 1$. $f'(x) = 0 \Rightarrow 2\sqrt{1-x} = 1 \Rightarrow \sqrt{1-x} = \frac{1}{2} \Rightarrow 1-x = \frac{1}{4} \Rightarrow x = \frac{3}{4}$. f' does not exist at $x = 1$, but we can't have a local maximum or minimum at an endpoint.

First Derivative Test: $f'(x) > 0 \Rightarrow x < \frac{3}{4}$ and $f'(x) < 0 \Rightarrow \frac{3}{4} < x < 1$. Since f' changes from positive to negative at $x = \frac{3}{4}$, $f(\frac{3}{4}) = \frac{5}{4}$ is a local maximum value.

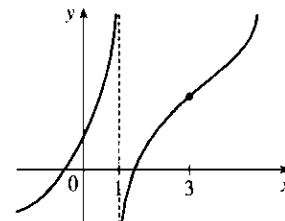
Second Derivative Test: $f''(x) = -\frac{1}{2}(-\frac{1}{2})(1-x)^{-3/2}(-1) = -\frac{1}{4(\sqrt{1-x})^3}$. $f''(\frac{3}{4}) = -2 < 0 \Rightarrow$

$f(\frac{3}{4}) = \frac{5}{4}$ is a local maximum value.

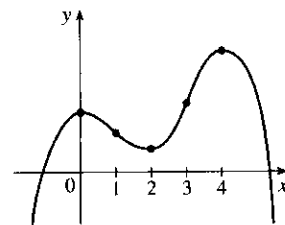
Preference: The First Derivative Test may be slightly easier to apply in this case.

20. (a) $f(x) = x^4(x-1)^3 \Rightarrow$
 $f'(x) = x^4 \cdot 3(x-1)^2 + (x-1)^3 \cdot 4x^3 = x^3(x-1)^2[3x + 4(x-1)] = x^3(x-1)^2(7x-4)$
 The critical numbers are 0, 1, and $\frac{4}{7}$.
 (b) $f''(x) = 3x^2(x-1)^2(7x-4) + x^3 \cdot 2(x-1)(7x-4) + x^3(x-1)^2 \cdot 7$
 $= x^2(x-1)[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)]$
 Now $f''(0) = f''(1) = 0$, so the Second Derivative Test gives no information for $x = 0$ or $x = 1$.
 $f''(\frac{4}{7}) = (\frac{4}{7})^2(\frac{4}{7}-1)[0 + 0 + 7(\frac{4}{7})(\frac{4}{7}-1)] = (\frac{4}{7})^2(-\frac{3}{7})(4)(-\frac{3}{7}) > 0$, so there is a local minimum at $x = \frac{4}{7}$.
 (c) f' is positive on $(-\infty, 0)$, negative on $(0, \frac{4}{7})$, positive on $(\frac{4}{7}, 1)$, and positive on $(1, \infty)$. So f has a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and no local maximum or minimum at $x = 1$.
 21. (a) By the Second Derivative Test, if $f'(2) = 0$ and $f''(2) = -5 < 0$, f has a local maximum at $x = 2$.
 (b) If $f'(6) = 0$, we know that f has a horizontal tangent at $x = 6$. Knowing that $f''(6) = 0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y = (x-6)^4$, $y = -(x-6)^4$, and $y = (x-6)^3$ all equal zero for $x = 6$, but the first has a local minimum at $x = 6$, the second has a local maximum at $x = 6$, and the third has an inflection point at $x = 6$.

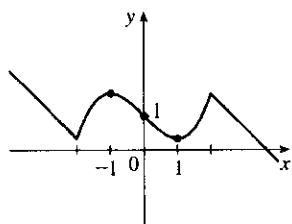
22. $f'(x) > 0$ for all $x \neq 1$ with vertical asymptote $x = 1$, so f is increasing on $(-\infty, 1)$ and $(1, \infty)$. $f''(x) > 0$ if $x < 1$ or $x > 3$, and $f''(x) < 0$ if $1 < x < 3$, so f is concave upward on $(-\infty, 1)$ and $(3, \infty)$, and concave downward on $(1, 3)$. There is an inflection point when $x = 3$.



23. $f'(0) = f'(2) = f'(4) = 0 \Rightarrow$ horizontal tangents at $x = 0, 2, 4$.
 $f'(x) > 0$ if $x < 0$ or $2 < x < 4 \Rightarrow f$ is increasing on $(-\infty, 0)$
 and $(2, 4)$. $f'(x) < 0$ if $0 < x < 2$ or $x > 4 \Rightarrow f$ is decreasing
 on $(0, 2)$ and $(4, \infty)$. $f''(x) > 0$ if $1 < x < 3 \Rightarrow f$ is concave
 upward on $(1, 3)$. $f''(x) < 0$ if $x < 1$ or $x > 3 \Rightarrow f$ is concave
 downward on $(-\infty, 1)$ and $(3, \infty)$. There are inflection points when
 $x = 1$ and 3 .

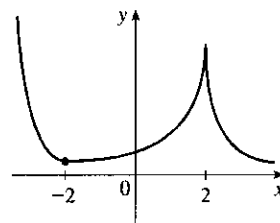
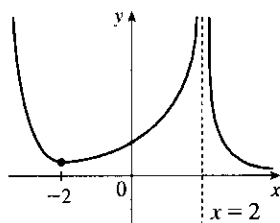


24.

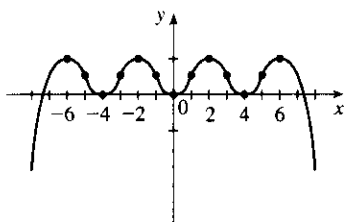


$f'(1) = f'(-1) = 0 \Rightarrow$ horizontal tangents at $x = \pm 1$.
 $f'(x) < 0$ if $|x| < 1 \Rightarrow f$ is decreasing on $(-1, 1)$. $f'(x) > 0$ if
 $1 < |x| < 2 \Rightarrow f$ is increasing on $(-2, -1)$ and $(1, 2)$.
 $f'(x) = -1$ if $|x| > 2 \Rightarrow$ the graph of f has constant slope -1
 on $(-\infty, -2)$ and $(2, \infty)$. $f''(x) < 0$ if $-2 < x < 0 \Rightarrow$
 f is concave downward on $(-2, 0)$. Inflection point $(0, 1)$.

25. $f'(x) > 0$ if $|x| < 2 \Rightarrow f$ is increasing on $(-2, 2)$. $f'(x) < 0$ if $|x| > 2 \Rightarrow f$ is decreasing on $(-\infty, -2)$
 and $(2, \infty)$. $f'(-2) = 0 \Rightarrow$ horizontal tangent at $x = -2$. $\lim_{x \rightarrow 2} |f'(x)| = \infty \Rightarrow$ there is a vertical asymp-
 tote or vertical tangent (cusp) at $x = 2$. $f''(x) > 0$ if $x \neq 2 \Rightarrow f$ is concave upward on $(-\infty, 2)$ and $(2, \infty)$.



26.

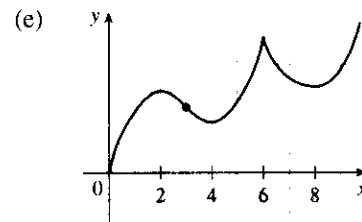


$f(0) = f'(0) = 0 \Rightarrow$ the graph of f passes through the origin
 and has a horizontal tangent there. $f'(2) = f'(4) = f'(6) = 0 \Rightarrow$
 horizontal tangents at $x = 2, 4, 6$. $f'(x) > 0$ if $0 < x < 2$ or
 $4 < x < 6 \Rightarrow f$ increasing on $(0, 2)$ and $(4, 6)$. $f'(x) < 0$ if
 $2 < x < 4$ or $x > 6 \Rightarrow f$ decreasing on $(2, 4)$ and $(6, \infty)$.

$f''(x) > 0$ if $0 < x < 1$ or $3 < x < 5 \Rightarrow f$ is CU on $(0, 1)$

and $(3, 5)$. $f''(x) < 0$ if $1 < x < 3$ or $x > 5 \Rightarrow f$ is CD on $(1, 3)$ and $(5, \infty)$. $f(-x) = f(x) \Rightarrow f$ is
 even and the graph is symmetric about the y -axis.

27. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative,
 that is, on $(2, 4)$ and $(6, 8)$.
 (b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where
 f' changes from negative to positive, at $x = 4$ and at $x = 8$.
 (c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$
 and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that
 is, on $(0, 3)$.
 (d) There is a point of inflection where f changes from being CD to
 being CU, that is, at $x = 3$.

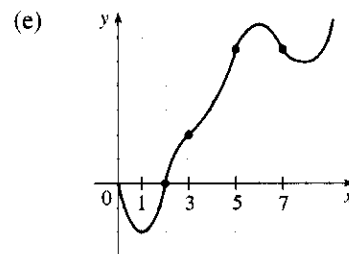


28. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

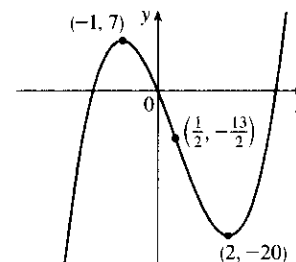
(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.



29. (a) $f(x) = 2x^3 - 3x^2 - 12x \Rightarrow f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$. $f'(x) > 0 \Leftrightarrow x < -1$ or $x > 2$ and $f'(x) < 0 \Leftrightarrow -1 < x < 2$. So f is increasing on $(-\infty, -1)$ and $(2, \infty)$, and f is decreasing on $(-1, 2)$.

(b) Since f changes from increasing to decreasing at $x = -1$, $f(-1) = 7$ is a local maximum value. Since f changes from decreasing to increasing at $x = 2$, $f(2) = -20$ is a local minimum value.

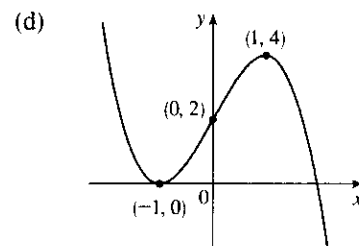
(c) $f''(x) = 6(2x - 1) \Rightarrow f''(x) > 0$ on $(\frac{1}{2}, \infty)$ and $f''(x) < 0$ on $(-\infty, \frac{1}{2})$. So f is concave upward on $(\frac{1}{2}, \infty)$ and concave downward on $(-\infty, \frac{1}{2})$. There is a change in concavity at $x = \frac{1}{2}$, and we have an inflection point at $(\frac{1}{2}, -\frac{13}{2})$.



30. (a) $f(x) = 2 + 3x - x^3 \Rightarrow f'(x) = 3 - 3x^2 = -3(x^2 - 1) = -3(x + 1)(x - 1)$. $f'(x) > 0 \Leftrightarrow -1 < x < 1$ and $f'(x) < 0 \Leftrightarrow x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) $f(-1) = 0$ is a local minimum value and $f(1) = 4$ is a local maximum value.

(c) $f''(x) = -6x \Rightarrow f''(x) > 0$ on $(-\infty, 0)$ and $f''(x) < 0$ on $(0, \infty)$. So f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. There is an inflection point at $(0, 2)$.



31. (a) $f(x) = x^4 - 6x^2 \Rightarrow f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

Interval	$4x$	$x^2 - 3$	$f'(x)$	f
$x < -\sqrt{3}$	-	+	-	decreasing on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	-	-	+	increasing on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	+	-	-	decreasing on $(0, \sqrt{3})$
$x > \sqrt{3}$	+	+	+	increasing on $(\sqrt{3}, \infty)$

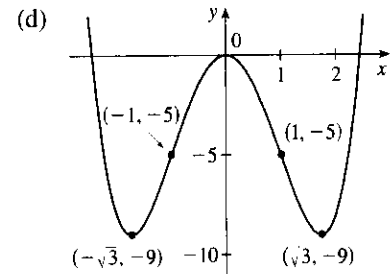
(b) Local minimum values $f(\pm\sqrt{3}) = -9$,

local maximum value $f(0) = 0$

(c) $f''(x) = 12x^2 - 12 = 12(x^2 - 1) > 0 \Leftrightarrow x^2 > 1 \Leftrightarrow$

$|x| > 1 \Leftrightarrow x > 1 \text{ or } x < -1$, so f is CU on $(-\infty, -1)$, $(1, \infty)$

and CD on $(-1, 1)$. Inflection points at $(\pm 1, -5)$



32. (a) $g(x) = 200 + 8x^3 + x^4 \Rightarrow g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$.

$g'(x) > 0 \Leftrightarrow x > -6$ ($x \neq 0$) and $g'(x) < 0 \Leftrightarrow x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

(b) $g(-6) = -232$ is a local minimum value.

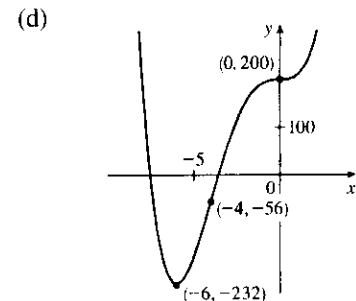
There is no local maximum value.

(c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when

$x = 0$. $g''(x) > 0 \Leftrightarrow x < -4$ or $x > 0$ and $g''(x) < 0 \Leftrightarrow$

$-4 < x < 0$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on

$(-4, 0)$. Inflection points at $(-4, -56)$ and $(0, 200)$



33. (a) $h(x) = 3x^5 - 5x^3 + 3 \Rightarrow h'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 0$ when $x = 0, \pm 1$. Since $15x^2$ is

nonnegative, $h'(x) > 0 \Leftrightarrow x^2 > 1 \Leftrightarrow |x| > 1 \Leftrightarrow x > 1 \text{ or } x < -1$, so h is increasing on

$(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$, with a horizontal tangent at $x = 0$.

(b) Local maximum value $h(-1) = 5$, local minimum value $h(1) = 1$

(c) $h''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

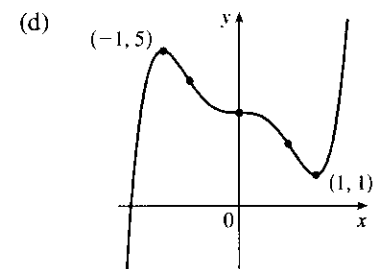
$$= 60x\left(x + \frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow$$

$h''(x) > 0$ when $x > \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} < x < 0$, so h is CU on

$(-\frac{1}{\sqrt{2}}, 0)$ and $(\frac{1}{\sqrt{2}}, \infty)$ and CD on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(0, \frac{1}{\sqrt{2}})$.

Inflection points at $(0, 3)$ and $(\pm \frac{1}{\sqrt{2}}, 3 \mp \frac{7}{8}\sqrt{2})$ [about

$(-0.71, 4.24)$ and $(0.71, 1.76)$].



34. (a) $h(x) = (x^2 - 1)^3 \Rightarrow h'(x) = 6x(x^2 - 1)^2 \geq 0 \Leftrightarrow x > 0$ ($x \neq 1$), so h is increasing on $(0, \infty)$ and

decreasing on $(-\infty, 0)$.

(b) $h(0) = -1$ is a local minimum value.

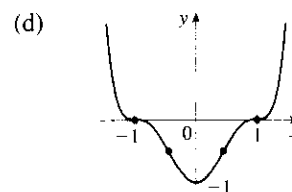
(c) $h''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1) = 6(x^2 - 1)(5x^2 - 1)$. The roots ± 1 and $\pm \frac{1}{\sqrt{5}}$ divide $\mathbb{R}$ into five intervals.

Interval	$x^2 - 1$	$5x^2 - 1$	$h''(x)$	Concavity
$x < -1$	+	+	+	upward
$-1 < x < -\frac{1}{\sqrt{5}}$	-	+	-	downward
$-\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$	-	-	+	upward
$\frac{1}{\sqrt{5}} < x < 1$	-	+	-	downward
$x > 1$	+	+	+	upward

From the table, we see that h is CU on $(-\infty, -1)$,

$(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ and $(1, \infty)$, and CD on $(-1, -\frac{1}{\sqrt{5}})$ and $(\frac{1}{\sqrt{5}}, 1)$.

Inflection points at $(\pm 1, 0)$ and $(\pm \frac{1}{\sqrt{5}}, -\frac{64}{125})$



35. (a) $A(x) = x\sqrt{x+3} \Rightarrow$

$$A'(x) = x \cdot \frac{1}{2}(x+3)^{-1/2} + \sqrt{x+3} \cdot 1 = \frac{x}{2\sqrt{x+3}} + \sqrt{x+3} = \frac{x+2(x+3)}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}.$$

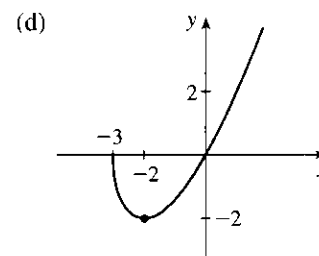
The domain of A is $[-3, \infty)$. $A'(x) > 0$ for $x > -2$ and $A'(x) < 0$ for $-3 < x < -2$, so A is increasing on $(-2, \infty)$ and decreasing on $(-3, -2)$.

(b) $A(-2) = -2$ is a local minimum value.

$$\begin{aligned} \text{(c) } A''(x) &= \frac{2\sqrt{x+3} \cdot 3 - (3x+6) \cdot \frac{1}{\sqrt{x+3}}}{(2\sqrt{x+3})^2} \\ &= \frac{6(x+3) - (3x+6)}{4(x+3)^{3/2}} = \frac{3x+12}{4(x+3)^{3/2}} = \frac{3(x+4)}{4(x+3)^{3/2}} \end{aligned}$$

$A''(x) > 0$ for all $x > -3$, so A is concave upward on $(-3, \infty)$.

There is no inflection point.



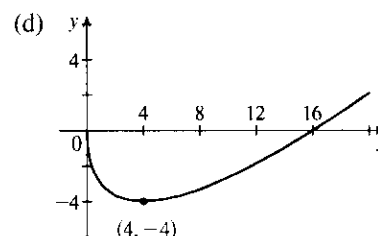
36. (a) $G(x) = x - 4\sqrt{x} \Rightarrow G'(x) = 1 - \frac{2}{\sqrt{x}} = \frac{1}{\sqrt{x}}(\sqrt{x} - 2)$. $G'(x) > 0 \Leftrightarrow x > 4$ and $G'(x) < 0 \Leftrightarrow 0 < x < 4$, so G is decreasing on $(0, 4)$ and increasing on $(4, \infty)$.

(b) Local minimum $G(4) = -4$. No local maximum.

(c) $G'(x) = 1 - 2x^{-1/2} \Rightarrow G''(x) = x^{-3/2} = 1/\sqrt{x^3}$, so

$G''(x) > 0$ for $x > 0$. Thus, G is CU on $(0, \infty)$.

G has no IPs.



$$37. (a) C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \Rightarrow C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \frac{4(x+1)}{3\sqrt[3]{x^2}}.$$

$C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

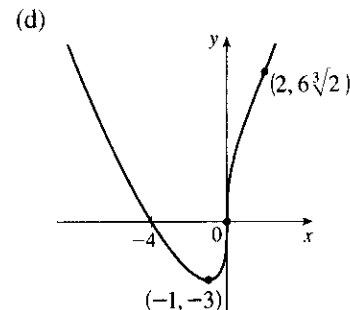
$$(c) C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \frac{4(x-2)}{9\sqrt[3]{x^5}}.$$

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so

C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$

and $(2, \infty)$. There are inflection points at $(0, 0)$ and

$$(2, 6\sqrt[3]{2}) \approx (2, 7.56).$$



$$38. (a) B(x) = 3x^{2/3} - x \Rightarrow B'(x) = 2x^{-1/3} - 1 = \frac{2}{\sqrt[3]{x}} - 1 = \frac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}. B'(x) > 0 \text{ if } 0 < x < 8 \text{ and}$$

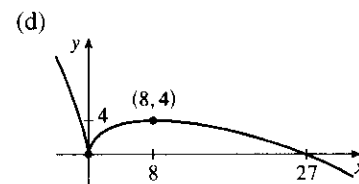
$B'(x) < 0$ if $x < 0$ or $x > 8$, so B is decreasing on $(-\infty, 0)$ and $(8, \infty)$, and B is increasing on $(0, 8)$.

(b) $B(0) = 0$ is a local minimum value.

$B(8) = 4$ is a local maximum value.

$$(c) B''(x) = -\frac{2}{3}x^{-4/3} = \frac{-2}{3x^{4/3}}, \text{ so } B''(x) < 0 \text{ for all } x \neq 0. B \text{ is}$$

concave downward on $(-\infty, 0)$ and $(0, \infty)$. There is no inflection point.



$$39. (a) f(\theta) = 2 \cos \theta - \cos 2\theta, 0 \leq \theta \leq 2\pi.$$

$$f'(\theta) = -2 \sin \theta + 2 \sin 2\theta = -2 \sin \theta + 2(2 \sin \theta \cos \theta) = 2 \sin \theta (2 \cos \theta - 1).$$

Interval	$\sin \theta$	$2 \cos \theta - 1$	$f'(\theta)$	f
$0 < \theta < \frac{\pi}{3}$	+	+	+	increasing on $(0, \frac{\pi}{3})$
$\frac{\pi}{3} < \theta < \pi$	+	-	-	decreasing on $(\frac{\pi}{3}, \pi)$
$\pi < \theta < \frac{5\pi}{3}$	-	-	+	increasing on $(\pi, \frac{5\pi}{3})$
$\frac{5\pi}{3} < \theta < 2\pi$	-	+	-	decreasing on $(\frac{5\pi}{3}, 2\pi)$

(b) $f(\frac{\pi}{3}) = \frac{3}{2}$ and $f(\frac{5\pi}{3}) = \frac{3}{2}$ are local maximum values and $f(\pi) = -3$ is a local minimum value.

$$(c) f'(\theta) = -2 \sin \theta + 2 \sin 2\theta \Rightarrow$$

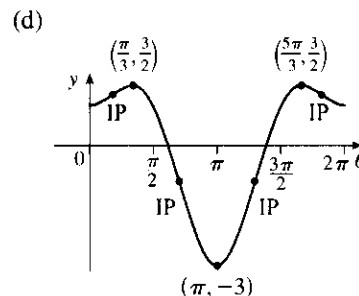
$$f''(\theta) = -2 \cos \theta + 4 \cos 2\theta = -2 \cos \theta + 4(2 \cos^2 \theta - 1) \\ = 2(4 \cos^2 \theta - \cos \theta - 2)$$

$$f''(\theta) = 0 \Leftrightarrow \cos \theta = \frac{1 \pm \sqrt{33}}{8} \Leftrightarrow \theta = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$

$$\Leftrightarrow \theta = \cos^{-1}\left(\frac{1 + \sqrt{33}}{8}\right) \approx 0.5678,$$

$$2\pi - \cos^{-1}\left(\frac{1 + \sqrt{33}}{8}\right) \approx 5.7154, \cos^{-1}\left(\frac{1 - \sqrt{33}}{8}\right) \approx 2.2057, \text{ or } 2\pi - \cos^{-1}\left(\frac{1 - \sqrt{33}}{8}\right) \approx 4.0775.$$

Denote these four values of θ by $\theta_1, \theta_4, \theta_2$, and θ_3 , respectively. Then f is CU on $(0, \theta_1)$, CD on (θ_1, θ_2) ,



CU on (θ_2, θ_3) , CD on (θ_3, θ_4) , and CU on $(\theta_4, 2\pi)$. To find the *exact* y -coordinate for $\theta = \theta_1$, we have

$$\begin{aligned} f(\theta_1) &= 2 \cos \theta_1 - \cos 2\theta_1 = 2 \cos \theta_1 - (2 \cos^2 \theta_1 - 1) = 2 \left(\frac{1 + \sqrt{33}}{8} \right) - 2 \left(\frac{1 + \sqrt{33}}{8} \right)^2 + 1 \\ &= \frac{1}{4} + \frac{1}{4} \sqrt{33} - \frac{1}{32} - \frac{1}{16} \sqrt{33} - \frac{33}{32} + 1 = \frac{3}{16} + \frac{3}{16} \sqrt{33} = \frac{3}{16} (1 + \sqrt{33}) = y_1 \approx 1.26. \end{aligned}$$

Similarly, $f(\theta_2) = \frac{3}{16} (1 - \sqrt{33}) = y_2 \approx -0.89$. So f has inflection points at (θ_1, y_1) , (θ_2, y_2) , (θ_3, y_2) , and (θ_4, y_1) .

40. (a) $f(t) = t + \cos t, -2\pi \leq t \leq 2\pi \Rightarrow$

$$f'(t) = 1 - \sin t \geq 0 \text{ for all } t \text{ and } f'(t) = 0 \text{ when}$$

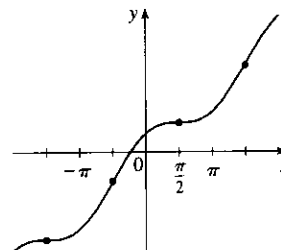
$$\sin t = 1 \Leftrightarrow t = -\frac{3\pi}{2} \text{ or } \frac{\pi}{2}, \text{ so } f \text{ is increasing}$$

$$\text{on } (-2\pi, 2\pi).$$

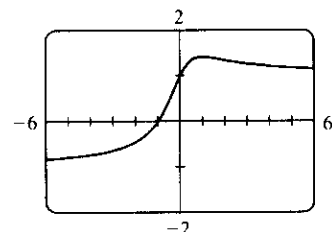
(b) No maximum or minimum

(c) $f''(t) = -\cos t > 0 \Leftrightarrow t \in (-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$, so f is CU on these intervals and CD on $(-2\pi, -\frac{3\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, and $(\frac{3\pi}{2}, 2\pi)$. Points of inflection at $(\pm\frac{3\pi}{2}, \pm\frac{3\pi}{2})$ and $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$

(d)



41. (a)



From the graph, we get an estimate of $f(1) \approx 1.41$ as a local maximum value, and no local minimum value.

$$f(x) = \frac{x+1}{\sqrt{x^2+1}} \Rightarrow f'(x) = \frac{1-x}{(x^2+1)^{3/2}}$$

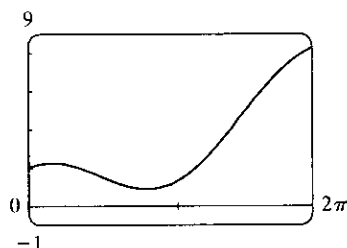
$$f'(x) = 0 \Leftrightarrow x = 1. f(1) = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ is the exact value.}$$

(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f' , which we can do by finding the critical numbers of f' .

$$f''(x) = \frac{2x^2 - 3x - 1}{(x^2 + 1)^{5/2}} = 0 \Leftrightarrow x = \frac{3 \pm \sqrt{17}}{4}. x = \frac{3 + \sqrt{17}}{4} \text{ corresponds to the minimum value of } f'.$$

$$\text{The maximum value of } f' \text{ is at } \left(\frac{3 - \sqrt{17}}{4}, \sqrt{\frac{7}{6} - \frac{\sqrt{17}}{6}} \right) \approx (-0.28, 0.69).$$

42. (a)



From the graph, we get estimates of $f(2.61) \approx 0.89$ as a local and absolute minimum, $f(0.53) \approx 2.26$ as a local maximum, and $f(2\pi) \approx 8.28$ as an absolute maximum. $f(x) = x + 2 \cos x$

$$(0 \leq x \leq 2\pi) \Rightarrow f'(x) = 1 - 2 \sin x. f'(x) = 0 \Leftrightarrow$$

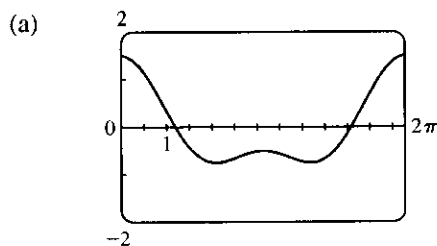
$$\sin x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}. f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3} \text{ is the exact value of the local maximum, } f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3} \text{ is the exact value of the local}$$

and absolute minimum, and $f(2\pi) = 2\pi + 2$ is the exact value of the absolute maximum.

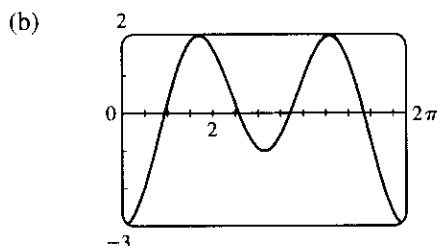
(b) From the graph in part (a), f increases most rapidly somewhere between $x = 4.5$ and $x = 5$. Now f increases most rapidly when $f'(x) = 1 - 2 \sin x$ has its maximum value. $f''(x) = -2 \cos x = 0 \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$.

$$f'(0) = f'(2\pi) = 1, f'\left(\frac{\pi}{2}\right) = -1, \text{ and } f'\left(\frac{3\pi}{2}\right) = 3. \text{ The maximum value of } f' \text{ occurs at } \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right).$$

43. $f(x) = \cos x + \frac{1}{2} \cos 2x \Rightarrow f'(x) = -\sin x - \sin 2x \Rightarrow f''(x) = -\cos x - 2 \cos 2x$



From the graph of f , it seems that f is CD on $(0, 1)$, CU on $(1, 2.5)$, CD on $(2.5, 3.7)$, CU on $(3.7, 5.3)$, and CD on $(5.3, 2\pi)$. The points of inflection appear to be at $(1, 0.4)$, $(2.5, -0.6)$, $(3.7, -0.6)$, and $(5.3, 0.4)$.



From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0, 0.94)$, CU on $(0.94, 2.57)$, CD on $(2.57, 3.71)$, CU on $(3.71, 5.35)$, and CD on $(5.35, 2\pi)$. Refined estimates of the inflection points are $(0.94, 0.44)$, $(2.57, -0.63)$, $(3.71, -0.63)$, and $(5.35, 0.44)$.

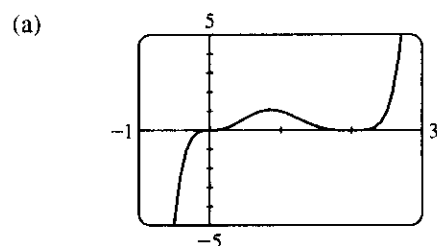
44. $f(x) = x^3(x-2)^4 \Rightarrow$

$$f'(x) = x^3 \cdot 4(x-2)^3 + (x-2)^4 \cdot 3x^2 = x^2(x-2)^3[4x + 3(x-2)] = x^2(x-2)^3(7x-6) \Rightarrow$$

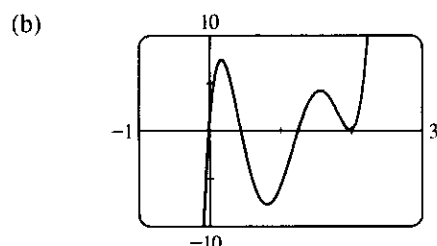
$$f''(x) = (2x)(x-2)^3(7x-6) + x^2 \cdot 3(x-2)^2(7x-6) + x^2(x-2)^3(7)$$

$$= x(x-2)^2[2(x-2)(7x-6) + 3x(7x-6) + 7x(x-2)]$$

$$= x(x-2)^2[42x^2 - 72x + 24] = 6x(x-2)^2(7x^2 - 12x + 4)$$



From the graph of f , it seems that f is CD on $(-\infty, 0)$, CU on $(0, 0.5)$, CD on $(0.5, 1.3)$, and CU on $(1.3, \infty)$. The points of inflection appear to be at $(0, 0)$, $(0.5, 0.5)$, and $(1.3, 0.6)$.



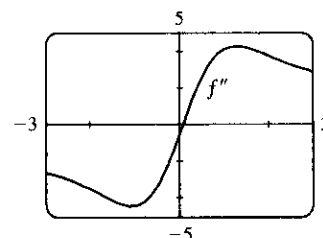
From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-\infty, 0)$, CU on $(0, 0.45)$, CD on $(0.45, 1.26)$, and CU on $(1.26, \infty)$. Refined estimates of the inflection points are $(0, 0)$, $(0.45, 0.53)$, and $(1.26, 0.60)$.

45. In Maple, we define f and then use the command

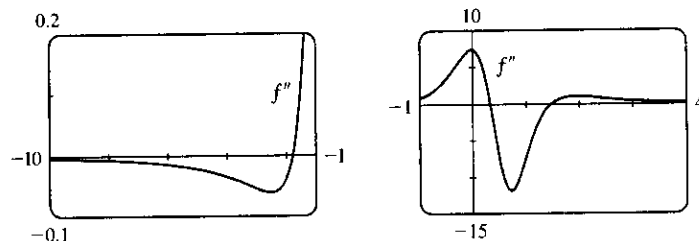
`plot(diff(diff(f,x),x),x=-3..3);` In Mathematica, we

define f and then use `Plot[Dt[Dt[f,x],x],{x,-3,3}]`. We

see that $f'' > 0$ for $x > 0.1$ and $f'' < 0$ for $x < 0.1$. So f is concave up on $(0.1, \infty)$ and concave down on $(-\infty, 0.1)$.

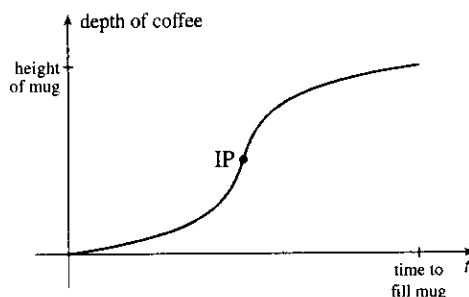


46. It appears that f'' is positive (and thus f is concave up) on $(-1.8, 0.3)$ and $(1.5, \infty)$ and negative (so f is concave down) on $(-\infty, -1.8)$ and $(0.3, 1.5)$.

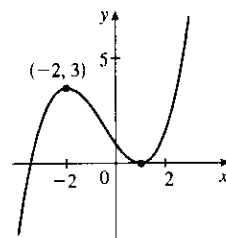


47. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

48. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



49. $f(x) = \tan x - x \Rightarrow f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $(0, \frac{\pi}{2})$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \Rightarrow \tan x - x > 0 \Rightarrow \tan x > x$ for $0 < x < \frac{\pi}{2}$.
50. Let $f(x) = 2\sqrt{x} - 3 + 1/x$. Then $f'(x) = 1/\sqrt{x} - 1/x^2 > 0$ for $x > 1$ since for $x > 1$, $x^2 > x > \sqrt{x}$. Hence, f is increasing, so for $x > 1$, $f(x) > f(1) = 0$ or $2\sqrt{x} - 3 + 1/x > 0$ for $x > 1$. Hence, $2\sqrt{x} > 3 - 1/x$ for $x > 1$.
51. $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$. We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$.
52. $f(x) = x^3 + ax^2 + bx + 2 \Rightarrow f'(x) = 3x^2 + 2ax + b$. If $x = -3$ is an extremum, then $f'(-3) = 27 - 6a + b = 0 \Leftrightarrow b = 6a - 27$. If $x = -1$ is an extremum, then $f'(-1) = 3 - 2a + b = 0 \Leftrightarrow b = 2a - 3$. So $b = 2a - 3$ and $b = 6a - 27 \Rightarrow b = 9, a = 6$. Then $f'(x) = 3x^2 + 12x + 9 = 3(x + 1)(x + 3)$ and the First Derivative Test shows that f has a local maximum when $x = -3$ and a local minimum when $x = -1$.



53. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I , let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J , we see that f is increasing on J , so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I .

54. (a) We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I , then $f'' > 0$ on I . If f and g are CU on I , then $f'' > 0$ and $g'' > 0$ on I , so $(f + g)'' = f'' + g'' > 0$ on $I \Rightarrow f + g$ is CU on I .

(b) Since f is positive and CU on I , $f > 0$ and $f'' > 0$ on I . So $g(x) = [f(x)]^2 \Rightarrow g' = 2ff' \Rightarrow g'' = 2f'f' + 2ff'' = 2(f')^2 + 2ff'' > 0 \Rightarrow g$ is CU on I .

55. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f > 0$, $f' \geq 0$, $f'' > 0$, $g > 0$, $g' \geq 0$, $g'' > 0$ on I . Then $(fg)' = f'g + fg' \Rightarrow (fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I .

(b) In part (a), if f and g are both decreasing instead of increasing, then $f' \leq 0$ and $g' \leq 0$ on I , so we still have $2f'g' \geq 0$ on I . Thus, $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I as in part (a).

(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I , so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I . Thus, fg is CU on I .

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I . Thus, fg is CD on I .

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I .

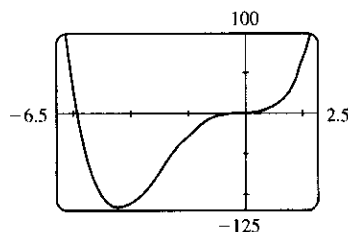
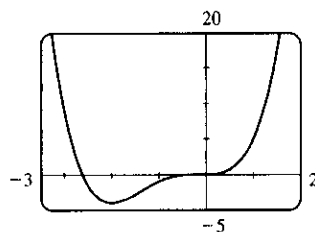
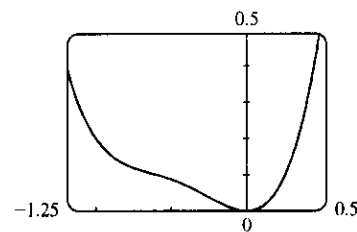
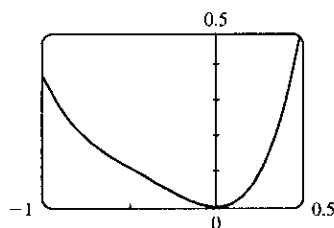
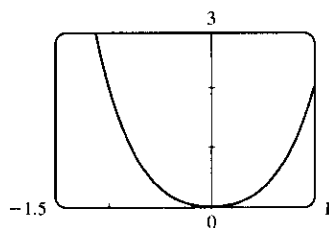
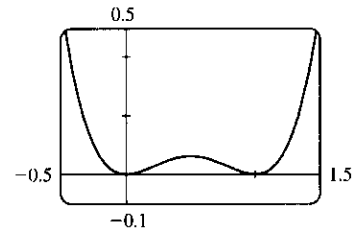
56. Since f and g are CU on $(-\infty, \infty)$, $f'' > 0$ and $g'' > 0$ on $(-\infty, \infty)$.

$$h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x) \Rightarrow$$

$$h''(x) = f''(g(x))g'(x)g'(x) + f'(g(x))g''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))g''(x) > 0 \text{ if } f' > 0.$$

So h is CU if f is increasing.

57. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c \Rightarrow f''(x) = 6ax + 2b$. So f is CU when $6ax + 2b > 0 \Leftrightarrow x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x -intercepts x_1, x_2 and x_3 , then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us $f(x) = a[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3]$. Equating the coefficients of the x^2 -terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x -coordinate of the point of inflection is $-\frac{b}{3a} = -\frac{-a(x_1 + x_2 + x_3)}{3a} = \frac{x_1 + x_2 + x_3}{3}$.
58. $P(x) = x^4 + cx^3 + x^2 \Rightarrow P'(x) = 4x^3 + 3cx^2 + 2x \Rightarrow P''(x) = 12x^2 + 6cx + 2$. The graph of $P''(x)$ is a parabola. If $P''(x)$ has two roots, then it changes sign twice and so has two inflection points. This happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2 - 4 \cdot 12 \cdot 2 > 0 \Leftrightarrow 36c^2 - 96 > 0 \Leftrightarrow |c| > \frac{2\sqrt{6}}{3} \approx 1.63$. If $36c^2 - 96 = 0 \Leftrightarrow c = \pm \frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2 - 96 < 0 \Leftrightarrow |c| < \frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.

 $c = 6$  $c = 3$  $c = 1.8$  $c = \frac{2\sqrt{6}}{3}$  $c = 0$  $c = -2$

For large positive c , the graph of f has two inflection points and a large dip to the left of the y -axis. As c decreases, the graph of f becomes flatter for $x < 0$, and eventually the dip rises above the x -axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

59. By hypothesis $g = f'$ is differentiable on an open interval containing c . Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.
60. $f(x) = x^4 \Rightarrow f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2 \Rightarrow f''(0) = 0$. For $x < 0$, $f''(x) > 0$, so f is CU on $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is also CU on $(0, \infty)$. Since f does not change concavity at 0, $(0, 0)$ is not an inflection point.

61. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\sqrt{x^2} \Rightarrow g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2|x| \Rightarrow g''(x) = 2x(x^2)^{-1/2} = \frac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.
62. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f'' = (f')'$ changes from negative to positive at c . So by the First Derivative Test, f' has a local minimum at $x = c$ and thus cannot change sign there, so f has no maximum or minimum at c . But since f'' changes from negative to positive at c , f has a point of inflection at c (it changes from concave down to concave up).

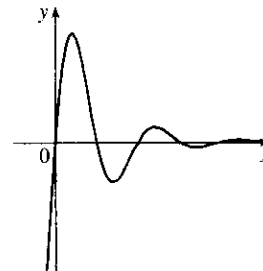
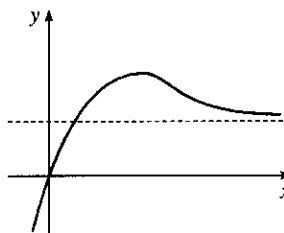
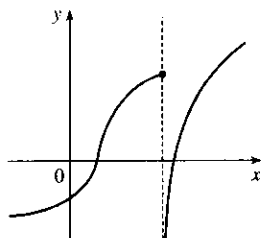
4.4 Limits at Infinity; Horizontal Asymptotes

1. (a) As x becomes large, the values of $f(x)$ approach 5.

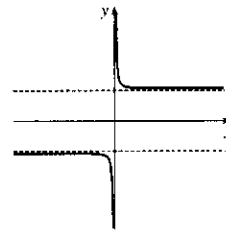
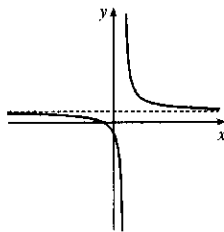
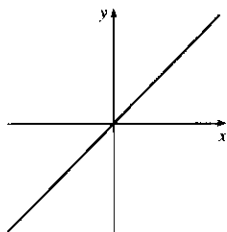
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.



- (b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote

One horizontal asymptote

Two horizontal asymptotes

3. (a) $\lim_{x \rightarrow 2} f(x) = \infty$

(c) $\lim_{x \rightarrow -1^+} f(x) = -\infty$

(e) $\lim_{x \rightarrow -\infty} f(x) = 2$

(b) $\lim_{x \rightarrow -1^-} f(x) = \infty$

(d) $\lim_{x \rightarrow \infty} f(x) = 1$

(f) Vertical: $x = -1, x = 2$; Horizontal: $y = 1, y = 2$

4. (a) $\lim_{x \rightarrow \infty} g(x) = 2$

(b) $\lim_{x \rightarrow -\infty} g(x) = -2$

(c) $\lim_{x \rightarrow 3} g(x) = \infty$

(d) $\lim_{x \rightarrow 0} g(x) = -\infty$

(e) $\lim_{x \rightarrow -2^+} g(x) = -\infty$

(f) Vertical: $x = -2, x = 0, x = 3$; Horizontal: $y = -2, y = 2$

5. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0, f(1) = 0.5, f(2) = 1, f(3) = 1.125, f(4) = 1, f(5) = 0.78125, f(6) = 0.5625, f(7) = 0.3828125, f(8) = 0.25, f(9) = 0.158203125, f(10) = 0.09765625, f(20) \approx 0.00038147, f(50) \approx 2.2204 \times 10^{-12}, f(100) \approx 7.8886 \times 10^{-27}$.
It appears that $\lim_{x \rightarrow \infty} (x^2/2^x) = 0$.

6. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10,000]$ by $[0, 0.2]$, we estimate that $\lim_{x \rightarrow \infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

From the table, we estimate that $\lim_{x \rightarrow \infty} f(x) = 0.1353$

(to four decimal places.)

7. $\lim_{x \rightarrow \infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8} = \lim_{x \rightarrow \infty} \frac{(3x^2 - x + 4)/x^2}{(2x^2 + 5x - 8)/x^2}$

[divide both the numerator and denominator by x^2
(the highest power of x that
appears in the denominator)]

$$= \frac{\lim_{x \rightarrow \infty} (3 - 1/x + 4/x^2)}{\lim_{x \rightarrow \infty} (2 + 5/x - 8/x^2)}$$

[Limit Law 5]

$$= \frac{\lim_{x \rightarrow \infty} 3 - \lim_{x \rightarrow \infty} (1/x) + \lim_{x \rightarrow \infty} (4/x^2)}{\lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} (5/x) - \lim_{x \rightarrow \infty} (8/x^2)}$$

[Limit Laws 1 and 2]

$$= \frac{3 - \lim_{x \rightarrow \infty} (1/x) + 4 \lim_{x \rightarrow \infty} (1/x^2)}{2 + 5 \lim_{x \rightarrow \infty} (1/x) - 8 \lim_{x \rightarrow \infty} (1/x^2)}$$

[Limit Laws 7 and 3]

$$= \frac{3 - 0 + 4(0)}{2 + 5(0) - 8(0)}$$

[Theorem 4]

$$= \frac{3}{2}$$

8. $\lim_{x \rightarrow \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}} = \sqrt{\lim_{x \rightarrow \infty} \frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

[Limit Law 11]

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{12 - 5/x^2 + 2/x^3}{1/x^3 + 4/x + 3}}$$

[divide by x^3]

$$= \sqrt{\frac{\lim_{x \rightarrow \infty} (12 - 5/x^2 + 2/x^3)}{\lim_{x \rightarrow \infty} (1/x^3 + 4/x + 3)}}$$

[Limit Law 5]

$$\begin{aligned}
&= \sqrt{\frac{\lim_{x \rightarrow \infty} 12 - \lim_{x \rightarrow \infty} (5/x^2) + \lim_{x \rightarrow \infty} (2/x^3)}{\lim_{x \rightarrow \infty} (1/x^3) + \lim_{x \rightarrow \infty} (4/x) + \lim_{x \rightarrow \infty} 3}} \quad [\text{Limit Laws 1 and 2}] \\
&= \sqrt{\frac{12 - 5 \lim_{x \rightarrow \infty} (1/x^2) + 2 \lim_{x \rightarrow \infty} (1/x^3)}{\lim_{x \rightarrow \infty} (1/x^3) + 4 \lim_{x \rightarrow \infty} (1/x) + 3}} \quad [\text{Limit Laws 7 and 3}] \\
&= \sqrt{\frac{12 - 5(0) + 2(0)}{0 + 4(0) + 3}} \quad [\text{Theorem 4}] \\
&= \sqrt{\frac{12}{3}} = \sqrt{4} = 2
\end{aligned}$$

$$9. \lim_{x \rightarrow \infty} \frac{1}{2x+3} = \lim_{x \rightarrow \infty} \frac{1/x}{(2x+3)/x} = \frac{\lim_{x \rightarrow \infty} (1/x)}{\lim_{x \rightarrow \infty} (2+3/x)} = \frac{\lim_{x \rightarrow \infty} (1/x)}{\lim_{x \rightarrow \infty} 2+3 \lim_{x \rightarrow \infty} (1/x)} = \frac{0}{2+3(0)} = \frac{0}{2} = 0$$

$$10. \lim_{x \rightarrow \infty} \frac{3x+5}{x-4} = \lim_{x \rightarrow \infty} \frac{(3x+5)/x}{(x-4)/x} = \lim_{x \rightarrow \infty} \frac{3+5/x}{1-4/x} = \frac{\lim_{x \rightarrow \infty} 3+5 \lim_{x \rightarrow \infty} \frac{1}{x}}{\lim_{x \rightarrow \infty} 1-4 \lim_{x \rightarrow \infty} \frac{1}{x}} = \frac{3+5(0)}{1-4(0)} = 3$$

$$\begin{aligned}
11. \lim_{x \rightarrow -\infty} \frac{1-x-x^2}{2x^2-7} &= \lim_{x \rightarrow -\infty} \frac{(1-x-x^2)/x^2}{(2x^2-7)/x^2} = \frac{\lim_{x \rightarrow -\infty} (1/x^2 - 1/x - 1)}{\lim_{x \rightarrow -\infty} (2 - 7/x^2)} \\
&= \frac{\lim_{x \rightarrow -\infty} (1/x^2) - \lim_{x \rightarrow -\infty} (1/x) - \lim_{x \rightarrow -\infty} 1}{\lim_{x \rightarrow -\infty} 2 - 7 \lim_{x \rightarrow -\infty} (1/x^2)} = \frac{0 - 0 - 1}{2 - 7(0)} = -\frac{1}{2}
\end{aligned}$$

$$12. \lim_{y \rightarrow \infty} \frac{2-3y^2}{5y^2+4y} = \lim_{y \rightarrow \infty} \frac{(2-3y^2)/y^2}{(5y^2+4y)/y^2} = \frac{\lim_{y \rightarrow \infty} (2/y^2 - 3)}{\lim_{y \rightarrow \infty} (5+4/y)} = \frac{2 \lim_{y \rightarrow \infty} (1/y^2) - \lim_{y \rightarrow \infty} 3}{\lim_{y \rightarrow \infty} 5+4 \lim_{y \rightarrow \infty} (1/y)} = \frac{2(0)-3}{5+4(0)} = -\frac{3}{5}$$

13. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\begin{aligned}
\lim_{x \rightarrow \infty} \frac{x^3+5x}{2x^3-x^2+4} &= \lim_{x \rightarrow \infty} \frac{\frac{x^3+5x}{x^3}}{\frac{2x^3-x^2+4}{x^3}} = \lim_{x \rightarrow \infty} \frac{1+\frac{5}{x^2}}{2-\frac{1}{x}+\frac{4}{x^3}} = \frac{\lim_{x \rightarrow \infty} \left(1+\frac{5}{x^2}\right)}{\lim_{x \rightarrow \infty} \left(2-\frac{1}{x}+\frac{4}{x^3}\right)} \\
&= \frac{\lim_{x \rightarrow \infty} 1+5 \lim_{x \rightarrow \infty} \frac{1}{x^2}}{\lim_{x \rightarrow \infty} 2-\lim_{x \rightarrow \infty} \frac{1}{x}+4 \lim_{x \rightarrow \infty} \frac{1}{x^3}} = \frac{1+5(0)}{2-0+4(0)} = \frac{1}{2}
\end{aligned}$$

$$14. \lim_{t \rightarrow -\infty} \frac{t^2+2}{t^3+t^2-1} = \lim_{t \rightarrow -\infty} \frac{(t^2+2)/t^3}{(t^3+t^2-1)/t^3} = \lim_{t \rightarrow -\infty} \frac{1/t+2/t^3}{1+1/t-1/t^3} = \frac{0+0}{1+0-0} = 0$$

15. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4 .

$$\begin{aligned}\lim_{u \rightarrow \infty} \frac{4u^4 + 5}{(u^2 - 2)(2u^2 - 1)} &= \lim_{u \rightarrow \infty} \frac{4u^4 + 5}{2u^4 - 5u^2 + 2} = \lim_{u \rightarrow \infty} \frac{\frac{4u^4 + 5}{u^4}}{\frac{2u^4 - 5u^2 + 2}{u^4}} = \lim_{u \rightarrow \infty} \frac{4 + \frac{5}{u^4}}{2 - \frac{5}{u^2} + \frac{2}{u^4}} \\&= \frac{\lim_{u \rightarrow \infty} \left(4 + \frac{5}{u^4}\right)}{\lim_{u \rightarrow \infty} \left(2 - \frac{5}{u^2} + \frac{2}{u^4}\right)} = \frac{\lim_{u \rightarrow \infty} 4 + 5 \lim_{u \rightarrow \infty} \frac{1}{u^4}}{\lim_{u \rightarrow \infty} 2 - 5 \lim_{u \rightarrow \infty} \frac{1}{u^2} + 2 \lim_{u \rightarrow \infty} \frac{1}{u^4}} = \frac{4 + 5(0)}{2 - 5(0) + 2(0)} \\&= \frac{4}{2} = 2\end{aligned}$$

$$16. \lim_{x \rightarrow \infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x \rightarrow \infty} \frac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}} = \lim_{x \rightarrow \infty} \frac{1+2/x}{\sqrt{9+1/x^2}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}$$

$$\begin{aligned}17. \lim_{x \rightarrow \infty} \frac{\sqrt{9x^6-x}}{x^3+1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3} = \frac{\lim_{x \rightarrow \infty} \sqrt{(9x^6-x)/x^6}}{\lim_{x \rightarrow \infty} (1+1/x^3)} \quad [\text{since } x^3 = \sqrt{x^6} \text{ for } x > 0] \\&= \frac{\lim_{x \rightarrow \infty} \sqrt{9-1/x^5}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} (1/x^3)} = \frac{\sqrt{\lim_{x \rightarrow \infty} 9 - \lim_{x \rightarrow \infty} (1/x^5)}}{1+0} \\&= \sqrt{9-0} = 3\end{aligned}$$

$$\begin{aligned}18. \lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6-x}}{x^3+1} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3} = \frac{\lim_{x \rightarrow -\infty} -\sqrt{(9x^6-x)/x^6}}{\lim_{x \rightarrow -\infty} (1+1/x^3)} \quad [\text{since } x^3 = -\sqrt{x^6} \text{ for } x < 0] \\&= \frac{\lim_{x \rightarrow -\infty} -\sqrt{9-1/x^5}}{\lim_{x \rightarrow -\infty} 1 + \lim_{x \rightarrow -\infty} (1/x^3)} = \frac{-\sqrt{\lim_{x \rightarrow -\infty} 9 - \lim_{x \rightarrow -\infty} (1/x^5)}}{1+0} \\&= -\sqrt{9-0} = -3\end{aligned}$$

$$19. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x \rightarrow -\infty} \frac{x/\sqrt{x^2}}{\sqrt{x^2+4}/\sqrt{x^2}} = \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{1+4/x^2}} \quad [\text{since } -x = |x| = \sqrt{x^2} \text{ for } x < 0] = -1$$

$$\begin{aligned}20. \lim_{x \rightarrow \infty} (\sqrt{x^4+6x^2} - x^2) &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^4+6x^2} - x^2}{1} \cdot \frac{\sqrt{x^4+6x^2} + x^2}{\sqrt{x^4+6x^2} + x^2} = \lim_{x \rightarrow \infty} \frac{(x^4+6x^2) - x^4}{\sqrt{x^4+6x^2} + x^2} \\&= \lim_{x \rightarrow \infty} \frac{6x^2}{\sqrt{x^4+6x^2} + x^2} = \lim_{x \rightarrow \infty} \frac{6x^2/x^2}{(\sqrt{x^4+6x^2} + x^2)/\sqrt{x^4}} \\&= \lim_{x \rightarrow \infty} \frac{6}{\sqrt{1+6/x^2} + 1} = \frac{6}{1+1} = 3\end{aligned}$$

$$\begin{aligned}21. \lim_{x \rightarrow \infty} (\sqrt{9x^2+x} - 3x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{9x^2+x} - 3x)(\sqrt{9x^2+x} + 3x)}{\sqrt{9x^2+x} + 3x} = \lim_{x \rightarrow \infty} \frac{(\sqrt{9x^2+x})^2 - (3x)^2}{\sqrt{9x^2+x} + 3x} \\&= \lim_{x \rightarrow \infty} \frac{(9x^2+x) - 9x^2}{\sqrt{9x^2+x} + 3x} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{9x^2+x} + 3x} \cdot \frac{1/x}{1/x} \\&= \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{9x^2/x^2 + x/x^2} + 3x/x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{9+1/x} + 3} = \frac{1}{\sqrt{9+0} + 3} = \frac{1}{3+3} = \frac{1}{6}\end{aligned}$$

$$\begin{aligned}
 22. \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 2x}) &= \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 2x}) \left[\frac{x - \sqrt{x^2 + 2x}}{x - \sqrt{x^2 + 2x}} \right] = \lim_{x \rightarrow -\infty} \frac{x^2 - (x^2 + 2x)}{x - \sqrt{x^2 + 2x}} \\
 &= \lim_{x \rightarrow -\infty} \frac{-2x}{x - \sqrt{x^2 + 2x}} = \lim_{x \rightarrow -\infty} \frac{-2}{1 + \sqrt{1 + 2/x}} = \frac{-2}{1 + \sqrt{1 + 2(0)}} = -1
 \end{aligned}$$

Note: In dividing numerator and denominator by x , we used the fact that for $x < 0$, $x = -\sqrt{x^2}$.

$$\begin{aligned}
 23. \lim_{x \rightarrow \infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx}) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + ax} - \sqrt{x^2 + bx})(\sqrt{x^2 + ax} + \sqrt{x^2 + bx})}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} \\
 &= \lim_{x \rightarrow \infty} \frac{(x^2 + ax) - (x^2 + bx)}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} = \lim_{x \rightarrow \infty} \frac{[(a - b)x]/x}{(\sqrt{x^2 + ax} + \sqrt{x^2 + bx})/\sqrt{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{a - b}{\sqrt{1 + a/x} + \sqrt{1 + b/x}} = \frac{a - b}{\sqrt{1 + 0} + \sqrt{1 + 0}} = \frac{a - b}{2}
 \end{aligned}$$

24. $\lim_{x \rightarrow \infty} \cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1 .

25. $\sqrt{x}$ is large when x is large, so $\lim_{x \rightarrow \infty} \sqrt{x} = \infty$.

26. $\sqrt[3]{x}$ is large negative when x is large negative, so $\lim_{x \rightarrow -\infty} \sqrt[3]{x} = -\infty$.

27. $\lim_{x \rightarrow \infty} (x - \sqrt{x}) = \lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x} - 1) = \infty$ since $\sqrt{x} \rightarrow \infty$ and $\sqrt{x} - 1 \rightarrow \infty$ as $x \rightarrow \infty$.

$$\begin{aligned}
 28. \lim_{x \rightarrow \infty} \frac{x^3 - 2x + 3}{5 - 2x^2} &= \lim_{x \rightarrow \infty} \frac{(x^3 - 2x + 3)/x^2}{(5 - 2x^2)/x^2} \quad [\text{divide by the highest power of } x \text{ in the denominator}] \\
 &= \lim_{x \rightarrow \infty} \frac{x - 2/x + 3/x^2}{5/x^2 - 2} = -\infty \text{ because } x - 2/x + 3/x^2 \rightarrow \infty \text{ and } 5/x^2 - 2 \rightarrow -2 \text{ as } x \rightarrow \infty.
 \end{aligned}$$

29. $\lim_{x \rightarrow -\infty} (x^4 + x^5) = \lim_{x \rightarrow -\infty} x^5(\frac{1}{x} + 1)$ [factor out the largest power of x] $= -\infty$ because $x^5 \rightarrow -\infty$ and $1/x + 1 \rightarrow 1$ as $x \rightarrow -\infty$.

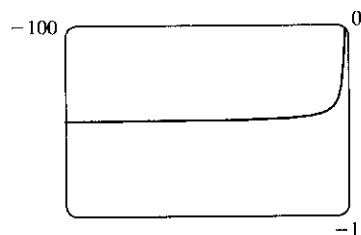
30. $\lim_{x \rightarrow \infty} (x^2 - x^4) = \lim_{x \rightarrow \infty} x^2(1 - x^2) = -\infty$ since $x^2 \rightarrow \infty$ and $1 - x^2 \rightarrow -\infty$.

$$\begin{aligned}
 31. \lim_{x \rightarrow \infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4} &= \lim_{x \rightarrow \infty} \frac{(x + x^3 + x^5)/x^4}{(1 - x^2 + x^4)/x^4} \quad [\text{divide by the highest power of } x \text{ in the denominator}] \\
 &= \lim_{x \rightarrow \infty} \frac{1/x^3 + 1/x + x}{1/x^4 - 1/x^2 + 1} = \infty
 \end{aligned}$$

because $(1/x^3 + 1/x + x) \rightarrow \infty$ and $(1/x^4 - 1/x^2 + 1) \rightarrow 1$ as $x \rightarrow \infty$.

32. If $t = \frac{1}{x}$, then $\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{1}{t} \sin t = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1$.

33. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x \rightarrow -\infty} f(x)$ to be -0.5 .

(b)

x	$f(x)$
-10,000	-0.4999625
-100,000	-0.4999962
-1,000,000	-0.4999996

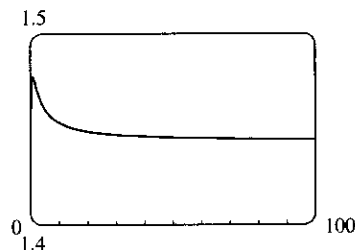
From the table, we estimate the limit to be -0.5 .

$$\begin{aligned}
 \text{(c)} \quad \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x) &= \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x) \left[\frac{\sqrt{x^2 + x + 1} - x}{\sqrt{x^2 + x + 1} - x} \right] = \lim_{x \rightarrow -\infty} \frac{(x^2 + x + 1) - x^2}{\sqrt{x^2 + x + 1} - x} \\
 &= \lim_{x \rightarrow -\infty} \frac{(x+1)(1/x)}{(\sqrt{x^2 + x + 1} - x)(1/x)} = \lim_{x \rightarrow -\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1} \\
 &= \frac{1 + 0}{-\sqrt{1 + 0 + 0} - 1} = -\frac{1}{2}
 \end{aligned}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x , with $x < 0$, we get

$$\frac{1}{x} \sqrt{x^2 + x + 1} = -\frac{1}{\sqrt{x^2}} \sqrt{x^2 + x + 1} = -\sqrt{1 + (1/x) + (1/x^2)}.$$

34. (a)



From the graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}, \text{ we estimate}$$

(to one decimal place) the value of $\lim_{x \rightarrow \infty} f(x)$ to be 1.4.

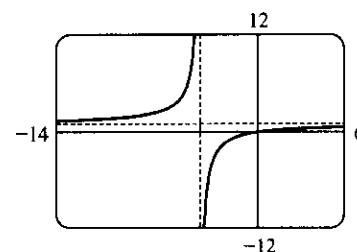
(b)

x	$f(x)$
10,000	1.44339
100,000	1.44338
1,000,000	1.44338

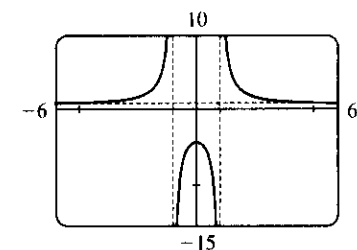
From the table, we estimate (to four decimal places) the limit to be 1.4434.

$$\begin{aligned}
 \text{(c)} \quad \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1})(\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1})}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{(3x^2 + 8x + 6) - (3x^2 + 3x + 1)}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{(5x + 5)(1/x)}{(\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1})(1/x)} \\
 &= \lim_{x \rightarrow \infty} \frac{5 + 5/x}{\sqrt{3 + 8/x + 6/x^2} + \sqrt{3 + 3/x + 1/x^2}} = \frac{5}{\sqrt{3} + \sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376
 \end{aligned}$$

35. $\lim_{x \rightarrow \pm\infty} \frac{x}{x+4} = \lim_{x \rightarrow \pm\infty} \frac{1}{1+4/x} = \frac{1}{1+0} = 1$, so $y = 1$ is a horizontal asymptote. $\lim_{x \rightarrow -4^-} \frac{x}{x+4} = \infty$ and $\lim_{x \rightarrow -4^+} \frac{x}{x+4} = -\infty$, so $x = -4$ is a vertical asymptote. The graph confirms these calculations.



36. Since $x^2 - 1 \rightarrow 0$ as $x \rightarrow \pm 1$ and $y < 0$ for $-1 < x < 1$ and $y > 0$ for $x < -1$ and $x > 1$, we have $\lim_{x \rightarrow 1^-} \frac{x^2 + 4}{x^2 - 1} = -\infty$, $\lim_{x \rightarrow 1^+} \frac{x^2 + 4}{x^2 - 1} = \infty$, $\lim_{x \rightarrow -1^-} \frac{x^2 + 4}{x^2 - 1} = \infty$, and $\lim_{x \rightarrow -1^+} \frac{x^2 + 4}{x^2 - 1} = -\infty$, so $x = 1$ and $x = -1$ are vertical asymptotes. Also $\lim_{x \rightarrow \pm\infty} \frac{x^2 + 4}{x^2 - 1} = \lim_{x \rightarrow \pm\infty} \frac{1 + 4/x^2}{1 - 1/x^2} = \frac{1+0}{1-0} = 1$, so $y = 1$ is a horizontal asymptote. The graph confirms these calculations.



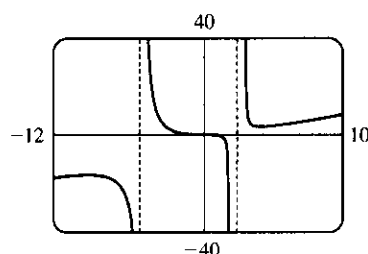
37. $\lim_{x \rightarrow \pm\infty} \frac{x^3}{x^2 + 3x - 10} = \lim_{x \rightarrow \pm\infty} \frac{x}{1 + (3/x) - (10/x^2)} = \pm\infty$, so there is no horizontal asymptote.

$$\lim_{x \rightarrow 2^+} \frac{x^3}{x^2 + 3x - 10} = \lim_{x \rightarrow 2^+} \frac{x^3}{(x+5)(x-2)} = \infty, \text{ since}$$

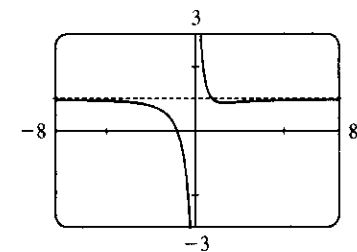
$$\frac{x^3}{(x+5)(x-2)} > 0 \text{ for } x > 2. \text{ Similarly, } \lim_{x \rightarrow 2^-} \frac{x^3}{x^2 + 3x - 10} = -\infty,$$

$$\lim_{x \rightarrow -5^-} \frac{x^3}{x^2 + 3x - 10} = -\infty, \text{ and } \lim_{x \rightarrow -5^+} \frac{x^3}{x^2 + 3x - 10} = \infty, \text{ so}$$

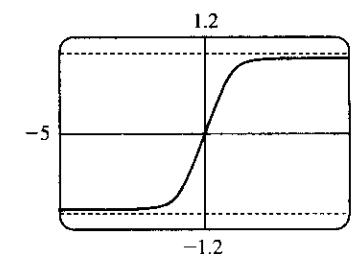
$x = 2$ and $x = -5$ are vertical asymptotes. The graph confirms these calculations.



38. $\lim_{x \rightarrow \pm\infty} \frac{x^3 + 1}{x^3 + x} = \lim_{x \rightarrow \pm\infty} \frac{1 + 1/x^3}{1 + 1/x^2} = 1$, so $y = 1$ is a horizontal asymptote. Since $y = \frac{x^3 + 1}{x^3 + x} = \frac{x^3 + 1}{x(x^2 + 1)}$ > 0 for $x > 0$ and $y < 0$ for $-1 < x < 0$, $\lim_{x \rightarrow 0^+} \frac{x^3 + 1}{x^3 + x} = \infty$ and $\lim_{x \rightarrow 0^-} \frac{x^3 + 1}{x^3 + x} = -\infty$, so $x = 0$ is a vertical asymptote.



39. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt[4]{x^4 + 1}} \cdot \frac{1/x}{1/\sqrt[4]{x^4}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt[4]{1 + \frac{1}{x^4}}} = \frac{1}{\sqrt[4]{1+0}} = 1$ and $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt[4]{x^4 + 1}} \cdot \frac{1/x}{-1/\sqrt[4]{x^4}} = \lim_{x \rightarrow -\infty} \frac{1}{-\sqrt[4]{1 + \frac{1}{x^4}}} = \frac{1}{-\sqrt[4]{1+0}} = -1$, so $y = \pm 1$ are horizontal asymptotes. There is no vertical asymptote.



$$40. \lim_{x \rightarrow \infty} \frac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x \rightarrow \infty} \frac{1-9/x}{\sqrt{4+(3/x)+(2/x^2)}} = \frac{1-0}{\sqrt{4+0+0}} = \frac{1}{2}.$$

Using the fact that $\sqrt{x^2} = |x| = -x$ for $x < 0$, we divide the numerator by $-x$ and the denominator by $\sqrt{x^2}$.

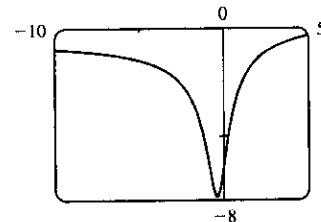
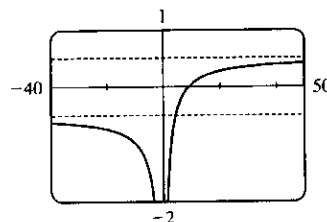
$$\text{Thus, } \lim_{x \rightarrow -\infty} \frac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x \rightarrow -\infty} \frac{-1+9/x}{\sqrt{4+(3/x)+(2/x^2)}} = \frac{-1+0}{\sqrt{4+0+0}} = -\frac{1}{2}.$$

The horizontal asymptotes are $y = \pm \frac{1}{2}$. The

polynomial $4x^2 + 3x + 2$ is positive for all x ,

so the denominator never approaches zero,

and thus there is no vertical asymptote.



41. Let's look for a rational function.

$$(1) \lim_{x \rightarrow \pm\infty} f(x) = 0 \Rightarrow \text{degree of numerator} < \text{degree of denominator}$$

$$(2) \lim_{x \rightarrow 0} f(x) = -\infty \Rightarrow \text{there is a factor of } x^2 \text{ in the denominator (not just } x, \text{ since that would produce a sign change at } x = 0), \text{ and the function is negative near } x = 0.$$

$$(3) \lim_{x \rightarrow 3} f(x) = \infty \text{ and } \lim_{x \rightarrow 3^+} f(x) = -\infty \Rightarrow \text{vertical asymptote at } x = 3; \text{ there is a factor of } (x-3) \text{ in the denominator.}$$

$$(4) f(2) = 0 \Rightarrow 2 \text{ is an } x\text{-intercept; there is at least one factor of } (x-2) \text{ in the numerator.}$$

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits

$$\text{gives us } f(x) = \frac{2-x}{x^2(x-3)} \text{ as one possibility.}$$

42. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility

$$\text{is } f(x) = \frac{x^2}{(x-1)(x-3)}.$$

43. $y = \frac{1-x}{1+x}$ has domain $(-\infty, -1) \cup (-1, \infty)$.

$$\lim_{x \rightarrow \pm\infty} \frac{1-x}{1+x} = \lim_{x \rightarrow \pm\infty} \frac{1/x-1}{1/x+1} = \frac{0-1}{0+1} = -1, \text{ so } y = -1 \text{ is a HA.}$$

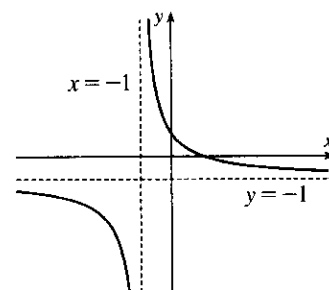
The line $x = -1$ is a VA.

$$y' = \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} = \frac{-2}{(1+x)^2} < 0 \text{ for } x \neq -1. \text{ Thus,}$$

$(-\infty, -1)$ and $(-1, \infty)$ are intervals of decrease.

$$y'' = -2 \cdot \frac{-2(1+x)}{[(1+x)^2]^2} = \frac{4}{(1+x)^3} < 0 \text{ for } x < -1 \text{ and } y'' > 0 \text{ for } x > -1, \text{ so the curve is CD on } (-\infty, -1)$$

and CU on $(-1, \infty)$. Since $x = -1$ is not in the domain, there is no IP.



44. $y = \frac{1+2x^2}{1+x^2}$ has domain $\mathbb{R}$.

$$\lim_{x \rightarrow \pm\infty} \frac{1+2x^2}{1+x^2} = \lim_{x \rightarrow \pm\infty} \frac{1/x^2 + 2}{1/x^2 + 1} = \frac{0+2}{0+1} = 2, \text{ so } y = 2 \text{ is a HA.}$$

$$\text{There is no VA. } y' = \frac{(1+x^2)(4x) - (1+2x^2)(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2} > 0$$

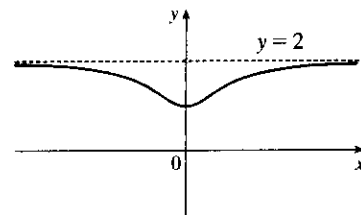
$$\Leftrightarrow x > 0,$$

and $y' < 0 \Leftrightarrow x < 0$. Thus, y is increasing on $(0, \infty)$ and y is decreasing on $(-\infty, 0)$. There is a local (and

$$\text{absolute) minimum at } (0, 1). \quad y'' = \frac{(1+x^2)^2(2) - (2x) \cdot 2(1+x^2)(2x)}{[(1+x^2)^2]^2} = \frac{2-6x^2}{(1+x^2)^3} = 0 \Leftrightarrow x = \pm \frac{1}{\sqrt{3}}.$$

$$y'' > 0 \Leftrightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}, \text{ so the curve is CU on } \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and CD on } \left(-\infty, -\frac{1}{\sqrt{3}}\right) \text{ and } \left(\frac{1}{\sqrt{3}}, \infty\right).$$

$$\text{There are IP at } \left(\pm \frac{1}{\sqrt{3}}, \frac{5}{4}\right).$$



45. $\lim_{x \rightarrow \pm\infty} \frac{x}{x^2+1} = \lim_{x \rightarrow \pm\infty} \frac{1/x}{1+1/x^2} = \frac{0}{1+0} = 0$, so $y = 0$ is a

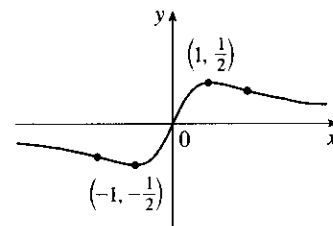
horizontal asymptote.

$$y' = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0 \text{ when } x = \pm 1 \text{ and } y' > 0 \Leftrightarrow$$

$$x^2 < 1 \Leftrightarrow -1 < x < 1, \text{ so } y \text{ is increasing on } (-1, 1) \text{ and decreasing on } (-\infty, -1) \text{ and } (1, \infty).$$

$$y'' = \frac{(1+x^2)^2(-2x) - (1-x^2)2(x^2+1)2x}{(1+x^2)^4} = \frac{2x(x^2-3)}{(1+x^2)^3} > 0 \Leftrightarrow x > \sqrt{3} \text{ or } -\sqrt{3} < x < 0, \text{ so } y \text{ is}$$

$$\text{CU on } (\sqrt{3}, \infty) \text{ and } (-\sqrt{3}, 0) \text{ and CD on } (-\infty, -\sqrt{3}) \text{ and } (0, \sqrt{3}).$$



46. $y = \frac{x}{\sqrt{x^2+1}} = \frac{x/|x|}{\sqrt{1+1/x^2}}$ has domain $\mathbb{R}$. As $x \rightarrow \pm\infty$, $y \rightarrow \pm 1$, so

$$y = \pm 1 \text{ are HA. There is no VA. } y = x(x^2+1)^{-1/2} \Rightarrow$$

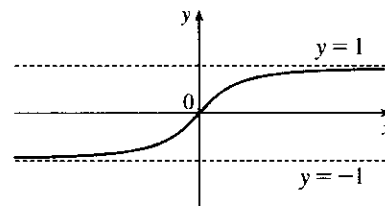
$$y' = x(-\frac{1}{2})(x^2+1)^{-3/2}(2x) + (x^2+1)^{-1/2}(1)$$

$$= (x^2+1)^{-3/2}[-x^2 + (x^2+1)]$$

$$= (x^2+1)^{-3/2} > 0 \text{ for all } x$$

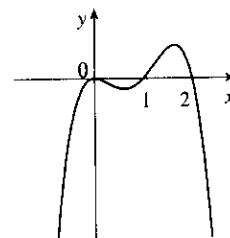
$$\text{Thus, } y \text{ is increasing for all } x. \quad y'' = (-\frac{3}{2})(x^2+1)^{-5/2}(2x) = \frac{-3x}{(x^2+1)^{5/2}} > 0 \text{ for } x < 0. \text{ So the curve is CU}$$

$$\text{on } (-\infty, 0) \text{ and CD on } (0, \infty). \text{ There is an inflection point at } (0, 0).$$



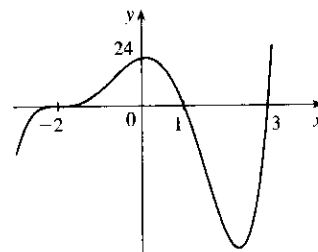
47. $y = f(x) = x^2(x - 2)(1 - x)$. The y -intercept is $f(0) = 0$, and the x -intercepts occur when $y = 0 \Rightarrow x = 0, 1$, and 2 . Notice that, since x^2 is always positive, the graph does not cross the x -axis at 0 , but does cross the x -axis at 1 and 2 .

$\lim_{x \rightarrow \infty} x^2(x - 2)(1 - x) = -\infty$, since the first two factors are large positive and the third large negative when x is large positive. $\lim_{x \rightarrow -\infty} x^2(x - 2)(1 - x) = -\infty$ because the first and third factors are large positive and the second large negative as $x \rightarrow -\infty$.

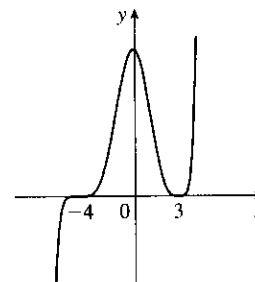


48. $y = (2 + x)^3(1 - x)(3 - x)$. As $x \rightarrow \infty$, the first factor is large positive, and the second and third factors are large negative. Therefore, $\lim_{x \rightarrow \infty} f(x) = \infty$. As $x \rightarrow -\infty$, the first factor is large negative, and the second and third factors are large positive. Therefore, $\lim_{x \rightarrow -\infty} f(x) = -\infty$. Now the y -intercept is

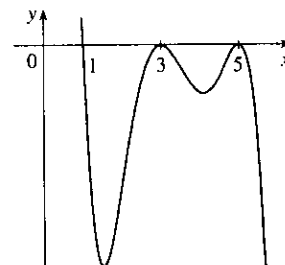
$f(0) = (2)^3(1)(3) = 24$ and the x -intercepts are the solutions to $f(x) = 0$
 $\Rightarrow x = -2, 1$ and 3 , and the graph crosses the x -axis at all of these points.



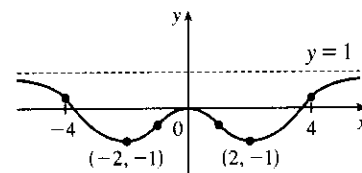
49. $y = f(x) = (x + 4)^5(x - 3)^4$. The y -intercept is $f(0) = 4^5(-3)^4 = 82,944$. The x -intercepts occur when $y = 0 \Rightarrow x = -4, 3$. Notice that the graph does not cross the x -axis at 3 because $(x - 3)^4$ is always positive, but does cross the x -axis at -4 . $\lim_{x \rightarrow \infty} (x + 4)^5(x - 3)^4 = \infty$ since both factors are large positive when x is large positive. $\lim_{x \rightarrow -\infty} (x + 4)^5(x - 3)^4 = -\infty$ since the first factor is large negative and the second factor is large positive when x is large negative.



50. $y = (1 - x)(x - 3)^2(x - 5)^2$. As $x \rightarrow \infty$, the first factor approaches $-\infty$ while the second and third factors approach ∞ . Therefore, $\lim_{x \rightarrow \infty} f(x) = -\infty$. As $x \rightarrow -\infty$, the factors all approach ∞ . Therefore, $\lim_{x \rightarrow -\infty} f(x) = \infty$. Now the y -intercept is $f(0) = (1)(-3)^2(-5)^2 = 225$ and the x -intercepts are the solutions to $f(x) = 0 \Rightarrow x = 1, 3$, and 5 . Notice that $f(x)$ does not change sign at $x = 3$ or $x = 5$ because the factors $(x - 3)^2$ and $(x - 5)^2$ are always positive, so the graph does not cross the x -axis at $x = 3$ or $x = 5$, but does cross the x -axis at $x = 1$.

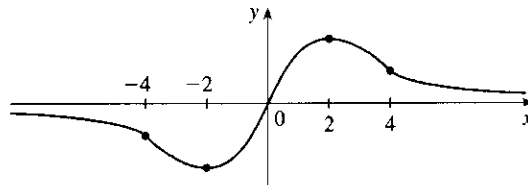


51. First we plot the points which are known to be on the graph: $(2, -1)$ and $(0, 0)$. We can also draw a short line segment of slope 0 at $x = 2$, since we are given that $f'(2) = 0$. Now we know that $f'(x) < 0$ (that is, the function is decreasing) on $(0, 2)$, and that $f''(x) < 0$ on $(0, 1)$ and $f''(x) > 0$ on $(1, 2)$. So we must join the points $(0, 0)$ and $(2, -1)$ in



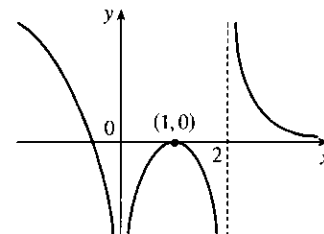
such a way that the curve is concave down on $(0, 1)$ and concave up on $(1, 2)$. The curve must be concave up and increasing on $(2, 4)$ and concave down and increasing toward $y = 1$ on $(4, \infty)$. Now we just need to reflect the curve in the y -axis, since we are given that f is an even function [the condition that $f(-x) = f(x)$ for all x].

52. The diagram shows one possible function satisfying all of the given conditions.

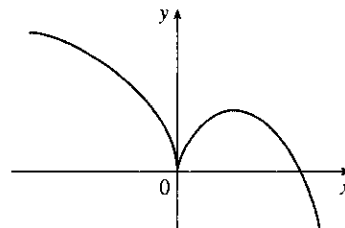


53. We are given that $f(1) = f'(1) = 0$. So we can draw a short horizontal line at the point $(1, 0)$ to represent this situation. We are given that $x = 0$ and $x = 2$ are vertical asymptotes, with $\lim_{x \rightarrow 0^-} f(x) = -\infty$, $\lim_{x \rightarrow 2^+} f(x) = \infty$ and $\lim_{x \rightarrow 2^-} f(x) = -\infty$, so we can draw the parts of the curve which approach these asymptotes.

On the interval $(-\infty, 0)$, the graph is concave down, and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$. Between the asymptotes the graph is concave down. On the interval $(2, \infty)$ the graph is concave up, and $f(x) \rightarrow 0$ as $x \rightarrow \infty$, so $y = 0$ is a horizontal asymptote. The diagram shows one possible function satisfying all of the given conditions.

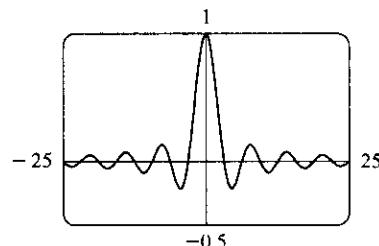


54. The diagram shows one possible function satisfying all of the given conditions.

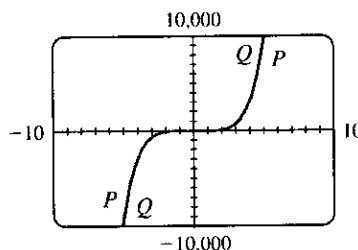
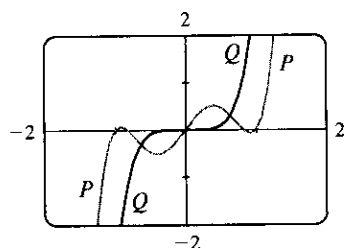


55. (a) Since $-1 \leq \sin x \leq 1$ for all x , $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ for $x > 0$. As $x \rightarrow \infty$, $-1/x \rightarrow 0$ and $1/x \rightarrow 0$, so by the Squeeze Theorem, $(\sin x)/x \rightarrow 0$. Thus, $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$.

- (b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n . Thus, the graph crosses the asymptote an infinite number of times.



56. (a) In both viewing rectangles, $\lim_{x \rightarrow \infty} P(x) = \lim_{x \rightarrow \infty} Q(x) = \infty$ and $\lim_{x \rightarrow -\infty} P(x) = \lim_{x \rightarrow -\infty} Q(x) = -\infty$. In the larger viewing rectangle, P and Q become less distinguishable.



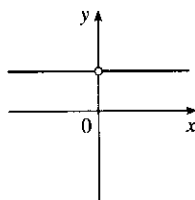
$$(b) \lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \lim_{x \rightarrow \infty} \frac{3x^5 - 5x^3 + 2x}{3x^5} = \lim_{x \rightarrow \infty} \left(1 - \frac{5}{3} \cdot \frac{1}{x^2} + \frac{2}{3} \cdot \frac{1}{x^4} \right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \Rightarrow$$

P and Q have the same end behavior.

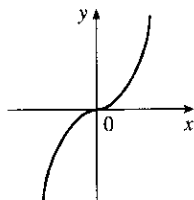
57. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

- (a) If $\deg P < \deg Q$, then the numerator $\rightarrow 0$ but the denominator doesn't. So $\lim_{x \rightarrow \infty} [P(x)/Q(x)] = 0$.
- (b) If $\deg P > \deg Q$, then the numerator $\rightarrow \pm\infty$ but the denominator doesn't, so $\lim_{x \rightarrow \infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

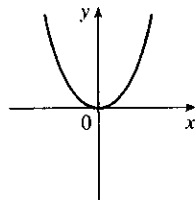
58.



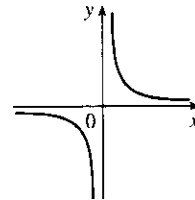
(i) $n = 0$



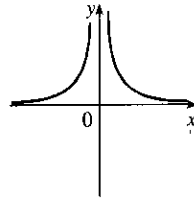
(ii) $n > 0$ (n odd)



(iii) $n > 0$ (n even)



(iv) $n < 0$ (n odd)



(v) $n < 0$ (n even)

From these sketches we see that

$$(a) \lim_{x \rightarrow 0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$$

$$(b) \lim_{x \rightarrow 0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0, n \text{ odd} \\ \infty & \text{if } n < 0, n \text{ even} \end{cases}$$

$$(c) \lim_{x \rightarrow \infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$$(d) \lim_{x \rightarrow -\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0, n \text{ odd} \\ \infty & \text{if } n > 0, n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$$

59. $\lim_{x \rightarrow \infty} \frac{4x-1}{x} = \lim_{x \rightarrow \infty} \left(4 - \frac{1}{x} \right) = 4$, and $\lim_{x \rightarrow \infty} \frac{4x^2+3x}{x^2} = \lim_{x \rightarrow \infty} \left(4 + \frac{3}{x} \right) = 4$. Therefore, by the Squeeze

Theorem, $\lim_{x \rightarrow \infty} f(x) = 4$.

60. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will

$$\text{be } C(t) = \frac{750t}{5000 + 25t} = \frac{30t}{200 + t} \frac{\text{g}}{\text{L}}.$$

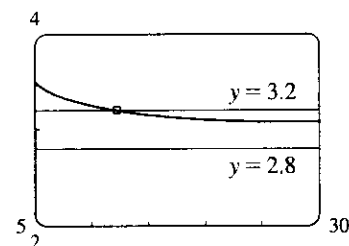
- (b) $\lim_{t \rightarrow \infty} C(t) = \lim_{t \rightarrow \infty} \frac{30t}{200 + t} = \lim_{t \rightarrow \infty} \frac{30t/t}{200/t + t/t} = \frac{30}{0 + 1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

61. $\left| \frac{6x^2 + 5x - 3}{2x^2 - 1} - 3 \right| < 0.2 \Leftrightarrow 2.8 < \frac{6x^2 + 5x - 3}{2x^2 - 1} < 3.2$. So

we graph the three parts of this inequality on the same screen, and

find that the curve $y = \frac{6x^2 + 5x - 3}{2x^2 - 1}$ seems to lie between the lines

$y = 2.8$ and $y = 3.2$ whenever $x > 12.8$. So we can choose $N = 13$ (or any larger number) so that the inequality holds whenever $x \geq N$.



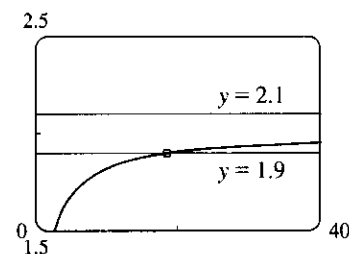
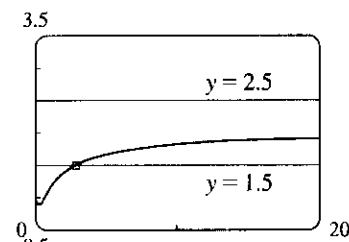
62. For $\varepsilon = 0.5$, we must find N such that whenever $x \geq N$, we have

$$\left| \frac{\sqrt{4x^2 + 1}}{x + 1} - 2 \right| < 0.5 \Leftrightarrow 1.5 < \frac{\sqrt{4x^2 + 1}}{x + 1} < 2.5.$$

We graph the three parts of this inequality on the same screen, and find that it holds whenever $x \geq 3$. So we choose $N = 3$ (or any larger

number). For $\varepsilon = 0.1$, we must have $1.9 < \frac{\sqrt{4x^2 + 1}}{x + 1} < 2.1$, and

the graphs show that this holds whenever $x \geq 19$. So we choose $N = 19$ (or any larger number).



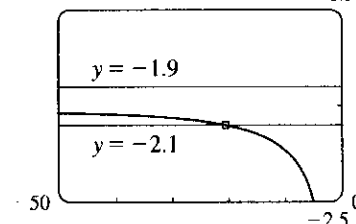
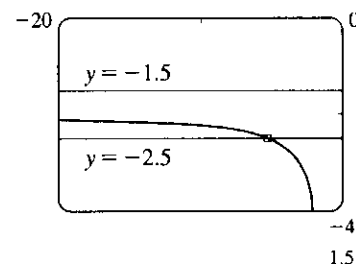
63. For $\varepsilon = 0.5$, we need to find N such that $\left| \frac{\sqrt{4x^2 + 1}}{x + 1} - (-2) \right| < 0.5$

$$\Leftrightarrow -2.5 < \frac{\sqrt{4x^2 + 1}}{x + 1} < -1.5 \text{ whenever } x \leq N.$$

We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \leq -6$. So we choose $N = -6$ (or any smaller number).

For $\varepsilon = 0.1$, we need $-2.1 < \frac{\sqrt{4x^2 + 1}}{x + 1} < -1.9$ whenever

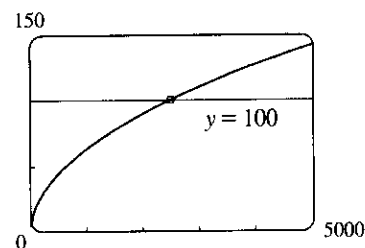
$x \leq N$. From the graph, it seems that this inequality holds for $x \leq -22$. So we choose $N = -22$ (or any smaller number).



64. We need N such that $\frac{2x+1}{\sqrt{x+1}} > 100$ whenever $x \geq N$. From the

graph, we see that this inequality holds for $x \geq 2500$. So we choose

$N = 2500$ (or any larger number).



65. (a) $1/x^2 < 0.0001 \Leftrightarrow x^2 > 1/0.0001 = 10,000 \Leftrightarrow x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \Leftrightarrow x^2 > 1/\varepsilon \Leftrightarrow x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

$$\text{Then } x > N \Rightarrow x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \left| \frac{1}{x^2} - 0 \right| = \frac{1}{x^2} < \varepsilon, \text{ so } \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0.$$

66. (a) $1/\sqrt{x} < 0.0001 \Leftrightarrow \sqrt{x} > 1/0.0001 = 10^4 \Leftrightarrow x > 10^8$

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \Leftrightarrow \sqrt{x} > 1/\varepsilon \Leftrightarrow x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

$$\text{Then } x > N \Rightarrow x > \frac{1}{\varepsilon^2} \Rightarrow \left| \frac{1}{\sqrt{x}} - 0 \right| = \frac{1}{\sqrt{x}} < \varepsilon, \text{ so } \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0.$$

67. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \Leftrightarrow x < -1/\varepsilon$.

Take $N = -1/\varepsilon$. Then $x < N \Rightarrow x < -1/\varepsilon \Rightarrow |(1/x) - 0| = -1/x < \varepsilon$, so $\lim_{x \rightarrow -\infty} (1/x) = 0$.

68. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow x^3 > M$. Now $x^3 > M \Leftrightarrow x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \Rightarrow x^3 > M$, so $\lim_{x \rightarrow \infty} x^3 = \infty$.

69. Suppose that $\lim_{x \rightarrow \infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that

$$|f(x) - L| < \varepsilon \text{ whenever } x > N. \text{ If } t = 1/x, \text{ then } x > N \Leftrightarrow 0 < 1/x < 1/N \Leftrightarrow 0 < t < 1/N. \text{ Thus,}$$

for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$.

This proves that $\lim_{t \rightarrow 0^+} f(1/t) = L = \lim_{x \rightarrow \infty} f(x)$.

Now suppose that $\lim_{x \rightarrow -\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that

$$|f(x) - L| < \varepsilon \text{ whenever } x < N. \text{ If } t = 1/x, \text{ then } x < N \Leftrightarrow 1/N < 1/x < 0 \Leftrightarrow 1/N < t < 0. \text{ Thus, for}$$

every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$.

This proves that $\lim_{t \rightarrow 0^-} f(1/t) = L = \lim_{x \rightarrow -\infty} f(x)$.

70. **Definition** Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x \rightarrow -\infty} f(x) = -\infty$ means that for every

negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\lim_{x \rightarrow -\infty} (1 + x^3) = -\infty$. Given a negative number M , we need a negative number

N such that $x < N \Rightarrow 1 + x^3 < M$. Now $1 + x^3 < M \Leftrightarrow x^3 < M - 1 \Leftrightarrow x < \sqrt[3]{M - 1}$. Thus, we

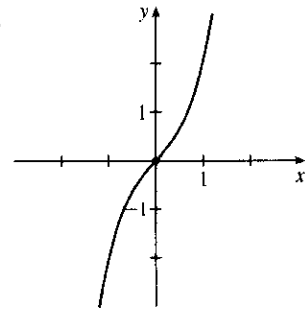
take $N = \sqrt[3]{M - 1}$ and find that $x < N \Rightarrow 1 + x^3 < M$. This proves that $\lim_{x \rightarrow -\infty} (1 + x^3) = -\infty$.

4.5 Summary of Curve Sketching

1. $y = f(x) = x^3 + x = x(x^2 + 1)$ A. f is a polynomial, so $D = \mathbb{R}$.

B. x -intercept = 0, y -intercept = $f(0) = 0$ C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. D. f is a polynomial, so there is no asymptote. E. $f'(x) = 3x^2 + 1 > 0$, so f is increasing on $(-\infty, \infty)$. F. There is no critical number and hence, no local maximum or minimum value. G. $f''(x) = 6x > 0$ on $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Since the concavity changes at $x = 0$, there is an inflection point at $(0, 0)$.

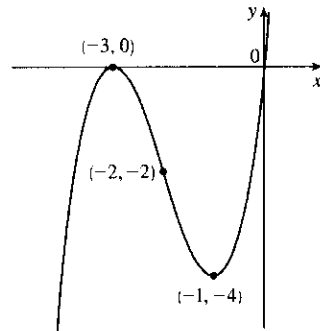
H.



2. $y = f(x) = x^3 + 6x^2 + 9x = x(x+3)^2$ A. $D = \mathbb{R}$ B. x -intercepts are -3 and 0 , y -intercept = 0 C. No symmetry D. No asymptote

E. $f'(x) = 3x^2 + 12x + 9 = 3(x+1)(x+3) < 0 \Leftrightarrow -3 < x < -1$, so f is decreasing on $(-3, -1)$ and increasing on $(-\infty, -3)$ and $(-1, \infty)$. F. Local maximum value $f(-3) = 0$, local minimum value $f(-1) = -4$ G. $f''(x) = 6x + 12 = 6(x+2) > 0 \Leftrightarrow x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$. IP at $(-2, -2)$

H.



3. $y = f(x) = 2 - 15x + 9x^2 - x^3 = -(x-2)(x^2 - 7x + 1)$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 2$; x -intercepts: $f(x) = 0 \Rightarrow x = 2$ or (by the quadratic formula) $x = \frac{7 \pm \sqrt{45}}{2} \approx 0.15, 6.85$

C. No symmetry D. No asymptote

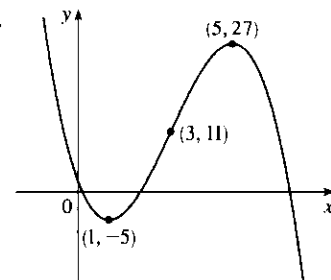
$$\begin{aligned} \text{E. } f'(x) &= -15 + 18x - 3x^2 = -3(x^2 - 6x + 5) \\ &= -3(x-1)(x-5) > 0 \Leftrightarrow 1 < x < 5 \end{aligned}$$

so f is increasing on $(1, 5)$ and decreasing on $(-\infty, 1)$ and $(5, \infty)$.

F. Local maximum value $f(5) = 27$, local minimum value $f(1) = -5$

G. $f''(x) = 18 - 6x = -6(x-3) > 0 \Leftrightarrow x < 3$, so f is CU on $(-\infty, 3)$ and CD on $(3, \infty)$. IP at $(3, 11)$

H.



4. $y = f(x) = 8x^2 - x^4 = x^2(8 - x^2)$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Rightarrow$

$x = 0, \pm 2\sqrt{2} (\approx \pm 2.83)$ C. $f(-x) = f(x)$, so f is even and symmetric about the y -axis. D. No asymptote

$$\text{E. } f'(x) = 16x - 4x^3 = 4x(4 - x^2) = 4x(2+x)(2-x) > 0 \Leftrightarrow$$

$x < -2$ or $0 < x < 2$, so f is increasing on $(-\infty, -2)$ and $(0, 2)$ and decreasing on $(-2, 0)$ and $(2, \infty)$. F. Local maximum value

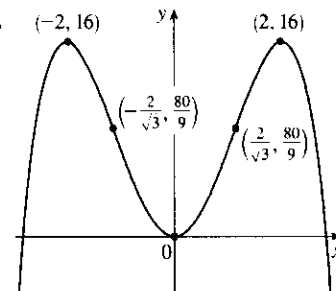
$f(\pm 2) = 16$, local minimum value $f(0) = 0$

$$\text{G. } f''(x) = 16 - 12x^2 = 4(4 - 3x^2) = 0 \Leftrightarrow x = \pm \frac{2}{\sqrt{3}}.$$

$f''(x) > 0 \Leftrightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$, so f is CU on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ and

CD on $(-\infty, -\frac{2}{\sqrt{3}})$ and $(\frac{2}{\sqrt{3}}, \infty)$. IP at $(\pm \frac{2}{\sqrt{3}}, \frac{80}{9})$

H.



5. $y = f(x) = x^4 + 4x^3 = x^3(x + 4)$ A. $D = \mathbb{R}$ B. y -intercept:

$f(0) = 0$; x -intercepts: $f(x) = 0 \Leftrightarrow x = -4, 0$ C. No symmetry

D. No asymptote E. $f'(x) = 4x^3 + 12x^2 = 4x^2(x + 3) > 0 \Leftrightarrow$

$x > -3$, so f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$.

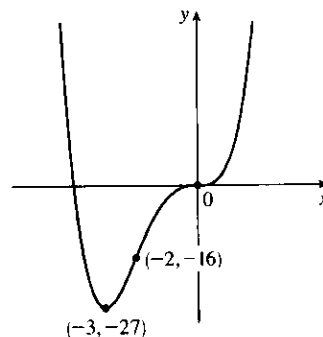
F. Local minimum value $f(-3) = -27$, no local maximum

G. $f''(x) = 12x^2 + 24x = 12x(x + 2) < 0 \Leftrightarrow -2 < x < 0$,

so f is CD on $(-2, 0)$ and CU on $(-\infty, -2)$ and $(0, \infty)$.

IP at $(0, 0)$ and $(-2, -16)$

H.



6. $y = f(x) = x(x + 2)^3$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$;

x -intercepts: $f(x) = 0 \Leftrightarrow x = -2, 0$ C. No symmetry D. No asymptote

E. $f'(x) = 3x(x + 2)^2 + (x + 2)^3 = (x + 2)^2 [3x + (x + 2)] = (x + 2)^2 (4x + 2)$. $f'(x) > 0 \Leftrightarrow x > -\frac{1}{2}$,

and $f'(x) < 0 \Leftrightarrow x < -2$ or $-2 < x < -\frac{1}{2}$, so f is increasing on $(-\frac{1}{2}, \infty)$ and decreasing on $(-\infty, -2)$

and $(-2, -\frac{1}{2})$. [Hence f is decreasing on $(-\infty, -\frac{1}{2})$ by the analogue of Exercise 4.3.53 for decreasing functions.]

F. Local minimum value $f(-\frac{1}{2}) = -\frac{27}{16}$, no local maximum

G. $f''(x) = (x + 2)^2(4) + (4x + 2)(2)(x + 2)$

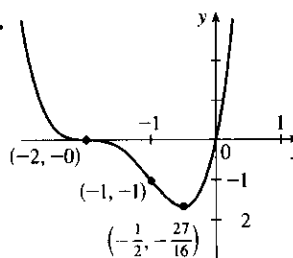
$$= 2(x + 2)[(x + 2)(2) + 4x + 2]$$

$$= 2(x + 2)(6x + 6) = 12(x + 1)(x + 2)$$

$f''(x) < 0 \Leftrightarrow -2 < x < -1$, so f is CD on $(-2, -1)$ and CU on

$(-\infty, -2)$ and $(-1, \infty)$. IP at $(-2, 0)$ and $(-1, -1)$

H.



7. $y = f(x) = 2x^5 - 5x^2 + 1$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 1$ C. No symmetry D. No asymptote

E. $f'(x) = 10x^4 - 10x = 10x(x^3 - 1) = 10x(x - 1)(x^2 + x + 1)$, so $f'(x) < 0 \Leftrightarrow 0 < x < 1$ and

$f'(x) > 0 \Leftrightarrow x < 0$ or $x > 1$. Thus, f is increasing on $(-\infty, 0)$ and $(1, \infty)$ and decreasing on $(0, 1)$.

F. Local maximum value $f(0) = 1$, local minimum value $f(1) = -2$

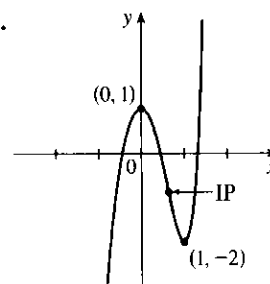
G. $f''(x) = 40x^3 - 10 = 10(4x^3 - 1)$ so $f''(x) = 0 \Leftrightarrow x = 1/\sqrt[3]{4}$.

$f''(x) > 0 \Leftrightarrow x > 1/\sqrt[3]{4}$ and $f''(x) < 0 \Leftrightarrow x < 1/\sqrt[3]{4}$,

so f is CD on $(-\infty, 1/\sqrt[3]{4})$ and CU on $(1/\sqrt[3]{4}, \infty)$.

IP at $\left(\frac{1}{\sqrt[3]{4}}, 1 - \frac{9}{2(\sqrt[3]{4})^2}\right) \approx (0.630, -0.786)$

H.



8. $y = f(x) = 20x^3 - 3x^5$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Leftrightarrow$

$$-3x^3(x^2 - \frac{20}{3}) = 0 \Leftrightarrow x = 0 \text{ or } \pm\sqrt{20/3} \approx \pm 2.582 \quad \text{C. } f(-x) = -f(x), \text{ so } f \text{ is odd;}$$

the curve is symmetric about the origin. D. No asymptote

E. $f'(x) = 60x^2 - 15x^4 = -15x^2(x^2 - 4) = -15x^2(x+2)(x-2)$, so $f'(x) > 0 \Leftrightarrow -2 < x < 0$ or

$0 < x < 2$ and $f'(x) < 0 \Leftrightarrow x < -2$ or $x > 2$. Thus, f is increasing on $(-2, 0)$ and $(0, 2)$ [hence on $(-2, 2)$ by Exercise 4.3.53] and f is decreasing on $(-\infty, -2)$ and $(2, \infty)$.

F. Local minimum value $f(-2) = -64$, local maximum

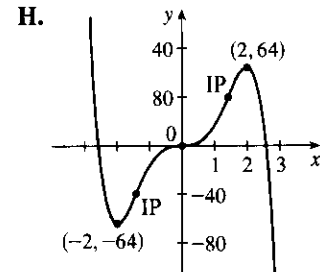
value $f(2) = 64$ G. $f''(x) = 120x - 60x^3 = -60x(x^2 - 2)$.

$$f''(x) > 0 \Leftrightarrow x < -\sqrt{2} \text{ or } 0 < x < \sqrt{2}; f''(x) < 0 \Leftrightarrow$$

$$-\sqrt{2} < x < 0 \text{ or } x > \sqrt{2}. \text{ Thus, } f \text{ is CU on } (-\infty, -\sqrt{2})$$

and $(0, \sqrt{2})$, and f is CD on $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$. IP at

$$(-\sqrt{2}, -28\sqrt{2}) \approx (-1.414, -39.598), (0, 0), \text{ and } (\sqrt{2}, 28\sqrt{2})$$



9. $y = f(x) = x/(x-1)$ A. $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ B. x -intercept = 0,

y -intercept = $f(0) = 0$ C. No symmetry D. $\lim_{x \rightarrow \pm\infty} \frac{x}{x-1} = 1$, so $y = 1$ is a HA.

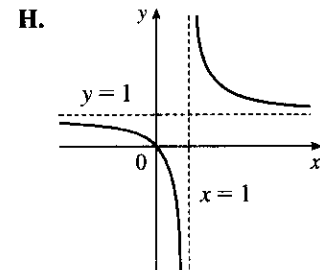
$$\lim_{x \rightarrow 1^-} \frac{x}{x-1} = -\infty, \lim_{x \rightarrow 1^+} \frac{x}{x-1} = \infty, \text{ so } x = 1 \text{ is a VA.}$$

E. $f'(x) = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2} < 0$ for $x \neq 1$, so f is decreasing

on $(-\infty, 1)$ and $(1, \infty)$. F. No extreme values

G. $f''(x) = \frac{2}{(x-1)^3} > 0 \Leftrightarrow x > 1$, so f is CU on $(1, \infty)$ and CD

on $(-\infty, 1)$. No IP



10. $y = x/(x-1)^2$ A. $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ B. x -intercept = 0, y -intercept = $f(0) = 0$

C. No symmetry D. $\lim_{x \rightarrow \pm\infty} \frac{x}{(x-1)^2} = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 1} \frac{x}{(x-1)^2} = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \frac{(x-1)^2(1) - x(2)(x-1)}{(x-1)^4} = \frac{-x-1}{(x-1)^3}$. This is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on

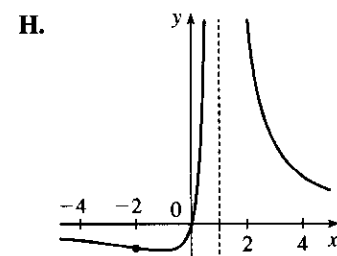
$(-1, 1)$, so $f(x)$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

F. Local minimum value $f(-1) = -\frac{1}{4}$, no local maximum.

G. $f''(x) = \frac{(x-1)^3(-1) + (x+1)(3)(x-1)^2}{(x-1)^6} = \frac{2(x+2)}{(x-1)^4}$. This is

negative on $(-\infty, -2)$, and positive on $(-2, 1)$ and $(1, \infty)$. So f is CD

on $(-\infty, -2)$ and CU on $(-2, 1)$ and $(1, \infty)$. IP at $(-2, -\frac{2}{9})$



11. $y = f(x) = 1/(x^2 - 9)$ A. $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

B. y -intercept $= f(0) = -\frac{1}{9}$, no x -intercept C. $f(-x) = f(x) \Rightarrow f$ is even; the curve is symmetric about

the y -axis. D. $\lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 3^-} \frac{1}{x^2 - 9} = -\infty$, $\lim_{x \rightarrow 3^+} \frac{1}{x^2 - 9} = \infty$,

$\lim_{x \rightarrow -3^-} \frac{1}{x^2 - 9} = \infty$, $\lim_{x \rightarrow -3^+} \frac{1}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

E. $f'(x) = -\frac{2x}{(x^2 - 9)^2} > 0 \Leftrightarrow x < 0$ ($x \neq -3$) so f is increasing

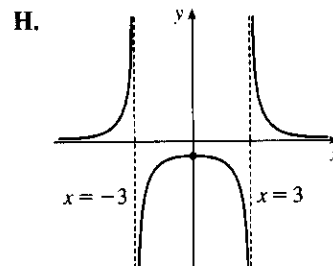
on $(-\infty, -3)$ and $(-3, 0)$ and decreasing on $(0, 3)$ and $(3, \infty)$.

F. Local maximum value $f(0) = -\frac{1}{9}$.

G. $y'' = \frac{-2(x^2 - 9)^2 + (2x)2(x^2 - 9)(2x)}{(x^2 - 9)^4} = \frac{6(x^2 + 3)}{(x^2 - 9)^3} > 0 \Leftrightarrow$

$x^2 > 9 \Leftrightarrow x > 3$ or $x < -3$, so f is CU on $(-\infty, -3)$ and $(3, \infty)$

and CD on $(-3, 3)$. No IP



12. $y = f(x) = x/(x^2 - 9)$ A. $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ B. x -intercept $= 0$,

y -intercept $= f(0) = 0$. C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin.

D. $\lim_{x \rightarrow \pm\infty} \frac{x}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 3^+} \frac{x}{x^2 - 9} = \infty$, $\lim_{x \rightarrow 3^-} \frac{x}{x^2 - 9} = -\infty$,

$\lim_{x \rightarrow -3^+} \frac{x}{x^2 - 9} = \infty$, $\lim_{x \rightarrow -3^-} \frac{x}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

E. $f'(x) = \frac{(x^2 - 9) - x(2x)}{(x^2 - 9)^2} = -\frac{x^2 + 9}{(x^2 - 9)^2} < 0$ ($x \neq \pm 3$) so f is decreasing on $(-\infty, -3)$, $(-3, 3)$,

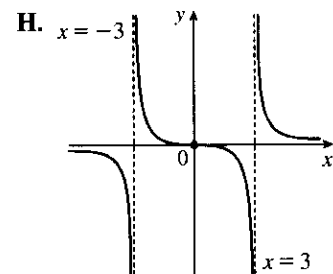
and $(3, \infty)$. F. No extreme values

G. $f''(x) = -\frac{2x(x^2 - 9)^2 - (x^2 + 9) \cdot 2(x^2 - 9)(2x)}{(x^2 - 9)^4}$

$= \frac{2x(x^2 + 27)}{(x^2 - 9)^3} > 0$ when $-3 < x < 0$ or $x > 3$,

so f is CU on $(-3, 0)$ and $(3, \infty)$; CD on $(-\infty, -3)$ and $(0, 3)$.

IP at $(0, 0)$



13. $y = f(x) = x/(x^2 + 9)$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. D. $\lim_{x \rightarrow \pm\infty} [x/(x^2 + 9)] = 0$, so

$y = 0$ is a HA; no VA E. $f'(x) = \frac{(x^2 + 9)(1) - x(2x)}{(x^2 + 9)^2} = \frac{9 - x^2}{(x^2 + 9)^2} = \frac{(3 + x)(3 - x)}{(x^2 + 9)^2} > 0 \Leftrightarrow$

$-3 < x < 3$, so f is increasing on $(-3, 3)$ and decreasing on $(-\infty, -3)$ and $(3, \infty)$.

F. Local minimum value $f(-3) = -\frac{1}{6}$, local maximum value $f(3) = \frac{1}{6}$

$$\begin{aligned} \text{G. } f''(x) &= \frac{(x^2 + 9)^2(-2x) - (9 - x^2) \cdot 2(x^2 + 9)(2x)}{[(x^2 + 9)^2]^2} \\ &= \frac{(2x)(x^2 + 9)[-(x^2 + 9) - 2(9 - x^2)]}{(x^2 + 9)^4} \\ &= \frac{2x(x^2 - 27)}{(x^2 + 9)^3} = 0 \Leftrightarrow x = 0, \pm\sqrt{27} = \pm 3\sqrt{3} \end{aligned}$$

$f''(x) > 0 \Leftrightarrow -3\sqrt{3} < x < 0$ or $x > 3\sqrt{3}$, so f is CU on

$(-3\sqrt{3}, 0)$ and $(3\sqrt{3}, \infty)$, and CD on $(-\infty, -3\sqrt{3})$ and $(0, 3\sqrt{3})$.

There are three inflection points: $(0, 0)$ and $(\pm 3\sqrt{3}, \pm \frac{1}{12}\sqrt{3})$.

14. $y = f(x) = x^2/(x^2 + 9)$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. $f(-x) = f(x)$, so f is even and symmetric about the y -axis. D. $\lim_{x \rightarrow \pm\infty} [x^2/(x^2 + 9)] = 1$, so $y = 1$

is a HA; no VA E. $f'(x) = \frac{(x^2 + 9)(2x) - x^2(2x)}{(x^2 + 9)^2} = \frac{18x}{(x^2 + 9)^2} > 0 \Leftrightarrow x > 0$, so f is increasing on

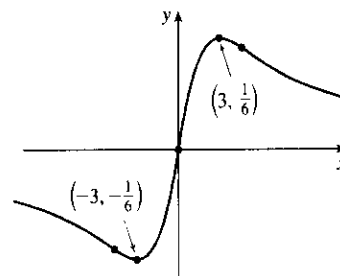
$(0, \infty)$ and decreasing on $(-\infty, 0)$. F. Local minimum value $f(0) = 0$; no local maximum

$$\begin{aligned} \text{G. } f''(x) &= \frac{(x^2 + 9)^2(18) - 18x \cdot 2(x^2 + 9) \cdot 2x}{[(x^2 + 9)^2]^2} = \frac{18(x^2 + 9)[(x^2 + 9) - 4x^2]}{(x^2 + 9)^4} = \frac{18(9 - 3x^2)}{(x^2 + 9)^3} \\ &= \frac{-54(x + \sqrt{3})(x - \sqrt{3})}{(x^2 + 9)^3} > 0 \Leftrightarrow -\sqrt{3} < x < \sqrt{3} \end{aligned}$$

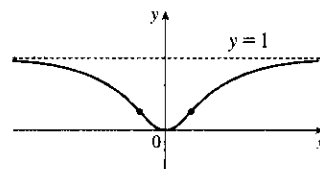
so f is CU on $(-\sqrt{3}, \sqrt{3})$ and CD on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$.

There are two inflection points: $(\pm\sqrt{3}, \frac{1}{4})$.

H.



H.



15. $y = f(x) = \frac{x-1}{x^2}$ A. $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ B. No y -intercept; x -intercept: $f(x) = 0 \Leftrightarrow$

$x = 1$ C. No symmetry D. $\lim_{x \rightarrow \pm\infty} \frac{x-1}{x^2} = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 0} \frac{x-1}{x^2} = -\infty$, so $x = 0$ is a VA.

$$\text{E. } f'(x) = \frac{x^2 \cdot 1 - (x-1) \cdot 2x}{(x^2)^2} = \frac{-x^2 + 2x}{x^4} = \frac{-(x-2)}{x^3}, \text{ so } f'(x) > 0 \Leftrightarrow 0 < x < 2 \text{ and}$$

$f'(x) < 0 \Leftrightarrow x < 0$ or $x > 2$. Thus, f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$.

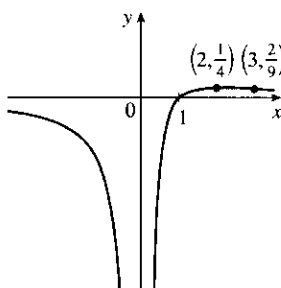
F. No local minimum, local maximum value $f(2) = \frac{1}{4}$.

$$\text{G. } f''(x) = \frac{x^3 \cdot (-1) - [-(x-2)] \cdot 3x^2}{(x^3)^2} = \frac{2x^3 - 6x^2}{x^6} = \frac{2(x-3)}{x^4}.$$

$f''(x)$ is negative on $(-\infty, 0)$ and $(0, 3)$ and positive on $(3, \infty)$, so f is

CD on $(-\infty, 0)$ and $(0, 3)$ and CU on $(3, \infty)$. IP at $(3, \frac{2}{9})$

H.



16. $y = f(x) = \frac{x^2 - 2}{x^4}$ A. $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ B. No y -intercept; x -intercepts: $f(x) = 0$

$\Leftrightarrow x = \pm\sqrt{2}$ C. $f(-x) = f(x)$, so f is even; the curve is symmetric about the y -axis.

D. $\lim_{x \rightarrow \pm\infty} \frac{x^2 - 2}{x^4} = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 0} \frac{x^2 - 2}{x^4} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \frac{x^4 \cdot 2x - (x^2 - 2)(4x^3)}{(x^4)^2} = \frac{-2x^5 + 8x^3}{x^8} = \frac{-2(x^2 - 4)}{x^5} = \frac{-2(x+2)(x-2)}{x^5}$.

$f'(x)$ is negative on $(-2, 0)$ and $(2, \infty)$ and positive on $(-\infty, -2)$ and $(0, 2)$, so f is decreasing on $(-2, 0)$ and $(2, \infty)$ and increasing on $(-\infty, -2)$ and $(0, 2)$. F. Local maximum value $f(\pm 2) = \frac{1}{8}$, no local minimum.

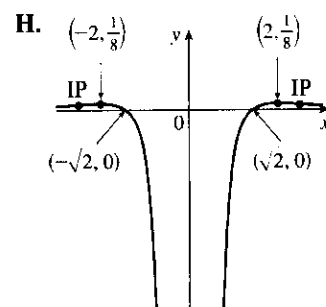
G. $f''(x) = \frac{x^5 \cdot (-4x) + 2(x^2 - 4) \cdot 5x^4}{(x^5)^2} = \frac{2x^4[-2x^2 + 5(x^2 - 4)]}{x^{10}} = \frac{2(3x^2 - 20)}{x^6}$

$f''(x)$ is positive on $(-\infty, -\sqrt{\frac{20}{3}})$ and $(\sqrt{\frac{20}{3}}, \infty)$ and negative on

$(-\sqrt{\frac{20}{3}}, 0)$ and $(0, \sqrt{\frac{20}{3}})$, so f is CU on $(-\infty, -\sqrt{\frac{20}{3}})$ and

$(\sqrt{\frac{20}{3}}, \infty)$ and CD on $(-\sqrt{\frac{20}{3}}, 0)$ and $(0, \sqrt{\frac{20}{3}})$.

IP at $(\pm\sqrt{\frac{20}{3}}, \frac{21}{200}) \approx (\pm 2.5820, 0.105)$



17. $y = f(x) = \frac{x^2}{x^2 + 3} = \frac{(x^2 + 3) - 3}{x^2 + 3} = 1 - \frac{3}{x^2 + 3}$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercepts:

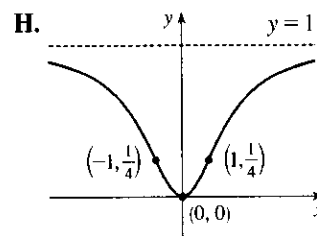
$f(x) = 0 \Leftrightarrow x = 0$ C. $f(-x) = f(x)$, so f is even; the graph is symmetric about the y -axis.

D. $\lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2 + 3} = 1$, so $y = 1$ is a HA. No VA. E. Using the Reciprocal Rule,

$f'(x) = -3 \cdot \frac{-2x}{(x^2 + 3)^2} = \frac{6x}{(x^2 + 3)^2}$. $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$, so f is decreasing

on $(-\infty, 0)$ and increasing on $(0, \infty)$. F. Local minimum value $f(0) = 0$, no local maximum.

G. $f''(x) = \frac{(x^2 + 3)^2 \cdot 6 - 6x \cdot 2(x^2 + 3) \cdot 2x}{[(x^2 + 3)^2]^2}$
 $= \frac{6(x^2 + 3)[(x^2 + 3) - 4x^2]}{(x^2 + 3)^4}$
 $= \frac{6(3 - 3x^2)}{(x^2 + 3)^3} = \frac{-18(x+1)(x-1)}{(x^2 + 3)^3}$



$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $(\pm 1, \frac{1}{4})$

18. $y = f(x) = \frac{x^3 - 1}{x^3 + 1}$ A. $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ B. x -intercept = 1,

y -intercept = $f(0) = -1$ C. No symmetry D. $\lim_{x \rightarrow \pm\infty} \frac{x^3 - 1}{x^3 + 1} = \lim_{x \rightarrow \pm\infty} \frac{1 - 1/x^3}{1 + 1/x^3} = 1$, so $y = 1$ is a HA.

$\lim_{x \rightarrow -1^-} \frac{x^3 - 1}{x^3 + 1} = \infty$ and $\lim_{x \rightarrow -1^+} \frac{x^3 - 1}{x^3 + 1} = -\infty$, so $x = -1$ is a VA.

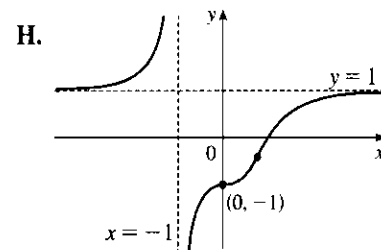
E. $f'(x) = \frac{(x^3 + 1)(3x^2) - (x^3 - 1)(3x^2)}{(x^3 + 1)^2} = \frac{6x^2}{(x^3 + 1)^2} > 0$ ($x \neq -1$) so f is increasing on $(-\infty, -1)$

and $(-1, \infty)$. F. No extreme values

G. $y'' = \frac{12x(x^3 + 1)^2 - 6x^2 \cdot 2(x^3 + 1) \cdot 3x^2}{(x^3 + 1)^4}$
 $= \frac{12x(1 - 2x^3)}{(x^3 + 1)^3} > 0 \Leftrightarrow x < -1$ or $0 < x < \frac{1}{\sqrt[3]{2}}$,

so f is CU on $(-\infty, -1)$ and $(0, \frac{1}{\sqrt[3]{2}})$ and CD on $(-1, 0)$ and $(\frac{1}{\sqrt[3]{2}}, \infty)$.

IP at $(0, -1)$, $(\frac{1}{\sqrt[3]{2}}, -\frac{1}{3})$



19. $y = f(x) = x\sqrt{5-x}$ A. The domain is $\{x \mid 5 - x \geq 0\} = (-\infty, 5]$ B. y -intercept: $f(0) = 0$;

x -intercepts: $f(x) = 0 \Leftrightarrow x = 0, 5$ C. No symmetry D. No asymptote

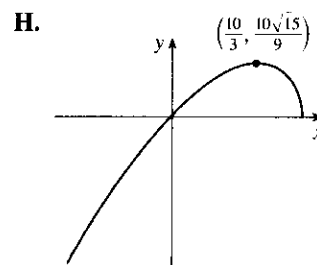
E. $f'(x) = x \cdot \frac{1}{2}(5-x)^{-1/2}(-1) + (5-x)^{1/2} \cdot 1 = \frac{1}{2}(5-x)^{-1/2}[-x + 2(5-x)] = \frac{10-3x}{2\sqrt{5-x}} > 0 \Leftrightarrow$

$x < \frac{10}{3}$, so f is increasing on $(-\infty, \frac{10}{3})$ and decreasing on $(\frac{10}{3}, 5)$.

F. Local maximum value $f(\frac{10}{3}) = \frac{10}{9}\sqrt{15} \approx 4.3$; no local minimum

G. $f''(x) = \frac{2(5-x)^{1/2}(-3) - (10-3x) \cdot 2(\frac{1}{2})(5-x)^{-1/2}(-1)}{(2\sqrt{5-x})^2}$
 $= \frac{(5-x)^{-1/2}[-6(5-x) + (10-3x)]}{4(5-x)} = \frac{3x-20}{4(5-x)^{3/2}}$

$f''(x) < 0$ for $x < 5$, so f is CD on $(-\infty, 5)$. No IP



20. $y = f(x) = 2\sqrt{x} - x$ A. $D = [0, \infty)$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Rightarrow$

$2\sqrt{x} = x \Rightarrow 4x = x^2 \Rightarrow 4x - x^2 = 0 \Rightarrow x(4-x) = 0 \Rightarrow x = 0, 4$ C. No symmetry

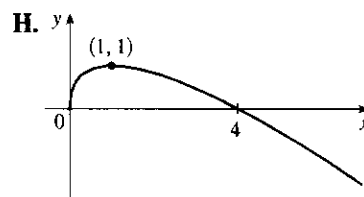
D. No asymptote E. $f'(x) = \frac{1}{\sqrt{x}} - 1 = \frac{1}{\sqrt{x}}(1 - \sqrt{x})$. This is positive for $x < 1$ and

negative for $x > 1$, so f is increasing on $(0, 1)$ and decreasing on $(1, \infty)$.

F. Local maximum value $f(1) = 1$, no local minimum.

G. $f''(x) = (x^{-1/2} - 1)' = -\frac{1}{2}x^{-3/2} = \frac{-1}{2x^{3/2}} < 0$ for $x > 0$,

so f is CD on $(0, \infty)$. No IP



21. $y = f(x) = \sqrt{x^2 + 1} - x$ A. $D = \mathbb{R}$ B. No x -intercept, y -intercept = 1 C. No symmetry

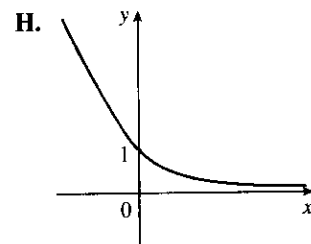
D. $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + 1} - x) = \infty$ and

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1} + x} = 0,$$

so $y = 0$ is a HA. E. $f'(x) = \frac{x}{\sqrt{x^2 + 1}} - 1 = \frac{x - \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \Rightarrow$

$f'(x) < 0$, so f is decreasing on $\mathbb{R}$. F. No extreme values

G. $f''(x) = \frac{1}{(x^2 + 1)^{3/2}} > 0$, so f is CU on $\mathbb{R}$. No IP



22. $y = f(x) = \sqrt{x/(x-5)}$ A. $D = \{x \mid x/(x-5) \geq 0\} = (-\infty, 0] \cup (5, \infty)$. B. Intercepts are 0.

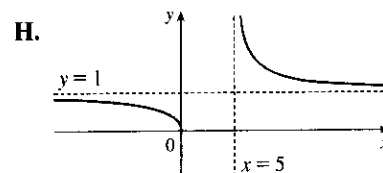
C. No symmetry D. $\lim_{x \rightarrow \pm\infty} \sqrt{\frac{x}{x-5}} = \lim_{x \rightarrow \pm\infty} \sqrt{\frac{1}{1-5/x}} = 1$, so $y = 1$ is a HA. $\lim_{x \rightarrow 5^+} \sqrt{\frac{x}{x-5}} = \infty$,

so $x = 5$ is a VA. E. $f'(x) = \frac{1}{2} \left(\frac{x}{x-5} \right)^{-1/2} \cdot \frac{(-5)}{(x-5)^2} = -\frac{5}{2} [x(x-5)^3]^{-1/2} < 0$, so f is decreasing on

$(-\infty, 0)$ and $(5, \infty)$. F. No extreme values

G. $f''(x) = \frac{5}{4} [x(x-5)^3]^{-3/2} (x-5)^2 (4x-5) > 0$ for $x > 5$, and

$f''(x) < 0$ for $x < 0$, so f is CU on $(5, \infty)$ and CD on $(-\infty, 0)$. No IP



23. $y = f(x) = x/\sqrt{x^2 + 1}$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Rightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

$$\begin{aligned} \text{D. } \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{x^2 + 1}/x} = \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{x^2 + 1}/\sqrt{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 1/x^2}} \\ &= \frac{1}{\sqrt{1 + 0}} = 1 \end{aligned}$$

and

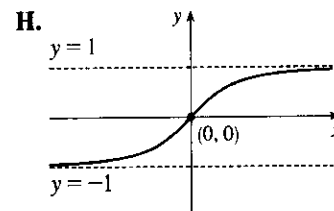
$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow -\infty} \frac{x/x}{\sqrt{x^2 + 1}/x} = \lim_{x \rightarrow -\infty} \frac{x/x}{\sqrt{x^2 + 1}/(-\sqrt{x^2})} \\ &= \lim_{x \rightarrow -\infty} \frac{1}{-\sqrt{1 + 1/x^2}} = \frac{1}{-\sqrt{1 + 0}} = -1 \end{aligned}$$

so $y = \pm 1$ are HA. No VA.

E. $f'(x) = \frac{\sqrt{x^2 + 1} - x \cdot \frac{2x}{2\sqrt{x^2 + 1}}}{[(x^2 + 1)^{1/2}]^2} = \frac{x^2 + 1 - x^2}{(x^2 + 1)^{3/2}} = \frac{1}{(x^2 + 1)^{3/2}} > 0$ for all x , so f is increasing on $\mathbb{R}$.

F. No extreme values

$$G. f''(x) = -\frac{3}{2}(x^2 + 1)^{-5/2} \cdot 2x = \frac{-3x}{(x^2 + 1)^{5/2}}, \text{ so } f''(x) > 0 \text{ for}$$

 $x < 0 \text{ and } f''(x) < 0 \text{ for } x > 0. \text{ Thus, } f \text{ is CU on } (-\infty, 0) \text{ and}$
CD on $(0, \infty)$. IP at $(0, 0)$


$$24. y = f(x) = x\sqrt{2-x^2} \quad A. D = [-\sqrt{2}, \sqrt{2}] \quad B. y\text{-intercept: } f(0) = 0; x\text{-intercepts: } f(x) = 0 \Rightarrow$$

 $x = 0, \pm\sqrt{2}. \quad C. f(-x) = -f(x), \text{ so } f \text{ is odd; the graph is symmetric about the origin.} \quad D. \text{ No asymptote}$

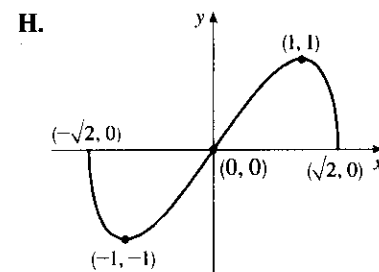
$$E. f'(x) = x \cdot \frac{-x}{\sqrt{2-x^2}} + \sqrt{2-x^2} = \frac{-x^2 + 2 - x^2}{\sqrt{2-x^2}} = \frac{2(1+x)(1-x)}{\sqrt{2-x^2}}. f'(x) \text{ is negative for}$$

 $-\sqrt{2} < x < -1 \text{ and } 1 < x < \sqrt{2}, \text{ and positive for } -1 < x < 1, \text{ so } f \text{ is decreasing on } (-\sqrt{2}, -1) \text{ and } (1, \sqrt{2})$
and increasing on $(-1, 1)$. F. Local minimum value $f(-1) = -1$, local maximum value $f(1) = 1$.

$$G. f''(x) = \frac{\sqrt{2-x^2}(-4x) - (2-2x^2)\frac{-x}{\sqrt{2-x^2}}}{[(2-x^2)^{1/2}]^2}$$

$$= \frac{(2-x^2)(-4x) + (2-2x^2)x}{(2-x^2)^{3/2}}$$

$$= \frac{2x^3 - 6x}{(2-x^2)^{3/2}} = \frac{2x(x^2 - 3)}{(2-x^2)^{3/2}}$$



Since $x^2 - 3 < 0$ for x in $[-\sqrt{2}, \sqrt{2}]$, $f''(x) > 0$ for $-\sqrt{2} < x < 0$ and $f''(x) < 0$ for $0 < x < \sqrt{2}$. Thus, f is CU on $(-\sqrt{2}, 0)$ and CD on $(0, \sqrt{2})$. The only IP is $(0, 0)$.

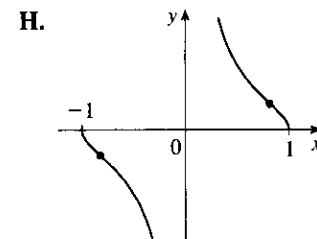
$$25. y = f(x) = \sqrt{1-x^2}/x \quad A. D = \left\{ x \mid |x| \leq 1, x \neq 0 \right\} = [-1, 0) \cup (0, 1] \quad B. x\text{-intercepts } \pm 1, \text{ no } y\text{-intercept}$$

$$C. f(-x) = -f(x), \text{ so the curve is symmetric about } (0, 0). \quad D. \lim_{x \rightarrow 0^+} \frac{\sqrt{1-x^2}}{x} = \infty, \lim_{x \rightarrow 0^-} \frac{\sqrt{1-x^2}}{x} = -\infty,$$

$$\text{so } x = 0 \text{ is a VA.} \quad E. f'(x) = \frac{(-x^2/\sqrt{1-x^2}) - \sqrt{1-x^2}}{x^2} = -\frac{1}{x^2\sqrt{1-x^2}} < 0, \text{ so } f \text{ is decreasing on } (-1, 0)$$

and $(0, 1)$. F. No extreme values

$$G. f''(x) = \frac{2-3x^2}{x^3(1-x^2)^{3/2}} > 0 \Leftrightarrow -1 < x < -\sqrt{\frac{2}{3}} \text{ or}$$

 $0 < x < \sqrt{\frac{2}{3}}, \text{ so } f \text{ is CU on } (-1, -\sqrt{\frac{2}{3}}) \text{ and } (0, \sqrt{\frac{2}{3}}) \text{ and CD on}$
 $(-\sqrt{\frac{2}{3}}, 0) \text{ and } (\sqrt{\frac{2}{3}}, 1). \text{ IP at } (\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}})$


26. $y = f(x) = x/\sqrt{x^2 - 1}$ A. $D = (-\infty, -1) \cup (1, \infty)$ B. No intercepts C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. D. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}} = 1$ and $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - 1}} = -1$, so $y = \pm 1$ are HA.

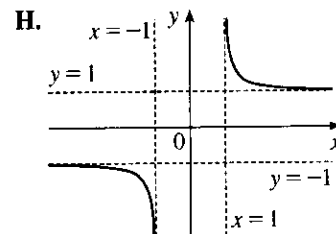
$\lim_{x \rightarrow 1^+} f(x) = +\infty$ and $\lim_{x \rightarrow -1^-} f(x) = -\infty$, so $x = \pm 1$ are VA.

E. $f'(x) = \frac{\sqrt{x^2 - 1} - x \cdot \frac{x}{\sqrt{x^2 - 1}}}{[(x^2 - 1)^{1/2}]^2} = \frac{x^2 - 1 - x^2}{(x^2 - 1)^{3/2}} = \frac{-1}{(x^2 - 1)^{3/2}} < 0$, so f is decreasing

on $(-\infty, -1)$ and $(1, \infty)$. F. No extreme values

G. $f''(x) = (-1)(-\frac{3}{2})(x^2 - 1)^{-5/2} \cdot 2x = \frac{3x}{(x^2 - 1)^{5/2}}$.

$f''(x) < 0$ on $(-\infty, -1)$ and $f''(x) > 0$ on $(1, \infty)$, so f is CD on $(-\infty, -1)$ and CU on $(1, \infty)$. No IP



27. $y = f(x) = x - 3x^{1/3}$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Rightarrow x = 3x^{1/3} \Rightarrow x^3 = 27x \Rightarrow x^3 - 27x = 0 \Rightarrow x(x^2 - 27) = 0 \Rightarrow x = 0, \pm 3\sqrt{3}$ C. $f(-x) = -f(x)$, so f is odd;

the graph is symmetric about the origin. D. No asymptote E. $f'(x) = 1 - x^{-2/3} = 1 - \frac{1}{x^{2/3}} = \frac{x^{2/3} - 1}{x^{2/3}}$.

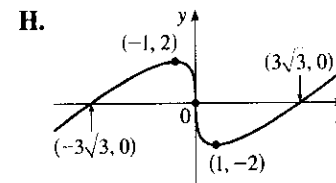
$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 0)$ and $(0, 1)$ [hence decreasing on $(-1, 1)$ since f is

continuous on $(-1, 1)$]. F. Local maximum value $f(-1) = 2$, local

minimum value $f(1) = -2$ G. $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$

and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$.

IP at $(0, 0)$



28. $y = f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x - 5)$ A. $D = \mathbb{R}$ B. x -intercepts 0, 5; y -intercept 0 C. No symmetry

D. $\lim_{x \rightarrow \pm\infty} x^{2/3}(x - 5) = \pm\infty$, so there is no asymptote

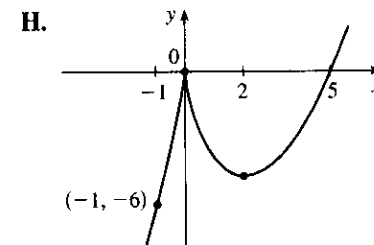
E. $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x - 2) > 0 \Leftrightarrow x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$, $(2, \infty)$ and decreasing on $(0, 2)$.

F. Local maximum value $f(0) = 0$, local minimum value $f(2) = -3\sqrt[3]{4}$

G. $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x + 1) > 0 \Leftrightarrow$

$x > -1$, so f is CU on $(-1, 0)$ and $(0, \infty)$, CD on $(-\infty, -1)$.

IP at $(-1, -6)$



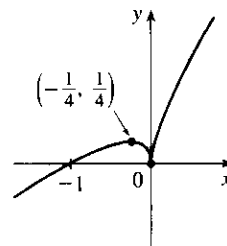
29. $y = f(x) = x + \sqrt{|x|}$ A. $D = \mathbb{R}$ B. x -intercepts 0, -1; y -intercept 0 C. No symmetry

D. $\lim_{x \rightarrow \infty} (x + \sqrt{|x|}) = \infty$, $\lim_{x \rightarrow -\infty} (x + \sqrt{|x|}) = -\infty$. No asymptote E. For $x > 0$, $f(x) = x + \sqrt{x} \Rightarrow f'(x) = 1 + \frac{1}{2\sqrt{x}} > 0$, so f increases on $(0, \infty)$. For $x < 0$, $f(x) = x + \sqrt{-x} \Rightarrow f'(x) = 1 - \frac{1}{2\sqrt{-x}} > 0$

$\Leftrightarrow 2\sqrt{-x} > 1 \Leftrightarrow -x > \frac{1}{4} \Leftrightarrow x < -\frac{1}{4}$, so f increases on $(-\infty, -\frac{1}{4})$ and decreases on $(-\frac{1}{4}, 0)$.

F. Local maximum value $f(-\frac{1}{4}) = \frac{1}{4}$, local minimum value $f(0) = 0$

H.



G. For $x > 0$, $f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(x) < 0$, so f is CD on

$(0, \infty)$. For $x < 0$, $f''(x) = -\frac{1}{4}(-x)^{-3/2} \Rightarrow f''(x) < 0$, so f is

CD on $(-\infty, 0)$. No IP

30. $y = f(x) = \sqrt[3]{(x^2 - 1)^2} = (x^2 - 1)^{2/3}$ A. $D = \mathbb{R}$ B. x -intercepts ± 1 , y -intercept 1 C. $f(-x) = f(x)$, so

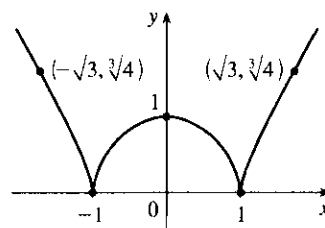
the curve is symmetric about the y -axis. D. $\lim_{x \rightarrow \pm\infty} (x^2 - 1)^{2/3} = \infty$, no asymptote

E. $f'(x) = \frac{4}{3}x(x^2 - 1)^{-1/3} \Rightarrow f'(x) > 0 \Leftrightarrow x > 1$ or $-1 < x < 0$, $f'(x) < 0 \Leftrightarrow x < -1$ or

$0 < x < 1$. So f is increasing on $(-1, 0)$, $(1, \infty)$ and decreasing on $(-\infty, -1)$, $(0, 1)$. F. Local minimum

values $f(-1) = f(1) = 0$, local maximum value $f(0) = 1$

H.



G. $f''(x) = \frac{4}{3}(x^2 - 1)^{-1/3} + \frac{4}{3}x(-\frac{1}{3})(x^2 - 1)^{-4/3}(2x)$
 $= \frac{4}{9}(x^2 - 3)(x^2 - 1)^{-4/3} > 0 \Leftrightarrow |x| > \sqrt{3}$

so f is CU on $(-\infty, -\sqrt{3})$, $(\sqrt{3}, \infty)$ and CD on $(-\sqrt{3}, -1)$,

$(-1, 1)$, $(1, \sqrt{3})$. IPs at $(\pm\sqrt{3}, \sqrt[3]{4})$

31. $y = f(x) = 3 \sin x - \sin^3 x$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Rightarrow$

$\sin x (3 - \sin^2 x) = 0 \Rightarrow \sin x = 0$ [since $\sin^2 x \leq 1 < 3$] $\Rightarrow x = n\pi$, n an integer.

C. $f(-x) = -f(x)$, so f is odd; the graph (shown for $-2\pi \leq x \leq 2\pi$) is symmetric about the origin and periodic with period 2π . D. No asymptote E. $f'(x) = 3 \cos x - 3 \sin^2 x \cos x = 3 \cos x (1 - \sin^2 x) = 3 \cos^3 x$.

$f'(x) > 0 \Leftrightarrow \cos x > 0 \Leftrightarrow x \in (2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2})$ for each integer n , and $f'(x) < 0 \Leftrightarrow$

$\cos x < 0 \Leftrightarrow x \in (2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2})$ for each integer n . Thus, f is increasing on $(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2})$

for each integer n , and f is decreasing on $(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2})$ for each integer n .

F. f has local maximum values $f(2n\pi + \frac{\pi}{2}) = 2$ and local minimum values $f(2n\pi + \frac{3\pi}{2}) = -2$.

G. $f''(x) = -9 \sin x \cos^2 x = -9 \sin x (1 - \sin^2 x) = -9 \sin x (1 - \sin x)(1 + \sin x)$. $f''(x) < 0 \Leftrightarrow$

$\sin x > 0$ and $\sin x \neq \pm 1 \Leftrightarrow x \in (2n\pi, 2n\pi + \frac{\pi}{2}) \cup (2n\pi + \frac{\pi}{2}, 2n\pi + \pi)$ for some integer n .

$f''(x) > 0 \Leftrightarrow \sin x < 0$ and $\sin x \neq \pm 1 \Leftrightarrow x \in ((2n-1)\pi, (2n-1)\pi + \frac{\pi}{2}) \cup ((2n-1)\pi + \frac{\pi}{2}, 2n\pi)$

for some integer n . Thus, f is CD on the intervals $(2n\pi, (2n + \frac{1}{2})\pi)$ and

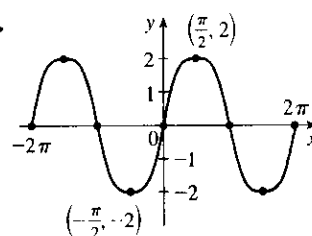
$((2n + \frac{1}{2})\pi, (2n + 1)\pi)$ [hence CD on the intervals $(2n\pi, (2n + 1)\pi)$]

for each integer n , and f is CU on the intervals $((2n-1)\pi, (2n - \frac{1}{2})\pi)$

and $((2n - \frac{1}{2})\pi, 2n\pi)$ [hence CU on the intervals $((2n-1)\pi, 2n\pi)$] for

each integer n . f has inflection points at $(n\pi, 0)$ for each integer n .

H.



32. $y = f(x) = \sin x - \tan x$ A. $D = \{x \mid x \neq (2n + 1)\frac{\pi}{2}\}$ B. $y = 0 \Leftrightarrow \sin x = \tan x = \frac{\sin x}{\cos x} \Leftrightarrow$

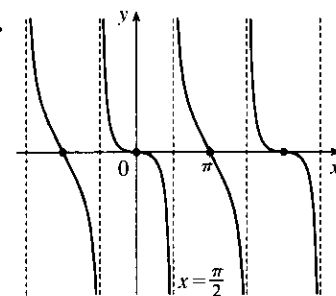
$\sin x = 0$ or $\cos x = 1 \Leftrightarrow x = n\pi$ (x -intercepts), y -intercept $= f(0) = 0$ C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. Also periodic with period 2π D. $\lim_{x \rightarrow (\pi/2)^-} (\sin x - \tan x) = -\infty$ and

$\lim_{x \rightarrow (\pi/2)^+} (\sin x - \tan x) = \infty$, so $x = n\pi + \frac{\pi}{2}$ are VA.

E. $f'(x) = \cos x - \sec^2 x \leq 0$, so f decreases on each interval in its domain, that is, on $((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2})$. F. No extreme values

G. $f''(x) = -\sin x - 2\sec^2 x \tan x = -\sin x (1 + 2\sec^3 x)$. Note that $1 + 2\sec^3 x \neq 0$ since $\sec^3 x \neq -\frac{1}{2}$. $f''(x) > 0$ for $-\frac{\pi}{2} < x < 0$ and $\frac{3\pi}{2} < x < 2\pi$, so f is CU on $((n - \frac{1}{2})\pi, n\pi)$ and CD on $(n\pi, (n + \frac{1}{2})\pi)$. f has IPs at $(n\pi, 0)$. Note also that $f'(0) = 0$, but $f'(\pi) = -2$.

H.



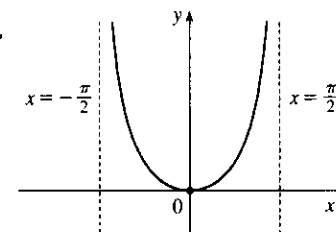
33. $y = f(x) = x \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ A. $D = (-\frac{\pi}{2}, \frac{\pi}{2})$ B. Intercepts are 0 C. $f(-x) = f(x)$, so the curve is symmetric about the y -axis. D. $\lim_{x \rightarrow (\pi/2)^-} x \tan x = \infty$ and $\lim_{x \rightarrow -(\pi/2)^+} x \tan x = \infty$, so $x = \frac{\pi}{2}$ and

$x = -\frac{\pi}{2}$ are VA. E. $f'(x) = \tan x + x \sec^2 x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, H.

so f increases on $(0, \frac{\pi}{2})$ and decreases on $(-\frac{\pi}{2}, 0)$.

F. Absolute and local minimum value $f(0) = 0$.

G. $y'' = 2\sec^2 x + 2x \tan x \sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $(-\frac{\pi}{2}, \frac{\pi}{2})$. No IP



34. $y = f(x) = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ A. $D = (-\frac{\pi}{2}, \frac{\pi}{2})$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(0) = 0$

$\Leftrightarrow 2x = \tan x \Leftrightarrow x = 0$ or $x \approx \pm 1.17$ C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the

origin. D. $\lim_{x \rightarrow (-\pi/2)^+} (2x - \tan x) = \infty$ and $\lim_{x \rightarrow (\pi/2)^-} (2x - \tan x) = -\infty$, so $x = \pm \frac{\pi}{2}$ are VA. No HA.

E. $f'(x) = 2 - \sec^2 x < 0 \Leftrightarrow |\sec x| > \sqrt{2}$ and $f'(x) > 0 \Leftrightarrow |\sec x| < \sqrt{2}$, so f is decreasing on $(-\frac{\pi}{2}, -\frac{\pi}{4})$, increasing on $(-\frac{\pi}{4}, \frac{\pi}{4})$, and decreasing again on $(\frac{\pi}{4}, \frac{\pi}{2})$ **F.** Local maximum

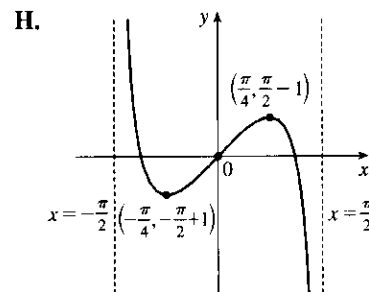
value $f(\frac{\pi}{4}) = \frac{\pi}{2} - 1$, local minimum value $f(-\frac{\pi}{4}) = -\frac{\pi}{2} + 1$

G. $f''(x) = -2 \sec x \cdot \sec x \tan x = -2 \tan x \sec^2 x$
 $= -2 \tan x (\tan^2 x + 1)$

so $f''(x) > 0 \Leftrightarrow \tan x < 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$, and $f''(x) < 0 \Leftrightarrow$

$\tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$. Thus, f is CU on $(-\frac{\pi}{2}, 0)$ and CD

on $(0, \frac{\pi}{2})$. f has an IP at $(0, 0)$.



35. $y = f(x) = \frac{1}{2}x - \sin x$, $0 < x < 3\pi$ **A.** $D = (0, 3\pi)$ **B.** No y -intercept. The x -intercept, approximately 1.9, can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{2} - \cos x > 0 \Leftrightarrow$

$\cos x < \frac{1}{2} \Leftrightarrow \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$, so f is increasing on $(\frac{\pi}{3}, \frac{5\pi}{3})$ and $(\frac{7\pi}{3}, 3\pi)$ and decreasing

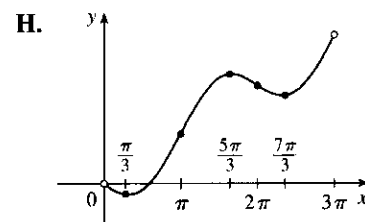
on $(0, \frac{\pi}{3})$ and $(\frac{5\pi}{3}, \frac{7\pi}{3})$. **F.** Local minimum value $f(\frac{\pi}{3}) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$,

local maximum value $f(\frac{5\pi}{3}) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$, local minimum value

$f(\frac{7\pi}{3}) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2}$ **G.** $f''(x) = \sin x > 0 \Leftrightarrow 0 < x < \pi$ or

$2\pi < x < 3\pi$, so f is CU on $(0, \pi)$ and $(2\pi, 3\pi)$ and CD on $(\pi, 2\pi)$.

IPs at $(\pi, \frac{\pi}{2})$ and $(2\pi, \pi)$.



36. $y = f(x) = \cos^2 x - 2 \sin x$ **A.** $D = \mathbb{R}$ **B.** y -intercept: $f(0) = 1$ **C.** No symmetry, but f has period 2π .

D. No asymptote **E.** $y' = 2 \cos x (-\sin x) - 2 \cos x = -2 \cos x (\sin x + 1)$. $y' = 0 \Leftrightarrow \cos x = 0$ or $\sin x = -1 \Leftrightarrow x = (2n+1)\frac{\pi}{2}$. $y' > 0$ when $\cos x < 0$ since $\sin x + 1 \geq 0$ for all x . So $y' > 0$ and f is increasing on $((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2})$; $y' < 0$ and f is decreasing on $((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2})$. **F.** Local

maximum values $f((4n+3)\frac{\pi}{2}) = 2$, local minimum values $f((4n+1)\frac{\pi}{2}) = -2$

G. $y' = -2 \cos x (\sin x + 1) = -\sin 2x - 2 \cos x \Rightarrow$

$y'' = -2 \cos 2x + 2 \sin x = -2(1 - 2 \sin^2 x) + 2 \sin x$

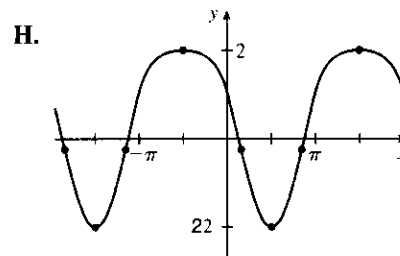
$= 4 \sin^2 x + 2 \sin x - 2 = 2(2 \sin x - 1)(\sin x + 1)$

$y'' = 0 \Leftrightarrow \sin x = \frac{1}{2}$ or $-1 \Rightarrow x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$, or

$\frac{3\pi}{2} + 2n\pi$. $y'' > 0$ and f is CU on $(\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi)$;

$y'' \leq 0$ and f is CD on $(\frac{5\pi}{6} + 2n\pi, \frac{\pi}{6} + 2(n+1)\pi)$.

IPs at $(\frac{\pi}{6} + 2n\pi, -\frac{1}{4})$ and $(\frac{5\pi}{6} + 2n\pi, -\frac{1}{4})$.



37. $y = f(x) = \sin 2x - 2 \sin x$ A. $D = \mathbb{R}$ B. y -intercept $= f(0) = 0$. $y = 0 \Leftrightarrow$

$$2 \sin x = \sin 2x = 2 \sin x \cos x \Leftrightarrow \sin x = 0 \text{ or } \cos x = 1 \Leftrightarrow x = n\pi \text{ (x-intercepts)}$$

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$.

Note: f is periodic with period 2π , so we determine E–G for $-\pi \leq x \leq \pi$. D. No asymptotes

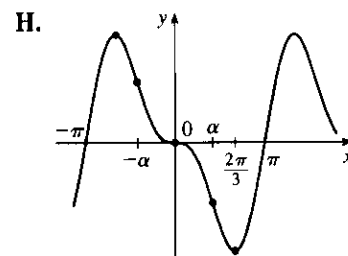
$$\begin{aligned} \text{E. } f'(x) &= 2 \cos 2x - 2 \cos x = 2(2 \cos^2 x - 1 - \cos x) = 2(2 \cos x + 1)(\cos x - 1) > 0 \Leftrightarrow \cos x < -\frac{1}{2} \\ &\Leftrightarrow -\pi < x < -\frac{2\pi}{3} \text{ or } \frac{2\pi}{3} < x < \pi, \text{ so } f \text{ is increasing on } (-\pi, -\frac{2\pi}{3}), (\frac{2\pi}{3}, \pi) \text{ and decreasing on } (-\frac{2\pi}{3}, \frac{2\pi}{3}). \end{aligned}$$

$$\text{F. Local maximum value } f(-\frac{2\pi}{3}) = \frac{3\sqrt{3}}{2},$$

$$\text{local minimum value } f(\frac{2\pi}{3}) = -\frac{3\sqrt{3}}{2}$$

G. $f''(x) = -4 \sin 2x + 2 \sin x = 2 \sin x (1 - 4 \cos x) = 0$ when $x = 0, \pm\pi$ or $\cos x = \frac{1}{4}$. If $\alpha = \cos^{-1} \frac{1}{4}$, then f is CU on $(-\alpha, 0)$ and (α, π) and CD on $(-\pi, -\alpha)$ and $(0, \alpha)$.

IPs at $(0, 0), (\pm\pi, 0), (\alpha, -\frac{3\sqrt{15}}{8}), (-\alpha, \frac{3\sqrt{15}}{8})$.

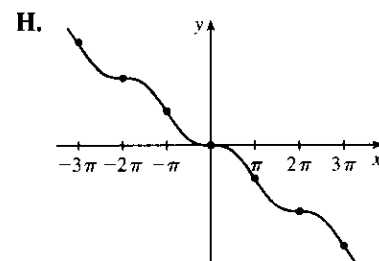


38. $f(x) = \sin x - x$ A. $D = \mathbb{R}$ B. x -intercept $= 0 = y$ -intercept

C. $f(-x) = \sin(-x) - (-x) = -(\sin x - x) = -f(x)$, so f is odd.

D. No asymptote E. $f'(x) = \cos x - 1 \leq 0$ for all x , so f is decreasing on $(-\infty, \infty)$. F. No extreme values G. $f''(x) = -\sin x \Rightarrow$

$f''(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow (2n-1)\pi < x < 2n\pi$, so f is CU on $((2n-1)\pi, 2n\pi)$ and CD on $(2n\pi, (2n+1)\pi)$, n an integer. Points of inflection occur when $x = n\pi$.



$$39. y = f(x) = \frac{\sin x}{1 + \cos x} \left[\begin{array}{l} \text{when} \\ \cos x \neq -1 \end{array} \frac{\sin x}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} = \frac{\sin x (1 - \cos x)}{\sin^2 x} = \frac{1 - \cos x}{\sin x} = \csc x - \cot x \right]$$

A. The domain of f is the set of all real numbers except odd integer multiples of π . B. y -intercept: $f(0) = 0$; x -intercepts: $x = n\pi$, n an even integer. C. $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π .

D. When n is an odd integer, $\lim_{x \rightarrow (n\pi)^-} f(x) = \infty$ and $\lim_{x \rightarrow (n\pi)^+} f(x) = -\infty$, so $x = n\pi$ is a VA for each odd integer n . No HA.

$$\text{E. } f'(x) = \frac{(1 + \cos x) \cdot \cos x - \sin x(-\sin x)}{(1 + \cos x)^2} = \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}. f'(x) > 0 \text{ for all } x \text{ except odd}$$

multiples of π , so f is increasing on $((2k-1)\pi, (2k+1)\pi)$ for each integer k . F. No extreme values

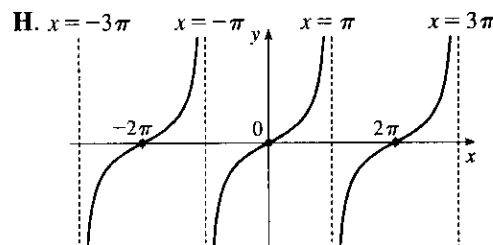
$$G. f''(x) = \frac{\sin x}{(1 + \cos x)^2} > 0 \Rightarrow \sin x > 0 \Rightarrow$$

$x \in (2k\pi, (2k+1)\pi)$ and $f''(x) < 0$ on $((2k-1)\pi, 2k\pi)$

for each integer k . f is CU on $(2k\pi, (2k+1)\pi)$ and CD on

$((2k-1)\pi, 2k\pi)$ for each integer k . f has IPs at $(2k\pi, 0)$

for each integer k .



40. $y = f(x) = \cos x / (2 + \sin x)$ A. $D = \mathbb{R}$ Note: f is periodic with period 2π , so we determine B–G on $[0, 2\pi]$.

B. x -intercepts $\frac{\pi}{2}, \frac{3\pi}{2}$, y -intercept $= f(0) = \frac{1}{2}$ C. No symmetry other than periodicity D. No asymptote

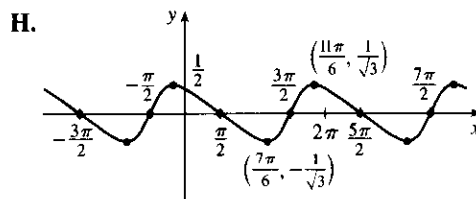
$$E. f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x (\cos x)}{(2 + \sin x)^2} = -\frac{2 \sin x + 1}{(2 + \sin x)^2}. f'(x) > 0 \Leftrightarrow 2 \sin x + 1 < 0 \Leftrightarrow$$

$\sin x < -\frac{1}{2} \Leftrightarrow \frac{7\pi}{6} < x < \frac{11\pi}{6}$, so f is increasing on $(\frac{7\pi}{6}, \frac{11\pi}{6})$ and decreasing on $(0, \frac{7\pi}{6})$, $(\frac{11\pi}{6}, 2\pi)$.

F. Local minimum value $f(\frac{7\pi}{6}) = -\frac{1}{\sqrt{3}}$, local maximum value $f(\frac{11\pi}{6}) = \frac{1}{\sqrt{3}}$

$$G. f''(x) = -\frac{(2 + \sin x)^2 (2 \cos x) - (2 \sin x + 1) 2(2 + \sin x) \cos x}{(2 + \sin x)^4} = -\frac{2 \cos x (1 - \sin x)}{(2 + \sin x)^3} > 0 \Leftrightarrow$$

$\cos x < 0 \Leftrightarrow \frac{\pi}{2} < x < \frac{3\pi}{2}$, so f is CU on $(\frac{\pi}{2}, \frac{3\pi}{2})$ and CD on $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$. IP at $(\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$



$$41. y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2 = -\frac{W}{24EI}x^2(x^2 - 2Lx + L^2) \\ = -\frac{W}{24EI}x^2(x - L)^2 = cx^2(x - L)^2$$

where $c = -\frac{W}{24EI}$ is a negative constant and $0 \leq x \leq L$. We sketch

$f(x) = cx^2(x - L)^2$ for $c = -1$. $f(0) = f(L) = 0$.

$f'(x) = cx^2[2(x - L)] + (x - L)^2(2cx) = 2cx(x - L)[x + (x - L)] = 2cx(x - L)(2x - L)$. So for

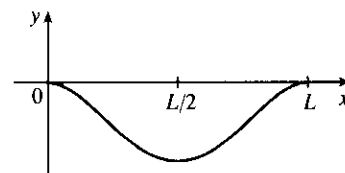
$0 < x < L$, $f'(x) > 0 \Leftrightarrow x(x - L)(2x - L) < 0$ (since $c < 0$) $\Leftrightarrow L/2 < x < L$ and $f'(x) < 0 \Leftrightarrow$

$0 < x < L/2$. So f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at $(L/2, f(L/2)) = (L/2, cL^4/16)$.

$$f'(x) = 2c[x(x - L)(2x - L)] \Rightarrow$$

$$f''(x) = 2c[1(x - L)(2x - L) + x(1)(2x - L) + x(x - L)(2)] = 2c(6x^2 - 6Lx + L^2) = 0 \Leftrightarrow$$

$$x = \frac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L, \text{ and these are the } x\text{-coordinates of the two inflection points.}$$

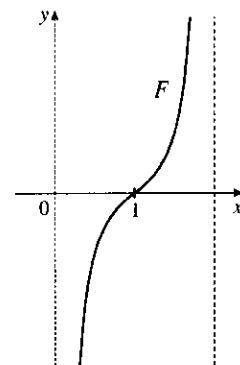


42. $F(x) = -\frac{k}{x^2} + \frac{k}{(x-2)^2}$, where $k > 0$ and $0 < x < 2$. For $0 < x < 2$,

$$x - 2 < 0, \text{ so } F'(x) = \frac{2k}{x^3} - \frac{2k}{(x-2)^3} > 0 \text{ and } F \text{ is increasing.}$$

$$\lim_{x \rightarrow 0^+} F(x) = -\infty \text{ and } \lim_{x \rightarrow 2^-} F(x) = \infty, \text{ so } x = 0 \text{ and } x = 2 \text{ are vertical}$$

asymptotes. Notice that when the middle particle is at $x = 1$, the net force acting on it is 0. When $x > 1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x = 2$, the force on it rapidly becomes very large. When $x < 1$, the net force is negative, so it acts to the left. If the particle approaches 0, the force becomes very large to the left.



43. $y = \frac{x^2 + 1}{x + 1}$. Long division gives us:

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2 + 1} \\ \underline{x^2 + x} \\ -x + 1 \\ \underline{-x - 1} \\ 2 \end{array}$$

$$\text{Thus, } y = f(x) = \frac{x^2 + 1}{x + 1} = x - 1 + \frac{2}{x + 1} \text{ and } f(x) - (x - 1) = \frac{2}{x + 1} = \frac{\frac{2}{x}}{1 + \frac{1}{x}} \quad [\text{for } x \neq 0] \rightarrow 0$$

as $x \rightarrow \pm\infty$. So the line $y = x - 1$ is a slant asymptote (SA).

44. $y = \frac{2x^3 + x^2 + x + 3}{x^2 + 2x}$. Long division gives us:

$$\begin{array}{r} 2x-3 \\ x^2+2x \overline{) 2x^3 + x^2 + x + 3} \\ \underline{2x^3 + 4x^2} \\ -3x^2 + x \\ \underline{-3x^2 - 6x} \\ 7x + 3 \end{array}$$

$$\text{Thus, } y = f(x) = \frac{2x^3 + x^2 + x + 3}{x^2 + 2x} = 2x - 3 + \frac{7x + 3}{x^2 + 2x} \text{ and } f(x) - (2x - 3) = \frac{7x + 3}{x^2 + 2x} = \frac{\frac{7}{x} + \frac{3}{x^2}}{1 + \frac{2}{x}}$$

[for $x \neq 0$] $\rightarrow 0$ as $x \rightarrow \pm\infty$. So the line $y = 2x - 3$ is a SA.

45. $y = \frac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$. Long division gives us:

$$\begin{array}{r} 2x - 2 \\ 2x^2 + x - 3 \overline{) 4x^3 - 2x^2 + 5} \\ \underline{4x^3 + 2x^2 - 6x} \\ -4x^2 + 6x + 5 \\ \underline{-4x^2 - 2x + 6} \\ 8x - 1 \end{array}$$

Thus, $y = f(x) = \frac{4x^3 - 2x^2 + 5}{2x^2 + x - 3} = 2x - 2 + \frac{8x - 1}{2x^2 + x - 3}$ and

$$f(x) - (2x - 2) = \frac{8x - 1}{2x^2 + x - 3} = \frac{\frac{8}{x} - \frac{1}{x^2}}{2 + \frac{1}{x} - \frac{3}{x^2}} \quad [\text{for } x \neq 0] \rightarrow 0 \text{ as } x \rightarrow \pm\infty. \text{ So the line } y = 2x - 2 \text{ is}$$

a SA.

46. $y = \frac{5x^4 + x^2 + x}{x^3 - x^2 + 2}$. Long division gives us:

$$\begin{array}{r} 5x + 5 \\ x^3 - x^2 + 2 \overline{) 5x^4 + x^2 + x} \\ \underline{5x^4 - 5x^3 + 10x} \\ 5x^3 + x^2 - 9x \\ \underline{5x^3 - 5x^2 + 10} \\ 6x^2 - 9x - 10 \end{array}$$

Thus, $y = f(x) = \frac{5x^4 + x^2 + x}{x^3 - x^2 + 2} = 5x + 5 + \frac{6x^2 - 9x - 10}{x^3 - x^2 + 2}$ and

$$f(x) - (5x + 5) = \frac{6x^2 - 9x - 10}{x^3 - x^2 + 2} = \frac{\frac{6}{x} - \frac{9}{x^2} - \frac{10}{x^3}}{1 - \frac{1}{x} + \frac{2}{x^3}} \quad [\text{for } x \neq 0] \rightarrow 0 \text{ as } x \rightarrow \pm\infty. \text{ So the line } y = 5x + 5$$

is a SA.

47. $y = f(x) = \frac{-2x^2 + 5x - 1}{2x - 1} = -x + 2 + \frac{1}{2x - 1}$ A. $D = \{x \in \mathbb{R} \mid x \neq \frac{1}{2}\} = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

B. y -intercept: $f(0) = 1$; x -intercepts: $f(x) = 0 \Rightarrow -2x^2 + 5x - 1 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{17}}{-4} \Rightarrow$

$x \approx 0.22, 2.28$. C. No symmetry

D. $\lim_{x \rightarrow (1/2)^-} f(x) = -\infty$ and $\lim_{x \rightarrow (1/2)^+} f(x) = \infty$, so $x = \frac{1}{2}$ is a VA.

$\lim_{x \rightarrow \pm\infty} [f(x) - (-x + 2)] = \lim_{x \rightarrow \pm\infty} \frac{1}{2x - 1} = 0$, so the line $y = -x + 2$ is a SA.

E. $f'(x) = -1 - \frac{2}{(2x-1)^2} < 0$ for $x \neq \frac{1}{2}$, so f is decreasing

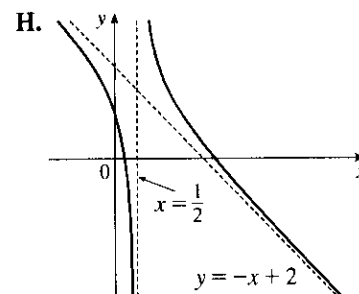
on $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$. F. No extreme values

G. $f'(x) = -1 - 2(2x-1)^{-2} \Rightarrow$

$f''(x) = -2(-2)(2x-1)^{-3}(2) = \frac{8}{(2x-1)^3}$, so $f''(x) > 0$ when

$x > \frac{1}{2}$ and $f''(x) < 0$ when $x < \frac{1}{2}$. Thus, f is CU on $(\frac{1}{2}, \infty)$ and CD

on $(-\infty, \frac{1}{2})$. No IP



48. $y = f(x) = \frac{x^2 + 12}{x - 2} = x + 2 + \frac{16}{x - 2}$ A. $D = \{x \in \mathbb{R} \mid x \neq 2\} = (-\infty, 2) \cup (2, \infty)$

B. y -intercept: $f(0) = -6$; no x -intercepts. C. No symmetry D. $\lim_{x \rightarrow 2^-} f(x) = -\infty$ and $\lim_{x \rightarrow 2^+} f(x) = \infty$,

so $x = 2$ is a VA. $\lim_{x \rightarrow \pm\infty} [f(x) - (x + 2)] = \lim_{x \rightarrow \pm\infty} \frac{16}{x - 2} = 0$, so the line $y = x + 2$ is a slant asymptote.

E. $f'(x) = 1 - \frac{16}{(x-2)^2} = \frac{x^2 - 4x - 12}{(x-2)^2} = \frac{(x-6)(x+2)}{(x-2)^2}$, so $f'(x) > 0$ when $x < -2$ or $x > 6$ and

$f'(x) < 0$ when $-2 < x < 2$ or $2 < x < 6$. Thus, f is increasing on $(-\infty, -2)$ and $(6, \infty)$ and decreasing on $(-2, 2)$ and $(2, 6)$.

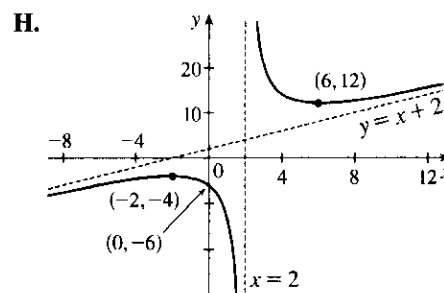
F. Local maximum value $f(-2) = -4$,

local minimum value $f(6) = 12$

G. $f''(x) = 16(-2)(x-2)^{-3} = \frac{32}{(x-2)^3}$, so $f''(x) > 0$ for

$x > 2$ and $f''(x) < 0$ for $x < 2$. f is CU on $(2, \infty)$ and CD

on $(-\infty, 2)$. No IP



49. $y = f(x) = (x^2 + 4)/x = x + 4/x$ A. $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ B. No intercept

C. $f(-x) = -f(x) \Rightarrow$ symmetry about the origin D. $\lim_{x \rightarrow \infty} (x + 4/x) = \infty$ but $f(x) - x = 4/x \rightarrow 0$ as

$x \rightarrow \pm\infty$, so $y = x$ is a slant asymptote. $\lim_{x \rightarrow 0^+} (x + 4/x) = \infty$ and

$\lim_{x \rightarrow 0^-} (x + 4/x) = -\infty$, so $x = 0$ is a VA. E. $f'(x) = 1 - 4/x^2 > 0$

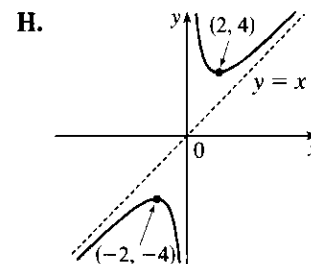
$\Leftrightarrow x^2 > 4 \Leftrightarrow x > 2$ or $x < -2$, so f is increasing on $(-\infty, -2)$

and $(2, \infty)$ and decreasing on $(-2, 0)$ and $(0, 2)$.

F. Local maximum value $f(-2) = -4$, local minimum value $f(2) = 4$

G. $f''(x) = 8/x^3 > 0 \Leftrightarrow x > 0$ so f is CU on $(0, \infty)$ and CD

on $(-\infty, 0)$. No IP



50. $y = f(x) = \frac{x^2 + x + 1}{x} = x + 1 + \frac{1}{x}$ A. $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ B. No intercept

(x -intercepts would occur when $x^2 + x + 1 = 0$ but this equation has no real roots since $b^2 - 4ac = -3 < 0$.)

C. No symmetry D. $\lim_{x \rightarrow \infty} (x + 1 + 1/x) = \pm\infty$, so no HA. But $(x + 1 + 1/x) - (x + 1) = 1/x \rightarrow 0$ as

$x \rightarrow \pm\infty$, so $y = x + 1$ is a slant asymptote. Also $\lim_{x \rightarrow 0^+} (x + 1 + 1/x) = \infty$, $\lim_{x \rightarrow 0^-} (x + 1 + 1/x) = -\infty$, so

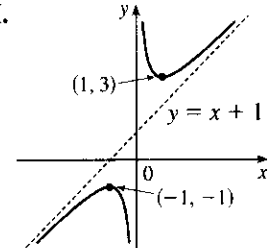
$x = 0$ is a VA. E. $f'(x) = 1 - 1/x^2 > 0$ when $x^2 > 1 \Leftrightarrow x > 1$ or

$x < -1$; $f'(x) < 0 \Leftrightarrow -1 < x < 1$. So f is increasing on $(-\infty, -1)$,

$(1, \infty)$ and decreasing on $(-1, 0)$, $(0, 1)$. F. $f(1) = 3$ is a local

minimum, $f(-1) = -1$ is a local maximum. G. $f''(x) = 2/x^3 > 0$

$\Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP



51. $y = f(x) = \frac{2x^3 + x^2 + 1}{x^2 + 1} = 2x + 1 + \frac{-2x}{x^2 + 1}$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 1$; x -intercept: $f(x) = 0$

$\Rightarrow 0 = 2x^3 + x^2 + 1 = (x + 1)(2x^2 - x + 1) \Rightarrow x = -1$ C. No symmetry D. No VA

$\lim_{x \rightarrow \pm\infty} [f(x) - (2x + 1)] = \lim_{x \rightarrow \pm\infty} \frac{-2x}{x^2 + 1} = \lim_{x \rightarrow \pm\infty} \frac{-2/x}{1 + 1/x^2} = 0$, so the line $y = 2x + 1$ is a slant asymptote.

$$\begin{aligned} \text{E. } f'(x) &= 2 + \frac{(x^2 + 1)(-2) - (-2x)(2x)}{(x^2 + 1)^2} = \frac{2(x^4 + 2x^2 + 1) - 2x^2 - 2 + 4x^2}{(x^2 + 1)^2} \\ &= \frac{2x^4 + 6x^2}{(x^2 + 1)^2} = \frac{2x^2(x^2 + 3)}{(x^2 + 1)^2} \end{aligned}$$

so $f'(x) > 0$ if $x \neq 0$. Thus, f is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at 0, f is increasing on $\mathbb{R}$. F. No extreme values

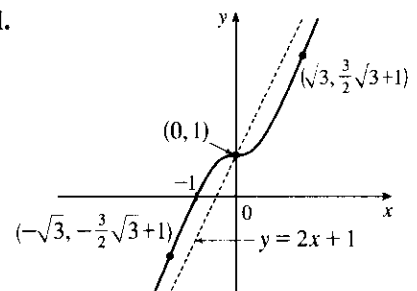
$$\begin{aligned} \text{G. } f''(x) &= \frac{(x^2 + 1)^2 \cdot (8x^3 + 12x) - (2x^4 + 6x^2) \cdot 2(x^2 + 1)(2x)}{[(x^2 + 1)^2]^2} \quad \text{H.} \\ &= \frac{4x(x^2 + 1)[(x^2 + 1)(2x^2 + 3) - 2x^4 - 6x^2]}{(x^2 + 1)^4} \\ &= \frac{4x(-x^2 + 3)}{(x^2 + 1)^3} \end{aligned}$$

so $f''(x) > 0$ for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and $f''(x) < 0$ for

$-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. f is CU on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$,

and CD on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. There are three IPs: $(0, 1)$, $(-\sqrt{3}, -\frac{3}{2}\sqrt{3} + 1) \approx (-1.73, -1.60)$, and

$(\sqrt{3}, \frac{3}{2}\sqrt{3} + 1) \approx (1.73, 3.60)$.



52. $y = f(x) = \frac{(x + 1)^3}{(x - 1)^2} = \frac{x^3 + 3x^2 + 3x + 1}{x^2 - 2x + 1} = x + 5 + \frac{12x - 4}{(x - 1)^2}$

A. $D = \{x \in \mathbb{R} \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ B. y -intercept: $f(0) = 1$;

x -intercept: $f(x) = 0 \Rightarrow x = -1$ C. No symmetry D. $\lim_{x \rightarrow 1} f(x) = \infty$, so $x = 1$ is a VA.

$$\lim_{x \rightarrow \pm\infty} [f(x) - (x + 5)] = \lim_{x \rightarrow \pm\infty} \frac{12x - 4}{x^2 - 2x + 1} = \lim_{x \rightarrow \pm\infty} \frac{\frac{12}{x} - \frac{4}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} = 0, \text{ so the line } y = x + 5 \text{ is a SA.}$$

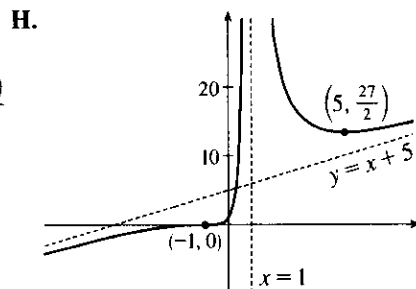
$$\begin{aligned} \text{E. } f'(x) &= \frac{(x-1)^2 \cdot 3(x+1)^2 - (x+1)^3 \cdot 2(x-1)}{[(x-1)^2]^2} \\ &= \frac{(x-1)(x+1)^2[3(x-1) - 2(x+1)]}{(x-1)^4} = \frac{(x+1)^2(x-5)}{(x-1)^3} \end{aligned}$$

so $f'(x) > 0$ when $x < -1$, $-1 < x < 1$, or $x > 5$, and $f'(x) < 0$ when $1 < x < 5$. f is increasing on $(-\infty, 1)$ and $(5, \infty)$ and decreasing on $(1, 5)$.

F. Local minimum value $f(5) = \frac{216}{16} = \frac{27}{2}$, no local maximum

$$\begin{aligned} \text{G. } f''(x) &= \frac{(x-1)^3[(x-1)^2 + (x-5) \cdot 2(x+1)] - (x+1)^2(x-5) \cdot 3(x-1)^2}{[(x-1)^3]^2} \\ &= \frac{(x-1)^2(x+1)\{(x-1)[(x+1) + 2(x-5)] - 3(x+1)(x-5)\}}{(x-1)^6} \\ &= \frac{(x+1)\{(x-1)[3x-9] - 3(x^2-4x-5)\}}{(x-1)^4} = \frac{(x+1)(24)}{(x-1)^4} \end{aligned}$$

so $f''(x) > 0$ if $-1 < x < 1$ or $x > 1$, and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 1)$ and $(1, \infty)$ and CD on $(-\infty, -1)$. IP at $(-1, 0)$



$$53. y = f(x) = \sqrt{4x^2 + 9} \Rightarrow f'(x) = \frac{4x}{\sqrt{4x^2 + 9}} \Rightarrow$$

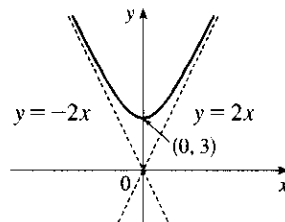
$$f''(x) = \frac{\sqrt{4x^2 + 9} \cdot 4 - 4x \cdot 4x/\sqrt{4x^2 + 9}}{4x^2 + 9} = \frac{4(4x^2 + 9) - 16x^2}{(4x^2 + 9)^{3/2}} = \frac{36}{(4x^2 + 9)^{3/2}}. \text{ } f \text{ is defined on}$$

$(-\infty, \infty)$. $f(-x) = f(x)$, so f is even, which means its graph is symmetric about the y -axis. The y -intercept is $f(0) = 3$. There are no x -intercepts since $f(x) > 0$ for all x .

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{4x^2 + 9} - 2x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{4x^2 + 9} - 2x)(\sqrt{4x^2 + 9} + 2x)}{\sqrt{4x^2 + 9} + 2x} \\ &= \lim_{x \rightarrow \infty} \frac{(4x^2 + 9) - 4x^2}{\sqrt{4x^2 + 9} + 2x} = \lim_{x \rightarrow \infty} \frac{9}{\sqrt{4x^2 + 9} + 2x} = 0 \end{aligned}$$

$$\text{and, similarly, } \lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 9} + 2x) = \lim_{x \rightarrow -\infty} \frac{9}{\sqrt{4x^2 + 9} - 2x} = 0,$$

so $y = \pm 2x$ are slant asymptotes. f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ with local minimum $f(0) = 3$. $f''(x) > 0$ for all x , so f is CU on $\mathbb{R}$.



$$54. y = f(x) = \sqrt{x^2 + 4x} = \sqrt{x(x+4)}. \quad x(x+4) \geq 0 \Leftrightarrow x \leq -4 \text{ or } x \geq 0, \text{ so } D = (-\infty, -4] \cup [0, \infty).$$

y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Rightarrow x = -4, 0$.

$$\begin{aligned}\sqrt{x^2+4x} \mp (x+2) &= \frac{\sqrt{x^2+4x} \mp (x+2)}{1} \cdot \frac{\sqrt{x^2+4x} \pm (x+2)}{\sqrt{x^2+4x} \pm (x+2)} = \frac{(x^2+4x) - (x^2+4x+4)}{\sqrt{x^2+4x} \pm (x+2)} \\ &= \frac{-4}{\sqrt{x^2+4x} \pm (x+2)}\end{aligned}$$

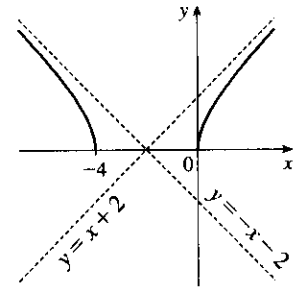
so $\lim_{x \rightarrow \pm\infty} [f(x) \mp (x+2)] = 0$. Thus, the graph of f approaches the slant asymptote $y = x + 2$ as $x \rightarrow \infty$ and it

approaches the slant asymptote $y = -(x + 2)$ as $x \rightarrow -\infty$. $f'(x) = \frac{x+2}{\sqrt{x^2+4x}}$, so $f'(x) < 0$ for $x < -4$

and $f'(x) > 0$ for $x > 0$; that is, f is decreasing on $(-\infty, -4)$ and increasing on $(0, \infty)$. There are no local extreme values.

$$\begin{aligned} f'(x) &= (x+2)(x^2+4x)^{-1/2} \Rightarrow \\ f''(x) &= (x+2) \cdot \left(-\frac{1}{2}\right)(x^2+4x)^{-3/2} \cdot (2x+4) + (x^2+4x)^{-1/2} \\ &= (x^2+4x)^{-3/2} [-(x+2)^2 + (x^2+4x)] \\ &= -4(x^2+4x)^{-3/2} < 0 \text{ on } D \end{aligned}$$

so f is CD on $(-\infty, -4)$ and $(0, \infty)$. No IP



55. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$. Now

$$\lim_{x \rightarrow \infty} \left[\frac{b}{a} \sqrt{x^2 - a^2} - \frac{b}{a} x \right] = \frac{b}{a} \cdot \lim_{x \rightarrow \infty} (\sqrt{x^2 - a^2} - x) \frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2} + x} = \frac{b}{a} \cdot \lim_{x \rightarrow \infty} \frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0,$$

which shows that $y = \frac{b}{a}x$ is a slant asymptote. Similarly,

$$\lim_{x \rightarrow \infty} \left[-\frac{b}{a} \sqrt{x^2 - a^2} - \left(-\frac{b}{a} x \right) \right] = -\frac{b}{a} \cdot \lim_{x \rightarrow \infty} \frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0, \text{ so } y = -\frac{b}{a}x \text{ is a slant asymptote.}$$

56. $f(x) - x^2 = \frac{x^3 + 1}{x} - x^2 = \frac{x^3 + 1 - x^3}{x} = \frac{1}{x}$, and $\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$. Therefore, $\lim_{x \rightarrow \pm\infty} [f(x) - x^2] = 0$, and so

the graph of f is asymptotic to that of $y = x^2$. For purposes of differentiation, we will use $f(x) = x^2 + 1/x$.

A. $D = \{x \mid x \neq 0\}$ **B.** No y -intercept; to find the x -intercept, we set $y = 0 \Leftrightarrow x = -1$.

C. No symmetry D. $\lim_{x \rightarrow 0^+} \frac{x^3 + 1}{x} = \infty$ and $\lim_{x \rightarrow 0^-} \frac{x^3 + 1}{x} = -\infty$,

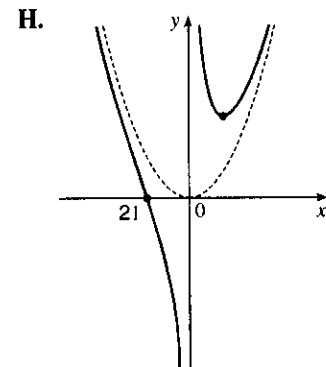
so $x = 0$ is a vertical asymptote. Also, the graph is asymptotic to the

parabola $y = x^2$, as shown above. E. $f'(x) = 2x - 1/x^2 > 0 \Leftrightarrow$

$x > \frac{1}{\sqrt[3]{2}}$, so f is increasing on $(\frac{1}{\sqrt[3]{2}}, \infty)$ and decreasing on $(-\infty, 0)$ and

$\left(0, \frac{1}{\sqrt[3]{2}}\right)$. **F.** Local minimum value $f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{3\sqrt[3]{3}}{2}$, no local maximum

G. $f''(x) = 2 + 2/x^3 > 0 \Leftrightarrow x < -1$ or $x > 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$, and CD on $(-1, 0)$. IP at $(-1, 0)$



57. $\lim_{x \rightarrow \pm\infty} [f(x) - x^3] = \lim_{x \rightarrow \pm\infty} \frac{x^4 + 1}{x} - \frac{x^4}{x} = \lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$, so the graph of f is asymptotic to that of $y = x^3$.

A. $D = \{x \mid x \neq 0\}$ B. No intercept C. f is symmetric about the origin. D. $\lim_{x \rightarrow 0^-} \left(x^3 + \frac{1}{x}\right) = -\infty$ and

$\lim_{x \rightarrow 0^+} \left(x^3 + \frac{1}{x}\right) = \infty$, so $x = 0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to

that of $y = x^3$. E. $f'(x) = 3x^2 - 1/x^2 > 0 \Leftrightarrow x^4 > \frac{1}{3} \Leftrightarrow$

$|x| > \frac{1}{\sqrt[4]{3}}$, so f is increasing on $(-\infty, -\frac{1}{\sqrt[4]{3}})$ and $(\frac{1}{\sqrt[4]{3}}, \infty)$ and

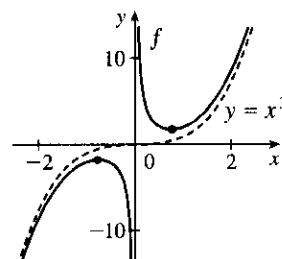
decreasing on $(-\frac{1}{\sqrt[4]{3}}, 0)$ and $(0, \frac{1}{\sqrt[4]{3}})$. F. Local maximum value

$f(-\frac{1}{\sqrt[4]{3}}) = -4 \cdot 3^{-5/4}$, local minimum value $f(\frac{1}{\sqrt[4]{3}}) = 4 \cdot 3^{-5/4}$

G. $f''(x) = 6x + 2/x^3 > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and

CD on $(-\infty, 0)$. No IP

H.



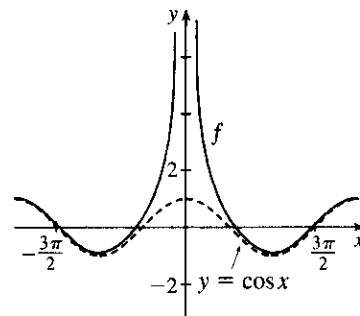
58. $\lim_{x \rightarrow \pm\infty} [f(x) - \cos x] = \lim_{x \rightarrow \pm\infty} 1/x^2 = 0$, so the graph of f is

asymptotic to that of $\cos x$. The intercepts can only be found

approximately. $f(x) = f(-x)$, so f is even. $\lim_{x \rightarrow 0} \left(\cos x + \frac{1}{x^2}\right) = \infty$, so

$x = 0$ is a vertical asymptote. We don't need to calculate the derivatives,

since we know the asymptotic behavior of the curve.

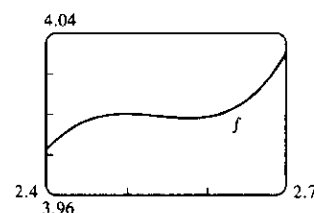
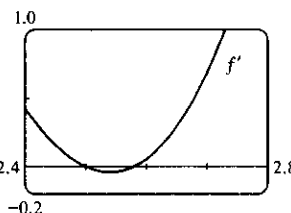
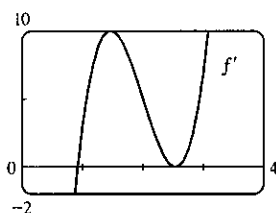
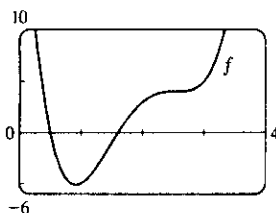


4.6 Graphing with Calculus and Calculators

$$1. f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29 \Rightarrow f'(x) = 16x^3 - 96x^2 + 178x - 95 \Rightarrow$$

$$f''(x) = 48x^2 - 192x + 178. f(x) = 0 \Leftrightarrow x \approx 0.5, 1.60; f'(x) = 0 \Leftrightarrow x \approx 0.92, 2.5, 2.58 \text{ and}$$

$$f''(x) = 0 \Leftrightarrow x \approx 1.46, 2.54.$$

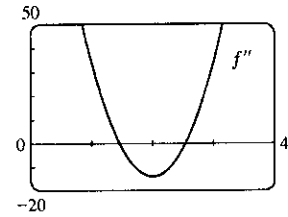


From the graphs of f' , we estimate that $f' < 0$ and that f is decreasing on $(-\infty, 0.92)$ and $(2.5, 2.58)$, and that

$f' > 0$ and f is increasing on $(0.92, 2.5)$ and $(2.58, \infty)$ with local minimum values $f(0.92) \approx -5.12$ and

$f(2.58) \approx 3.998$ and local maximum value $f(2.5) = 4$. The graphs of f' make it clear that f has a maximum and a minimum near $x = 2.5$, shown more clearly in the fourth graph.

From the graph of f'' , we estimate that $f'' > 0$ and that f is CU on $(-\infty, 1.46)$ and $(2.54, \infty)$, and that $f'' < 0$ and f is CD on $(1.46, 2.54)$. There are inflection points at about $(1.46, -1.40)$ and $(2.54, 3.999)$.

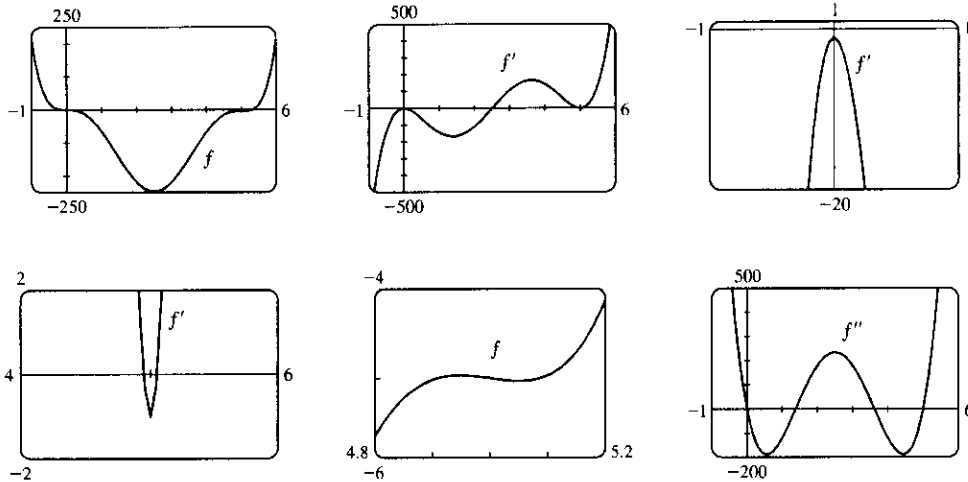


$$2. f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x \Rightarrow$$

$$f'(x) = 6x^5 - 75x^4 + 300x^3 - 375x^2 - 1 \Rightarrow f''(x) = 30x^4 - 300x^3 + 900x^2 - 750x.$$

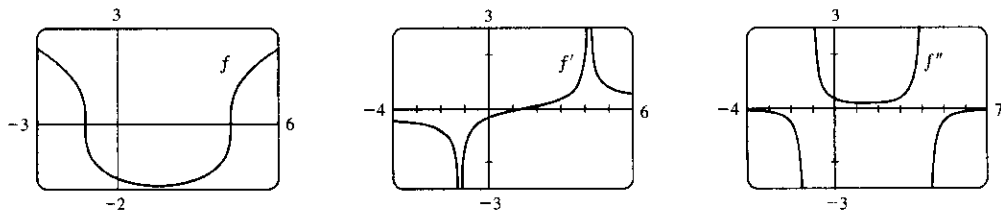
$$f(x) = 0 \Leftrightarrow x = 0 \text{ or } x \approx 5.33; f'(x) = 0 \Leftrightarrow x \approx 2.50, 4.95, \text{ or } 5.05;$$

$$f''(x) = 0 \Leftrightarrow x = 0, 5 \text{ or } x \approx 1.38, 3.62.$$



From the graphs of f' , we estimate that f is decreasing on $(-\infty, 2.50)$, increasing on $(2.50, 4.95)$, decreasing on $(4.95, 5.05)$, and increasing on $(5.05, \infty)$, with local minimum values $f(2.50) \approx -246.6$ and $f(5.05) \approx -5.03$, and local maximum value $f(4.95) \approx -4.965$ (notice the second graph of f). From the graph of f'' , we estimate that f is CU on $(-\infty, 0)$, CD on $(0, 1.38)$, CU on $(1.38, 3.62)$, CD on $(3.62, 5)$, and CU on $(5, \infty)$. There are inflection points at $(0, 0)$ and $(5, -5)$, and at about $(1.38, -126.38)$ and $(3.62, -128.62)$.

$$3. f(x) = \sqrt[3]{x^2 - 3x - 5} \Rightarrow f'(x) = \frac{1}{3} \frac{2x - 3}{(x^2 - 3x - 5)^{2/3}} \Rightarrow f''(x) = -\frac{2}{9} \frac{x^2 - 3x + 24}{(x^2 - 3x - 5)^{5/3}}$$



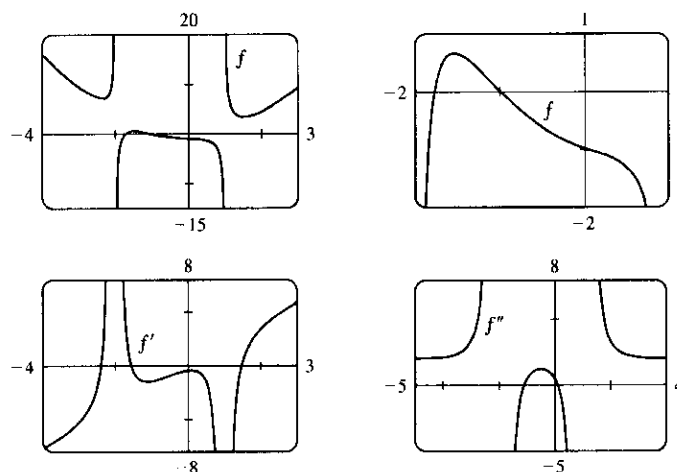
Note: With some CAS's, including Maple, it is necessary to define $f(x) = \frac{x^2 - 3x - 5}{|x^2 - 3x - 5|^{1/3}}$,

since the CAS does not compute real cube roots of negative numbers. We estimate from the graph of f' that f is

increasing on $(1.5, \infty)$, and decreasing on $(-\infty, 1.5)$. f has no maximum. Minimum value: $f(1.5) \approx -1.9$. From the graph of f'' , we estimate that f is CU on $(-1.2, 4.2)$ and CD on $(-\infty, -1.2)$ and $(4.2, \infty)$. IP at $(-1.2, 0)$ and $(4.2, 0)$.

$$4. f(x) = \frac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2} \Rightarrow f'(x) = 2 \frac{x^5 + 2x^4 - 3x^3 - 4x^2 + 2x - 1}{(x^2 + x - 2)^2} \Rightarrow$$

$$f''(x) = 2 \frac{x^6 + 3x^5 - 3x^4 - 11x^3 + 12x^2 + 18x - 2}{(x^2 + x - 2)^3}$$

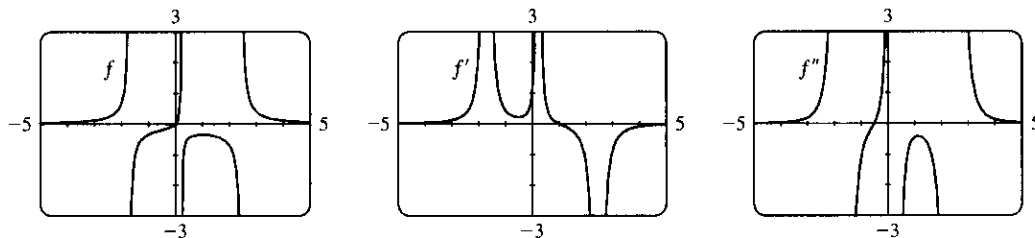


We estimate from the graph of f' that f is increasing on $(-2.4, -2)$, $(-2, -1.5)$ and $(1.5, \infty)$ and decreasing on $(-\infty, -2.4)$, $(-1.5, 1)$ and $(1, 1.5)$. Local maximum value: $f(-1.5) \approx 0.7$.

Local minimum values: $f(-2.4) \approx 7.2$, $f(1.5) \approx 3.4$. From the graph of f'' , we estimate that f is CU on $(-\infty, -2)$, $(-1.1, 0.1)$ and $(1, \infty)$ and CD on $(-2, -1.1)$ and $(0.1, 1)$. f has IP at $(-1.1, 0.2)$ and $(0.1, -1.1)$.

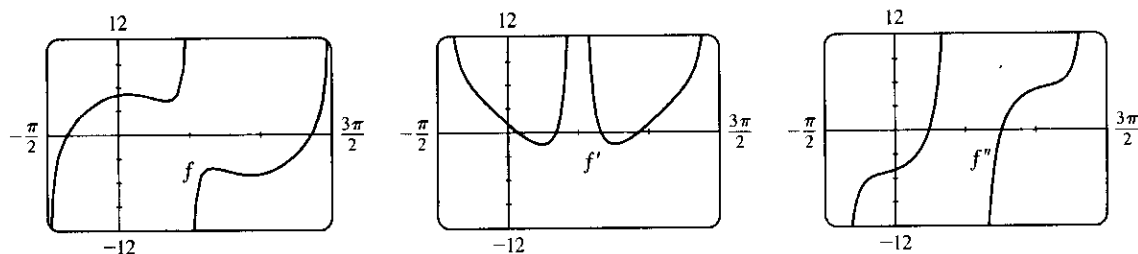
$$5. f(x) = \frac{x}{x^3 - x^2 - 4x + 1} \Rightarrow f'(x) = \frac{-2x^3 + x^2 + 1}{(x^3 - x^2 - 4x + 1)^2} \Rightarrow$$

$$f''(x) = \frac{2(3x^5 - 3x^4 + 5x^3 - 6x^2 + 3x + 4)}{(x^3 - x^2 - 4x + 1)^3}$$



We estimate from the graph of f that $y = 0$ is a horizontal asymptote, and that there are vertical asymptotes at $x = -1.7$, $x = 0.24$, and $x = 2.46$. From the graph of f' , we estimate that f is increasing on $(-\infty, -1.7)$, $(-1.7, 0.24)$, and $(0.24, 1)$, and that f is decreasing on $(1, 2.46)$ and $(2.46, \infty)$. There is a local maximum value at $f(1) = -\frac{1}{3}$. From the graph of f'' , we estimate that f is CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, and $(2.46, \infty)$, and that f is CD on $(-1.7, -0.506)$ and $(0.24, 2.46)$. There is an inflection point at $(-0.506, -0.192)$.

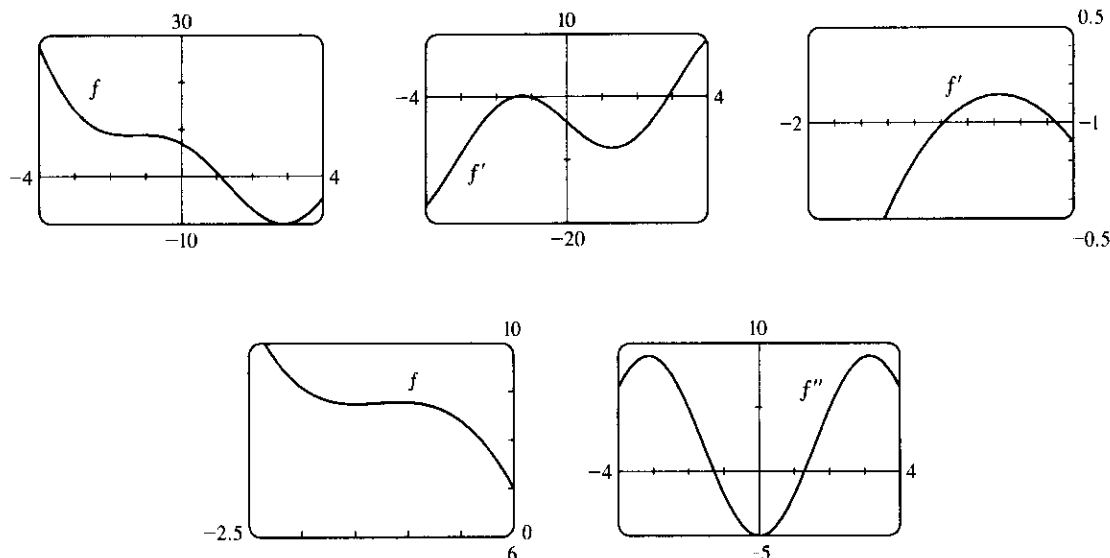
6. $f(x) = \tan x + 5 \cos x \Rightarrow f'(x) = \sec^2 x - 5 \sin x \Rightarrow f''(x) = 2 \sec^2 x \tan x - 5 \cos x$. Since f is periodic with period 2π , and defined for all x except odd multiples of $\frac{\pi}{2}$, we graph f and its derivatives on $[-\frac{\pi}{2}, \frac{3\pi}{2}]$.



We estimate from the graph of f' that f is increasing on $(-\frac{\pi}{2}, 0.21)$, $(1.07, \frac{\pi}{2})$, $(\frac{\pi}{2}, 2.07)$, and $(2.93, \frac{3\pi}{2})$, and decreasing on $(0.21, 1.07)$ and $(2.07, 2.93)$. Local minimum values: $f(1.07) \approx 4.23$, $f(2.93) \approx -5.10$. Local maximum values: $f(0.21) \approx 5.10$, $f(2.07) \approx -4.23$.

From the graph of f'' , we estimate that f is CU on $(0.76, \frac{\pi}{2})$ and $(2.38, \frac{3\pi}{2})$, and CD on $(-\frac{\pi}{2}, 0.76)$ and $(\frac{\pi}{2}, 2.38)$. f has IP at $(0.76, 4.57)$ and $(2.38, -4.57)$.

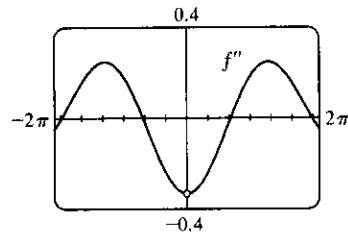
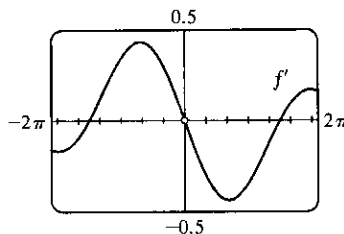
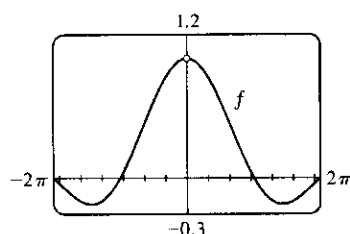
7. $f(x) = x^2 - 4x + 7 \cos x$, $-4 \leq x \leq 4$. $f'(x) = 2x - 4 - 7 \sin x \Rightarrow f''(x) = 2 - 7 \cos x$.
 $f(x) = 0 \Leftrightarrow x \approx 1.10$; $f'(x) = 0 \Leftrightarrow x \approx -1.49, -1.07$, or 2.89 ; $f''(x) = 0 \Leftrightarrow x = \pm \cos^{-1}(\frac{2}{7}) \approx \pm 1.28$.



From the graphs of f' , we estimate that f is decreasing ($f' < 0$) on $(-4, -1.49)$, increasing on $(-1.49, -1.07)$, decreasing on $(-1.07, 2.89)$, and increasing on $(2.89, 4)$, with local minimum values $f(-1.49) \approx 8.75$ and $f(2.89) \approx -9.99$ and local maximum value $f(-1.07) \approx 8.79$ (notice the second graph of f). From the graph of f'' , we estimate that f is CU ($f'' > 0$) on $(-4, -1.28)$, CD on $(-1.28, 1.28)$, and CU on $(1.28, 4)$. There are inflection points at about $(-1.28, 8.77)$ and $(1.28, -1.48)$.

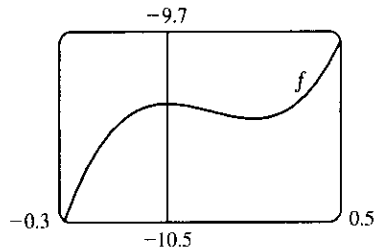
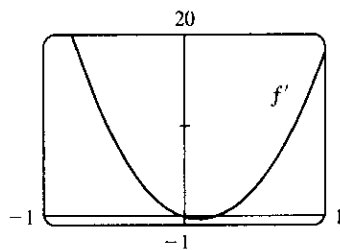
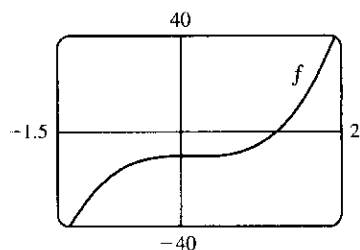
$$8. f(x) = \frac{\sin x}{x}, -2\pi \leq x \leq 2\pi. f'(x) = \frac{x \cos x - \sin x}{x^2} \Rightarrow$$

$$f''(x) = \frac{x^2(\cos x - x \sin x - \cos x) - (x \cos x - \sin x)(2x)}{(x^2)^2} = \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}$$



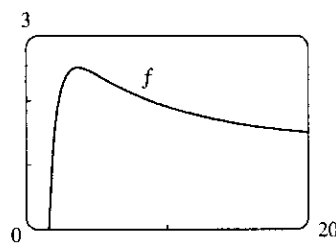
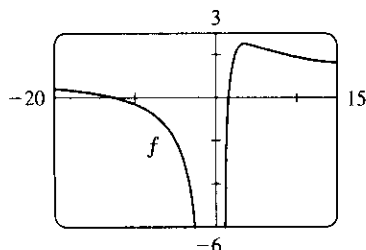
f is an even function with domain $(-\infty, 0) \cup (0, \infty)$. There is no y -intercept, but $\lim_{x \rightarrow 0} f(x) = 1$. The x -intercepts are $-2\pi, -\pi, 0, \pi$, and 2π . From the graph of f' , we estimate that f is decreasing on $(-2\pi, -4.49)$, increasing on $(-4.49, 0)$, decreasing on $(0, 4.49)$, and increasing on $(4.49, 2\pi)$. Thus, f has local minima of $f(\pm 4.49) \approx -0.22$. From the graph of f'' , we estimate that f is CD on $(-2\pi, -5.94)$, CU on $(-5.94, -2.08)$, CD on $(-2.08, 0)$ and $(0, 2.08)$, CU on $(2.08, 5.94)$, and CD on $(5.94, 2\pi)$. f has IPs at approximately $(\pm 5.94, -0.06)$ and $(\pm 2.08, 0.42)$.

$$9. f(x) = 8x^3 - 3x^2 - 10 \Rightarrow f'(x) = 24x^2 - 6x \Rightarrow f''(x) = 48x - 6$$



From the graphs, it appears that $f(x) = 8x^3 - 3x^2 - 10$ increases on $(-\infty, 0)$ and $(0.25, \infty)$ and decreases on $(0, 0.25)$; that f has a local maximum value of $f(0) = -10.0$ and a local minimum value of $f(0.25) \approx -10.1$; that f is CU on $(0.1, \infty)$ and CD on $(-\infty, 0.1)$; and that f has an IP at $(0.1, -10)$. To find the exact values, note that $f'(x) = 24x^2 - 6x = 6x(4x - 1)$, which is positive (f is increasing) for $(-\infty, 0)$ and $(\frac{1}{4}, \infty)$, and negative (f is decreasing) on $(0, \frac{1}{4})$. By the FDT, f has a local maximum at $x = 0$: $f(0) = -10$; and f has a local minimum at $\frac{1}{4}$: $f(\frac{1}{4}) = \frac{1}{8} - \frac{3}{16} - 10 = -\frac{161}{16}$. $f''(x) = 48x - 6 = 6(8x - 1)$, which is positive (f is CU) on $(\frac{1}{8}, \infty)$ and negative (f is CD) on $(-\infty, \frac{1}{8})$. f has an IP at $(\frac{1}{8}, f(\frac{1}{8})) = (\frac{1}{8}, -\frac{321}{32})$.

10.



From the graphs, it appears that f increases on $(0, 3.6)$ and decreases on $(-\infty, 0)$ and $(3.6, \infty)$; that f has a local

maximum of $f(3.6) \approx 2.5$ and no local minima; that f is CU on $(5.5, \infty)$ and CD on $(-\infty, 0)$ and $(0, 5.5)$; and

that f has an IP at $(5.5, 2.3)$. $f(x) = \frac{x^2 + 11x - 20}{x^2} = 1 + \frac{11}{x} - \frac{20}{x^2} \Rightarrow$

$f'(x) = -11x^{-2} + 40x^{-3} = -x^{-3}(11x - 40)$, which is positive (f is increasing) on $(0, \frac{40}{11})$, and negative (f is decreasing) on $(-\infty, 0)$ and on $(\frac{40}{11}, \infty)$. By the FDT, f has a local maximum at $x = \frac{40}{11}$:

$f(\frac{40}{11}) = \frac{(\frac{40}{11})^2 + 11(\frac{40}{11}) - 20}{(\frac{40}{11})^2} = \frac{1600 + 11 \cdot 11 \cdot 40 - 20 \cdot 121}{1600} = \frac{201}{80}$; and f has no local minimum.

$f'(x) = -11x^{-2} + 40x^{-3} \Rightarrow f''(x) = 22x^{-3} - 120x^{-4} = 2x^{-4}(11x - 60)$, which is positive (f is CU) on $(\frac{60}{11}, \infty)$, and negative (f is CD) on $(-\infty, 0)$ and $(0, \frac{60}{11})$. f has an IP at $(\frac{60}{11}, f(\frac{60}{11})) = (\frac{60}{11}, \frac{211}{90})$.

11. From the graph, it appears that f increases on $(-2.1, 2.1)$ and decreases on $(-3, -2.1)$ and $(2.1, 3)$; that f has a local maximum of $f(2.1) \approx 4.5$ and a local minimum of $f(-2.1) \approx -4.5$; that f is CU on $(-3, 0)$ and CD on $(0, 3)$, and that f has an IP at $(0, 0)$. $f(x) = x\sqrt{9-x^2} \Rightarrow$

$f'(x) = \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} = \frac{9-2x^2}{\sqrt{9-x^2}}$, which is positive

(f is increasing) on $(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$ and negative (f is decreasing) on $(-3, -\frac{3\sqrt{2}}{2})$ and $(\frac{3\sqrt{2}}{2}, 3)$. By the FDT,

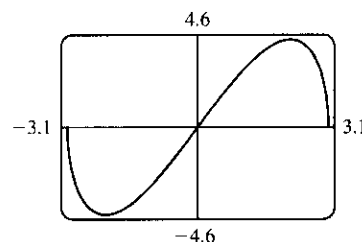
f has a local maximum value of $f(\frac{3\sqrt{2}}{2}) = \frac{3\sqrt{2}}{2} \sqrt{9 - (\frac{3\sqrt{2}}{2})^2} = \frac{9}{2}$; and f has a local minimum value of

$f(-\frac{3\sqrt{2}}{2}) = -\frac{9}{2}$ (since f is an odd function). $f'(x) = \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} \Rightarrow$

$$f''(x) = \frac{\sqrt{9-x^2}(-2x) + x^2(\frac{1}{2})(9-x^2)^{-1/2}(-2x)}{9-x^2} - x(9-x^2)^{-1/2} = \frac{-2x - x^3(9-x^2)^{-1} - x}{\sqrt{9-x^2}}$$

$$= \frac{-3x}{\sqrt{9-x^2}} - \frac{x^3}{(9-x^2)^{3/2}} = \frac{x(2x^2-27)}{(9-x^2)^{3/2}}$$

which is positive (f is CU) on $(-3, 0)$ and negative (f is CD) on $(0, 3)$. f has an IP at $(0, 0)$.

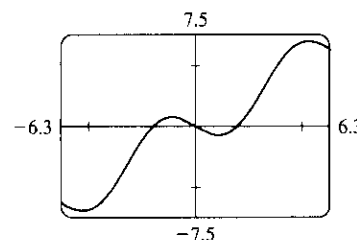


12. From the graph, it appears that f increases on $(-5.2, -1.0)$ and $(1.0, 5.2)$ and decreases on $(-2\pi, -5.2)$, $(-1.0, 1.0)$, and $(5.2, 2\pi)$; that f has local maximum values of $f(-1.0) \approx 0.7$ and $f(5.2) \approx 7.0$ and local minimum values of $f(-5.2) \approx -7.0$ and $f(1.0) \approx -0.7$; that f is CU on $(-2\pi, -3.1)$ and $(0, 3.1)$ and CD on $(-3.1, 0)$ and $(3.1, 2\pi)$, and that f has IP at $(0, 0)$, $(-3.1, -3.1)$ and $(3.1, 3.1)$. $f(x) = x - 2\sin x \Rightarrow$

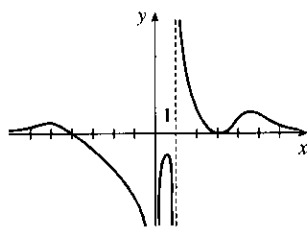
$f'(x) = 1 - 2\cos x$, which is positive (f is increasing) when $\cos x < \frac{1}{2}$, that is, on $(-\frac{5\pi}{3}, -\frac{\pi}{3})$ and $(\frac{\pi}{3}, \frac{5\pi}{3})$, and negative (f is decreasing) on $(-2\pi, -\frac{5\pi}{3})$, $(-\frac{\pi}{3}, \frac{\pi}{3})$, and $(\frac{5\pi}{3}, 2\pi)$. By the FDT, f has local maximum values

of $f(-\frac{\pi}{3}) = \frac{\pi}{3} + \sqrt{3}$ and $f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3}$, and local minimum values of $f(-\frac{5\pi}{3}) = -\frac{5\pi}{3} - \sqrt{3}$ and

$f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$. $f'(x) = 1 - 2\cos x \Rightarrow f''(x) = 2\sin x$, which is positive (f is CU) on $(-2\pi, -\pi)$ and $(0, \pi)$ and negative (f is CD) on $(-\pi, 0)$ and $(\pi, 2\pi)$. f has IP at $(0, 0)$, $(-\pi, -\pi)$ and (π, π) .



13.

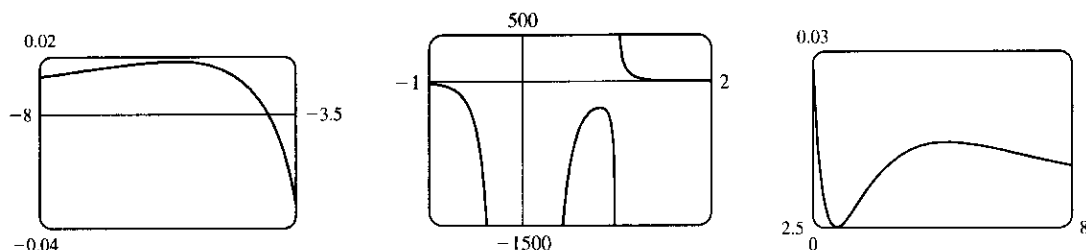


$f(x) = \frac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since

$$\lim_{x \rightarrow 0} f(x) = -\infty, \lim_{x \rightarrow 1^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow 1^+} f(x) = \infty.$$

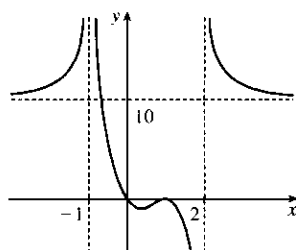
$$\begin{aligned} f(x) &= \frac{\frac{x+4}{x^4} \cdot \frac{(x-3)^2}{x^2}}{\frac{x-1}{x^3}} \quad \left[\text{dividing numerator and denominator by } x^3 \right] \\ &= \frac{(1+4/x)(1-3/x)^2}{x(x-1)} \rightarrow 0 \text{ as } x \rightarrow \pm\infty, \text{ so } f \text{ is asymptotic} \end{aligned}$$

to the x -axis. Since f is undefined at $x = 0$, it has no y -intercept. $f(x) = 0 \Rightarrow (x+4)(x-3)^2 = 0 \Rightarrow x = -4$ or $x = 3$, so f has x -intercepts -4 and 3 . Note, however, that the graph of f is only tangent to the x -axis and does not cross it at $x = 3$, since f is positive as $x \rightarrow 3^-$ and as $x \rightarrow 3^+$.



From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x -axis at $x = 3$) that the minimum value is $f(3) = 0$.

14.



$f(x) = \frac{10x(x-1)^4}{(x-2)^3(x+1)^2}$ has VA at $x = -1$ and at $x = 2$ since

$$\lim_{x \rightarrow -1} f(x) = \infty, \lim_{x \rightarrow 2^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow 2^+} f(x) = \infty.$$

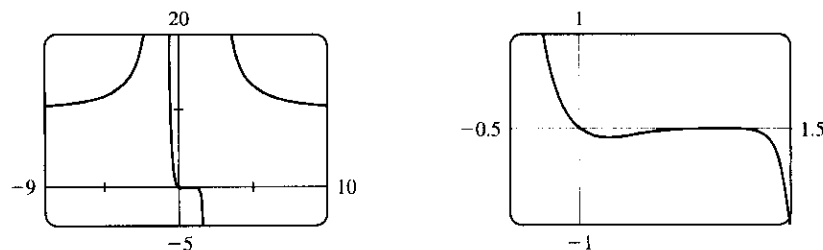
$$f(x) = \frac{10(1-1/x)^4}{(1-2/x)^3(1+1/x)^2} \rightarrow 10 \text{ as } x \rightarrow \pm\infty, \text{ so } f \text{ is asymptotic to}$$

the line $y = 10$. $f(0) = 0$, so f has a y -intercept at 0 . $f(x) = 0 \Rightarrow$

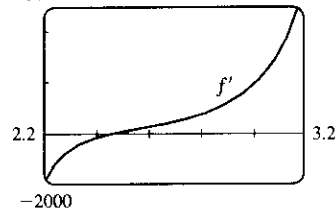
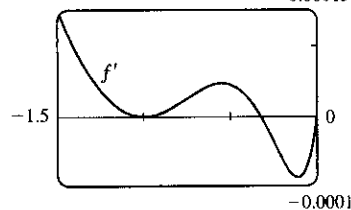
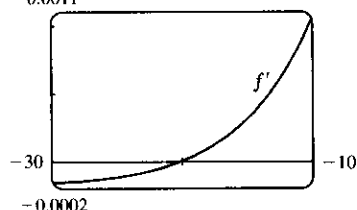
$$10x(x-1)^4 = 0 \Rightarrow x = 0 \text{ or } x = 1. \text{ So } f \text{ has } x\text{-intercepts } 0 \text{ and } 1.$$

Note, however, that f does not change sign at $x = 1$, so the graph is tangent to the x -axis and does not cross it.

We know (since the graph is tangent to the x -axis at $x = 1$) that the maximum value is $f(1) = 0$. From the graphs it appears that the minimum value is about $f(0.2) = -0.1$.

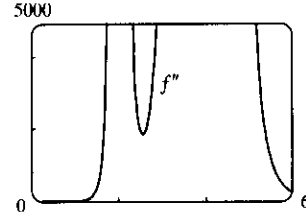
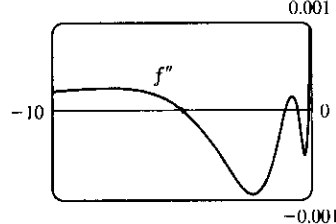
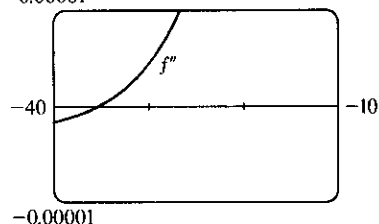


15. $f(x) = \frac{x^2(x+1)^3}{(x-2)^3(x+4)^4} \Rightarrow f'(x) = -\frac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x+4)^5}$ (from CAS).



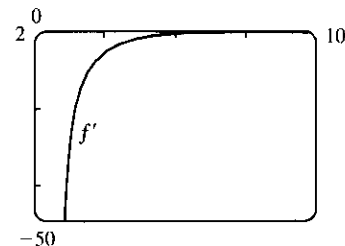
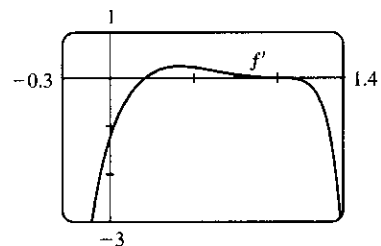
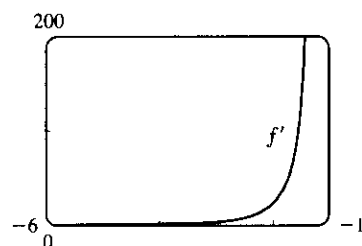
From the graphs of f' , it seems that the critical points which indicate extrema occur at $x \approx -20, -0.3$, and 2.5 , as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.)

We differentiate again, obtaining $f''(x) = 2 \frac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x+4)^6}$.

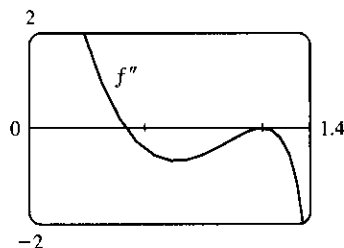
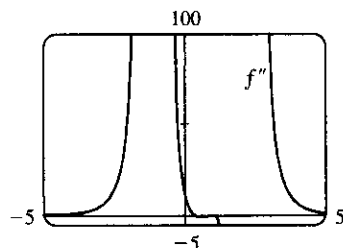


From the graphs of f'' , it appears that f is CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$ and $(4, \infty)$ and CD on $(-\infty, -35.3)$, $(-5.0, -1)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y -coordinates of the inflection points, and find that these points are approximately $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.000066)$.

16. $f(x) = \frac{10x(x-1)^4}{(x-2)^3(x+1)^2} \Rightarrow f'(x) = -20 \frac{(x-1)^3(5x-1)}{(x-2)^4(x+1)^3}$ (from CAS).



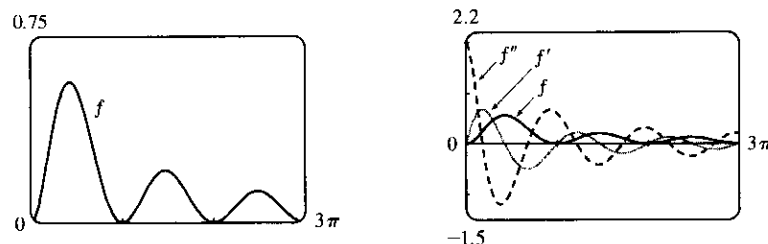
From the graphs of f' , we estimate that f is increasing on $(-\infty, -1)$ and $(0.2, 1)$ and decreasing on $(-1, 0.2)$, $(1, 2)$ and $(2, \infty)$. Differentiating $f'(x)$, we get $f''(x) = 60 \frac{(x-1)^2(5x^3-8x^2+17x-6)}{(x-2)^5(x+1)^4}$.



From the graphs of f'' , it seems that f is CU on $(-\infty, -1.0)$, $(-1.0, 0.4)$ and $(2.0, \infty)$, and CD on $(0.4, 2)$. There is an inflection point at about $(0.4, -0.06)$.

17. $y = f(x) = \frac{\sin^2 x}{\sqrt{x^2 + 1}}$ with $0 \leq x \leq 3\pi$. From a CAS, $y' = \frac{\sin x [2(x^2 + 1) \cos x - x \sin x]}{(x^2 + 1)^{3/2}}$ and

$$y'' = \frac{(4x^4 + 6x^2 + 5)\cos^2 x - 4x(x^2 + 1)\sin x \cos x - 2x^4 - 2x^2 - 3}{(x^2 + 1)^{5/2}}.$$

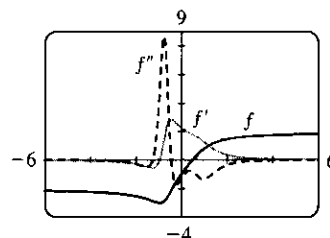


From the graph of f' and the formula for y' , we determine that $y' = 0$ when $x = \pi, 2\pi, 3\pi$, or $x \approx 1.3, 4.6$, or 7.8 . So f is increasing on $(0, 1.3)$, $(\pi, 4.6)$, and $(2\pi, 7.8)$. f is decreasing on $(1.3, \pi)$, $(4.6, 2\pi)$, and $(7.8, 3\pi)$. Local maximum values: $f(1.3) \approx 0.6$, $f(4.6) \approx 0.21$, and $f(7.8) \approx 0.13$. Local minimum values: $f(\pi) = f(2\pi) = 0$. From the graph of f'' , we see that $y'' = 0 \Leftrightarrow x \approx 0.6, 2.1, 3.8, 5.4, 7.0$, or 8.6 . So f is CU on $(0, 0.6)$, $(2.1, 3.8)$, $(5.4, 7.0)$, and $(8.6, 3\pi)$. f is CD on $(0.6, 2.1)$, $(3.8, 5.4)$, and $(7.0, 8.6)$. There are IP at $(0.6, 0.25)$, $(2.1, 0.31)$, $(3.8, 0.10)$, $(5.4, 0.11)$, $(7.0, 0.061)$, and $(8.6, 0.065)$.

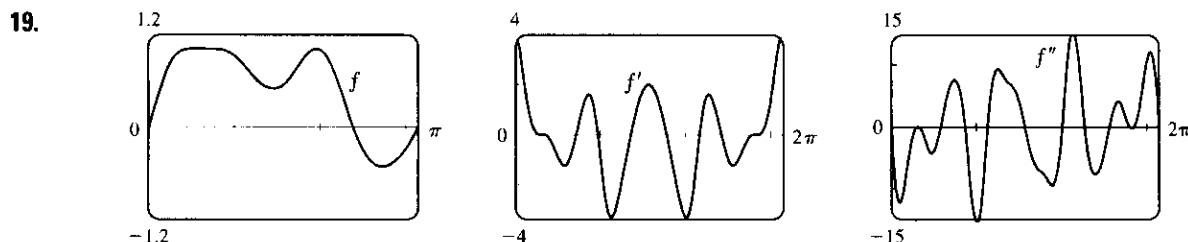
18. $f(x) = \frac{2x - 1}{\sqrt[3]{x^4 + x + 1}} \Rightarrow$

$$f'(x) = \frac{4x^3 + 6x + 9}{4(x^4 + x + 1)^{5/4}} \Rightarrow$$

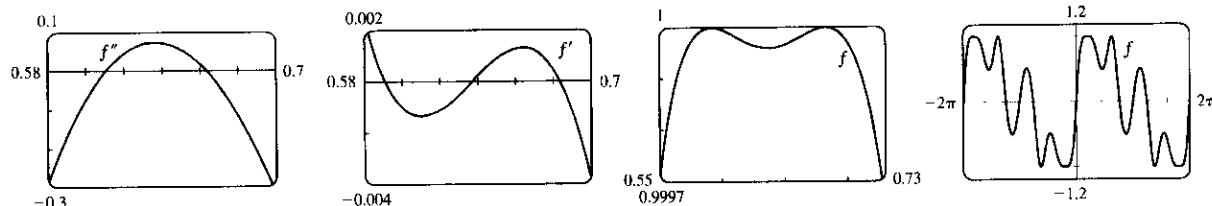
$$f''(x) = -\frac{32x^6 + 96x^4 + 152x^3 - 48x^2 + 6x + 21}{16(x^4 + x + 1)^{9/4}}$$



From the graph of f' , f appears to be decreasing on $(-\infty, -0.94)$ and increasing on $(-0.94, \infty)$. There is a local minimum value of $f(-0.94) \approx -3.01$. From the graph of f'' , f appears to be CU on $(-1.25, -0.44)$ and CD on $(-\infty, -1.25)$ and $(-0.44, \infty)$. There are inflection points at $(-1.25, -2.87)$ and $(-0.44, -2.14)$.

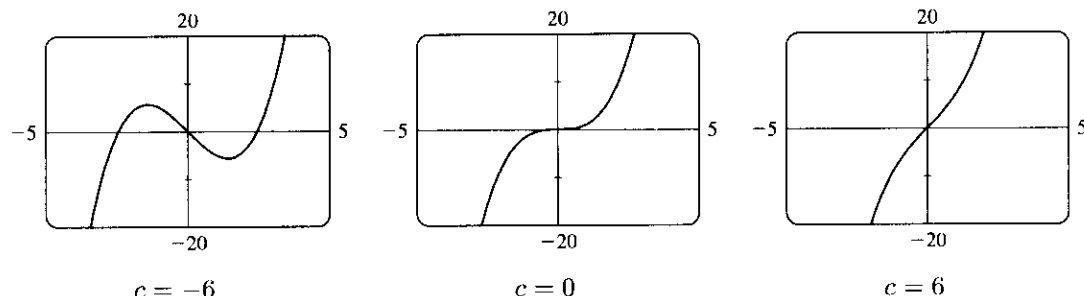


From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)](1 + 3 \cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x -axis at about $x = 0.7$. The graph of $f'' = -[\sin(x + \sin 3x)](1 + 3 \cos 3x)^2 + \cos(x + \sin 3x)(-9 \sin 3x)$ is even more interesting near this x -value: it seems to just touch the x -axis.



If we zoom in on this place on the graph of f'' , we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum value is roughly $f(0.64) = 0.99996$. There are also a maximum value of about $f(1.96) = 1$ and minimum values of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π , and it is also rotationally symmetric about all points of the form $((2n + 1)\pi, 0)$, n an integer.

20. $f(x) = x^3 + cx = x(x^2 + c) \Rightarrow f'(x) = 3x^2 + c \Rightarrow f''(x) = 6x$



x -intercepts: When $c \geq 0$, 0 is the only x -intercept. When $c < 0$, the x -intercepts are 0 and $\pm\sqrt{-c}$.

y -intercept = $f(0) = 0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. The origin is the only inflection point.

If $c > 0$, then $f'(x) > 0$ for all x , so f is increasing and has no local maximum or minimum.

If $c = 0$, then $f'(x) \geq 0$ with equality at $x = 0$, so again f is increasing and has no local maximum or minimum.

If $c < 0$, then $f'(x) = 3[x^2 - (-c/3)] = 3(x + \sqrt{-c/3})(x - \sqrt{-c/3})$,

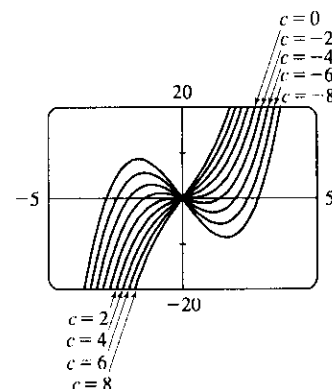
so $f'(x) > 0$ on $(-\infty, -\sqrt{-c/3})$ and $(\sqrt{-c/3}, \infty)$; $f'(x) < 0$ on

$(-\sqrt{-c/3}, \sqrt{-c/3})$. It follows that $f(-\sqrt{-c/3}) = -\frac{2}{3}c\sqrt{-c/3}$ is

a local maximum value and $f(\sqrt{-c/3}) = \frac{2}{3}c\sqrt{-c/3}$ is a local

minimum value. As c decreases (toward more negative values), the local maximum and minimum move further apart.

There is no absolute maximum or minimum value. The only transitional value of c corresponding to a change in character of the graph is $c = 0$.



21. $f(x) = x^4 + cx^2 = x^2(x^2 + c)$. Note that f is an even function. For $c \geq 0$, the only x -intercept is the point $(0, 0)$.

We calculate $f'(x) = 4x^3 + 2cx = 4x(x^2 + \frac{1}{2}c) \Rightarrow f''(x) = 12x^2 + 2c$. If $c \geq 0$, $x = 0$ is the only critical point and there is no inflection point. As we can see from the examples, there is no change in the basic shape of the graph for $c \geq 0$; it merely becomes steeper as c increases. For $c = 0$, the graph is the simple curve

$y = x^4$. For $c < 0$, there are x -intercepts at 0 and at $\pm\sqrt{-c}$. Also,

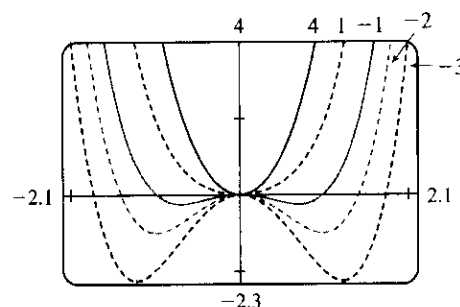
there is a maximum at $(0, 0)$, and there are minima at

$(\pm\sqrt{-\frac{1}{2}c}, -\frac{1}{4}c^2)$. As $c \rightarrow -\infty$, the x -coordinates of these minima

get larger in absolute value, and the minimum points move

downward. There are inflection points at $(\pm\sqrt{-\frac{1}{6}c}, -\frac{5}{36}c^2)$, which

also move away from the origin as $c \rightarrow -\infty$.



22. We need only consider the function $f(x) = x^2\sqrt{c^2 - x^2}$ for $c \geq 0$, because if c is replaced by $-c$, the function is unchanged. For $c = 0$, the graph consists of the single point $(0, 0)$. The domain of f is $[-c, c]$, and the graph of f is symmetric about the y -axis.

$$f'(x) = 2x\sqrt{c^2 - x^2} + x^2 \frac{-2x}{2\sqrt{c^2 - x^2}} = 2x\sqrt{c^2 - x^2} - \frac{x^3}{\sqrt{c^2 - x^2}} = \frac{2x(c^2 - x^2) - x^3}{\sqrt{c^2 - x^2}} = -\frac{3x(x^2 - \frac{2}{3}c^2)}{\sqrt{c^2 - x^2}}.$$

So we see that all members of the family of curves have horizontal tangents at $x = 0$, since $f'(0) = 0$ for all $c > 0$.

Also, the tangents to all the curves become very steep as $x \rightarrow \pm c$, since $\lim_{x \rightarrow -c^+} f'(x) = \infty$ and

$\lim_{x \rightarrow c^-} f'(x) = -\infty$. We set $f'(x) = 0 \Leftrightarrow x = 0$ or $x^2 - \frac{2}{3}c^2 = 0$, so the absolute maximum values

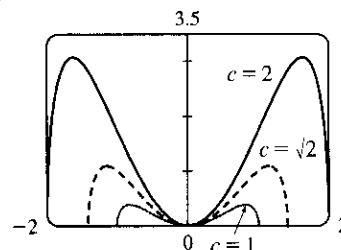
are $f(\pm\sqrt{\frac{2}{3}c}) = \frac{2}{3\sqrt{3}}c^3$.

$$f''(x) = \frac{(-9x^2 + 2c^2)\sqrt{c^2 - x^2} - (-3x^3 + 2c^2x)(-x/\sqrt{c^2 - x^2})}{c^2 - x^2} = \frac{6x^4 - 9c^2x^2 + 2c^4}{(c^2 - x^2)^{3/2}}.$$

Using the quadratic formula, we find that $f''(x) = 0 \Leftrightarrow x^2 = \frac{9c^2 \pm c^2\sqrt{33}}{12}$. Since $-c < x < c$, we take

$x^2 = \frac{9 - \sqrt{33}}{12}c^2$, so the inflection points are $(\pm\sqrt{\frac{9 - \sqrt{33}}{12}}c, \frac{(9 - \sqrt{33})(\sqrt{33} - 3)}{144}c^3)$.

From these calculations we can see that the maxima and the points of inflection get both horizontally and vertically further from the origin as c increases. Since all of the functions have two maxima and two inflection points, we see that the basic shape of the curve does not change as c changes.



23. Note that $c = 0$ is a transitional value at which the graph consists of the x -axis. Also, we can see that if we

substitute $-c$ for c , the function $f(x) = \frac{cx}{1 + c^2x^2}$ will be reflected in the x -axis, so we investigate only positive

values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim_{x \rightarrow \pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c . We calculate

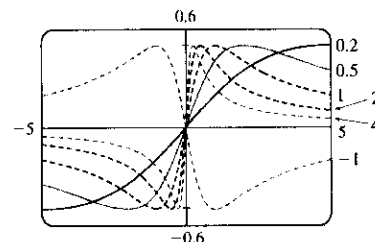
$$f'(x) = \frac{(1 + c^2 x^2)c - cx(2c^2 x)}{(1 + c^2 x^2)^2} = -\frac{c(c^2 x^2 - 1)}{(1 + c^2 x^2)^2}. \quad f'(x) = 0 \Leftrightarrow c^2 x^2 - 1 = 0 \Leftrightarrow x = \pm 1/c. \text{ So there}$$

is an absolute maximum value of $f(1/c) = \frac{1}{2}$ and an absolute minimum value of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c , but the maximum points move closer to the y -axis as c increases.

$$\begin{aligned} f''(x) &= \frac{(-2c^3 x)(1 + c^2 x^2)^2 - (-c^3 x^2 + c)[2(1 + c^2 x^2)(2c^2 x)]}{(1 + c^2 x^2)^4} \\ &= \frac{(-2c^3 x)(1 + c^2 x^2) + (c^3 x^2 - c)(4c^2 x)}{(1 + c^2 x^2)^3} = \frac{2c^3 x(c^2 x^2 - 3)}{(1 + c^2 x^2)^3} \end{aligned}$$

$f''(x) = 0 \Leftrightarrow x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $(\pm\sqrt{3}/c, \pm\sqrt{3}/4)$.

Again, the y -coordinate of the inflection points does not depend on c , but as c increases, both inflection points approach the y -axis.



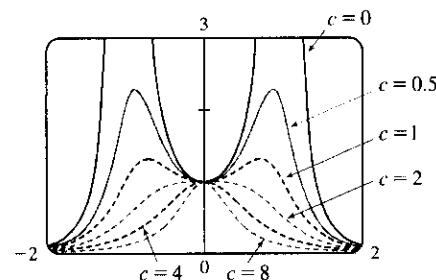
24. Note that $f(x) = \frac{1}{(1-x^2)^2 + cx^2}$ is an even function, and also that $\lim_{x \rightarrow \pm\infty} f(x) = 0$ for any value of c , so $y = 0$

is a horizontal asymptote. We calculate the derivatives:

$$\begin{aligned} f'(x) &= \frac{-4(1-x^2)x + 2cx}{[(1-x^2)^2 + cx^2]^2} = \frac{4x[x^2 + (\frac{1}{2}c - 1)]}{[(1-x^2)^2 + cx^2]^2}, \text{ and} \\ f''(x) &= 2 \frac{10x^6 + (9c-18)x^4 + (3c^2-12c+6)x^2 + 2-c}{[x^4 + (c-2)x^2 + 1]^3}. \end{aligned}$$

We first consider the case $c > 0$. Then the denominator of f' is

positive, that is, $(1-x^2)^2 + cx^2 > 0$ for all x , so f has domain $\mathbb{R}$ and also $f > 0$. If $\frac{1}{2}c - 1 \geq 0$; that is, $c \geq 2$, then the only critical point is $f(0) = 1$, a maximum. Graphing a few examples for $c \geq 2$ shows that there are two IP which approach the y -axis as $c \rightarrow \infty$.

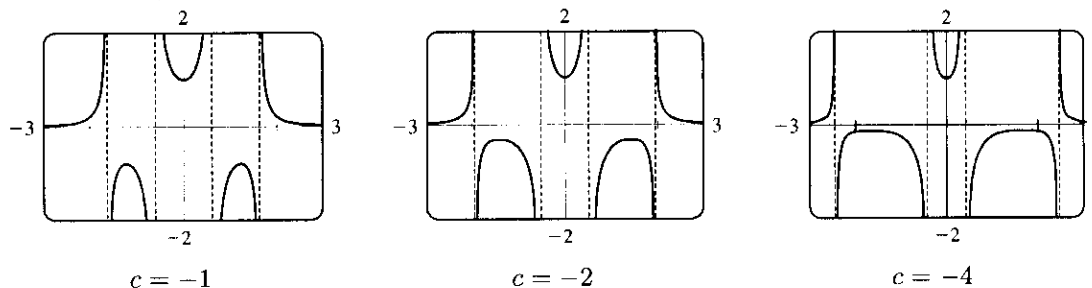


$c = 2$ and $c = 0$ are transitional values of c at which the shape of the curve changes. For $0 < c < 2$, there are three critical points: $f(0) = 1$, a minimum value, and $f(\pm\sqrt{1 - \frac{1}{2}c}) = \frac{1}{c(1 - c/4)}$, both maximum values. As c decreases from 2 to 0, the maximum values get larger and larger, and the x -values at which they occur go from 0 to ± 1 . Graphs show that there are four inflection points for $0 < c < 2$, and that they get farther away from the origin, both vertically and horizontally, as $c \rightarrow 0^+$. For $c = 0$, the function is simply asymptotic to the x -axis and to the lines $x = \pm 1$, approaching $+\infty$ from both sides of each. The y -intercept is 1, and $(0, 1)$ is a local minimum. There are no inflection points. Now if $c < 0$, we can write

$f(x) = \frac{1}{(1-x^2)^2 + cx^2} = \frac{1}{(1-x^2)^2 - (\sqrt{-c}x)^2} = \frac{1}{(x^2 - \sqrt{-c}x - 1)(x^2 + \sqrt{-c}x - 1)}$. So f has vertical asymptotes where $x^2 \pm \sqrt{-c}x - 1 = 0 \Leftrightarrow x = (-\sqrt{-c} \pm \sqrt{4-c})/2$ or $x = (\sqrt{-c} \pm \sqrt{4-c})/2$. As c

decreases, the two exterior asymptotes move away from the origin, while the two interior ones move toward it. We graph a few examples to see the behavior of the graph near the asymptotes, and the nature of the critical points

$$x = 0 \text{ and } x = \pm\sqrt{1 - \frac{1}{2}c}:$$



We see that there is one local minimum value, $f(0) = 1$, and there are two local maximum values,

$f\left(\pm\sqrt{1 - \frac{1}{2}c}\right) = \frac{1}{c(1 - c/4)}$ as before. As c decreases, the x -values at which these maxima occur get larger, and the maximum values themselves approach 0, though they are always negative.

25. $f(x) = cx + \sin x \Rightarrow f'(x) = c + \cos x \Rightarrow f''(x) = -\sin x$

$f(-x) = -f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

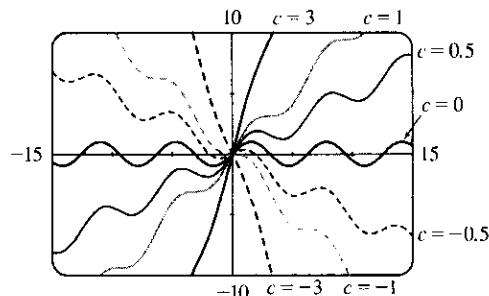
$$f(x) = 0 \Leftrightarrow \sin x = -cx, \text{ so } 0 \text{ is always an } x\text{-intercept.}$$

$f'(x) = 0 \Leftrightarrow \cos x = -c$, so there is no critical number when $|c| > 1$. If $|c| \leq 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x = -c$ in the interval $[0, \pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c = 1$, $x_1 = 0$; when $c = 0$, $x = \frac{\pi}{2}$; and when $c = -1$, $x_1 = \pi$.)

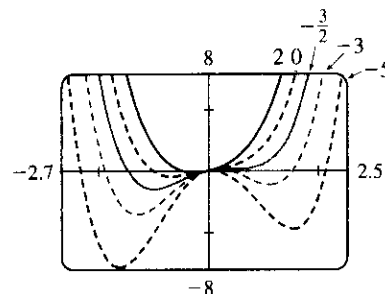
$f''(x) < 0 \Leftrightarrow \sin x > 0$, so f is CD on intervals of the form $(2n\pi, (2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi, 2n\pi)$. The inflection points of f are the points $(2n\pi, 2n\pi c)$, where n is an integer.

If $c \geq 1$, then $f'(x) \geq 0$ for all x , so f is increasing and has no extremum. If $c \leq -1$, then $f'(x) \leq 0$ for all x , so f is decreasing and has no extremum. If $|c| < 1$, then $f'(x) > 0 \Leftrightarrow \cos x > -c \Leftrightarrow x$ is in an interval of the form $(2n\pi - x_1, 2n\pi + x_1)$ for some integer n . These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi + x_1, 2(n+1)\pi - x_1)$. Thus, f has local maxima at the points $2n\pi + x_1$, where f has the values $c(2n\pi + x_1) + \sin x_1 = c(2n\pi + x_1) + \sqrt{1 - c^2}$, and f has local minima at the points $2n\pi - x_1$, where we have $f(2n\pi - x_1) = c(2n\pi - x_1) - \sin x_1 = c(2n\pi - x_1) - \sqrt{1 - c^2}$.

The transitional values of c are -1 and 1 . The inflection points move vertically, but not horizontally, when c changes. When $|c| \geq 1$, there is no extremum. For $|c| < 1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c = 0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.

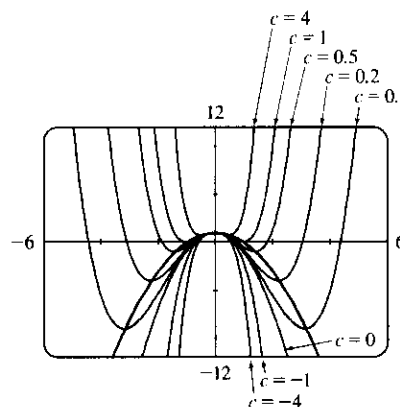


26. For $c = 0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x = 0$, so that there are two inflection points for any $c < 0$. This can be seen algebraically by calculating the second derivative: $f(x) = x^4 + cx^2 + x \Rightarrow f'(x) = 4x^3 + 2cx + 1 \Rightarrow f''(x) = 12x^2 + 2c$. Thus, $f''(x) > 0$ when $c > 0$. For $c < 0$, there are inflection points when $x = \pm\sqrt{-\frac{1}{6}c}$. For $c = 0$, the graph has one critical number, at the absolute minimum somewhere around $x = -0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0, we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x = 1$ and $x = 2$. Consequently, there is also a maximum near $x = 0$. After a bit of experimentation, we find that at $c = -1.5$, there appear to be two critical numbers: the absolute minimum at about $x = -1$, and a horizontal tangent with no extremum at about $x = 0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c = -3$ and with $c = -5$. To prove this algebraically, we calculate $f'(x) = 4x^3 + 2cx + 1$. Now if we substitute our value of $c = -1.5$, the formula for $f'(x)$ becomes $4x^3 - 3x + 1 = (x + 1)(2x - 1)^2$. This has a double root at $x = \frac{1}{2}$, indicating that the function has two critical points: $x = -1$ and $x = \frac{1}{2}$, just as we had guessed from the graph.



27. (a) $f(x) = cx^4 - 2x^2 + 1$. For $c = 0$, $f(x) = -2x^2 + 1$, a parabola whose vertex, $(0, 1)$, is the absolute maximum. For $c > 0$, $f(x) = cx^4 - 2x^2 + 1$ opens upward with two minimum points. As $c \rightarrow 0$, the minimum points spread apart and move downward; they are below the x -axis for $0 < c < 1$ and above for $c > 1$. For $c < 0$, the graph opens downward, and has an absolute maximum at $x = 0$ and no local minimum.

- (b) $f'(x) = 4cx^3 - 4x = 4cx(x^2 - 1/c)$ ($c \neq 0$). If $c \leq 0$, 0 is the only critical number. $f''(x) = 12cx^2 - 4$, so $f''(0) = -4$ and there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. If $c > 0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. $f''(\pm 1/\sqrt{c}) = 12 - 4 = 8 > 0$, so there is a local minimum at $x = \pm 1/\sqrt{c}$. Here $f(\pm 1/\sqrt{c}) = c(1/c^2) - 2/c + 1 = -1/c + 1$. But $(\pm 1/\sqrt{c}, -1/c + 1)$ lies on $y = 1 - x^2$ since $1 - (\pm 1/\sqrt{c})^2 = 1 - 1/c$.



28. (a) $f(x) = 2x^3 + cx^2 + 2x \Rightarrow f'(x) = 6x^2 + 2cx + 2 = 2(3x^2 + cx + 1)$. $f'(x) = 0 \Leftrightarrow x = \frac{-c \pm \sqrt{c^2 - 12}}{6}$. So f has critical points $\Leftrightarrow c^2 - 12 \geq 0 \Leftrightarrow |c| \geq 2\sqrt{3}$. For $c = \pm 2\sqrt{3}$, $f'(x) \geq 0$ on $(-\infty, \infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2 - 12 > 0$, then

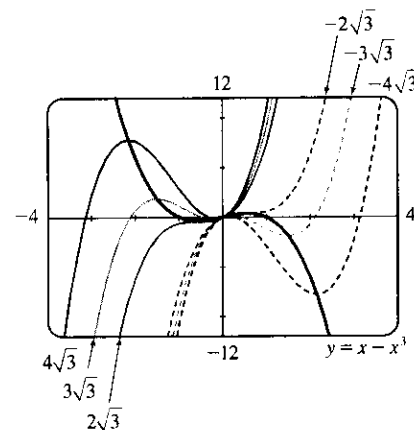
f' changes from positive to negative at $x = \frac{-c - \sqrt{c^2 - 12}}{6}$ and from negative to positive at $x = \frac{-c + \sqrt{c^2 - 12}}{6}$. So f has a local maximum at $x = \frac{-c - \sqrt{c^2 - 12}}{6}$ and a local minimum at $x = \frac{-c + \sqrt{c^2 - 12}}{6}$.

(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0) = 0 \Rightarrow$

$$3x_0^2 + cx_0 + 1 = 0 \Leftrightarrow c = \frac{-1 - 3x_0^2}{x_0}. \text{ Now}$$

$$\begin{aligned} f(x_0) &= 2x_0^3 + cx_0^2 + 2x_0 = 2x_0^3 + x_0^2 \left(\frac{-1 - 3x_0^2}{x_0} \right) + 2x_0 \\ &= 2x_0^3 - x_0 - 3x_0^3 + 2x_0 = x_0 - x_0^3 \end{aligned}$$

So the point is $(x_0, y_0) = (x_0, x_0 - x_0^3)$; that is, the point lies on the curve $y = x - x^3$.



4.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y . Then $x + y = 23$, so $y = 23 - x$. Call the product P . Then

$P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since

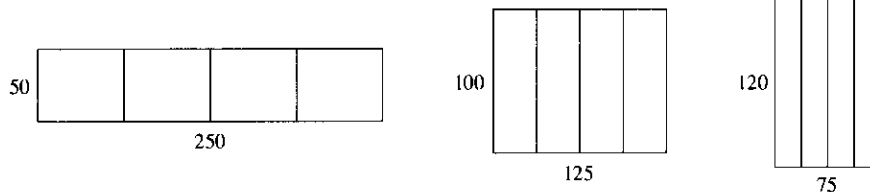
$P'(x) = 23 - 2x$, we see that $P'(x) = 0 \Leftrightarrow x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is

$P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

Or: Note that $P''(x) = -2 < 0$ for all x , so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

2. The two numbers are $x + 100$ and x . Minimize $f(x) = (x + 100)x = x^2 + 100x$. $f'(x) = 2x + 100 = 0 \Rightarrow x = -50$. Since $f''(x) = 2 > 0$, there is an absolute minimum at $x = -50$. The two numbers are 50 and -50 .
3. The two numbers are x and $\frac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \frac{100}{x}$. $f'(x) = 1 - \frac{100}{x^2} = \frac{x^2 - 100}{x^2}$. The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.
4. Let $x > 0$ and let $f(x) = x + 1/x$. We wish to minimize $f(x)$. Now $f'(x) = 1 - \frac{1}{x^2} = \frac{1}{x^2}(x^2 - 1) = \frac{1}{x^2}(x + 1)(x - 1)$, so the only critical number in $(0, \infty)$ is 1. $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, so f has an absolute minimum at $x = 1$, and $f(1) = 2$. Or: $f''(x) = 2/x^3 > 0$ for all $x > 0$, so f is concave upward everywhere and the critical point $(1, 2)$ must correspond to a local minimum for f .
5. If the rectangle has dimensions x and y , then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is $A = xy = x(50 - x)$. We wish to maximize the function $A(x) = x(50 - x) = 50x - x^2$, where $0 < x < 50$. Since $A'(x) = 50 - 2x = -2(x - 25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute maximum at $x = 25$, and $A(25) = 25^2 = 625$ m². The dimensions of the rectangle that maximize its area are $x = y = 25$ m. (The rectangle is a square.)
6. If the rectangle has dimensions x and y , then its area is $xy = 1000$ m², so $y = 1000/x$. The perimeter $P = 2x + 2y = 2x + 2000/x$. We wish to minimize the function $P(x) = 2x + 2000/x$ for $x > 0$. $P'(x) = 2 - 2000/x^2 = (2/x^2)(x^2 - 1000)$, so the only critical number in the domain of P is $x = \sqrt{1000}$. $P''(x) = 4000/x^3 > 0$, so P is concave upward throughout its domain and $P(\sqrt{1000}) = 4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x = y = \sqrt{1000} = 10\sqrt{10}$ m. (The rectangle is a square.)

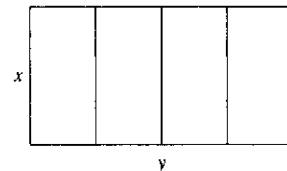
7. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft². There appears to be a maximum area of at least 12,500 ft².

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



(c) Area $A = \text{length} \times \text{width} = y \cdot x$

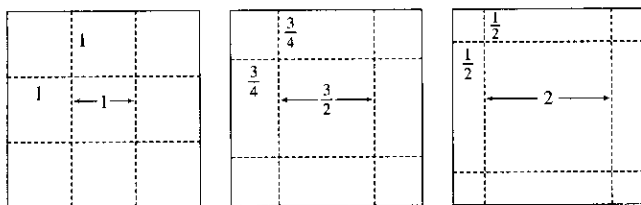
(d) Length of fencing = 750 $\Rightarrow 5x + 2y = 750$

(e) $5x + 2y = 750 \Rightarrow y = 375 - \frac{5}{2}x \Rightarrow A(x) = (375 - \frac{5}{2}x)x = 375x - \frac{5}{2}x^2$

(f) $A'(x) = 375 - 5x = 0 \Rightarrow x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$.

Then $y = \frac{375}{2} = 187.5$. The largest area is $75(\frac{375}{2}) = 14,062.5$ ft². These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

8. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3 . There appears to be a maximum volume of at least 2 ft^3 .

(c) Volume $V = \text{length} \times \text{width} \times \text{height} \Rightarrow V = y \cdot y \cdot x = xy^2$

(d) Length of cardboard = 3 $\Rightarrow x + y + x = 3 \Rightarrow y + 2x = 3$

(e) $y + 2x = 3 \Rightarrow y = 3 - 2x \Rightarrow V(x) = x(3 - 2x)^2$

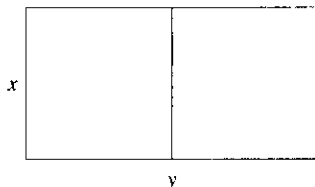
(f) $V(x) = x(3 - 2x)^2 \Rightarrow$

$$V'(x) = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 \cdot 1 = (3 - 2x)[-4x + (3 - 2x)] = (3 - 2x)(-6x + 3),$$

so the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$. Now $0 \leq x \leq \frac{3}{2}$ and $V(0) = V(\frac{3}{2}) = 0$, so the maximum is

$$V(\frac{1}{2}) = (\frac{1}{2})(2)^2 = 2 \text{ ft}^3, \text{ which is the value found from our third figure in part (a).}$$

9.



$xy = 1.5 \times 10^6$, so $y = 1.5 \times 10^6/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2(1.5 \times 10^6/x) = 3x + 3 \times 10^6/x = F(x)$.

$F'(x) = 3 - 3 \times 10^6/x^2 = 3(x^2 - 10^6)/x^2$. The critical number is $x = 10^3$ and $F'(x) < 0$ for $0 < x < 10^3$ and $F'(x) > 0$ if $x > 10^3$, so the absolute minimum occurs when $x = 10^3$ and $y = 1.5 \times 10^3$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

10. Let b be the length of the base of the box and h the height. The volume is $32,000 = b^2h \Rightarrow h = 32,000/b^2$.

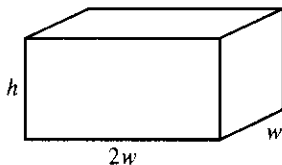
The surface area of the open box is $S = b^2 + 4hb = b^2 + 4(32,000/b^2)b = b^2 + 4(32,000)/b$. So

$S'(b) = 2b - 4(32,000)/b^2 = 2(b^3 - 64,000)/b^2 = 0 \Leftrightarrow b = \sqrt[3]{64,000} = 40$. This gives an absolute minimum since $S'(b) < 0$ if $0 < b < 40$ and $S'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

11. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \Rightarrow h = (1200 - b^2)/(4b)$. The volume is $V = b^2h = b^2(1200 - b^2)/(4b) = 300b - b^3/4 \Rightarrow V'(b) = 300 - \frac{3}{4}b^2$.

$V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 280). If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2(10) = 4000 \text{ cm}^3$.

12.



$V = lwh \Rightarrow 10 = (2w)(w)h = 2w^2h$, so $h = 5/w^2$. The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh$, so

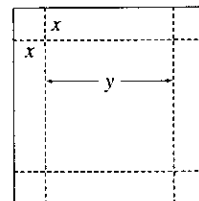
$$C(w) = 20w^2 + 36w(5/w^2) = 20w^2 + 180/w.$$

$$C'(w) = 40w - 180/w^2 = 40(w^3 - \frac{9}{2})/w^2 \Rightarrow w = \sqrt[3]{\frac{9}{2}} \text{ is the}$$

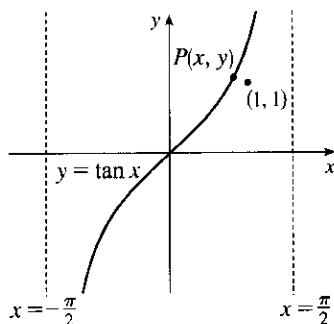
critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and

$$C'(w) > 0 \text{ for } w > \sqrt[3]{\frac{9}{2}}. \quad C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx \$163.54.$$

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.

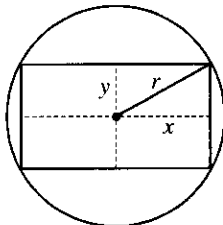


18.



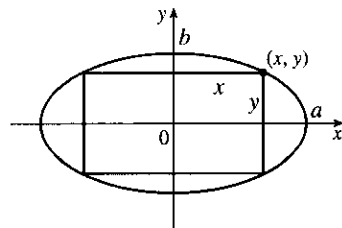
The distance d from $(1, 1)$ to an arbitrary point $P(x, y)$ on the curve $y = \tan x$ is $d = \sqrt{(x-1)^2 + (y-1)^2}$ and the square of the distance is $S = d^2 = (x-1)^2 + (\tan x - 1)^2$.
 $S' = 2(x-1) + 2(\tan x - 1) \sec^2 x$. Graphing S' on $(-\frac{\pi}{2}, \frac{\pi}{2})$ gives us a zero at $x \approx 0.82$, and so $\tan x \approx 1.08$. The point on $y = \tan x$ that is closest to $(1, 1)$ is approximately $(0.82, 1.08)$.

19.



The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now $A'(x) = 4\left(\sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}$. The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.
 $y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}r$ and $2y = \sqrt{2}r$.

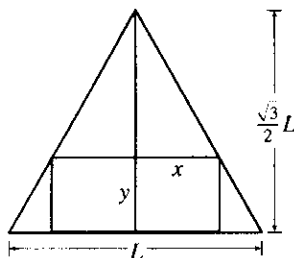
20.



The area of the rectangle is $(2x)(2y) = 4xy$. Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives $y = \frac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\frac{b}{a}x\sqrt{a^2 - x^2}$.
 $A'(x) = \frac{4b}{a}\left[x \cdot \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2} \cdot 1\right]$
 $= \frac{4b}{a}(a^2 - x^2)^{-1/2}[-x^2 + a^2 - x^2] = \frac{4b}{a\sqrt{a^2 - x^2}}[a^2 - 2x^2]$

So the critical number is $x = \frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right) = 2ab$.

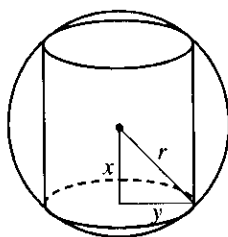
21.



The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2 + (L/2)^2 = L^2 \Rightarrow h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2 \Rightarrow h = \frac{\sqrt{3}}{2}L$. Using similar triangles, $\frac{\frac{\sqrt{3}}{2}L - y}{x} = \frac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3} \Rightarrow \sqrt{3}x = \frac{\sqrt{3}}{2}L - y \Rightarrow y = \frac{\sqrt{3}}{2}L - \sqrt{3}x \Rightarrow y = \frac{\sqrt{3}}{2}(L - 2x)$.

The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}x(L - 2x) = \sqrt{3}Lx - 2\sqrt{3}x^2$, where $0 \leq x \leq L/2$. Now $0 = A'(x) = \sqrt{3}L - 4\sqrt{3}x \Rightarrow x = \sqrt{3}L/(4\sqrt{3}) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

27.



The cylinder has surface area

 $2(\text{area of the base}) + (\text{lateral surface area})$

$$= 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height}) = 2\pi y^2 + 2\pi y(2x).$$

Now $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$, so the surface area is

$$\begin{aligned} S(x) &= 2\pi(r^2 - x^2) + 4\pi x \sqrt{r^2 - x^2}, \quad 0 \leq x \leq r \\ &= 2\pi r^2 - 2\pi x^2 + 4\pi(x \sqrt{r^2 - x^2}) \end{aligned}$$

$$\text{Thus, } S'(x) = 0 - 4\pi x + 4\pi \left[x \cdot \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2} \cdot 1 \right]$$

$$= 4\pi \left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \right] = 4\pi \cdot \frac{-x \sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}}$$

$$S'(x) = 0 \Rightarrow x \sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (*) \Rightarrow (x \sqrt{r^2 - x^2})^2 = (r^2 - 2x^2)^2 \Rightarrow$$

$$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \Rightarrow r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \Rightarrow 5x^4 - 5r^2x^2 + r^4 = 0.$$

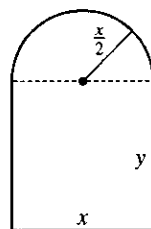
This is a quadratic equation in x^2 . By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10} r^2$, but we reject the root with the + sign since it doesn't satisfy (*). [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}} r$. Since

$S(0) = S(r) = 0$, the maximum surface area occurs at the critical number and $x^2 = \frac{5 - \sqrt{5}}{10} r^2 \Rightarrow$

$$y^2 = r^2 - \frac{5 - \sqrt{5}}{10} r^2 = \frac{5 + \sqrt{5}}{10} r^2 \Rightarrow \text{the surface area is}$$

$$\begin{aligned} 2\pi \left(\frac{5 + \sqrt{5}}{10} r^2 \right) + 4\pi \sqrt{\frac{5 - \sqrt{5}}{10}} \sqrt{\frac{5 + \sqrt{5}}{10}} r^2 &= \pi r^2 \left[2 \cdot \frac{5 + \sqrt{5}}{10} + 4 \frac{\sqrt{(5 - \sqrt{5})(5 + \sqrt{5})}}{10} \right] = \pi r^2 \left[\frac{5 + \sqrt{5}}{5} + \frac{2\sqrt{20}}{5} \right] = \\ \pi r^2 \left[\frac{5 + \sqrt{5} + 2 \cdot 2\sqrt{5}}{5} \right] &= \pi r^2 \left[\frac{5 + 5\sqrt{5}}{5} \right] = \pi r^2 (1 + \sqrt{5}). \end{aligned}$$

28.



$$\text{Perimeter} = 30 \Rightarrow 2y + x + \pi \left(\frac{x}{2} \right) = 30 \Rightarrow$$

$$y = \frac{1}{2} \left(30 - x - \frac{\pi x}{2} \right) = 15 - \frac{x}{2} - \frac{\pi x}{4}. \text{ The area is the area of the}$$

rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi \left(\frac{x}{2} \right)^2$, so

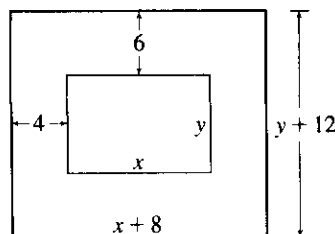
$$A(x) = x \left(15 - \frac{x}{2} - \frac{\pi x}{4} \right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2.$$

$$A'(x) = 15 - \left(1 + \frac{\pi}{4} \right)x = 0 \Rightarrow x = \frac{15}{1 + \pi/4} = \frac{60}{4 + \pi}. \quad A''(x) = - \left(1 + \frac{\pi}{4} \right) < 0, \text{ so this gives a}$$

$$\text{maximum. The dimensions are } x = \frac{60}{4 + \pi} \text{ ft and } y = 15 - \frac{30}{4 + \pi} - \frac{15\pi}{4 + \pi} = \frac{60 + 15\pi - 30 - 15\pi}{4 + \pi} = \frac{30}{4 + \pi} \text{ ft,}$$

so the height of the rectangle is half the base.

29.



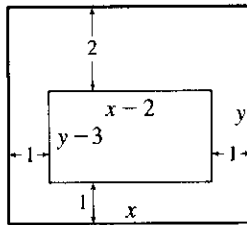
$$xy = 384 \Rightarrow y = 384/x. \text{ Total area is}$$

$$A(x) = (8 + x)(12 + 384/x) = 12(40 + x + 256/x), \text{ so}$$

$$A'(x) = 12(1 - 256/x^2) = 0 \Rightarrow x = 16. \text{ There is an absolute}$$

minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$. When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

30.



$xy = 180$, so $y = 180/x$. The printed area is

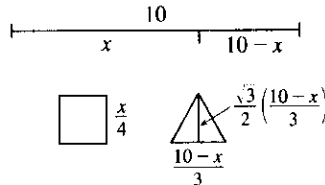
$$(x-2)(y-3) = (x-2)(180/x-3) = 186 - 3x - 360/x = A(x).$$

$$A'(x) = -3 + 360/x^2 = 0 \text{ when } x^2 = 120 \Rightarrow x = 2\sqrt{30}.$$

This gives an absolute maximum since $A'(x) > 0$ for $0 < x < 2\sqrt{30}$ and

$A'(x) < 0$ for $x > 2\sqrt{30}$. When $x = 2\sqrt{30}$, $y = 180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

31.



Let x be the length of the wire used for the square. The total area is

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2} \left(\frac{10-x}{3}\right) \frac{\sqrt{3}}{2} \left(\frac{10-x}{3}\right) \\ = \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}(10-x)^2, \quad 0 \leq x \leq 10$$

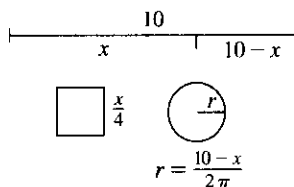
$$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10-x) = 0 \Leftrightarrow \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \Leftrightarrow x = \frac{40\sqrt{3}}{9+4\sqrt{3}}.$$

$$A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81, A(10) = \frac{100}{16} = 6.25 \text{ and } A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72, \text{ so}$$

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.35$ m.

32.



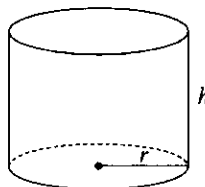
$$\text{Total area is } A(x) = \left(\frac{x}{4}\right)^2 + \pi \left(\frac{10-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi},$$

$$0 \leq x \leq 10. \quad A'(x) = \frac{x}{8} - \frac{10-x}{2\pi} = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{5}{\pi} = 0 \Rightarrow$$

$$x = 40/(4+\pi). \quad A(0) = 25/\pi \approx 7.96, A(10) = 6.25, \text{ and}$$

$A(40/(4+\pi)) \approx 3.5$, so the maximum occurs when $x = 0$ m and the minimum occurs when $x = 40/(4+\pi)$ m.

33.



The volume is $V = \pi r^2 h$ and the surface area is

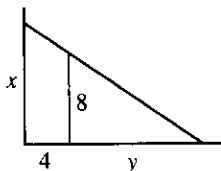
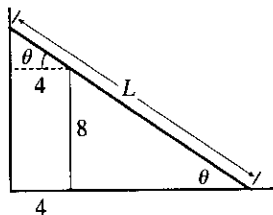
$$S(r) = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r \left(\frac{V}{\pi r^2}\right) = \pi r^2 + \frac{2V}{r}.$$

$$S'(r) = 2\pi r - \frac{2V}{r^2} = 0 \Rightarrow 2\pi r^3 = 2V \Rightarrow r = \sqrt[3]{\frac{V}{\pi}} \text{ cm.}$$

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\frac{V}{\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\frac{V}{\pi}}$. When

$$r = \sqrt[3]{\frac{V}{\pi}}, h = \frac{V}{\pi r^2} = \frac{V}{\pi(V/\pi)^{2/3}} = \sqrt[3]{\frac{V}{\pi}} \text{ cm.}$$

34.



$$L = 8 \csc \theta + 4 \sec \theta, 0 < \theta < \frac{\pi}{2},$$

$$\frac{dL}{d\theta} = -8 \csc \theta \cot \theta + 4 \sec \theta \tan \theta = 0 \text{ when}$$

$$\sec \theta \tan \theta = 2 \csc \theta \cot \theta \Leftrightarrow \tan^3 \theta = 2 \Leftrightarrow \tan \theta = \sqrt[3]{2} \Leftrightarrow \theta = \tan^{-1} \sqrt[3]{2}.$$

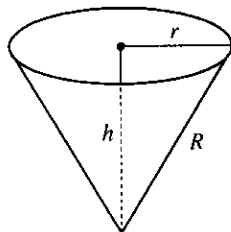
$dL/d\theta < 0$ when $0 < \theta < \tan^{-1} \sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1} \sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when

$\theta = \tan^{-1} \sqrt[3]{2}$, and the shortest ladder has length

$$L = 8 \frac{\sqrt{1 + 2^{2/3}}}{2^{1/3}} + 4 \sqrt{1 + 2^{2/3}} \approx 16.65 \text{ ft.}$$

Another method: Minimize $L^2 = x^2 + (4 + y)^2$, where $\frac{x}{4 + y} = \frac{8}{y}$.

35.



$$h^2 + r^2 = R^2 \Rightarrow V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} (R^2 - h^2) h = \frac{\pi}{3} (R^2 h - h^3).$$

$V'(h) = \frac{\pi}{3} (R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}} R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}} R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}} R$. The maximum volume is

$$V\left(\frac{1}{\sqrt{3}} R\right) = \frac{\pi}{3} \left(\frac{1}{\sqrt{3}} R^3 - \frac{1}{3\sqrt{3}} R^3 \right) = \frac{2}{9\sqrt{3}} \pi R^3.$$

36. The volume and surface area of a cone with radius r and height h are given by $V = \frac{1}{3} \pi r^2 h$ and $S = \pi r \sqrt{r^2 + h^2}$.

We'll minimize $A = S^2$ subject to $V = 27$. $V = 27 \Rightarrow \frac{1}{3} \pi r^2 h = 27 \Rightarrow r^2 = \frac{81}{\pi h}$ (1).

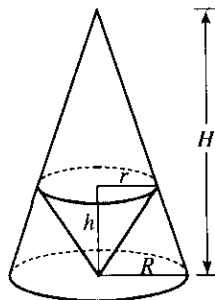
$$A = \pi^2 r^2 (r^2 + h^2) = \pi^2 \left(\frac{81}{\pi h} \right) \left(\frac{81}{\pi h} + h^2 \right) = \frac{81^2}{h^2} + 81\pi h, \text{ so } A' = 0 \Rightarrow \frac{-2 \cdot 81^2}{h^3} + 81\pi = 0 \Rightarrow$$

$$81\pi = \frac{2 \cdot 81^2}{h^3} \Rightarrow h^3 = \frac{162}{\pi} \Rightarrow h = \sqrt[3]{\frac{162}{\pi}} = 3 \sqrt[3]{\frac{6}{\pi}} \approx 3.722. \text{ From (1),}$$

$$r^2 = \frac{81}{\pi h} = \frac{81}{\pi \cdot 3 \sqrt[3]{6/\pi}} = \frac{27}{\sqrt[3]{6\pi^2}} \Rightarrow r = \frac{3 \sqrt[3]{3}}{\sqrt[3]{6\pi^2}} \approx 2.632. A'' = 6 \cdot 81^2 / h^4 > 0, \text{ so } A \text{ and hence } S \text{ has an}$$

absolute minimum at these values of r and h .

37.



By similar triangles, $\frac{H}{R} = \frac{H-h}{r}$ (1). The volume of the inner cone is

$$V = \frac{1}{3} \pi r^2 h, \text{ so we'll solve (1) for } h. \frac{Hr}{R} = H - h \Rightarrow$$

$$h = H - \frac{Hr}{R} = \frac{HR - Hr}{R} = \frac{H}{R} (R - r) \quad (2).$$

$$\text{Thus, } V(r) = \frac{\pi}{3} r^2 \cdot \frac{H}{R} (R - r) = \frac{\pi H}{3R} (Rr^2 - r^3) \Rightarrow$$

$$V'(r) = \frac{\pi H}{3R} (2Rr - 3r^2) = \frac{\pi H}{3R} r (2R - 3r).$$

$$V'(r) = 0 \Rightarrow r = 0 \text{ or } 2R = 3r \Rightarrow r = \frac{2}{3}R \text{ and from (2), } h = \frac{H}{R}\left(R - \frac{2}{3}R\right) = \frac{H}{R}\left(\frac{1}{3}R\right) = \frac{1}{3}H.$$

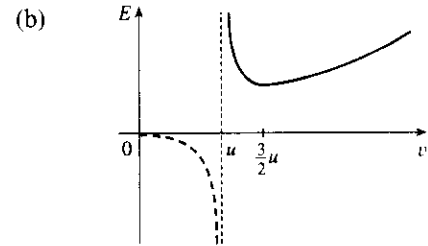
$V'(r)$ changes from positive to negative at $r = \frac{2}{3}R$, so the inner cone has a maximum volume of

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right) = \frac{4}{27} \cdot \frac{1}{3}\pi R^2 H, \text{ which is approximately 15\% of the volume of the larger cone.}$$

$$38. (a) E(v) = \frac{aLv^3}{v-u} \Rightarrow E'(v) = aL \frac{(v-u)3v^2 - v^3}{(v-u)^2} = 0$$

$$\text{when } 2v^3 = 3uv^2 \Rightarrow 2v = 3u \Rightarrow v = \frac{3}{2}u.$$

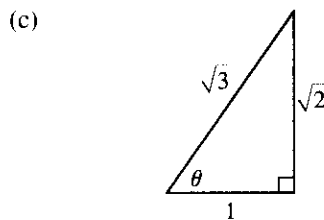
The First Derivative Test shows that this value of v gives the minimum value of E .



$$39. S = 6sh - \frac{3}{2}s^2 \cot \theta + 3s^2 \frac{\sqrt{3}}{2} \csc \theta$$

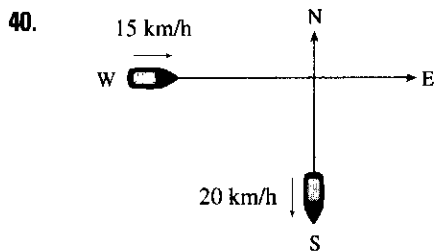
$$(a) \frac{dS}{d\theta} = \frac{3}{2}s^2 \csc^2 \theta - 3s^2 \frac{\sqrt{3}}{2} \csc \theta \cot \theta \text{ or } \frac{3}{2}s^2 \csc \theta (\csc \theta - \sqrt{3} \cot \theta).$$

$$(b) \frac{dS}{d\theta} = 0 \text{ when } \csc \theta - \sqrt{3} \cot \theta = 0 \Rightarrow \frac{1}{\sin \theta} - \sqrt{3} \frac{\cos \theta}{\sin \theta} = 0 \Rightarrow \cos \theta = \frac{1}{\sqrt{3}}. \text{ The First Derivative Test shows that the minimum surface area occurs when } \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ.$$



If $\cos \theta = \frac{1}{\sqrt{3}}$, then $\cot \theta = \frac{1}{\sqrt{2}}$ and $\csc \theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$\begin{aligned} S &= 6sh - \frac{3}{2}s^2 \frac{1}{\sqrt{2}} + 3s^2 \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{\sqrt{2}} = 6sh - \frac{3}{2\sqrt{2}}s^2 + \frac{9}{2\sqrt{2}}s^2 \\ &= 6sh + \frac{6}{2\sqrt{2}}s^2 = 6s\left(h + \frac{1}{\sqrt{2}}s\right) \end{aligned}$$



Let t be the time, in hours, after 2:00 P.M. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. If $D(t)$ is the distance between the boats at time t , we minimize $f(t) = [D(t)]^2 = 20^2 t^2 + 15^2 (t-1)^2$.

$$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0 \text{ when } t = \frac{450}{1250} = 0.36 \text{ h.}$$

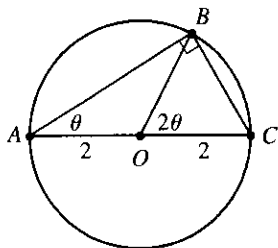
$0.36 \text{ h} \times \frac{60 \text{ min}}{\text{h}} = 21.6 \text{ min} = 21 \text{ min } 36 \text{ s.}$ Since $f''(t) > 0$, this gives a minimum, so the boats are closest together at 2:21:36 P.M.

$$41. \text{ Here } T(x) = \frac{\sqrt{x^2 + 25}}{6} + \frac{5-x}{8}, 0 \leq x \leq 5 \Rightarrow T'(x) = \frac{x}{6\sqrt{x^2 + 25}} - \frac{1}{8} = 0 \Leftrightarrow 8x = 6\sqrt{x^2 + 25}$$

$$\Leftrightarrow 16x^2 = 9(x^2 + 25) \Leftrightarrow x = \frac{15}{\sqrt{7}}. \text{ But } \frac{15}{\sqrt{7}} > 5, \text{ so } T \text{ has no critical number. Since } T(0) \approx 1.46 \text{ and}$$

$T(5) \approx 1.18$, he should row directly to B .

42.



In isosceles triangle AOB , $\angle O = 180^\circ - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $4 \cos \theta$ while the distance walked is the length of arc $BC = 2(2\theta) = 4\theta$. The time taken is given by

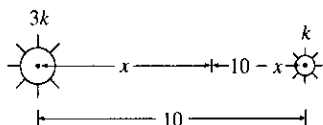
$$T(\theta) = \frac{4 \cos \theta}{2} + \frac{4\theta}{4} = 2 \cos \theta + \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

$$T'(\theta) = -2 \sin \theta + 1 = 0 \Leftrightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$$

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T ; that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$. $T(0) = 2$,

$T(\frac{\pi}{6}) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T(\frac{\pi}{2}) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta) = -2 \cos \theta < 0$ for $0 \leq \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

43.



The total illumination is $I(x) = \frac{3k}{x^2} + \frac{k}{(10-x)^2}$, $0 < x < 10$. Then

$$I'(x) = \frac{-6k}{x^3} + \frac{2k}{(10-x)^3} = 0 \Rightarrow 6k(10-x)^3 = 2kx^3 \Rightarrow$$

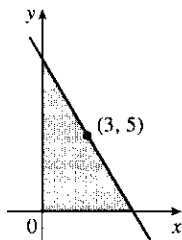
$$3(10-x)^3 = x^3 \Rightarrow \sqrt[3]{3}(10-x) = x \Rightarrow 10\sqrt[3]{3} - \sqrt[3]{3}x = x$$

$$\Rightarrow 10\sqrt[3]{3} = x + \sqrt[3]{3}x \Rightarrow 10\sqrt[3]{3} = (1 + \sqrt[3]{3})x \Rightarrow$$

$$x = \frac{10\sqrt[3]{3}}{1 + \sqrt[3]{3}} \approx 5.9 \text{ ft. This gives a minimum since } I''(x) > 0 \text{ for}$$

$$0 < x < 10.$$

44.



The line with slope m (where $m < 0$) through $(3, 5)$ has equation $y - 5 = m(x - 3)$ or $y = mx + (5 - 3m)$. The y -intercept is $5 - 3m$ and the x -intercept is $-5/m + 3$.

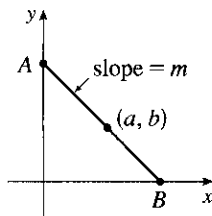
So the triangle has area $A(m) = \frac{1}{2}(5 - 3m)(-5/m + 3) = 15 - 25/(2m) - \frac{9}{2}m$.

$$\text{Now } A'(m) = \frac{25}{2m^2} - \frac{9}{2} = 0 \Leftrightarrow m^2 = \frac{25}{9} \Rightarrow m = -\frac{5}{3} \text{ (since } m < 0).$$

$$A''(m) = -\frac{25}{m^3} > 0, \text{ so there is an absolute minimum when } m = -\frac{5}{3}. \text{ Thus, an}$$

$$\text{equation of the line is } y - 5 = -\frac{5}{3}(x - 3) \text{ or } y = -\frac{5}{3}x + 10.$$

45.



Every line segment in the first quadrant passing through (a, b) with endpoints on the x - and y -axes satisfies an equation of the form $y - b = m(x - a)$, where $m < 0$. By setting $x = 0$ and then $y = 0$, we find its endpoints, $A(0, b - am)$ and $B(a - \frac{b}{m}, 0)$.

$$\text{The distance } d \text{ from } A \text{ to } B \text{ is given by } d = \sqrt{[(a - \frac{b}{m}) - 0]^2 + [0 - (b - am)]^2}.$$

It follows that the square of the length of the line segment, as a function of m , is given by

$$S(m) = \left(a - \frac{b}{m}\right)^2 + (am - b)^2 = a^2 - \frac{2ab}{m} + \frac{b^2}{m^2} + a^2m^2 - 2abm + b^2. \text{ Thus,}$$

$$\begin{aligned} S'(m) &= \frac{2ab}{m^2} - \frac{2b^2}{m^3} + 2a^2m - 2ab = \frac{2}{m^3}(abm - b^2 + a^2m^4 - abm^3) \\ &= \frac{2}{m^3}[b(am - b) + am^3(am - b)] = \frac{2}{m^3}(am - b)(b + am^3) \end{aligned}$$

Thus, $S'(m) = 0 \Leftrightarrow m = b/a$ or $m = -\sqrt[3]{\frac{b}{a}}$. Since $b/a > 0$ and $m < 0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since

$\frac{2}{m^3} < 0$, we see that $S'(m) < 0$ for $m < -\sqrt[3]{\frac{b}{a}}$ and $S'(m) > 0$ for $m > -\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute

minimum value when $m = -\sqrt[3]{\frac{b}{a}}$. That value is

$$\begin{aligned} S\left(-\sqrt[3]{\frac{b}{a}}\right) &= \left(a + b\sqrt[3]{\frac{a}{b}}\right)^2 + \left(-a\sqrt[3]{\frac{b}{a}} - b\right)^2 = \left(a + \sqrt[3]{ab^2}\right)^2 + \left(\sqrt[3]{a^2b} + b\right)^2 \\ &= a^2 + 2a^{4/3}b^{2/3} + a^{2/3}b^{4/3} + a^{4/3}b^{2/3} + 2a^{2/3}b^{4/3} + b^2 = a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2 \end{aligned}$$

The last expression is of the form $x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3$ with $x = a^{2/3}$ and $y = b^{2/3}$,

so we can write it as $(a^{2/3} + b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S} = (a^{2/3} + b^{2/3})^{3/2}$.

46. $y = 1 + 40x^3 - 3x^5 \Rightarrow y' = 120x^2 - 15x^4$, so the tangent line to the curve at $x = a$ has slope

$$m(a) = 120a^2 - 15a^4. \text{ Now } m'(a) = 240a - 60a^3 = -60a(a^2 - 4) = -60a(a + 2)(a - 2), \text{ so } m'(a) > 0$$

for $a < -2$, and $0 < a < 2$, and $m'(a) < 0$ for $-2 < a < 0$ and $a > 2$. Thus, m is increasing on $(-\infty, -2)$,

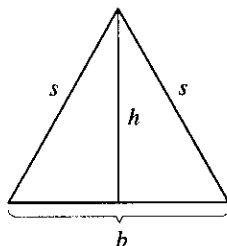
decreasing on $(-2, 0)$, increasing on $(0, 2)$, and decreasing on $(2, \infty)$. Clearly, $m(a) \rightarrow -\infty$ as $a \rightarrow \pm\infty$, so the

maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal

$$120 \cdot 2^2 - 15 \cdot 2^4 = 480 - 240 = 240. \text{ So } 240 \text{ is the largest slope, and it occurs at the points } (-2, -223) \text{ and}$$

$(2, 225)$. Note: $a = 0$ corresponds to a local minimum of m .

47.



$$\text{Here } s^2 = h^2 + b^2/4, \text{ so } h^2 = s^2 - b^2/4. \text{ The area is } A = \frac{1}{2}b\sqrt{s^2 - b^2/4}.$$

$$\text{Let the perimeter be } p, \text{ so } 2s + b = p \text{ or } s = (p - b)/2 \Rightarrow$$

$$A(b) = \frac{1}{2}b\sqrt{(p - b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4. \text{ Now}$$

$$A'(b) = \frac{\sqrt{p^2 - 2pb}}{4} - \frac{bp/4}{\sqrt{p^2 - 2pb}} = \frac{-3pb + p^2}{4\sqrt{p^2 - 2pb}}. \text{ Therefore, } A'(b) = 0 \Rightarrow$$

$$-3pb + p^2 = 0 \Rightarrow b = p/3. \text{ Since } A'(b) > 0 \text{ for } b < p/3 \text{ and } A'(b) < 0 \text{ for } b > p/3, \text{ there is an absolute}$$

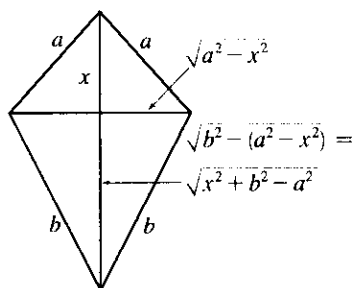
maximum when $b = p/3$. But then $2s + p/3 = p$, so $s = p/3 \Rightarrow s = b \Rightarrow$ the triangle is equilateral.

48. See the figure. The area is given by

$$A(x) = \frac{1}{2}(2\sqrt{a^2 - x^2})x + \frac{1}{2}(2\sqrt{a^2 - x^2})(\sqrt{x^2 + b^2 - a^2}) = \sqrt{a^2 - x^2}(x + \sqrt{x^2 + b^2 - a^2}) \text{ for}$$

$$0 \leq x \leq a. \text{ Now } A'(x) = \sqrt{a^2 - x^2} \left(1 + \frac{x}{\sqrt{x^2 + b^2 - a^2}} \right) + (x + \sqrt{x^2 + b^2 - a^2}) \frac{-x}{\sqrt{a^2 - x^2}} = 0 \Leftrightarrow$$

$$\frac{x}{\sqrt{a^2 - x^2}}(x + \sqrt{x^2 + b^2 - a^2}) = \sqrt{a^2 - x^2} \left(\frac{x + \sqrt{x^2 + b^2 - a^2}}{\sqrt{x^2 + b^2 - a^2}} \right).$$



Except for the trivial case where $x = 0$, $a = b$ and $A(x) = 0$, we have

$x + \sqrt{x^2 + b^2 - a^2} > 0$. Hence, cancelling this factor gives

$$\begin{aligned} \frac{x}{\sqrt{a^2 - x^2}} &= \frac{\sqrt{a^2 - x^2}}{\sqrt{x^2 + b^2 - a^2}} \Rightarrow x\sqrt{x^2 + b^2 - a^2} = a^2 - x^2 \Rightarrow \\ x^2(x^2 + b^2 - a^2) &= a^4 - 2a^2x^2 + x^4 \Rightarrow x^2(b^2 - a^2) = a^4 - 2a^2x^2 \\ \Rightarrow x^2(b^2 + a^2) &= a^4 \Rightarrow x = \frac{a^2}{\sqrt{a^2 + b^2}}. \end{aligned}$$

Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0) = a\sqrt{b^2 - a^2}$, $A(a) = 0$ and

$$\begin{aligned} A\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right) &= \sqrt{a^2 - \left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)^2} \left[\frac{a^2}{\sqrt{a^2 + b^2}} + \sqrt{\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)^2 + b^2 - a^2} \right] \\ &= \frac{ab}{\sqrt{a^2 + b^2}} \left[\frac{a^2}{\sqrt{a^2 + b^2}} + \frac{b^2}{\sqrt{a^2 + b^2}} \right] = \frac{ab(a^2 + b^2)}{a^2 + b^2} = ab \end{aligned}$$

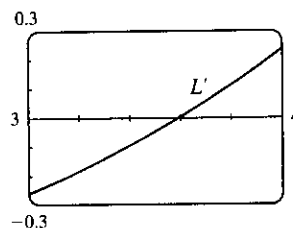
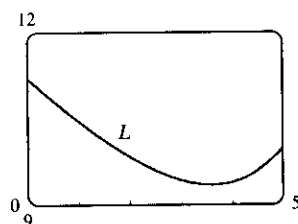
Since $b \geq \sqrt{b^2 - a^2}$, $A(a^2/\sqrt{a^2 + b^2}) \geq A(0)$. So there is an absolute maximum when $x = \frac{a^2}{\sqrt{a^2 + b^2}}$. In this

case the horizontal piece should be $\frac{2ab}{\sqrt{a^2 + b^2}}$ and the vertical piece should be $\frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}$.

49. Note that $|AD| = |AP| + |PD| \Rightarrow 5 = x + |PD| \Rightarrow |PD| = 5 - x$. Using the Pythagorean Theorem for $\triangle PDB$ and $\triangle PDC$ gives us

$$\begin{aligned} L(x) &= |AP| + |BP| + |CP| = x + \sqrt{(5-x)^2 + 2^2} + \sqrt{(5-x)^2 + 3^2} \\ &= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34} \end{aligned}$$

$$\Rightarrow L'(x) = 1 + \frac{x-5}{\sqrt{x^2 - 10x + 29}} + \frac{x-5}{\sqrt{x^2 - 10x + 34}}.$$

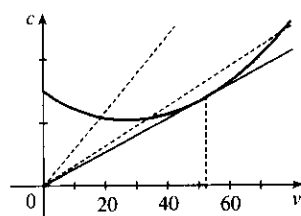


From the graphs of L and L' , it seems that the minimum value of L is about $L(3.59) = 9.35$ m.

50. We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then

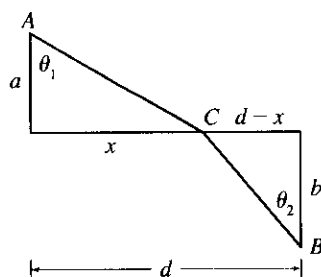
$\frac{c}{v} = \frac{\text{gallons/hour}}{\text{miles/hour}} = \frac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G . To find the

minimum, we calculate $\frac{dG}{dv} = \frac{d}{dv} \left(\frac{c}{v} \right) = \frac{v \frac{dc}{dv} - c \frac{dv}{dv}}{v^2} = \frac{v \frac{dc}{dv} - c}{v^2}$.



This is 0 when $v \frac{dc}{dv} - c = 0 \Leftrightarrow \frac{dc}{dv} = \frac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi/h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

51.



The total time is

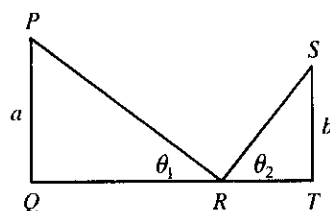
$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B) \\ = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2}, \quad 0 < x < d$$

$$T'(x) = \frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{d-x}{v_2 \sqrt{b^2 + (d-x)^2}} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \Rightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$.

[Note: $T''(x) > 0$]

52.



If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a \csc \theta_1 + b \csc \theta_2$.

Differentiating with respect to θ_1 , and setting $\frac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a \csc \theta_1 \cot \theta_1 - b \csc \theta_2 \cot \theta_2 \frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\frac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a \cot \theta_1 + b \cot \theta_2$.

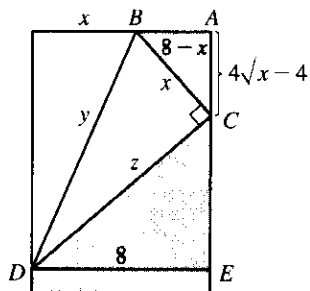
Differentiating this equation implicitly with respect to θ_1 , we get $-a \csc^2 \theta_1 - b \csc^2 \theta_2 \frac{d\theta_2}{d\theta_1} = 0 \Rightarrow$

$\frac{d\theta_2}{d\theta_1} = -\frac{a \csc^2 \theta_1}{b \csc^2 \theta_2}$. We substitute this into the expression for $\frac{df}{d\theta_1}$ to get

$$-a \csc \theta_1 \cot \theta_1 - b \csc \theta_2 \cot \theta_2 \left(-\frac{a \csc^2 \theta_1}{b \csc^2 \theta_2} \right) = 0 \Leftrightarrow -a \csc \theta_1 \cot \theta_1 + a \frac{\csc^2 \theta_1 \cot \theta_2}{\csc \theta_2} = 0 \Leftrightarrow$$

$\cot \theta_1 \csc \theta_2 = \csc \theta_1 \cot \theta_2 \Leftrightarrow \frac{\cot \theta_1}{\csc \theta_1} = \frac{\cot \theta_2}{\csc \theta_2} \Leftrightarrow \cos \theta_1 = \cos \theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

53.



$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so

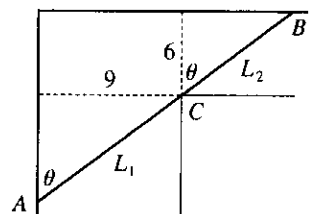
$$z/8 = x/(4\sqrt{x-4}) \Rightarrow z = 2x/\sqrt{x-4}. \text{ Thus, we minimize}$$

$$f(x) = y^2 = x^2 + 4x^2/(x-4) = x^3/(x-4), \quad 4 < x \leq 8.$$

$$f'(x) = \frac{(x-4)(3x^2) - x^3}{(x-4)^2} = \frac{x^2[3(x-4) - x]}{(x-4)^2} = \frac{2x^2(x-6)}{(x-4)^2} = 0$$

when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

54.



Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C . As $\theta \rightarrow 0$ or $\theta \rightarrow \frac{\pi}{2}$, we have $L \rightarrow \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

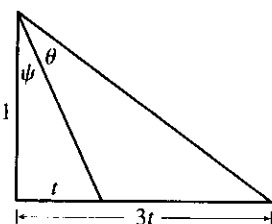
$$\text{From the diagram, } L = L_1 + L_2 = 9 \csc \theta + 6 \sec \theta \Rightarrow dL/d\theta = -9 \csc \theta \cot \theta + 6 \sec \theta \tan \theta = 0 \text{ when}$$

$$6 \sec \theta \tan \theta = 9 \csc \theta \cot \theta \Leftrightarrow \tan^3 \theta = \frac{9}{6} = 1.5 \Leftrightarrow \tan \theta = \sqrt[3]{1.5}. \text{ Then } \sec^2 \theta = 1 + \left(\frac{3}{2}\right)^{2/3} \text{ and}$$

$$\csc^2 \theta = 1 + \left(\frac{3}{2}\right)^{-2/3}, \text{ so the longest pipe has length } L = 9 \left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6 \left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07 \text{ ft.}$$

$$\text{Or, use } \theta = \tan^{-1}(\sqrt[3]{1.5}) \approx 0.852 \Rightarrow L = 9 \csc \theta + 6 \sec \theta \approx 21.07 \text{ ft.}$$

55.



It suffices to maximize $\tan \theta$. Now

$$\frac{3t}{1} = \tan(\psi + \theta) = \frac{\tan \psi + \tan \theta}{1 - \tan \psi \tan \theta} = \frac{t + \tan \theta}{1 - t \tan \theta}.$$

$$\text{So } 3t(1 - t \tan \theta) = t + \tan \theta \Rightarrow 2t = (1 + 3t^2) \tan \theta \Rightarrow$$

$$\tan \theta = \frac{2t}{1 + 3t^2}. \text{ Let } f(t) = \tan \theta = \frac{2t}{1 + 3t^2} \Rightarrow$$

$$f'(t) = \frac{2(1 + 3t^2) - 2t(6t)}{(1 + 3t^2)^2} = \frac{2(1 - 3t^2)}{(1 + 3t^2)^2} = 0 \Leftrightarrow$$

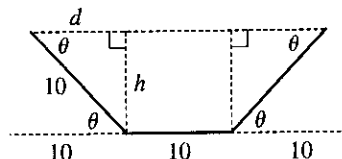
$$1 - 3t^2 = 0 \Leftrightarrow t = \frac{1}{\sqrt{3}} \text{ since } t \geq 0.$$

Now $f'(t) > 0$ for $0 \leq t < \frac{1}{\sqrt{3}}$ and $f'(t) < 0$ for $t > \frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t = \frac{1}{\sqrt{3}}$

$$\text{and } \tan \theta = \frac{2(1/\sqrt{3})}{1 + 3(1/\sqrt{3})^2} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}. \text{ Substituting for } t \text{ and } \theta \text{ in } 3t = \tan(\psi + \theta) \text{ gives us}$$

$$\sqrt{3} = \tan\left(\psi + \frac{\pi}{6}\right) \Rightarrow \psi = \frac{\pi}{6}.$$

56.

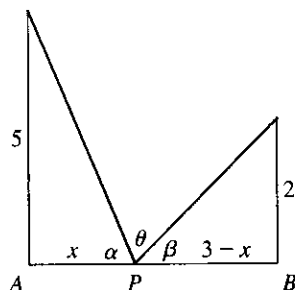


We maximize the cross-sectional area

$$\begin{aligned}
 A(\theta) &= 10h + 2\left(\frac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta) \\
 &= 100(\sin\theta + \sin\theta\cos\theta), \quad 0 \leq \theta \leq \frac{\pi}{2} \\
 A'(\theta) &= 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(\cos\theta + 2\cos^2\theta - 1) = 100(2\cos\theta - 1)(\cos\theta + 1) \\
 &= 0 \text{ when } \cos\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}. \quad (\cos\theta \neq -1 \text{ since } 0 \leq \theta \leq \frac{\pi}{2}.)
 \end{aligned}$$

Now $A(0) = 0$, $A(\frac{\pi}{2}) = 100$ and $A(\frac{\pi}{3}) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.

57.



From the figure, $\tan\alpha = \frac{5}{x}$ and $\tan\beta = \frac{2}{3-x}$. Since

$$\alpha + \beta + \theta = 180^\circ = \pi, \quad \theta = \pi - \tan^{-1}\left(\frac{5}{x}\right) - \tan^{-1}\left(\frac{2}{3-x}\right) \Rightarrow$$

$$\begin{aligned}
 \frac{d\theta}{dx} &= -\frac{1}{1 + \left(\frac{5}{x}\right)^2} \left(-\frac{5}{x^2}\right) - \frac{1}{1 + \left(\frac{2}{3-x}\right)^2} \left[\frac{2}{(3-x)^2}\right] \\
 &= \frac{x^2}{x^2 + 25} \cdot \frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2 + 4} \cdot \frac{2}{(3-x)^2}.
 \end{aligned}$$

$$\text{Now } \frac{d\theta}{dx} = 0 \Rightarrow \frac{5}{x^2 + 25} = \frac{2}{x^2 - 6x + 13} \Rightarrow 2x^2 + 50 = 5x^2 - 30x + 65 \Rightarrow$$

$$3x^2 - 30x + 15 = 0 \Rightarrow x^2 - 10x + 5 = 0 \Rightarrow x = 5 \pm 2\sqrt{5}. \text{ We reject the root with the } + \text{ sign,}$$

since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

58. Let x be the distance from the observer to the wall. Then, from the given figure,

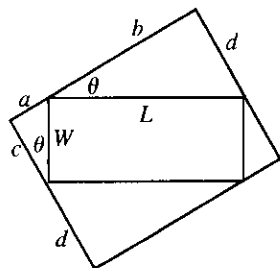
$$\theta = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right), \quad x > 0 \Rightarrow$$

$$\begin{aligned}
 \frac{d\theta}{dx} &= \frac{1}{1 + [(h+d)/x]^2} \left[-\frac{h+d}{x^2}\right] - \frac{1}{1 + (d/x)^2} \left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2 + (h+d)^2} + \frac{d}{x^2 + d^2} \\
 &= \frac{d[x^2 + (h+d)^2] - (h+d)(x^2 + d^2)}{[x^2 + (h+d)^2](x^2 + d^2)} = \frac{h^2d + hd^2 - hx^2}{[x^2 + (h+d)^2](x^2 + d^2)} = 0 \Leftrightarrow
 \end{aligned}$$

$$hx^2 = h^2d + hd^2 \Leftrightarrow x^2 = hd + d^2 \Leftrightarrow x = \sqrt{d(h+d)}. \text{ Since } d\theta/dx > 0 \text{ for all } x < \sqrt{d(h+d)} \text{ and}$$

$$d\theta/dx < 0 \text{ for all } x > \sqrt{d(h+d)}, \text{ the absolute maximum occurs when } x = \sqrt{d(h+d)}.$$

59.



In the small triangle with sides a and c and hypotenuse W , $\sin \theta = \frac{a}{W}$ and $\cos \theta = \frac{c}{W}$. In the triangle with sides b and d and hypotenuse L , $\sin \theta = \frac{d}{L}$ and $\cos \theta = \frac{b}{L}$. Thus, $a = W \sin \theta$, $c = W \cos \theta$, $d = L \sin \theta$, and $b = L \cos \theta$, so the area of the circumscribed rectangle is

$$\begin{aligned} A(\theta) &= (a+b)(c+d) = (W \sin \theta + L \cos \theta)(W \cos \theta + L \sin \theta) \\ &= W^2 \sin \theta \cos \theta + WL \sin^2 \theta + LW \cos^2 \theta + L^2 \sin \theta \cos \theta \\ &= LW \sin^2 \theta + LW \cos^2 \theta + (L^2 + W^2) \sin \theta \cos \theta \\ &= LW(\sin^2 \theta + \cos^2 \theta) + (L^2 + W^2) \cdot \frac{1}{2} \cdot 2 \sin \theta \cos \theta \\ &= LW + \frac{1}{2}(L^2 + W^2) \sin 2\theta, \quad 0 \leq \theta \leq \frac{\pi}{2} \end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \Leftrightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$. So the maximum area is $A(\frac{\pi}{4}) = LW + \frac{1}{2}(L^2 + W^2) = \frac{1}{2}(L^2 + 2LW + W^2) = \frac{1}{2}(L+W)^2$.

60. (a) Let D be the point such that $a = |AD|$. From the figure, $\sin \theta = \frac{b}{|BC|} \Rightarrow |BC| = b \csc \theta$ and

$$\cos \theta = \frac{|BD|}{|BC|} = \frac{a - |AB|}{|BC|} \Rightarrow |BC| = (a - |AB|) \sec \theta. \text{ Eliminating } |BC| \text{ gives}$$

$$(a - |AB|) \sec \theta = b \csc \theta \Rightarrow b \cot \theta = a - |AB| \Rightarrow |AB| = a - b \cot \theta. \text{ The total resistance is}$$

$$R(\theta) = C \frac{|AB|}{r_1^4} + C \frac{|BC|}{r_2^4} = C \left(\frac{a - b \cot \theta}{r_1^4} + \frac{b \csc \theta}{r_2^4} \right).$$

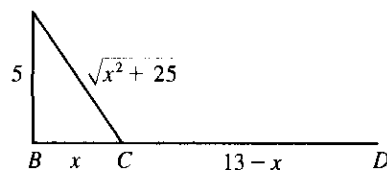
$$(b) R'(\theta) = C \left(\frac{b \csc^2 \theta}{r_1^4} - \frac{b \csc \theta \cot \theta}{r_2^4} \right) = bC \csc \theta \left(\frac{\csc \theta}{r_1^4} - \frac{\cot \theta}{r_2^4} \right).$$

$$R'(\theta) = 0 \Leftrightarrow \frac{\csc \theta}{r_1^4} = \frac{\cot \theta}{r_2^4} \Leftrightarrow \frac{r_2^4}{r_1^4} = \frac{\cot \theta}{\csc \theta} = \cos \theta.$$

$$R'(\theta) > 0 \Leftrightarrow \frac{\csc \theta}{r_1^4} > \frac{\cot \theta}{r_2^4} \Rightarrow \cos \theta < \frac{r_2^4}{r_1^4} \text{ and } R'(\theta) < 0 \text{ when } \cos \theta > \frac{r_2^4}{r_1^4}, \text{ so there is an absolute minimum when } \cos \theta = r_2^4/r_1^4.$$

$$(c) \text{ When } r_2 = \frac{2}{3}r_1, \text{ we have } \cos \theta = \left(\frac{2}{3}\right)^4, \text{ so } \theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79^\circ.$$

61. (a)



If k = energy/km over land, then

energy/km over water = $1.4k$. So the total energy is

$$E = 1.4k \sqrt{25 + x^2} + k(13 - x), \quad 0 \leq x \leq 13,$$

$$\text{and so } \frac{dE}{dx} = \frac{1.4kx}{(25 + x^2)^{1/2}} - k.$$

Set $\frac{dE}{dx} = 0$: $1.4kx = k(25 + x^2)^{1/2} \Rightarrow 1.96x^2 = x^2 + 25 \Rightarrow 0.96x^2 = 25 \Rightarrow x = \frac{5}{\sqrt{0.96}} \approx 5.1$.

Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$,

$E(13) \approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B .

- (b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the

distance of the flight. $E = W\sqrt{25 + x^2} + L(13 - x) \Rightarrow \frac{dE}{dx} = \frac{Wx}{\sqrt{25 + x^2}} - L = 0$ when

$\frac{W}{L} = \frac{\sqrt{25 + x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B .

- (c) For flight direct to D , $x = 13$, so from part (b), $W/L = \frac{\sqrt{25 + 13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B . But note that $\lim_{x \rightarrow 0^+} (W/L) = \infty$, so if the point at which E is a minimum is close to B , then W/L is large.

- (d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $(c)(4) = 1 \cdot (25 + 4^2)^{1/2} \Rightarrow c = \sqrt{41}/4 \approx 1.6$.

62. (a) $I(x) \propto \frac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

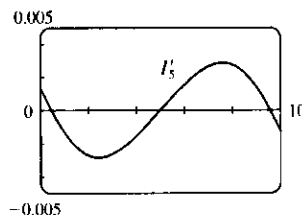
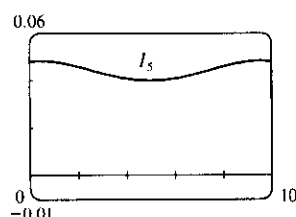
$$I(x) = \frac{k}{x^2 + d^2} + \frac{k}{(10 - x)^2 + d^2} = \frac{k}{x^2 + d^2} + \frac{k}{x^2 - 20x + 100 + d^2}.$$

- (b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience

we take $k = 1$. $I'(x) = -\frac{2x}{(x^2 + d^2)^2} - \frac{2(x - 10)}{(x^2 - 20x + 100 + d^2)^2}$.

Substituting $d = 5$ into the equations for $I(x)$ and $I'(x)$, we get

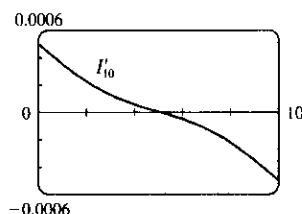
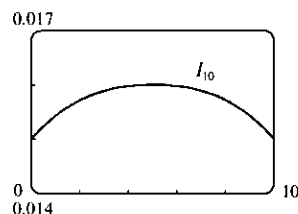
$$I_5(x) = \frac{1}{x^2 + 25} + \frac{1}{x^2 - 20x + 125} \quad \text{and} \quad I'_5(x) = -\frac{2x}{(x^2 + 25)^2} - \frac{2(x - 10)}{(x^2 - 20x + 125)^2}$$



From the graphs, it appears that $I_5(x)$ has a minimum at $x = 5$ m.

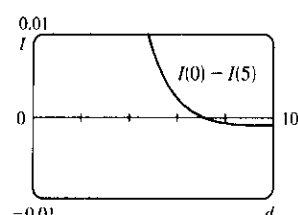
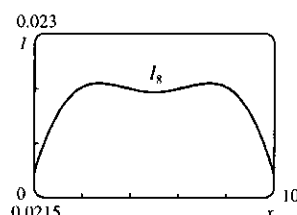
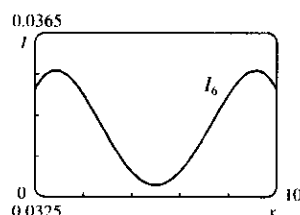
(c) Substituting $d = 10$ into the equations for $I(x)$ and $I'(x)$ gives $I_{10}(x) = \frac{1}{x^2 + 100} + \frac{1}{x^2 - 20x + 200}$ and

$$I'_{10}(x) = -\frac{2x}{(x^2 + 100)^2} - \frac{2(x - 10)}{(x^2 - 20x + 200)^2}.$$



From the graphs, it seems that for $d = 10$, the intensity is minimized at the endpoints, that is, $x = 0$ and $x = 10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x = 5$ with $d = 5$) to the endpoints ($x = 0$ and $x = 10$ with $d = 10$).



So we try $d = 6$ (see the first figure) and we see that the minimum value still occurs at $x = 5$. Next, we let $d = 8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8, we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d , we solve $I(0) = I(5)$ (with $k = 1$):

$$\frac{1}{d^2} + \frac{1}{100 + d^2} = \frac{1}{25 + d^2} + \frac{1}{25 + d^2} = \frac{2}{25 + d^2} \Rightarrow$$

$$(25 + d^2)(100 + d^2) + d^2(25 + d^2) = 2d^2(100 + d^2) \Rightarrow$$

$$2500 + 125d^2 + d^4 + 25d^2 + d^4 = 200d^2 + 2d^4 \Rightarrow 2500 = 50d^2 \Rightarrow d^2 = 50 \Rightarrow$$

$d = 5\sqrt{2} \approx 7.071$ (for $0 \leq d \leq 10$). The third figure, a graph of $I(0) - I(5)$ with d independent, confirms that

$I(0) - I(5) = 0$, that is, $I(0) = I(5)$, when $d = 5\sqrt{2}$. Thus, the point of minimal illumination changes

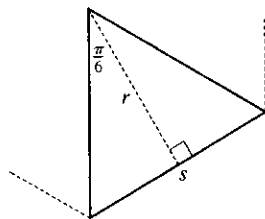
abruptly from the midpoint to the endpoints when $d = 5\sqrt{2}$.

APPLIED PROJECT The Shape of a Can

1. In this case, the amount of metal used in the making of each top or bottom is $(2r)^2 = 4r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2(4r^2)$. But $V = \pi r^2 h \Leftrightarrow h = V/\pi r^2$. Substituting this expression for h in A gives $A = 2V/r + 8r^2$. Differentiating A with respect to r , we get $dA/dr = -2V/r^2 + 16r = 0 \Rightarrow$

$$16r^3 = 2V = 2\pi r^2 h \Leftrightarrow \frac{h}{r} = \frac{8}{\pi} \approx 2.55. \text{ This gives a minimum because } \frac{d^2 A}{dr^2} = 16 + \frac{4V}{r^3} > 0.$$

2.



We need to find the area of metal used up by each end, that is, the area of each hexagon. We subdivide the hexagon into six congruent triangles, each sharing one side (s in the diagram) with the hexagon. We calculate the length of $s = 2r \tan \frac{\pi}{6} = \frac{2}{\sqrt{3}}r$, so the area of each triangle is $\frac{1}{2}sr = \frac{1}{\sqrt{3}}r^2$, and the total area of the hexagon is $6 \cdot \frac{1}{\sqrt{3}}r^2 = 2\sqrt{3}r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2 \cdot 2\sqrt{3}r^2$.

Substituting for h as in Problem 1 and differentiating, we get $\frac{dA}{dr} = -\frac{2V}{r^2} + 8\sqrt{3}r$. Setting this equal to 0, we get

$$8\sqrt{3}r^3 = 2V = 2\pi r^2 h \Rightarrow \frac{h}{r} = \frac{4\sqrt{3}}{\pi} \approx 2.21. \text{ Again this minimizes } A \text{ because } \frac{d^2A}{dr^2} = 8\sqrt{3} + \frac{4V}{r^3} > 0.$$

3. Let $C = 4\sqrt{3}r^2 + 2\pi rh + k(4\pi r + h) = 4\sqrt{3}r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) + k\left(4\pi r + \frac{V}{\pi r^2}\right)$. Then

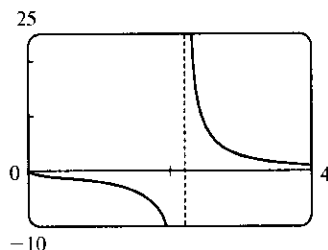
$$\frac{dC}{dr} = 8\sqrt{3}r - \frac{2V}{r^2} + 4k\pi - \frac{2kV}{\pi r^3}. \text{ Setting this equal to 0, dividing by 2 and substituting } \frac{V}{r^2} = \pi h \text{ and}$$

$$\frac{V}{\pi r^3} = \frac{h}{r} \text{ in the second and fourth terms respectively, we get } 0 = 4\sqrt{3}r - \pi h + 2k\pi - \frac{kh}{r} \Leftrightarrow$$

$$k\left(2\pi - \frac{h}{r}\right) = \pi h - 4\sqrt{3}r \Rightarrow \frac{k}{r} \frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}} = 1. \text{ We now multiply by } \frac{\sqrt[3]{V}}{k}, \text{ noting that}$$

$$\frac{\sqrt[3]{V}}{k} \frac{k}{r} = \sqrt[3]{\frac{V}{r^3}} = \sqrt[3]{\frac{\pi h}{r}}, \text{ and get } \frac{\sqrt[3]{V}}{k} = \sqrt[3]{\frac{\pi h}{r}} \cdot \frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}.$$

4.



Let $\sqrt[3]{V}/k = T$ and $h/r = x$ so that $T(x) = \frac{2\pi - x}{\pi x - 4\sqrt{3}}$. We see

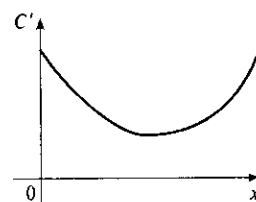
from the graph of T that when the ratio $\sqrt[3]{V}/k$ is large; that is, either the volume of the can is large or the cost of joining (proportional to k) is small, the optimum value of h/r is about 2.21, but when $\sqrt[3]{V}/k$ is small, indicating small volume or expensive joining, the optimum value of h/r is larger. (The part of the graph for $\sqrt[3]{V}/k < 0$ has no physical meaning, but confirms the location of the asymptote.)

5. Our conclusion is usually true in practice. But there are exceptions, such as cans of tuna, which may have to do with the shape of a reasonable slice of tuna. And for a comfortable grip on a soda or beer can, the geometry of the human hand is a restriction on the radius. Other possible considerations are packaging, transportation and stocking constraints, aesthetic appeal and other marketing concerns. Also, there may be better models than ours which prescribe a differently shaped can in special circumstances.

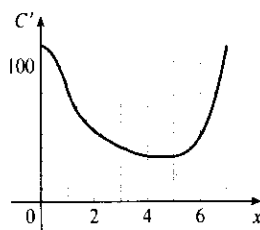
4.8 Applications to Business and Economics

1. (a) $C(0)$ represents the fixed costs of production, such as rent, utilities, machinery etc., which are incurred even when nothing is produced.
- (b) The inflection point is the point at which $C''(x)$ changes from negative to positive; that is, the marginal cost $C'(x)$ changes from decreasing to increasing. Thus, the marginal cost is minimized at the inflection point.

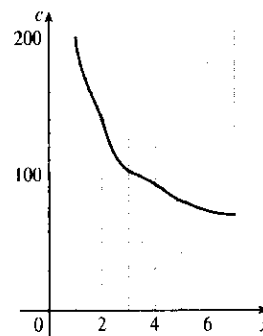
- (c) The marginal cost function is $C'(x)$. We graph it as in Example 1 in Section 3.2.



2. (a) We graph C' as in Example 1 in Section 3.2.



- (b) By reading values of $C(x)$ from its graph, we can plot $c(x) = C(x)/x$.

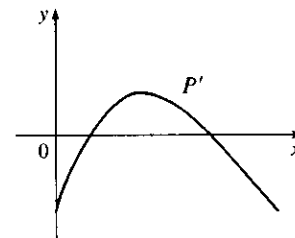
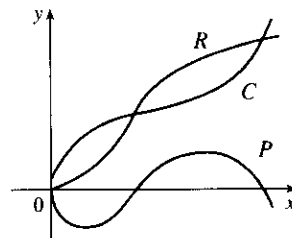
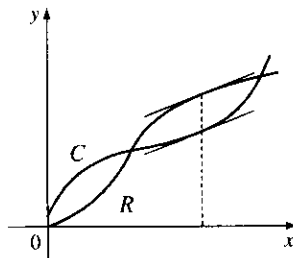


- (c) Since the graph in part (b) is decreasing, we estimate that the minimum value of $c(x)$ occurs at $x = 7$. The average cost and the marginal cost are equal at that value. See the box preceding Example 1.
3. $c(x) = 21.4 - 0.002x$ and $c(x) = C(x)/x \Rightarrow C(x) = 21.4x - 0.002x^2$. $C'(x) = 21.4 - 0.004x$ and $C'(1000) = 17.4$. This means that the cost of producing the 1001st unit is about \$17.40.

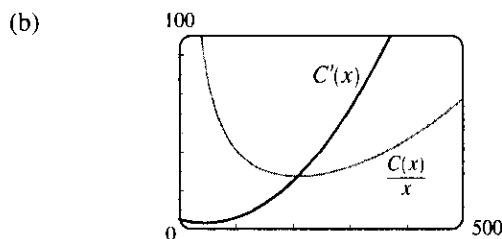
4. (a) Profit is maximized when the marginal revenue is equal to the marginal cost; that is, when R and C have equal slopes. See the box preceding Example 2.

- (b) $P(x) = R(x) - C(x)$ is sketched.

- (c) The marginal profit function is defined as $P'(x)$.



5. (a) The cost function is $C(x) = 40,000 + 300x + x^2$, so the cost at a production level of 1000 is $C(1000) = \$1,340,000$. The average cost function is $c(x) = \frac{C(x)}{x} = \frac{40,000}{x} + 300 + x$ and $c(1000) = \$1340/\text{unit}$. The marginal cost function is $C'(x) = 300 + 2x$ and $C'(1000) = \$2300/\text{unit}$.
- (b) See the box preceding Example 1. We must have $C'(x) = c(x) \Leftrightarrow 300 + 2x = \frac{40,000}{x} + 300 + x \Leftrightarrow x = \frac{40,000}{x} \Rightarrow x^2 = 40,000 \Rightarrow x = \sqrt{40,000} = 200$. This gives a minimum value of the average cost function $c(x)$ since $c''(x) = \frac{80,000}{x^3} > 0$.
- (c) The minimum average cost is $c(200) = \$700/\text{unit}$.
6. (a) $C(x) = 25,000 + 120x + 0.1x^2$, $C(1000) = \$245,000$. $c(x) = \frac{C(x)}{x} = \frac{25,000}{x} + 120 + 0.1x$, $c(1000) = \$245/\text{unit}$. $C'(x) = 120 + 0.2x$, $C'(1000) = \$320/\text{unit}$.
- (b) We must have $C'(x) = c(x) \Leftrightarrow 120 + 0.2x = \frac{25,000}{x} + 120 + 0.1x \Leftrightarrow 0.1x = \frac{25,000}{x} \Rightarrow 0.1x^2 = 25,000 \Rightarrow x = \sqrt{250,000} = 500$. This gives a minimum since $c''(x) = \frac{50,000}{x^3} > 0$.
- (c) The minimum average cost is $c(500) = \$220.00/\text{unit}$.
7. (a) $C(x) = 16,000 + 200x + 4x^{3/2}$, $C(1000) = 16,000 + 200,000 + 40,000\sqrt{10} \approx 216,000 + 126,491$, so $C(1000) \approx \$342,491$. $c(x) = C(x)/x = \frac{16,000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$, $C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.
- (b) We must have $C'(x) = c(x) \Leftrightarrow 200 + 6x^{1/2} = \frac{16,000}{x} + 200 + 4x^{1/2} \Leftrightarrow 2x^{3/2} = 16,000 \Leftrightarrow x = (8,000)^{2/3} = 400$ units. To check that this is a minimum, we calculate $c'(x) = \frac{-16,000}{x^2} + \frac{2}{\sqrt{x}} = \frac{2}{x^2}(x^{3/2} - 8000)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$, and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [Note: $c''(x)$ is not positive for all $x > 0$.]
- (c) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.
8. (a) $C(x) = 10,000 + 340x - 0.3x^2 + 0.0001x^3$, $C(1000) = \$150,000$. $c(x) = C(x)/x = \frac{10,000}{x} + 340 - 0.3x + 0.0001x^2$, $c(1000) = \$150/\text{unit}$. $C'(x) = 340 - 0.6x + 0.0003x^2$, $C'(1000) = \$40/\text{unit}$.
- (b) We must have $C'(x) = c(x) \Leftrightarrow 340 - 0.6x + 0.0003x^2 = \frac{10,000}{x} + 340 - 0.3x + 0.0001x^2 \Leftrightarrow 0.0002x^2 = \frac{10,000}{x} + 0.3x \Leftrightarrow 0.0002x^3 - 0.3x^2 - 10,000 = 0 \Leftrightarrow x^3 - 1500x^2 - 50,000,000 = 0 \Rightarrow x \approx 1521.60 \approx 1522$ units. This gives a minimum since $c''(x) = \frac{20,000}{x^3} + 0.0002 > 0$.
- (c) The minimum average cost is about $c(1521.60) \approx \$121.62/\text{unit}$.
9. (a) $C(x) = 3700 + 5x - 0.04x^2 + 0.0003x^3 \Rightarrow C'(x) = 5 - 0.08x + 0.0009x^2$ (marginal cost). $c(x) = \frac{C(x)}{x} = \frac{3700}{x} + 5 - 0.04x + 0.0003x^2$ (average cost).

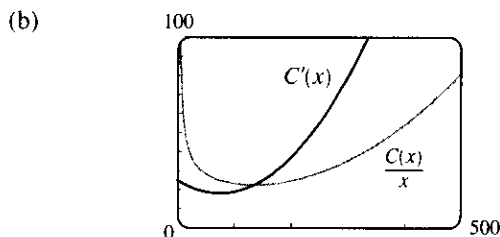


The graphs intersect at (208.51, 27.45), so the production level that minimizes average cost is about 209 units.

(c) $c'(x) = -\frac{3700}{x^2} - 0.04 + 0.0006x = 0 \Rightarrow 3700 + 0.04x^2 - 0.0006x^3 = 0 \Rightarrow x_1 \approx 208.51$.
 $c(x_1) \approx \$27.45/\text{unit}$.

(d) The marginal cost is given by $C'(x)$, so to find its minimum value we'll find the derivative of C' ; that is, C'' .
 $C''(x) = -0.08 + 0.0018x = 0 \Rightarrow x_1 = \frac{800}{18} = 44.4\bar{4}$. $C'(x_1) = \$3.22/\text{unit}$.
 $C'''(x) = 0.0018 > 0$ for all x , so this is the minimum marginal cost. C''' is the second derivative of C' .

10. (a) $C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3 \Rightarrow C'(x) = 25 - 0.18x + 0.0012x^2$ (marginal cost).
 $c(x) = \frac{C(x)}{x} = \frac{339}{x} + 25 - 0.09x + 0.0004x^2$ (average cost).



The graphs intersect at (135.56, 22.65), so the production level that minimizes average cost is about 136 units.

(c) $c'(x) = -\frac{339}{x^2} - 0.09 + 0.0008x = 0 \Rightarrow x_1 \approx 135.56$. $c(x_1) \approx \$22.65/\text{unit}$.

(d) $C''(x) = -0.18 + 0.0024x = 0 \Rightarrow x = \frac{1800}{24} = 75$. $C'(75) = \$18.25/\text{unit}$.
 $C'''(x) = 0.0024 > 0$ for all x , so this is the minimum marginal cost.

11. $C(x) = 680 + 4x + 0.01x^2$, $p(x) = 12 \Rightarrow R(x) = xp(x) = 12x$. If the profit is maximum, then
 $R'(x) = C'(x) \Rightarrow 12 = 4 + 0.02x \Rightarrow 0.02x = 8 \Rightarrow x = 400$. The profit is maximized if $P''(x) < 0$,
but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now
 $R''(x) = 0 < 0.02 = C''(x)$, so $x = 400$ gives a maximum.

12. $C(x) = 680 + 4x + 0.01x^2$, $p(x) = 12 - x/500$. Then $R(x) = xp(x) = 12x - x^2/500$. If the profit is maximum, then $R'(x) = C'(x)$ [See the box preceding Example 2.] $\Leftrightarrow 12 - x/250 = 4 + 0.02x \Leftrightarrow 8 = 0.024x \Leftrightarrow x = 8/0.024 = \frac{1000}{3}$. The profit is maximized if $P''(x) < 0$, but since
 $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now
 $R''(x) = -\frac{1}{250} < 0.02 = C''(x)$, so $x = \frac{1000}{3}$ gives a maximum.

13. $C(x) = 1450 + 36x - x^2 + 0.001x^3$, $p(x) = 60 - 0.01x$. Then $R(x) = xp(x) = 60x - 0.01x^2$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 60 - 0.02x = 36 - 2x + 0.003x^2 \Rightarrow 0.003x^2 - 1.98x - 24 = 0$. By the quadratic formula, $x = \frac{1.98 \pm \sqrt{(-1.98)^2 + 4(0.003)(24)}}{2(0.003)} = \frac{1.98 \pm \sqrt{4.2084}}{0.006}$. Since $x > 0$,
 $x \approx (1.98 + 2.05)/0.006 \approx 672$. Now $R''(x) = -0.02$ and $C''(x) = -2 + 0.006x \Rightarrow C''(672) = 2.032$
 $\Rightarrow R''(672) < C''(672) \Rightarrow$ there is a maximum at $x = 672$.

14. $C(x) = 16,000 + 500x - 1.6x^2 + 0.004x^3$, $p(x) = 1700 - 7x$. Then $R(x) = xp(x) = 1700x - 7x^2$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 1700 - 14x = 500 - 3.2x + 0.012x^2 \Leftrightarrow 0.012x^2 + 10.8x - 1200 = 0 \Leftrightarrow x^2 + 900x - 100,000 = 0 \Leftrightarrow (x + 1000)(x - 100) = 0 \Leftrightarrow x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -14 < -3.2 + 0.024x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

15. $C(x) = 0.001x^3 - 0.3x^2 + 6x + 900$. The marginal cost is $C'(x) = 0.003x^2 - 0.6x + 6$.

$C'(x)$ is increasing when $C''(x) > 0 \Leftrightarrow 0.006x - 0.6 > 0 \Leftrightarrow x > 0.6/0.006 = 100$. So $C'(x)$ starts to increase when $x = 100$.

16. $C(x) = 0.0002x^3 - 0.25x^2 + 4x + 1500$. The marginal cost is $C'(x) = 0.0006x^2 - 0.50x + 4$.

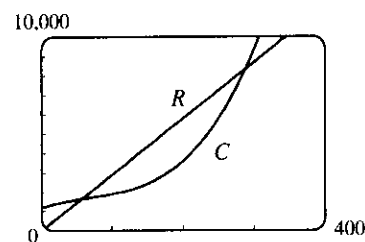
$C'(x)$ is increasing when $C''(x) > 0 \Leftrightarrow 0.0012x - 0.5 > 0 \Leftrightarrow x > 0.5/0.0012 \approx 417$. So $C'(x)$ starts to increase when $x = 417$.

17. (a) $C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$.

$$R(x) = xp(x) = 29x - 0.00021x^2.$$

Since the profit is maximized when $R'(x) = C'(x)$,

we examine the curves R and C in the figure, looking for x -values at which the slopes of the tangent lines are equal. It appears that $x = 200$ is a good estimate.



- (b) $R'(x) = C'(x) \Rightarrow 29 - 0.00042x = 12 - 0.2x + 0.0015x^2 \Rightarrow 0.0015x^2 - 0.19958x - 17 = 0 \Rightarrow x \approx 192.06$ (for $x > 0$). As in Exercise 11, $R''(x) < C''(x) \Rightarrow -0.00042 < -0.2 + 0.003x \Leftrightarrow 0.003x > 0.19958 \Leftrightarrow x > 66.5$. Our value of 192 is in this range, so we have a maximum profit when we produce 192 yards of fabric.

18. (a) Cost = setup cost + manufacturing cost $\Rightarrow C(x) = 500 + m(x) = 500 + 20x - 5x^{3/4} + 0.01x^2$.

We can solve $x(p) = 320 - 7.7p$ for p in terms of x to find the demand (or price) function.

$$x = 320 - 7.7p \Rightarrow 7.7p = 320 - x \Rightarrow p(x) = \frac{320 - x}{7.7}, \quad R(x) = xp(x) = \frac{320x - x^2}{7.7}.$$

- (b) $C'(x) = R'(x) \Rightarrow 20 - \frac{15}{4}x^{-1/4} + 0.02x = \frac{320 - 2x}{7.7} \Rightarrow x \approx 81.53$ planes, and $p(x) = \$30.97$ million. The maximum profit associated with these values is about \$463.59 million.

19. (a) We are given that the demand function p is linear and $p(27,000) = 10$, $p(33,000) = 8$, so the slope is

$$\frac{10 - 8}{27,000 - 33,000} = -\frac{1}{3000} \text{ and an equation of the line is } y - 10 = \left(-\frac{1}{3000}\right)(x - 27,000) \Rightarrow y = p(x) = -\frac{1}{3000}x + 19 = 19 - (x/3000).$$

- (b) The revenue is $R(x) = xp(x) = 19x - (x^2/3000) \Rightarrow R'(x) = 19 - (x/1500) = 0$ when $x = 28,500$.

Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28,500 \Rightarrow$ the price is $p(28,500) = \$9.50$.

20. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \Leftrightarrow y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14$
 $\Rightarrow x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \13 .

21. (a) As in Example 3, we see that the demand function p is linear. We are given that $p(1000) = 450$ and deduce that $p(1100) = 440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is
 $\frac{440 - 450}{1100 - 1000} = -\frac{1}{10}$, so an equation is $p - 450 = -\frac{1}{10}(x - 1000)$ or $p(x) = -\frac{1}{10}x + 550$.

(b) $R(x) = xp(x) = -\frac{1}{10}x^2 + 550x$. $R'(x) = -\frac{1}{5}x + 550 = 0$ when $x = 5(550) = 2750$.
 $p(2750) = 275$, so the rebate should be $450 - 275 = \$175$.

(c) $C(x) = 68,000 + 150x \Rightarrow$
 $P(x) = R(x) - C(x) = -\frac{1}{10}x^2 + 550x - 68,000 - 150x = -\frac{1}{10}x^2 + 400x - 68,000$,
 $P'(x) = -\frac{1}{5}x + 400 = 0$ when $x = 2000$. $p(2000) = 350$. Therefore, the rebate to maximize profits should be $450 - 350 = \$100$.

22. Let x denote the number of \$10 increases in rent. Then the price is $p(x) = 800 + 10x$, and the number of units occupied is $100 - x$. Now the revenue is

$$\begin{aligned} R(x) &= (\text{rental price per unit}) \times (\text{number of units rented}) \\ &= (800 + 10x)(100 - x) = -10x^2 + 200x + 80,000 \text{ for } 0 \leq x \leq 100 \Rightarrow \end{aligned}$$

$R'(x) = -20x + 200 = 0 \Leftrightarrow x = 10$. This is a maximum since $R''(x) = -20 < 0$ for all x . Now we must check the value of $R(x) = (800 + 10x)(100 - x)$ at $x = 10$ and at the endpoints of the domain to see which value of x gives the maximum value of R . $R(0) = 80,000$, $R(10) = (900)(90) = 81,000$, and
 $R(100) = (1800)(0) = 0$. Thus, the maximum revenue of \$81,000/week occurs when 90 units are occupied at a rent of \$900/week.

23. If the reorder quantity is x , then the manager places $\frac{800}{x}$ orders per year. Storage costs for the year are

$\frac{1}{2}x \cdot 4 = 2x$ dollars. Handling costs are \$100 per delivery, for a total of $\frac{800}{x} \cdot 100 = \frac{80,000}{x}$ dollars. The total costs for the year are $C(x) = 2x + \frac{80,000}{x}$. To minimize $C(x)$, we calculate

$C'(x) = 2 - \frac{80,000}{x^2} = \frac{2}{x^2}(x^2 - 40,000)$. This is negative when $x < 200$, zero when $x = 200$, and positive when $x > 200$, so C is decreasing on $(0, 200)$ and increasing on $(200, \infty)$, reaching its absolute minimum when $x = 200$. Thus, the optimal reorder quantity is 200 cases. The manager will place 4 orders per year for a total cost of $C(200) = \$800$.

24. She will have A/n dollars after each withdrawal and 0 dollars just before the next withdrawal, so her average cash balance at any given time is $\frac{1}{2}(A/n + 0) = A/(2n)$. The transaction costs for n withdrawals are nT . The lost interest cost on the average cash balance is $[A/(2n)] \cdot R$. Thus, the total cost for n transactions is

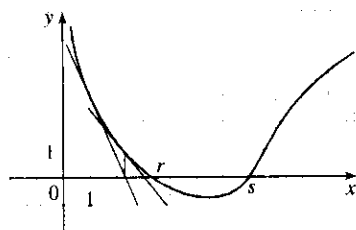
$$C(n) = nT + \frac{AR}{2n}. \text{ Now } C'(n) = T - \frac{AR}{2n^2} \text{ and } C'(n) = 0 \Rightarrow \frac{AR}{2n^2} = T \Rightarrow n^2 = \frac{AR}{2T} \Rightarrow$$

$n = \sqrt{\frac{AR}{2T}}$, the value of n that minimizes total costs since $C''(n) = -\frac{AR}{n^3} < 0$. Thus, the optimal average cash

$$\text{balance is } \frac{A}{2n} = \frac{A\sqrt{2T}}{2\sqrt{AR}} = \frac{\sqrt{AT}}{\sqrt{2R}} = \sqrt{\frac{AT}{2R}}.$$

4.9 Newton's Method

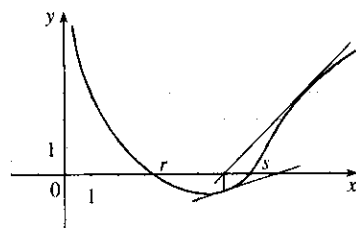
1. (a)



The tangent line at $x = 1$ intersects the x -axis at $x \approx 2.3$, so $x_2 \approx 2.3$. The tangent line at $x = 2.3$ intersects the x -axis at $x \approx 3$, so $x_3 \approx 3.0$.

(b) $x_1 = 5$ would *not* be a better first approximation than $x_1 = 1$ since the tangent line is nearly horizontal. In fact, the second approximation for $x_1 = 5$ appears to be to the left of $x = 1$.

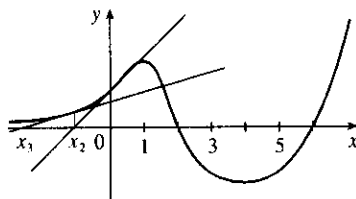
2.



The tangent line at $x = 9$ intersects the x -axis at $x \approx 6.0$, so $x_2 \approx 6.0$. The tangent line at $x = 6.0$ intersects the x -axis at $x \approx 8.0$, so $x_3 \approx 8.0$.

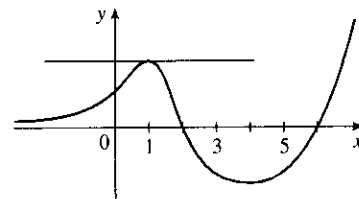
3. Since $x_1 = 3$ and $y = 5x - 4$ is tangent to $y = f(x)$ at $x = 3$, we simply need to find where the tangent line intersects the x -axis. $y = 0 \Rightarrow 5x_2 - 4 = 0 \Rightarrow x_2 = \frac{4}{5}$.

4. (a)



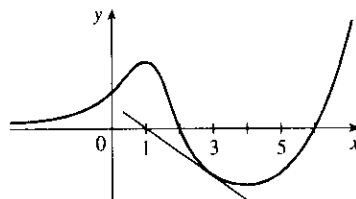
If $x_1 = 0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.

(b)



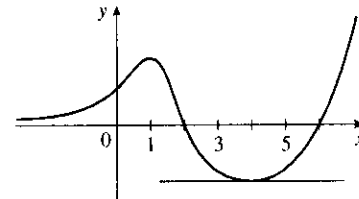
If $x_1 = 1$, the tangent line is horizontal and Newton's method fails.

(c)



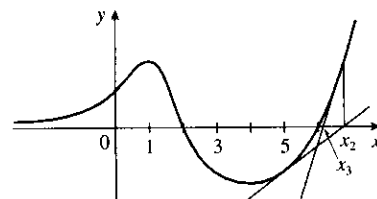
If $x_1 = 3$, then $x_2 = 1$ and we have the same situation as in part (b). Newton's method fails again.

(d)



If $x_1 = 4$, the tangent line is horizontal and Newton's method fails.

(e)



If $x_1 = 5$, then x_2 is greater than 6, x_3 gets closer to 6, and the sequence of approximations converges to 6. Newton's method succeeds!

$$5. f(x) = x^3 + 2x - 4 \Rightarrow f'(x) = 3x^2 + 2, \text{ so } x_{n+1} = x_n - \frac{x_n^3 + 2x_n - 4}{3x_n^2 + 2}. \text{ Now } x_1 = 1 \Rightarrow$$

$$x_2 = 1 - \frac{1 + 2 - 4}{3 \cdot 1^2 + 2} = 1 - \frac{-1}{5} = 1.2 \Rightarrow x_3 = 1.2 - \frac{(1.2)^3 + 2(1.2) - 4}{3(1.2)^2 + 2} \approx 1.1797.$$

$$6. f(x) = x^3 - x^2 - 1 \Rightarrow f'(x) = 3x^2 - 2x, \text{ so } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}.$$

$$\text{Now } x_1 = 1 \Rightarrow x_2 = 1 - \frac{1 - 1 - 1}{3 - 2} = 2 \Rightarrow x_3 = 2 - \frac{2^3 - 2^2 - 1}{3 \cdot 2^2 - 2 \cdot 2} = 1.625.$$

$$7. f(x) = x^4 - 20 \Rightarrow f'(x) = 4x^3, \text{ so } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 20}{4x_n^3}.$$

$$\text{Now } x_1 = 2 \Rightarrow x_2 = 2 - \frac{2^4 - 20}{4(2)^3} = 2.125 \Rightarrow x_3 = 2.125 - \frac{(2.125)^4 - 20}{4(2.125)^3} \approx 2.1148.$$

$$8. f(x) = x^5 + 2 \Rightarrow f'(x) = 5x^4, \text{ so } x_{n+1} = x_n - \frac{x_n^5 + 2}{5x_n^4}. \text{ Now } x_1 = -1 \Rightarrow$$

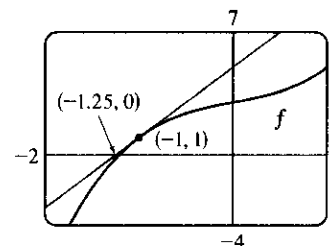
$$x_2 = -1 - \frac{(-1)^5 + 2}{5 \cdot (-1)^4} = -1 - \frac{1}{5} = -1.2 \Rightarrow x_3 = -1.2 - \frac{(-1.2)^5 + 2}{5(-1.2)^4} \approx -1.1529.$$

$$9. f(x) = x^3 + x + 3 \Rightarrow f'(x) = 3x^2 + 1, \text{ so}$$

$$x_{n+1} = x_n - \frac{x_n^3 + x_n + 3}{3x_n^2 + 1}. \text{ Now } x_1 = -1 \Rightarrow$$

$$x_2 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{-1 - 1 + 3}{3 + 1} = -1 - \frac{1}{4} = -1.25.$$

Newton's method follows the tangent line at $(-1, 1)$ down to its intersection with the x -axis at $(-1.25, 0)$, giving the second approximation $x_2 = -1.25$.

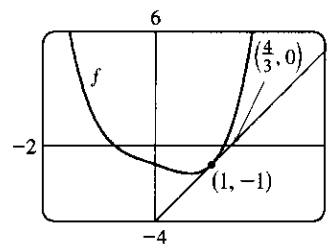


$$10. f(x) = x^4 - x - 1 \Rightarrow f'(x) = 4x^3 - 1, \text{ so}$$

$$x_{n+1} = x_n - \frac{x_n^4 - x_n - 1}{4x_n^3 - 1}. \text{ Now } x_1 = 1 \Rightarrow$$

$$x_2 = 1 - \frac{1^4 - 1 - 1}{4 \cdot 1^3 - 1} = 1 - \frac{-1}{3} = \frac{4}{3}. \text{ Newton's method follows the}$$

tangent line at $(1, -1)$ up to its intersection with the x -axis at $(\frac{4}{3}, 0)$, giving the second approximation $x_2 = \frac{4}{3}$.



$$11. \text{ To approximate } x = \sqrt[3]{30} \text{ (so that } x^3 = 30), \text{ we can take } f(x) = x^3 - 30. \text{ So } f'(x) = 3x^2, \text{ and thus,}$$

$$x_{n+1} = x_n - \frac{x_n^3 - 30}{3x_n^2}. \text{ Since } \sqrt[3]{27} = 3 \text{ and } 27 \text{ is close to } 30, \text{ we'll use } x_1 = 3. \text{ We need to find approximations}$$

$$\text{until they agree to eight decimal places. } x_1 = 3 \Rightarrow x_2 \approx 3.11111111, x_3 \approx 3.10723734,$$

$$x_4 \approx 3.10723251 \approx x_5. \text{ So } \sqrt[3]{30} \approx 3.10723251, \text{ to eight decimal places.}$$

Here is a quick and easy method for finding the iterations for Newton's method on a programmable calculator. (The screens shown are from the TI-83 Plus, but the method is similar on other calculators.) Assign $f(x) = x^3 - 30$ to Y_1 , and $f'(x) = 3x^2$ to Y_2 . Now store $x_1 = 3$ in X and then enter $X - Y_1/Y_2 \rightarrow X$ to get $x_2 = 3.\bar{1}$. By successively pressing the ENTER key, you get the approximations $x_3, x_4, \dots$

```

Plot1 Plot2 Plot3
Y1=X^3-30
Y2=3X^2
Y3=
Y4=
Y5=
Y6=
Y7=

```

```

3→X
X-Y1/Y2→X
3.111111111
3.107237339
3.107232506
3.107232506

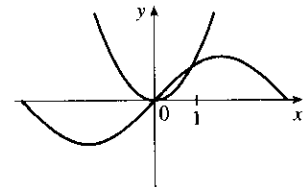
```

In Derive, load the utility file SOLVE. Enter $\text{NEWTON}(x^3 - 30, x, 3)$ and then APPROXIMATE to get $[3, 3.11111111, 3.10723733, 3.10723250, 3.10723250]$. You can request a specific iteration by adding a fourth argument. For example, $\text{NEWTON}(x^3 - 30, x, 3, 2)$ gives $[3, 3.11111111, 3.10723733]$.

In Maple, make the assignments $f := x \rightarrow x^3 - 30$; $g := x \rightarrow x - f(x)/D(f)(x)$; and $x := 3$;. Repeatedly execute the command $x := g(x)$; to generate successive approximations.

In Mathematica, make the assignments $f[x_] := x^3 - 30$, $g[x_] := x - f[x]/f'[x]$, and $x = 3$. Repeatedly execute the command $x = g[x]$ to generate successive approximations.

12. $f(x) = x^7 - 1000 \Rightarrow f'(x) = 7x^6$, so $x_{n+1} = x_n - \frac{x_n^7 - 1000}{7x_n^6}$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \Rightarrow x_2 \approx 2.76739173$, $x_3 \approx 2.69008741$, $x_4 \approx 2.68275645$, $x_5 \approx 2.68269580 \approx x_6$. So $\sqrt[7]{1000} \approx 2.68269580$, to eight decimal places.
13. $f(x) = 2x^3 - 6x^2 + 3x + 1 \Rightarrow f'(x) = 6x^2 - 12x + 3 \Rightarrow x_{n+1} = x_n - \frac{2x_n^3 - 6x_n^2 + 3x_n + 1}{6x_n^2 - 12x_n + 3}$. We need to find approximations until they agree to six decimal places. $x_1 = 2.5 \Rightarrow x_2 \approx 2.285714$, $x_3 \approx 2.228824$, $x_4 \approx 2.224765$, $x_5 \approx 2.224745 \approx x_6$. So the root is 2.224745, to six decimal places.
14. $f(x) = x^4 + x - 4 \Rightarrow f'(x) = 4x^3 + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 + x_n - 4}{4x_n^3 + 1}$. $x_1 = 1.5 \Rightarrow x_2 \approx 1.323276$, $x_3 \approx 1.285346$, $x_4 \approx 1.283784$, $x_5 \approx 1.283782 \approx x_6$. So the root is 1.283782, to six decimal places.
15. $\sin x = x^2$, so $f(x) = \sin x - x^2 \Rightarrow f'(x) = \cos x - 2x \Rightarrow x_{n+1} = x_n - \frac{\sin x_n - x_n^2}{\cos x_n - 2x_n}$. From the figure, the positive root of $\sin x = x^2$ is near 1. $x_1 = 1 \Rightarrow x_2 \approx 0.891396$, $x_3 \approx 0.876985$, $x_4 \approx 0.876726 \approx x_5$. So the positive root is 0.876726, to six decimal places.

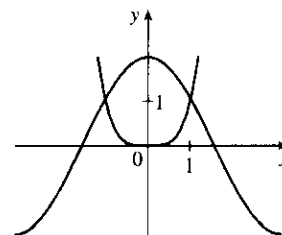


16. $2 \cos x = x^4$, so $f(x) = 2 \cos x - x^4 \Rightarrow f'(x) = -2 \sin x - 4x^3$

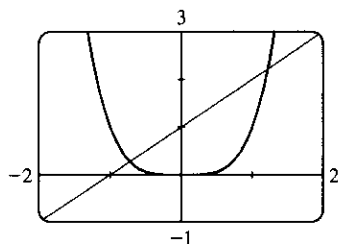
$$\Rightarrow x_{n+1} = x_n - \frac{2 \cos x_n - x_n^4}{-2 \sin x_n - 4x_n^3}. \text{ From the figure, the positive root}$$

of $2 \cos x = x^4$ is near 1. $x_1 = 1 \Rightarrow x_2 \approx 1.014184$,

$x_3 \approx 1.013958 \approx x_4$. So the positive root is 1.013958, to six decimal places.



17.



From the graph, we see that there appear to be points of intersection near

$x = -0.7$ and $x = 1.2$. Solving $x^4 = 1 + x$ is the same as solving

$$f(x) = x^4 - x - 1 = 0. f(x) = x^4 - x - 1 \Rightarrow f'(x) = 4x^3 - 1,$$

$$\text{so } x_{n+1} = x_n - \frac{x_n^4 - x_n - 1}{4x_n^3 - 1}.$$

$$x_1 = -0.7$$

$$x_2 \approx -0.725253$$

$$x_3 \approx -0.724493$$

$$x_4 \approx -0.724492 \approx x_5$$

$$x_1 = 1.2$$

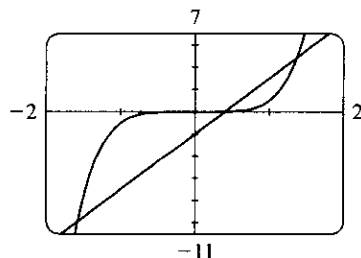
$$x_2 \approx 1.221380$$

$$x_3 \approx 1.220745$$

$$x_4 \approx 1.220744 \approx x_5$$

To six decimal places, the roots of the equation are -0.724492 and 1.220744 .

18.



From the graph, we see that reasonable first approximations are $x = 0.5$

and $x = \pm 1.5$. $f(x) = x^5 - 5x + 2 \Rightarrow f'(x) = 5x^4 - 5$, so

$$x_{n+1} = x_n - \frac{x_n^5 - 5x_n + 2}{5x_n^4 - 5}.$$

$$x_1 = -1.5$$

$$x_2 \approx -1.593846$$

$$x_3 \approx -1.582241$$

$$x_4 \approx -1.582036 \approx x_5$$

$$x_1 = 0.5$$

$$x_2 \approx 0.4$$

$$x_3 \approx 0.402102 \approx x_4$$

$$x_1 = 1.5$$

$$x_2 \approx 1.396923$$

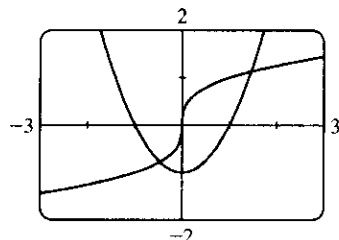
$$x_3 \approx 1.373078$$

$$x_4 \approx 1.371885$$

$$x_5 \approx 1.371882 \approx x_6$$

To six decimal places, the roots are -1.582036 , 0.402102 , and 1.371882 .

19.



From the graph, we see that there appear to be points of intersection near

$x = -0.5$ and $x = 1.5$. Solving $\sqrt[3]{x} = x^2 - 1$ is the same as solving

$$f(x) = \sqrt[3]{x} - x^2 + 1 = 0. f(x) = \sqrt[3]{x} - x^2 + 1 \Rightarrow$$

$$f'(x) = \frac{1}{3}x^{-2/3} - 2x, \text{ so } x_{n+1} = x_n - \frac{\sqrt[3]{x_n} - x_n^2 + 1}{\frac{1}{3}x_n^{-2/3} - 2x_n}.$$

$$x_1 = -0.5$$

$$x_2 \approx -0.471421$$

$$x_3 \approx -0.471074 \approx x_4$$

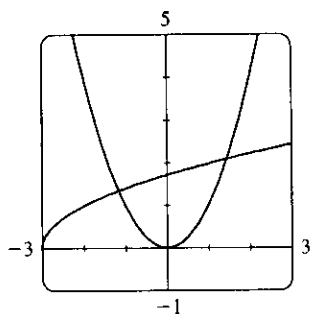
$$x_1 = 1.5$$

$$x_2 \approx 1.461653$$

$$x_3 \approx 1.461070 \approx x_4$$

To six decimal places, the roots are -0.471074 and 1.461070 .

20.



From the graph, we see that there appear to be points of intersection near

$x = -1.2$ and $x = 1.5$. Solving $\sqrt{x+3} = x^2$ is the same as solving

$$f(x) = x^2 - \sqrt{x+3} = 0. \quad f'(x) = 2x - \frac{1}{2\sqrt{x+3}} \Rightarrow$$

$$f'(x) = 2x - \frac{1}{2\sqrt{x+3}}, \text{ so } x_{n+1} = x_n - \frac{x_n^2 - \sqrt{x_n+3}}{2x_n - 1/(2\sqrt{x_n+3})}.$$

$$\begin{array}{ll} x_1 = -1.2 & x_1 = 1.5 \\ x_2 \approx -1.164526 & x_2 \approx 1.453449 \\ x_3 \approx -1.164035 \approx x_4 & x_3 \approx 1.452627 \approx x_4 \end{array}$$

To six decimal places, the roots of the equation are -1.164035 and 1.452627 .

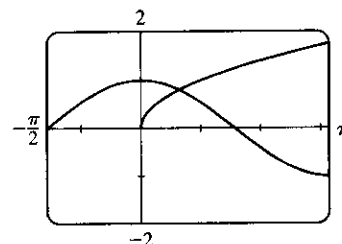
21. From the graph, there appears to be a point of intersection near $x = 0.6$.

Solving $\cos x = \sqrt{x}$ is the same as solving $f(x) = \cos x - \sqrt{x} = 0$.

$$f(x) = \cos x - \sqrt{x} \Rightarrow f'(x) = -\sin x - 1/(2\sqrt{x}), \text{ so}$$

$$x_{n+1} = x_n - \frac{\cos x_n - \sqrt{x_n}}{-\sin x_n - 1/(2\sqrt{x_n})}. \text{ Now } x_1 = 0.6 \Rightarrow$$

$x_2 \approx 0.641928, x_3 \approx 0.641714 \approx x_4$. To six decimal places, the root of the equation is 0.641714 .



22. From the graph, there appears to be a point of intersection near $x = 0.7$.

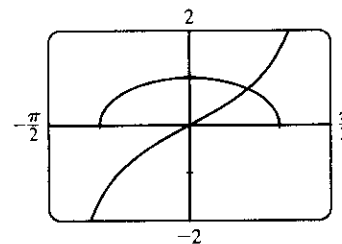
Solving $\tan x = \sqrt{1-x^2}$ is the same as solving

$$f(x) = \tan x - \sqrt{1-x^2} = 0. \quad f'(x) = \tan x - \sqrt{1-x^2} \Rightarrow$$

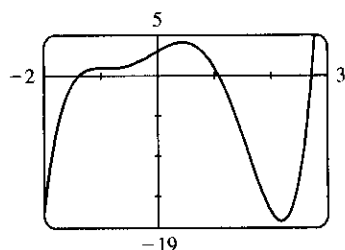
$$f'(x) = \sec^2 x + x/\sqrt{1-x^2}, \text{ so } x_{n+1} = x_n - \frac{\tan x_n - \sqrt{1-x_n^2}}{\sec^2 x_n + x_n/\sqrt{1-x_n^2}}.$$

$$x_1 = 0.7 \Rightarrow x_2 \approx 0.652356, x_3 \approx 0.649895, x_4 \approx 0.649889 \approx x_5.$$

To six decimal places, the root of the equation is 0.649889 .



23.



$$f(x) = x^5 - x^4 - 5x^3 - x^2 + 4x + 3 \Rightarrow$$

$$f'(x) = 5x^4 - 4x^3 - 15x^2 - 2x + 4 \Rightarrow$$

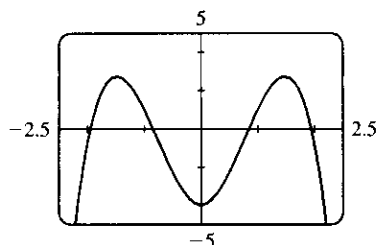
$$x_{n+1} = x_n - \frac{x_n^5 - x_n^4 - 5x_n^3 - x_n^2 + 4x_n + 3}{5x_n^4 - 4x_n^3 - 15x_n^2 - 2x_n + 4}. \text{ From the graph of } f,$$

there appear to be roots near $-1.4, 1.1,$ and 2.7 .

$$\begin{array}{lll} x_1 = -1.4 & x_1 = 1.1 & x_1 = 2.7 \\ x_2 \approx -1.39210970 & x_2 \approx 1.07780402 & x_2 \approx 2.72046250 \\ x_3 \approx -1.39194698 & x_3 \approx 1.07739442 & x_3 \approx 2.71987870 \\ x_4 \approx -1.39194691 \approx x_5 & x_4 \approx 1.07739428 \approx x_5 & x_4 \approx 2.71987822 \approx x_5 \end{array}$$

To eight decimal places, the roots of the equation are $-1.39194691, 1.07739428,$ and 2.71987822 .

24.



Solving $x^2(4 - x^2) = \frac{4}{x^2 + 1}$ is the same as solving

$$f(x) = 4x^2 - x^4 - \frac{4}{x^2 + 1} = 0. \quad f'(x) = 8x - 4x^3 + \frac{8x}{(x^2 + 1)^2} \Rightarrow$$

$$x_{n+1} = x_n - \frac{4x_n^2 - x_n^4 - 4/(x_n^2 + 1)}{8x_n - 4x_n^3 + 8x_n/(x_n^2 + 1)^2}. \quad \text{From the graph of } f(x),$$

there appear to be roots near $x = \pm 1.9$ and $x = \pm 0.8$. Since f is even, we only need to find the positive roots.

$$x_1 = 0.8$$

$$x_1 = 1.9$$

$$x_2 \approx 0.84287645$$

$$x_2 \approx 1.94689103$$

$$x_3 \approx 0.84310820$$

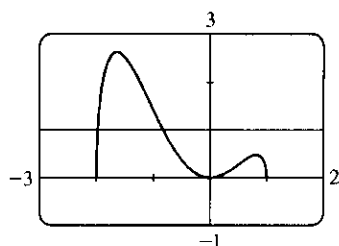
$$x_3 \approx 1.94383891$$

$$x_4 \approx 0.84310821 \approx x_5$$

$$x_4 \approx 1.94382538 \approx x_5$$

To eight decimal places, the roots of the equation are ± 0.84310821 and ± 1.94382538 .

25.



From the graph, $y = x^2\sqrt{2 - x - x^2}$ and $y = 1$ intersect twice, at

$$x \approx -2 \text{ and at } x \approx -1. \quad f(x) = x^2\sqrt{2 - x - x^2} - 1 \Rightarrow$$

$$\begin{aligned} f'(x) &= x^2 \cdot \frac{1}{2}(2 - x - x^2)^{-1/2}(-1 - 2x) + (2 - x - x^2)^{1/2} \cdot 2x \\ &= \frac{1}{2}x(2 - x - x^2)^{-1/2} [x(-1 - 2x) + 4(2 - x - x^2)] \\ &= \frac{x(8 - 5x - 6x^2)}{2\sqrt{(2+x)(1-x)}}, \end{aligned}$$

so $x_{n+1} = x_n - \frac{x_n^2\sqrt{2 - x_n - x_n^2} - 1}{\frac{x_n(8 - 5x_n - 6x_n^2)}{2\sqrt{(2+x_n)(1-x_n)}}}$. Trying $x_1 = -2$ won't work because $f'(-2)$ is undefined, so we'll

try $x_1 = -1.95$.

$$x_1 = -1.95$$

$$x_1 = -0.8$$

$$x_2 \approx -1.98580357$$

$$x_2 \approx -0.82674444$$

$$x_3 \approx -1.97899778$$

$$x_3 \approx -0.82646236$$

$$x_4 \approx -1.97807848$$

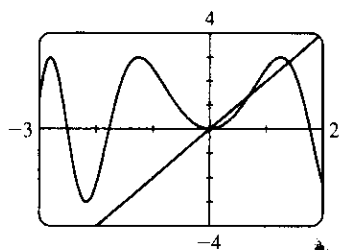
$$x_4 \approx -0.82646233 \approx x_5$$

$$x_5 \approx -1.97806682$$

$$x_6 \approx -1.97806681 \approx x_7$$

To eight decimal places, the roots of the equation are -1.97806681 and -0.82646233 .

26.



From the equations $y = 3 \sin(x^2)$ and $y = 2x$ and the graph, we deduce that one root of the equation $3 \sin(x^2) = 2x$ is $x = 0$. We also see that the graphs intersect at approximately $x = 0.7$ and $x = 1.4$.

$$f(x) = 3 \sin(x^2) - 2x \Rightarrow f'(x) = 3 \cos(x^2) \cdot 2x - 2, \text{ so}$$

$$x_{n+1} = x_n - \frac{3 \sin(x_n^2) - 2x_n}{6x_n \cos(x_n^2) - 2}.$$

$$x_1 = 0.7$$

$$x_1 = 1.4$$

$$x_2 \approx 0.69303689$$

$$x_2 \approx 1.39530295$$

$$x_3 \approx 0.69299996 \approx x_4$$

$$x_3 \approx 1.39525078$$

$$x_4 \approx 1.39525077 \approx x_5$$

To eight decimal places, the roots of the equation are 0.69299996 and 1.39525077 .

27. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x$, so Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

- (b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

28. (a) $f(x) = \frac{1}{x} - a \Rightarrow f'(x) = -\frac{1}{x^2}$, so $x_{n+1} = x_n - \frac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$.

- (b) Using (a) with $a = 1.6894$ and $x_1 = \frac{1}{2} = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$. So $1/1.6894 \approx 0.588789$.

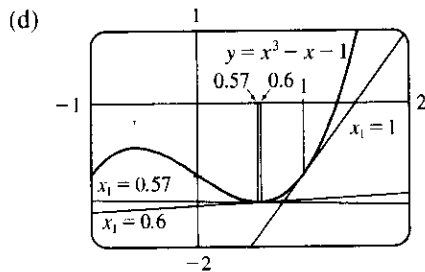
29. $f(x) = x^3 - 3x + 6 \Rightarrow f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

30. $x^3 - x = 1 \Leftrightarrow x^3 - x - 1 = 0$. $f(x) = x^3 - x - 1 \Rightarrow f'(x) = 3x^2 - 1$, so $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

(a) $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.347826$, $x_4 \approx 1.325200$, $x_5 \approx 1.324718 \approx x_6$

(b) $x_1 = 0.6$, $x_2 = 17.9$, $x_3 \approx 11.946802$, $x_4 \approx 7.985520$, $x_5 \approx 5.356909$, $x_6 \approx 3.624996$, $x_7 \approx 2.505589$, $x_8 \approx 1.820129$, $x_9 \approx 1.461044$, $x_{10} \approx 1.339323$, $x_{11} \approx 1.324913$, $x_{12} \approx 1.324718 \approx x_{13}$

(c) $x_1 = 0.57$, $x_2 \approx -54.165455$, $x_3 \approx -36.114293$, $x_4 \approx -24.082094$, $x_5 \approx -16.063387$, $x_6 \approx -10.721483$, $x_7 \approx -7.165534$, $x_8 \approx -4.801704$, $x_9 \approx -3.233425$, $x_{10} \approx -2.193674$, $x_{11} \approx -1.496867$, $x_{12} \approx -0.997546$, $x_{13} \approx -0.496305$, $x_{14} \approx -2.894162$, $x_{15} \approx -1.967962$, $x_{16} \approx -1.341355$, $x_{17} \approx -0.870187$, $x_{18} \approx -0.249949$, $x_{19} \approx -1.192219$, $x_{20} \approx -0.731952$, $x_{21} \approx 0.355213$, $x_{22} \approx -1.753322$, $x_{23} \approx -1.189420$, $x_{24} \approx -0.729123$, $x_{25} \approx 0.377844$, $x_{26} \approx -1.937872$, $x_{27} \approx -1.320350$, $x_{28} \approx -0.851919$, $x_{29} \approx -0.200959$, $x_{30} \approx -1.119386$, $x_{31} \approx -0.654291$, $x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.



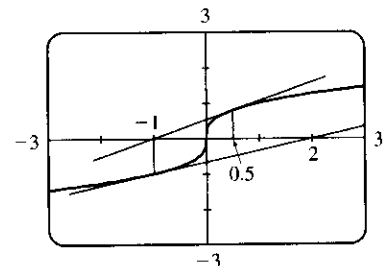
From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the root. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

31. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0. In the figure, we have $x_1 = 0.5$,

$$x_2 = -2(0.5) = -1, \text{ and } x_3 = -2(-1) = 2.$$

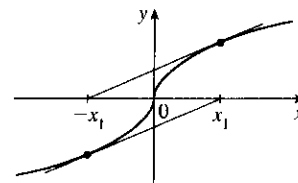


32. According to Newton's Method, for
- $x_n > 0$
- ,

$$x_{n+1} = x_n - \frac{\sqrt{x_n}}{1/(2\sqrt{x_n})} = x_n - 2x_n = -x_n \text{ and for } x_n < 0,$$

$$x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{1/(2\sqrt{-x_n})} = x_n - [-2(-x_n)] = -x_n. \text{ So we can}$$

see that after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the root.



33. (a)
- $f(x) = 3x^4 - 28x^3 + 6x^2 + 24x \Rightarrow f'(x) = 12x^3 - 84x^2 + 12x + 24 \Rightarrow$

$$f''(x) = 36x^2 - 168x + 12. \text{ Now to solve } f'(x) = 0, \text{ try } x_1 = \frac{1}{2} \Rightarrow x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = \frac{2}{3} \Rightarrow$$

$$x_3 \approx 0.6455 \Rightarrow x_4 \approx 0.6452 \Rightarrow x_5 \approx 0.6452. \text{ Now try } x_1 = 6 \Rightarrow x_2 = 7.12 \Rightarrow$$

$$x_3 \approx 6.8353 \Rightarrow x_4 \approx 6.8102 \Rightarrow x_5 \approx 6.8100. \text{ Finally try } x_1 = -0.5 \Rightarrow x_2 \approx -0.4571 \Rightarrow$$

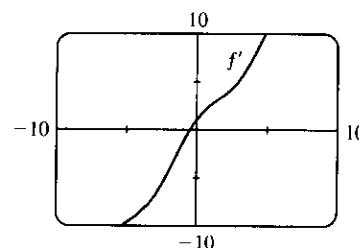
$$x_3 \approx -0.4552 \Rightarrow x_4 \approx -0.4552. \text{ Therefore, } x = -0.455, 6.810 \text{ and } 0.645 \text{ are all critical numbers correct to three decimal places.}$$

- (b) $f(-1) = 13, f(7) = -1939, f(6.810) \approx -1949.07, f(-0.455) \approx -6.912, f(0.645) \approx 10.982$. Therefore, $f(6.810) \approx -1949.07$ is the absolute minimum correct to two decimal places.

34. $f(x) = x^2 + \sin x \Rightarrow f'(x) = 2x + \cos x$. $f'(x)$ exists for all x , so to find the minimum of f , we can examine the zeros of f' . From the graph of f' , we see that a good choice for x_1 is $x_1 = -0.5$. Use $g(x) = 2x + \cos x$ and $g'(x) = 2 - \sin x$ to obtain $x_2 \approx -0.450627$,

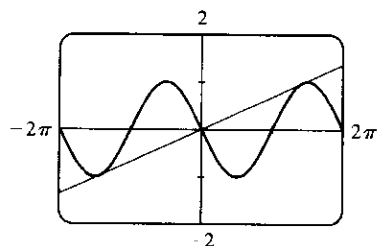
$$x_3 \approx -0.450184 \approx x_4. \text{ Since } f''(x) = 2 - \sin x > 0 \text{ for all } x,$$

$$f(-0.450184) \approx -0.232466 \text{ is the absolute minimum.}$$



35. $y = x^3 + \cos x \Rightarrow y' = 3x^2 - \sin x \Rightarrow y'' = 6x - \cos x \Rightarrow y''' = 6 + \sin x$. Now to solve $y'' = 0$, try $x_1 = 0$, and then $x_2 = x_1 - \frac{y''(x_1)}{y'''(x_1)} = \frac{1}{6} \Rightarrow x_3 \approx 0.164419 \approx x_4$. For $x < 0.164419$, $y'' < 0$, and for $x > 0.164419$, $y'' > 0$. Therefore, the point of inflection, correct to six decimal places, is $(0.164419, y(0.164419)) \approx (0.164419, 0.990958)$.

- 36.



$f(x) = -\sin x \Rightarrow f'(x) = -\cos x$. At $x = a$, the slope of the tangent line is $f'(a) = -\cos a$. The line through the origin and $(a, f(a))$ is $y = \frac{-\sin a - 0}{a - 0}x$. If this line is to be tangent to f at $x = a$, then its

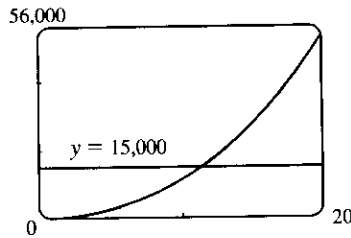
$$\text{slope must equal } f'(a). \text{ Thus, } \frac{-\sin a}{a} = -\cos a \Rightarrow \tan a = a.$$

To solve this equation using Newton's method, let $g(x) = \tan x - x$,

$$g'(x) = \sec^2 x - 1, \text{ and } x_{n+1} = x_n - \frac{\tan x_n - x_n}{\sec^2 x_n - 1} \text{ with } x_1 = 4.5 \text{ (estimated from the figure). } x_2 \approx 4.493614,$$

$$x_3 \approx 4.493410, x_4 \approx 4.493409 \approx x_5. \text{ Thus, the slope of the line that has the largest slope is } f'(x_5) \approx 0.217234.$$

37.



The volume of the silo, in terms of its radius, is

$$V(r) = \pi r^2(30) + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = 30\pi r^2 + \frac{2}{3}\pi r^3.$$

From a graph of V , we see that $V(r) = 15,000$ at $r \approx 11$ ft. Now we use Newton's method to solve the equation $V(r) - 15,000 = 0$.

$\frac{dV}{dr} = 60\pi r + 2\pi r^2$, so $r_{n+1} = r_n - \frac{30\pi r_n^2 + \frac{2}{3}\pi r_n^3 - 15,000}{60\pi r_n + 2\pi r_n^2}$. Taking $r_1 = 11$, we get $r_2 \approx 11.2853$, $r_3 \approx 11.2807 \approx r_4$. So in order for the silo to hold $15,000 \text{ ft}^3$ of grain, its radius must be about 11.2807 ft.

38. Let the radius of the circle be r . Using $s = r\theta$, we have $5 = r\theta$ and so $r = 5/\theta$. From the Law of Cosines we get

$$4^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos \theta \Leftrightarrow 16 = 2r^2(1 - \cos \theta) = 2(5/\theta)^2(1 - \cos \theta).$$

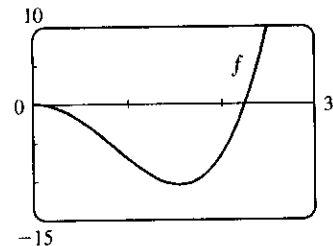
Multiplying by θ^2 gives $16\theta^2 = 50(1 - \cos \theta)$, so we take

$$f(\theta) = 16\theta^2 + 50 \cos \theta - 50 \text{ and } f'(\theta) = 32\theta - 50 \sin \theta. \text{ The formula}$$

for Newton's method is $\theta_{n+1} = \theta_n - \frac{16\theta_n^2 + 50 \cos \theta_n - 50}{32\theta_n - 50 \sin \theta_n}$. From the

graph of f , we can use $\theta_1 = 2.2$, giving us $\theta_2 \approx 2.2662$,

$\theta_3 \approx 2.2622 \approx \theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130^\circ$.



39. In this case, $A = 18,000$, $R = 375$, and $n = 5(12) = 60$. So the formula $A = \frac{R}{i} [1 - (1+i)^{-n}]$ becomes

$$18,000 = \frac{375}{x} [1 - (1+x)^{-60}] \Leftrightarrow 48x = 1 - (1+x)^{-60} \quad [\text{multiply each term by } (1+x)^{60}] \Leftrightarrow$$

$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0. \text{ Let the LHS be called } f(x), \text{ so that}$$

$$\begin{aligned} f'(x) &= 48x(60)(1+x)^{59} + 48(1+x)^{60} - 60(1+x)^{59} \\ &= 12(1+x)^{59} [4x(60) + 4(1+x) - 5] = 12(1+x)^{59} (244x - 1) \end{aligned}$$

$$x_{n+1} = x_n - \frac{48x_n(1+x_n)^{60} - (1+x_n)^{60} + 1}{12(1+x_n)^{59} (244x_n - 1)}. \text{ An interest rate of } 1\% \text{ per month seems like a reasonable}$$

estimate for $x = i$. So let $x_1 = 1\% = 0.01$, and we get $x_2 \approx 0.0082202$, $x_3 \approx 0.0076802$, $x_4 \approx 0.0076291$, $x_5 \approx 0.0076286 \approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

40. (a) $p(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 \Rightarrow$

$$p'(x) = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1-r)x + 2(1-r). \text{ So we use}$$

$$x_{n+1} = x_n - \frac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1-r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1-r)x_n + 2(1-r)}. \text{ We substitute in the value}$$

$r \approx 3.04042 \times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the Sun, so we take $x_1 = 0.95$ as our first approximation, and get $x_2 \approx 0.96682$,

$$x_3 \approx 0.97770, x_4 \approx 0.98451, x_5 \approx 0.98830, x_6 \approx 0.98976, x_7 \approx 0.98998, x_8 \approx 0.98999 \approx x_9.$$

So, to five decimal places, L_1 is located 0.98999 AU from the Sun (or 0.01001 AU from Earth).

(b) In this case we use Newton's method with the function

$$p(x) - 2rx^2 = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \Rightarrow$$

$$[p(x) - 2rx^2]' = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r). \text{ So}$$

$$x_{n+1} = x_n - \frac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}.$$
 Again, we substitute

$r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the Sun and, judging from the result of part (a), probably less than 0.02 AU from Earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the Sun (or 0.01008 AU from Earth).

4.10 Antiderivatives

$$1. f(x) = 6x^2 - 8x + 3 \Rightarrow F(x) = 6 \frac{x^{2+1}}{2+1} - 8 \frac{x^{1+1}}{1+1} + 3x + C = 2x^3 - 4x^2 + 3x + C$$

$$\text{Check: } F'(x) = 2 \cdot 3x^2 - 4 \cdot 2x + 3 + 0 = 6x^2 - 8x + 3 = f(x)$$

$$2. f(x) = 4 + x^2 - 5x^3 \Rightarrow F(x) = 4x + \frac{1}{3}x^3 - \frac{5}{4}x^4 + C$$

$$3. f(x) = 1 - x^3 + 5x^5 - 3x^7 \Rightarrow F(x) = x - \frac{x^{3+1}}{3+1} + 5 \frac{x^{5+1}}{5+1} - 3 \frac{x^{7+1}}{7+1} + C = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 - \frac{3}{8}x^8 + C$$

$$4. f(x) = x^{20} + 4x^{10} + 8 \Rightarrow F(x) = \frac{1}{21}x^{21} + \frac{4}{11}x^{11} + 8x + C$$

$$5. f(x) = 5x^{1/4} - 7x^{3/4} \Rightarrow F(x) = 5 \frac{x^{1/4+1}}{\frac{1}{4}+1} - 7 \frac{x^{3/4+1}}{\frac{3}{4}+1} + C = 5 \frac{x^{5/4}}{5/4} - 7 \frac{x^{7/4}}{7/4} + C = 4x^{5/4} - 4x^{7/4} + C$$

$$6. f(x) = 2x + 3x^{1.7} \Rightarrow F(x) = x^2 + \frac{3}{2.7}x^{2.7} + C = x^2 + \frac{10}{9}x^{2.7} + C$$

$$7. f(x) = 6\sqrt{x} - \frac{8}{\sqrt{x}} = 6x^{1/2} - x^{-1/2} \Rightarrow$$

$$F(x) = 6 \frac{x^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{-1/2+1}}{-1/2+1} + C = 6 \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} + C = 4x^{3/2} - \frac{6}{7}x^{7/6} + C$$

$$8. f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4} = x^{3/4} + x^{4/3} \Rightarrow F(x) = \frac{x^{7/4}}{7/4} + \frac{x^{7/3}}{7/3} + C = \frac{4}{7}x^{7/4} + \frac{3}{7}x^{7/3} + C$$

$$9. f(x) = \frac{10}{x^9} = 10x^{-9} \text{ has domain } (-\infty, 0) \cup (0, \infty), \text{ so } F(x) = \begin{cases} \frac{10x^{-8}}{-8} + C_1 = -\frac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\frac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$$

See Example 1 for a similar problem.

$$10. g(x) = \frac{5 - 4x^3 + 2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2 \text{ has domain } (-\infty, 0) \cup (0, \infty), \text{ so}$$

$$G(x) = \begin{cases} 5 \frac{x^{-5}}{-5} - 4 \frac{x^{-2}}{-2} + 2x + C_1 = -\frac{1}{x^5} + \frac{2}{x^2} + 2x + C_1 & \text{if } x < 0 \\ -\frac{1}{x^5} + \frac{2}{x^2} + 2x + C_2 & \text{if } x > 0 \end{cases}$$

$$11. f(u) = \frac{u^4 + 3\sqrt{u}}{u^2} = \frac{u^4}{u^2} + \frac{3u^{1/2}}{u^2} = u^2 + 3u^{-3/2} \Rightarrow$$

$$F(u) = \frac{u^3}{3} + 3 \frac{u^{-3/2+1}}{-3/2+1} + C = \frac{1}{3}u^3 + 3 \frac{u^{-1/2}}{-1/2} + C = \frac{1}{3}u^3 - \frac{6}{\sqrt{u}} + C$$

12. $f(x) = x^{2/3} + 2x^{-1/3}$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$F(x) = \begin{cases} \frac{x^{5/3}}{5/3} + \frac{2x^{2/3}}{2/3} + C_1 = \frac{3}{5}x^{5/3} + 3x^{2/3} + C_1 & \text{if } x > 0 \\ \frac{3}{5}x^{5/3} + 3x^{2/3} + C_2 & \text{if } x < 0 \end{cases}$$

13. $h(x) = x^3 + 5 \sin x \Rightarrow H(x) = \frac{1}{4}x^4 + 5(-\cos x) + C = \frac{1}{4}x^4 - 5 \cos x + C$

14. $f(t) = 3 \cos t - 4 \sin t \Rightarrow F(t) = 3(\sin t) - 4(-\cos t) + C = 3 \sin t + 4 \cos t + C$

15. $f(t) = 4\sqrt{t} - \sec t \tan t \Rightarrow F(t) = \frac{4}{3/2}t^{3/2} - \sec t + C = \frac{8}{3}t^{3/2} - \sec t + C_n$ on the interval $(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})$.

16. $f(\theta) = 6\theta^2 - 7 \sec^2 \theta \Rightarrow F(\theta) = 2\theta^3 - 7 \tan \theta + C_n$ on the interval $(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})$.

17. $f''(x) = 6x + 12x^2 \Rightarrow f'(x) = 6 \cdot \frac{x^2}{2} + 12 \cdot \frac{x^3}{3} + C = 3x^2 + 4x^3 + C \Rightarrow$

$$f(x) = 3 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^4}{4} + Cx + D = x^3 + x^4 + Cx + D \quad [C \text{ and } D \text{ are just arbitrary constants}]$$

18. $f''(x) = 2 + x^3 + x^6 \Rightarrow f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + C \Rightarrow f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Cx + D$

19. $f''(x) = 1 + x^{4/5} \Rightarrow f'(x) = x + \frac{5}{9}x^{9/5} + C \Rightarrow$

$$f(x) = \frac{1}{2}x^2 + \frac{5}{9} \cdot \frac{5}{14}x^{14/5} + Cx + D = \frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + Cx + D$$

20. $f''(x) = \cos x \Rightarrow f'(x) = \sin x + C \Rightarrow f(x) = -\cos x + Cx + D$

21. $f'''(t) = 60t^2 \Rightarrow f''(t) = 20t^3 + C \Rightarrow f'(t) = 5t^4 + Ct + D \Rightarrow f(t) = t^5 + \frac{1}{2}Ct^2 + Dt + E$

22. $f'''(t) = t - \sqrt{t} \Rightarrow f''(t) = \frac{1}{2}t^2 - \frac{2}{3}t^{3/2} + C \Rightarrow f'(t) = \frac{1}{6}t^3 - \frac{4}{15}t^{5/2} + Ct + D \Rightarrow$

$$f(t) = \frac{1}{24}t^4 - \frac{8}{105}t^{7/2} + \frac{1}{2}Ct^2 + Dt + E$$

23. $f'(x) = 1 - 6x \Rightarrow f(x) = x - 3x^2 + C. f(0) = C \text{ and } f(0) = 8 \Rightarrow C = 8, \text{ so } f(x) = x - 3x^2 + 8.$

24. $f'(x) = 8x^3 + 12x + 3 \Rightarrow f(x) = 2x^4 + 6x^2 + 3x + C. f(1) = 11 + C \text{ and } f(1) = 6 \Rightarrow$
 $11 + C = 6 \Rightarrow C = -5, \text{ so } f(x) = 2x^4 + 6x^2 + 3x - 5.$

25. $f'(x) = \sqrt{x}(6 + 5x) = 6x^{1/2} + 5x^{3/2} \Rightarrow f(x) = 4x^{3/2} + 2x^{5/2} + C.$

$$f(1) = 6 + C \text{ and } f(1) = 10 \Rightarrow C = 4, \text{ so } f(x) = 4x^{3/2} + 2x^{5/2} + 4.$$

26. $f'(x) = 2x - 3/x^4 = 2x - 3x^{-4} \Rightarrow f(x) = x^2 + x^{-3} + C$ because we're given that $x > 0$.

$$f(1) = 2 + C \text{ and } f(1) = 3 \Rightarrow C = 1, \text{ so } f(x) = x^2 + 1/x^3 + 1.$$

27. $f'(t) = 2 \cos t + \sec^2 t \Rightarrow f(t) = 2 \sin t + \tan t + C$ because $-\pi/2 < t < \pi/2$.

$$f(\frac{\pi}{3}) = 2(\sqrt{3}/2) + \sqrt{3} + C = 2\sqrt{3} + C \text{ and } f(\frac{\pi}{3}) = 4 \Rightarrow C = 4 - 2\sqrt{3}, \text{ so}$$

$$f(t) = 2 \sin t + \tan t + 4 - 2\sqrt{3}.$$

28. $f'(x) = 3x^{-2} \Rightarrow f(x) = \begin{cases} -3/x + C_1 & \text{if } x > 0 \\ -3/x + C_2 & \text{if } x < 0 \end{cases} \quad f(1) = -3 + C_1 = 0 \Rightarrow C_1 = 3,$

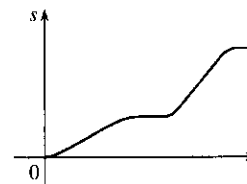
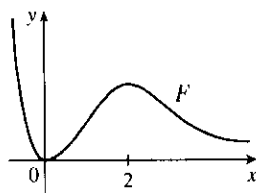
$$f(-1) = 3 + C_2 = 0 \Rightarrow C_2 = -3. \text{ So } f(x) = \begin{cases} -3/x + 3 & \text{if } x > 0 \\ -3/x - 3 & \text{if } x < 0 \end{cases}$$

29. $f''(x) = 24x^2 + 2x + 10 \Rightarrow f'(x) = 8x^3 + x^2 + 10x + C. f'(1) = 8 + 1 + 10 + C \text{ and } f'(1) = -3 \Rightarrow$
 $19 + C = -3 \Rightarrow C = -22, \text{ so } f'(x) = 8x^3 + x^2 + 10x - 22 \text{ and hence,}$

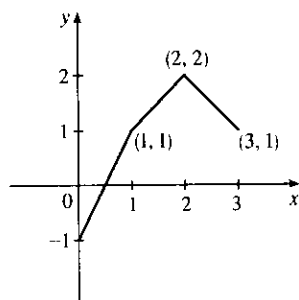
$$f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + D. f(1) = 2 + \frac{1}{3} + 5 - 22 + D \text{ and } f(1) = 5 \Rightarrow D = 22 - \frac{7}{3} = \frac{59}{3}.$$

$$\text{so } f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + \frac{59}{3}.$$

30. $f''(x) = 4 - 6x - 40x^3 \Rightarrow f'(x) = 4x - 3x^2 - 10x^4 + C$. $f'(0) = C$ and $f'(0) = 1 \Rightarrow C = 1$, so $f'(x) = 4x - 3x^2 - 10x^4 + 1$ and hence, $f(x) = 2x^2 - x^3 - 2x^5 + x + D$. $f(0) = D$ and $f(0) = 2 \Rightarrow D = 2$, so $f(x) = 2x^2 - x^3 - 2x^5 + x + 2$.
31. $f''(\theta) = \sin \theta + \cos \theta \Rightarrow f'(\theta) = -\cos \theta + \sin \theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \Rightarrow C = 5$, so $f'(\theta) = -\cos \theta + \sin \theta + 5$ and hence, $f(\theta) = -\sin \theta - \cos \theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \Rightarrow D = 4$, so $f(\theta) = -\sin \theta - \cos \theta + 5\theta + 4$.
32. $f''(t) = 3/\sqrt{t} = 3t^{-1/2} \Rightarrow f'(t) = 6t^{1/2} + C$. $f'(4) = 12 + C$ and $f'(4) = 7 \Rightarrow C = -5$, so $f'(t) = 6t^{1/2} - 5$ and hence, $f(t) = 4t^{3/2} - 5t + D$. $f(4) = 32 - 20 + D$ and $f(4) = 20 \Rightarrow D = 8$, so $f(t) = 4t^{3/2} - 5t + 8$.
33. $f''(x) = 2 - 12x \Rightarrow f'(x) = 2x - 6x^2 + C \Rightarrow f(x) = x^2 - 2x^3 + Cx + D$. $f(0) = D$ and $f(0) = 9 \Rightarrow D = 9$. $f(2) = 4 - 16 + 2C + 9 = 2C - 3$ and $f(2) = 15 \Rightarrow 2C = 18 \Rightarrow C = 9$, so $f(x) = x^2 - 2x^3 + 9x + 9$.
34. $f''(x) = 20x^3 + 12x^2 + 4 \Rightarrow f'(x) = 5x^4 + 4x^3 + 4x + C \Rightarrow f(x) = x^5 + x^4 + 2x^2 + Cx + D$. $f(0) = D$ and $f(0) = 8 \Rightarrow D = 8$. $f(1) = 1 + 1 + 2 + C + 8 = C + 12$ and $f(1) = 5 \Rightarrow C = -7$, so $f(x) = x^5 + x^4 + 2x^2 - 7x + 8$.
35. $f''(x) = 2 + \cos x \Rightarrow f'(x) = 2x + \sin x + C \Rightarrow f(x) = x^2 - \cos x + Cx + D$. $f(0) = -1 + D$ and $f(0) = -1 \Rightarrow D = 0$. $f(\frac{\pi}{2}) = \pi^2/4 + (\frac{\pi}{2})C$ and $f(\frac{\pi}{2}) = 0 \Rightarrow (\frac{\pi}{2})C = -\pi^2/4 \Rightarrow C = -\frac{\pi}{2}$, so $f(x) = x^2 - \cos x - (\frac{\pi}{2})x$.
36. $f'''(x) = \sin x \Rightarrow f''(x) = -\cos x + C \Rightarrow 1 = f''(0) = -1 + C \Rightarrow C = 2$, so $f''(x) = -\cos x + 2 \Rightarrow f'(x) = -\sin x + 2x + D \Rightarrow 1 = f'(0) = D \Rightarrow f'(x) = -\sin x + 2x + 1 \Rightarrow f(x) = \cos x + x^2 + x + E \Rightarrow 1 = f(0) = 1 + E \Rightarrow E = 0$, so $f(x) = \cos x + x^2 + x$.
37. Given $f'(x) = 2x + 1$, we have $f(x) = x^2 + x + C$. Since f passes through $(1, 6)$, $f(1) = 6 \Rightarrow 1^2 + 1 + C = 6 \Rightarrow C = 4$. Therefore, $f(x) = x^2 + x + 4$ and $f(2) = 2^2 + 2 + 4 = 10$.
38. $f'(x) = x^3 \Rightarrow f(x) = \frac{1}{4}x^4 + C$. $x + y = 0 \Rightarrow y = -x \Rightarrow m = -1$. Now $m = f'(x) \Rightarrow -1 = x^3 \Rightarrow x = -1 \Rightarrow y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of f . From f , $1 = \frac{1}{4}(-1)^4 + C \Rightarrow C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.
39. b is the antiderivative of f . For small x , f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x , so only b can be f 's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.
40. We know right away that c cannot be f 's antiderivative, since the slope of c is not zero at the x -value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f .
41. The graph of F will have a minimum at 0 and a maximum at 2, since $f = F'$ goes from negative to positive at $x = 0$, and from positive to negative at $x = 2$.
42. The position function is the antiderivative of the velocity function, so its graph has to be horizontal where the velocity function is equal to 0.



43.



$$f'(x) = \begin{cases} 2 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x \leq 3 \end{cases} \Rightarrow f(x) = \begin{cases} 2x + C & \text{if } 0 \leq x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x \leq 3 \end{cases}$$

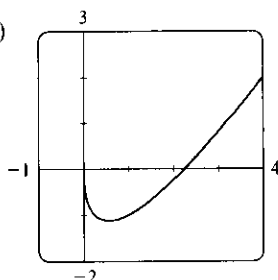
$f(0) = -1 \Rightarrow 2(0) + C = -1 \Rightarrow C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$. The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that f is continuous. We can include the point $x = 1$ on

either the first or the second part of f . The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x \leq 3$ is -1 , so we get to $(3, 1)$. $f(3) = 1 \Rightarrow -3 + E = 1 \Rightarrow E = 4$. Thus,

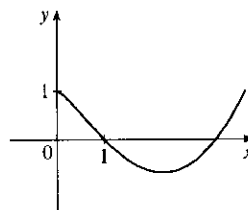
$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \leq x \leq 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$ or at $x = 2$.

44. (a)

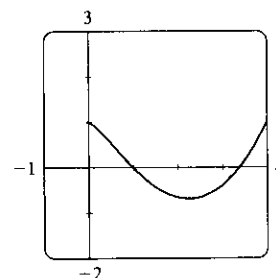


(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. f has a minimum at about $x = 0.5$, so its derivative is zero there. f is decreasing on $(0, 0.5)$, so its derivative is negative and hence, F is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2, \infty)$, f is positive and increasing, so F is increasing and CU.

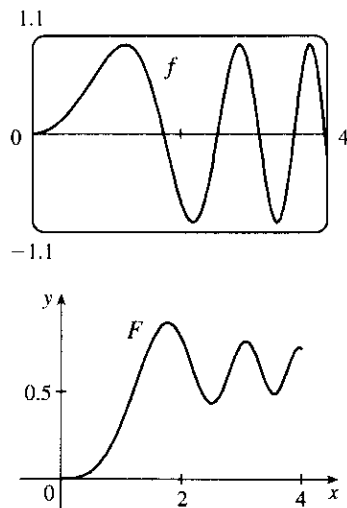


(c) $f(x) = 2x - 3\sqrt{x} \Rightarrow$
 $F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C$. $F(0) = C$ and
 $F(0) = 1 \Rightarrow C = 1$, so
 $F(x) = x^2 - 2x^{3/2} + 1$.

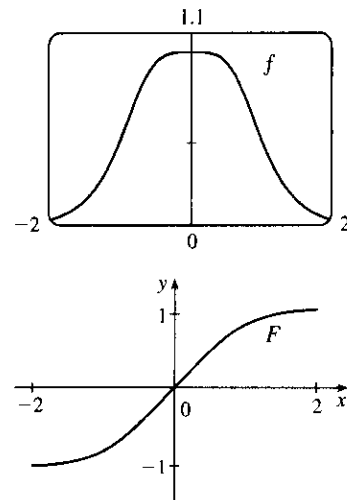
(d)



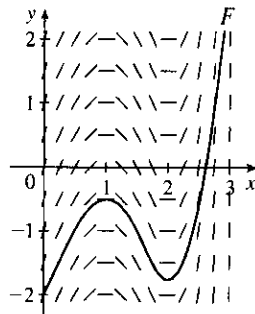
45. $f(x) = \sin(x^2)$, $0 \leq x \leq 4$



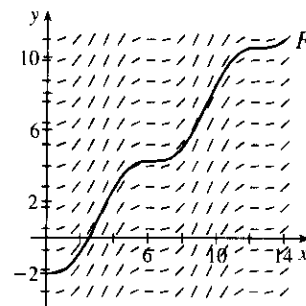
46. $f(x) = 1/(x^4 + 1)$



47.



48.



49.

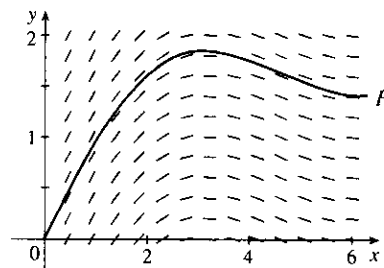
x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	-0.100
4.0	-0.189
4.5	-0.217
5.0	-0.192
5.5	-0.128
6.0	-0.047

We compute slopes [values of $f(x) = (\sin x)/x$ for

$0 < x < 2\pi$] as in the table [$\lim_{x \rightarrow 0^+} f(x) = 1$] and draw a direction field as in Example 6. Then we use the direction

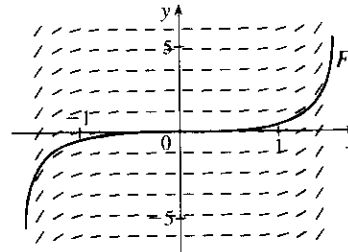
field to graph F starting at $(0, 0)$.



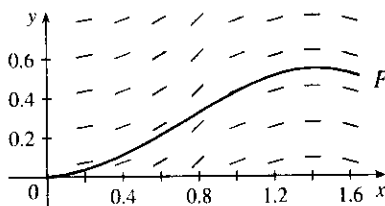
50.

x	$f(x)$
0	0
± 0.2	0.041
± 0.4	0.169
± 0.6	0.410
± 0.8	0.824
± 1.0	1.557
± 1.2	3.087
± 1.4	8.117
± 1.5	21.152

We compute slopes [values of $f(x) = x \tan x$ for $-\pi/2 < x < \pi/2$] as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$ and extending in both directions. Note that if f is an even function, then the antiderivative F that passes through the origin is an odd function.

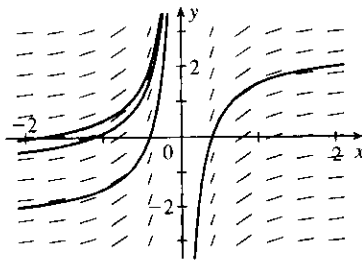


51.



Remember that the given table values of f are the slopes of F at any x . For example, at $x = 1.4$, the slope of F is $f(1.4) = 0$.

52. (a)



(b) The general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = \begin{cases} -1/x + C_1 & \text{if } x < 0 \\ -1/x + C_2 & \text{if } x > 0 \end{cases} \quad \text{since } f(x) \text{ is not defined at } x = 0.$$

The graph of the general antiderivatives of $f(x)$ looks like the graph in part (a), as expected.

53. $v(t) = s'(t) = \sin t - \cos t \Rightarrow s(t) = -\cos t - \sin t + C$. $s(0) = -1 + C$ and $s(0) = 0 \Rightarrow C = 1$, so $s(t) = -\cos t - \sin t + 1$.

54. $v(t) = s'(t) = 1.5\sqrt{t} \Rightarrow s(t) = t^{3/2} + C$. $s(4) = 8 + C$ and $s(4) = 10 \Rightarrow C = 2$, so $s(t) = t^{3/2} + 2$.

55. $a(t) = v'(t) = t - 2 \Rightarrow v(t) = \frac{1}{2}t^2 - 2t + C$. $v(0) = C$ and $v(0) = 3 \Rightarrow C = 3$, so $v(t) = \frac{1}{2}t^2 - 2t + 3$ and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + D$. $s(0) = D$ and $s(0) = 1 \Rightarrow D = 1$, and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + 1$.

56. $a(t) = v'(t) = \cos t + \sin t \Rightarrow v(t) = \sin t - \cos t + C \Rightarrow 5 = v(0) = -1 + C \Rightarrow C = 6$, so $v(t) = \sin t - \cos t + 6 \Rightarrow s(t) = -\cos t - \sin t + 6t + D \Rightarrow 0 = s(0) = -1 + D \Rightarrow D = 1$, so $s(t) = -\cos t - \sin t + 6t + 1$.

57. $a(t) = v'(t) = 10 \sin t + 3 \cos t \Rightarrow v(t) = -10 \cos t + 3 \sin t + C \Rightarrow s(t) = -10 \sin t - 3 \cos t + Ct + D$. $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \Rightarrow D = 3$ and $C = \frac{6}{\pi}$. Thus, $s(t) = -10 \sin t - 3 \cos t + \frac{6}{\pi}t + 3$.

58. $a(t) = v'(t) = 10 + 3t - 3t^2 \Rightarrow v(t) = 10t + \frac{3}{2}t^2 - t^3 + C \Rightarrow s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + Ct + D \Rightarrow$
 $0 = s(0) = D$ and $10 = s(2) = 20 + 4 - 4 + 2C \Rightarrow C = -5$, so $s(t) = -5t + 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4$.
59. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.
 $v'(t) = a(t) = -9.8 \Rightarrow v(t) = -9.8t + C$. Now $v(0) = 0 \Rightarrow C = 0$, so $v(t) = -9.8t \Rightarrow$
 $s(t) = -4.9t^2 + D$. Last, $s(0) = 450 \Rightarrow D = 450 \Rightarrow s(t) = 450 - 4.9t^2$.
- (b) The stone reaches the ground when $s(t) = 0$. $450 - 4.9t^2 = 0 \Rightarrow t^2 = 450/4.9 \Rightarrow$
 $t_1 = \sqrt{450/4.9} \approx 9.58$ s.
- (c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.
- (d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \Rightarrow$
 $0 + C = -5 \Rightarrow v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \Rightarrow D = 450 \Rightarrow$
 $s(t) = -4.9t^2 - 5t + 450$. Solving $s(t) = 0$ by using the quadratic formula gives us
 $t = (5 \pm \sqrt{8845})/(-9.8) \Rightarrow t_1 \approx 9.09$ s.
60. $v'(t) = a(t) = a \Rightarrow v(t) = at + C$ and $v_0 = v(0) = C \Rightarrow v(t) = at + v_0 \Rightarrow$
 $s(t) = \frac{1}{2}at^2 + v_0t + D \Rightarrow s_0 = s(0) = D \Rightarrow s(t) = \frac{1}{2}at^2 + v_0t + s_0$
61. By Exercise 60 with $a = -9.8$, $s(t) = -4.9t^2 + v_0t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So
 $[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6v_0t + v_0^2 = v_0^2 + 96.04t^2 - 19.6v_0t = v_0^2 - 19.6(-4.9t^2 + v_0t)$.
 But $-4.9t^2 + v_0t$ is just $s(t)$ without the s_0 term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$.
62. For the first ball, $s_1(t) = -16t^2 + 48t + 432$ from Example 8. For the second ball, $a(t) = -32 \Rightarrow$
 $v(t) = -32t + C$, but $v(1) = -32(1) + C = 24 \Rightarrow C = 56$, so $v(t) = -32t + 56 \Rightarrow$
 $s(t) = -16t^2 + 56t + D$, but $s(1) = -16(1)^2 + 56(1) + D = 432 \Rightarrow D = 392$, and
 $s_2(t) = -16t^2 + 56t + 392$. The balls pass each other when $s_1(t) = s_2(t) \Rightarrow$
 $-16t^2 + 48t + 432 = -16t^2 + 56t + 392 \Leftrightarrow 8t = 40 \Leftrightarrow t = 5$ s.
Another solution: From Exercise 60, we have $s_1(t) = -16t^2 + 48t + 432$ and $s_2(t) = -16t^2 + 24t + 432$.
 We now want to solve $s_1(t) = s_2(t - 1) \Rightarrow -16t^2 + 48t + 432 = -16(t - 1)^2 + 24(t - 1) + 432 \Rightarrow$
 $48t = 32t - 16 + 24t - 24 \Rightarrow 40 = 8t \Rightarrow t = 5$ s.
63. Using Exercise 60 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is
 $s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t$ and $v(t) = -120 \Rightarrow -32t = -120 \Rightarrow t = 3.75$, so
 $0 = s(3.75) = -16(3.75)^2 + h \Rightarrow h = 16(3.75)^2 = 225$ ft.
64. (a) $EIy'' = mg(L - x) + \frac{1}{2}\rho g(L - x)^2 \Rightarrow EIy' = -\frac{1}{2}mg(L - x)^2 - \frac{1}{6}\rho g(L - x)^3 + C \Rightarrow$
 $EIy = \frac{1}{6}mg(L - x)^3 + \frac{1}{24}\rho g(L - x)^4 + Cx + D$. Since the left end of the board is fixed, we must have
 $y = y' = 0$ when $x = 0$. Thus, $0 = -\frac{1}{2}mgL^2 - \frac{1}{6}\rho gL^3 + C$ and $0 = \frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4 + D$. It follows that

$$EIy = \frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + \left(\frac{1}{2}mgL^2 + \frac{1}{6}\rho gL^3\right)x - \left(\frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4\right) \text{ and}$$

$$f(x) = y = \frac{1}{EI} \left[\frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + \left(\frac{1}{2}mgL^2 + \frac{1}{6}\rho gL^3\right)x - \left(\frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4\right) \right]$$

- (b) $f(L) < 0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$\begin{aligned} -f(L) &= \frac{-1}{EI} \left[\frac{1}{2}mgL^3 + \frac{1}{6}\rho gL^4 - \frac{1}{6}mgL^3 - \frac{1}{24}\rho gL^4 \right] \\ &= \frac{-1}{EI} \left(\frac{1}{3}mgL^3 + \frac{1}{8}\rho gL^4 \right) = -\frac{gL^3}{EI} \left(\frac{m}{3} + \frac{\rho L}{8} \right) \end{aligned}$$

Note: This is positive because g is negative.

65. Marginal cost $= 1.92 - 0.002x = C'(x) \Rightarrow C(x) = 1.92x - 0.001x^2 + K$. But

$$C(1) = 1.92 - 0.001 + K = 562 \Rightarrow K = 560.081. \text{ Therefore, } C(x) = 1.92x - 0.001x^2 + 560.081 \Rightarrow$$

$$C(100) = 742.081, \text{ so the cost of producing 100 items is } \$742.08.$$

66. Let the mass, measured from one end, be $m(x)$. Then $m(0) = 0$ and $\rho = \frac{dm}{dx} = x^{-1/2} \Rightarrow m(x) = 2x^{1/2} + C$

$$\text{and } m(0) = C = 0, \text{ so } m(x) = 2\sqrt{x}. \text{ Thus, the mass of the 100-centimeter rod is } m(100) = 2\sqrt{100} = 20 \text{ g.}$$

67. Taking the upward direction to be positive we have that for $0 \leq t \leq 10$ (using the subscript 1 to refer to

$$0 \leq t \leq 10), a_1(t) = -(9 - 0.9t) = v_1'(t) \Rightarrow v_1(t) = -9t + 0.45t^2 + v_0, \text{ but } v_1(0) = v_0 = -10 \Rightarrow$$

$$v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \Rightarrow s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0. \text{ But } s_1(0) = 500 = s_0 \Rightarrow$$

$$s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500. \quad s_1(10) = -450 + 150 - 100 + 500 = 100, \text{ so it takes more}$$

than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \Rightarrow$

$$v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \Rightarrow v(t) = -55. \text{ At 55 ft/s, it will take}$$

$$100/55 \approx 1.8 \text{ s to fall the last 100 ft. Hence, the total time is } 10 + \frac{100}{55} = \frac{130}{11} \approx 11.8 \text{ s.}$$

68. $v'(t) = a(t) = -22$. The initial velocity is 50 mi/h $= \frac{50 \cdot 5280}{3600} = \frac{220}{3}$ ft/s, so $v(t) = -22t + \frac{220}{3}$. The car stops

$$\text{when } v(t) = 0 \Leftrightarrow t = \frac{220}{3 \cdot 22} = \frac{10}{3}. \text{ Since } s(t) = -11t^2 + \frac{220}{3}t, \text{ the distance covered is}$$

$$s\left(\frac{10}{3}\right) = -11\left(\frac{10}{3}\right)^2 + \frac{220}{3} \cdot \frac{10}{3} = \frac{1100}{9} = 122.\bar{2} \text{ ft.}$$

69. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity (after 5 seconds) is

$$50 \text{ mi/h} = 50 \cdot \frac{5280}{3600} = \frac{220}{3} \text{ ft/s. So } v(t) = kt + C \text{ and } v(0) = 44 \Rightarrow C = 44. \text{ Thus, } v(t) = kt + 44 \Rightarrow$$

$$v(5) = 5k + 44. \text{ But } v(5) = \frac{220}{3}, \text{ so } 5k + 44 = \frac{220}{3} \Rightarrow 5k = \frac{88}{3} \Rightarrow k = \frac{88}{15} \approx 5.87 \text{ ft/s}^2.$$

70. $a(t) = -16 \Rightarrow v(t) = -16t + v_0$ where v_0 is the car's speed (in ft/s) when the brakes were applied. The car

$$\text{stops when } -16t + v_0 = 0 \Leftrightarrow t = \frac{1}{16}v_0. \text{ Now } s(t) = \frac{1}{2}(-16)t^2 + v_0t = -8t^2 + v_0t. \text{ The car travels 200 ft in}$$

$$\text{the time that it takes to stop, so } s\left(\frac{1}{16}v_0\right) = 200 \Rightarrow 200 = -8\left(\frac{1}{16}v_0\right)^2 + v_0\left(\frac{1}{16}v_0\right) = \frac{1}{32}v_0^2 \Rightarrow$$

$$v_0^2 = 32 \cdot 200 = 6400 \Rightarrow v_0 = 80 \text{ ft/s } (54.54 \text{ mi/h}).$$

71. Let the acceleration be $a(t) = k \text{ km/h}^2$. We have $v(0) = 100 \text{ km/h}$ and we can take the initial position $s(0)$ to be 0. We want the time t_f for which $v(t) = 0$ to satisfy $s(t) < 0.08 \text{ km}$. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$, where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$. Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so
- $$s(t_f) = \frac{1}{2}k \left(-\frac{100}{k}\right)^2 + 100 \left(-\frac{100}{k}\right) = 10,000 \left(\frac{1}{2k} - \frac{1}{k}\right) = -\frac{5,000}{k}.$$
- The condition $s(t_f)$ must satisfy is
- $$-\frac{5,000}{k} < 0.08 \Rightarrow -\frac{5,000}{0.08} > k \quad [k \text{ is negative}] \Rightarrow k < -62,500 \text{ km/h}^2, \text{ or equivalently,}$$
- $$k < -\frac{3125}{648} \approx -4.82 \text{ m/s}^2.$$

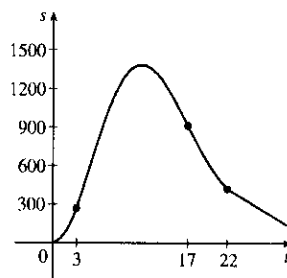
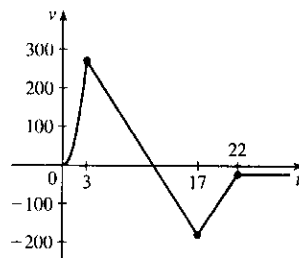
72. (a) For $0 \leq t \leq 3$ we have $a(t) = 60t \Rightarrow v(t) = 30t^2 + C \Rightarrow v(0) = 0 = C \Rightarrow v(t) = 30t^2$, so $s(t) = 10t^3 + C \Rightarrow s(0) = 0 = C \Rightarrow s(t) = 10t^3$. Note that $v(3) = 270$ and $s(3) = 270$. For $3 < t \leq 17$: $a(t) = -g = -32 \text{ ft/s}^2 \Rightarrow v(t) = -32(t-3) + C \Rightarrow v(3) = 270 = C \Rightarrow v(t) = -32(t-3) + 270 \Rightarrow s(t) = -16(t-3)^2 + 270(t-3) + C \Rightarrow s(3) = 270 = C \Rightarrow s(t) = -16(t-3)^2 + 270(t-3) + 270$. Note that $v(17) = -178$ and $s(17) = 914$. For $17 < t \leq 22$: The velocity increases linearly from -178 ft/s to -18 ft/s during this period, so $\frac{\Delta v}{\Delta t} = \frac{-18 - (-178)}{22 - 17} = \frac{160}{5} = 32$. Thus, $v(t) = 32(t-17) - 178 \Rightarrow s(t) = 16(t-17)^2 - 178(t-17) + 914$ and $s(22) = 424 \text{ ft}$. For $t > 22$: $v(t) = -18 \Rightarrow s(t) = -18(t-22) + C$. But $s(22) = 424 = C \Rightarrow s(t) = -18(t-22) + 424$.

Therefore, until the rocket lands, we have

$$v(t) = \begin{cases} 30t^2 & \text{if } 0 \leq t \leq 3 \\ -32(t-3) + 270 & \text{if } 3 < t \leq 17 \\ 32(t-17) - 178 & \text{if } 17 < t \leq 22 \\ -18 & \text{if } t > 22 \end{cases}$$

and

$$s(t) = \begin{cases} 10t^3 & \text{if } 0 \leq t \leq 3 \\ -16(t-3)^2 + 270(t-3) + 270 & \text{if } 3 < t \leq 17 \\ 16(t-17)^2 - 178(t-17) + 914 & \text{if } 17 < t \leq 22 \\ -18(t-22) + 424 & \text{if } t > 22 \end{cases}$$



- (b) To find the maximum height, set $v(t)$ on $3 < t \leq 17$ equal to 0. $-32(t-3) + 270 = 0 \Rightarrow t_1 = 11.4375 \text{ s}$ and the maximum height is $s(t_1) = -16(t_1-3)^2 + 270(t_1-3) + 270 = 1409.0625 \text{ ft}$.
- (c) To find the time to land, set $s(t) = -18(t-22) + 424 = 0$. Then $t-22 = \frac{424}{18} = 23.\bar{5}$, so $t \approx 45.6 \text{ s}$.

73. (a) First note that $90 \text{ mi/h} = 90 \times \frac{5280}{3600} \text{ ft/s} = 132 \text{ ft/s}$. Then $a(t) = 4 \text{ ft/s}^2 \Rightarrow v(t) = 4t + C$, but $v(0) = 0 \Rightarrow C = 0$. Now $4t = 132$ when $t = \frac{132}{4} = 33 \text{ s}$, so it takes 33 s to reach 132 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 2t^2$, $0 \leq t \leq 33$. So $s(33) = 2178 \text{ ft}$. 15 minutes = $15(60) = 900 \text{ s}$, so for $33 < t \leq 933$ we have $v(t) = 132 \text{ ft/s} \Rightarrow s(933) = 132(900) + 2178 = 120,978 \text{ ft} = 22.9125 \text{ mi}$.
- (b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900 - 66 = 834 \text{ s}$ it travels at 132 ft/s, so the distance traveled is $132 \cdot 834 = 110,088 \text{ ft}$. Thus, the total distance is $2178 + 110,088 + 2178 = 114,444 \text{ ft} = 21.675 \text{ mi}$.
- (c) $45 \text{ mi} = 45(5280) = 237,600 \text{ ft}$. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have 233,244 ft at 132 ft/s for a trip of $233,244/132 = 1767 \text{ s}$ at 90 mi/h. The total time is $1767 + 2(33) = 1833 \text{ s} = 30 \text{ min } 33 \text{ s} = 30.55 \text{ min}$.
- (d) $37.5(60) = 2250 \text{ s}$. $2250 - 2(33) = 2184 \text{ s}$ at maximum speed. $2184(132) + 2(2178) = 292,644 \text{ total feet}$ or $292,644/5280 = 55.425 \text{ mi}$.

4 Review

CONCEPT CHECK

- A function f has an **absolute maximum** at $x = c$ if $f(c)$ is the largest function value on the entire domain of f , whereas f has a **local maximum** at c if $f(c)$ is the largest function value when x is near c . See Figure 4 in Section 4.1.
- (a) See Theorem 4.1.3.
(b) See the Closed Interval Method before Example 8 in Section 4.1.
- (a) See Theorem 4.1.4.
(b) See Definition 4.1.6.
- (a) See Rolle's Theorem at the beginning of Section 4.2.
(b) See the Mean Value Theorem in Section 4.2. Geometric interpretation—there is some point P on the graph of a function f [on the interval (a, b)] where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.
- (a) See the I/D Test before Example 1 in Section 4.3.
(b) See the Concavity Test before Example 4 in Section 4.3.
- (a) See the First Derivative Test after Example 1 in Section 4.3.
(b) See the Second Derivative Test before Example 6 in Section 4.3.
(c) See the note before Example 7 in Section 4.3.
- (a) See Definitions 4.4.1 and 4.4.5.
(b) See Definitions 4.4.2 and 4.4.6.
(c) See Definition 4.4.7.
(d) See Definition 4.4.3.
- Without calculus you could get misleading graphs that fail to show the most interesting features of a function. See the discussion following Figure 3 in Section 4.5 and the first paragraph in Section 4.6.

9. (a) See Figure 3 in Section 4.9.

$$(b) x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$(c) x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

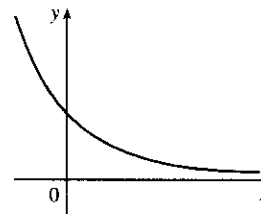
(d) Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0.

10. (a) See the definition at the beginning of Section 4.10.

(b) If F_1 and F_2 are both antiderivatives of f on an interval I , then they differ by a constant.

TRUE-FALSE QUIZ

- False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.
- False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.
- False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.
- True. By the Mean Value Theorem, $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{0}{2} = 0$. Note that $|c| < 1 \Leftrightarrow c \in (-1, 1)$.
- True. This is an example of part (b) of the I/D Test.
- False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all c .
- False. $f'(x) = g'(x) \Rightarrow f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.
- False. Assume there is a function f such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all x , a contradiction.
- True. The graph of one such function is sketched.



- False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the x -axis—at $x = b$, say. But since $f''(x) > 0$ for all x , the graph of f must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all x .
- True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ (since f and g are increasing on I), so $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$.
- False. $f(x) = x$ and $g(x) = 2x$ are both increasing on $(0, 1)$, but $f(x) - g(x) = -x$ is not increasing on $(0, 1)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0, 1)$. But $f(x)g(x) = x(x - 1)$ is not increasing on $(0, 1)$.
14. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ (since f and g are both positive and increasing). Hence, $f(x_1)g(x_1) < f(x_2)g(x_1) < f(x_2)g(x_2)$. So fg is increasing on I .
15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ (f is increasing) $\Rightarrow$
 $\frac{1}{f(x_1)} > \frac{1}{f(x_2)}$ (f is positive) $\Rightarrow g(x_1) > g(x_2) \Rightarrow g(x) = 1/f(x)$ is decreasing on I .
16. False. The most general antiderivative is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ (see Example 1 in Section 4.10).
17. True. By the Mean Value Theorem, there exists a number c in $(0, 1)$ such that
 $f(1) - f(0) = f'(c)(1 - 0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

EXERCISES

1. $f(x) = 10 + 27x - x^3$, $0 \leq x \leq 4$. $f'(x) = 27 - 3x^2 = -3(x^2 - 9) = -3(x + 3)(x - 3) = 0$ only when $x = 3$ (since -3 is not in the domain). $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$, so $f(3) = 64$ is a local maximum value. Checking the endpoints, we find $f(0) = 10$ and $f(4) = 54$. Thus, $f(0) = 10$ is the absolute minimum value and $f(3) = 64$ is the absolute maximum value.
2. $f(x) = x - \sqrt{x}$, $0 \leq x \leq 4$. $f'(x) = 1 - 1/(2\sqrt{x}) = 0 \Leftrightarrow 2\sqrt{x} = 1 \Rightarrow x = \frac{1}{4}$. $f'(x)$ does not exist $\Leftrightarrow x = 0$. $f'(x) < 0$ for $0 < x < \frac{1}{4}$ and $f'(x) > 0$ for $\frac{1}{4} < x < 4$, so $f(\frac{1}{4}) = -\frac{1}{4}$ is a local and absolute minimum value. $f(0) = 0$ and $f(4) = 2$, so $f(4) = 2$ is the absolute maximum value.
3. $f(x) = \frac{x}{x^2 + x + 1}$, $-2 \leq x \leq 0$. $f'(x) = \frac{(x^2 + x + 1)(1) - x(2x + 1)}{(x^2 + x + 1)^2} = \frac{1 - x^2}{(x^2 + x + 1)^2} = 0 \Leftrightarrow x = -1$ (since 1 is not in the domain). $f'(x) < 0$ for $-2 < x < -1$ and $f'(x) > 0$ for $-1 < x < 0$, so $f(-1) = -1$ is a local and absolute minimum value. $f(-2) = -\frac{2}{3}$ and $f(0) = 0$, so $f(0) = 0$ is an absolute maximum value.
4. $f(x) = (x^2 + 2x)^3$, $[-2, 1]$. $f'(x) = 3(x^2 + 2x)^2(2x + 2) = 6(x + 1)x^2(x + 2)^2$, so the only critical numbers in the interior of the domain are $x = -1, 0$. $f'(x) < 0$ for $-2 < x < -1$ and $f'(x) > 0$ for $-1 < x < 0$ and $0 < x < 1$, so f is decreasing on $(-2, -1)$ and increasing on $(-1, 1)$. Thus, $f(-1) = -1$ is a local minimum value. $f(-2) = 0$ and $f(1) = 27$, so the local minimum value is the absolute minimum value and $f(1) = 27$ is the absolute maximum value.
5. $f(x) = x + \sin 2x$, $[0, \pi]$. $f'(x) = 1 + 2\cos 2x = 0 \Leftrightarrow \cos 2x = -\frac{1}{2} \Leftrightarrow 2x = \frac{2\pi}{3}$ or $\frac{4\pi}{3} \Leftrightarrow x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. $f''(x) = -4\sin 2x$, so $f''(\frac{\pi}{3}) = -4\sin \frac{2\pi}{3} = -2\sqrt{3} < 0$ and $f''(\frac{2\pi}{3}) = -4\sin \frac{4\pi}{3} = 2\sqrt{3} > 0$, so $f(\frac{\pi}{3}) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 1.91$ is a local maximum value and $f(\frac{2\pi}{3}) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.23$ is a local minimum value. Also $f(0) = 0$ and $f(\pi) = \pi$, so $f(0) = 0$ is the absolute minimum value and $f(\pi) = \pi$ is the absolute maximum value.
6. $f(x) = \sin x + \cos^2 x$, $[0, \pi]$. $f'(x) = \cos x - 2\cos x \sin x = \cos x(1 - 2\sin x)$, so $f'(x) = 0$ for x in $(0, \pi) \Leftrightarrow \cos x = 0$ or $\sin x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$. $f'(x) = \cos x - \sin 2x \Rightarrow f''(x) = -\sin x - 2\cos 2x$, so $f''(\frac{\pi}{6}) = -\frac{1}{2} - 2(\frac{1}{2}) = -\frac{3}{2}$, $f''(\frac{\pi}{2}) = -1 - 2(-1) = 1$, and $f''(\frac{5\pi}{6}) = -\frac{1}{2} - 2(\frac{1}{2}) = -\frac{3}{2}$. Thus, $f(\frac{\pi}{6}) = \frac{5}{4}$ and $f(\frac{5\pi}{6}) = \frac{5}{4}$ are local maxima and $f(\frac{\pi}{2}) = 1$ is a local minimum. $f(0) = 1$ and $f(\pi) = 1$, so f has its absolute minimum value of 1 at $0, \frac{\pi}{2}$, and π . f attains its absolute maximum value of $\frac{5}{4}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

$$7. \lim_{x \rightarrow \infty} \frac{3x^4 + x - 5}{6x^4 - 2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x^3} - \frac{5}{x^4}}{6 - \frac{2}{x^2} + \frac{1}{x^4}} = \frac{3 + 0 + 0}{6 - 0 + 0} = \frac{1}{2}$$

$$8. \lim_{t \rightarrow \infty} \frac{t^3 - t + 2}{(2t - 1)(t^2 + t + 1)} = \lim_{t \rightarrow \infty} \frac{1 - 1/t^2 + 2/t^3}{(2 - 1/t)(1 + 1/t + 1/t^2)} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

$$9. \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{3x - 1} = \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}/\sqrt{x^2}}{(3x - 1)/\sqrt{x^2}} = \lim_{x \rightarrow -\infty} \frac{\sqrt{4 + 1/x^2}}{-3 + 1/x} \quad [\text{since } -x = |x| = \sqrt{x^2} \text{ for } x < 0]$$

$$= \frac{2}{-3 + 0} = -\frac{2}{3}$$

$$10. \lim_{x \rightarrow -\infty} (x^2 + x^3) = \lim_{x \rightarrow -\infty} x^2(1 + x) = -\infty \text{ since } x^2 \rightarrow \infty \text{ and } 1 + x \rightarrow -\infty \text{ as } x \rightarrow -\infty.$$

$$11. \lim_{x \rightarrow \infty} (\sqrt{4x^2 + 3x} - 2x) = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 3x} - 2x}{1} \cdot \frac{\sqrt{4x^2 + 3x} + 2x}{\sqrt{4x^2 + 3x} + 2x} = \lim_{x \rightarrow \infty} \frac{(4x^2 + 3x) - 4x^2}{\sqrt{4x^2 + 3x} + 2x}$$

$$= \lim_{x \rightarrow \infty} \frac{3x}{\sqrt{4x^2 + 3x} + 2x} = \lim_{x \rightarrow \infty} \frac{3x/\sqrt{x^2}}{(\sqrt{4x^2 + 3x} + 2x)/\sqrt{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3}{\sqrt{4 + 3/x} + 2} \quad [\text{since } x = |x| = \sqrt{x^2} \text{ for } x > 0]$$

$$= \frac{3}{2 + 2} = \frac{3}{4}$$

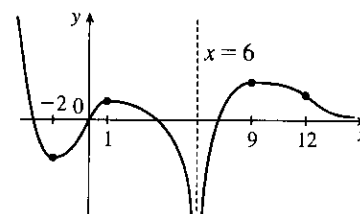
$$12. 0 \leq \sin^4 x \leq 1, \text{ so } 0 \leq \frac{\sin^4 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}. \text{ Since } \lim_{x \rightarrow \infty} 0 = 0 \text{ and } \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0, \lim_{x \rightarrow \infty} \frac{\sin^4 x}{\sqrt{x}} = 0 \text{ by the Squeeze Theorem.}$$

$$13. f(0) = 0, f'(-2) = f'(1) = f'(9) = 0, \lim_{x \rightarrow \infty} f(x) = 0,$$

$$\lim_{x \rightarrow 6} f(x) = -\infty, f'(x) < 0 \text{ on } (-\infty, -2), (1, 6), \text{ and } (9, \infty),$$

$$f'(x) > 0 \text{ on } (-2, 1) \text{ and } (6, 9), f''(x) > 0 \text{ on } (-\infty, 0)$$

$$\text{and } (12, \infty), f''(x) < 0 \text{ on } (0, 6) \text{ and } (6, 12)$$



$$14. \text{ For } 0 < x < 1, f'(x) = 2x, \text{ so } f(x) = x^2 + C. \text{ Since } f(0) = 0,$$

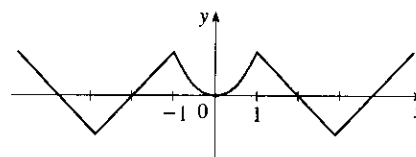
$$f(x) = x^2 \text{ on } [0, 1]. \text{ For } 1 < x < 3, f'(x) = -1, \text{ so}$$

$$f(x) = -x + D. 1 = f(1) = -1 + D \Rightarrow D = 2, \text{ so}$$

$$f(x) = 2 - x. \text{ For } x > 3, f'(x) = 1, \text{ so } f(x) = x + E.$$

$$-1 = f(3) = 3 + E \Rightarrow E = -4, \text{ so } f(x) = x - 4. \text{ Since } f$$

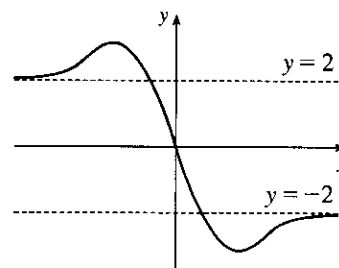
$$\text{is even, its graph is symmetric about the } y\text{-axis.}$$



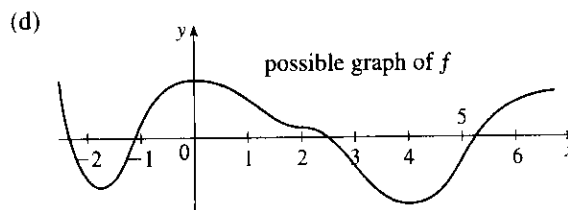
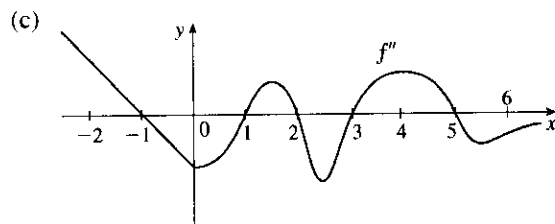
$$15. f \text{ is odd, } f'(x) < 0 \text{ for } 0 < x < 2, \quad f'(x) > 0 \text{ for } x > 2,$$

$$f''(x) > 0 \text{ for } 0 < x < 3, \quad f''(x) < 0 \text{ for } x > 3,$$

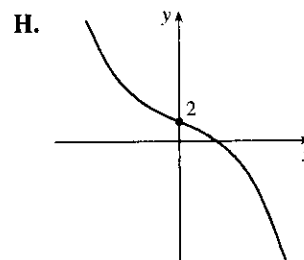
$$\lim_{x \rightarrow \infty} f(x) = -2$$



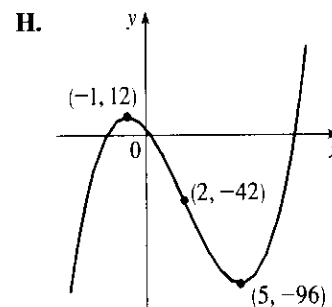
16. (a) Using the Test for Monotonic Functions we know that f is increasing on $(-2, 0)$ and $(4, \infty)$ because $f' > 0$ on $(-2, 0)$ and $(4, \infty)$, and that f is decreasing on $(-\infty, -2)$ and $(0, 4)$ because $f' < 0$ on $(-\infty, -2)$ and $(0, 4)$.
- (b) Using the First Derivative Test, we know that f has a local maximum at $x = 0$ because f' changes from positive to negative at $x = 0$, and that f has a local minimum at $x = 4$ because f' changes from negative to positive at $x = 4$.



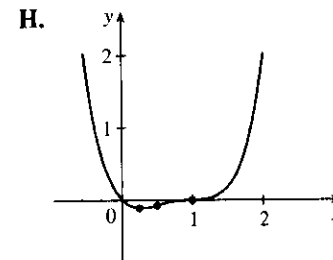
17. $y = f(x) = 2 - 2x - x^3$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 2$.
The x -intercept (approximately 0.770917) can be found using Newton's Method. C. No symmetry D. No asymptote
E. $f'(x) = -2 - 3x^2 = -(3x^2 + 2) < 0$, so f is decreasing on $\mathbb{R}$.
F. No extreme value G. $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so f is CD on $(0, \infty)$ and CU on $(-\infty, 0)$.
There is an IP at $(0, 2)$.



18. $y = f(x) = x^3 - 6x^2 - 15x + 4$ A. $D = \mathbb{R}$
B. y -intercept: $f(0) = 4$; x -intercepts: $f(x) = 0 \Rightarrow x \approx -2.09, 0.24, 7.85$ C. No symmetry D. No asymptote
E. $f'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x + 1)(x - 5)$, so f is increasing on $(-\infty, -1)$, decreasing on $(-1, 5)$, and increasing on $(5, \infty)$. F. Local maximum value $f(-1) = 12$, local minimum value $f(5) = -96$. G. $f''(x) = 6x - 12 = 6(x - 2)$, so f is CD on $(-\infty, 2)$ and CU on $(2, \infty)$. There is an IP at $(2, -42)$.



19. $y = f(x) = x^4 - 3x^3 + 3x^2 - x = x(x - 1)^3$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Leftrightarrow x = 0$ or $x = 1$ C. No symmetry D. f is a polynomial function and hence, it has no asymptote.
E. $f'(x) = 4x^3 - 9x^2 + 6x - 1$. Since the sum of the coefficients is 0, 1 is a root of f' , so $f'(x) = (x - 1)(4x^2 - 5x + 1) = (x - 1)^2(4x - 1)$. $f'(x) < 0 \Rightarrow x < \frac{1}{4}$, so f is decreasing on $(-\infty, \frac{1}{4})$ and f is increasing on $(\frac{1}{4}, \infty)$. F. $f'(x)$ does not change sign at $x = 1$, so there is not a local extremum there. $f(\frac{1}{4}) = -\frac{27}{256}$ is a local minimum value. G. $f''(x) = 12x^2 - 18x + 6 = 6(2x - 1)(x - 1)$.
 $f''(x) = 0 \Leftrightarrow x = \frac{1}{2}$ or 1 . $f''(x) < 0 \Leftrightarrow \frac{1}{2} < x < 1 \Rightarrow f$ is CD on $(\frac{1}{2}, 1)$ and CU on $(-\infty, \frac{1}{2})$ and $(1, \infty)$. There are inflection points at $(\frac{1}{2}, -\frac{1}{16})$ and $(1, 0)$.



20. $y = f(x) = \frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)}$ A. $D = \{x \mid x \neq \pm 1\}$ B. y -intercept: $f(0) = 1$; no x -intercept

C. $f(-x) = f(x)$, so f is even and the graph of f is symmetric about the y -axis. D. Vertical asymptotes:

$x = \pm 1$. Horizontal asymptote: $y = 0$ E. $y' = \frac{2x}{(1-x^2)^2} = 0 \Leftrightarrow x = 0$, so f is decreasing on $(-\infty, -1)$

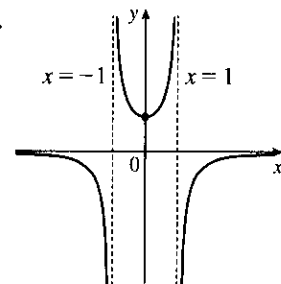
and $(-1, 0)$, and increasing on $(0, 1)$ and $(1, \infty)$.

F. Local minimum value $f(0) = 1$; no local maximum

G. $f''(x) = \frac{(1-x^2)^2 \cdot 2 - 2x \cdot 2(1-x^2)(-2x)}{(1-x^2)^4}$
 $= \frac{2(1-x^2) + 8x^2}{(1-x^2)^3} = \frac{6x^2 + 2}{(1-x^2)^3} < 0 \Leftrightarrow x^2 > 1,$

so f is CD on $(-\infty, -1)$ and $(1, \infty)$, and CU on $(-1, 1)$. No IP

H.



21. $y = f(x) = \frac{1}{x(x-3)^2}$ A. $D = \{x \mid x \neq 0, 3\} = (-\infty, 0) \cup (0, 3) \cup (3, \infty)$ B. No intercepts.

C. No symmetry. D. $\lim_{x \rightarrow \pm\infty} \frac{1}{x(x-3)^2} = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 0^+} \frac{1}{x(x-3)^2} = \infty,$

$\lim_{x \rightarrow 0^-} \frac{1}{x(x-3)^2} = -\infty, \lim_{x \rightarrow 3^-} \frac{1}{x(x-3)^2} = \infty, \lim_{x \rightarrow 3^+} \frac{1}{x(x-3)^2} = -\infty,$ so $x = 0$ and $x = 3$ are VA.

E. $f'(x) = -\frac{(x-3)^2 + 2x(x-3)}{x^2(x-3)^4} = \frac{3(1-x)}{x^2(x-3)^3} \Rightarrow$

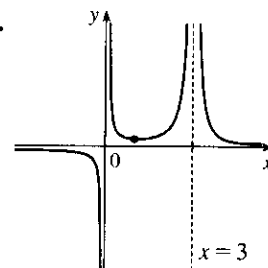
$f'(x) > 0 \Leftrightarrow 1 < x < 3$, so f is increasing on $(1, 3)$ and decreasing on $(-\infty, 0)$, $(0, 1)$, and $(3, \infty)$. F. Local minimum value $f(1) = \frac{1}{4}$

G. $f''(x) = \frac{6(2x^2 - 4x + 3)}{x^3(x-3)^4}$. Note that $2x^2 - 4x + 3 > 0$ for all x

since it has negative discriminant. So $f''(x) > 0 \Leftrightarrow x > 0 \Rightarrow f$ is

CU on $(0, 3)$ and $(3, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



22. $y = f(x) = \frac{1}{x} + \frac{1}{x+1} = \frac{2x+1}{x(x+1)}$ A. $D = \{x \mid x \neq 0, -1\}$ B. No y -intercept, x -intercept $= -\frac{1}{2}$

C. No symmetry D. $\lim_{x \rightarrow \pm\infty} f(x) = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 0^+} \frac{2x+1}{x(x+1)} = \infty, \lim_{x \rightarrow 0^-} \frac{2x+1}{x(x+1)} = -\infty,$

$\lim_{x \rightarrow -1^+} \frac{2x+1}{x(x+1)} = \infty, \lim_{x \rightarrow -1^-} \frac{2x+1}{x(x+1)} = -\infty$, so $x = 0, x = -1$ are VA.

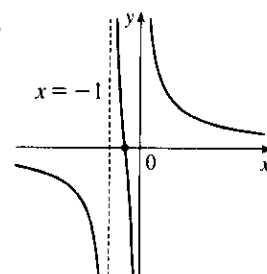
E. $f'(x) = -\frac{1}{x^2} - \frac{1}{(x+1)^2} < 0$, so f is decreasing on $(-\infty, -1)$,

$(-1, 0)$ and $(0, \infty)$. F. No extreme values

G. $f''(x) = \frac{2}{x^3} + \frac{2}{(x+1)^3} = \frac{2(2x+1)(x^2+x+1)}{x^3(x+1)^3}$.

$f''(x) > 0 \Leftrightarrow x > 0$ or $-1 < x < -\frac{1}{2}$, so f is CU on $(0, \infty)$ and $(-1, -\frac{1}{2})$ and CD on $(-\infty, -1)$ and $(-\frac{1}{2}, 0)$. IP at $(-\frac{1}{2}, 0)$

H.



23. $y = f(x) = \frac{x^2}{x+8} = x - 8 + \frac{64}{x+8}$ A. $D = \{x \mid x \neq -8\}$ B. Intercepts are 0 C. No symmetry

D. $\lim_{x \rightarrow \infty} \frac{x^2}{x+8} = \infty$, but $f(x) - (x - 8) = \frac{64}{x+8} \rightarrow 0$ as $x \rightarrow \infty$, so $y = x - 8$ is a slant asymptote.

$\lim_{x \rightarrow -8^+} \frac{x^2}{x+8} = \infty$ and $\lim_{x \rightarrow -8^-} \frac{x^2}{x+8} = -\infty$, so $x = -8$ is a VA.

E. $f'(x) = 1 - \frac{64}{(x+8)^2} = \frac{x(x+16)}{(x+8)^2} > 0 \Leftrightarrow x > 0$ or $x < -16$,

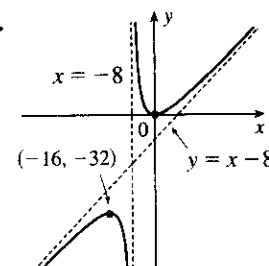
so f is increasing on $(-\infty, -16)$ and $(0, \infty)$ and decreasing on

$(-16, -8)$ and $(-8, 0)$. F. Local maximum value $f(-16) = -32$,

local minimum value $f(0) = 0$ G. $f''(x) = 128/(x+8)^3 > 0 \Leftrightarrow$

$x > -8$, so f is CU on $(-8, \infty)$ and CD on $(-\infty, -8)$. No IP

H.



24. $y = f(x) = x + \sqrt{1-x}$ A. $D = \{x \mid x \leq 1\} = (-\infty, 1]$ B. y -intercept = 1; x -intercepts occur when

$x + \sqrt{1-x} = 0 \Rightarrow \sqrt{1-x} = -x \Rightarrow 1-x = x^2 \Rightarrow x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$, but

the larger root is extraneous, so the only x -intercept is $\frac{-1-\sqrt{5}}{2}$. C. No symmetry D. No asymptote

E. $f'(x) = 1 - 1/(2\sqrt{1-x}) = 0 \Leftrightarrow 2\sqrt{1-x} = 1 \Leftrightarrow$

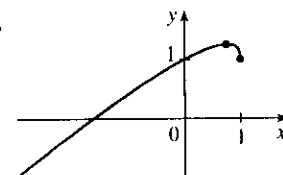
$1-x = \frac{1}{4} \Leftrightarrow x = \frac{3}{4}$ and $f'(x) > 0 \Leftrightarrow x < \frac{3}{4}$, so f is increasing

on $(-\infty, \frac{3}{4})$, decreasing on $(\frac{3}{4}, 1)$. F. Local maximum value $f(\frac{3}{4}) = \frac{5}{4}$

G. $f''(x) = -\frac{1}{4(1-x)^{3/2}} < 0 \Leftrightarrow x < 1$, so f is CD on $(-\infty, 1)$.

No IP

H.



25. $y = f(x) = x\sqrt{2+x}$ A. $D = [-2, \infty)$ B. y -intercept: $f(0) = 0$; x -intercepts: -2 and 0 C. No

symmetry D. No asymptote E. $f'(x) = \frac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \frac{1}{2\sqrt{2+x}} [x + 2(2+x)] = \frac{3x+4}{2\sqrt{2+x}} = 0$

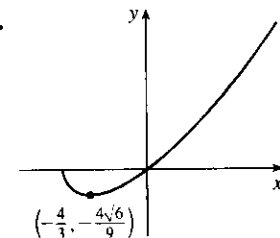
when $x = -\frac{4}{3}$, so f is decreasing on $(-2, -\frac{4}{3})$ and increasing on $(-\frac{4}{3}, \infty)$. F. Local minimum value

$f(-\frac{4}{3}) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$, no local maximum

G. $f''(x) = \frac{2\sqrt{2+x} \cdot 3 - (3x+4)\frac{1}{\sqrt{2+x}}}{4(2+x)}$
 $= \frac{6(2+x) - (3x+4)}{4(2+x)^{3/2}} = \frac{3x+8}{4(2+x)^{3/2}}$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP

H.



26. $y = f(x) = \sqrt{x} - \sqrt[3]{x}$ A. $D = [0, \infty)$ B. y -intercept 0; x -intercepts 0, 1

C. No symmetry D. $\lim_{x \rightarrow \infty} (x^{1/2} - x^{1/3}) = \lim_{x \rightarrow \infty} [x^{1/3}(x^{1/6} - 1)] = \infty$, no asymptote

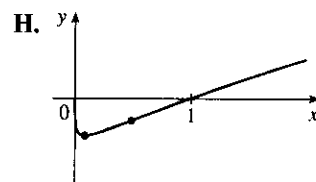
E. $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-2/3} = \frac{3x^{1/6} - 2}{6x^{2/3}} > 0 \Leftrightarrow 3x^{1/6} > 2 \Leftrightarrow x > (\frac{2}{3})^6$, so f is increasing on

$(\frac{2}{3})^6, \infty)$ and decreasing on $(0, (\frac{2}{3})^6)$. F. $f((\frac{2}{3})^6) = -\frac{4}{27}$ is a local minimum value.

G. $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{2}{9}x^{-5/3} = \frac{8 - 9x^{1/6}}{36x^{5/3}} > 0 \Leftrightarrow x^{1/6} < \frac{8}{9}$

$\Leftrightarrow x < (\frac{8}{9})^6$, so f is CU on $(0, (\frac{8}{9})^6)$ and CD on $((\frac{8}{9})^6, \infty)$.

IP at $((\frac{8}{9})^6, -\frac{64}{729})$



27. $y = f(x) = \sin^2 x - 2 \cos x$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = -2$ C. $f(-x) = f(x)$, so

f is symmetric with respect to the y -axis. f has period 2π . D. No asymptote

E. $y' = 2 \sin x \cos x + 2 \sin x = 2 \sin x (\cos x + 1)$. $y' = 0 \Leftrightarrow \sin x = 0$ or $\cos x = -1 \Leftrightarrow x = n\pi$ or $x = (2n+1)\pi$. $y' > 0$ when $\sin x > 0$, since $\cos x + 1 \geq 0$ for all x . Therefore, $y' > 0$ (and so f is increasing)

on $(2n\pi, (2n+1)\pi)$; $y' < 0$ (and so f is decreasing) on $((2n-1)\pi, 2n\pi)$. F. Local maximum values are

$f((2n+1)\pi) = 2$; local minimum values are $f(2n\pi) = -2$. G. $y' = \sin 2x + 2 \sin x \Rightarrow$

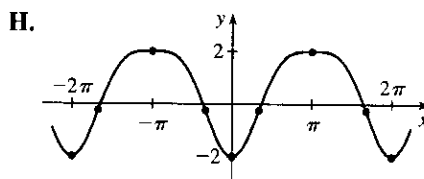
$$y'' = 2 \cos 2x + 2 \cos x = 2(2 \cos^2 x - 1) + 2 \cos x = 4 \cos^2 x + 2 \cos x - 2$$

$$= 2(2 \cos^2 x + \cos x - 1) = 2(2 \cos x - 1)(\cos x + 1)$$

$y'' = 0 \Leftrightarrow \cos x = \frac{1}{2}$ or $-1 \Leftrightarrow x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$. $y'' > 0$ (and so f is CU) on

$(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3})$; $y'' \leq 0$ (and so f is CD) on $(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{5\pi}{3})$. There are inflection points

at $(2n\pi \pm \frac{\pi}{3}, -\frac{1}{4})$.



28. $y = f(x) = 4x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ A. $D = (-\frac{\pi}{2}, \frac{\pi}{2})$. B. y -intercept $= f(0) = 0$ C. $f(-x) = -f(x)$,

so the curve is symmetric about $(0, 0)$. D. $\lim_{x \rightarrow \pi/2^-} (4x - \tan x) = -\infty$, $\lim_{x \rightarrow -\pi/2^+} (4x - \tan x) = \infty$, so $x = \frac{\pi}{2}$

and $x = -\frac{\pi}{2}$ are VA. E. $f'(x) = 4 - \sec^2 x > 0 \Leftrightarrow \sec x < 2 \Leftrightarrow$

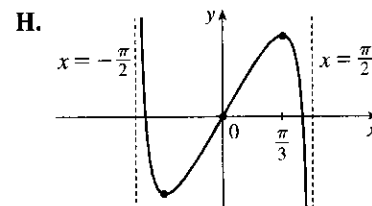
$\cos x > \frac{1}{2} \Leftrightarrow -\frac{\pi}{3} < x < \frac{\pi}{3}$, so f is increasing on $(-\frac{\pi}{3}, \frac{\pi}{3})$ and

decreasing on $(-\frac{\pi}{2}, -\frac{\pi}{3})$ and $(\frac{\pi}{3}, \frac{\pi}{2})$. F. $f(\frac{\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$ is a

local maximum value, $f(-\frac{\pi}{3}) = \sqrt{3} - \frac{4\pi}{3}$ is a local minimum value.

G. $f''(x) = -2\sec^2 x \tan x > 0 \Leftrightarrow \tan x < 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$,

so f is CU on $(-\frac{\pi}{2}, 0)$ and CD on $(0, \frac{\pi}{2})$. IP at $(0, 0)$



$$29. f(x) = \frac{x^2 - 1}{x^3} \Rightarrow f'(x) = \frac{x^3(2x) - (x^2 - 1)3x^2}{x^6} = \frac{3 - x^2}{x^4} \Rightarrow$$

$$f''(x) = \frac{x^4(-2x) - (3 - x^2)4x^3}{x^8} = \frac{2x^2 - 12}{x^5}$$

Estimates: From the graphs of f' and f'' , it appears that f is increasing on

$(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$;

f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of

about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD

on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about

$(-2.45, -0.34)$ and $(2.45, 0.34)$.

Exact: Now $f'(x) = \frac{3 - x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is

increasing on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$; and $f'(x)$ is negative (and so f is

decreasing) on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$.

f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of

$f(\sqrt{3}) = \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^3} = \frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on

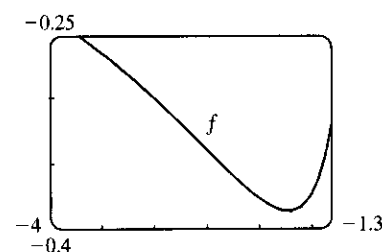
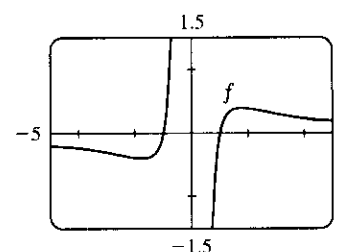
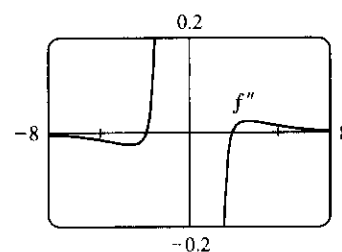
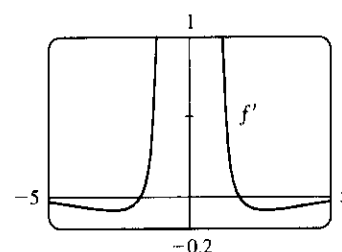
the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local

minimum of $f(-\sqrt{3}) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \frac{2x^2 - 12}{x^5}$ is positive (so

f is CU) on $(-\sqrt{6}, 0)$ and $(\sqrt{6}, \infty)$, and negative (so f is CD) on

$(-\infty, -\sqrt{6})$ and $(0, \sqrt{6})$. There are IP at $(\sqrt{6}, \frac{5\sqrt{6}}{36})$ and

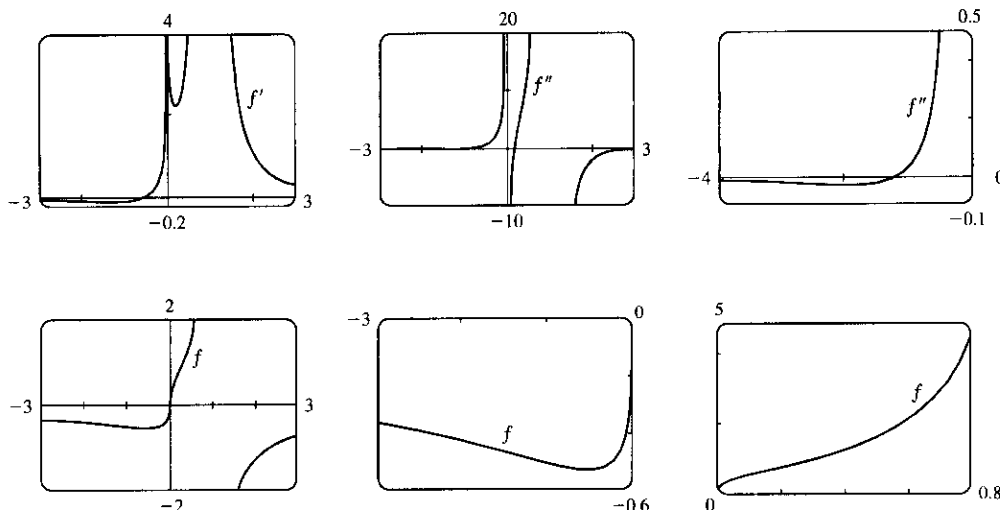
$(-\sqrt{6}, -\frac{5\sqrt{6}}{36})$.



$$30. f(x) = \frac{\sqrt[3]{x}}{1-x} = x^{1/3}(1-x)^{-1} \Rightarrow$$

$$f'(x) = x^{1/3}(-1)(1-x)^{-2}(-1) + (1-x)^{-1}\left(\frac{1}{3}\right)x^{-2/3} = \frac{x^{-2/3}}{3} \frac{1+2x}{(x-1)^2} \Rightarrow$$

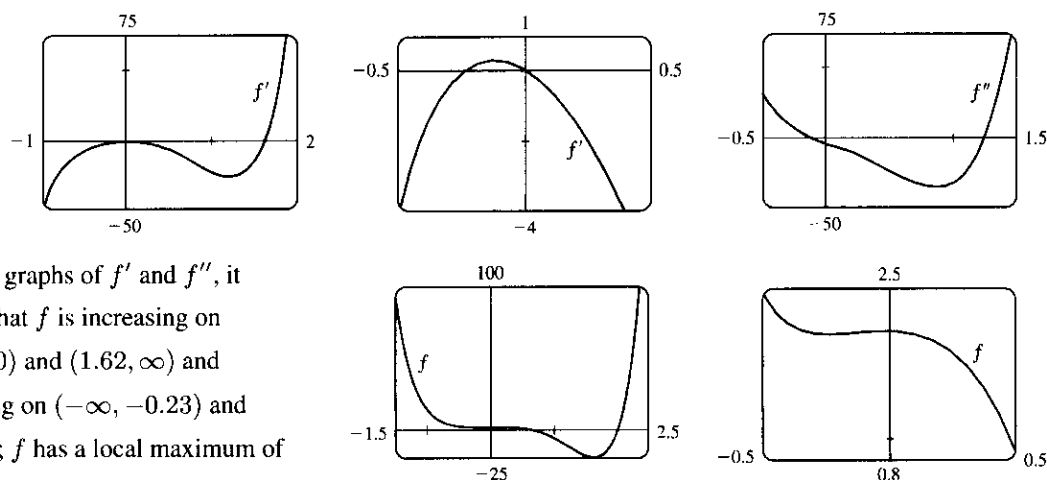
$$f''(x) = \frac{x^{-2/3}}{3} \frac{(x-1)^2(2) - (1+2x)(2)(x-1)}{(x-1)^4} + \frac{1+2x}{(x-1)^2} \left(\frac{-2x^{-5/3}}{9} \right) = -\frac{2x^{-5/3}}{9} \frac{5x^2+5x-1}{(x-1)^3}$$



From the graphs, it appears that f is increasing on $(-0.50, 1)$ and $(1, \infty)$, with a vertical asymptote at $x = 1$, and decreasing on $(-\infty, -0.50)$; f has no local maximum, but a local minimum of about $f(-0.50) = -0.53$; f is CU on $(-1.17, 0)$ and $(0.17, 1)$ and CD on $(-\infty, -1.17)$, $(0, 0.17)$ and $(1, \infty)$; and f has inflection points at about $(-1.17, -0.49)$, $(0, 0)$ and $(0.17, 0.67)$. Note also that $\lim_{x \rightarrow \pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

$$31. f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \Rightarrow f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \Rightarrow$$

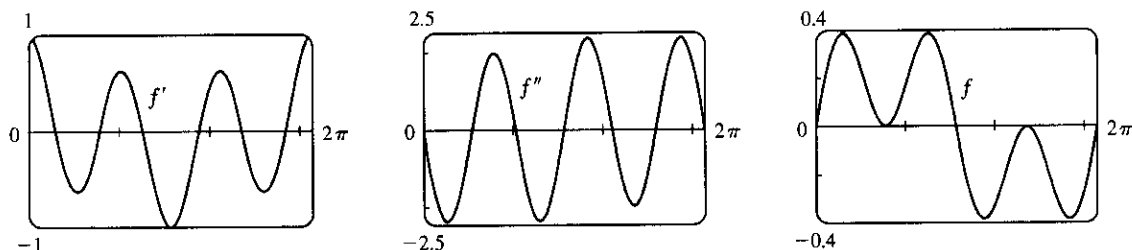
$$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$$



From the graphs of f' and f'' , it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of about $f(0) = 2$ and local minima

of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.

$$32. f(x) = \sin x \cos^2 x \Rightarrow f'(x) = \cos^3 x - 2 \sin^2 x \cos x \Rightarrow f''(x) = -7 \sin x \cos^2 x + 2 \sin^3 x$$



From the graphs of f' and f'' , it appears that f is increasing on $(0, 0.62)$, $(1.57, 2.53)$, $(3.76, 4.71)$ and $(5.67, 2\pi)$ and decreasing on $(0.62, 1.57)$, $(2.53, 3.76)$ and $(4.71, 5.67)$; f has local maxima of about $f(0.62) = f(2.53) = 0.38$ and $f(4.71) = 0$ and local minima of about $f(1.57) = 0$ and $f(3.76) = f(5.67) = -0.38$; f is CU on $(1.08, 2.06)$, $(3.14, 4.22)$ and $(5.20, 2\pi)$ and CD on $(0, 1.08)$, $(2.06, 3.14)$ and $(4.22, 5.20)$; and f has inflection points at about $(0, 0)$, $(1.08, 0.20)$, $(2.06, 0.20)$, $(3.14, 0)$, $(4.22, -0.20)$, $(5.20, -0.20)$ and $(2\pi, 0)$.

33. $f(x) = x^{101} + x^{51} + x - 1 = 0$. Since f is continuous and $f(0) = -1$ and $f(1) = 2$, the equation has at least one root in $(0, 1)$, by the Intermediate Value Theorem. Suppose the equation has two roots, a and b , with $a < b$.

Then $f(a) = 0 = f(b)$, so by the Mean Value Theorem, there is a number x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0, \text{ so } f' \text{ has a root in } (a, b). \text{ But this is impossible since}$$

$$f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1 \text{ for all } x.$$

34. By the Mean Value Theorem, $f'(c) = \frac{f(4) - f(0)}{4 - 0} \Leftrightarrow 4f'(c) = f(4) - 1$ for some c with $0 < c < 4$. Since

$$2 \leq f'(c) \leq 5, \text{ we have } 4(2) \leq 4f'(c) \leq 4(5) \Leftrightarrow 4(2) \leq f(4) - 1 \leq 4(5) \Leftrightarrow 8 \leq f(4) - 1 \leq 20 \Leftrightarrow 9 \leq f(4) \leq 21.$$

35. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a

$$\text{number } c \text{ in } (32, 33) \text{ such that } f'(c) = \frac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2, \text{ but } \frac{1}{5}c^{-4/5} > 0 \Rightarrow \sqrt[5]{33} - 2 > 0$$

$$\Rightarrow \sqrt[5]{33} > 2. \text{ Also } f' \text{ is decreasing, so that } f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \Rightarrow$$

$$0.0125 > f'(c) = \sqrt[5]{33} - 2 \Rightarrow \sqrt[5]{33} < 2.0125. \text{ Therefore, } 2 < \sqrt[5]{33} < 2.0125.$$

36. For $(1, 6)$ to be on the curve $y = x^3 + ax^2 + bx + 1$, we have that $6 = 1 + a + b + 1 \Rightarrow b = 4 - a$. Now

$$y' = 3x^2 + 2ax + b \text{ and } y'' = 6x + 2a. \text{ Also, for } (1, 6) \text{ to be an inflection point it must be true that}$$

$$y''(1) = 6(1) + 2a = 0 \Rightarrow a = -3 \Rightarrow b = 4 - (-3) = 7. \text{ Note that with } a = -3, \text{ we have}$$

$$y'' = 6x - 6 = 6(x - 1), \text{ so } y'' \text{ changes sign at } x = 1, \text{ proving that } (1, 6) \text{ is a point of inflection. [This does not follow from the fact that } y''(1) = 0.]$$

37. (a) $g(x) = f(x^2) \Rightarrow g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \neq 0$, we must have

$$f'(x^2) > 0 \text{ for } x \neq 0, \text{ so } g'(x) = 0 \Leftrightarrow x = 0. \text{ Now } g'(x) \text{ changes sign (from negative to positive) at}$$

$x = 0$, since one of its factors, $f'(x^2)$, is positive for all x , and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

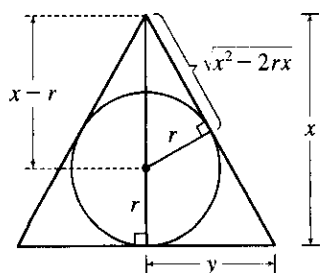
- (b) $g'(x) = 2xf'(x^2) \Rightarrow g''(x) = 2[xf''(x^2)(2x) + f'(x^2)] = 4x^2f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \neq 0$, $f''(x^2) > 0$ (since f is CU for $x > 0$), and $f'(x^2) > 0$ for all $x \neq 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \neq 0$. Whether $g''(0)$ is positive or 0 doesn't matter (since the sign of g'' does not change there); g is concave upward on $\mathbb{R}$.

38. Call the two integers x and y . Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \Leftrightarrow y = 125$. $P''(y) = -8 < 0$, so $P(125) = 62,500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

39. If $B = 0$, the line is vertical and the distance from $x = -\frac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \frac{C}{A}\right| = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so we minimize $f(x) = (x - x_1)^2 + \left(-\frac{A}{B}x - \frac{C}{B} - y_1\right)^2 \Rightarrow$
 $f'(x) = 2(x - x_1) + 2\left(-\frac{A}{B}x - \frac{C}{B} - y_1\right)\left(-\frac{A}{B}\right)$. $f'(x) = 0 \Rightarrow x = \frac{B^2x_1 - AB y_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \frac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives
 $f(x) = \frac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is $\sqrt{f(x)} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

40. On the hyperbola $xy = 8$, if $d(x)$ is the distance from the point $(x, y) = (x, 8/x)$ to the point $(3, 0)$, then $[d(x)]^2 = (x - 3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x - 3) - 128/x^3 = 0 \Rightarrow x^4 - 3x^3 - 64 = 0 \Rightarrow (x - 4)(x^3 + x^2 + 4x + 16) = 0 \Rightarrow x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

41.



By similar triangles, $\frac{y}{x} = \frac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

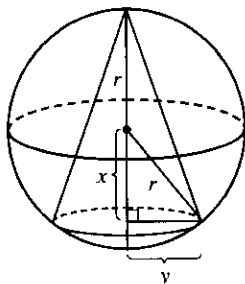
$$A(x) = \frac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2 - 2rx}} \Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx}$$

$$= \frac{rx^2(x - 3r)}{(x^2 - 2rx)^{3/2}} = 0 \text{ when } x = 3r.$$

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = r(9r^2)/(\sqrt{3}r) = 3\sqrt{3}r^2$.

42.



The volume of the cone is

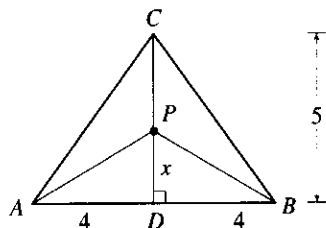
$$V = \frac{1}{3}\pi y^2(r+x) = \frac{1}{3}\pi(r^2 - x^2)(r+x), \quad -r \leq x \leq r.$$

$$\begin{aligned} V'(x) &= \frac{\pi}{3} [(r^2 - x^2)(1) + (r+x)(-2x)] \\ &= \frac{\pi}{3} [(r+x)(r-x-2x)] = \frac{\pi}{3}(r+x)(r-3x) \\ &= 0 \text{ when } x = -r \text{ or } x = r/3. \end{aligned}$$

 Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$

$$\text{and the volume is } V\left(\frac{r}{3}\right) = \frac{\pi}{3}\left(r^2 - \frac{r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81}.$$

43.



We minimize

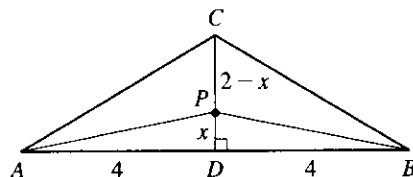
$$L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2 + 16} + (5 - x),$$

$$0 \leq x \leq 5. \quad L'(x) = 2x/\sqrt{x^2 + 16} - 1 = 0 \Leftrightarrow$$

$$2x = \sqrt{x^2 + 16} \Leftrightarrow 4x^2 = x^2 + 16 \Leftrightarrow x = \frac{4}{\sqrt{3}}.$$

$L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

44.


 If $|CD| = 2$, the last part of $L(x)$ changes from $(5 - x)$ to

$$(2 - x) \text{ with } 0 \leq x \leq 2. \text{ But we still get } L'(x) = 0 \Leftrightarrow$$

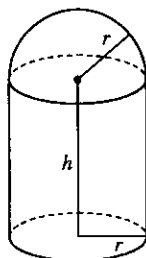
$$x = \frac{4}{\sqrt{3}}, \text{ which isn't in the interval } [0, 2]. \text{ Now } L(0) = 10$$

and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

$$45. \quad v = K\sqrt{\frac{L}{C} + \frac{C}{L}} \Rightarrow \frac{dv}{dL} = \frac{K}{2\sqrt{(L/C) + (C/L)}}\left(\frac{1}{C} - \frac{C}{L^2}\right) = 0 \Leftrightarrow \frac{1}{C} = \frac{C}{L^2} \Leftrightarrow L^2 = C^2 \Leftrightarrow$$

$L = C$. This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

46.


 We minimize the surface area $S = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi rh$.

Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for h , we get $h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \frac{V}{\pi r^2} - \frac{2}{3}r$, so

$$S(r) = 3\pi r^2 + 2\pi r\left[\frac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \frac{2V}{r}.$$

$$S'(r) = -\frac{2V}{r^2} + \frac{10}{3}\pi r = \frac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \Leftrightarrow \frac{10}{3}\pi r^3 = 2V$$

$$\Leftrightarrow r^3 = \frac{3V}{5\pi} \Leftrightarrow r = \sqrt[3]{\frac{3V}{5\pi}}. \text{ This gives an absolute minimum since } S'(r) < 0 \text{ for } 0 < r < \sqrt[3]{\frac{3V}{5\pi}} \text{ and}$$

$$S'(r) > 0 \text{ for } r > \sqrt[3]{\frac{3V}{5\pi}}. \text{ Thus, } h = \frac{V - \frac{2}{3}\pi \cdot \frac{3V}{5\pi}}{\pi \sqrt[3]{\frac{3V}{5\pi}}} = \frac{(V - \frac{2}{5}V) \sqrt[3]{(5\pi)^2}}{\pi \sqrt[3]{(3V)^2}} = \frac{3V \sqrt[3]{(5\pi)^2}}{5\pi \sqrt[3]{(3V)^2}} = \sqrt[3]{\frac{3V}{5\pi}} = r.$$

47. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11,000 + 1000(x)$. Now the revenue per game is

$$\begin{aligned} R(x) &= (\text{price per person}) \times (\text{number of people per game}) \\ &= (12 - x)(11,000 + 1000x) = -1000x^2 + 1000x + 132,000 \end{aligned}$$

for $0 \leq x \leq 4$ (since the seating capacity is 15,000) $\Rightarrow R'(x) = -2000x + 1000 = 0 \Leftrightarrow x = 0.5$.

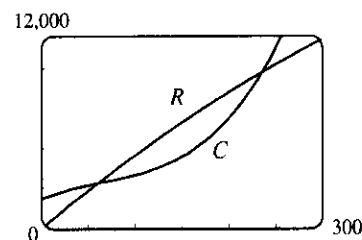
This is a maximum since $R''(x) = -2000 < 0$ for all x . Now we must check the value of

$R(x) = (12 - x)(11,000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R . $R(0) = (12)(11,000) = 132,000$, $R(0.5) = (11.5)(11,500) = 132,250$, and $R(4) = (8)(15,000) = 120,000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

48. (a) $C'(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and

$R(x) = xp(x) = 48.2x - 0.03x^2$. The profit is maximized when $C'(x) = R'(x)$.

From the figure, we estimate that the tangents are parallel when $x \approx 160$.



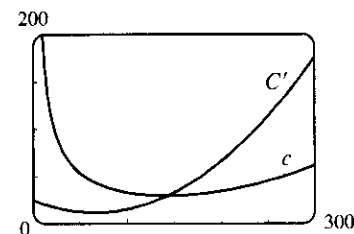
- (b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x) \Rightarrow$

$$0.003x^2 - 0.34x - 23.2 = 0 \Rightarrow x_1 \approx 161.3 \ (x > 0). \quad R''(x) = -0.06 \text{ and}$$

$C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57 \Rightarrow$ profit is maximized by producing 161 units.

- (c) $c(x) = \frac{C(x)}{x} = \frac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost.

Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection. From the figure, $C'(x) = c(x) \Leftrightarrow x \approx 144$.



49. $f(x) = x^5 - x^4 + 3x^2 - 3x - 2 \Rightarrow f'(x) = 5x^4 - 4x^3 + 6x - 3$, so

$$x_{n+1} = x_n - \frac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}. \text{ Now } x_1 = 1 \Rightarrow x_2 = 1.5 \Rightarrow x_3 \approx 1.343860 \Rightarrow$$

$x_4 \approx 1.300320 \Rightarrow x_5 \approx 1.297396 \Rightarrow x_6 \approx 1.297383 \approx x_7$, so the root in $[1, 2]$ is 1.297383, to six decimal places.

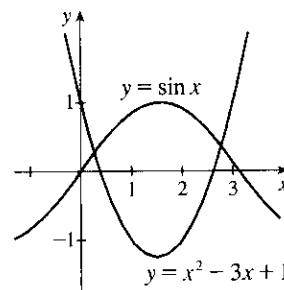
50. Graphing $y = \sin x$ and $y = x^2 - 3x + 1$ shows that there are two roots, one about 0.3 and the other about 2.8.

$$f(x) = \sin x - x^2 + 3x - 1 \Rightarrow f'(x) = \cos x - 2x + 3 \Rightarrow$$

$$x_{n+1} = x_n - \frac{\sin x_n - x_n^2 + 3x_n - 1}{\cos x_n - 2x_n + 3}. \text{ Now } x_1 = 0.3 \Rightarrow$$

$$x_2 \approx 0.268552 \Rightarrow x_3 \approx 0.268881 \approx x_4 \text{ and } x_1 = 2.8 \Rightarrow$$

$x_2 \approx 2.770354 \Rightarrow x_3 \approx 2.770058 \approx x_4$, so to six decimal places, the roots are 0.268881 and 2.770058.



51. $f(t) = \cos t + t - t^2 \Rightarrow f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists

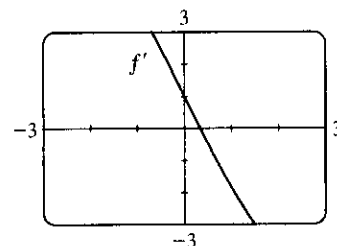
for all t , so to find the maximum of f , we can examine the zeros of f' .

From the graph of f' , we see that a good choice for t_1 is $t_1 = 0.3$.

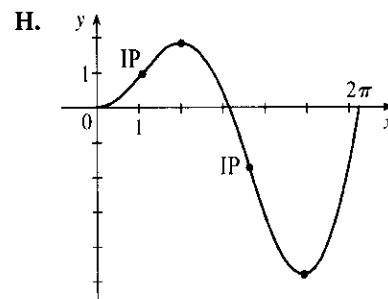
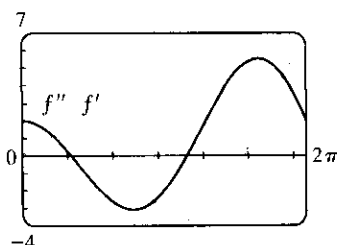
Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain

$$t_2 \approx 0.33535293, t_3 \approx 0.33541803 \approx t_4. \text{ Since}$$

$f''(t) = -\cos t - 2 < 0$ for all t , $f(0.33541803) \approx 1.16718557$ is the absolute maximum.



52. $y = f(x) = x \sin x$, $0 \leq x \leq 2\pi$. **A.** $D = [0, 2\pi]$ **B.** y -intercept: $f(0) = 0$; x -intercepts: $f(x) = 0 \Leftrightarrow x = 0$ or $\sin x = 0 \Leftrightarrow x = 0, \pi$, or 2π . **C.** There is no symmetry on D , but if f is defined for all real numbers x , then f is an even function. **D.** No asymptote **E.** $f'(x) = x \cos x + \sin x$. To find critical numbers in $(0, 2\pi)$, we graph f' and see that there are two critical numbers, about 2 and 4.9. To find them more precisely, we use Newton's method, setting $g(x) = f'(x) = x \cos x + \sin x$, so that $g'(x) = f''(x) = 2 \cos x - x \sin x$ and
- $$x_{n+1} = x_n - \frac{x_n \cos x_n + \sin x_n}{2 \cos x_n - x_n \sin x_n}. \quad x_1 = 2 \Rightarrow x_2 \approx 2.029048, x_3 \approx 2.028758 \approx x_4 \text{ and } x_1 = 4.9 \Rightarrow$$
- $$x_2 \approx 4.913214, x_3 \approx 4.913180 \approx x_4, \text{ so the critical numbers, to six decimal places, are } r_1 = 2.028758 \text{ and } r_2 = 4.913180.$$
- By checking sample values of f' in $(0, r_1)$, (r_1, r_2) , and $(r_2, 2\pi)$, we see that f is increasing on $(0, r_1)$, decreasing on (r_1, r_2) , and increasing on $(r_2, 2\pi)$. **F.** Local maximum value $f(r_1) \approx 1.819706$, local minimum value $f(r_2) \approx -4.814470$. **G.** $f''(x) = 2 \cos x - x \sin x$. To find points where $f''(x) = 0$, we graph f'' and find that $f''(x) = 0$ at about 1 and 3.6. To find the values more precisely, we use Newton's method. Set
- $$h(x) = f''(x) = 2 \cos x - x \sin x. \text{ Then } h'(x) = -3 \sin x - x \cos x, \text{ so } x_{n+1} = x_n - \frac{2 \cos x_n - x_n \sin x_n}{-3 \sin x_n - x_n \cos x_n}.$$
- $$x_1 = 1 \Rightarrow x_2 \approx 1.078028, x_3 \approx 1.076874 \approx x_4 \text{ and } x_1 = 3.6 \Rightarrow x_2 \approx 3.643996, x_3 \approx 3.643597 \approx x_4,$$
- so the zeros of f'' , to six decimal places, are $r_3 = 1.076874$ and $r_4 = 3.643597$. By checking sample values of f'' in $(0, r_3)$, (r_3, r_4) , and $(r_4, 2\pi)$, we see that f is CU on $(0, r_3)$, CD on (r_3, r_4) , and CU on $(r_4, 2\pi)$. f has inflection points at $(r_3, f(r_3) \approx 0.948166)$ and $(r_4, f(r_4) \approx -1.753240)$.



$$53. f'(x) = \sqrt{x^5} - 4/\sqrt[5]{x} = x^{5/2} - 4x^{-1/5} \Rightarrow f(x) = \frac{2}{7}x^{7/2} - 4\left(\frac{5}{4}x^{4/5}\right) + C = \frac{2}{7}x^{7/2} - 5x^{4/5} + C$$

$$54. f'(x) = 8x - 3\sec^2 x \Rightarrow f(x) = 8\left(\frac{1}{2}x^2\right) - 3\tan x + C_n = 4x^2 - 3\tan x + C_n \text{ on the interval } \left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right).$$

$$55. f'(t) = 2t - 3\sin t \Rightarrow f(t) = t^2 + 3\cos t + C.$$

$$f(0) = 3 + C \text{ and } f(0) = 5 \Rightarrow C = 2, \text{ so } f(t) = t^2 + 3\cos t + 2.$$

$$56. f'(u) = \frac{u^2 + \sqrt{u}}{u} = u + u^{-1/2} \Rightarrow f(u) = \frac{1}{2}u^2 + 2u^{1/2} + C.$$

$$f(1) = \frac{1}{2} + 2 + C \text{ and } f(1) = 3 \Rightarrow C = \frac{1}{2}, \text{ so } f(u) = \frac{1}{2}u^2 + 2\sqrt{u} + \frac{1}{2}.$$

$$57. f''(x) = 1 - 6x + 48x^2 \Rightarrow f'(x) = x - 3x^2 + 16x^3 + C. \quad f'(0) = C \text{ and } f'(0) = 2 \Rightarrow C = 2, \text{ so } f'(x) = x - 3x^2 + 16x^3 + 2 \text{ and hence, } f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D. \quad f(0) = D \text{ and } f(0) = 1 \Rightarrow D = 1, \text{ so } f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1.$$

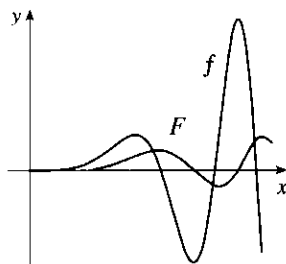
$$58. f''(x) = 2x^3 + 3x^2 - 4x + 5 \Rightarrow f'(x) = \frac{1}{2}x^4 + x^3 - 2x^2 + 5x + C \Rightarrow$$

$$f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 + Cx + D. \quad f(0) = D \text{ and } f(0) = 2 \Rightarrow D = 2.$$

$$f(1) = \frac{1}{10} + \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + C + 2 \text{ and } f(1) = 0 \Rightarrow C = -\frac{6}{60} - \frac{15}{60} + \frac{40}{60} - \frac{150}{60} - \frac{120}{60} = -\frac{251}{60}, \text{ so}$$

$$f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - \frac{251}{60}x + 2.$$

$$59. f(x) = x^2 \sin(x^2), \quad 0 \leq x \leq \pi$$



$$60. f(x) = x^4 + x^3 + cx^2 \Rightarrow f'(x) = 4x^3 + 3x^2 + 2cx. \text{ This is 0 when } x(4x^2 + 3x + 2c) = 0 \Leftrightarrow$$

$x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the roots of this last equation are

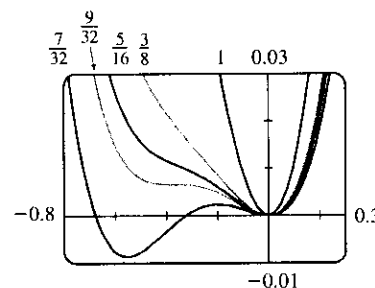
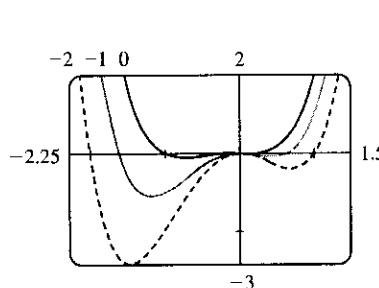
$$x = \frac{-3 \pm \sqrt{9 - 32c}}{8}. \text{ Now if } 9 - 32c < 0 \Leftrightarrow c > \frac{9}{32}, \text{ then } (0, 0) \text{ is the only critical point, a minimum.}$$

If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the root with the + sign coincides with the critical point at $x = 0$. For $0 < c < \frac{9}{32}$, there is a minimum at

$$x = -\frac{3}{8} - \frac{\sqrt{9 - 32c}}{8}, \text{ a maximum at } x = -\frac{3}{8} + \frac{\sqrt{9 - 32c}}{8}, \text{ and a minimum at } x = 0. \text{ For } c = 0, \text{ there is a}$$

minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\frac{3}{8} \pm \frac{\sqrt{9-32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$. The roots of this equation are $x = \frac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \leq 0 \Leftrightarrow c \geq \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\frac{1}{4} \pm \frac{\sqrt{9-24c}}{12}$.

Value of c	No. of CP	No. of IP
$c < 0$	3	2
$c = 0$	2	2
$0 < c < \frac{9}{32}$	3	2
$c = \frac{9}{32}$	2	2
$\frac{9}{32} < c < \frac{3}{8}$	1	2
$c \geq \frac{3}{8}$	1	0



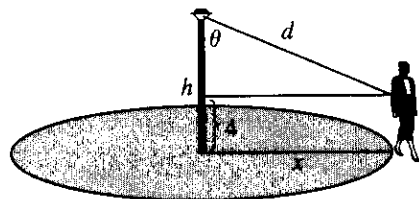
61. Choosing the positive direction to be upward, we have $a(t) = -9.8 \Rightarrow v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0 \Rightarrow v(t) = -9.8t = s'(t) \Rightarrow s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \Rightarrow s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \Rightarrow t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \Rightarrow v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995$ m/s. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst*.

62. Let $s_A(t)$ and $s_B(t)$ be the position functions for cars A and B and let $f(t) = s_A(t) - s(t)$. Since A passed B twice, there must be three values of t such that $f(t) = 0$. Then by three applications of Rolle's Theorem (see Exercise 4.2.22), there is a number c such that $f''(c) = 0$. So $s_A''(c) = s_B''(c)$, that is, A and B had equal accelerations at $t = c$. We assume that f is continuous on $[0, T]$ and twice differentiable on $(0, T)$, where T is the total time of the race.

$$\begin{aligned}
 63. (a) I &= \frac{k \cos \theta}{d^2} = \frac{k(h/d)}{d^2} = k \frac{h}{d^3} = k \frac{h}{(\sqrt{40^2 + h^2})^3} = k \frac{h}{(1600 + h^2)^{3/2}} \Rightarrow \\
 \frac{dI}{dh} &= k \frac{(1600 + h^2)^{3/2} - h \cdot \frac{3}{2} (1600 + h^2)^{1/2} \cdot 2h}{[(1600 + h^2)^{3/2}]^2} = \frac{k(1600 + h^2)^{1/2} (1600 + h^2 - 3h^2)}{(1600 + h^2)^3} \\
 &= \frac{k(1600 - 2h^2)}{(1600 + h^2)^{5/2}} \quad [k \text{ is the constant of proportionality}]
 \end{aligned}$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \Rightarrow h^2 = 800 \Rightarrow h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)



$$\frac{dx}{dt} = 4 \text{ ft/s}$$

$$I = \frac{k \cos \theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3} = \frac{k(h-4)}{[(h-4)^2 + x^2]^{3/2}} = k(h-4)[(h-4)^2 + x^2]^{-3/2}$$

$$\begin{aligned} \frac{dI}{dt} &= \frac{dI}{dx} \cdot \frac{dx}{dt} = k(h-4)\left(-\frac{3}{2}\right)[(h-4)^2 + x^2]^{-5/2} \cdot 2x \cdot \frac{dx}{dt} \\ &= k(h-4)(-3x)[(h-4)^2 + x^2]^{-5/2} \cdot 4 = \frac{-12xk(h-4)}{[(h-4)^2 + x^2]^{5/2}} \end{aligned}$$

$$\left. \frac{dI}{dt} \right|_{x=40} = -\frac{480k(h-4)}{[(h-4)^2 + 1600]^{5/2}}$$

□ PROBLEMS PLUS

- Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since f has period 2π . $f'(x) = \cos x + \sin x = 0 \Leftrightarrow \cos x = -\sin x \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating f at its critical numbers and endpoints, we get $f(0) = -1$, $f(\frac{3\pi}{4}) = \sqrt{2}$, $f(\frac{7\pi}{4}) = -\sqrt{2}$, and $f(2\pi) = -1$. So f has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2} \Rightarrow |\sin x - \cos x| \leq \sqrt{2}$.
- $x^2 y^2 (4 - x^2)(4 - y^2) = x^2(4 - x^2)y^2(4 - y^2) = f(x)f(y)$, where $f(t) = t^2(4 - t^2)$. We will show that $0 \leq f(t) \leq 4$ for $|t| \leq 2$, which gives $0 \leq f(x)f(y) \leq 16$ for $|x| \leq 2$ and $|y| \leq 2$.
 $f(t) = 4t^2 - t^4 \Rightarrow f'(t) = 8t - 4t^3 = 4t(2 - t^2) = 0 \Rightarrow t = 0$ or $\pm\sqrt{2}$.
 $f(0) = 0$, $f(\pm\sqrt{2}) = 2(4 - 2) = 4$, and $f(2) = 0$. So 0 is the absolute minimum value of $f(t)$ on $[-2, 2]$ and 4 is the absolute maximum value of $f(t)$ on $[-2, 2]$. We conclude that $0 \leq f(t) \leq 4$ for $|t| \leq 2$ and hence, $0 \leq f(x)f(y) \leq 4^2$ or $0 \leq x^2(4 - x^2)y^2(4 - y^2) \leq 16$.
- First we show that $x(1 - x) \leq \frac{1}{4}$ for all x . Let $f(x) = x(1 - x) = x - x^2$. Then $f'(x) = 1 - 2x$. This is 0 when $x = \frac{1}{2}$ and $f'(x) > 0$ for $x < \frac{1}{2}$, $f'(x) < 0$ for $x > \frac{1}{2}$, so the absolute maximum of f is $f(\frac{1}{2}) = \frac{1}{4}$. Thus, $x(1 - x) \leq \frac{1}{4}$ for all x .
 Now suppose that the given assertion is false, that is, $a(1 - b) > \frac{1}{4}$ and $b(1 - a) > \frac{1}{4}$. Multiply these inequalities: $a(1 - b)b(1 - a) > \frac{1}{16} \Rightarrow [a(1 - a)][b(1 - b)] > \frac{1}{16}$. But we know that $a(1 - a) \leq \frac{1}{4}$ and $b(1 - b) \leq \frac{1}{4} \Rightarrow [a(1 - a)][b(1 - b)] \leq \frac{1}{16}$. Thus, we have a contradiction, so the given assertion is proved.
- Let $P(a, 1 - a^2)$ be the point of contact. The equation of the tangent line at P is $y - (1 - a^2) = (-2a)(x - a) \Rightarrow y - 1 + a^2 = -2ax + 2a^2 \Rightarrow y = -2ax + a^2 + 1$. To find the x -intercept, put $y = 0$: $2ax = a^2 + 1 \Rightarrow x = \frac{a^2 + 1}{2a}$. To find the y -intercept, put $x = 0$: $y = a^2 + 1$. Therefore, the area of the triangle is $\frac{1}{2} \left(\frac{a^2 + 1}{2a} \right) (a^2 + 1) = \frac{(a^2 + 1)^2}{4a}$. Therefore, we minimize the function $A(a) = \frac{(a^2 + 1)^2}{4a}$, $0 < a \leq 1$.
 $A'(a) = \frac{(4a)2(a^2 + 1)(2a) - (a^2 + 1)^2(4)}{16a^2} = \frac{(a^2 + 1)[4a^2 - (a^2 + 1)]}{4a^2} = \frac{(a^2 + 1)(3a^2 - 1)}{4a^2}$.
 $A'(a) = 0$ when $3a^2 - 1 = 0 \Rightarrow a = \frac{1}{\sqrt{3}}$. $A'(a) < 0$ for $a < \frac{1}{\sqrt{3}}$, $A'(a) > 0$ for $a > \frac{1}{\sqrt{3}}$. So by the First Derivative Test, there is an absolute minimum when $a = \frac{1}{\sqrt{3}}$. The required point is $(\frac{1}{\sqrt{3}}, \frac{2}{3})$ and the corresponding minimum area is $A(\frac{1}{\sqrt{3}}) = \frac{4\sqrt{3}}{9}$.
- Differentiating $x^2 + xy + y^2 = 12$ implicitly with respect to x gives $2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$. At a highest or lowest point, $\frac{dy}{dx} = 0 \Leftrightarrow y = -2x$. Substituting $-2x$ for y in the original equation gives $x^2 + x(-2x) + (-2x)^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. If $x = 2$, then $y = -2x = -4$, and if $x = -2$ then $y = 4$. Thus, the highest and lowest points are $(-2, 4)$ and $(2, -4)$.
- (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height H of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of H is concave upward. Therefore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

$$7. f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-2|}$$

$$= \begin{cases} \frac{1}{1-x} + \frac{1}{1-(x-2)} & \text{if } x < 0 \\ \frac{1}{1+x} + \frac{1}{1-(x-2)} & \text{if } 0 \leq x < 2 \\ \frac{1}{1+x} + \frac{1}{1+(x-2)} & \text{if } x \geq 2 \end{cases} \Rightarrow f'(x) = \begin{cases} \frac{1}{(1-x)^2} + \frac{1}{(3-x)^2} & \text{if } x < 0 \\ \frac{-1}{(1+x)^2} + \frac{1}{(3-x)^2} & \text{if } 0 < x < 2 \\ \frac{-1}{(1+x)^2} - \frac{1}{(x-1)^2} & \text{if } x > 2 \end{cases}$$

We see that $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 2$. For $0 < x < 2$, we have

$$f'(x) = \frac{1}{(3-x)^2} - \frac{1}{(x+1)^2} = \frac{(x^2 + 2x + 1) - (x^2 - 6x + 9)}{(3-x)^2(x+1)^2} = \frac{8(x-1)}{(3-x)^2(x+1)^2}, \text{ so } f'(x) < 0 \text{ for}$$

$0 < x < 1$, $f'(1) = 0$ and $f'(x) > 0$ for $1 < x < 2$. We have shown that $f'(x) > 0$ for $x < 0$; $f'(x) < 0$ for $0 < x < 1$; $f'(x) > 0$ for $1 < x < 2$; and $f'(x) < 0$ for $x > 2$. Therefore, by the First Derivative Test, the local maxima of f are at $x = 0$ and $x = 2$, where f takes the value $\frac{4}{3}$. Therefore, $\frac{4}{3}$ is the absolute maximum value of f .

8. If $f''(x) > 0$ for all x , then f' is increasing on $(-\infty, \infty)$, so $f'(0)$ must be greater than $f'(-1)$. But $f'(0) = 0 < \frac{1}{2} = f'(-1)$, so such a function cannot exist.

9. $A = (x_1, x_1^2)$ and $B = (x_2, x_2^2)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = (x, x^2)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB . Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$f(x) = \text{area}(A_1ABB_1) - \text{area}(A_1APP_1) - \text{area}(B_1BPP_1)$$

$$= \frac{1}{2}(x_1^2 + x_2^2)(x_2 - x_1) - \frac{1}{2}(x_1^2 + x^2)(x - x_1) - \frac{1}{2}(x^2 + x_2^2)(x_2 - x)$$

After expanding and canceling terms, we get

$$f(x) = \frac{1}{2}(x_2x_1^2 - x_1x_2^2 - x_1^2x + x_1x^2 - x_2x^2 + x_2x_1^2) = \frac{1}{2}[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)]$$

$$f'(x) = \frac{1}{2}[-x_1^2 + x_2^2 + 2x(x_1 - x_2)]. \quad f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0 \text{ since } x_2 > x_1.$$

$$f'(x) = 0 \Rightarrow 2x(x_1 - x_2) = x_1^2 - x_2^2 \Rightarrow x_P = \frac{1}{2}(x_1 + x_2).$$

$$f(x_P) = \frac{1}{2}(x_1^2[\frac{1}{2}(x_2 - x_1)] + x_2^2[\frac{1}{2}(x_2 - x_1)] + \frac{1}{4}(x_1 + x_2)^2(x_1 - x_2))$$

$$= \frac{1}{2}[\frac{1}{2}(x_2 - x_1)(x_1^2 + x_2^2) - \frac{1}{4}(x_2 - x_1)(x_1 + x_2)^2]$$

$$= \frac{1}{8}(x_2 - x_1)[2(x_1^2 + x_2^2) - (x_1^2 + 2x_1x_2 + x_2^2)]$$

$$= \frac{1}{8}(x_2 - x_1)(x_1^2 - 2x_1x_2 + x_2^2) = \frac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \frac{1}{8}(x_2 - x_1)(x_2 - x_1)^2$$

$$= \frac{1}{8}(x_2 - x_1)^3$$

To put this in terms of m and b , we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \Rightarrow x_1 = \frac{1}{2}(m - \sqrt{m^2 + 4b})$. Similarly, $x_2 = \frac{1}{2}(m + \sqrt{m^2 + 4b})$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}(\sqrt{m^2 + 4b})^3$, and is attained at the point $P(x_P, x_P^2) = P(\frac{1}{2}m, \frac{1}{4}m^2)$.

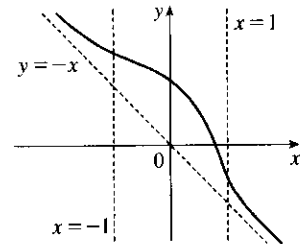
Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}[(x_2x_1^2 - x_1x_2^2) + (x_1x^2 - xx_1^2) + (xx_2^2 - x_2x^2)]$.

10. If $f'(x) < 0$ for all x , $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and

$\lim_{x \rightarrow \pm\infty} [f(x) + x] = 0$, then f is decreasing everywhere, concave up on

$(-\infty, -1)$ and $(1, \infty)$, concave down on $(-1, 1)$, and approaches the line

$y = -x$ as $x \rightarrow \pm\infty$. An example of such a graph is sketched.



11. $f(x) = (a^2 + a - 6) \cos 2x + (a - 2)x + \cos 1 \Rightarrow f'(x) = -(a^2 + a - 6) \sin 2x + (a - 2)$.

The derivative exists for all x , so the only possible critical points will occur where $f'(x) = 0 \Leftrightarrow$

$2(a - 2)(a + 3) \sin 2x = a - 2 \Leftrightarrow$ either $a = 2$ or $2(a + 3) \sin 2x = 1$, with the latter implying that

$\sin 2x = \frac{1}{2(a + 3)}$. Since the range of $\sin 2x$ is $[-1, 1]$, this equation has no solution whenever either

$\frac{1}{2(a + 3)} < -1$ or $\frac{1}{2(a + 3)} > 1$. Solving these inequalities, we get $-\frac{7}{2} < a < -\frac{5}{2}$.

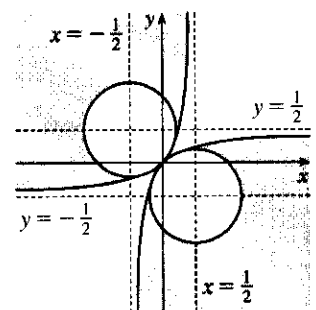
12. To sketch the region $\{(x, y) \mid 2xy \leq |x - y| \leq x^2 + y^2\}$, we consider two cases.

Case 1: $x \geq y$ This is the case in which (x, y) lies on or below the line $y = x$. The double inequality becomes $2xy \leq x - y \leq x^2 + y^2$. The right-hand inequality holds if and only if $x^2 - x + y^2 + y \geq 0 \Leftrightarrow (x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 \geq \frac{1}{2} \Leftrightarrow (x, y)$ lies on or outside the circle with radius $\frac{1}{\sqrt{2}}$ centered at $(\frac{1}{2}, -\frac{1}{2})$.

The left-hand inequality holds if and only if $2xy - x + y \leq 0 \Leftrightarrow xy - \frac{1}{2}x + \frac{1}{2}y \leq 0 \Leftrightarrow (x + \frac{1}{2})(y - \frac{1}{2}) \leq -\frac{1}{4} \Leftrightarrow (x, y)$ lies on or below the hyperbola $(x + \frac{1}{2})(y - \frac{1}{2}) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = \frac{1}{2}$ and $x = -\frac{1}{2}$ asymptotically.

Case 2: $y \geq x$ This is the case in which (x, y) lies on or above the line $y = x$. The double inequality becomes $2xy \leq y - x \leq x^2 + y^2$. The right-hand inequality holds if and only if $x^2 + x + y^2 - y \geq 0 \Leftrightarrow (x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 \geq \frac{1}{2} \Leftrightarrow (x, y)$ lies on or outside the circle of radius $\frac{1}{\sqrt{2}}$ centered at $(-\frac{1}{2}, \frac{1}{2})$. The left-hand inequality holds if and only if $2xy + x - y \leq 0 \Leftrightarrow xy + \frac{1}{2}x - \frac{1}{2}y \leq 0 \Leftrightarrow (x - \frac{1}{2})(y + \frac{1}{2}) \leq -\frac{1}{4} \Leftrightarrow (x, y)$ lies on or above the left-hand branch of the hyperbola $(x - \frac{1}{2})(y + \frac{1}{2}) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = -\frac{1}{2}$ and $x = \frac{1}{2}$ asymptotically. Therefore, the region of interest consists of the points on or above the left branch of the hyperbola $(x - \frac{1}{2})(y + \frac{1}{2}) = -\frac{1}{4}$ that are on or outside the circle

$(x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$, together with the points on or below the right branch of the hyperbola $(x + \frac{1}{2})(y - \frac{1}{2}) = -\frac{1}{4}$ that are on or outside the circle $(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 = \frac{1}{2}$. Note that the inequalities are unchanged when x and y are interchanged, so the region is symmetric about the line $y = x$. So we need only have analyzed case 1 and then reflected that region about the line $y = x$, instead of considering case 2.



13. (a) Let $y = |AD|$, $x = |AB|$, and $1/x = |AC|$, so that $|AB| \cdot |AC| = 1$.

We compute the area $\mathcal{A}$ of $\triangle ABC$ in two ways. First,

$$\mathcal{A} = \frac{1}{2} |AB| |AC| \sin \frac{2\pi}{3} = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}.$$

$$\mathcal{A} = (\text{area of } \triangle ABD) + (\text{area of } \triangle ACD)$$

$$= \frac{1}{2} |AB| |AD| \sin \frac{\pi}{3} + \frac{1}{2} |AD| |AC| \sin \frac{\pi}{3}$$

$$= \frac{1}{2} xy \frac{\sqrt{3}}{2} + \frac{1}{2} y(1/x) \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} y(x + 1/x)$$

Equating the two expressions for the area, we get $\frac{\sqrt{3}}{4} y \left(x + \frac{1}{x} \right) = \frac{\sqrt{3}}{4} \Leftrightarrow y = \frac{1}{x + 1/x} = \frac{x}{x^2 + 1}, x > 0$.

Another method: Use the Law of Sines on the triangles ABD and ABC . In $\triangle ABD$, we have

$$\angle A + \angle B + \angle D = 180^\circ \Leftrightarrow 60^\circ + \alpha + \angle D = 180^\circ \Leftrightarrow \angle D = 120^\circ - \alpha. \text{ Thus,}$$

$$\frac{x}{y} = \frac{\sin(120^\circ - \alpha)}{\sin \alpha} = \frac{\sin 120^\circ \cos \alpha - \cos 120^\circ \sin \alpha}{\sin \alpha} = \frac{\frac{\sqrt{3}}{2} \cos \alpha + \frac{1}{2} \sin \alpha}{\sin \alpha} \Rightarrow \frac{x}{y} = \frac{\sqrt{3}}{2} \cot \alpha + \frac{1}{2},$$

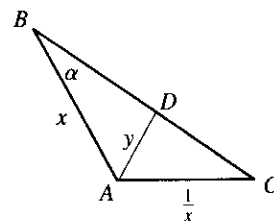
and by a similar argument with $\triangle ABC$, $\frac{\sqrt{3}}{2} \cot \alpha = x^2 + \frac{1}{2}$. Eliminating $\cot \alpha$ gives $\frac{x}{y} = (x^2 + \frac{1}{2}) + \frac{1}{2} \Rightarrow$

$$y = \frac{x}{x^2 + 1}, x > 0.$$

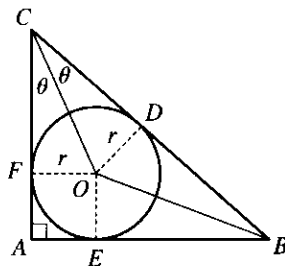
- (b) We differentiate our expression for y with respect to x to find the maximum:

$$\frac{dy}{dx} = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = 0 \text{ when } x = 1. \text{ This indicates a maximum by the First Derivative}$$

Test, since $y'(x) > 0$ for $0 < x < 1$ and $y'(x) < 0$ for $x > 1$, so the maximum value of y is $y(1) = \frac{1}{2}$.



14. (a)



From geometry, two tangents to a circle from a given point have the same length, so $|CF| = |CD|$, $|AE| = |AF|$, and $|BD| = |BE|$. Thus,

$$\begin{aligned} \frac{1}{2}(|BC| + |AC| - |AB|) &= \frac{1}{2}[(|BD| + |DC|) + (|AF| + |FC|) - (|AE| + |EB|)] \\ &= \frac{1}{2}[(|BD| + |CD|) + (|AF| + |CD|) - (|AF| + |BD|)] \\ &= \frac{1}{2}[2|CD|] = |CD| \end{aligned}$$

- (b) Using the result from part (a) and the fact that $a = |BC|$, we have $\tan \theta = \frac{r}{|CD|} \Rightarrow$

$$\frac{r}{\tan \theta} = |CD| = \frac{1}{2}(|AC| + |BC| - |AB|) = \frac{1}{2}(a \cos 2\theta + a - a \sin 2\theta) \Leftrightarrow$$

$$\begin{aligned} r &= \frac{1}{2} a \tan \theta (2 \cos^2 \theta - 1 + 1 - 2 \sin \theta \cos \theta) \\ &= \frac{1}{2} a (2 \sin \theta \cos \theta - 2 \sin^2 \theta) && \text{[in terms of } \theta] \\ &= \frac{1}{2} a (\sin 2\theta + \cos 2\theta - 1) && \text{[in terms of } 2\theta] \end{aligned}$$

- (c) We differentiate r with respect to θ and set $dr/d\theta$ equal to 0 to find the maximum values:

$$\begin{aligned} dr/d\theta &= \frac{1}{2} a (2 \cos 2\theta - 2 \sin 2\theta) = a (\cos 2\theta - \sin 2\theta). \text{ Since } 0 < \theta < \frac{\pi}{4}, dr/d\theta = 0 \Leftrightarrow \cos 2\theta = \sin 2\theta \\ \Leftrightarrow 1 &= \tan 2\theta \Leftrightarrow 2\theta = \frac{\pi}{4} \Leftrightarrow \theta = \frac{\pi}{8}. \text{ This gives a maximum by the First Derivative Test, since} \end{aligned}$$

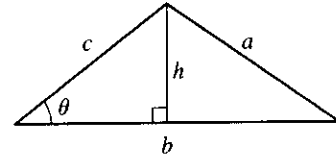
$dr/d\theta > 0$ for $0 < \theta < \frac{\pi}{8}$, and $dr/d\theta < 0$ for $\frac{\pi}{8} < \theta < \frac{\pi}{4}$. The maximum value is

$$r\left(\frac{\pi}{8}\right) = \frac{1}{2}a\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} - 1\right) = \frac{1}{2}(\sqrt{2} - 1)a \approx 0.207a.$$

15. (a) $A = \frac{1}{2}bh$ with $\sin \theta = h/c$, so $A = \frac{1}{2}bc \sin \theta$. But A is a constant, so differentiating this equation with respect to t , we

$$\text{get } \frac{dA}{dt} = 0 = \frac{1}{2} \left[bc \cos \theta \frac{d\theta}{dt} + b \frac{dc}{dt} \sin \theta + \frac{db}{dt} c \sin \theta \right]$$

$$\Rightarrow bc \cos \theta \frac{d\theta}{dt} = -\sin \theta \left[b \frac{dc}{dt} + c \frac{db}{dt} \right] \Rightarrow \frac{d\theta}{dt} = -\tan \theta \left[\frac{1}{c} \frac{dc}{dt} + \frac{1}{b} \frac{db}{dt} \right].$$



- (b) We use the Law of Cosines to get the length of side a in terms of those of b and c , and then

$$\text{we differentiate implicitly with respect to } t: a^2 = b^2 + c^2 - 2bc \cos \theta \Rightarrow$$

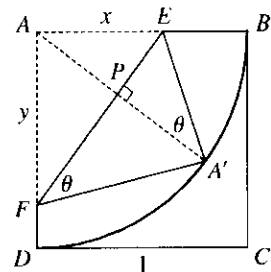
$$2a \frac{da}{dt} = 2b \frac{db}{dt} + 2c \frac{dc}{dt} - 2 \left[bc(-\sin \theta) \frac{d\theta}{dt} + b \frac{dc}{dt} \cos \theta + \frac{db}{dt} c \cos \theta \right] \Rightarrow$$

$$\frac{da}{dt} = \frac{1}{a} \left(b \frac{db}{dt} + c \frac{dc}{dt} + bc \sin \theta \frac{d\theta}{dt} - b \frac{dc}{dt} \cos \theta - c \frac{db}{dt} \cos \theta \right). \text{ Now we substitute our value of } a \text{ from the}$$

Law of Cosines and the value of $d\theta/dt$ from part (a), and simplify (primes signify differentiation by t):

$$\begin{aligned} \frac{da}{dt} &= \frac{bb' + cc' + bc \sin \theta [-\tan \theta (c'/c + b'/b)] - (bc' + cb')(\cos \theta)}{\sqrt{b^2 + c^2 - 2bc \cos \theta}} \\ &= \frac{bb' + cc' - [\sin^2 \theta (bc' + cb') + \cos^2 \theta (bc' + cb')]/\cos \theta}{\sqrt{b^2 + c^2 - 2bc \cos \theta}} = \frac{bb' + cc' - (bc' + cb')\sec \theta}{\sqrt{b^2 + c^2 - 2bc \cos \theta}} \end{aligned}$$

16. Let $x = |AE|$, $y = |AF|$ as shown. The area $\mathcal{A}$ of the $\triangle AEF$ is $\mathcal{A} = \frac{1}{2}xy$. We need to find a relationship between x and y , so that we can take the derivative $d\mathcal{A}/dx$ and then find the maximum and minimum areas. Now let A' be the point on which A ends up after the fold has been performed, and let P be the intersection of AA' and EF . Note that AA' is perpendicular to EF since we are reflecting A through the line EF to get to A' , and that $|AP| = |PA'|$ for the same reason. But $|AA'| = 1$, since AA' is a radius of the circle. Since $|AP| + |PA'| = |AA'|$,



we have $|AP| = \frac{1}{2}$. Another way to express the area of the triangle is

$$\mathcal{A} = \frac{1}{2}|EF||AP| = \frac{1}{2}\sqrt{x^2 + y^2}\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{x^2 + y^2}. \text{ Equating the two expressions for } \mathcal{A}, \text{ we get}$$

$$\frac{1}{2}xy = \frac{1}{4}\sqrt{x^2 + y^2} \Rightarrow 4x^2y^2 = x^2 + y^2 \Rightarrow y^2(4x^2 - 1) = x^2 \Rightarrow y = x/\sqrt{4x^2 - 1}.$$

(Note that we could also have derived this result from the similarity of $\triangle A'PE$ and $\triangle A'FE$; that is,

$$\frac{|A'P|}{|PE|} = \frac{|A'F|}{|A'E|} \Rightarrow \frac{\frac{1}{2}}{\sqrt{x^2 - (\frac{1}{2})^2}} = \frac{y}{x} \Rightarrow y = \frac{\frac{1}{2}x}{\sqrt{4x^2 - 1}/2} = \frac{x}{\sqrt{4x^2 - 1}}.)$$

Now we can substitute for y and calculate $\frac{d\mathcal{A}}{dx}$: $\mathcal{A} = \frac{1}{2} \frac{x^2}{\sqrt{4x^2 - 1}} \Rightarrow$

$$\frac{dA}{dx} = \frac{1}{2} \left[\frac{\sqrt{4x^2 - 1}(2x) - x^2(\frac{1}{2})(4x^2 - 1)^{-1/2}(8x)}{4x^2 - 1} \right]. \text{ This is 0 when } 2x\sqrt{4x^2 - 1} - 4x^3(4x^2 - 1)^{-1/2} = 0$$

$$\Leftrightarrow 2x(4x^2 - 1)^{-1/2} [(4x^2 - 1) - 2x^2] = 0 \Rightarrow (4x^2 - 1) - 2x^2 = 0 \quad (x > 0) \Leftrightarrow 2x^2 = 1 \Rightarrow$$

$x = \frac{1}{\sqrt{2}}$. So this is one possible value for an extremum. We must also test the endpoints of the interval over

which x ranges. The largest value that x can attain is 1, and the smallest value of x occurs when $y = 1 \Leftrightarrow$

$$1 = x/\sqrt{4x^2 - 1} \Leftrightarrow x^2 = 4x^2 - 1 \Leftrightarrow 3x^2 = 1 \Leftrightarrow x = \frac{1}{\sqrt{3}}. \text{ This will give the same value of } A \text{ as will}$$

$x = 1$, since the geometric situation is the same (reflected through the line $y = x$). We calculate

$$A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \frac{(1/\sqrt{2})^2}{\sqrt{4(1/\sqrt{2})^2 - 1}} = \frac{1}{4} = 0.25, \text{ and } A(1) = \frac{1}{2} \frac{1^2}{\sqrt{4(1)^2 - 1}} = \frac{1}{2\sqrt{3}} \approx 0.29. \text{ So the maximum area}$$

$$\text{is } A(1) = A\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}} \text{ and the minimum area is } A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}.$$

Another method: Use the angle θ (see diagram above) as a variable:

$$A = \frac{1}{2}xy = \frac{1}{2}\left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\csc\theta\right) = \frac{1}{8\sin\theta\cos\theta} = \frac{1}{4\sin 2\theta}. A \text{ is minimized when } \sin 2\theta \text{ is maximal, that is,}$$

$$\text{when } \sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}. \text{ Also note that } A'E = x = \frac{1}{2}\sec\theta \leq 1 \Rightarrow \sec\theta \leq 2 \Rightarrow$$

$$\cos\theta \geq \frac{1}{2} \Rightarrow \theta \leq \frac{\pi}{3}, \text{ and similarly, } A'F = y = \frac{1}{2}\csc\theta \leq 1 \Rightarrow \csc\theta \leq 2 \Rightarrow \sin\theta \leq \frac{1}{2} \Rightarrow \theta \geq \frac{\pi}{6}.$$

$$\text{As above, we find that } A \text{ is maximized at these endpoints: } A\left(\frac{\pi}{6}\right) = \frac{1}{4\sin\frac{\pi}{3}} = \frac{1}{2\sqrt{3}} = \frac{1}{4\sin\frac{2\pi}{3}} = A\left(\frac{\pi}{3}\right); \text{ and}$$

$$\text{minimized at } \theta = \frac{\pi}{4}: A\left(\frac{\pi}{4}\right) = \frac{1}{4\sin\frac{\pi}{2}} = \frac{1}{4}.$$

$$17. \text{ (a) Distance} = \text{rate} \times \text{time, so time} = \text{distance/rate. } T_1 = \frac{D}{c_1},$$

$$T_2 = \frac{2|PR|}{c_1} + \frac{|RS|}{c_2} = \frac{2h\sec\theta}{c_1} + \frac{D - 2h\tan\theta}{c_2}, T_3 = \frac{2\sqrt{h^2 + D^2/4}}{c_1} = \frac{\sqrt{4h^2 + D^2}}{c_1}.$$

$$\text{(b) } \frac{dT_2}{d\theta} = \frac{2h}{c_1} \cdot \sec\theta \tan\theta - \frac{2h}{c_2} \sec^2\theta = 0 \text{ when } 2h\sec\theta \left(\frac{1}{c_1} \tan\theta - \frac{1}{c_2} \sec\theta \right) = 0 \Rightarrow$$

$$\frac{1}{c_1} \frac{\sin\theta}{\cos\theta} - \frac{1}{c_2} \frac{1}{\cos\theta} = 0 \Rightarrow \frac{\sin\theta}{c_1 \cos\theta} = \frac{1}{c_2 \cos\theta} \Rightarrow \sin\theta = \frac{c_1}{c_2}. \text{ The First Derivative Test shows that}$$

this gives a minimum.

$$\text{(c) Using part (a) with } D = 1 \text{ and } T_1 = 0.26, \text{ we have } T_1 = \frac{D}{c_1} \Rightarrow$$

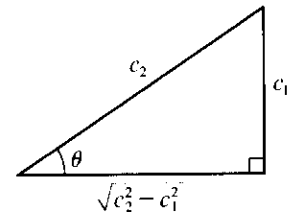
$$c_1 = \frac{1}{0.26} \approx 3.85 \text{ km/s. } T_3 = \frac{\sqrt{4h^2 + D^2}}{c_1} \Rightarrow 4h^2 + D^2 = T_3^2 c_1^2 \Rightarrow$$

$$h = \frac{1}{2} \sqrt{T_3^2 c_1^2 - D^2} = \frac{1}{2} \sqrt{(0.34)^2 (1/0.26)^2 - 1^2} \approx 0.42 \text{ km. To find } c_2, \text{ we use } \sin\theta = \frac{c_1}{c_2} \text{ from part (b)}$$

and $T_2 = \frac{2h \sec \theta}{c_1} + \frac{D - 2h \tan \theta}{c_2}$ from part (a). From the figure,

$$\sin \theta = \frac{c_1}{c_2} \Rightarrow \sec \theta = \frac{c_2}{\sqrt{c_2^2 - c_1^2}} \text{ and } \tan \theta = \frac{c_1}{\sqrt{c_2^2 - c_1^2}}, \text{ so}$$

$$T_2 = \frac{2hc_2}{c_1 \sqrt{c_2^2 - c_1^2}} + \frac{D \sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2 \sqrt{c_2^2 - c_1^2}}.$$



Using the values for T_2 [given as 0.32], h , c_1 , and D , we can graph

$$Y_1 = T_2 \text{ and } Y_2 = \frac{2hc_2}{c_1 \sqrt{c_2^2 - c_1^2}} + \frac{D \sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2 \sqrt{c_2^2 - c_1^2}} \text{ and find their intersection points. Doing so gives us}$$

$c_2 \approx 4.10$ and 7.66 , but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6^\circ$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

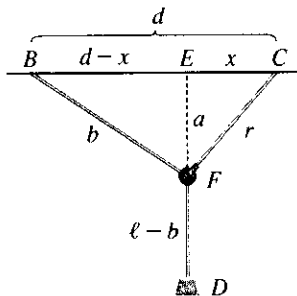
18. A straight line intersects the curve $y = f(x) = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points if and only if the graph of f has two inflection points. $f'(x) = 4x^3 + 3cx^2 + 24x - 5$ and $f''(x) = 12x^2 + 6cx + 24$.

$$f''(x) = 0 \Leftrightarrow x = \frac{-6c \pm \sqrt{(6c)^2 - 4(12)(24)}}{2(12)}. \text{ There are two distinct roots for } f''(x) = 0 \text{ (and hence two}$$

inflection points) if and only if the discriminant is positive; that is, $36c^2 - 1152 > 0 \Leftrightarrow c^2 > 32 \Leftrightarrow$

$|c| > \sqrt{32}$. Thus, the desired values of c are $c < -4\sqrt{2}$ or $c > 4\sqrt{2}$.

19.



Let $a = |EF|$ and $b = |BF|$ as shown in the figure.

Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned} |ED| &= |EF| + |FD| = a + \ell - b \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + a^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + (\sqrt{r^2 - x^2})^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2} \end{aligned}$$

$$\text{Let } f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}.$$

$$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) - \frac{1}{2}(d^2 + r^2 - 2dx)^{-1/2}(-2d) = \frac{-x}{\sqrt{r^2 - x^2}} + \frac{d}{\sqrt{d^2 + r^2 - 2dx}}.$$

$$f'(x) = 0 \Rightarrow \frac{x}{\sqrt{r^2 - x^2}} = \frac{d}{\sqrt{d^2 + r^2 - 2dx}} \Rightarrow \frac{x^2}{r^2 - x^2} = \frac{d^2}{d^2 + r^2 - 2dx} \Rightarrow$$

$$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \Rightarrow 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \Rightarrow$$

$$0 = 2dx^2(x - d) - r^2(x^2 - d^2) \Rightarrow 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \Rightarrow$$

$$0 = (x - d)[2dx^2 - r^2(x + d)]$$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x :

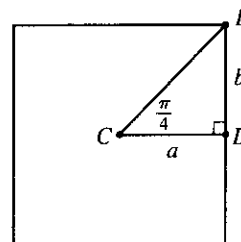
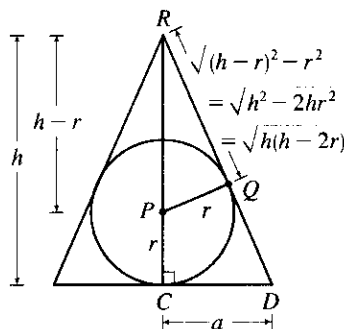
$$x = \frac{-(-r^2) \pm \sqrt{(-r^2)^2 - 4(2d)(-dr^2)}}{2(2d)} = \frac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}. \text{ Because } \sqrt{r^4 + 8d^2r^2} > r^2, \text{ the "negative"}$$

can be discarded. Thus,

$$\begin{aligned} x &= \frac{r^2 + \sqrt{r^4 + 8d^2r^2}}{4d} = \frac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \quad (r > 0) \\ &= \frac{r}{4d} \left(r + \sqrt{r^2 + 8d^2} \right) \end{aligned}$$

The maximum value of $|ED|$ occurs at this value of x .

20.



Let $a = \overline{CD}$ denote the distance from the center C of the base to the midpoint D of a side of the base.

$$\text{Since } \triangle PQR \text{ is similar to } \triangle DCR, \frac{a}{h} = \frac{r}{\sqrt{h(h-2r)}} \Rightarrow a = \frac{rh}{\sqrt{h(h-2r)}} = r \frac{\sqrt{h}}{\sqrt{h-2r}}.$$

Let b denote one-half the length of a side of the base. The area A of the base is

$$A = 8(\text{area of } \triangle CDE) = 8\left(\frac{1}{2}ab\right) = 4a\left(a \tan \frac{\pi}{4}\right) = 4a^2.$$

$$\text{The volume of the pyramid is } V = \frac{1}{3}Ah = \frac{1}{3}(4a^2)h = \frac{4}{3}\left(r \frac{\sqrt{h}}{\sqrt{h-2r}}\right)^2 h = \frac{4}{3}r^2 \frac{h^2}{h-2r}, \text{ with domain } h > 2r.$$

$$\text{Now } \frac{dV}{dh} = \frac{4}{3}r^2 \cdot \frac{(h-2r)(2h) - h^2(1)}{(h-2r)^2} = \frac{4}{3}r^2 \frac{h^2 - 4hr}{(h-2r)^2} = \frac{4}{3}r^2 \frac{h(h-4r)}{(h-2r)^2}$$

$$\begin{aligned} \text{and } \frac{d^2V}{dh^2} &= \frac{4}{3}r^2 \cdot \frac{(h-2r)^2(2h-4r) - (h^2-4hr)(2)(h-2r)(1)}{[(h-2r)^2]^2} \\ &= \frac{4}{3}r^2 \cdot \frac{2(h-2r)[(h^2-4hr+4r^2) - (h^2-4hr)]}{(h-2r)^2} \\ &= \frac{8}{3}r^2 \cdot \frac{4r^2}{(h-2r)^3} = \frac{32}{3}r^4 \cdot \frac{1}{(h-2r)^3}. \end{aligned}$$

The first derivative is equal to zero for $h = 4r$ and the second derivative is positive for $h > 2r$, so the volume of the pyramid is minimized when $h = 4r$.

To extend our solution to a regular n -gon, we make the following changes:

- (1) the number of sides of the base is n
- (2) the number of triangles in the base is $2n$
- (3) $\angle DCE = \frac{\pi}{n}$
- (4) $b = a \tan \frac{\pi}{n}$

We then obtain the following results:

$$A = na^2 \tan \frac{\pi}{n}, V = \frac{nr^2}{3} \cdot \tan\left(\frac{\pi}{n}\right) \cdot \frac{h^2}{h-2r}, \frac{dV}{dh} = \frac{nr^2}{3} \cdot \tan\left(\frac{\pi}{n}\right) \cdot \frac{h(h-4r)}{(h-2r)^2}, \text{ and}$$

$\frac{d^2V}{dh^2} = \frac{8nr^4}{3} \cdot \tan\left(\frac{\pi}{n}\right) \cdot \frac{1}{(h-2r)^3}$. Notice that the answer, $h = 4r$, is independent of the number of sides of the

base of the polygon!

21. $V = \frac{4}{3}\pi r^3 \Leftrightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. But $\frac{dV}{dt}$ is proportional to the surface area, so $\frac{dV}{dt} = k \cdot 4\pi r^2$ for some

constant k . Therefore, $4\pi r^2 \frac{dr}{dt} = k \cdot 4\pi r^2 \Leftrightarrow \frac{dr}{dt} = k = \text{constant}$. An antiderivative of k with respect to t is kt , so $r = kt + C$. When $t = 0$, the radius r must equal the original radius r_0 , so $C = r_0$, and $r = kt + r_0$. To find

$$k \text{ we use the fact that when } t = 3, r = 3k + r_0 \text{ and } V = \frac{1}{2}V_0 \rightarrow \frac{4}{3}\pi(3k + r_0)^3 = \frac{1}{2} \cdot \frac{4}{3}\pi r_0^3 \Rightarrow$$

$$(3k + r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow \quad 3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow \quad k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right). \text{ Since } r = kt + r_0,$$

$$r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0. \text{ When the snowball has melted completely we have } r = 0 \Rightarrow$$

$$\frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0 \text{ which gives } t = \frac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}. \text{ Hence, it takes } \frac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1} - 3 = \frac{3}{\sqrt[3]{2} - 1} \approx 11 \text{ h } 33 \text{ min}$$

longer.

- 22.** By ignoring the bottom hemisphere of the initial spherical bubble, we can rephrase the problem as follows: Prove that the maximum height of a stack of n hemispherical bubbles is $\sqrt{n}$ if the radius of the bottom hemisphere is 1. We proceed by induction. The case $n = 1$ is obvious since $\sqrt{1}$ is the height of the first hemisphere. Suppose the assertion is true for $n = k$ and let's suppose we have $k + 1$ hemispherical bubbles forming a stack of maximum height. Suppose the second hemisphere (counting from the bottom) has radius r . Then by our induction hypothesis

(scaled to the setting of a bottom hemisphere of radius r), the height of the stack formed by the top k bubbles is $\sqrt{k}r$. (If it were shorter, then the total stack of $k+1$ bubbles wouldn't have maximum height.)

The height of the whole stack is $H(r) = \sqrt{k}r + \sqrt{1-r^2}$. (See the figure.)

We want to choose r so as to maximize $H(r)$. Note that $0 < r < 1$. We calculate

$$H'(r) = \sqrt{k} - \frac{r}{\sqrt{1-r^2}} \text{ and } H''(r) = \frac{-1}{(1-r^2)^{3/2}}. \quad H'(r) = 0 \Leftrightarrow$$

$$r^2 = k(1-r^2) \Leftrightarrow (k+1)r^2 = k \Leftrightarrow r = \sqrt{\frac{k}{k+1}}. \text{ This is the only}$$

critical number in $(0, 1)$ and it represents a local maximum (hence an absolute

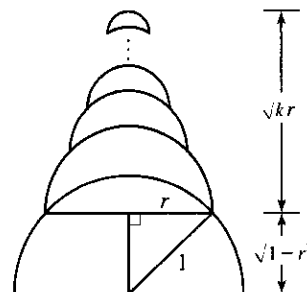
maximum) since $H''(r) < 0$ on $(0, 1)$. When $r = \sqrt{\frac{k}{k+1}}$,

$$H(r) = \sqrt{k} \frac{\sqrt{k}}{\sqrt{k+1}} + \sqrt{1 - \frac{k}{k+1}} = \frac{k}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} = \sqrt{k+1}. \text{ Thus, the assertion is true for } n = k+1$$

when it is true for $n = k$. By induction, it is true for all positive integers n .

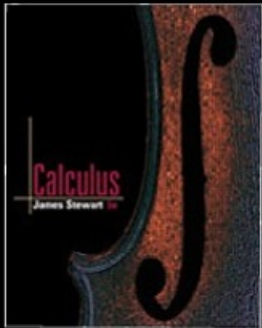
Note: In general, a maximally tall stack of n hemispherical bubbles consists of bubbles with

$$\text{radii } 1, \sqrt{\frac{n-1}{n}}, \sqrt{\frac{n-2}{n}}, \dots, \sqrt{\frac{2}{n}}, \sqrt{\frac{1}{n}}.$$

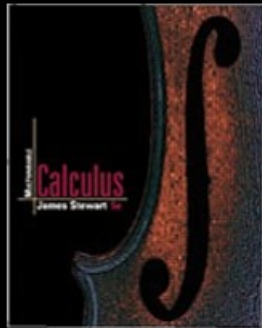


Chapter 5

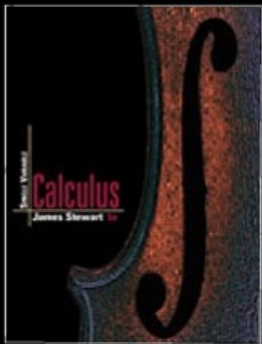
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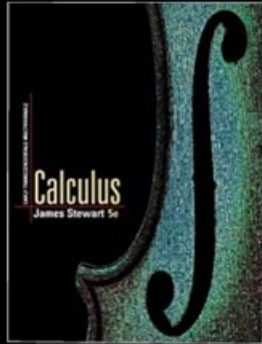
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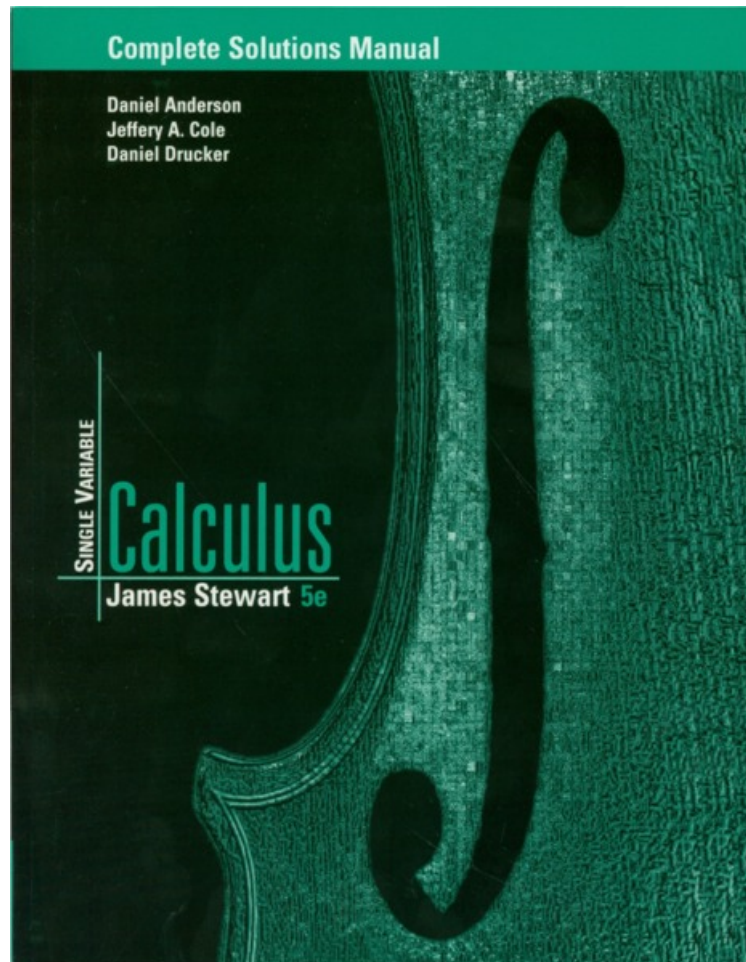
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5 □ INTEGRALS

5.1 Areas and Distances

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n = 5$.

$$\begin{aligned} L_5 &= \sum_{i=1}^5 f(x_{i-1}) \Delta x \quad [\Delta x = \frac{b-a}{n} = \frac{10-0}{5} = 2] \\ &= f(x_0) \cdot 2 + f(x_1) \cdot 2 + f(x_2) \cdot 2 + f(x_3) \cdot 2 + f(x_4) \cdot 2 \\ &= 2[f(0) + f(2) + f(4) + f(6) + f(8)] \\ &\approx 2(1 + 3 + 4.3 + 5.4 + 6.3) = 2(20) = 40 \end{aligned}$$

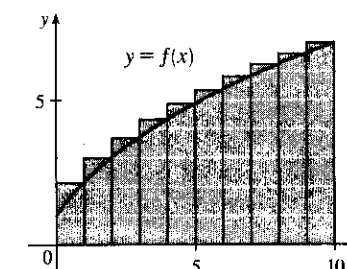
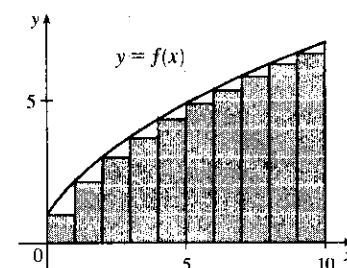
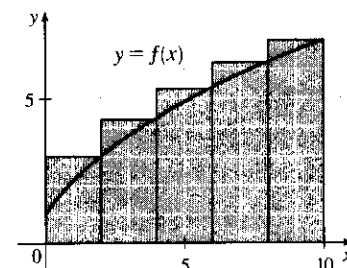
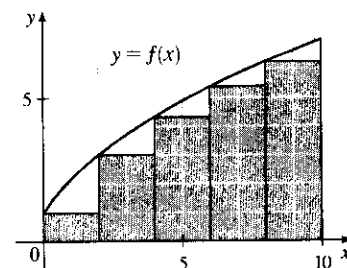
Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.

$$\begin{aligned} R_5 &= \sum_{i=1}^5 f(x_i) \Delta x \\ &= 2[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] \\ &= 2[f(2) + f(4) + f(6) + f(8) + f(10)] \\ &\approx 2(3 + 4.3 + 5.4 + 6.3 + 7) = 2(26) = 52 \end{aligned}$$

Comparing R_5 to L_5 , we see that we have added the area of the rightmost upper rectangle, $f(10) \cdot 2$, to the sum and subtracted the area of the leftmost lower rectangle, $f(0) \cdot 2$, from the sum.

$$\begin{aligned} \text{(b) } L_{10} &= \sum_{i=1}^{10} f(x_{i-1}) \Delta x \quad [\Delta x = \frac{10-0}{10} = 1] \\ &= 1[f(x_0) + f(x_1) + \cdots + f(x_9)] \\ &= f(0) + f(1) + \cdots + f(9) \\ &\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7 \\ &= 43.2 \end{aligned}$$

$$\begin{aligned} R_{10} &= \sum_{i=1}^{10} f(x_i) \Delta x = f(1) + f(2) + \cdots + f(10) \\ &= L_{10} + 1 \cdot f(10) - 1 \cdot f(0) \quad \left[\begin{array}{l} \text{add rightmost upper rectangle,} \\ \text{subtract leftmost lower rectangle} \end{array} \right] \\ &= 43.2 + 7 - 1 = 49.2 \end{aligned}$$

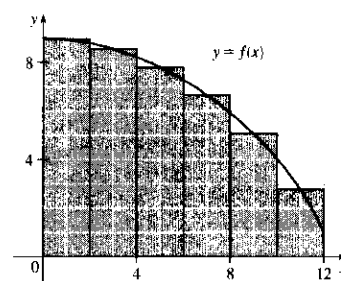
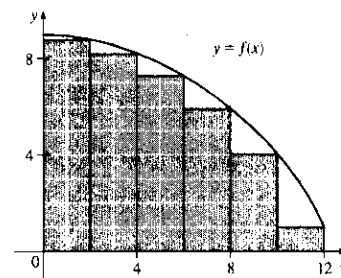
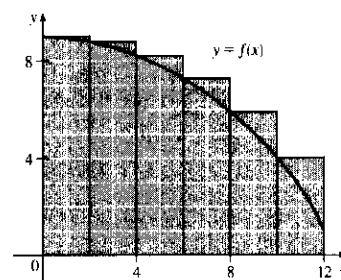


$$\begin{aligned}
 2. (a) (i) L_6 &= \sum_{i=1}^6 f(x_{i-1}) \Delta x \quad [\Delta x = \frac{12-0}{6} = 2] \\
 &= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] \\
 &= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)] \\
 &\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1) \\
 &= 2(43.3) = 86.6
 \end{aligned}$$

$$\begin{aligned}
 (ii) R_6 &= L_6 + 2 \cdot f(12) - 2 \cdot f(0) \\
 &\approx 86.6 + 2(1) - 2(9) = 70.6
 \end{aligned}$$

[Add area of rightmost lower rectangle
and subtract area of leftmost upper rectangle.]

$$\begin{aligned}
 (iii) M_6 &= \sum_{i=1}^6 f(x_i^*) \Delta x \\
 &= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)] \\
 &\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8) \\
 &= 2(39.7) = 79.4
 \end{aligned}$$



(b) Since f is decreasing, we obtain an *overestimate* by using *left* endpoints; that is, L_6 .

(c) Since f is decreasing, we obtain an *underestimate* by using *right* endpoints; that is, R_6 .

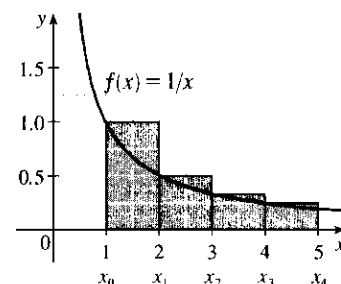
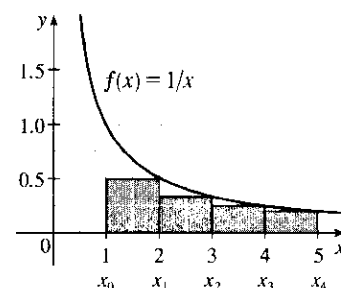
(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6 .

$$\begin{aligned}
 3. (a) R_4 &= \sum_{i=1}^4 f(x_i) \Delta x \quad [\Delta x = \frac{5-1}{4} = 1] \\
 &= f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1 \\
 &= f(2) + f(3) + f(4) + f(5) \\
 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\bar{3}
 \end{aligned}$$

Since f is decreasing on $[1, 5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4 .

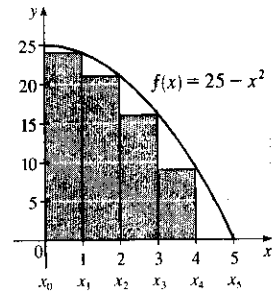
$$\begin{aligned}
 (b) L_4 &= \sum_{i=1}^4 f(x_{i-1}) \Delta x \\
 &= f(1) + f(2) + f(3) + f(4) \\
 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 2.08\bar{3}
 \end{aligned}$$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(1) \cdot 1 - f(5) \cdot 1$.



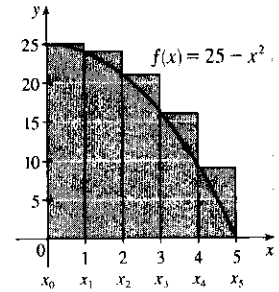
$$\begin{aligned}
 4. (a) R_5 &= \sum_{i=1}^5 f(x_i) \Delta x \quad [\Delta x = \frac{5-0}{5} = 1] \\
 &= f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1 + f(x_5) \cdot 1 \\
 &= f(1) + f(2) + f(3) + f(4) + f(5) \\
 &= 24 + 21 + 16 + 9 + 0 = 70
 \end{aligned}$$

Since f is decreasing on $[0, 5]$, R_5 is an underestimate.



$$\begin{aligned}
 (b) L_5 &= \sum_{i=1}^5 f(x_{i-1}) \Delta x \\
 &= f(0) + f(1) + f(2) + f(3) + f(4) \\
 &= 25 + 24 + 21 + 16 + 9 = 95
 \end{aligned}$$

L_5 is an overestimate.

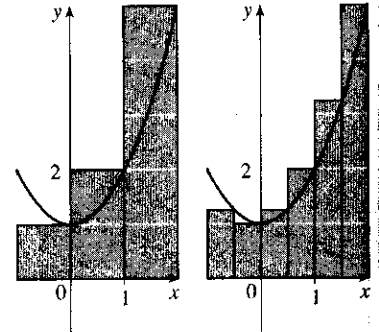


$$5. (a) f(x) = 1 + x^2 \text{ and } \Delta x = \frac{2 - (-1)}{3} = 1 \Rightarrow$$

$$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 8.$$

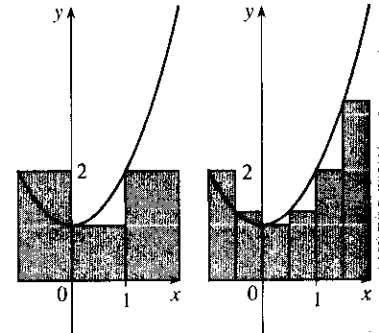
$$\Delta x = \frac{2 - (-1)}{6} = 0.5 \Rightarrow$$

$$\begin{aligned}
 R_6 &= 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)] \\
 &= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5) \\
 &= 0.5(13.75) = 6.875
 \end{aligned}$$



$$(b) L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) = 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 = 5$$

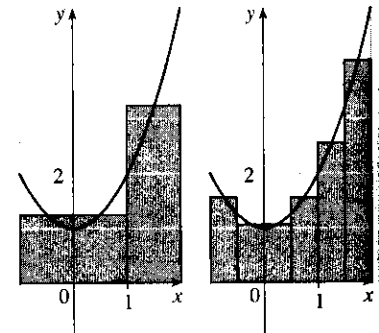
$$\begin{aligned}
 L_6 &= 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)] \\
 &= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25) \\
 &= 0.5(10.75) = 5.375
 \end{aligned}$$



$$(c) M_3 = 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5)$$

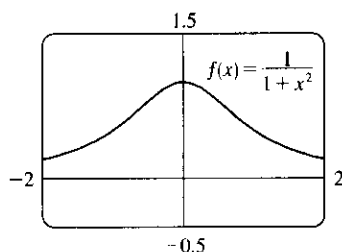
$$= 1 \cdot 1.25 + 1 \cdot 1.25 + 1 \cdot 3.25 = 5.75$$

$$\begin{aligned}
 M_6 &= 0.5[f(-0.75) + f(-0.25) + f(0.25) \\
 &\quad + f(0.75) + f(1.25) + f(1.75)] \\
 &= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625) \\
 &= 0.5(11.875) = 5.9375
 \end{aligned}$$



(d) M_6 appears to be the best estimate.

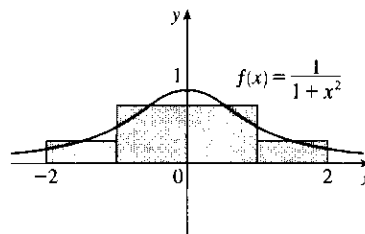
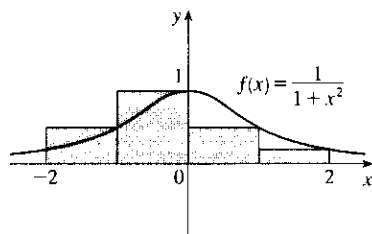
6. (a)



(b) $f(x) = 1/(1+x^2)$ and $\Delta x = \frac{2-(-2)}{4} = 1 \Rightarrow$

$$\begin{aligned} \text{(i) } R_4 &= \sum_{i=1}^4 f(x_i) \Delta x \\ &= f(-1) \cdot 1 + f(0) \cdot 1 \\ &\quad + f(1) \cdot 1 + f(2) \cdot 1 \\ &= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5} = \frac{11}{5} = 2.2 \end{aligned}$$

$$\begin{aligned} \text{(ii) } M_4 &= \sum_{i=1}^4 f(\bar{x}_i) \Delta x \quad [\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)] \\ &= f(-1.5) \cdot 1 + f(-0.5) \cdot 1 \\ &\quad + f(0.5) \cdot 1 + f(1.5) \cdot 1 \\ &= \frac{4}{13} + \frac{4}{5} + \frac{4}{5} + \frac{4}{13} = \frac{144}{65} \approx 2.2154 \end{aligned}$$



(c) $n = 8$, so $\Delta x = \frac{2-(-2)}{8} = \frac{1}{2}$.

$$\begin{aligned} R_8 &= \frac{1}{2} [f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)] \\ &= \frac{1}{2} \left[\frac{4}{13} + \frac{1}{2} + \frac{4}{5} + 1 + \frac{4}{5} + \frac{1}{2} + \frac{4}{13} + \frac{1}{5} \right] = \frac{287}{130} \approx 2.2077 \end{aligned}$$

$$\begin{aligned} M_8 &= \frac{1}{2} [f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)] \\ &= \frac{1}{2} \left[2 \left(\frac{16}{65} + \frac{16}{41} + \frac{16}{25} + \frac{16}{17} \right) \right] \approx 2.2176 \end{aligned}$$

7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let $SUM = 0$, $X_MIN = 0$, $X_MAX = \pi$, $N = 10$ (or 30 or 50, depending on which sum we are calculating), $DELTA_X = (X_MAX - X_MIN)/N$, and $RIGHT_ENDPOINT = X_MIN + DELTA_X$.

2 Repeat steps 2a, 2b in sequence until $RIGHT_ENDPOINT > X_MAX$.

2a Add $\sin(RIGHT_ENDPOINT)$ to SUM .

2b Add $DELTA_X$ to $RIGHT_ENDPOINT$.

At the end of this procedure, $(DELTA_X) \cdot (SUM)$ is equal to the answer we are looking for. We find that

$$R_{10} = \frac{\pi}{10} \sum_{i=1}^{10} \sin\left(\frac{i\pi}{10}\right) \approx 1.9835, R_{30} = \frac{\pi}{30} \sum_{i=1}^{30} \sin\left(\frac{i\pi}{30}\right) \approx 1.9982, \text{ and } R_{50} = \frac{\pi}{50} \sum_{i=1}^{50} \sin\left(\frac{i\pi}{50}\right) \approx 1.9993.$$

It appears that the exact area is 2.

Shown below is program SUMRIGHT and its output from a TI-83 Plus calculator. To generalize the program, we have input (rather than assigned) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1 , enabling us to evaluate any right sum merely by changing Y_1 and running the program.

```
PROGRAM: SUMRIGHT
:0→S
:Prompt Xmin
:Prompt Xmax
:Prompt N
: (Xmax-Xmin)/N→D
:Xmin+D→R
:For(I,1,N)
: S+Y1(R)→S
: R+D→R
:End
:D*S→Z
:Disp Z
```

```
PrgrmSUMRIGHT
Xmin=?0
Xmax=?π
N=?10
1.983523537
Done
```

8. We can use the algorithm from Exercise 7 with $X_MIN = 1$, $X_MAX = 2$, and $1 / (\text{RIGHT_ENDPOINT})^2$ instead

of $\sin(\text{RIGHT_ENDPOINT})$ in step 2a. We find that $R_{10} = \frac{1}{10} \sum_{i=1}^{10} \frac{1}{(1+i/10)^2} \approx 0.4640$,

$R_{30} = \frac{1}{30} \sum_{i=1}^{30} \frac{1}{(1+i/30)^2} \approx 0.4877$, and $R_{50} = \frac{1}{50} \sum_{i=1}^{50} \frac{1}{(1+i/50)^2} \approx 0.4926$. It appears that the exact area is $\frac{1}{2}$.

9. In Maple, we have to perform a number of steps before getting a numerical answer. After

loading the student package [command: `with(student);`] we use the command

`left_sum:=leftsum(x^(1/2),x=1..4,10 [or 30, or 50]);` which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have

a special command for these sums, so we must type them in manually. For example, the first left sum is given by

$(3/10) * \text{Sum}[\text{Sqrt}[1+3(i-1)/10], \{i, 1, 10\}]$, and we use the `N` command on the resulting output to get a numerical approximation.

In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves.

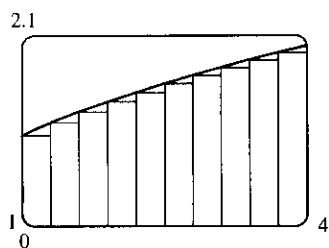
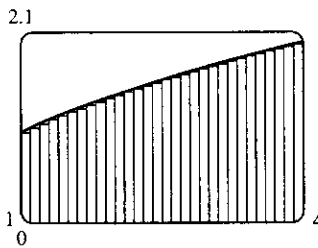
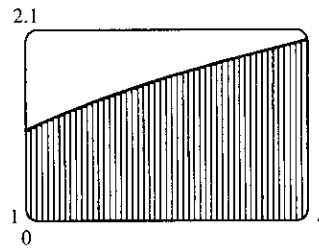
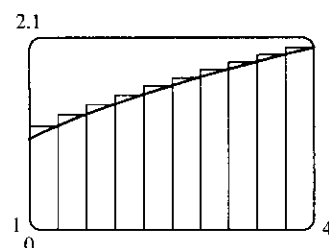
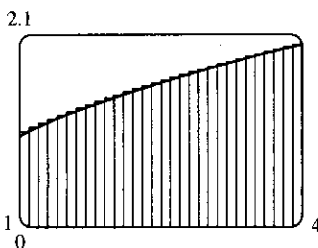
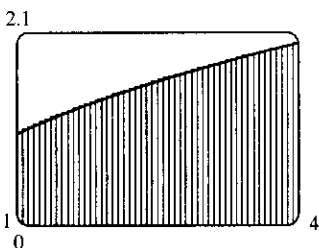
(We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x) = \sqrt{x}$, $1 \leq x \leq 4$, the left sums are of the form $L_n = \frac{3}{n} \sum_{i=1}^n \sqrt{1 + \frac{3(i-1)}{n}}$. Specifically,

$L_{10} \approx 4.5148$, $L_{30} \approx 4.6165$, and $L_{50} \approx 4.6366$. The right sums are of the form $R_n = \frac{3}{n} \sum_{i=1}^n \sqrt{1 + \frac{3i}{n}}$.

Specifically, $R_{10} \approx 4.8148$, $R_{30} \approx 4.7165$, and $R_{50} \approx 4.6966$.

- (b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.

left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$ right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$

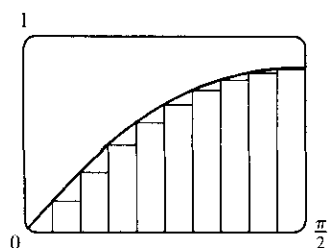
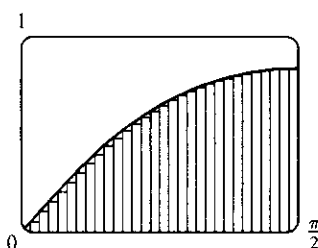
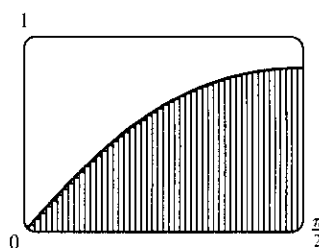
- (c) We know that since $\sqrt{x}$ is an increasing function on $(1, 4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $4.637 > 4.6$ and the right sum with $n = 50$ is about $4.697 < 4.7$, we conclude that $4.6 < L_{50} < \text{exact area} < R_{50} < 4.7$, so the exact area is between 4.6 and 4.7.

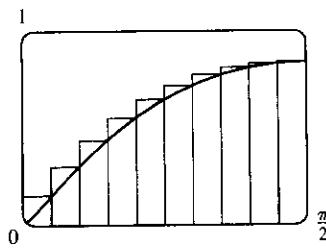
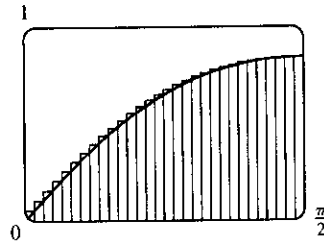
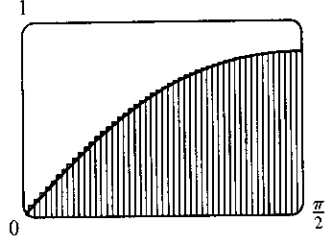
10. See the solution to Exercise 9 for the CAS commands for evaluating the sums.

- (a) With $f(x) = \sin(\sin x)$, $0 \leq x \leq \frac{\pi}{2}$, the left sums are of the form $L_n = \frac{\pi}{2n} \sum_{i=1}^n \sin\left(\sin \frac{\pi(i-1)}{2n}\right)$. In particular, $L_{10} \approx 0.8251$, $L_{30} \approx 0.8710$, and $L_{50} \approx 0.8799$. The right sums are of the form

$$R_n = \frac{\pi}{2n} \sum_{i=1}^n \sin\left(\sin \frac{\pi i}{2n}\right). \text{ In particular, } R_{10} \approx 0.9573, R_{30} \approx 0.9150, \text{ and } R_{50} \approx 0.9064.$$

- (b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.

left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$

right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$

(c) We know that since $\sin(\sin x)$ is an increasing function on $(0, \frac{\pi}{2})$ [this is true because its derivative, $-\cos(\sin x)(-\cos x)$, is positive on that interval], all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $0.8799 > 0.87$ and the right sum with $n = 50$ is about $0.9064 < 0.91$, we conclude that $0.87 < L_{50} < \text{exact area} < R_{50} < 0.91$, so the exact area is between 0.87 and 0.91.

- 11.** Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5) \\ = 0.5(69.4) = 34.7 \text{ ft}$$

$$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$$

- 12.** (a) $d \approx L_5 = (30 \text{ ft/s})(12 \text{ s}) + 28 \cdot 12 + 25 \cdot 12 + 22 \cdot 12 + 24 \cdot 12$

$$= (30 + 28 + 25 + 22 + 24) \cdot 12 = 129 \cdot 12 = 1548 \text{ ft}$$

(b) $d \approx R_5 = (28 + 25 + 22 + 24 + 27) \cdot 12 = 126 \cdot 12 = 1512$ ft

(c) The estimates are neither lower nor upper estimates since v is neither an increasing nor a decreasing function of t .

- 13.** Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2$ L.

Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70$ L.

- 14.** We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$$d = \sum_{i=1}^6 v(t_i) \Delta t_i = (185 \text{ ft/s})(10 \text{ s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54,694 \text{ ft}$$

- 15.** For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \\ \approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

- 16.** For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30-0}{6} = 5 \text{ s} = \frac{5}{3600} \text{ h} = \frac{1}{720} \text{ h}$.

$$M_6 = \frac{1}{720} [v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)]$$

$$= \frac{1}{720} (31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \frac{1}{720} (521.75) \approx 0.725 \text{ km}$$

For a very rough check on the above calculation, we can draw a line from $(0, 0)$ to $(30, 120)$ and calculate the area

of the triangle: $\frac{1}{2}(30)(120) = 1800$. Divide by 3600 to get 0.5, which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

17. $f(x) = \sqrt[4]{x}$, $1 \leq x \leq 16$. $\Delta x = (16 - 1)/n = 15/n$ and $x_i = 1 + i \Delta x = 1 + 15i/n$.

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt[4]{1 + \frac{15i}{n}} \cdot \frac{15}{n}.$$

18. $f(x) = 1 + x^4$, $2 \leq x \leq 5$. $\Delta x = (5 - 2)/n = 3/n$ and $x_i = 2 + i \Delta x = 2 + 3i/n$.

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + \left(2 + \frac{3i}{n} \right)^4 \right] \cdot \frac{3}{n}.$$

19. $f(x) = x \cos x$, $0 \leq x \leq \frac{\pi}{2}$. $\Delta x = (\frac{\pi}{2} - 0)/n = \frac{\pi}{2n}$ and $x_i = 0 + i \Delta x = \frac{\pi}{2n} i$.

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i\pi}{2n} \cos\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n}.$$

20. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left(5 + \frac{2i}{n} \right)^{10}$ can be interpreted as the area of the region lying under the graph of $y = (5 + x)^{10}$ on the

interval $[0, 2]$, since for $y = (5 + x)^{10}$ on $[0, 2]$ with $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_i = 0 + i \Delta x = \frac{2i}{n}$, and $x_i^* = x_i$, the

expression for the area is $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(5 + \frac{2i}{n} \right)^{10} \frac{2}{n}$. Note that the answer is not unique.

We could use $y = x^{10}$ on $[5, 7]$ or, in general, $y = ((5 - n) + x)^{10}$ on $[n, n + 2]$.

21. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{4n} \tan \frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval

$[0, \frac{\pi}{4}]$, since for $y = \tan x$ on $[0, \frac{\pi}{4}]$ with $\Delta x = \frac{\pi/4 - 0}{n} = \frac{\pi}{4n}$, $x_i = 0 + i \Delta x = \frac{i\pi}{4n}$, and $x_i^* = x_i$, the

expression for the area is $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \tan\left(\frac{i\pi}{4n}\right) \frac{\pi}{4n}$. Note that this answer is not unique,

since the expression for the area is the same for the function $y = \tan(x - k\pi)$ on the interval $[k\pi, k\pi + \frac{\pi}{4}]$, where k is any integer.

22. (a) $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = 0 + i \Delta x = \frac{i}{n}$. $A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n} \right)^3 \cdot \frac{1}{n}$.

$$(b) \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^3}{n^3} \cdot \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n i^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2} \right]^2 = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{4n^2} = \frac{1}{4} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 = \frac{1}{4}$$

23. (a) $y = f(x) = x^5$. $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and $x_i = 0 + i \Delta x = \frac{2i}{n}$.

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n} \right)^5 \cdot \frac{2}{n} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{32i^5}{n^5} \cdot \frac{2}{n} = \lim_{n \rightarrow \infty} \frac{64}{n^6} \sum_{i=1}^n i^5.$$

$$(b) \sum_{i=1}^n i^5 \stackrel{\text{CAS}}{=} \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

$$(c) \lim_{n \rightarrow \infty} \frac{64}{n^6} \cdot \frac{n^2(n+1)^2(2n^2+2n-1)}{12} = \frac{64}{12} \lim_{n \rightarrow \infty} \frac{(n^2+2n+1)(2n^2+2n-1)}{n^2 \cdot n^2} \\ = \frac{16}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) \left(2 + \frac{2}{n} - \frac{1}{n^2} \right) = \frac{16}{3} \cdot 1 \cdot 2 = \frac{32}{3}$$

24. (a) $y = f(x) = x^4 + 5x^2 + x$, $2 \leq x \leq 7 \Rightarrow \Delta x = \frac{7-2}{n} = \frac{5}{n}$, $x_i = 2 + i\Delta x = 2 + \frac{5i}{n} \Rightarrow$

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{5}{n} \sum_{i=1}^n \left[\left(2 + \frac{5i}{n}\right)^4 + 5 \left(2 + \frac{5i}{n}\right)^2 + \left(2 + \frac{5i}{n}\right) \right]$$

(b) $R_n = \frac{5}{n} \cdot \frac{4723n^4 + 7845n^3 + 3475n^2 - 125}{6n^3}$

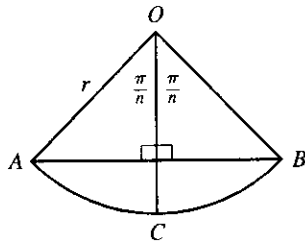
(c) $A = \lim_{n \rightarrow \infty} R_n = \frac{23,615}{6} = 3935.8\bar{3}$

25. $y = f(x) = \cos x$. $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and $x_i = 0 + i\Delta x = \frac{bi}{n}$.

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n} \stackrel{\text{CAS}}{=} \lim_{n \rightarrow \infty} \left[\frac{b \sin\left(b\left(\frac{1}{2n} + 1\right)\right)}{2n \sin\left(\frac{b}{2n}\right)} - \frac{b}{2n} \right] \stackrel{\text{CAS}}{=} \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin \frac{\pi}{2} = 1$. The value 1 is the *exact* area, whereas the value obtained in Example 3(b), 1.006, is just an *approximation*.

26. (a)



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB . Thus, $\triangle AOB$ is divided into two right triangles with legs of length

$$\frac{1}{2}(AB) = r \sin(\pi/n) \text{ and } r \cos(\pi/n).$$

$\triangle AOB$ has area $2 \cdot \frac{1}{2} [r \sin(\pi/n)] [r \cos(\pi/n)] = r^2 \sin(\pi/n) \cos(\pi/n) = \frac{1}{2} r^2 \sin(2\pi/n)$, so

$$A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2} n r^2 \sin(2\pi/n).$$

(b) To use Equation 3.5.2, $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function—in this case, $2\pi/n$.

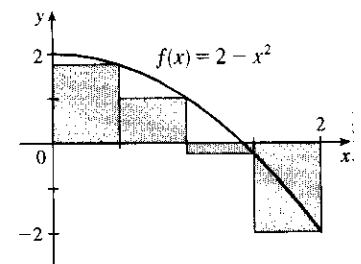
$$\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \frac{1}{2} n r^2 \sin(2\pi/n) = \lim_{n \rightarrow \infty} \frac{1}{2} n r^2 \frac{\sin(2\pi/n)}{2\pi/n} \cdot \frac{2\pi}{n} = \lim_{n \rightarrow \infty} \frac{\sin(2\pi/n)}{2\pi/n} \pi r^2. \text{ Let } \theta = \frac{2\pi}{n}.$$

$$\text{Then as } n \rightarrow \infty, \theta \rightarrow 0, \text{ so } \lim_{n \rightarrow \infty} \frac{\sin(2\pi/n)}{2\pi/n} \pi r^2 = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \pi r^2 = (1) \pi r^2 = \pi r^2.$$

5.2 The Definite Integral

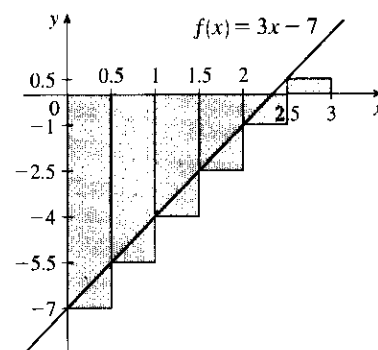
1. $R_4 = \sum_{i=1}^4 f(x_i) \Delta x$ [$x_i^* = x_i$ is a right endpoint and $\Delta x = 0.5$]
 $= 0.5 [f(0.5) + f(1) + f(1.5) + f(2)]$ [$f(x) = 2 - x^2$]
 $= 0.5 [1.75 + 1 + (-0.25) + (-2)]$
 $= 0.5(0.5) = 0.25$

The Riemann sum represents the sum of the areas of the two rectangles above the x -axis minus the sum of the areas of the two rectangles below the x -axis; that is, the *net area* of the rectangles with respect to the x -axis.



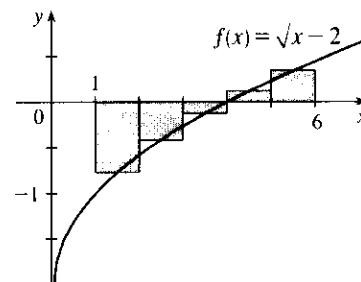
$$\begin{aligned}
 2. L_6 &= \sum_{i=1}^6 f(x_{i-1}) \Delta x \quad [x_i^* = x_{i-1} \text{ is a left endpoint and } \Delta x = 0.5] \\
 &= 0.5 [f(0) + f(0.5) + f(1) \\
 &\quad + f(1.5) + f(2) + f(2.5)] \quad [f(x) = 3x - 7] \\
 &= 0.5[-7 + (-5.5) + (-4) + (-2.5) + (-1) + 0.5] \\
 &= 0.5(-19.5) = -9.75
 \end{aligned}$$

The Riemann sum represents the area of the rectangle above the x -axis minus the area of the five rectangles below the x -axis.



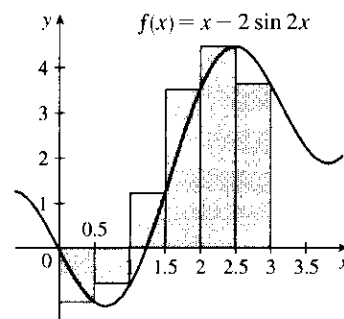
$$\begin{aligned}
 3. M_5 &= \sum_{i=1}^5 f(\bar{x}_i) \Delta x \quad [x_i^* = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) \text{ is a midpoint and } \Delta x = 1] \\
 &= 1 [f(1.5) + f(2.5) + f(3.5) \\
 &\quad + f(4.5) + f(5.5)] \quad [f(x) = \sqrt{x} - 2] \\
 &\approx -0.856759
 \end{aligned}$$

The Riemann sum represents the sum of the areas of the two rectangles above the x -axis minus the sum of the areas of the three rectangles below the x -axis.



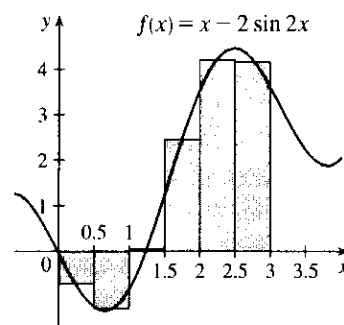
$$\begin{aligned}
 4. (a) R_6 &= \sum_{i=1}^6 f(x_i) \Delta x \quad [x_i^* = x_i \text{ is a right endpoint and } \Delta x = 0.5] \\
 &= 0.5 [f(0.5) + f(1) + f(1.5) + f(2) \\
 &\quad + f(2.5) + f(3)] \quad [f(x) = x - 2 \sin 2x] \\
 &\approx 5.353254
 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles above the x -axis minus the sum of the areas of the two rectangles below the x -axis.



$$\begin{aligned}
 (b) M_6 &= \sum_{i=1}^6 f(\bar{x}_i) \Delta x \quad [x_i^* = \bar{x}_i \text{ is a midpoint and } \Delta x = 0.5] \\
 &= 0.5 [f(0.25) + f(0.75) + f(1.25) + f(1.75) \\
 &\quad + f(2.25) + f(2.75)] \quad [f(x) = x - 2 \sin 2x] \\
 &\approx 4.458461
 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles above the x -axis minus the sum of the areas of the two rectangles below the x -axis.



$$5. \Delta x = (b - a)/n = (8 - 0)/4 = 8/4 = 2.$$

(a) Using the right endpoints to approximate $\int_0^8 f(x) dx$, we have

$$\sum_{i=1}^4 f(x_i) \Delta x = 2[f(2) + f(4) + f(6) + f(8)] \approx 2[1 + 2 + (-2) + 1] = 4.$$

(b) Using the left endpoints to approximate $\int_0^8 f(x) dx$, we have

$$\sum_{i=1}^4 f(x_{i-1}) \Delta x = 2[f(0) + f(2) + f(4) + f(6)] \approx 2[2 + 1 + 2 + (-2)] = 6.$$

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x) dx$, we have

$$\sum_{i=1}^4 f(\bar{x}_i) \Delta x = 2[f(1) + f(3) + f(5) + f(7)] \approx 2[3 + 2 + 1 + (-1)] = 10.$$

6. (a) Using the right endpoints to approximate $\int_{-3}^3 g(x) dx$, we have

$$\begin{aligned} \sum_{i=1}^6 g(x_i) \Delta x &= 1[g(-2) + g(-1) + g(0) + g(1) + g(2) + g(3)] \\ &\approx 1 - 0.5 - 1.5 - 1.5 - 0.5 + 2.5 = -0.5 \end{aligned}$$

(b) Using the left endpoints to approximate $\int_{-3}^3 g(x) dx$, we have

$$\begin{aligned} \sum_{i=1}^6 g(x_{i-1}) \Delta x &= 1[g(-3) + g(-2) + g(-1) + g(0) + g(1) + g(2)] \\ &\approx 2 + 1 - 0.5 - 1.5 - 1.5 - 0.5 = -1 \end{aligned}$$

(c) Using the midpoint of each subinterval to approximate $\int_{-3}^3 g(x) dx$, we have

$$\begin{aligned} \sum_{i=1}^6 g(\bar{x}_i) \Delta x &= 1[g(-2.5) + g(-1.5) + g(-0.5) + g(0.5) + g(1.5) + g(2.5)] \\ &\approx 1.5 + 0 - 1 - 1.75 - 1 + 0.5 = -1.75 \end{aligned}$$

7. Since f is increasing, $L_5 \leq \int_0^{25} f(x) dx \leq R_5$.

$$\begin{aligned} \text{Lower estimate} = L_5 &= \sum_{i=1}^5 f(x_{i-1}) \Delta x = 5[f(0) + f(5) + f(10) + f(15) + f(20)] \\ &= 5(-42 - 37 - 25 - 6 + 15) = 5(-95) = -475 \end{aligned}$$

$$\begin{aligned} \text{Upper estimate} = R_5 &= \sum_{i=1}^5 f(x_i) \Delta x = 5[f(5) + f(10) + f(15) + f(20) + f(25)] \\ &= 5(-37 - 25 - 6 + 15 + 36) = 5(-17) = -85 \end{aligned}$$

8. (a) Using the right endpoints to approximate $\int_0^6 f(x) dx$, we have

$$\sum_{i=1}^3 f(x_i) \Delta x = 2[f(2) + f(4) + f(6)] = 2(8.3 + 2.3 - 10.5) = 0.2$$

(b) Using the left endpoints to approximate $\int_0^6 f(x) dx$, we have

$$\sum_{i=1}^3 f(x_{i-1}) \Delta x = 2[f(0) + f(2) + f(4)] = 2(9.3 + 8.3 + 2.3) = 39.8$$

(c) Using the midpoint of each interval to approximate $\int_0^6 f(x) dx$, we have

$$\sum_{i=1}^3 f(\bar{x}_i) \Delta x = 2[f(1) + f(3) + f(5)] = 2(9.0 + 6.5 - 7.6) = 15.8.$$

The estimate using the right endpoints must be less than $\int_0^6 f(x) dx$, since if we take x_i^* to be the right endpoint x_i of each interval, then $f(x_i) \leq f(x)$ for all x on $[x_{i-1}, x_i]$, which implies that $f(x_i) \Delta x \leq \int_{x_{i-1}}^{x_i} f(x) dx$, and so

the sum $\sum_{i=1}^3 [f(x_i) \Delta x] \leq \sum_{i=1}^3 \left[\int_{x_{i-1}}^{x_i} f(x) dx \right] = \int_0^6 f(x) dx$. Similarly, if we take x_i^* to be the left endpoint x_{i-1}

of each interval, then $f(x_{i-1}) \geq f(x)$ for all x on $[x_{i-1}, x_i]$, and so $\sum_{i=1}^3 [f(x_{i-1}) \Delta x] \geq \int_0^6 f(x) dx$. We cannot

say anything about the midpoint estimate.

9. $\Delta x = (10 - 2)/4 = 2$, so the endpoints are 2, 4, 6, 8, and 10, and the midpoints are 3, 5, 7, and 9. The Midpoint Rule gives

$$\int_2^{10} \sqrt{x^3 + 1} \, dx \approx \sum_{i=1}^4 f(\bar{x}_i) \Delta x = 2(\sqrt{3^3 + 1} + \sqrt{5^3 + 1} + \sqrt{7^3 + 1} + \sqrt{9^3 + 1}) \approx 124.1644.$$

10. $\Delta x = (\pi - 0)/6 = \frac{\pi}{6}$, so the endpoints are 0, $\frac{\pi}{6}$, $\frac{2\pi}{6}$, $\frac{3\pi}{6}$, $\frac{4\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{6\pi}{6}$, and the midpoints are $\frac{\pi}{12}$, $\frac{3\pi}{12}$, $\frac{5\pi}{12}$, $\frac{7\pi}{12}$, $\frac{9\pi}{12}$, and $\frac{11\pi}{12}$. The Midpoint Rule gives

$$\int_0^{\pi} \sec(x/3) \, dx \approx \sum_{i=1}^6 f(\bar{x}_i) \Delta x = \frac{\pi}{6} \left(\sec \frac{\pi}{36} + \sec \frac{3\pi}{36} + \sec \frac{5\pi}{36} + \sec \frac{7\pi}{36} + \sec \frac{9\pi}{36} + \sec \frac{11\pi}{36} \right) \approx 3.9379.$$

11. $\Delta x = (1 - 0)/5 = 0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives

$$\int_0^1 \sin(x^2) \, dx \approx \sum_{i=1}^5 f(\bar{x}_i) \Delta x = 0.2 [\sin(0.1)^2 + \sin(0.3)^2 + \sin(0.5)^2 + \sin(0.7)^2 + \sin(0.9)^2] \approx 0.3084.$$

12. $\Delta x = \frac{5-1}{4} = 1$, so the endpoints are 1, 2, 3, 4, and 5, and the midpoints are 1.5, 2.5, 3.5, and 4.5. The Midpoint Rule gives

$$\int_1^5 \frac{x-1}{x+1} \, dx \approx \sum_{i=1}^4 f(\bar{x}_i) \Delta x = 1 \left[\frac{1.5-1}{1.5+1} + \frac{2.5-1}{2.5+1} + \frac{3.5-1}{3.5+1} + \frac{4.5-1}{4.5+1} \right] \approx 1.8205.$$

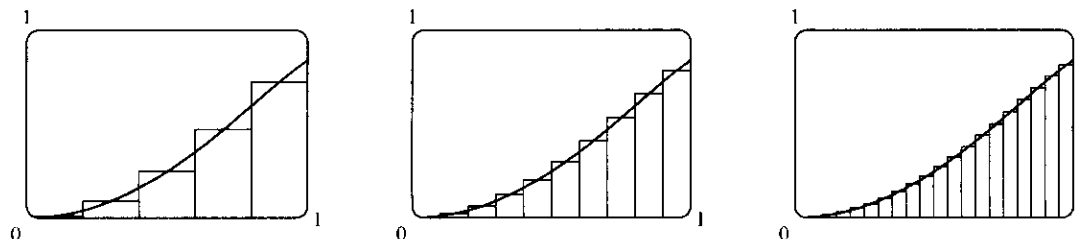
13. In Maple, we use the command `with(student)`; to load the sum and box commands, then

`m:=middlesum(sin(x^2), x=0..1, 5)`; which gives us the sum in summation notation, then

`M:=evalf(m)`; which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command

`middlebox(sin(x^2), x=0..1, 5)` generates the graph. Repeating for $n = 10$ and $n = 20$ gives

$M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.



14. See the solution to Exercise 5.1.7 for a possible algorithm to calculate the sums. With $\Delta x = (1 - 0)/100 = 0.01$ and subinterval endpoints 1, 1.01, 1.02, ..., 1.99, 2, we calculate that the left Riemann sum is

$$L_{100} = \sum_{i=1}^{100} \sin(x_{i-1}^2) \Delta x \approx 0.30607, \text{ and the right Riemann sum is } R_{100} = \sum_{i=1}^{100} \sin(x_i^2) \Delta x \approx 0.31448.$$

Since $f(x) = \sin(x^2)$ is an increasing function, we must have $L_{100} \leq \int_0^1 \sin(x^2) \, dx \leq R_{100}$, so

$0.306 < L_{100} \leq \int_0^1 \sin(x^2) \, dx \leq R_{100} < 0.315$. Therefore, the approximate value $0.3084 \approx 0.31$ in Exercise 11 must be accurate to two decimal places.

15. We'll create the table of values to approximate $\int_0^\pi \sin x \, dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1 = \sin x$, $X_{\min} = 0$, $X_{\max} = \pi$, and $n = 5, 10, 50$, and 100 .

The values of R_n appear to be approaching 2.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

16. $\int_0^2 \sqrt{1+x^4} \, dx$ with $n = 5, 10, 50$, and 100 .

n	L_n	R_n
5	3.080614	4.329856
10	3.354110	3.978731
50	3.591540	3.716464
100	3.622383	3.684845

The value of the integral lies between 3.622 and 3.685. Note that

$f(x) = \sqrt{1+x^4}$ is increasing on $(0, 2)$. We cannot make a similar statement for $\int_{-1}^2 \sqrt{1+x^4} \, dx$ since f is decreasing on $(-1, 0)$.

17. On $[0, \pi]$, $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i \sin x_i \Delta x = \int_0^\pi x \sin x \, dx$.
18. On $[1, 5]$, $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{x_i}{1+x_i} \Delta x = \int_1^5 \frac{x}{1+x} \, dx$.
19. On $[1, 8]$, $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{2x_i^* + (x_i^*)^2} \Delta x = \int_1^8 \sqrt{2x+x^2} \, dx$.
20. On $[0, 2]$, $\lim_{n \rightarrow \infty} \sum_{i=1}^n [4 - 3(x_i^*)^2 + 6(x_i^*)^5] \Delta x = \int_0^2 (4 - 3x^2 + 6x^5) \, dx$.
21. Note that $\Delta x = \frac{5 - (-1)}{n} = \frac{6}{n}$ and $x_i = -1 + i \Delta x = -1 + \frac{6i}{n}$.

$$\begin{aligned}
 \int_{-1}^5 (1+3x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + 3 \left(-1 + \frac{6i}{n} \right) \right] \frac{6}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{6}{n} \sum_{i=1}^n \left[-2 + \frac{18i}{n} \right] = \lim_{n \rightarrow \infty} \frac{6}{n} \left[\sum_{i=1}^n (-2) + \sum_{i=1}^n \frac{18i}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{6}{n} \left[-2n + \frac{18}{n} \sum_{i=1}^n i \right] = \lim_{n \rightarrow \infty} \frac{6}{n} \left[-2n + \frac{18}{n} \cdot \frac{n(n+1)}{2} \right] \\
 &= \lim_{n \rightarrow \infty} \left[-12 + \frac{108}{n^2} \cdot \frac{n(n+1)}{2} \right] = \lim_{n \rightarrow \infty} \left[-12 + 54 \frac{n+1}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[-12 + 54 \left(1 + \frac{1}{n} \right) \right] = -12 + 54 \cdot 1 = 42
 \end{aligned}$$

$$\begin{aligned}
22. \int_1^4 (x^2 + 2x - 5) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad [\Delta x = 3/n \text{ and } x_i = 1 + 3i/n] \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(1 + \frac{3i}{n}\right)^2 + 2\left(1 + \frac{3i}{n}\right) - 5 \right] \left(\frac{3}{n}\right) \\
&= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\sum_{i=1}^n \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 2 + \frac{6i}{n} - 5\right) \right] \\
&= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\sum_{i=1}^n \left(\frac{9}{n^2} \cdot i^2 + \frac{12}{n} \cdot i - 2\right) \right] \\
&= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\frac{9}{n^2} \sum_{i=1}^n i^2 + \frac{12}{n} \sum_{i=1}^n i - \sum_{i=1}^n 2 \right] \\
&= \lim_{n \rightarrow \infty} \left(\frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{36}{n^2} \cdot \frac{n(n+1)}{2} - \frac{6}{n} \cdot n \right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{9}{2} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} + 18 \cdot \frac{n+1}{n} - 6 \right) \\
&= \lim_{n \rightarrow \infty} \left[\frac{9}{2} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) + 18 \left(1 + \frac{1}{n}\right) - 6 \right] = \frac{9}{2} \cdot 1 \cdot 2 + 18 \cdot 1 - 6 = 21
\end{aligned}$$

23. Note that $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and $x_i = 0 + i \Delta x = \frac{2i}{n}$.

$$\begin{aligned}
\int_0^2 (2 - x^2) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 - \frac{4i^2}{n^2}\right) \left(\frac{2}{n}\right) \\
&= \lim_{n \rightarrow \infty} \frac{2}{n} \left[\sum_{i=1}^n 2 - \frac{4}{n^2} \sum_{i=1}^n i^2 \right] = \lim_{n \rightarrow \infty} \frac{2}{n} \left(2n - \frac{4}{n^2} \sum_{i=1}^n i^2 \right) \\
&= \lim_{n \rightarrow \infty} \left[4 - \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right] = \lim_{n \rightarrow \infty} \left(4 - \frac{4}{3} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} \right) \\
&= \lim_{n \rightarrow \infty} \left[4 - \frac{4}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \right] = 4 - \frac{4}{3} \cdot 1 \cdot 2 = \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
24. \int_0^5 (1 + 2x^3) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad [\Delta x = 5/n \text{ and } x_i = 5i/n] \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + 2 \cdot \frac{125i^3}{n^3}\right) \left(\frac{5}{n}\right) = \lim_{n \rightarrow \infty} \frac{5}{n} \left[\sum_{i=1}^n 1 + \frac{250}{n^3} \sum_{i=1}^n i^3 \right] \\
&= \lim_{n \rightarrow \infty} \frac{5}{n} \left(1 \cdot n + \frac{250}{n^3} \sum_{i=1}^n i^3 \right) = \lim_{n \rightarrow \infty} \left[5 + \frac{1250}{n^4} \cdot \frac{n^2(n+1)^2}{4} \right] \\
&= \lim_{n \rightarrow \infty} \left[5 + 312.5 \cdot \frac{(n+1)^2}{n^2} \right] = \lim_{n \rightarrow \infty} \left[5 + 312.5 \left(1 + \frac{1}{n}\right)^2 \right] \\
&= 5 + 312.5 = 317.5
\end{aligned}$$

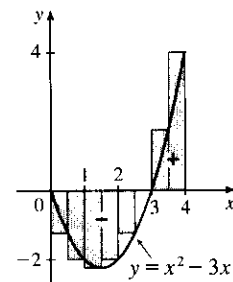
25. Note that $\Delta x = \frac{2-1}{n} = \frac{1}{n}$ and $x_i = 1 + i\Delta x = 1 + i(1/n) = 1 + i/n$.

$$\begin{aligned}
 \int_1^2 x^3 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{i}{n}\right)^3 \left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{n+i}{n}\right)^3 \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n^3 + 3n^2 i + 3ni^2 + i^3) = \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\sum_{i=1}^n n^3 + \sum_{i=1}^n 3n^2 i + \sum_{i=1}^n 3ni^2 + \sum_{i=1}^n i^3 \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[n \cdot n^3 + 3n^2 \sum_{i=1}^n i + 3n \sum_{i=1}^n i^2 + \sum_{i=1}^n i^3 \right] \\
 &= \lim_{n \rightarrow \infty} \left[1 + \frac{3}{n^2} \cdot \frac{n(n+1)}{2} + \frac{3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4} \right] \\
 &= \lim_{n \rightarrow \infty} \left[1 + \frac{3}{2} \cdot \frac{n+1}{n} + \frac{1}{2} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} + \frac{1}{4} \cdot \frac{(n+1)^2}{n^2} \right] \\
 &= \lim_{n \rightarrow \infty} \left[1 + \frac{3}{2} \left(1 + \frac{1}{n}\right) + \frac{1}{2} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) + \frac{1}{4} \left(1 + \frac{1}{n}\right)^2 \right] = 1 + \frac{3}{2} + \frac{1}{2} \cdot 2 + \frac{1}{4} = 3.75
 \end{aligned}$$

26. (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.

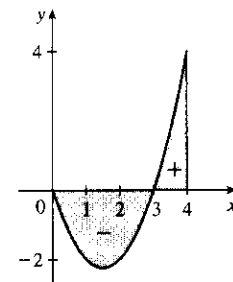
$$\begin{aligned}
 \int_0^4 (x^2 - 3x) dx &\approx \sum_{i=1}^8 f(x_i^*) \Delta x \\
 &= 0.5 \{ [0.5^2 - 3(0.5)] + [1.0^2 - 3(1.0)] + \cdots \\
 &\quad + [3.5^2 - 3(3.5)] + [4.0^2 - 3(4.0)] \} \\
 &= \frac{1}{2} \left(-\frac{5}{4} - 2 - \frac{9}{4} - 2 - \frac{5}{4} + 0 + \frac{7}{4} + 4 \right) = -1.5
 \end{aligned}$$

(b)



$$\begin{aligned}
 \text{(c)} \quad \int_0^4 (x^2 - 3x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{4i}{n} \right)^2 - 3 \left(\frac{4i}{n} \right) \right] \left(\frac{4}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[\frac{16}{n^2} \sum_{i=1}^n i^2 - \frac{12}{n} \sum_{i=1}^n i \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2} \cdot \frac{n(n+1)}{2} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{32}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - 24 \left(1 + \frac{1}{n} \right) \right] \\
 &= \frac{32}{3} \cdot 2 - 24 = -\frac{8}{3}
 \end{aligned}$$

- (d) $\int_0^4 (x^2 - 3x) dx = A_1 - A_2$,
where A_1 is the area marked + and
 A_2 is the area marked -.



$$\begin{aligned}
 27. \quad \int_a^b x dx &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n \left[a + \frac{b-a}{n} i \right] = \lim_{n \rightarrow \infty} \left[\frac{a(b-a)}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{a(b-a)}{n} n + \frac{(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2} \right] = a(b-a) + \lim_{n \rightarrow \infty} \frac{(b-a)^2}{2} \left(1 + \frac{1}{n} \right) \\
 &= a(b-a) + \frac{1}{2}(b-a)^2 = (b-a) \left(a + \frac{1}{2}b - \frac{1}{2}a \right) = (b-a) \frac{1}{2}(b+a) = \frac{1}{2}(b^2 - a^2)
 \end{aligned}$$

$$\begin{aligned}
 28. \int_a^b x^2 dx &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n \left[a + \frac{b-a}{n} i \right]^2 = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n \left[a^2 + 2a \frac{b-a}{n} i + \frac{(b-a)^2}{n^2} i^2 \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{(b-a)^3}{n^3} \sum_{i=1}^n i^2 + \frac{2a(b-a)^2}{n^2} \sum_{i=1}^n i + \frac{a^2(b-a)}{n} \sum_{i=1}^n 1 \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{(b-a)^3}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{2a(b-a)^2}{n^2} \frac{n(n+1)}{2} + \frac{a^2(b-a)}{n} n \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{(b-a)^3}{6} \cdot 1 \cdot \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + a(b-a)^2 \cdot 1 \cdot \left(1 + \frac{1}{n} \right) + a^2(b-a) \right] \\
 &= \frac{(b-a)^3}{3} + a(b-a)^2 + a^2(b-a) = \frac{b^3 - 3ab^2 + 3a^2b - a^3}{3} + ab^2 - 2a^2b + a^3 + a^2b - a^3 \\
 &= \frac{b^3}{3} - \frac{a^3}{3} - ab^2 + a^2b + ab^2 - a^2b = \frac{b^3 - a^3}{3}
 \end{aligned}$$

29. $f(x) = \frac{x}{1+x^5}$, $a = 2$, $b = 6$, and $\Delta x = \frac{6-2}{n} = \frac{4}{n}$. Using Equation 3, we get $x_i^* = x_i = 2 + i \Delta x = 2 + \frac{4i}{n}$,

$$\text{so } \int_2^6 \frac{x}{1+x^5} dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2 + \frac{4i}{n}}{1 + \left(2 + \frac{4i}{n} \right)^5} \cdot \frac{4}{n}.$$

30. $f(x) = x^2 \sin x$, $[a, b] = [0, 2\pi]$, $\Delta x = \frac{2\pi - 0}{n} = \frac{2\pi}{n}$, and $x_i = a + i \Delta x = \frac{2\pi i}{n}$, so

$$\int_0^{2\pi} x^2 \sin x dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2\pi i}{n} \right)^2 \sin \left(\frac{2\pi i}{n} \right) \right] \cdot \frac{2\pi}{n}.$$

31. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^\pi \sin 5x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n (\sin 5x_i) \left(\frac{\pi}{n} \right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\sin \frac{5\pi i}{n} \right) \frac{\pi}{n} \stackrel{\text{CAS}}{=} \pi \lim_{n \rightarrow \infty} \frac{1}{n} \cot \left(\frac{5\pi}{2n} \right) \stackrel{\text{CAS}}{=} \pi \left(\frac{2}{5\pi} \right) = \frac{2}{5}$$

32. $\Delta x = (10 - 2)/n = 8/n$ and $x_i^* = x_i = 2 + 8i/n$.

$$\begin{aligned}
 \int_2^{10} x^6 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 + \frac{8i}{n} \right)^6 \left(\frac{8}{n} \right) = 8 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(2 + \frac{8i}{n} \right)^6 \\
 &\stackrel{\text{CAS}}{=} 8 \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{64(58,593n^6 + 164,052n^5 + 131,208n^4 - 27,776n^2 + 2048)}{21n^5} \\
 &\stackrel{\text{CAS}}{=} 8 \left(\frac{1,249,984}{7} \right) = \frac{9,999,872}{7} \approx 1,428,553.1
 \end{aligned}$$

33. (a) Think of $\int_0^2 f(x) dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is

$$A = \frac{1}{2}(b + B)h, \text{ so } \int_0^2 f(x) dx = \frac{1}{2}(1 + 3)2 = 4.$$

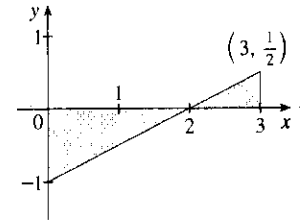
$$\begin{aligned}
 \text{(b) } \int_0^5 f(x) dx &= \int_0^2 f(x) dx + \int_2^3 f(x) dx + \int_3^5 f(x) dx \\
 &\quad \text{trapezoid} \quad \text{rectangle} \quad \text{triangle} \\
 &= \frac{1}{2}(1 + 3)2 + 3 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 3 = 4 + 3 + 3 = 10
 \end{aligned}$$

(c) $\int_5^7 f(x) dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x) dx = -\frac{1}{2} \cdot 2 \cdot 3 = -3.$

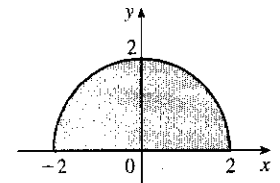
- (d) $\int_7^9 f(x) dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals $-\frac{1}{2}(B+b)h = -\frac{1}{2}(3+2)2 = -5$. Thus,
- $$\int_0^9 f(x) dx = \int_0^5 f(x) dx + \int_5^7 f(x) dx + \int_7^9 f(x) dx = 10 + (-3) + (-5) = 2.$$

34. (a) $\int_0^2 g(x) dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$ (area of a triangle)
 (b) $\int_2^6 g(x) dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ (negative of the area of a semicircle)
 (c) $\int_6^7 g(x) dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ (area of a triangle)
 $\int_0^7 g(x) dx = \int_0^2 g(x) dx + \int_2^6 g(x) dx + \int_6^7 g(x) dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

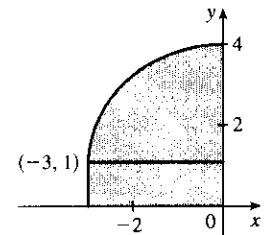
35. $\int_0^3 (\frac{1}{2}x - 1) dx$ can be interpreted as the area of the triangle above the x -axis minus the area of the triangle below the x -axis; that is,
- $$\frac{1}{2}(1)(\frac{1}{2}) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}.$$



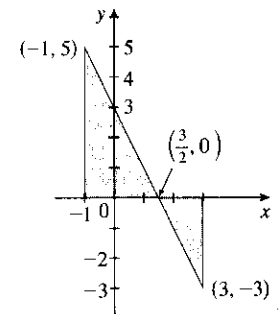
36. $\int_{-2}^2 \sqrt{4-x^2} dx$ can be interpreted as the area under the graph of $f(x) = \sqrt{4-x^2}$ between $x = -2$ and $x = 2$. This is equal to half the area of the circle with radius 2, so $\int_{-2}^2 \sqrt{4-x^2} dx = \frac{1}{2}\pi \cdot 2^2 = 2\pi$.



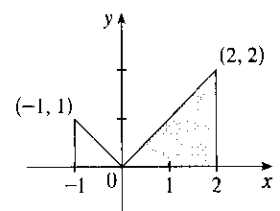
37. $\int_{-3}^0 (1 + \sqrt{9-x^2}) dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9-x^2}$ between $x = -3$ and $x = 0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so
- $$\int_{-3}^0 (1 + \sqrt{9-x^2}) dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi.$$



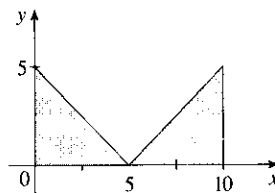
38. $\int_{-1}^3 (3-2x) dx$ can be interpreted as the area of the triangle above the x -axis minus the area of the triangle below the x -axis; that is,
- $$\frac{1}{2}(\frac{5}{2})(5) - \frac{1}{2}(\frac{3}{2})(3) = \frac{25}{4} - \frac{9}{4} = 4.$$



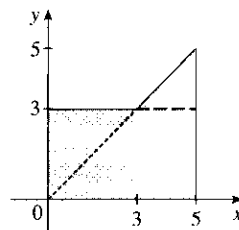
39. $\int_{-1}^2 |x| dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$.



40. $\int_0^{10} |x - 5| dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $2(\frac{1}{2})(5)(5) = 25$.



41. $\int_9^4 \sqrt{t} dt = -\int_4^9 \sqrt{t} dt$ [because we reversed the limits of integration]
 $= -\int_4^9 \sqrt{x} dx$ [we can use any letter without changing the value of the integral]
 $= -\frac{38}{3}$
42. $\int_1^1 x^2 \cos x dx = 0$ since the limits of integration are equal.
43. $\int_0^1 (5 - 6x^2) dx = \int_0^1 5 dx - 6 \int_0^1 x^2 dx = 5(1 - 0) - 6(\frac{1}{3}) = 5 - 2 = 3$
44. $\int_2^5 (1 + 3x^4) dx = \int_2^5 1 dx + \int_2^5 3x^4 dx = 1(5 - 2) + 3 \int_2^5 x^4 dx = 1(3) + 3(618.6) = 1858.8$
45. $\int_1^4 (2x^2 - 3x + 1) dx = 2 \int_1^4 x^2 dx - 3 \int_1^4 x dx + \int_1^4 1 dx$
 $= 2 \cdot \frac{1}{3}(4^3 - 1^3) - 3 \cdot \frac{1}{2}(4^2 - 1^2) + 1(4 - 1) = \frac{45}{2} = 22.5$
46. $\int_0^{\pi/2} (2 \cos x - 5x) dx = \int_0^{\pi/2} 2 \cos x dx - \int_0^{\pi/2} 5x dx = 2 \int_0^{\pi/2} \cos x dx - 5 \int_0^{\pi/2} x dx$
 $= 2(1) - 5 \frac{(\pi/2)^2 - 0^2}{2} = 2 - \frac{5\pi^2}{8}$
47. $\int_{-2}^2 f(x) dx + \int_2^5 f(x) dx - \int_{-2}^{-1} f(x) dx = \int_{-2}^5 f(x) dx + \int_{-1}^{-2} f(x) dx$ [by Property 5 and reversing limits]
 $= \int_{-1}^5 f(x) dx$ [Property 5]
48. $\int_1^4 f(x) dx = \int_1^5 f(x) dx - \int_4^5 f(x) dx = 12 - 3.6 = 8.4$
49. $\int_0^9 [2f(x) + 3g(x)] dx = 2 \int_0^9 f(x) dx + 3 \int_0^9 g(x) dx = 2(37) + 3(16) = 122$
50. If $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x) dx$ can be interpreted as the area of the shaded region, which consists of a 5-by-3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus,
 $\int_0^5 f(x) dx = 5(3) + \frac{1}{2}(2)(2) = 17$.
51. $0 \leq \sin x < 1$ on $[0, \frac{\pi}{4}]$, so $\sin^3 x \leq \sin^2 x$ on $[0, \frac{\pi}{4}]$. Hence, $\int_0^{\pi/4} \sin^3 x dx \leq \int_0^{\pi/4} \sin^2 x dx$ (Property 7).
52. $5 - x \geq 3 \geq x + 1$ on $[1, 2]$, so $\sqrt{5 - x} \geq \sqrt{x + 1}$ and $\int_1^2 \sqrt{5 - x} dx \geq \int_1^2 \sqrt{x + 1} dx$.
53. If $-1 \leq x \leq 1$, then $0 \leq x^2 \leq 1$ and $1 \leq 1 + x^2 \leq 2$, so $1 \leq \sqrt{1 + x^2} \leq \sqrt{2}$ and
 $1[1 - (-1)] \leq \int_{-1}^1 \sqrt{1 + x^2} dx \leq \sqrt{2}[1 - (-1)]$ [Property 8]; that is, $2 \leq \int_{-1}^1 \sqrt{1 + x^2} dx \leq 2\sqrt{2}$.
54. $\frac{1}{2} \leq \sin x \leq 1$ for $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}$, so $\frac{1}{2}(\frac{\pi}{2} - \frac{\pi}{6}) \leq \int_{\pi/6}^{\pi/2} \sin x dx \leq 1(\frac{\pi}{2} - \frac{\pi}{6})$ [Property 8]; that is,
 $\frac{\pi}{6} \leq \int_{\pi/6}^{\pi/2} \sin x dx \leq \frac{\pi}{3}$.
55. If $1 \leq x \leq 2$, then $\frac{1}{2} \leq \frac{1}{x} \leq 1$, so $\frac{1}{2}(2 - 1) \leq \int_1^2 \frac{1}{x} dx \leq 1(2 - 1)$ or $\frac{1}{2} \leq \int_1^2 \frac{1}{x} dx \leq 1$.



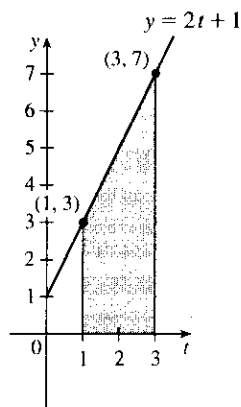
56. If $0 \leq x \leq 2$, then $0 \leq x^3 \leq 8$, so $1 \leq x^3 + 1 \leq 9$ and $1 \leq \sqrt{x^3 + 1} \leq 3$. Thus,
 $1(2 - 0) \leq \int_0^2 \sqrt{x^3 + 1} dx \leq 3(2 - 0)$ that is, $2 \leq \int_0^2 \sqrt{x^3 + 1} dx \leq 6$.
57. If $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$, then $1 \leq \tan x \leq \sqrt{3}$, so $1(\frac{\pi}{3} - \frac{\pi}{4}) \leq \int_{\pi/4}^{\pi/3} \tan x dx \leq \sqrt{3}(\frac{\pi}{3} - \frac{\pi}{4})$ or
 $\frac{\pi}{12} \leq \int_{\pi/4}^{\pi/3} \tan x dx \leq \frac{\pi}{12}\sqrt{3}$.
58. Let $f(x) = x^3 - 3x + 3$ for $0 \leq x \leq 2$. Then $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$, so f is decreasing on $(0, 1)$ and increasing on $(1, 2)$. f has the absolute minimum value $f(1) = 1$. Since $f(0) = 3$ and $f(2) = 5$, the absolute maximum value of f is $f(2) = 5$. Thus, $1 \leq x^3 - 3x + 3 \leq 5$ for x in $[0, 2]$. It follows from Property 8 that
 $1 \cdot (2 - 0) \leq \int_0^2 (x^3 - 3x + 3) dx \leq 5 \cdot (2 - 0)$; that is, $2 \leq \int_0^2 (x^3 - 3x + 3) dx \leq 10$.
59. For $-1 \leq x \leq 1$, $0 \leq x^4 \leq 1$ and $1 \leq \sqrt{1+x^4} \leq \sqrt{2}$, so $1[1 - (-1)] \leq \int_{-1}^1 \sqrt{1+x^4} dx \leq \sqrt{2}[1 - (-1)]$ or
 $2 \leq \int_{-1}^1 \sqrt{1+x^4} dx \leq 2\sqrt{2}$.
60. If $\frac{1}{4}\pi \leq x \leq \frac{3}{4}\pi$, then $\frac{\sqrt{2}}{2} \leq \sin x \leq 1$ and $\frac{1}{2} \leq \sin^2 x \leq 1$, so $\frac{1}{2}(\frac{3}{4}\pi - \frac{1}{4}\pi) \leq \int_{\pi/4}^{3\pi/4} \sin^2 x dx \leq 1(\frac{3}{4}\pi - \frac{1}{4}\pi)$;
that is, $\frac{1}{4}\pi \leq \int_{\pi/4}^{3\pi/4} \sin^2 x dx \leq \frac{1}{2}\pi$.
61. $\sqrt{x^4 + 1} \geq \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4 + 1} dx \geq \int_1^3 x^2 dx = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$.
62. $0 \leq \sin x \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$, so $x \sin x \leq x \Rightarrow \int_0^{\pi/2} x \sin x dx \leq \int_0^{\pi/2} x dx = \frac{1}{2}[(\frac{\pi}{2})^2 - 0^2] = \frac{\pi^2}{8}$.
63. Using a regular partition and right endpoints as in the proof of Property 2, we calculate
 $\int_a^b cf(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n cf(x_i) \Delta x_i = \lim_{n \rightarrow \infty} c \sum_{i=1}^n f(x_i) \Delta x_i = c \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i = c \int_a^b f(x) dx$.
64. As in the proof of Property 2, we write $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$. Now $f(x_i) \geq 0$ and $\Delta x \geq 0$, so
 $f(x_i) \Delta x \geq 0$ and therefore $\sum_{i=1}^n f(x_i) \Delta x \geq 0$. But the limit of nonnegative quantities is nonnegative by
Theorem 2.3.2, so $\int_a^b f(x) dx \geq 0$.
65. Since $-|f(x)| \leq f(x) \leq |f(x)|$, it follows from Property 7 that
 $-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx \Rightarrow \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$
- Note that the definite integral is a real number, and so the following property applies: $-a \leq b \leq a \Rightarrow |b| \leq a$
for all real numbers b and nonnegative numbers a .
66. $\left| \int_0^{2\pi} f(x) \sin 2x dx \right| \leq \int_0^{2\pi} |f(x) \sin 2x| dx$ [by Exercise 65] $= \int_0^{2\pi} |f(x)| |\sin 2x| dx \leq \int_0^{2\pi} |f(x)| dx$ by
Property 7, since $|\sin 2x| \leq 1 \Rightarrow |f(x)| |\sin 2x| \leq |f(x)|$.
67. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^4}{n^5} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^4 = \int_0^1 x^4 dx$
68. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + (i/n)^2} = \int_0^1 \frac{dx}{1 + x^2}$

69. Choose $x_i = 1 + \frac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1 + \frac{i-1}{n}\right)\left(1 + \frac{i}{n}\right)}$. Then

$$\begin{aligned}
 \int_1^2 x^{-2} dx &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{\left(1 + \frac{i-1}{n}\right)\left(1 + \frac{i}{n}\right)} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{(n+i-1)(n+i)} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{n+i-1} - \frac{1}{n+i} \right) \quad [\text{by the hint}] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{i=1}^n \frac{1}{n+i-1} - \sum_{i=1}^n \frac{1}{n+i} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} \right] - \left[\frac{1}{n+1} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} \right] \right) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{n} - \frac{1}{2n} \right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2} \right) = \frac{1}{2}
 \end{aligned}$$

DISCOVERY PROJECT Area Functions

1. (a)



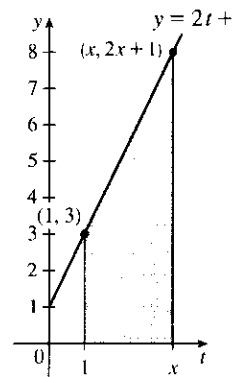
$$\begin{aligned}
 \text{Area of trapezoid} &= \frac{1}{2}(b_1 + b_2)h \\
 &= \frac{1}{2}(3 + 7)2 \\
 &= 10 \text{ square units}
 \end{aligned}$$

Or:

$$\begin{aligned}
 \text{Area of rectangle} + \text{area of triangle} \\
 &= b_r h_r + \frac{1}{2} b_t h_t \\
 &= (2)(3) + \frac{1}{2}(2)(4) = 10 \text{ square units}
 \end{aligned}$$

(c) $A'(x) = 2x + 1$. This is the y -coordinate of the point $(x, 2x + 1)$ on the given line.

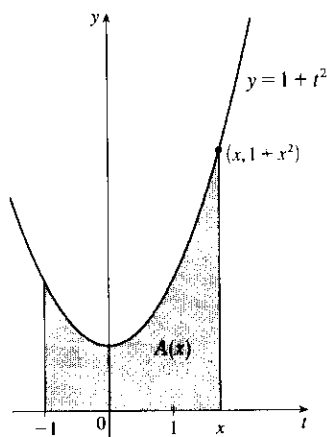
(b)



As in part (a),

$$\begin{aligned}
 A(x) &= \frac{1}{2}[3 + (2x + 1)](x - 1) \\
 &= \frac{1}{2}(2x + 4)(x - 1) \\
 &= (x + 2)(x - 1) \\
 &= x^2 + x - 2 \text{ square units}
 \end{aligned}$$

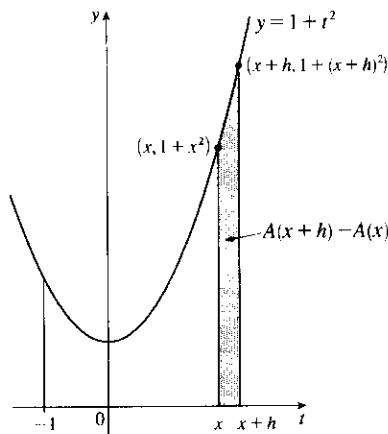
2. (a)



$$\begin{aligned} \text{(b) } A(x) &= \int_{-1}^x (1+t^2) dt = \int_{-1}^x 1 dt + \int_{-1}^x t^2 dt \quad [\text{Property 2}] \\ &= 1[x - (-1)] + \frac{x^3 - (-1)^3}{3} \quad \left[\begin{array}{l} \text{Property 1 and} \\ \text{Exercise 5.2.28} \end{array} \right] \\ &= x + 1 + \frac{1}{3}x^3 + \frac{1}{3} \\ &= \frac{1}{3}x^3 + x + \frac{4}{3} \end{aligned}$$

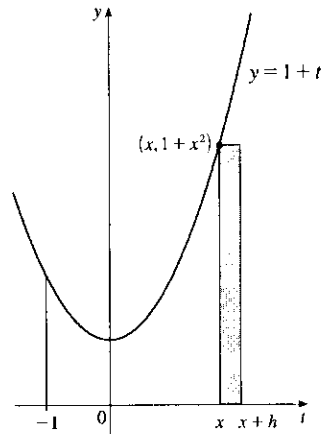
(c) $A'(x) = x^2 + 1$. This is the y -coordinate of the point $(x, 1 + x^2)$ on the given curve.

(d)



$A(x+h) - A(x)$ is the area under the curve $y = 1 + t^2$ from $t = x$ to $t = x+h$.

(e)



An approximating rectangle is shown in the figure.

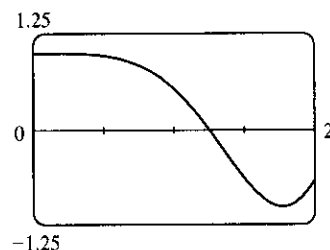
It has height $1 + x^2$, width h , and area $h(1 + x^2)$, so

$$A(x+h) - A(x) \approx h(1+x^2) \Rightarrow$$

$$\frac{A(x+h) - A(x)}{h} \approx 1 + x^2.$$

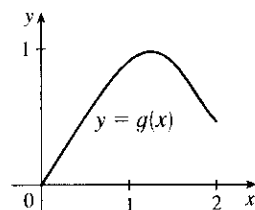
(f) Part (e) says that the average rate of change of A is approximately $1 + x^2$. As h approaches 0, the quotient approaches the instantaneous rate of change—namely, $A'(x)$. So the result of part (c), $A'(x) = x^2 + 1$, is geometrically plausible.

3. (a) $f(x) = \cos(x^2)$

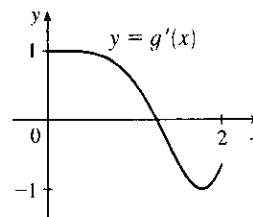


(b) $g(x)$ starts to decrease at that value of x where $\cos(t^2)$ changes from positive to negative; that is, at about $x = 1.25$.

- (c) $g(x) = \int_0^x \cos(t^2) dt$. Using an integration command, we find that $g(0) = 0$, $g(0.2) \approx 0.200$, $g(0.4) \approx 0.399$, $g(0.6) \approx 0.592$, $g(0.8) \approx 0.768$, $g(1.0) \approx 0.905$, $g(1.2) \approx 0.974$, $g(1.4) \approx 0.950$, $g(1.6) \approx 0.826$, $g(1.8) \approx 0.635$, and $g(2.0) \approx 0.461$.



- (d) We sketch the graph of g' using the method of Example 1 in Section 3.2. The graphs of $g'(x)$ and $f(x)$ look alike, so we guess that $g'(x) = f(x)$.



4. In Problems 1 and 2, we showed that if $g(x) = \int_a^x f(t) dt$, then $g'(x) = f(x)$, for the functions $f(t) = 2t + 1$ and $f(t) = 1 + t^2$. In Problem 3 we guessed that the same is true for $f(t) = \cos(t^2)$, based on visual evidence. So we conjecture that $g'(x) = f(x)$ for any continuous function f . This turns out to be true and is proved in Section 5.3 (the Fundamental Theorem of Calculus).

5.3 The Fundamental Theorem of Calculus

1. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

2. (a) $g(x) = \int_0^x f(t) dt$, so $g(0) = \int_0^0 f(t) dt = 0$.

$$g(1) = \int_0^1 f(t) dt = \frac{1}{2} \cdot 1 \cdot 1 \text{ [area of triangle]} = \frac{1}{2}.$$

$$g(2) = \int_0^2 f(t) dt = \int_0^1 f(t) dt + \int_1^2 f(t) dt \text{ [below the } x\text{-axis]} \\ = \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot 1 = 0.$$

$$g(3) = g(2) + \int_2^3 f(t) dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}.$$

$$g(4) = g(3) + \int_3^4 f(t) dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0.$$

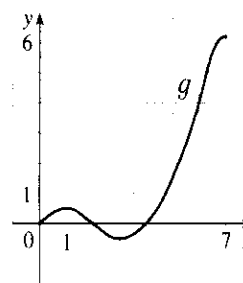
$$g(5) = g(4) + \int_4^5 f(t) dt = 0 + 1.5 = 1.5.$$

$$g(6) = g(5) + \int_5^6 f(t) dt = 1.5 + 2.5 = 4.$$

- (b) $g(7) = g(6) + \int_6^7 f(t) dt \approx 4 + 2.2 \text{ [estimate from the graph]} = 6.2$.

- (c) The answers from part (a) and part (b) indicate that g has a minimum at $x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

(d)



3. (a) $g(x) = \int_0^x f(t) dt$.

$$g(0) = \int_0^0 f(t) dt = 0$$

$$g(1) = \int_0^1 f(t) dt = 1 \cdot 2 = 2 \text{ [rectangle]},$$

$$g(2) = \int_0^2 f(t) dt = \int_0^1 f(t) dt + \int_1^2 f(t) dt = g(1) + \int_1^2 f(t) dt \\ = 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5 \text{ [rectangle plus triangle]},$$

$$g(3) = \int_0^3 f(t) dt = g(2) + \int_2^3 f(t) dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7,$$

$$g(6) = g(3) + \int_3^6 f(t) dt \text{ [the integral is negative since } f \text{ lies under the } x\text{-axis]} \\ = 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2\right) \right] = 7 - 4 = 3$$

 (b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep adding more area.

 (c) g has a maximum value when we start subtracting area; that is, at $x = 3$.

4. (a) $g(-3) = \int_{-3}^{-3} f(t) dt = 0$, $g(3) = \int_{-3}^3 f(t) dt = \int_{-3}^0 f(t) dt + \int_0^3 f(t) dt = 0$ by symmetry, since the area above the x -axis is the same as the area below the axis.

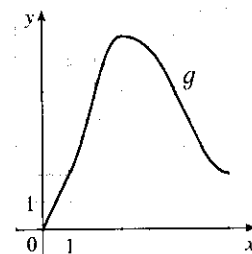
(b) From the graph, it appears that to the nearest $\frac{1}{2}$, $g(-2) = \int_{-3}^{-2} f(t) dt \approx 1$, $g(-1) = \int_{-3}^{-1} f(t) dt \approx 3\frac{1}{2}$, and $g(0) = \int_{-3}^0 f(t) dt \approx 5\frac{1}{2}$.

 (c) g is increasing on $(-3, 0)$ because as x increases from -3 to 0, we keep adding more area.

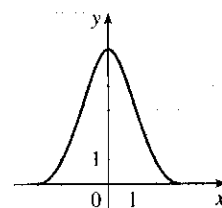
 (d) g has a maximum value when we start subtracting area; that is, at $x = 0$.

 (f) The graph of $g'(x)$ is the same as that of $f(x)$, as indicated by FTC1.

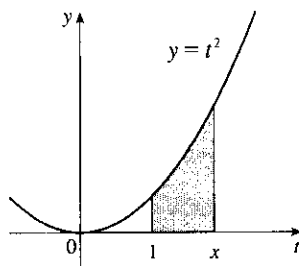
(d)



(e)



5.



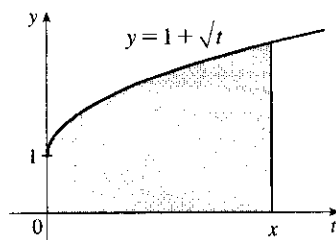
(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2 dt \Rightarrow$

$$g'(x) = f(x) = x^2.$$

(b) Using FTC2, $g(x) = \int_1^x t^2 dt = \left[\frac{1}{3}t^3 \right]_1^x = \frac{1}{3}x^3 - \frac{1}{3} \Rightarrow$

$$g'(x) = x^2.$$

6.



(a) By FTC1 with $f(t) = 1 + \sqrt{t}$ and $a = 0$,

$$g(x) = \int_0^x (1 + \sqrt{t}) dt \Rightarrow g'(x) = f(x) = 1 + \sqrt{x}.$$

(b) Using FTC2, $g(x) = \int_0^x (1 + \sqrt{t}) dt = \left[t + \frac{2}{3}t^{3/2} \right]_0^x = x + \frac{2}{3}x^{3/2}$

$$\Rightarrow g'(x) = 1 + x^{1/2} = 1 + \sqrt{x}.$$

7. $f(t) = \sqrt{1+2t}$ and $g(x) = \int_0^x \sqrt{1+2t} dt$, so by FTC1, $g'(x) = f(x) = \sqrt{1+2x}$.

8. $f(t) = (2+t^4)^5$ and $g(x) = \int_1^x (2+t^4)^5 dt$, so $g'(x) = f(x) = (2+x^4)^5$.

9. $f(t) = t^2 \sin t$ and $g(y) = \int_2^y t^2 \sin t dt$, so by FTC1, $g'(y) = f(y) = y^2 \sin y$.

10. $f(x) = \frac{1}{x+x^2}$ and $g(u) = \int_3^u \frac{1}{x+x^2} dx$, so $g'(u) = f(u) = \frac{1}{u+u^2}$.

$$11. F(x) = \int_x^2 \cos(t^2) dt = - \int_2^x \cos(t^2) dt \Rightarrow F'(x) = -\cos(x^2)$$

$$12. f(\theta) = \tan \theta \text{ and } F(x) = \int_x^{10} \tan \theta d\theta = - \int_{10}^x \tan \theta d\theta, \text{ so by FTC1, } F'(x) = -f(x) = -\tan x.$$

$$13. \text{ Let } u = \frac{1}{x}. \text{ Then } \frac{du}{dx} = -\frac{1}{x^2}. \text{ Also, } \frac{dh}{dx} = \frac{dh}{du} \frac{du}{dx}, \text{ so}$$

$$h'(x) = \frac{d}{dx} \int_2^{1/x} \sin^4 t dt = \frac{d}{du} \int_2^u \sin^4 t dt \cdot \frac{du}{dx} = \sin^4 u \frac{du}{dx} = \frac{-\sin^4(1/x)}{x^2}.$$

$$14. \text{ Let } u = x^2. \text{ Then } \frac{du}{dx} = 2x. \text{ Also, } \frac{dh}{dx} = \frac{dh}{du} \frac{du}{dx}, \text{ so}$$

$$h'(x) = \frac{d}{dx} \int_0^{x^2} \sqrt{1+r^3} dr = \frac{d}{du} \int_0^u \sqrt{1+r^3} dr \cdot \frac{du}{dx} = \sqrt{1+u^3}(2x) = 2x \sqrt{1+(x^2)^3} = 2x \sqrt{1+x^6}.$$

$$15. \text{ Let } u = \sqrt{x}. \text{ Then } \frac{du}{dx} = \frac{1}{2\sqrt{x}}. \text{ Also, } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}, \text{ so}$$

$$y' = \frac{d}{dx} \int_3^{\sqrt{x}} \frac{\cos t}{t} dt = \frac{d}{du} \int_3^u \frac{\cos t}{t} dt \cdot \frac{du}{dx} = \frac{\cos u}{u} \cdot \frac{1}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{2x}.$$

$$16. \text{ Let } u = \cos x. \text{ Then } \frac{du}{dx} = -\sin x. \text{ Also, } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}, \text{ so}$$

$$\begin{aligned} y' &= \frac{d}{dx} \int_1^{\cos x} (t + \sin t) dt = \frac{d}{du} \int_1^u (t + \sin t) dt \cdot \frac{du}{dx} \\ &= (u + \sin u) \cdot (-\sin x) = -\sin x [\cos x + \sin(\cos x)] \end{aligned}$$

$$17. \text{ Let } w = 1 - 3x. \text{ Then } \frac{dw}{dx} = -3. \text{ Also, } \frac{dy}{dx} = \frac{dy}{dw} \frac{dw}{dx}, \text{ so}$$

$$\begin{aligned} y' &= \frac{d}{dx} \int_{1-3x}^1 \frac{u^3}{1+u^2} du = \frac{d}{dw} \int_w^1 \frac{u^3}{1+u^2} du \cdot \frac{dw}{dx} \\ &= -\frac{d}{dw} \int_1^w \frac{u^3}{1+u^2} du \cdot \frac{dw}{dx} = -\frac{w^3}{1+w^2}(-3) = \frac{3(1-3x)^3}{1+(1-3x)^2} \end{aligned}$$

$$18. \text{ Let } u = \frac{1}{x^2}. \text{ Then } \frac{du}{dx} = -\frac{2}{x^3}. \text{ Also, } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}, \text{ so}$$

$$y' = \frac{d}{dx} \int_{1/x^2}^0 \sin^3 t dt = \frac{d}{du} \int_u^0 \sin^3 t dt \cdot \frac{du}{dx} = -\frac{d}{du} \int_0^u \sin^3 t dt \cdot \frac{du}{dx} = -\sin^3 u \left(-\frac{2}{x^3}\right) = \frac{2 \sin^3(1/x^2)}{x^3}.$$

$$19. \int_{-1}^3 x^5 dx = \left[\frac{x^6}{6} \right]_{-1}^3 = \frac{3^6}{6} - \frac{(-1)^6}{6} = \frac{729-1}{6} = \frac{364}{3}$$

$$20. \int_{-2}^5 6 dx = [6x]_{-2}^5 = 6[5 - (-2)] = 6(7) = 42$$

$$21. \int_2^8 (4x+3) dx = \left[\frac{4}{2}x^2 + 3x \right]_2^8 = (2 \cdot 8^2 + 3 \cdot 8) - (2 \cdot 2^2 + 3 \cdot 2) = 152 - 14 = 138$$

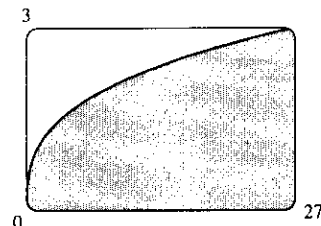
$$22. \int_0^4 (1+3y-y^2) dy = \left[y + \frac{3}{2}y^2 - \frac{1}{3}y^3 \right]_0^4 = \left(4 + \frac{3}{2} \cdot 16 - \frac{1}{3} \cdot 64 \right) - (0) = \frac{20}{3}$$

$$23. \int_0^1 x^{4/5} dx = \left[\frac{5}{9}x^{9/5} \right]_0^1 = \frac{5}{9} - 0 = \frac{5}{9}$$

$$24. \int_1^8 \sqrt[3]{x} dx = \int_1^8 x^{1/3} dx = \left[\frac{3}{4}x^{4/3} \right]_1^8 = \frac{3}{4}(8^{4/3} - 1^{4/3}) = \frac{3}{4}(2^4 - 1) = \frac{3}{4}(16 - 1) = \frac{3}{4}(15) = \frac{45}{4}$$

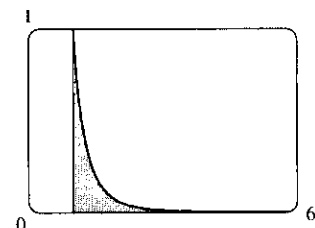
$$25. \int_1^2 \frac{3}{t^4} dt = 3 \int_1^2 t^{-4} dt = 3 \left[\frac{t^{-3}}{-3} \right]_1^2 = \frac{3}{-3} \left[\frac{1}{t^3} \right]_1^2 = -1 \left(\frac{1}{8} - 1 \right) = \frac{7}{8}$$

26. $\int_{-2}^3 x^{-5} dx$ does not exist because the function $f(x) = x^{-5}$ has an infinite discontinuity at $x = 0$; that is, f is discontinuous on the interval $[-2, 3]$.
27. $\int_{-5}^5 \frac{2}{x^3} dx$ does not exist because the function $f(x) = \frac{2}{x^3}$ has an infinite discontinuity at $x = 0$; that is, f is discontinuous on the interval $[-5, 5]$.
28. $\int_{\pi}^{2\pi} \cos \theta d\theta = [\sin \theta]_{\pi}^{2\pi} = \sin 2\pi - \sin \pi = 0 - 0 = 0$
29. $\int_0^2 x(2 + x^5) dx = \int_0^2 (2x + x^6) dx = [x^2 + \frac{1}{7}x^7]_0^2 = (4 + \frac{128}{7}) - (0 + 0) = \frac{156}{7}$
30. $\int_1^4 \frac{1}{\sqrt{x}} dx = \int_1^4 x^{-1/2} dx = \left[\frac{x^{1/2}}{1/2} \right]_1^4 = [2x^{1/2}]_1^4 = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2$
31. $\int_0^{\pi/4} \sec^2 t dt = [\tan t]_0^{\pi/4} = \tan \frac{\pi}{4} - \tan 0 = 1 - 0 = 1$
32. $\int_0^1 (3 + x\sqrt{x}) dx = \int_0^1 (3 + x^{3/2}) dx = [3x + \frac{2}{5}x^{5/2}]_0^1 = [(3 + \frac{2}{5}) - 0] = \frac{17}{5}$
33. $\int_{\pi}^{2\pi} \csc^2 \theta d\theta$ does not exist because the function $f(\theta) = \csc^2 \theta$ has infinite discontinuities at $\theta = \pi$ and $\theta = 2\pi$; that is, f is discontinuous on the interval $[\pi, 2\pi]$.
34. $\int_0^{\pi/6} \csc \theta \cot \theta d\theta$ does not exist because the function $f(\theta) = \csc \theta \cot \theta$ has an infinite discontinuity at $\theta = 0$; that is, f is discontinuous on the interval $[0, \frac{\pi}{6}]$.
35. $\int_0^2 f(x) dx = \int_0^1 x^4 dx + \int_1^2 x^5 dx = [\frac{1}{5}x^5]_0^1 + [\frac{1}{6}x^6]_1^2 = (\frac{1}{5} - 0) + (\frac{64}{6} - \frac{1}{6}) = 10.7$
36. $\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^0 x dx + \int_0^{\pi} \sin x dx = [\frac{1}{2}x^2]_{-\pi}^0 - [\cos x]_0^{\pi} = (0 - \frac{\pi^2}{2}) - (\cos \pi - \cos 0)$
 $= -\frac{\pi^2}{2} - (-1 - 1) = 2 - \frac{\pi^2}{2}$
37. From the graph, it appears that the area is about 60. The actual area is
 $\int_0^{27} x^{1/3} dx = [\frac{3}{4}x^{4/3}]_0^{27} = \frac{3}{4} \cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.



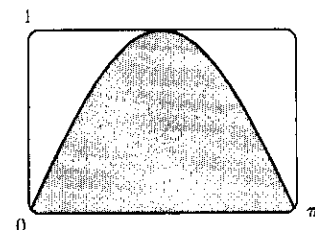
38. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

$$\int_1^6 x^{-4} dx = \left[\frac{x^{-3}}{-3} \right]_1^6 = \left[\frac{-1}{3x^3} \right]_1^6 = -\frac{1}{3 \cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318.$$



39. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is

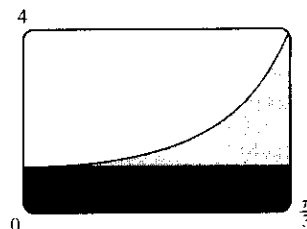
$$\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = -(-1) + 1 = 2.$$



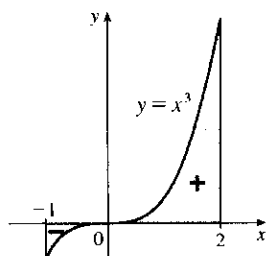
40. Splitting up the region as shown, we estimate that the area under the graph

is $\frac{\pi}{3} + \frac{1}{4}(3 \cdot \frac{\pi}{3}) \approx 1.8$. The actual area is

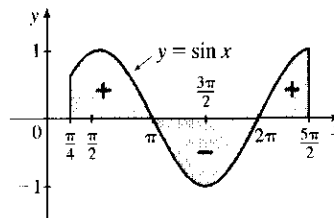
$$\int_0^{\pi/3} \sec^2 x \, dx = [\tan x]_0^{\pi/3} = \sqrt{3} - 0 = \sqrt{3} \approx 1.73.$$



41. $\int_{-1}^2 x^3 \, dx = [\frac{1}{4}x^4]_{-1}^2 = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$



42. $\int_{\pi/4}^{5\pi/2} \sin x \, dx = [-\cos x]_{\pi/4}^{5\pi/2} = 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$



43. $g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1} \, du = \int_{2x}^0 \frac{u^2 - 1}{u^2 + 1} \, du + \int_0^{3x} \frac{u^2 - 1}{u^2 + 1} \, du = -\int_0^{2x} \frac{u^2 - 1}{u^2 + 1} \, du + \int_0^{3x} \frac{u^2 - 1}{u^2 + 1} \, du \Rightarrow$

$$g'(x) = -\frac{(2x)^2 - 1}{(2x)^2 + 1} \cdot \frac{d}{dx}(2x) + \frac{(3x)^2 - 1}{(3x)^2 + 1} \cdot \frac{d}{dx}(3x) = -2 \cdot \frac{4x^2 - 1}{4x^2 + 1} + 3 \cdot \frac{9x^2 - 1}{9x^2 + 1}$$

44. $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}} \, dt = \int_{\tan x}^1 \frac{dt}{\sqrt{2+t^4}} + \int_1^{x^2} \frac{dt}{\sqrt{2+t^4}} = -\int_1^{\tan x} \frac{dt}{\sqrt{2+t^4}} + \int_1^{x^2} \frac{dt}{\sqrt{2+t^4}} \Rightarrow$

$$g'(x) = \frac{-1}{\sqrt{2+\tan^4 x}} \frac{d}{dx}(\tan x) + \frac{1}{\sqrt{2+x^8}} \frac{d}{dx}(x^2) = -\frac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}$$

45. $y = \int_{\sqrt{x}}^{x^3} \sqrt{t} \sin t \, dt = \int_{\sqrt{x}}^1 \sqrt{t} \sin t \, dt + \int_1^{x^3} \sqrt{t} \sin t \, dt = -\int_1^{\sqrt{x}} \sqrt{t} \sin t \, dt + \int_1^{x^3} \sqrt{t} \sin t \, dt \Rightarrow$

$$y' = -\frac{1}{2\sqrt{x}} (\sin \sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) + x^{3/2} \sin(x^3) \cdot \frac{d}{dx}(x^3) = -\frac{\sqrt{x} \sin \sqrt{x}}{2\sqrt{x}} + x^{3/2} \sin(x^3) (3x^2)$$

$$= 3x^{7/2} \sin(x^3) - \frac{\sin \sqrt{x}}{2\sqrt{x}}$$

46. $y = \int_{\cos x}^{5x} \cos(u^2) \, du = \int_0^{5x} \cos(u^2) \, du - \int_0^{\cos x} \cos(u^2) \, du \Rightarrow$

$$y' = \cos(25x^2) \cdot \frac{d}{dx}(5x) - \cos(\cos^2 x) \cdot \frac{d}{dx}(\cos x) = \cos(25x^2) \cdot 5 - \cos(\cos^2 x) \cdot (-\sin x)$$

$$= 5 \cos(25x^2) + \sin x \cos(\cos^2 x)$$

47. $F(x) = \int_1^x f(t) \, dt \Rightarrow F'(x) = f(x) = \int_1^{x^2} \frac{\sqrt{1+u^4}}{u} \, du \left[\text{since } f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u} \, du \right] \Rightarrow$

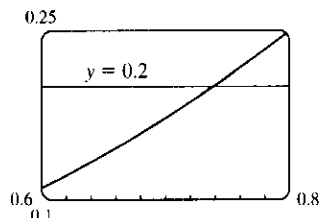
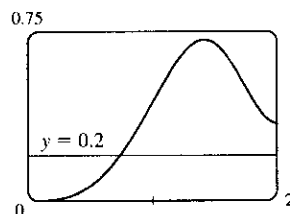
$$F''(x) = f'(x) = \frac{\sqrt{1+(x^2)^4}}{x^2} \cdot \frac{d}{dx}(x^2) = \frac{\sqrt{1+x^8}}{x^2} \cdot 2x = \frac{2\sqrt{1+x^8}}{x}. \text{ So } F''(2) = \sqrt{1+2^8} = \sqrt{257}.$$

48. For the curve to be concave upward, we must have $y'' > 0$. $y = \int_0^x \frac{1}{1+t+t^2} \, dt \Rightarrow y' = \frac{1}{1+x+x^2} \Rightarrow$

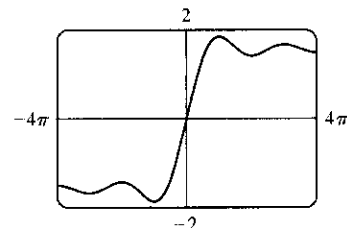
$$y'' = \frac{-(1+2x)}{(1+x+x^2)^2}. \text{ For this expression to be positive, we must have } (1+2x) < 0, \text{ since } (1+x+x^2)^2 > 0 \text{ for}$$

all x . $(1+2x) < 0 \Leftrightarrow x < -\frac{1}{2}$. Thus, the curve is concave upward on $(-\infty, -\frac{1}{2})$.

49. (a) The Fresnel function $S(x) = \int_0^x \sin(\frac{\pi}{2}t^2) dt$ has local maximum values where $0 = S'(x) = \sin(\frac{\pi}{2}x^2)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow x^2 = 2(2n-1) \Leftrightarrow x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow x^2 = 4n \Leftrightarrow x = -2\sqrt{n}$. S' does not change sign at $x = 0$.
- (b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos(\frac{\pi}{2}x^2)(2\frac{\pi}{2}x) = \pi x \cos(\frac{\pi}{2}x^2)$. For $x > 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) > 0 \Leftrightarrow 0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $(2n - \frac{1}{2})\pi < \frac{\pi}{2}x^2 < (2n + \frac{1}{2})\pi$, n any integer $\Leftrightarrow 0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x < 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) < 0 \Leftrightarrow (2n - \frac{3}{2})\pi < \frac{\pi}{2}x^2 < (2n - \frac{1}{2})\pi$, n any integer $\Leftrightarrow 4n-3 < x^2 < 4n-1 \Leftrightarrow \sqrt{4n-3} < |x| < \sqrt{4n-1} \Rightarrow \sqrt{4n-3} < -x < \sqrt{4n-1} \Rightarrow -\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $(-\sqrt{4n-1}, -\sqrt{4n-3})$, n any positive integer. To summarize: S is concave upward on the intervals $(0, 1)$, $(-\sqrt{3}, -1)$, $(\sqrt{3}, \sqrt{5})$, $(-\sqrt{7}, -\sqrt{5})$, $(\sqrt{7}, 3)$,
- (c) In Maple, we use `plot({int(sin(Pi*t^2/2), t=0..x), 0.2}, x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2], {t, 0, x}], 0.2}, {x, 0, 2}]`. In Derive, we load the utility file `FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin(\frac{\pi}{2}t^2) dt = 0.2$ at $x \approx 0.74$.



50. (a) In Maple, we should start by setting `si:=int(sin(t)/t, t=0..x);`. In Mathematica, the command is `si=Integrate[Sin[t]/t, {t, 0, x}]`. Note that both systems recognize this function; Maple calls it `Si(x)` and Mathematica calls it `SinIntegral[x]`. In Maple, the command to generate the graph is `plot(si, x=-4*Pi..4*Pi);`. In Mathematica, it is `Plot[si, {x, -4*Pi, 4*Pi}]`. In Derive, we load the utility file `EXP_INT` and plot `SI(x)`.



- (b) $Si(x)$ has local maximum values where $Si'(x)$ changes from positive to negative, passing through 0. From the Fundamental Theorem we know that $Si'(x) = \frac{d}{dx} \int_0^x \frac{\sin t}{t} dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x . For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of $Si'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \dots$
- (c) To find the first inflection point, we solve $Si''(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a root finder gives the value $x \approx 4.4934$. To find the y -coordinate of the inflection point, we evaluate $Si(4.4934) \approx 1.6556$. So the coordinates of the first

inflection point to the right of the origin are about (4.4934, 1.6556). Alternatively, we could graph $S''(x)$ and estimate the first positive x -value at which it changes sign.

- (d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim_{x \rightarrow \pm\infty} \text{Si}(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2}$. Since $\text{Si}(x)$ is an odd function,

$\lim_{x \rightarrow -\infty} \text{Si}(x) = -\frac{\pi}{2}$. So $\text{Si}(x)$ has the horizontal asymptotes $y = \pm \frac{\pi}{2}$.

- (e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in Exercise 49(c), we graph $y = \text{Si}(x)$ and $y = 1$ on the same screen to see where they intersect.

51. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9 . g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7 . There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

- (b) We can see from the graph that $\left| \int_0^1 f \, dt \right| < \left| \int_1^3 f \, dt \right| < \left| \int_3^5 f \, dt \right| < \left| \int_5^7 f \, dt \right| < \left| \int_7^9 f \, dt \right|$.

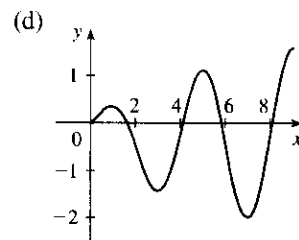
So $g(1) = \left| \int_0^1 f \, dt \right|$, $g(5) = \int_0^5 f \, dt = g(1) - \left| \int_1^3 f \, dt \right| + \left| \int_3^5 f \, dt \right|$, and

$g(9) = \int_0^9 f \, dt = g(5) - \left| \int_5^7 f \, dt \right| + \left| \int_7^9 f \, dt \right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

- (c) g is concave downward on those intervals where $g'' < 0$. But

$g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on

(approximately) $(\frac{1}{2}, 2)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.



52. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8$, and 10 . g has local maxima at $x = 2$ and 6 (since $f = g'$ changes from positive to negative there) and local minima at $x = 4$ and 8 . There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

- (b) We can see from the graph that $\left| \int_0^2 f \, dt \right| > \left| \int_2^4 f \, dt \right| > \left| \int_4^6 f \, dt \right| > \left| \int_6^8 f \, dt \right| > \left| \int_8^{10} f \, dt \right|$.

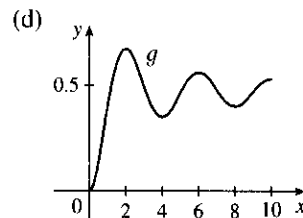
So $g(2) = \left| \int_0^2 f \, dt \right|$, $g(6) = \int_0^6 f \, dt = g(2) - \left| \int_2^4 f \, dt \right| + \left| \int_4^6 f \, dt \right|$, and

$g(10) = \int_0^{10} f \, dt = g(6) - \left| \int_6^8 f \, dt \right| + \left| \int_8^{10} f \, dt \right|$. Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

- (c) g is concave downward on those intervals where $g'' < 0$. But

$g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$

and $(9, 10)$. So g is concave downward on these intervals.



53. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^3}{n^4} = \lim_{n \rightarrow \infty} \frac{1-0}{n} \sum_{i=1}^n \left(\frac{i}{n} \right)^3 = \int_0^1 x^3 \, dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$

54. $\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}} \right) = \lim_{n \rightarrow \infty} \frac{1-0}{n} \sum_{i=1}^n \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x} \, dx = \left[\frac{2x^{3/2}}{3} \right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$

55. Suppose $h < 0$. Since f is continuous on $[x + h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x + h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x + h, x]$. By Property 8 of integrals, $m(-h) \leq \int_{x+h}^x f(t) dt \leq M(-h)$; that is,

$$f(u)(-h) \leq -\int_{x+h}^x f(t) dt \leq f(v)(-h). \text{ Since } -h > 0, \text{ we can divide this inequality by } -h:$$

$$f(u) \leq \frac{1}{h} \int_x^{x+h} f(t) dt \leq f(v). \text{ By Equation 2, } \frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt \text{ for } h \neq 0, \text{ and hence}$$

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v), \text{ which is Equation 3 in the case where } h < 0.$$

$$\begin{aligned} 56. \frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt &= \frac{d}{dx} \left[\int_{g(x)}^a f(t) dt + \int_a^{h(x)} f(t) dt \right] \quad (\text{where } a \text{ is in the domain of } f) \\ &= \frac{d}{dx} \left[-\int_a^{g(x)} f(t) dt \right] + \frac{d}{dx} \left[\int_a^{h(x)} f(t) dt \right] = -f(g(x)) g'(x) + f(h(x)) h'(x) \\ &= f(h(x)) h'(x) - f(g(x)) g'(x) \end{aligned}$$

57. (a) Let $f(x) = \sqrt{x} \Rightarrow f'(x) = 1/(2\sqrt{x}) > 0$ for $x > 0 \Rightarrow f$ is increasing on $(0, \infty)$. If $x \geq 0$, then $x^3 \geq 0$, so $1 + x^3 \geq 1$ and since f is increasing, this means that $f(1 + x^3) \geq f(1) \Rightarrow \sqrt{1 + x^3} \geq 1$ for $x \geq 0$. Next let $g(t) = t^2 - t \Rightarrow g'(t) = 2t - 1 \Rightarrow g'(t) > 0$ when $t \geq 1$. Thus, g is increasing on $(1, \infty)$. And since $g(1) = 0$, $g(t) \geq 0$ when $t \geq 1$. Now let $t = \sqrt{1 + x^3}$, where $x \geq 0$. $\sqrt{1 + x^3} \geq 1$ (from above) $\Rightarrow t \geq 1 \Rightarrow g(t) \geq 0 \Rightarrow (1 + x^3) - \sqrt{1 + x^3} \geq 0$ for $x \geq 0$. Therefore, $1 \leq \sqrt{1 + x^3} \leq 1 + x^3$ for $x \geq 0$.

$$\begin{aligned} \text{(b) From part (a) and Property 7: } \int_0^1 1 dx &\leq \int_0^1 \sqrt{1 + x^3} dx \leq \int_0^1 (1 + x^3) dx \Leftrightarrow \\ [x]_0^1 &\leq \int_0^1 \sqrt{1 + x^3} dx \leq [x + \frac{1}{4}x^4]_0^1 \Leftrightarrow 1 \leq \int_0^1 \sqrt{1 + x^3} dx \leq 1 + \frac{1}{4} = 1.25. \end{aligned}$$

58. (a) If $x < 0$, then $g(x) = \int_0^x f(t) dt = \int_0^x 0 dt = 0$.

$$\text{If } 0 \leq x \leq 1, \text{ then } g(x) = \int_0^x f(t) dt = \int_0^x t dt = \left[\frac{1}{2}t^2 \right]_0^x = \frac{1}{2}x^2.$$

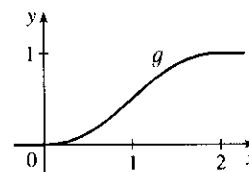
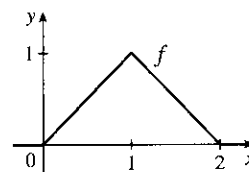
If $1 < x \leq 2$, then

$$\begin{aligned} g(x) &= \int_0^x f(t) dt = \int_0^1 f(t) dt + \int_1^x f(t) dt \\ &= g(1) + \int_1^x (2 - t) dt = \frac{1}{2}(1)^2 + \left[2t - \frac{1}{2}t^2 \right]_1^x \\ &= \frac{1}{2} + (2x - \frac{1}{2}x^2) - (2 - \frac{1}{2}) = 2x - \frac{1}{2}x^2 - 1. \end{aligned}$$

$$\text{If } x > 2, \text{ then } g(x) = \int_0^x f(t) dt = g(2) + \int_2^x 0 dt = 1 + 0 = 1. \text{ So}$$

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \leq x \leq 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)



- (c) f is not differentiable at its corners at $x = 0, 1$, and 2 . f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$. g is differentiable on $(-\infty, \infty)$.

59. Using FTC1, we differentiate both sides of $6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$ to get $\frac{f(x)}{x^2} = 2 \frac{1}{2\sqrt{x}} \Rightarrow f(x) = x^{3/2}$.

To find a , we substitute $x = a$ in the original equation to obtain $6 + \int_a^a \frac{f(t)}{t^2} dt = 2\sqrt{a} \Rightarrow 6 + 0 = 2\sqrt{a} \Rightarrow 3 = \sqrt{a} \Rightarrow a = 9$.

60. By the Fundamental Theorem of Calculus, $\int_1^2 (h')'(u) du = h'(2) - h'(1) = 5 - 2 = 3$. The other information is unnecessary.

61. (a) Let $F(t) = \int_0^t f(s) ds$. Then, by FTC1, $F'(t) = f(t) = \text{rate of depreciation}$, so $F(t)$ represents the loss in value over the interval $[0, t]$.

(b) $C(t) = \frac{1}{t} \left[A + \int_0^t f(s) ds \right] = \frac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \frac{1}{t} \left[A + \int_0^t f(s) ds \right]$. Using FTC1, we have $C'(t) = -\frac{1}{t^2} \left[A + \int_0^t f(s) ds \right] + \frac{1}{t} f(t)$.

$$C'(t) = 0 \Rightarrow t f(t) = A + \int_0^t f(s) ds \Rightarrow f(t) = \frac{1}{t} \left[A + \int_0^t f(s) ds \right] = C(t).$$

62. (a) $C(t) = \frac{1}{t} \int_0^t [f(s) + g(s)] ds$. Using FTC1 and the Product Rule, we have

$$C'(t) = \frac{1}{t} [f(t) + g(t)] - \frac{1}{t^2} \int_0^t [f(s) + g(s)] ds. \text{ Set } C'(t) = 0:$$

$$\frac{1}{t} [f(t) + g(t)] - \frac{1}{t^2} \int_0^t [f(s) + g(s)] ds = 0 \Rightarrow [f(t) + g(t)] - \frac{1}{t} \int_0^t [f(s) + g(s)] ds = 0 \Rightarrow$$

$$[f(t) + g(t)] - C(t) = 0 \Rightarrow C(t) = f(t) + g(t).$$

(b) For $0 \leq t \leq 30$, we have $D(t) = \int_0^t \left(\frac{V}{15} - \frac{V}{450}s \right) ds = \left[\frac{V}{15}s - \frac{V}{900}s^2 \right]_0^t = \frac{V}{15}t - \frac{V}{900}t^2$.

$$\text{So } D(t) = V \Rightarrow \frac{V}{15}t - \frac{V}{900}t^2 = V \Rightarrow 60t - t^2 = 900 \Rightarrow t^2 - 60t + 900 = 0 \Rightarrow$$

$$(t - 30)^2 = 0 \Rightarrow t = 30. \text{ So the length of time } T \text{ is 30 months.}$$

(c) $C(t) = \frac{1}{t} \int_0^t \left(\frac{V}{15} - \frac{V}{450}s + \frac{V}{12,900}s^2 \right) ds = \frac{1}{t} \left[\frac{V}{15}s - \frac{V}{900}s^2 + \frac{V}{38,700}s^3 \right]_0^t$

$$= \frac{1}{t} \left(\frac{V}{15}t - \frac{V}{900}t^2 + \frac{V}{38,700}t^3 \right) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38,700}t^2 \Rightarrow$$

$$C'(t) = -\frac{V}{900} + \frac{V}{19,350}t = 0 \text{ when } \frac{1}{19,350}t = \frac{1}{900} \Rightarrow t = 21.5.$$

$$C(21.5) = \frac{V}{15} - \frac{V}{900}(21.5) + \frac{V}{38,700}(21.5)^2 \approx 0.05472V, C(0) = \frac{V}{15} \approx 0.06667V, \text{ and}$$

$$C(30) = \frac{V}{15} - \frac{V}{900}(30) + \frac{V}{38,700}(30)^2 \approx 0.05659V, \text{ so the absolute minimum is } C(21.5) \approx 0.05472V.$$

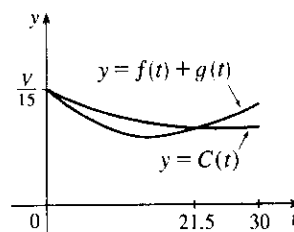
(d) As in part (c), we have $C(t) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38,700}t^2$, so $C(t) = f(t) + g(t)$

$$\Leftrightarrow \frac{V}{15} - \frac{V}{900}t + \frac{V}{38,700}t^2 = \frac{V}{15} - \frac{V}{450}t + \frac{V}{12,900}t^2 \Leftrightarrow$$

$$t^2 \left(\frac{1}{12,900} - \frac{1}{38,700} \right) = t \left(\frac{1}{450} - \frac{1}{900} \right) \Leftrightarrow$$

$$t = \frac{1/900}{2/38,700} = \frac{43}{2} = 21.5. \text{ This is the value of } t \text{ that we obtained as the critical}$$

number of C in part (c), so we have verified the result of (a) in this case.



$$63. \int_1^9 \frac{1}{2x} dx = \frac{1}{2} \int_1^9 \frac{1}{x} dx = \frac{1}{2} [\ln |x|]_1^9 = \frac{1}{2} (\ln 9 - \ln 1) = \frac{1}{2} \ln 9 - 0 = \ln 9^{1/2} = \ln 3$$

$$64. \int_0^1 10^x dx = \left[\frac{10^x}{\ln 10} \right]_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$$

$$65. \int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}} dt = 6 \int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}} dt = 6 [\sin^{-1} t]_{1/2}^{\sqrt{3}/2} = 6 \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) - \sin^{-1} \left(\frac{1}{2} \right) \right]$$

$$= 6 \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = 6 \left(\frac{\pi}{6} \right) = \pi$$

$$66. \int_0^1 \frac{4}{t^2+1} dt = 4 \int_0^1 \frac{1}{1+t^2} dt = 4 [\tan^{-1} t]_0^1 = 4 (\tan^{-1} 1 - \tan^{-1} 0) = 4 \left(\frac{\pi}{4} - 0 \right) = \pi$$

$$67. \int_{-1}^1 e^{u+1} du = [e^{u+1}]_{-1}^1 = e^2 - e^0 = e^2 - 1 \quad [\text{or start with } e^{u+1} = e^u e^1]$$

$$68. \int_1^2 \frac{4+u^2}{u^3} du = \int_1^2 (4u^{-3} + u^{-1}) du = \left[-\frac{4}{2} u^{-2} + \ln |u| \right]_1^2 = \left[-\frac{2}{u^2} + \ln u \right]_1^2$$

$$= \left(-\frac{1}{2} + \ln 2 \right) - \left(-2 + \ln 1 \right) = \frac{3}{2} + \ln 2$$

5.4 Indefinite Integrals and the Net Change Theorem

$$1. \frac{d}{dx} [\sqrt{x^2+1} + C] = \frac{d}{dx} [(x^2+1)^{1/2} + C] = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

$$2. \frac{d}{dx} [x \sin x + \cos x + C] = x \cos x + (\sin x) \cdot 1 - \sin x = x \cos x$$

$$3. \frac{d}{dx} \left[\frac{x}{a^2 \sqrt{a^2-x^2}} + C \right] = \frac{1}{a^2} \frac{\sqrt{a^2-x^2} - x(-x)/\sqrt{a^2-x^2}}{a^2-x^2} = \frac{1}{a^2} \frac{(a^2-x^2) + x^2}{(a^2-x^2)^{3/2}} = \frac{1}{\sqrt{(a^2-x^2)^3}}$$

$$4. \frac{d}{dx} \left[-\frac{\sqrt{x^2+a^2}}{a^2 x} + C \right] = -\frac{1}{a^2} \frac{d}{dx} \left[\frac{\sqrt{x^2+a^2}}{x} \right] = -\frac{x(x/\sqrt{x^2+a^2}) - \sqrt{x^2+a^2} \cdot 1}{a^2 x^2}$$

$$= -\frac{x^2 - (x^2+a^2)}{a^2 x^2 \sqrt{x^2+a^2}} = \frac{1}{x^2 \sqrt{x^2+a^2}}$$

$$5. \int x^{-3/4} dx = \frac{x^{-3/4+1}}{-3/4+1} + C = \frac{x^{1/4}}{1/4} + C = 4x^{1/4} + C$$

$$6. \int \sqrt[3]{x} dx = \int x^{1/3} dx = \frac{x^{4/3}}{4/3} + C = \frac{3}{4} x^{4/3} + C$$

$$7. \int (x^3 + 6x + 1) dx = \frac{x^4}{4} + 6\frac{x^2}{2} + x + C = \frac{1}{4}x^4 + 3x^2 + x + C$$

$$8. \int x(1 + 2x^4) dx = \int (x + 2x^5) dx = \frac{x^2}{2} + 2\frac{x^6}{6} + C = \frac{1}{2}x^2 + \frac{1}{3}x^6 + C$$

$$9. \int (1-t)(2+t^2) dt = \int (2-2t+t^2-t^3) dt = 2t - 2\frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + C = 2t - t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$$

$$10. \int \left(u^2 + 1 + \frac{1}{u^2}\right) du = \int (u^2 + 1 + u^{-2}) du = \frac{u^3}{3} + u + \frac{u^{-1}}{-1} + C = \frac{1}{3}u^3 + u - \frac{1}{u} + C$$

$$11. \int (2 - \sqrt{x})^2 dx = \int (4 - 4\sqrt{x} + x) dx = 4x - 4\frac{x^{3/2}}{3/2} + \frac{x^2}{2} + C = 4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2 + C$$

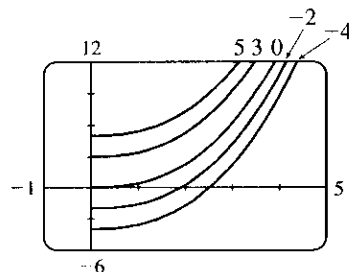
$$12. \int (\sin \theta + 3 \cos \theta) d\theta = -\cos \theta + 3 \sin \theta + C$$

$$13. \int \frac{\sin x}{1 - \sin^2 x} dx = \int \frac{\sin x}{\cos^2 x} dx = \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} dx = \int \sec x \tan x dx = \sec x + C$$

$$14. \int \frac{\sin 2x}{\sin x} dx = \int \frac{2 \sin x \cos x}{\sin x} dx = \int 2 \cos x dx = 2 \sin x + C$$

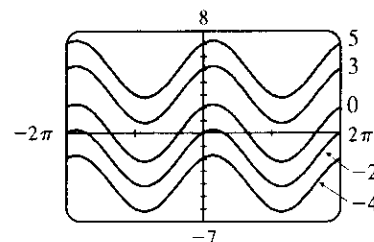
$$15. \int x\sqrt{x} dx = \int x^{3/2} dx = \frac{2}{5}x^{5/2} + C.$$

The members of the family in the figure correspond to $C = 5, 3, 0, -2$, and -4 .



$$16. \int (\cos x - 2 \sin x) dx = \sin x + 2 \cos x + C.$$

The members of the family in the figure correspond to $C = 5, 3, 0, -2$, and -4 .



$$17. \int_0^2 (6x^2 - 4x + 5) dx = \left[6 \cdot \frac{1}{3}x^3 - 4 \cdot \frac{1}{2}x^2 + 5x\right]_0^2 = [2x^3 - 2x^2 + 5x]_0^2 = (16 - 8 + 10) - 0 = 18$$

$$18. \int_1^3 (1 + 2x - 4x^3) dx = \left[x + 2 \cdot \frac{1}{2}x^2 - 4 \cdot \frac{1}{4}x^4\right]_1^3 = [x + x^2 - x^4]_1^3 \\ = (3 + 9 - 81) - (1 + 1 - 1) = -69 - 1 = -70$$

$$19. \int_{-3}^0 (5y^4 - 6y^2 + 14) dy = \left[5\left(\frac{1}{5}y^5\right) - 6\left(\frac{1}{3}y^3\right) + 14y\right]_{-3}^0 = [y^5 - 2y^3 + 14y]_{-3}^0 \\ = 0 - (-243 + 54 - 42) = 231$$

$$20. \int_{-2}^0 (u^5 - u^3 + u^2) du = \left[\frac{1}{6}u^6 - \frac{1}{4}u^4 + \frac{1}{3}u^3\right]_{-2}^0 = 0 - \left(\frac{32}{3} - 4 - \frac{8}{3}\right) = -4$$

$$21. \int_{-2}^2 (3u + 1)^2 du = \int_{-2}^2 (9u^2 + 6u + 1) du = \left[9 \cdot \frac{1}{3}u^3 + 6 \cdot \frac{1}{2}u^2 + u\right]_{-2}^2 = [3u^3 + 3u^2 + u]_{-2}^2 \\ = (24 + 12 + 2) - (-24 + 12 - 2) = 38 - (-14) = 52$$

$$22. \int_0^4 (2v + 5)(3v - 1) dv = \int_0^4 (6v^2 + 13v - 5) dv = \left[6 \cdot \frac{1}{3}v^3 + 13 \cdot \frac{1}{2}v^2 - 5v\right]_0^4 = \left[2v^3 + \frac{13}{2}v^2 - 5v\right]_0^4 \\ = (128 + 104 - 20) - 0 = 212$$

$$23. \int_1^4 \sqrt{t}(1+t) dt = \int_1^4 (t^{1/2} + t^{3/2}) dt = \left[\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2}\right]_1^4 = \left(\frac{16}{3} + \frac{64}{5}\right) - \left(\frac{2}{3} + \frac{2}{5}\right) = \frac{14}{3} + \frac{62}{5} = \frac{256}{15}$$

$$24. \int_0^9 \sqrt{2t} \, dt = \int_0^9 \sqrt{2} t^{1/2} \, dt = \left[\sqrt{2} \cdot \frac{2}{3} t^{3/2} \right]_0^9 = \sqrt{2} \cdot \frac{2}{3} \cdot 27 - 0 = 18\sqrt{2}$$

$$25. \int_{-2}^{-1} \left(4y^3 + \frac{2}{y^3} \right) dy = \left[4 \cdot \frac{1}{4} y^4 + 2 \cdot \frac{1}{-2} y^{-2} \right]_{-2}^{-1} = \left[y^4 - \frac{1}{y^2} \right]_{-2}^{-1} = (1 - 1) - (16 - \frac{1}{4}) = -\frac{63}{4}$$

$$26. \int_1^2 \frac{y + 5y^7}{y^3} dy = \int_1^2 (y^{-2} + 5y^4) dy = \left[-y^{-1} + 5 \cdot \frac{1}{5} y^5 \right]_1^2 = \left[-\frac{1}{y} + y^5 \right]_1^2 = (-\frac{1}{2} + 32) - (-1 + 1) = \frac{63}{2}$$

$$27. \int_0^1 x(\sqrt[3]{x} + \sqrt[4]{x}) \, dx = \int_0^1 (x^{4/3} + x^{5/4}) \, dx = \left[\frac{3}{7} x^{7/3} + \frac{4}{9} x^{9/4} \right]_0^1 = \left(\frac{3}{7} + \frac{4}{9} \right) - 0 = \frac{55}{63}$$

$$28. \int_1^2 \left(x + \frac{1}{x} \right)^2 dx = \int_1^2 (x^2 + 2 + x^{-2}) \, dx = \left[\frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} \right]_1^2 = \left[\frac{x^3}{3} + 2x - \frac{1}{x} \right]_1^2 \\ = \left(\frac{8}{3} + 4 - \frac{1}{2} \right) - \left(\frac{1}{3} + 2 - 1 \right) = \frac{29}{6}$$

$$29. \int_1^4 \sqrt{5/x} \, dx = \sqrt{5} \int_1^4 x^{-1/2} \, dx = \sqrt{5} [2\sqrt{x}]_1^4 = \sqrt{5} (2 \cdot 2 - 2 \cdot 1) = 2\sqrt{5}$$

$$30. \int_1^9 \frac{3x-2}{\sqrt{x}} \, dx = \int_1^9 (3x^{1/2} - 2x^{-1/2}) \, dx = \left[3 \cdot \frac{2}{3} x^{3/2} - 2 \cdot 2x^{1/2} \right]_1^9 = [2x^{3/2} - 4x^{1/2}]_1^9 \\ = (54 - 12) - (2 - 4) = 44$$

$$31. \int_0^\pi (4 \sin \theta - 3 \cos \theta) \, d\theta = [-4 \cos \theta - 3 \sin \theta]_0^\pi = (4 - 0) - (-4 - 0) = 8$$

$$32. \int_{\pi/4}^{\pi/3} \sec \theta \tan \theta \, d\theta = [\sec \theta]_{\pi/4}^{\pi/3} = \sec \frac{\pi}{3} - \sec \frac{\pi}{4} = 2 - \sqrt{2}$$

$$33. \int_0^{\pi/4} \frac{1 + \cos^2 \theta}{\cos^2 \theta} \, d\theta = \int_0^{\pi/4} \left(\frac{1}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} \right) d\theta = \int_0^{\pi/4} (\sec^2 \theta + 1) \, d\theta \\ = [\tan \theta + \theta]_0^{\pi/4} = \left(\tan \frac{\pi}{4} + \frac{\pi}{4} \right) - (0 + 0) = 1 + \frac{\pi}{4}$$

$$34. \int_0^{\pi/3} \frac{\sin \theta + \sin \theta \tan^2 \theta}{\sec^2 \theta} \, d\theta = \int_0^{\pi/3} \frac{\sin \theta (1 + \tan^2 \theta)}{\sec^2 \theta} \, d\theta = \int_0^{\pi/3} \frac{\sin \theta \sec^2 \theta}{\sec^2 \theta} \, d\theta = \int_0^{\pi/3} \sin \theta \, d\theta \\ = [-\cos \theta]_0^{\pi/3} = -\frac{1}{2} - (-1) = \frac{1}{2}$$

$$35. \int_1^{64} \frac{1 + \sqrt[3]{x}}{\sqrt{x}} \, dx = \int_1^{64} \left(\frac{1}{x^{1/2}} + \frac{x^{1/3}}{x^{1/2}} \right) dx = \int_1^{64} (x^{-1/2} + x^{(1/3) - (1/2)}) \, dx = \int_1^{64} (x^{-1/2} + x^{-1/6}) \, dx \\ = \left[2x^{1/2} + \frac{6}{5} x^{5/6} \right]_1^{64} = \left(16 + \frac{192}{5} \right) - \left(2 + \frac{6}{5} \right) = 14 + \frac{186}{5} = \frac{256}{5}$$

$$36. \int_0^1 (1 + x^2)^3 \, dx = \int_0^1 (1 + 3x^2 + 3x^4 + x^6) \, dx = \left[x + x^3 + \frac{3}{5} x^5 + \frac{1}{7} x^7 \right]_0^1 = \left(1 + 1 + \frac{3}{5} + \frac{1}{7} \right) - 0 = \frac{96}{35}$$

$$37. \int_0^1 (\sqrt[4]{x^5} + \sqrt[5]{x^4}) \, dx = \int_0^1 (x^{5/4} + x^{4/5}) \, dx = \left[\frac{x^{9/4}}{9/4} + \frac{x^{9/5}}{9/5} \right]_0^1 = \left[\frac{4}{9} x^{9/4} + \frac{5}{9} x^{9/5} \right]_0^1 = \frac{4}{9} + \frac{5}{9} - 0 = 1$$

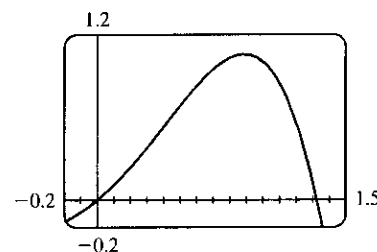
$$38. \int_1^8 \frac{x-1}{\sqrt[3]{x^2}} \, dx = \int_1^8 (x^{1/3} - x^{-2/3}) \, dx = \left[\frac{x^{4/3}}{4/3} - \frac{x^{1/3}}{1/3} \right]_1^8 = \left[\frac{3}{4} x^{4/3} - 3x^{1/3} \right]_1^8 \\ = \left(\frac{3}{4} \cdot 16 - 3 \cdot 2 \right) - \left(\frac{3}{4} - 3 \right) = \frac{33}{4}$$

$$39. \int_{-1}^2 (x - 2|x|) \, dx = \int_{-1}^0 [x - 2(-x)] \, dx + \int_0^2 [x - 2(x)] \, dx = \int_{-1}^0 3x \, dx + \int_0^2 (-x) \, dx = 3 \left[\frac{1}{2} x^2 \right]_{-1}^0 - \left[\frac{1}{2} x^2 \right]_0^2 \\ = 3(0 - \frac{1}{2}) - (2 - 0) = -\frac{7}{2} = -3.5$$

$$40. \int_0^{3\pi/2} |\sin x| \, dx = \int_0^\pi \sin x \, dx + \int_\pi^{3\pi/2} (-\sin x) \, dx = [-\cos x]_0^\pi + [\cos x]_\pi^{3\pi/2} \\ = [1 - (-1)] + [0 - (-1)] = 2 + 1 = 3$$

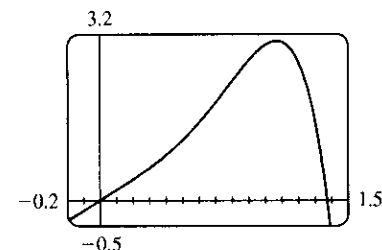
41. The graph shows that $y = x + x^2 - x^4$ has x -intercepts at $x = 0$ and at $x = a \approx 1.32$. So the area of the region that lies under the curve and above the x -axis is

$$\begin{aligned}\int_0^a (x + x^2 - x^4) dx &= \left[\frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^a \\ &= \left(\frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{5}a^5 \right) - 0 \\ &\approx 0.84\end{aligned}$$



42. The graph shows that $y = 2x + 3x^4 - 2x^6$ has x -intercepts at $x = 0$ and at $x = a \approx 1.37$. So the area of the region that lies under the curve and above the x -axis is

$$\begin{aligned}\int_0^a (2x + 3x^4 - 2x^6) dx &= \left[x^2 + \frac{3}{5}x^5 - \frac{2}{7}x^7 \right]_0^a \\ &= \left(a^2 + \frac{3}{5}a^5 - \frac{2}{7}a^7 \right) - 0 \\ &\approx 2.18\end{aligned}$$



43. $A = \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{1}{3}y^3 \right]_0^2 = \left(4 - \frac{8}{3} \right) - 0 = \frac{4}{3}$
44. $y = \sqrt[4]{x} \Rightarrow x = y^4$, so $A = \int_0^1 y^4 dy = \left[\frac{1}{5}y^5 \right]_0^1 = \frac{1}{5}$.
45. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t . We know from the Net Change Theorem that $\int_5^{10} w'(t) dt = w(10) - w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.
46. $\int_a^b I(t) dt = \int_a^b Q'(t) dt = Q(b) - Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t = a$ to $t = b$.
47. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t . [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120} r(t) dt = -\int_0^{120} V'(t) dt = -[V(120) - V(0)] = V(0) - V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).
48. By the Net Change Theorem, $\int_0^{15} n'(t) dt = n(15) - n(0) = n(15) - 100$ represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t) dt = n(15)$ represents the total bee population after 15 weeks.
49. By the Net Change Theorem, $\int_{1000}^{5000} R'(x) dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.
50. The slope of the trail is the rate of change of the elevation E , so $f(x) = E'(x)$. By the Net Change Theorem, $\int_3^5 f(x) dx = \int_3^5 E'(x) dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ miles and $x = 5$ miles from the start of the trail.
51. In general, the unit of measurement for $\int_a^b f(x) dx$ is the product of the unit for $f(x)$ and the unit for x . Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x) dx$ are newton-meters. (A newton-meter is abbreviated N-m and is called a joule.)
52. The units for $a(x)$ are pounds per foot and the units for x are feet, so the units for da/dx are pounds per foot per foot, denoted (lb/ft)/ft. The unit of measurement for $\int_2^8 a(x) dx$ is the product of pounds per foot and feet; that is, pounds.

$$53. (a) \text{ displacement} = \int_0^3 (3t - 5) dt = \left[\frac{3}{2}t^2 - 5t \right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2} \text{ m}$$

$$(b) \text{ distance traveled} = \int_0^3 |3t - 5| dt = \int_0^{5/3} (5 - 3t) dt + \int_{5/3}^3 (3t - 5) dt$$

$$= \left[5t - \frac{3}{2}t^2 \right]_0^{5/3} + \left[\frac{3}{2}t^2 - 5t \right]_{5/3}^3 = \frac{25}{3} - \frac{3}{2} \cdot \frac{25}{9} + \frac{27}{2} - 15 - \left(\frac{3}{2} \cdot \frac{25}{9} - \frac{25}{3} \right) = \frac{41}{6} \text{ m}$$

$$54. (a) \text{ displacement} = \int_1^6 (t^2 - 2t - 8) dt = \left[\frac{1}{3}t^3 - t^2 - 8t \right]_1^6 = (72 - 36 - 48) - \left(\frac{1}{3} - 1 - 8 \right) = -\frac{10}{3} \text{ m}$$

$$(b) \text{ distance traveled} = \int_1^6 |t^2 - 2t - 8| dt = \int_1^6 |(t - 4)(t + 2)| dt$$

$$= \int_1^4 (-t^2 + 2t + 8) dt + \int_4^6 (t^2 - 2t - 8) dt = \left[-\frac{1}{3}t^3 + t^2 + 8t \right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t \right]_4^6$$

$$= \left(-\frac{64}{3} + 16 + 32 \right) - \left(-\frac{1}{3} + 1 + 8 \right) + (72 - 36 - 48) - \left(\frac{64}{3} - 16 - 32 \right) = \frac{98}{3} \text{ m}$$

$$55. (a) v'(t) = a(t) = t + 4 \Rightarrow v(t) = \frac{1}{2}t^2 + 4t + C \Rightarrow v(0) = C = 5 \Rightarrow v(t) = \frac{1}{2}t^2 + 4t + 5 \text{ m/s}$$

$$(b) \text{ distance traveled} = \int_0^{10} |v(t)| dt = \int_0^{10} \left| \frac{1}{2}t^2 + 4t + 5 \right| dt = \int_0^{10} \left(\frac{1}{2}t^2 + 4t + 5 \right) dt$$

$$= \left[\frac{1}{6}t^3 + 2t^2 + 5t \right]_0^{10} = \frac{500}{3} + 200 + 50 = 416\frac{2}{3} \text{ m}$$

$$56. (a) v'(t) = a(t) = 2t + 3 \Rightarrow v(t) = t^2 + 3t + C \Rightarrow v(0) = C = -4 \Rightarrow v(t) = t^2 + 3t - 4$$

$$(b) \text{ distance traveled} = \int_0^3 |t^2 + 3t - 4| dt = \int_0^3 |(t + 4)(t - 1)| dt$$

$$= \int_0^1 (-t^2 - 3t + 4) dt + \int_1^3 (t^2 + 3t - 4) dt$$

$$= \left[-\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t \right]_0^1 + \left[\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t \right]_1^3$$

$$= \left(-\frac{1}{3} - \frac{3}{2} + 4 \right) + \left(9 + \frac{27}{2} - 12 \right) - \left(\frac{1}{3} + \frac{3}{2} - 4 \right) = \frac{89}{6} \text{ m}$$

$$57. \text{ Since } m'(x) = \rho(x), m = \int_0^4 \rho(x) dx = \int_0^4 (9 + 2\sqrt{x}) dx = \left[9x + \frac{4}{3}x^{3/2} \right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3} \text{ kg.}$$

58. By the Net Change Theorem, the amount of water that flows from the tank is

$$\int_0^{10} r(t) dt = \int_0^{10} (200 - 4t) dt = [200t - 2t^2]_0^{10} = (2000 - 200) - 0 = 1800 \text{ liters.}$$

59. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t) dt$. We use the Midpoint Rule for $0 \leq t \leq 100$ with $n = 5$. Note that the length of each of the five time intervals is

$$20 \text{ seconds} = \frac{20}{3600} \text{ hour} = \frac{1}{180} \text{ hour. So the distance traveled is}$$

$$\int_0^{100} v(t) dt \approx \frac{1}{180} [v(10) + v(30) + v(50) + v(70) + v(90)]$$

$$= \frac{1}{180} (38 + 58 + 51 + 53 + 47)$$

$$= \frac{247}{180} \approx 1.4 \text{ miles}$$

60. (a) By the Net Change Theorem, the total amount spewed into the atmosphere is

$$Q(6) - Q(0) = \int_0^6 r(t) dt = Q(6) \text{ since } Q(0) = 0. \text{ The rate } r(t) \text{ is positive, so } Q \text{ is an increasing function.}$$

$$\text{Thus, an upper estimate for } Q(6) \text{ is } R_6 \text{ and a lower estimate for } Q(6) \text{ is } L_6. \quad \Delta t = \frac{b-a}{n} = \frac{6-0}{6} = 1.$$

$$R_6 = \sum_{i=1}^6 r(t_i) \Delta t = 10 + 24 + 36 + 46 + 54 + 60 = 230 \text{ tonnes.}$$

$$L_6 = \sum_{i=1}^6 r(t_{i-1}) \Delta t = R_6 + r(0) - r(6) = 230 + 2 - 60 = 172 \text{ tonnes.}$$

$$(b) \Delta t = \frac{b-a}{n} = \frac{6-0}{3} = 2.$$

$$Q(6) \approx M_3 = 2[r(1) + r(3) + r(5)] = 2(10 + 36 + 54) = 2(100) = 200 \text{ tonnes.}$$

61. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards

$$\text{is } C(4000) - C(2000) = \int_{2000}^{4000} C'(x) dx.$$

$$\begin{aligned} \int_{2000}^{4000} C'(x) dx &= \int_{2000}^{4000} (3 - 0.01x + 0.000006x^2) dx \\ &= [3x - 0.005x^2 + 0.000002x^3]_{2000}^{4000} = 60,000 - 2,000 = \$58,000 \end{aligned}$$

62. By the Net Change Theorem, the amount of water after four days is

$$\begin{aligned} 25,000 + \int_0^4 r(t) dt &\approx 25,000 + M_4 \\ &= 25,000 + \frac{4-0}{4} [r(0.5) + r(1.5) + r(2.5) + r(3.5)] \\ &\approx 25,000 + [1500 + 1770 + 740 + (-690)] = 28,320 \text{ liters} \end{aligned}$$

63. (a) We can find the area between the Lorenz curve and the line $y = x$ by subtracting the area under $y = L(x)$ from the area under $y = x$. Thus,

$$\begin{aligned} \text{coefficient of inequality} &= \frac{\text{area between Lorenz curve and line } y = x}{\text{area under line } y = x} = \frac{\int_0^1 [x - L(x)] dx}{\int_0^1 x dx} \\ &= \frac{\int_0^1 [x - L(x)] dx}{[x^2/2]_0^1} = \frac{\int_0^1 [x - L(x)] dx}{1/2} = 2 \int_0^1 [x - L(x)] dx \end{aligned}$$

(b) $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x \Rightarrow L(50\%) = L(\frac{1}{2}) = \frac{5}{48} + \frac{7}{24} = \frac{19}{48} = 0.3958\bar{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\begin{aligned} \text{coefficient of inequality} &= 2 \int_0^1 [x - L(x)] dx = 2 \int_0^1 (x - \frac{5}{12}x^2 - \frac{7}{12}x) dx \\ &= 2 \int_0^1 (\frac{5}{12}x - \frac{5}{12}x^2) dx = 2 \int_0^1 \frac{5}{12}(x - x^2) dx \\ &= \frac{5}{6} [\frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1 = \frac{5}{6} (\frac{1}{2} - \frac{1}{3}) = \frac{5}{6} (\frac{1}{6}) = \frac{5}{36} \end{aligned}$$

64. (a) From Exercise 4.1.66(a), $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

$$(b) h(125) - h(0) = \int_0^{125} v(t) dt = [0.000365t^4 - 0.03851t^3 + 12.490845t^2 - 21.26872t]_0^{125} \approx 206,407 \text{ ft}$$

$$\begin{aligned} 65. \int_1^e \frac{x^2 + x + 1}{x} dx &= \int_1^e \left(x + 1 + \frac{1}{x} \right) dx = [\frac{1}{2}x^2 + x + \ln|x|]_1^e \\ &= (\frac{1}{2}e^2 + e + \ln e) - (\frac{1}{2} + 1 + \ln 1) = \frac{1}{2}e^2 + e - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 66. \int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx &= \int_4^9 \left(x + 2 + \frac{1}{x} \right) dx = [\frac{1}{2}x^2 + 2x + \ln|x|]_4^9 \\ &= \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) = \frac{85}{2} + \ln \frac{9}{4} \end{aligned}$$

$$67. \int \left(x^2 + 1 + \frac{1}{x^2 + 1} \right) dx = \frac{x^3}{3} + x + \tan^{-1} x + C$$

$$\begin{aligned} 68. B = 3A &\Rightarrow \int_0^b e^x dx = 3 \int_0^a e^x dx \Rightarrow [e^x]_0^b = 3[e^x]_0^a \Rightarrow e^b - 1 = 3(e^a - 1) \Rightarrow \\ e^b &= 3e^a - 2 \Rightarrow b = \ln(3e^a - 2) \end{aligned}$$

5.5 The Substitution Rule

1. Let $u = 3x$. Then $du = 3 dx$, so $dx = \frac{1}{3} du$. Thus,

$\int \cos 3x dx = \int \cos u \left(\frac{1}{3} du\right) = \frac{1}{3} \int \cos u du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin 3x + C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,

$$\frac{d}{dx} \left(\frac{1}{3} \sin 3x + C \right) = \frac{1}{3} (\cos 3x) \cdot 3 = \cos 3x, \text{ the desired result.}$$

2. Let $u = 4 + x^2$. Then $du = 2x dx$ and $x dx = \frac{1}{2} du$, so

$$\int x(4 + x^2)^{10} dx = \int u^{10} \left(\frac{1}{2} du\right) = \frac{1}{2} \cdot \frac{1}{11} u^{11} + C = \frac{1}{22} (4 + x^2)^{11} + C.$$

3. Let $u = x^3 + 1$. Then $du = 3x^2 dx$ and $x^2 dx = \frac{1}{3} du$, so

$$\int x^2 \sqrt{x^3 + 1} dx = \int \sqrt{u} \left(\frac{1}{3} du\right) = \frac{1}{3} \frac{u^{3/2}}{3/2} + C = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C = \frac{2}{9} (x^3 + 1)^{3/2} + C.$$

4. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$ and $\frac{1}{\sqrt{x}} dx = 2 du$, so

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int \sin u (2 du) = 2(-\cos u) + C = -2 \cos \sqrt{x} + C.$$

5. Let $u = 1 + 2x$. Then $du = 2 dx$ and $dx = \frac{1}{2} du$, so

$$\int \frac{4}{(1 + 2x)^3} dx = 4 \int u^{-3} \left(\frac{1}{2} du\right) = 2 \frac{u^{-2}}{-2} + C = -\frac{1}{u^2} + C = -\frac{1}{(1 + 2x)^2} + C.$$

6. Let $u = \cos \theta$. Then $du = -\sin \theta d\theta$ and $\sin \theta d\theta = -du$, so

$$\int \cos^4 \theta \sin \theta d\theta = \int u^4 (-du) = -\frac{1}{5} u^5 + C = -\frac{1}{5} \cos^5 \theta + C.$$

7. Let $u = x^2 + 3$. Then $du = 2x dx$, so $\int 2x(x^2 + 3)^4 dx = \int u^4 du = \frac{1}{5} u^5 + C = \frac{1}{5} (x^2 + 3)^5 + C$.

8. Let $u = x^3 + 5$. Then $du = 3x^2 dx$ and $x^2 dx = \frac{1}{3} du$, so

$$\int x^2 (x^3 + 5)^9 dx = \int u^9 \left(\frac{1}{3} du\right) = \frac{1}{3} \cdot \frac{1}{10} u^{10} + C = \frac{1}{30} (x^3 + 5)^{10} + C.$$

9. Let $u = 3x - 2$. Then $du = 3 dx$ and $dx = \frac{1}{3} du$, so

$$\int (3x - 2)^{20} dx = \int u^{20} \left(\frac{1}{3} du\right) = \frac{1}{3} \cdot \frac{1}{21} u^{21} + C = \frac{1}{63} (3x - 2)^{21} + C.$$

10. Let $u = 2 - x$. Then $du = -dx$ and $dx = -du$, so

$$\int (2 - x)^6 dx = \int u^6 (-du) = -\frac{1}{7} u^7 + C = -\frac{1}{7} (2 - x)^7 + C.$$

11. Let $u = 1 + x + 2x^2$. Then $du = (1 + 4x) dx$, so

$$\int \frac{1 + 4x}{\sqrt{1 + x + 2x^2}} dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1 + x + 2x^2} + C.$$

12. Let $u = x^2 + 1$. Then $du = 2x dx$ and $x dx = \frac{1}{2} du$, so

$$\int \frac{x}{(x^2 + 1)^2} dx = \int u^{-2} \left(\frac{1}{2} du\right) = \frac{1}{2} \cdot \frac{-1}{u} + C = \frac{-1}{2u} + C = \frac{-1}{2(x^2 + 1)} + C.$$

13. Let $u = 2y + 1$. Then $du = 2 dy$ and $dy = \frac{1}{2} du$, so

$$\int \frac{3}{(2y + 1)^5} dy = \int 3u^{-5} \left(\frac{1}{2} du\right) = \frac{3}{2} \cdot \frac{1}{-4} u^{-4} + C = \frac{-3}{8(2y + 1)^4} + C.$$

14. Let $u = 5t + 4$. Then $du = 5 dt$ and $dt = \frac{1}{5} du$, so

$$\int \frac{1}{(5t+4)^{2.7}} dt = \int u^{-2.7} \left(\frac{1}{5} du\right) = \frac{1}{5} \cdot \frac{1}{-1.7} u^{-1.7} + C = \frac{-1}{8.5} u^{-1.7} + C = \frac{-2}{17(5t+4)^{1.7}} + C.$$

15. Let $u = 4 - t$. Then $du = -dt$ and $dt = -du$, so

$$\int \sqrt{4-t} dt = \int u^{1/2} (-du) = -\frac{2}{3} u^{3/2} + C = -\frac{2}{3} (4-t)^{3/2} + C.$$

16. Let $u = 2y^4 - 1$. Then $du = 8y^3 dy$ and $y^3 dy = \frac{1}{8} du$, so

$$\int y^3 \sqrt{2y^4 - 1} dy = \int u^{1/2} \left(\frac{1}{8} du\right) = \frac{1}{8} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{12} (2y^4 - 1)^{3/2} + C.$$

17. Let $u = \pi t$. Then $du = \pi dt$ and $dt = \frac{1}{\pi} du$, so

$$\int \sin \pi t dt = \int \sin u \left(\frac{1}{\pi} du\right) = \frac{1}{\pi} (-\cos u) + C = -\frac{1}{\pi} \cos \pi t + C.$$

18. Let $u = 2\theta$. Then $du = 2 d\theta$ and $d\theta = \frac{1}{2} du$, so

$$\int \sec 2\theta \tan 2\theta d\theta = \int \sec u \tan u \left(\frac{1}{2} du\right) = \frac{1}{2} \sec u + C = \frac{1}{2} \sec 2\theta + C.$$

19. Let $u = \sqrt{t}$. Then $du = \frac{dt}{2\sqrt{t}}$ and $\frac{1}{\sqrt{t}} dt = 2 du$, so

$$\int \frac{\cos \sqrt{t}}{\sqrt{t}} dt = \int \cos u (2 du) = 2 \sin u + C = 2 \sin \sqrt{t} + C.$$

20. Let $u = 1 + x^{3/2}$. Then $du = \frac{3}{2} x^{1/2} dx$ and $\sqrt{x} dx = \frac{2}{3} du$, so

$$\int \sqrt{x} \sin(1 + x^{3/2}) dx = \int \sin u \left(\frac{2}{3} du\right) = \frac{2}{3} \cdot (-\cos u) + C = -\frac{2}{3} \cos(1 + x^{3/2}) + C.$$

21. Let $u = \sin \theta$. Then $du = \cos \theta d\theta$, so $\int \cos \theta \sin^6 \theta d\theta = \int u^6 du = \frac{1}{7} u^7 + C = \frac{1}{7} \sin^7 \theta + C.$

22. Let $u = 1 + \tan \theta$. Then $du = \sec^2 \theta d\theta$, so

$$\int (1 + \tan \theta)^5 \sec^2 \theta d\theta = \int u^5 du = \frac{1}{6} u^6 + C = \frac{1}{6} (1 + \tan \theta)^6 + C.$$

23. Let $u = 1 + z^3$. Then $du = 3z^2 dz$ and $z^2 dz = \frac{1}{3} du$, so

$$\int \frac{z^2}{\sqrt[3]{1+z^3}} dz = \int u^{-1/3} \left(\frac{1}{3} du\right) = \frac{1}{3} \cdot \frac{3}{2} u^{2/3} + C = \frac{1}{2} (1 + z^3)^{2/3} + C.$$

24. Let $u = ax^2 + 2bx + c$. Then $du = 2(ax + b) dx$ and $(ax + b) dx = \frac{1}{2} du$, so

$$\int \frac{(ax+b) dx}{\sqrt{ax^2+2bx+c}} = \int \frac{\frac{1}{2} du}{\sqrt{u}} = \frac{1}{2} \int u^{-1/2} du = u^{1/2} + C = \sqrt{ax^2+2bx+c} + C.$$

25. Let $u = \cot x$. Then $du = -\csc^2 x dx$ and $\csc^2 x dx = -du$, so

$$\int \sqrt{\cot x} \csc^2 x dx = \int \sqrt{u} (-du) = -\frac{u^{3/2}}{3/2} + C = -\frac{2}{3} (\cot x)^{3/2} + C.$$

26. Let $u = \frac{\pi}{x}$. Then $du = -\frac{\pi}{x^2} dx$ and $\frac{1}{x^2} dx = -\frac{1}{\pi} du$, so

$$\int \frac{\cos(\pi/x)}{x^2} dx = \int \cos u \left(-\frac{1}{\pi} du\right) = -\frac{1}{\pi} \sin u + C = -\frac{1}{\pi} \sin \frac{\pi}{x} + C.$$

27. Let $u = \sec x$. Then $du = \sec x \tan x dx$, so

$$\int \sec^3 x \tan x dx = \int \sec^2 x (\sec x \tan x) dx = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} \sec^3 x + C.$$

28. Let $u = x^3 + 1$. Then $x^3 = u - 1$ and $du = 3x^2 dx$, so

$$\begin{aligned}\int \sqrt[3]{x^3 + 1} x^5 dx &= \int \sqrt[3]{x^3 + 1} \cdot x^3 \cdot x^2 dx = \int u^{1/3} (u - 1) \left(\frac{1}{3} du\right) = \frac{1}{3} \int (u^{4/3} - u^{1/3}) du \\ &= \frac{1}{3} \left(\frac{3}{7} u^{7/3} - \frac{3}{4} u^{4/3} \right) + C = \frac{1}{7} (x^3 + 1)^{7/3} - \frac{1}{4} (x^3 + 1)^{4/3} + C\end{aligned}$$

29. Let $u = b + cx^{a+1}$. Then $du = (a + 1)cx^a dx$, so

$$\int x^a \sqrt{b + cx^{a+1}} dx = \int u^{1/2} \frac{1}{(a + 1)c} du = \frac{1}{(a + 1)c} \left(\frac{2}{3} u^{3/2} \right) + C = \frac{2}{3c(a + 1)} (b + cx^{a+1})^{3/2} + C.$$

30. Let $u = \cos t$. Then $du = -\sin t dt$ and $\sin t dt = -du$, so

$$\int \sin t \sec^2(\cos t) dt = \int \sec^2 u \cdot (-du) = -\tan u + C = -\tan(\cos t) + C.$$

31. Let $u = x + 2$. Then $du = dx$, so

$$\begin{aligned}\int \frac{x}{\sqrt[4]{x+2}} dx &= \int \frac{u-2}{\sqrt[4]{u}} du = \int (u^{3/4} - 2u^{-1/4}) du = \frac{4}{7} u^{7/4} - 2 \cdot \frac{4}{3} u^{3/4} + C \\ &= \frac{4}{7} (x+2)^{7/4} - \frac{8}{3} (x+2)^{3/4} + C\end{aligned}$$

32. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

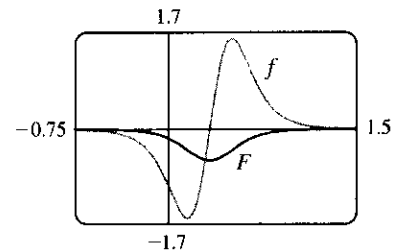
$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x}} dx &= \int \frac{(1-u)^2}{\sqrt{u}} (-du) = -\int \frac{1-2u+u^2}{\sqrt{u}} du = -\int (u^{-1/2} - 2u^{1/2} + u^{3/2}) du \\ &= -\left(2u^{1/2} - 2 \cdot \frac{2}{3} u^{3/2} + \frac{2}{5} u^{5/2} \right) + C = -2\sqrt{1-x} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C\end{aligned}$$

In Exercises 33–36, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

33. $f(x) = \frac{3x-1}{(3x^2-2x+1)^4}.$

$u = 3x^2 - 2x + 1 \Rightarrow du = (6x - 2) dx = 2(3x - 1) dx$, so

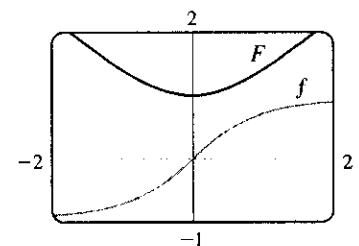
$$\begin{aligned}\int \frac{3x-1}{(3x^2-2x+1)^4} dx &= \int \frac{1}{u^4} \left(\frac{1}{2} du \right) = \frac{1}{2} \int u^{-4} du \\ &= -\frac{1}{6} u^{-3} + C = -\frac{1}{6(3x^2-2x+1)^3} + C\end{aligned}$$



Notice that at $x = \frac{1}{3}$, f changes from negative to positive, and F has a local minimum.

34. $f(x) = \frac{x}{\sqrt{x^2+1}}, u = x^2 + 1 \Rightarrow du = 2x dx$, so

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+1}} dx &= \int \frac{1}{\sqrt{u}} \left(\frac{1}{2} du \right) = \frac{1}{2} \int u^{-1/2} du \\ &= u^{1/2} + C = \sqrt{x^2+1} + C.\end{aligned}$$

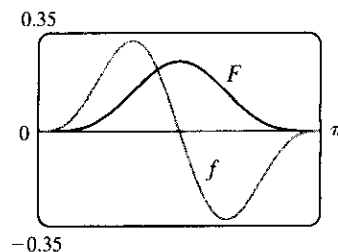


Note that at $x = 0$, f changes from negative to positive and F has a local minimum.

35. $f(x) = \sin^3 x \cos x$. $u = \sin x \Rightarrow du = \cos x dx$, so

$$\int \sin^3 x \cos x dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$$

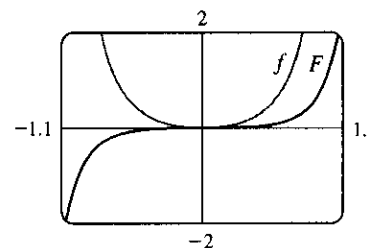
Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π , so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.



36. $f(\theta) = \tan^2 \theta \sec^2 \theta$. $u = \tan \theta \Rightarrow du = \sec^2 \theta d\theta$, so

$$\int \tan^2 \theta \sec^2 \theta d\theta = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 \theta + C$$

Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a horizontal tangent.



37. Let $u = x - 1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Thus, $\int_0^2 (x-1)^{25} dx = \int_{-1}^1 u^{25} du = 0$ by Theorem 6(b), since $f(u) = u^{25}$ is an odd function.

38. Let $u = 4 + 3x$, so $du = 3 dx$. When $x = 0$, $u = 4$; when $x = 7$, $u = 25$. Thus,

$$\int_0^7 \sqrt{4+3x} dx = \int_4^{25} \sqrt{u} \left(\frac{1}{3} du\right) = \frac{1}{3} \left[\frac{u^{3/2}}{3/2} \right]_4^{25} = \frac{2}{9} (25^{3/2} - 4^{3/2}) = \frac{2}{9} (125 - 8) = \frac{234}{9} = 26$$

39. Let $u = 1 + 2x^3$, so $du = 6x^2 dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Thus,

$$\int_0^1 x^2 (1 + 2x^3)^5 dx = \int_1^3 u^5 \left(\frac{1}{6} du\right) = \frac{1}{6} \left[\frac{u^6}{6} \right]_1^3 = \frac{1}{36} (3^6 - 1^6) = \frac{1}{36} (729 - 1) = \frac{728}{36} = \frac{182}{9}$$

40. Let $u = x^2$, so $du = 2x dx$. When $x = 0$, $u = 0$; when $x = \sqrt{\pi}$, $u = \pi$. Thus,

$$\int_0^{\sqrt{\pi}} x \cos(x^2) dx = \int_0^{\pi} \cos u \left(\frac{1}{2} du\right) = \frac{1}{2} [\sin u]_0^{\pi} = \frac{1}{2} (\sin \pi - \sin 0) = \frac{1}{2} (0 - 0) = 0.$$

41. Let $u = t/4$, so $du = \frac{1}{4} dt$. When $t = 0$, $u = 0$; when $t = \pi$, $u = \pi/4$. Thus,

$$\int_0^{\pi} \sec^2(t/4) dt = \int_0^{\pi/4} \sec^2 u (4 du) = 4 [\tan u]_0^{\pi/4} = 4 (\tan \frac{\pi}{4} - \tan 0) = 4(1 - 0) = 4.$$

42. Let $u = \pi t$, so $du = \pi dt$. When $t = \frac{1}{6}$, $u = \frac{\pi}{6}$; when $t = \frac{1}{2}$, $u = \frac{\pi}{2}$. Thus,

$$\int_{1/6}^{1/2} \csc \pi t \cot \pi t dt = \int_{\pi/6}^{\pi/2} \csc u \cot u \left(\frac{1}{\pi} du\right) = \frac{1}{\pi} [-\csc u]_{\pi/6}^{\pi/2} = -\frac{1}{\pi} (1 - 2) = \frac{1}{\pi}.$$

43. $\int_{-\pi/6}^{\pi/6} \tan^3 \theta d\theta = 0$ by Theorem 6(b), since $f(\theta) = \tan^3 \theta$ is an odd function.

44. $\int_0^2 \frac{dx}{(2x-3)^2}$ does not exist since $f(x) = \frac{1}{(2x-3)^2}$ has an infinite discontinuity at $x = \frac{3}{2}$.

45. Let $u = \cos \theta$, so $du = -\sin \theta d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Thus,

$$\int_0^{\pi/3} \frac{\sin \theta}{\cos^2 \theta} d\theta = \int_1^{1/2} \frac{-du}{u^2} = \int_{1/2}^1 u^{-2} du = \left[-\frac{1}{u} \right]_{1/2}^1 = -1 - (-2) = 1.$$

46. $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} dx = 0$ by Theorem 6(b), since $f(x) = \frac{x^2 \sin x}{1+x^6}$ is an odd function.

47. Let $u = 1 + 2x$, so $du = 2 dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3} \left(\frac{1}{2} du\right) = \left[\frac{1}{2} \cdot 3u^{1/3} \right]_1^{27} = \frac{3}{2} (3 - 1) = 3.$$

48. Let $u = \sin x$, so $du = \cos x \, dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x \sin(\sin x) \, dx = \int_0^1 \sin u \, du = [-\cos u]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

49. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x \sqrt{x-1} \, dx = \int_0^1 (u+1)\sqrt{u} \, du = \int_0^1 (u^{3/2} + u^{1/2}) \, du = \left[\frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}.$$

50. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u - 1)$ and $du = 2 \, dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\begin{aligned} \int_0^4 \frac{x \, dx}{\sqrt{1+2x}} &= \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}} \frac{du}{2} = \frac{1}{4} \int_1^9 (u^{1/2} - u^{-1/2}) \, du = \frac{1}{4} \left[\frac{2}{3} u^{3/2} - 2u^{1/2} \right]_1^9 \\ &= \frac{1}{4} \cdot \frac{2}{3} \left[u^{3/2} - 3u^{1/2} \right]_1^9 = \frac{1}{6} [(27 - 9) - (1 - 3)] = \frac{20}{6} = \frac{10}{3} \end{aligned}$$

51. $\int_0^4 \frac{dx}{(x-2)^3}$ does not exist since $f(x) = \frac{1}{(x-2)^3}$ has an infinite discontinuity at $x = 2$.

52. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x \, dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x \sqrt{a^2 - x^2} \, dx = \int_{a^2}^0 u^{1/2} \left(-\frac{1}{2} du\right) = \frac{1}{2} \int_0^{a^2} u^{1/2} \, du = \frac{1}{2} \cdot \left[\frac{2}{3} u^{3/2} \right]_0^{a^2} = \frac{1}{3} a^3.$$

53. Let $u = x^2 + a^2$, so $du = 2x \, dx$ and $x \, dx = \frac{1}{2} du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

$$\begin{aligned} \int_0^a x \sqrt{x^2 + a^2} \, dx &= \int_{a^2}^{2a^2} u^{1/2} \left(\frac{1}{2} du\right) = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_{a^2}^{2a^2} = \left[\frac{1}{3} u^{3/2} \right]_{a^2}^{2a^2} \\ &= \frac{1}{3} \left[(2a^2)^{3/2} - (a^2)^{3/2} \right] = \frac{1}{3} (2\sqrt{2} - 1) a^3 \end{aligned}$$

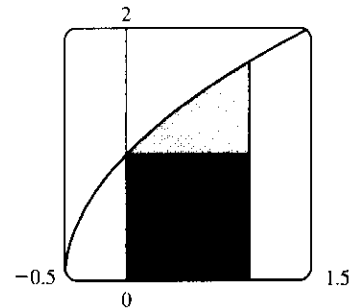
54. $\int_{-a}^a x \sqrt{x^2 + a^2} \, dx = 0$ by Theorem 6(b), since $f(x) = x \sqrt{x^2 + a^2}$ is an odd function.

55. From the graph, it appears that the area under the curve is about

$1 + (\text{a little more than } \frac{1}{2} \cdot 1 \cdot 0.7)$, or about 1.4. The exact area is given by

$A = \int_0^1 \sqrt{2x+1} \, dx$. Let $u = 2x+1$, so $du = 2 \, dx$. The limits change to $2 \cdot 0 + 1 = 1$ and $2 \cdot 1 + 1 = 3$, and

$$A = \int_1^3 \sqrt{u} \left(\frac{1}{2} du\right) = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_1^3 = \frac{1}{3} (3\sqrt{3} - 1) = \sqrt{3} - \frac{1}{3} \approx 1.399.$$



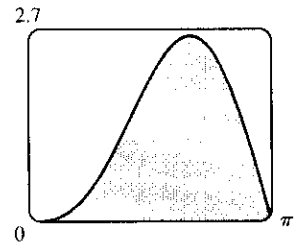
56. From the graph, it appears that the area under the curve is almost

$\frac{1}{2} \cdot \pi \cdot 2.6$, or about 4. The exact area is given by

$$\begin{aligned} A &= \int_0^\pi (2 \sin x - \sin 2x) \, dx = -2 [\cos x]_0^\pi - \int_0^\pi \sin 2x \, dx \\ &= -2(-1 - 1) - 0 = 4 \end{aligned}$$

Note: $\int_0^\pi \sin 2x \, dx = 0$ since it is clear from the graph of $y = \sin 2x$

that $\int_{\pi/2}^\pi \sin 2x \, dx = -\int_0^{\pi/2} \sin 2x \, dx$.



57. First write the integral as a sum of two integrals:

$I = \int_{-2}^2 (x+3)\sqrt{4-x^2} dx = I_1 + I_2 = \int_{-2}^2 x\sqrt{4-x^2} dx + \int_{-2}^2 3\sqrt{4-x^2} dx$. $I_1 = 0$ by Theorem 6(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3 \cdot \frac{1}{2}(\pi \cdot 2^2) = 6\pi$.

58. Let $u = x^2$. Then $du = 2x dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so

$I = \int_0^1 x\sqrt{1-x^4} dx = \frac{1}{2} \int_0^1 \sqrt{1-u^2} du$. But this integral can be interpreted as the area of a quarter-circle with radius 1. So $I = \frac{1}{2} \cdot \frac{1}{4}(\pi \cdot 1^2) = \frac{1}{8}\pi$.

59. The volume of inhaled air in the lungs at time t is

$$\begin{aligned} V(t) &= \int_0^t f(u) du = \int_0^t \frac{1}{2} \sin\left(\frac{2\pi}{5} u\right) du = \int_0^{2\pi t/5} \frac{1}{2} \sin v \left(\frac{5}{2\pi} dv\right) \quad [\text{substitute } v = \frac{2\pi}{5} u, dv = \frac{2\pi}{5} du] \\ &= \frac{5}{4\pi} [-\cos v]_0^{2\pi t/5} = \frac{5}{4\pi} [-\cos(\frac{2\pi}{5} t) + 1] = \frac{5}{4\pi} [1 - \cos(\frac{2\pi}{5} t)] \text{ liters} \end{aligned}$$

60. Number of calculators $= x(4) - x(2) = \int_2^4 5000 [1 - 100(t+10)^{-2}] dt$
 $= 5000 [t + 100(t+10)^{-1}]_2^4 = 5000 [(4 + \frac{100}{14}) - (2 + \frac{100}{12})] \approx 4048$

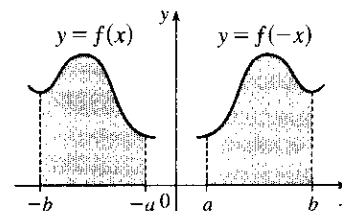
61. Let $u = 2x$. Then $du = 2 dx$, so $\int_0^2 f(2x) dx = \int_0^4 f(u) (\frac{1}{2} du) = \frac{1}{2} \int_0^4 f(u) du = \frac{1}{2}(10) = 5$.

62. Let $u = x^2$. Then $du = 2x dx$, so $\int_0^3 x f(x^2) dx = \int_0^9 f(u) (\frac{1}{2} du) = \frac{1}{2} \int_0^9 f(u) du = \frac{1}{2}(4) = 2$.

63. (a) Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x) dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u) du = \int_{-b}^{-a} f(x) dx$$

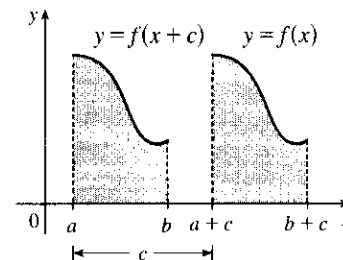
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f , and the limits of integration, about the y -axis.



(b) Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x+c) dx = \int_{a+c}^{b+c} f(u) du = \int_{a+c}^{b+c} f(x) dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f , and the limits of integration, by a distance c .



64. The area under the graph of $y = \sin \sqrt{x}$ from 0 to 4 is $A_1 = \int_0^4 \sin \sqrt{x} dx$. The area under the graph of $y = 2x \sin x$ from 0 to 2 is $A_2 = \int_0^2 2x \sin x dx$ [$u = x^2$, $du = 2x dx$, $\sqrt{u} = x$ for $0 \leq x \leq 2$] $= \int_0^4 \sin \sqrt{u} du$. Since the integration variable is immaterial, $A_1 = A_2$.

65. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$$\int_0^1 x^a (1-x)^b dx = \int_1^0 (1-u)^a u^b (-du) = \int_0^1 u^b (1-u)^a du = \int_0^1 x^b (1-x)^a dx.$$

66. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\begin{aligned}\int_0^\pi x f(\sin x) dx &= -\int_\pi^0 (\pi - u) f(\sin(\pi - u)) du = \int_0^\pi (\pi - u) f(\sin u) du \\ &= \pi \int_0^\pi f(\sin u) du - \int_0^\pi u f(\sin u) du = \pi \int_0^\pi f(\sin x) dx - \int_0^\pi x f(\sin x) dx \\ \Rightarrow 2 \int_0^\pi x f(\sin x) dx &= \pi \int_0^\pi f(\sin x) dx \quad \Rightarrow \quad \int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx.\end{aligned}$$

67. Let $u = 5 - 3x$. Then $du = -3 dx$ and $dx = -\frac{1}{3} du$, so

$$\int \frac{dx}{5 - 3x} = \int \frac{1}{u} \left(-\frac{1}{3} du\right) = -\frac{1}{3} \ln |u| + C = -\frac{1}{3} \ln |5 - 3x| + C.$$

68. Let $u = x^2 + 1$. Then $du = 2x dx$ and $x dx = \frac{1}{2} du$, so

$$\begin{aligned}\int \frac{x}{x^2 + 1} dx &= \int \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 1| + C = \frac{1}{2} \ln(x^2 + 1) + C \quad [\text{since } x^2 + 1 > 0] \\ \text{or } \ln(x^2 + 1)^{1/2} + C &= \ln \sqrt{x^2 + 1} + C.\end{aligned}$$

69. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} (\ln x)^3 + C$.

70. Let $u = \tan^{-1} x$. Then $du = \frac{dx}{1 + x^2}$, so $\int \frac{\tan^{-1} x}{1 + x^2} dx = \int u du = \frac{u^2}{2} + C = \frac{(\tan^{-1} x)^2}{2} + C$.

71. Let $u = 1 + e^x$. Then $du = e^x dx$, so $\int e^x \sqrt{1 + e^x} dx = \int \sqrt{u} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1 + e^x}$. Then $u^2 = 1 + e^x$ and $2u du = e^x dx$, so

$$\int e^x \sqrt{1 + e^x} dx = \int u \cdot 2u du = \frac{2}{3} u^3 + C = \frac{2}{3} (1 + e^x)^{3/2} + C.$$

72. Let $u = \cos t$. Then $du = -\sin t dt$ and $\sin t dt = -du$, so

$$\int e^{\cos t} \sin t dt = \int e^u (-du) = -e^u + C = -e^{\cos t} + C.$$

73. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln |u| + C = \ln |\ln x| + C$.

74. Let $u = e^x + 1$. Then $du = e^x dx$, so $\int \frac{e^x}{e^x + 1} dx = \int \frac{du}{u} = \ln |u| + C = \ln(e^x + 1) + C$.

75. $\int \cot x dx = \int \frac{\cos x}{\sin x} dx$. Let $u = \sin x$. Then $du = \cos x dx$, so

$$\int \cot x dx = \int \frac{1}{u} du = \ln |u| + C = \ln |\sin x| + C.$$

76. Let $u = \cos x$. Then $du = -\sin x dx$ and $\sin x dx = -du$, so

$$\int \frac{\sin x}{1 + \cos^2 x} dx = \int \frac{-du}{1 + u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

77. Let $u = 1 + x^2$. Then $du = 2x dx$, so

$$\begin{aligned}\int \frac{1 + x}{1 + x^2} dx &= \int \frac{1}{1 + x^2} dx + \int \frac{x}{1 + x^2} dx = \tan^{-1} x + \int \frac{\frac{1}{2} du}{u} = \tan^{-1} x + \frac{1}{2} \ln |u| + C \\ &= \tan^{-1} x + \frac{1}{2} \ln |1 + x^2| + C = \tan^{-1} x + \frac{1}{2} \ln(1 + x^2) + C \quad [\text{since } 1 + x^2 > 0].\end{aligned}$$

78. Let $u = x^2$. Then $du = 2x dx$, so $\int \frac{x}{1+x^4} dx = \int \frac{\frac{1}{2} du}{1+u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(x^2) + C$.

79. Let $u = 1/x$, so $du = -1/x^2 dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2} dx = \int_1^{1/2} e^u (-du) = -[e^u]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

80. Let $u = -x^2$, so $du = -2x dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 x e^{-x^2} dx = \int_0^{-1} e^u \left(-\frac{1}{2} du\right) = -\frac{1}{2} [e^u]_0^{-1} = -\frac{1}{2} (e^{-1} - e^0) = \frac{1}{2} (1 - 1/e).$$

81. Let $u = \ln x$, so $du = \frac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$, $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x \sqrt{\ln x}} = \int_1^4 u^{-1/2} du = 2[u^{1/2}]_1^4 = 2(2 - 1) = 2.$$

82. Let $u = \sin^{-1} x$, so $du = \frac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} u du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

83. $\frac{x \sin x}{1 + \cos^2 x} = x \cdot \frac{\sin x}{2 - \sin^2 x} = x f(\sin x)$, where $f(t) = \frac{t}{2 - t^2}$. By Exercise 66,

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$$

Let $u = \cos x$. Then $du = -\sin x dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\begin{aligned} \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx &= -\frac{\pi}{2} \int_1^{-1} \frac{du}{1 + u^2} = \frac{\pi}{2} \int_{-1}^1 \frac{du}{1 + u^2} = \frac{\pi}{2} [\tan^{-1} u]_{-1}^1 \\ &= \frac{\pi}{2} [\tan^{-1} 1 - \tan^{-1}(-1)] = \frac{\pi}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \right] = \frac{\pi^2}{4} \end{aligned}$$

5 Review

CONCEPT CHECK

- (a) $\sum_{i=1}^n f(x_i^*) \Delta x$ is an expression for a Riemann sum of a function f .
 x_i^* is a point in the i th subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subintervals.
 (b) See Figure 1 in Section 5.2.
 (c) In Section 5.2, see Figure 3 and the paragraph beside it.
- (a) See Definition 5.2.2.
 (b) See Figure 2 in Section 5.2.
 (c) In Section 5.2, see Figure 4 and the paragraph above it.
- See the Fundamental Theorem of Calculus after Example 8 in Section 5.3.

4. (a) See the Net Change Theorem after Example 5 in Section 5.4.
 (b) $\int_{t_1}^{t_2} r(t) dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2 .
5. (a) $\int_{60}^{120} v(t) dt$ represents the change in position of the particle from $t = 60$ to $t = 120$ seconds.
 (b) $\int_{60}^{120} |v(t)| dt$ represents the total distance traveled by the particle from $t = 60$ to 120 seconds.
 (c) $\int_{60}^{120} a(t) dt$ represents the change in the velocity of the particle from $t = 60$ to $t = 120$ seconds.
6. (a) $\int f(x) dx$ is the family of functions $\{F \mid F' = f\}$. Any two such functions differ by a constant.
 (b) The connection is given by the Evaluation Theorem: $\int_a^b f(x) dx = [F(x)]_a^b$ if f is continuous.
7. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.
8. See the Substitution Rule (5.5.4). This says that it is permissible to operate with the dx after an integral sign as if it were a differential.

 TRUE-FALSE QUIZ

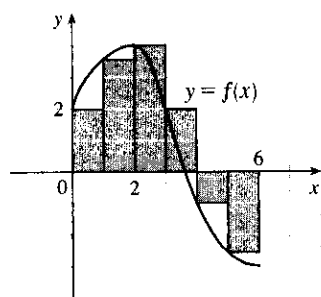
1. True by Property 2 of the Integral in Section 5.2.
2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.
3. True by Property 3 of the Integral in Section 5.2.
4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$,

$$\int_0^1 x f(x) dx = \int_0^1 x dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2} \text{ (a constant) while } x \int_0^1 1 dx = x [x]_0^1 = x \cdot 1 = x \text{ (a variable).}$$
5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2} dx = \int_0^1 x dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.
6. True by the Net Change Theorem.
7. True by Comparison Property 7 of the Integral in Section 5.2.
8. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but
 $f'(x) = 0 < 1 = g'(x)$ for $x \in (0, 1)$.
9. True. The integrand is an odd function that is continuous on $[-1, 1]$, so the result follows from Theorem 5.5.6(b).
10. True.
$$\int_{-5}^5 (ax^2 + bx + c) dx = \int_{-5}^5 (ax^2 + c) dx + \int_{-5}^5 bx dx$$

$$= 2 \int_0^5 (ax^2 + c) dx \text{ [by 5.5.6(a)] } + 0 \text{ [by 5.5.6(b)]}$$
11. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 6 of Integrals.)
12. False. See the remarks and Figure 4 before Example 1 in Section 5.2, and notice that $y = x - x^3 < 0$ for $1 < x \leq 2$.
13. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.
14. True by FTC1.

EXERCISES

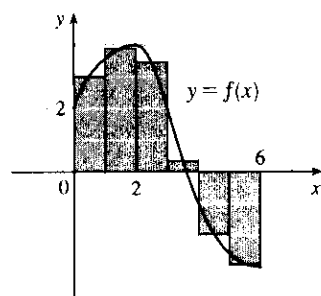
1. (a)



$$\begin{aligned}
 L_6 &= \sum_{i=1}^6 f(x_{i-1}) \Delta x \quad [\Delta x = \frac{6-0}{6} = 1] \\
 &= f(x_0) \cdot 1 + f(x_1) \cdot 1 + f(x_2) \cdot 1 \\
 &\quad + f(x_3) \cdot 1 + f(x_4) \cdot 1 + f(x_5) \cdot 1 \\
 &\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8
 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles above the x -axis minus the sum of the areas of the two rectangles below the x -axis.

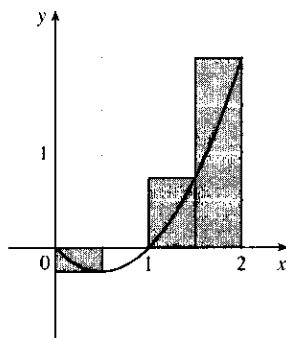
(b)



$$\begin{aligned}
 M_6 &= \sum_{i=1}^6 f(\bar{x}_i) \Delta x \quad [\Delta x = \frac{6-0}{6} = 1] \\
 &= f(\bar{x}_1) \cdot 1 + f(\bar{x}_2) \cdot 1 + f(\bar{x}_3) \cdot 1 \\
 &\quad + f(\bar{x}_4) \cdot 1 + f(\bar{x}_5) \cdot 1 + f(\bar{x}_6) \cdot 1 \\
 &= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5) \\
 &\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7
 \end{aligned}$$

The Riemann sum represents the sum of the areas of the four rectangles above the x -axis minus the sum of the areas of the two rectangles below the x -axis.

2. (a)



$$f(x) = x^2 - x \text{ and } \Delta x = \frac{2-0}{4} = 0.5 \Rightarrow$$

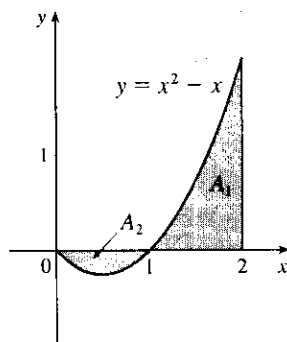
$$\begin{aligned}
 R_4 &= 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2) \\
 &= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25
 \end{aligned}$$

The Riemann sum represents the sum of the areas of the two rectangles above the x -axis minus the area of the rectangle below the x -axis. (The second rectangle vanishes.)

$$\begin{aligned}
 \text{(b)} \quad \int_0^2 (x^2 - x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad [\Delta x = 2/n \text{ and } x_i = 2i/n] \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{4i^2}{n^2} - \frac{2i}{n} \right) \left(\frac{2}{n} \right) = \lim_{n \rightarrow \infty} \frac{2}{n} \left[\frac{4}{n^2} \sum_{i=1}^n i^2 - \frac{2}{n} \sum_{i=1}^n i \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2} \cdot \frac{n(n+1)}{2} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{4}{3} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} - 2 \cdot \frac{n+1}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - 2 \left(1 + \frac{1}{n} \right) \right] = \frac{4}{3} \cdot 1 \cdot 2 - 2 \cdot 1 = \frac{2}{3}
 \end{aligned}$$

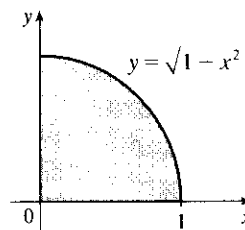
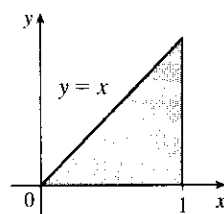
$$\text{(c)} \quad \int_0^2 (x^2 - x) dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_0^2 = \left(\frac{8}{3} - 2 \right) = \frac{2}{3}$$

(d)



$\int_0^2 (x^2 - x) dx = A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.

$$3. \int_0^1 (x + \sqrt{1-x^2}) dx = \int_0^1 x dx + \int_0^1 \sqrt{1-x^2} dx = I_1 + I_2.$$



I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle. Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.

$$4. \text{ On } [0, \pi], \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin x_i \Delta x = \int_0^\pi \sin x dx = [-\cos x]_0^\pi = -(-1) - (-1) = 2.$$

$$5. \int_0^6 f(x) dx = \int_0^4 f(x) dx + \int_4^6 f(x) dx \Rightarrow 10 = 7 + \int_4^6 f(x) dx \Rightarrow \int_4^6 f(x) dx = 10 - 7 = 3$$

$$6. (a) \int_1^5 (x + 2x^5) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad \left[\Delta x = \frac{5-1}{n} = \frac{4}{n}, x_i = 1 + \frac{4i}{n} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(1 + \frac{4i}{n} \right) + 2 \left(1 + \frac{4i}{n} \right)^5 \right] \cdot \frac{4}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{1305n^4 + 3126n^3 + 2080n^2 - 256}{n^3} \cdot \frac{4}{n} = 5220$$

$$(b) \int_1^5 (x + 2x^5) dx = \left[\frac{1}{2}x^2 + \frac{2}{6}x^6 \right]_1^5 = \left(\frac{25}{2} + \frac{15,625}{3} \right) - \left(\frac{1}{2} + \frac{1}{3} \right) = 12 + 5208 = 5220$$

7. First note that either a or b must be the graph of $\int_0^x f(t) dt$, since $\int_0^0 f(t) dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t) dt$.

$$8. (a) \text{ By FTC2, we have } \int_0^{\pi/2} \frac{d}{dx} \left(\sin \frac{x}{2} \cos \frac{x}{3} \right) dx = \left[\sin \frac{x}{2} \cos \frac{x}{3} \right]_0^{\pi/2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - 0 \cdot 1 = \frac{\sqrt{6}}{4}.$$

$$(b) \frac{d}{dx} \int_0^{\pi/2} \sin \frac{x}{2} \cos \frac{x}{3} dx = 0, \text{ since the definite integral is a constant.}$$

$$(c) \frac{d}{dx} \int_x^{\pi/2} \sin \frac{t}{2} \cos \frac{t}{3} dt = \frac{d}{dx} \left(- \int_{\pi/2}^x \sin \frac{t}{2} \cos \frac{t}{3} dt \right) = - \frac{d}{dx} \int_{\pi/2}^x \sin \frac{t}{2} \cos \frac{t}{3} dt = - \sin \frac{x}{2} \cos \frac{x}{3},$$

by FTC1.

$$9. \int_1^2 (8x^3 + 3x^2) dx = \left[8 \cdot \frac{1}{4} x^4 + 3 \cdot \frac{1}{3} x^3 \right]_1^2 = [2x^4 + x^3]_1^2 = (2 \cdot 2^4 + 2^3) - (2 + 1) = 40 - 3 = 37$$

$$10. \int_0^T (x^4 - 8x + 7) dx = \left[\frac{1}{5} x^5 - 4x^2 + 7x \right]_0^T = \left(\frac{1}{5} T^5 - 4T^2 + 7T \right) - 0 = \frac{1}{5} T^5 - 4T^2 + 7T$$

$$11. \int_0^1 (1 - x^9) dx = \left[x - \frac{1}{10} x^{10} \right]_0^1 = \left(1 - \frac{1}{10} \right) - 0 = \frac{9}{10}$$

12. Let $u = 1 - x$, so $du = -dx$ and $dx = -du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. Thus,

$$\int_0^1 (1 - x)^9 dx = \int_1^0 u^9 (-du) = \int_0^1 u^9 du = \frac{1}{10} [u^{10}]_0^1 = \frac{1}{10} (1 - 0) = \frac{1}{10}.$$

$$13. \int_1^9 \frac{\sqrt{u} - 2u^2}{u} du = \int_1^9 (u^{-1/2} - 2u) du = \left[2u^{1/2} - u^2 \right]_1^9 = (6 - 81) - (2 - 1) = -76$$

$$14. \int_0^1 (\sqrt[4]{u} + 1)^2 du = \int_0^1 (u^{1/2} + 2u^{1/4} + 1) du = \left[\frac{2}{3} u^{3/2} + \frac{8}{5} u^{5/4} + u \right]_0^1 = \left(\frac{2}{3} + \frac{8}{5} + 1 \right) - 0 = \frac{49}{15}$$

15. Let $u = y^2 + 1$, so $du = 2y dy$ and $y dy = \frac{1}{2} du$. When $y = 0$, $u = 1$; when $y = 1$, $u = 2$. Thus,

$$\int_0^1 y(y^2 + 1)^5 dy = \int_1^2 u^5 \left(\frac{1}{2} du \right) = \frac{1}{2} \left[\frac{1}{6} u^6 \right]_1^2 = \frac{1}{12} (64 - 1) = \frac{63}{12} = \frac{21}{4}.$$

16. Let $u = 1 + y^3$, so $du = 3y^2 dy$ and $y^2 dy = \frac{1}{3} du$. When $y = 0$, $u = 1$; when $y = 2$, $u = 9$. Thus,

$$\int_0^2 y^2 \sqrt{1 + y^3} dy = \int_1^9 u^{1/2} \left(\frac{1}{3} du \right) = \frac{1}{3} \left[\frac{2}{3} u^{3/2} \right]_1^9 = \frac{2}{9} (27 - 1) = \frac{52}{9}.$$

17. $\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \frac{1}{(t-4)^2}$ has an infinite discontinuity at $t = 4$; that is, f is discontinuous on the interval $[1, 5]$.

18. Let $u = 3\pi t$, so $du = 3\pi dt$. When $t = 0$, $u = 0$; when $t = 1$, $u = 3\pi$. Thus,

$$\int_0^1 \sin(3\pi t) dt = \int_0^{3\pi} \sin u \left(\frac{1}{3\pi} du \right) = \frac{1}{3\pi} [-\cos u]_0^{3\pi} = -\frac{1}{3\pi} (-1 - 1) = \frac{2}{3\pi}.$$

19. Let $u = v^3$, so $du = 3v^2 dv$. When $v = 0$, $u = 0$; when $v = 1$, $u = 1$. Thus,

$$\int_0^1 v^2 \cos(v^3) dv = \int_0^1 \cos u \left(\frac{1}{3} du \right) = \frac{1}{3} [\sin u]_0^1 = \frac{1}{3} (\sin 1 - 0) = \frac{1}{3} \sin 1.$$

20. $\int_{-1}^1 \frac{\sin x}{1 + x^2} dx = 0$ by Theorem 5.5.6(b), since $f(x) = \frac{\sin x}{1 + x^2}$ is an odd function.

21. Let $u = x^2 + 4x$. Then $du = (2x + 4) dx = 2(x + 2) dx$, so

$$\int \frac{x + 2}{\sqrt{x^2 + 4x}} dx = \int u^{-1/2} \left(\frac{1}{2} du \right) = \frac{1}{2} \cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2 + 4x} + C.$$

22. Let $u = 3t$. Then $du = 3 dt$, so $\int \csc^2 3t dt = \int \csc^2 u \left(\frac{1}{3} du \right) = \frac{1}{3} (-\cot u) + C = -\frac{1}{3} \cot 3t + C$.

23. Let $u = \sin \pi t$. Then $du = \pi \cos \pi t dt$, so $\int \sin \pi t \cos \pi t dt = \int u \left(\frac{1}{\pi} du \right) = \frac{1}{\pi} \cdot \frac{1}{2} u^2 + C = \frac{1}{2\pi} (\sin \pi t)^2 + C$.

24. Let $u = \cos x$. Then $du = -\sin x dx$, so

$$\int \sin x \cos(\cos x) dx = -\int \cos u du = -\sin u + C = -\sin(\cos x) + C.$$

25. Let $u = 2\theta$. Then $du = 2 d\theta$, so

$$\begin{aligned} \int_0^{\pi/8} \sec 2\theta \tan 2\theta d\theta &= \int_0^{\pi/4} \sec u \tan u \left(\frac{1}{2} du \right) = \frac{1}{2} [\sec u]_0^{\pi/4} = \frac{1}{2} (\sec \frac{\pi}{4} - \sec 0) \\ &= \frac{1}{2} (\sqrt{2} - 1) = \frac{1}{2} \sqrt{2} - \frac{1}{2} \end{aligned}$$

26. Let $u = 1 + \tan t$, so $du = \sec^2 t \, dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{4}$, $u = 2$. Thus,

$$\int_0^{\pi/4} (1 + \tan t)^3 \sec^2 t \, dt = \int_1^2 u^3 \, du = \left[\frac{1}{4} u^4 \right]_1^2 = \frac{1}{4} (2^4 - 1^4) = \frac{1}{4} (16 - 1) = \frac{15}{4}.$$

27. Since $x^2 - 4 < 0$ for $0 \leq x < 2$ and $x^2 - 4 > 0$ for $2 < x \leq 3$, we have $|x^2 - 4| = -(x^2 - 4) = 4 - x^2$ for $0 \leq x < 2$ and $|x^2 - 4| = x^2 - 4$ for $2 < x \leq 3$. Thus,

$$\begin{aligned} \int_0^3 |x^2 - 4| \, dx &= \int_0^2 (4 - x^2) \, dx + \int_2^3 (x^2 - 4) \, dx = \left[4x - \frac{x^3}{3} \right]_0^2 + \left[\frac{x^3}{3} - 4x \right]_2^3 \\ &= \left(8 - \frac{8}{3} \right) - 0 + (9 - 12) - \left(\frac{8}{3} - 8 \right) = \frac{16}{3} - 3 + \frac{16}{3} = \frac{32}{3} - 3 = \frac{23}{3} \end{aligned}$$

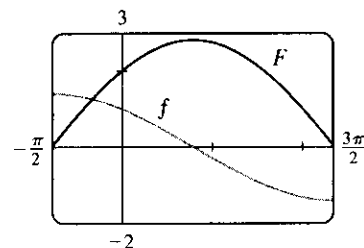
28. Since $\sqrt{x} - 1 < 0$ for $0 \leq x < 1$ and $\sqrt{x} - 1 > 0$ for $1 < x \leq 4$, we have $|\sqrt{x} - 1| = -(\sqrt{x} - 1) = 1 - \sqrt{x}$ for $0 \leq x < 1$ and $|\sqrt{x} - 1| = \sqrt{x} - 1$ for $1 < x \leq 4$. Thus,

$$\begin{aligned} \int_0^4 |\sqrt{x} - 1| \, dx &= \int_0^1 (1 - \sqrt{x}) \, dx + \int_1^4 (\sqrt{x} - 1) \, dx = \left[x - \frac{2}{3} x^{3/2} \right]_0^1 + \left[\frac{2}{3} x^{3/2} - x \right]_1^4 \\ &= \left(1 - \frac{2}{3} \right) - 0 + \left(\frac{16}{3} - 4 \right) - \left(\frac{2}{3} - 1 \right) = \frac{1}{3} + \frac{16}{3} - 4 + \frac{1}{3} = 6 - 4 = 2 \end{aligned}$$

In Exercises 29 and 30, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

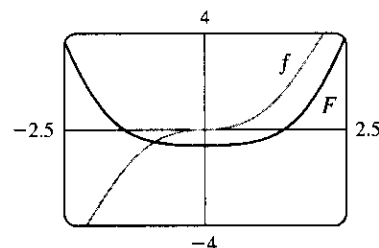
29. Let $u = 1 + \sin x$. Then $du = \cos x \, dx$, so

$$\int \frac{\cos x \, dx}{\sqrt{1 + \sin x}} = \int u^{-1/2} \, du = 2u^{1/2} + C = 2\sqrt{1 + \sin x} + C.$$



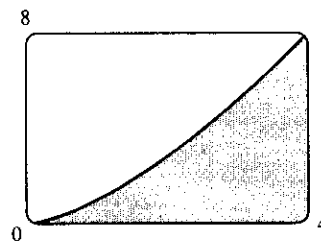
30. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x \, dx = \frac{1}{2} \, du$, so

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 + 1}} \, dx &= \int \frac{(u - 1)}{\sqrt{u}} \left(\frac{1}{2} \, du \right) = \frac{1}{2} \int (u^{1/2} - u^{-1/2}) \, du \\ &= \frac{1}{2} \left(\frac{2}{3} u^{3/2} - 2u^{1/2} \right) + C \\ &= \frac{1}{3} (x^2 + 1)^{3/2} - (x^2 + 1)^{1/2} + C \\ &= \frac{1}{3} (x^2 + 1)^{1/2} [(x^2 + 1) - 3] + C \\ &= \frac{1}{3} \sqrt{x^2 + 1} (x^2 - 2) + C \end{aligned}$$



31. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate

$$\int_0^4 x \sqrt{x} \, dx = \int_0^4 x^{3/2} \, dx = \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{2}{5} (4)^{5/2} = \frac{64}{5} = 12.8.$$

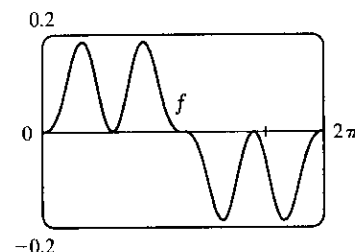


32. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \sin^3 x \, dx$ is equal to 0.

To evaluate the integral, we write the integral as

$$I = \int_0^{2\pi} \cos^2 x (1 - \cos^2 x) \sin x \, dx \text{ and let } u = \cos x \Rightarrow$$

$$du = -\sin x \, dx. \text{ Thus, } I = \int_1^{-1} u^2(1 - u^2)(-du) = 0.$$



33. By FTC1, $F(x) = \int_1^x \sqrt{1+t^4} \, dt \Rightarrow F'(x) = \sqrt{1+x^4}$.

34. $F(x) = \int_{\pi}^x \tan(s^2) \, ds \Rightarrow F'(x) = \tan(x^2)$

35. $g(x) = \int_0^{x^3} \frac{t \, dt}{\sqrt{1+t^3}}$. Let $y = g(u)$ and $u = x^3$.

$$\text{Then } g'(x) = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{u}{\sqrt{1+u^3}} 3x^2 = \frac{x^3}{\sqrt{1+x^9}} 3x^2 = \frac{3x^5}{\sqrt{1+x^9}}.$$

36. Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$. Also, $\frac{dg}{dx} = \frac{dg}{du} \frac{du}{dx}$, so

$$\begin{aligned} \frac{d}{dx} \int_1^{\cos x} \sqrt[3]{1-t^2} \, dt &= \frac{d}{du} \int_1^u \sqrt[3]{1-t^2} \, dt \cdot \frac{du}{dx} = \sqrt[3]{1-u^2} (-\sin x) \\ &= \sqrt[3]{1-\cos^2 x} (-\sin x) = -\sin x \sqrt[3]{\sin^2 x} = -(\sin x)^{5/3} \end{aligned}$$

37. $y = \int_{\sqrt{x}}^x \frac{\cos \theta}{\theta} \, d\theta = \int_1^x \frac{\cos \theta}{\theta} \, d\theta + \int_{\sqrt{x}}^1 \frac{\cos \theta}{\theta} \, d\theta = \int_1^x \frac{\cos \theta}{\theta} \, d\theta - \int_1^{\sqrt{x}} \frac{\cos \theta}{\theta} \, d\theta \Rightarrow$

$$y' = \frac{\cos x}{x} - \frac{\cos \sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{2 \cos x - \cos \sqrt{x}}{2x}$$

38. $y = \int_{2x}^{3x+1} \sin(t^4) \, dt = \int_{2x}^0 \sin(t^4) \, dt + \int_0^{3x+1} \sin(t^4) \, dt = \int_0^{3x+1} \sin(t^4) \, dt - \int_0^{2x} \sin(t^4) \, dt \Rightarrow$

$$y' = \sin[(3x+1)^4] \cdot \frac{d}{dx}(3x+1) - \sin[(2x)^4] \cdot \frac{d}{dx}(2x) = 3 \sin[(3x+1)^4] - 2 \sin[(2x)^4]$$

39. If $1 \leq x \leq 3$, then $\sqrt{1^2+3} \leq \sqrt{x^2+3} \leq \sqrt{3^2+3} \Rightarrow 2 \leq \sqrt{x^2+3} \leq 2\sqrt{3}$, so

$$2(3-1) \leq \int_1^3 \sqrt{x^2+3} \, dx \leq 2\sqrt{3}(3-1); \text{ that is, } 4 \leq \int_1^3 \sqrt{x^2+3} \, dx \leq 4\sqrt{3}.$$

40. If $3 \leq x \leq 5$, then $4 \leq x+1 \leq 6$ and $\frac{1}{6} \leq \frac{1}{x+1} \leq \frac{1}{4}$, so $\frac{1}{6}(5-3) \leq \int_3^5 \frac{1}{x+1} \, dx \leq \frac{1}{4}(5-3)$; that is,

$$\frac{1}{3} \leq \int_3^5 \frac{1}{x+1} \, dx \leq \frac{1}{2}.$$

41. $0 \leq x \leq 1 \Rightarrow 0 \leq \cos x \leq 1 \Rightarrow x^2 \cos x \leq x^2 \Rightarrow$

$$\int_0^1 x^2 \cos x \, dx \leq \int_0^1 x^2 \, dx = \frac{1}{3} [x^3]_0^1 = \frac{1}{3} \text{ [Property 7].}$$

42. On the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, x is increasing and $\sin x$ is decreasing, so $\frac{\sin x}{x}$ is decreasing. Therefore, the largest value

of $\frac{\sin x}{x}$ on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ is $\frac{\sin(\pi/4)}{\pi/4} = \frac{\sqrt{2}/2}{\pi/4} = \frac{2\sqrt{2}}{\pi}$. By Property 8 with $M = \frac{2\sqrt{2}}{\pi}$ we get

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x} \, dx \leq \frac{2\sqrt{2}}{\pi} \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

43. Let $f(x) = \sqrt{1+x^3}$ on $[0, 1]$. The Midpoint Rule with $n = 5$ gives

$$\begin{aligned}\int_0^1 \sqrt{1+x^3} dx &\approx \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \\ &= \frac{1}{5}[\sqrt{1+(0.1)^3} + \sqrt{1+(0.3)^3} + \cdots + \sqrt{1+(0.9)^3}] \approx 1.110\end{aligned}$$

44. (a) displacement $= \int_0^5 (t^2 - t) dt = [\frac{1}{3}t^3 - \frac{1}{2}t^2]_0^5 = \frac{125}{3} - \frac{25}{2} = \frac{175}{6} = 29.1\bar{6}$ meters

$$\begin{aligned}\text{(b) distance traveled} &= \int_0^5 |t^2 - t| dt = \int_0^1 |t(t-1)| dt + \int_1^5 (t^2 - t) dt \\ &= [\frac{1}{2}t^2 - \frac{1}{3}t^3]_0^1 + [\frac{1}{3}t^3 - \frac{1}{2}t^2]_1^5 \\ &= \frac{1}{2} - \frac{1}{3} - 0 + (\frac{125}{3} - \frac{25}{2}) - (\frac{1}{3} - \frac{1}{2}) = \frac{177}{6} = 29.5 \text{ meters}\end{aligned}$$

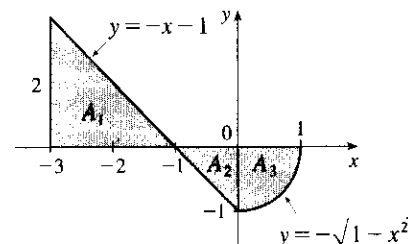
45. Note that $r(t) = b'(t)$, where $b(t)$ = the number of barrels of oil consumed up to time t . So, by the Net Change Theorem, $\int_0^3 r(t) dt = b(3) - b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2003.

46. Distance covered $= \int_0^{5.0} v(t) dt \approx M_5 = \frac{5.0-0}{5}[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5)]$
 $= 1(4.67 + 8.86 + 10.22 + 10.67 + 10.81) = 45.23$ m

47. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$. The increase in the bee population was

$$\begin{aligned}\int_0^{24} r(t) dt &\approx M_6 = 4[r(2) + r(6) + r(10) + r(14) + r(18) + r(22)] \\ &\approx 4[50 + 1000 + 7000 + 8550 + 1350 + 150] \\ &= 4(18,100) = 72,400\end{aligned}$$

48. $A_1 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$, $A_2 = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, and since $y = -\sqrt{1-x^2}$ for $0 \leq x \leq 1$ represents a quarter-circle with radius 1, $A_3 = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$. So
 $\int_{-3}^1 f(x) dx = A_1 - A_2 - A_3 = 2 - \frac{1}{2} - \frac{\pi}{4} = \frac{1}{4}(6 - \pi)$.



49. By the Fundamental Theorem of Calculus, we know that $F(x) = \int_a^x t^2 \sin(t^2) dt$ is an antiderivative of $f(x) = x^2 \sin(x^2)$. This integral cannot be expressed in any simpler form. Since $\int_a^a f dt = 0$ for any a , we can take $a = 1$, and then $F(1) = 0$, as required. So $F(x) = \int_1^x t^2 \sin(t^2) dt$ is the desired function.

50. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus,

$$\begin{aligned}C'(x) &= \frac{d}{dx} \left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt \right] = \cos\left(\frac{\pi}{2}x^2\right). \text{ This is positive when } \frac{\pi}{2}x^2 \text{ is in the interval} \\ &((2n - \frac{1}{2})\pi, (2n + \frac{1}{2})\pi), n \text{ any integer. This implies that } (2n - \frac{1}{2})\pi < \frac{\pi}{2}x^2 < (2n + \frac{1}{2})\pi \Leftrightarrow \\ &0 \leq |x| \leq 1 \text{ or } \sqrt{4n-1} < |x| < \sqrt{4n+1}, n \text{ any positive integer. So } C \text{ is increasing on the intervals } [-1, 1], \\ &[\sqrt{3}, \sqrt{5}], [-\sqrt{5}, -\sqrt{3}], [\sqrt{7}, 3], [-3, -\sqrt{7}], \dots\end{aligned}$$

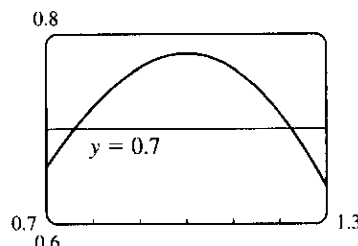
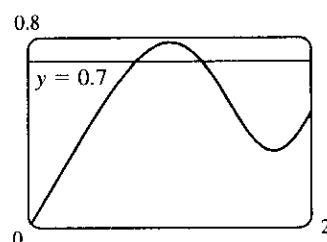
- (b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'' : $C'(x) = \cos(\frac{\pi}{2}x^2)$
 $\Rightarrow C''(x) = -\sin(\frac{\pi}{2}x^2)(\frac{\pi}{2} \cdot 2x) = -\pi x \sin(\frac{\pi}{2}x^2)$. For $x > 0$, this is positive where

$(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any positive integer $\Leftrightarrow \sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer.

Since there is a factor of $-x$ in C'' , the intervals of upward concavity for $x < 0$ are $(-\sqrt{2(2n+1)}, -2\sqrt{n})$,

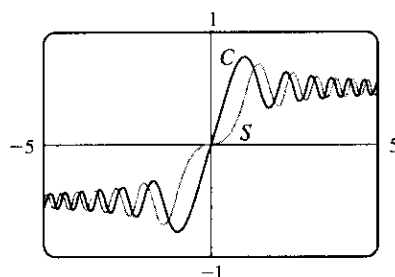
n any nonnegative integer. That is, C is concave upward on $(-\sqrt{2}, 0)$, $(\sqrt{2}, 2)$, $(-\sqrt{6}, -2)$, $(\sqrt{6}, 2\sqrt{2})$, $\dots$

(c)



From the graphs, we can determine that $\int_0^x \cos(\frac{\pi}{2}t^2) dt = 0.7$ at $x \approx 0.76$ and $x \approx 1.22$.

(d)



The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S 's flattens out near the origin, while C 's does not. Note that for $x > 0$, C is increasing where S is concave up, and C is decreasing where S is concave down. Similarly, S is increasing where C is concave down, and S is decreasing where C is concave up. For $x < 0$, these relationships are reversed; that is, C is increasing where S is concave down, and S is increasing where C is concave up. See Example 5.3.3 and Exercise 5.3.49 for a discussion of $S(x)$.

$$51. \int_0^x f(t) dt = x \sin x + \int_0^x \frac{f(t)}{1+t^2} dt \Rightarrow f(x) = x \cos x + \sin x + \frac{f(x)}{1+x^2} \text{ (by differentiation)} \Rightarrow$$

$$f(x) \left(1 - \frac{1}{1+x^2} \right) = x \cos x + \sin x \Rightarrow f(x) \left(\frac{x^2}{1+x^2} \right) = x \cos x + \sin x \Rightarrow$$

$$f(x) = \frac{1+x^2}{x^2} (x \cos x + \sin x)$$

52. From the given equation, $\int_a^x f(t) dt = \sin x - \frac{1}{2}$. Differentiating both sides using FTC1 gives $f(x) = \cos x$. We put $x = a$ into the first equation to get $0 = \sin a - \frac{1}{2}$, so $a = \frac{\pi}{6}$ satisfies the given equation.

53. Let $u = f(x)$ and $du = f'(x) dx$. So $2 \int_a^b f(x) f'(x) dx = 2 \int_{f(a)}^{f(b)} u du = [u^2]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

54. Let $F(x) = \int_2^x \sqrt{1+t^3} dt$. Then $F'(2) = \lim_{h \rightarrow 0} \frac{F(2+h) - F(2)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_2^{2+h} \sqrt{1+t^3} dt$, and

$$F'(x) = \sqrt{1+x^3}, \text{ so } \lim_{h \rightarrow 0} \frac{1}{h} \int_2^{2+h} \sqrt{1+t^3} dt = F'(2) = \sqrt{1+2^3} = \sqrt{9} = 3.$$

55. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x) dx = \int_1^0 f(u)(-du) = \int_0^1 f(u) du = \int_0^1 f(x) dx$.

$$56. \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\frac{1}{n} \right)^9 + \left(\frac{2}{n} \right)^9 + \left(\frac{3}{n} \right)^9 + \cdots + \left(\frac{n}{n} \right)^9 \right] = \lim_{n \rightarrow \infty} \frac{1-0}{n} \sum_{i=1}^n \left(\frac{i}{n} \right)^9 = \int_0^1 x^9 dx = \left[\frac{x^{10}}{10} \right]_0^1 = \frac{1}{10}$$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

□ PROBLEMS PLUS

1. Differentiating both sides of the equation $x \sin \pi x = \int_0^{x^2} f(t) dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin \pi x + \pi x \cos \pi x = 2xf(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi \cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi \cdot 1) = \frac{\pi}{2}$.

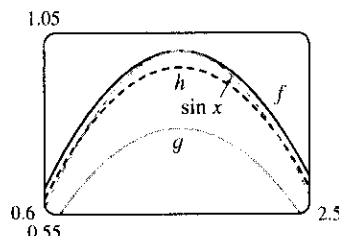
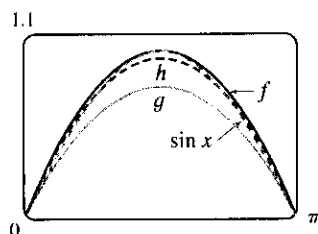
2. (a) Let $f(x) = ax^2 + bx + c$. $f(0) = f(\pi) = 0$, so we know that $f(0) = 0 \Rightarrow c = 0$, and $f(\pi) = 0 \Leftrightarrow a\pi^2 + b\pi = 0 \Leftrightarrow b = -a\pi$. So $f(x) = ax^2 - a\pi x$. Now we want the maximum value of $f(x)$ on $[0, \pi]$ to be the same as that of $\sin x$, that is, 1. So we find the value of x at which f has a maximum by differentiating and setting $f'(x) = 0 \Leftrightarrow 2ax - a\pi = 0 \Leftrightarrow x = \frac{\pi}{2}$. Now $f(\frac{\pi}{2}) = a(\frac{\pi}{2})^2 - a\pi(\frac{\pi}{2}) = -\frac{1}{4}\pi^2 a$. We set this equal to 1, in order to find a : $-\frac{1}{4}\pi^2 a = 1 \Leftrightarrow a = -\frac{4}{\pi^2} \Rightarrow b = -a\pi = \frac{4}{\pi}$. Thus, the desired function is $f(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x$.

Alternate solution (without calculus): Use $f(x) = ax(x - \pi)$.

- (b) Once again, $g(0) = g(\pi) = 0 \Rightarrow g(x) = ax^2 - a\pi x$. We want $g'(0) = [\frac{d}{dx}(\sin x)]_0 = \cos 0 = 1$. We calculated $g'(x)$ in part (a), so we set $g'(0) = 1 \Leftrightarrow 2a(0) - a\pi = 1 \Leftrightarrow a = -\frac{1}{\pi}$. We also want $g'(\pi) = [\frac{d}{dx}(\sin x)]_\pi = \cos \pi = -1$, so we check that $g'(\pi) = -1$ with $a = -\frac{1}{\pi}$: $g'(\pi) = 2(-\frac{1}{\pi})\pi - (-\frac{1}{\pi})\pi = -1$. Thus, the desired function is $g(x) = -\frac{1}{\pi}x^2 + x$.

- (c) Again, $h(x) = ax^2 - a\pi x$. Now we want the area under the curves of $h(x)$ and $\sin x$ to be the same; that is, $\int_0^\pi h(x) dx = \int_0^\pi \sin x dx = [-\cos x]_0^\pi = (1) - (-1) = 2$. We integrate h between 0 and π and set the result equal to 2: $\int_0^\pi (ax^2 - a\pi x) dx = [\frac{1}{3}ax^3 - \frac{1}{2}a\pi x^2]_0^\pi = \frac{1}{3}a\pi^3 - \frac{1}{2}a\pi^3 = -\frac{1}{6}a\pi^3 = 2 \Leftrightarrow a = -12/\pi^3$. Thus, the desired function is $h(x) = -\frac{12}{\pi^3}x^2 + \frac{12}{\pi^2}x$.

(d)



3. For $1 \leq x \leq 2$, we have $x^4 \leq 2^4 = 16$, so $1 + x^4 \leq 17$ and $\frac{1}{1+x^4} \geq \frac{1}{17}$. Thus,

$$\int_1^2 \frac{1}{1+x^4} dx \geq \int_1^2 \frac{1}{17} dx = \frac{1}{17}. \text{ Also } 1 + x^4 > x^4 \text{ for } 1 \leq x \leq 2, \text{ so } \frac{1}{1+x^4} < \frac{1}{x^4} \text{ and}$$

$$\int_1^2 \frac{1}{1+x^4} dx < \int_1^2 x^{-4} dx = \left[\frac{x^{-3}}{-3} \right]_1^2 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}. \text{ Thus, we have the estimate}$$

$$\frac{1}{17} \leq \int_1^2 \frac{1}{1+x^4} dx \leq \frac{7}{24}.$$

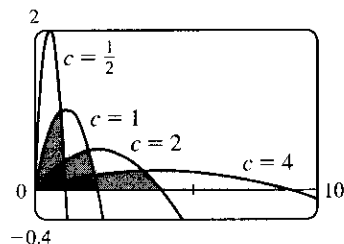
4. By FTC2, $\int_0^1 f'(x) dx = f(1) - f(0) = 1 - 0 = 1$.

5. Such a function cannot exist. $f'(x) > 3$ for all x means that f is differentiable (and hence continuous) for all x . So by FTC2, $\int_1^4 f'(x) dx = f(4) - f(1) = 7 - (-1) = 8$. However, if $f'(x) > 3$ for all x , then $\int_1^4 f'(x) dx \geq 3 \cdot (4 - 1) = 9$ by Comparison Property 8 in Section 5.2.

Another solution: By the Mean Value Theorem, there exists a number $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{7 - (-1)}{3} = \frac{8}{3} \Rightarrow 8 = 3f'(c). \text{ But } f'(x) > 3 \Rightarrow 3f'(c) > 9, \text{ so such a function cannot exist.}$$

6. (a)

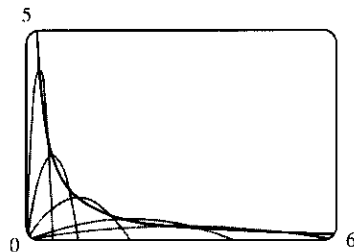


From the graph of $f(x) = \frac{2cx - x^2}{c^3}$, it appears that the areas are equal; that is, the area enclosed is independent of c .

- (b) We first find the x -intercepts of the curve, to determine the limits of integration: $y = 0 \Leftrightarrow 2cx - x^2 = 0 \Leftrightarrow x = 0$ or $x = 2c$. Now we integrate the function between these limits to find the enclosed area:

$$A = \int_0^{2c} \frac{2cx - x^2}{c^3} dx = \frac{1}{c^3} \left[cx^2 - \frac{1}{3}x^3 \right]_0^{2c} = \frac{1}{c^3} \left[c(2c)^2 - \frac{1}{3}(2c)^3 \right] = \frac{1}{c^3} \left[4c^3 - \frac{8}{3}c^3 \right] = \frac{4}{3}, \text{ a constant.}$$

(c)



The vertices of the family of parabolas seem to determine a branch of a hyperbola.

- (d) For a particular c , the vertex is the point where the maximum occurs. We have seen that the x -intercepts are 0 and $2c$, so by symmetry, the maximum occurs at $x = c$, and its value is $\frac{2c(c) - c^2}{c^3} = \frac{1}{c}$. So we are interested in the curve consisting of all points of the form $\left(c, \frac{1}{c}\right)$, $c > 0$. This is the part of the hyperbola $y = 1/x$ lying in the first quadrant.

7. $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} dt$, where $g(x) = \int_0^{\cos x} [1 + \sin(t^2)] dt$. Using FTC1 and the Chain Rule (twice) we have $f'(x) = \frac{1}{\sqrt{1+[g(x)]^3}} g'(x) = \frac{1}{\sqrt{1+[g(x)]^3}} [1 + \sin(\cos^2 x)] (-\sin x)$. Now $g\left(\frac{\pi}{2}\right) = \int_0^0 [1 + \sin(t^2)] dt = 0$, so $f'\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{1+0}} (1 + \sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1$.
8. If $f(x) = \int_0^x x^2 \sin(t^2) dt = x^2 \int_0^x \sin(t^2) dt$, then $f'(x) = x^2 \sin(x^2) + 2x \int_0^x \sin(t^2) dt$, by the Product Rule and FTC1.

9. $f(x) = 2 + x - x^2 = (-x + 2)(x + 1) = 0 \Leftrightarrow x = 2$ or $x = -1$. $f(x) \geq 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2) dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \geq 0$ in $[-1, 2]$.)

10. This sum can be interpreted as a Riemann sum, with the right endpoints of the subintervals as sample points and with $a = 0$, $b = 10,000$, and $f(x) = \sqrt{x}$. So we approximate

$$\sum_{i=1}^{10,000} \sqrt{i} \approx \lim_{n \rightarrow \infty} \frac{10,000}{n} \sum_{i=1}^n \sqrt{\frac{10,000i}{n}} = \int_0^{10,000} \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_0^{10,000} = \frac{2}{3} (1,000,000) \approx 666,667.$$

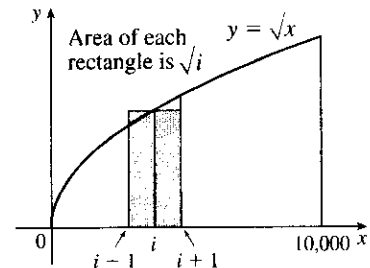
Alternate method: We can use graphical methods as follows:

From the figure we see that $\int_{i-1}^i \sqrt{x} dx < \sqrt{i} < \int_i^{i+1} \sqrt{x} dx$, so

$$\int_0^{10,000} \sqrt{x} dx < \sum_{i=1}^{10,000} \sqrt{i} < \int_1^{10,001} \sqrt{x} dx. \text{ Since}$$

$$\int \sqrt{x} dx = \frac{2}{3} x^{3/2} + C, \text{ we get } \int_0^{10,000} \sqrt{x} dx = 666,666.\bar{6} \text{ and}$$

$$\int_1^{10,001} \sqrt{x} dx = \frac{2}{3} \left[(10,001)^{3/2} - 1 \right] \approx 666,766.$$



Hence, $666,666.\bar{6} < \sum_{i=1}^{10,000} \sqrt{i} < 666,766$. We can estimate the sum by averaging these bounds:

$$\sum_{i=1}^{10,000} \sqrt{i} \approx \frac{666,666.\bar{6} + 666,766}{2} \approx 666,716. \text{ The actual value is about } 666,716.46.$$

11. (a) We can split the integral $\int_0^n \lfloor x \rfloor dx$ into the sum $\sum_{i=1}^n \left[\int_{i-1}^i \lfloor x \rfloor dx \right]$. But on each of the intervals $[i-1, i)$ of

integration, $\lfloor x \rfloor$ is a constant function, namely $i-1$. So the i th integral in the sum is equal to

$$(i-1)[i - (i-1)] = (i-1). \text{ So the original integral is equal to } \sum_{i=1}^n (i-1) = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}.$$

- (b) We can write $\int_a^b \lfloor x \rfloor dx = \int_0^b \lfloor x \rfloor dx - \int_0^a \lfloor x \rfloor dx$.

Now $\int_0^b \lfloor x \rfloor dx = \int_0^{\lfloor b \rfloor} \lfloor x \rfloor dx + \int_{\lfloor b \rfloor}^b \lfloor x \rfloor dx$. The first of these integrals is equal to $\frac{1}{2}(\lfloor b \rfloor - 1)\lfloor b \rfloor$, by part (a),

and since $\lfloor x \rfloor = \lfloor b \rfloor$ on $[\lfloor b \rfloor, b]$, the second integral is just $\lfloor b \rfloor(b - \lfloor b \rfloor)$. So

$$\int_0^b \lfloor x \rfloor dx = \frac{1}{2}(\lfloor b \rfloor - 1)\lfloor b \rfloor + \lfloor b \rfloor(b - \lfloor b \rfloor) = \frac{1}{2}\lfloor b \rfloor(2b - \lfloor b \rfloor - 1) \text{ and similarly}$$

$$\int_0^a \lfloor x \rfloor dx = \frac{1}{2}\lfloor a \rfloor(2a - \lfloor a \rfloor - 1). \text{ Therefore, } \int_a^b \lfloor x \rfloor dx = \frac{1}{2}\lfloor b \rfloor(2b - \lfloor b \rfloor - 1) - \frac{1}{2}\lfloor a \rfloor(2a - \lfloor a \rfloor - 1).$$

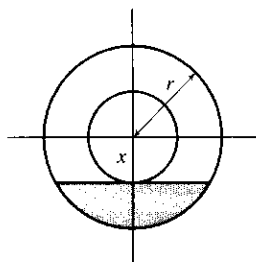
12. By FTC1, $\frac{d}{dx} \int_1^{\sin x} \sqrt{1+u^4} du = \sqrt{1+\sin^4 x}$. Again using FTC1,

$$\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4} du \right) dt = \frac{d}{dx} \int_1^{\sin x} \sqrt{1+u^4} du = \sqrt{1+\sin^4 x} \cos x.$$

13. Differentiating the equation $\int_0^x f(t) dt = [f(x)]^2$ using FTC1 gives $f(x) = 2f(x)f'(x) \Rightarrow$

$f(x)[2f'(x) - 1] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. $f'(x) = \frac{1}{2} \Rightarrow f(x) = \frac{1}{2}x + C$. To find C we substitute into the original equation to get $\int_0^x (\frac{1}{2}t + C) dt = (\frac{1}{2}x + C)^2 \Leftrightarrow \frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C = 0$, so $f(x) = \frac{1}{2}x$. Therefore, $f(x) = 0$ or $f(x) = \frac{1}{2}x$.

14.



Let x be the distance between the center of the disk and the surface of the liquid.

The wetted circular region has area $\pi r^2 - \pi x^2$ while the unexposed wetted region (shaded in the diagram) has area $2 \int_x^r \sqrt{r^2 - t^2} dt$, so the exposed wetted region has area $A(x) = \pi r^2 - \pi x^2 - 2 \int_x^r \sqrt{r^2 - t^2} dt$, $0 \leq x \leq r$. By FTC1, we have $A'(x) = -2\pi x + 2\sqrt{r^2 - x^2}$.

$$\text{Now } A'(x) > 0 \Rightarrow -2\pi x + 2\sqrt{r^2 - x^2} > 0 \Rightarrow \sqrt{r^2 - x^2} > \pi x \Rightarrow r^2 - x^2 > \pi^2 x^2 \Rightarrow$$

$$r^2 > \pi^2 x^2 + x^2 \Rightarrow r^2 > x^2(\pi^2 + 1) \Rightarrow x^2 < \frac{r^2}{\pi^2 + 1} \Rightarrow x < \frac{r}{\sqrt{\pi^2 + 1}}, \text{ and we'll call this value } x^*.$$

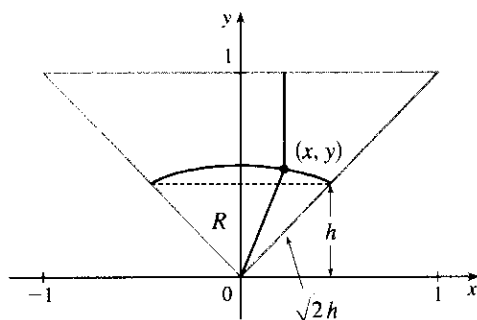
Since $A'(x) > 0$ for $0 < x < x^*$ and $A'(x) < 0$ for $x^* < x < r$, we have an absolute maximum when $x = x^*$.

15. Note that $\frac{d}{dx} \left(\int_0^x \left[\int_0^u f(t) dt \right] du \right) = \int_0^x f(t) dt$ by FTC1, while

$$\begin{aligned} \frac{d}{dx} \left[\int_0^x f(u)(x-u) du \right] &= \frac{d}{dx} \left[x \int_0^x f(u) du \right] - \frac{d}{dx} \left[\int_0^x f(u)u du \right] \\ &= \int_0^x f(u) du + xf(x) - f(x)x = \int_0^x f(u) du \end{aligned}$$

Hence, $\int_0^x f(u)(x-u) du = \int_0^x \left[\int_0^u f(t) dt \right] du + C$. Setting $x = 0$ gives $C = 0$.

16.



We restrict our attention to the triangle shown. A point in this triangle is closer to the side shown than to any other side, so if we find the area of the region R consisting of all points in the triangle that are closer to the center than to that side, we can multiply this area by 4 to find the total area. We find the equation of the set of points which are equidistant from the center and the side: the distance of the point (x, y) from the side is $1 - y$, and its distance from the center is $\sqrt{x^2 + y^2}$.

So the distances are equal if $\sqrt{x^2 + y^2} = 1 - y \Leftrightarrow x^2 + y^2 = 1 - 2y + y^2 \Leftrightarrow y = \frac{1}{2}(1 - x^2)$. Note that the area we are interested in is equal to the area of a triangle plus a crescent-shaped area. To find these areas, we

have to find the y -coordinate h of the horizontal line separating them. From the diagram, $1 - h = \sqrt{2}h \Leftrightarrow$

$h = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$. We calculate the areas in terms of h , and substitute afterward.

The area of the triangle is $\frac{1}{2}(2h)(h) = h^2$, and the area of the crescent-shaped section is

$\int_{-h}^h \left[\frac{1}{2}(1-x^2) - h \right] dx = 2 \int_0^h \left(\frac{1}{2} - h - \frac{1}{2}x^2 \right) dx = 2 \left[\left(\frac{1}{2} - h \right)x - \frac{1}{6}x^3 \right]_0^h = h - 2h^2 - \frac{1}{3}h^3$. So the area of

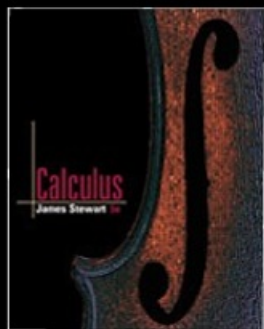
the whole region is

$$\begin{aligned} 4 \left[\left(h - 2h^2 - \frac{1}{3}h^3 \right) + h^2 \right] &= 4h \left(1 - h - \frac{1}{3}h^2 \right) = 4(\sqrt{2} - 1) \left[1 - (\sqrt{2} - 1) - \frac{1}{3}(\sqrt{2} - 1)^2 \right] \\ &= 4(\sqrt{2} - 1) \left(1 - \frac{1}{3}\sqrt{2} \right) = \frac{4}{3}(4\sqrt{2} - 5) \end{aligned}$$

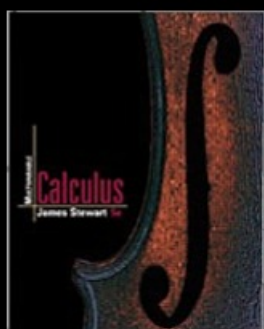
$$\begin{aligned} 17. \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \cdots + \sqrt{\frac{n}{n+n}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+2/n}} + \cdots + \frac{1}{\sqrt{1+1}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) \quad \left[\text{where } f(x) = \frac{1}{\sqrt{1+x}} \right] \\ &= \int_0^1 \frac{1}{\sqrt{1+x}} dx = [2\sqrt{1+x}]_0^1 = 2(\sqrt{2} - 1) \end{aligned}$$

Chapter 6

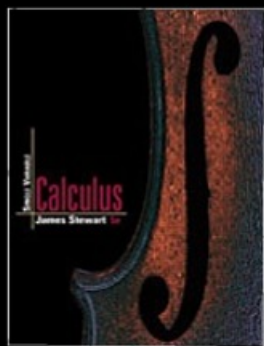
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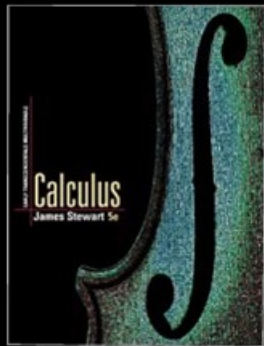
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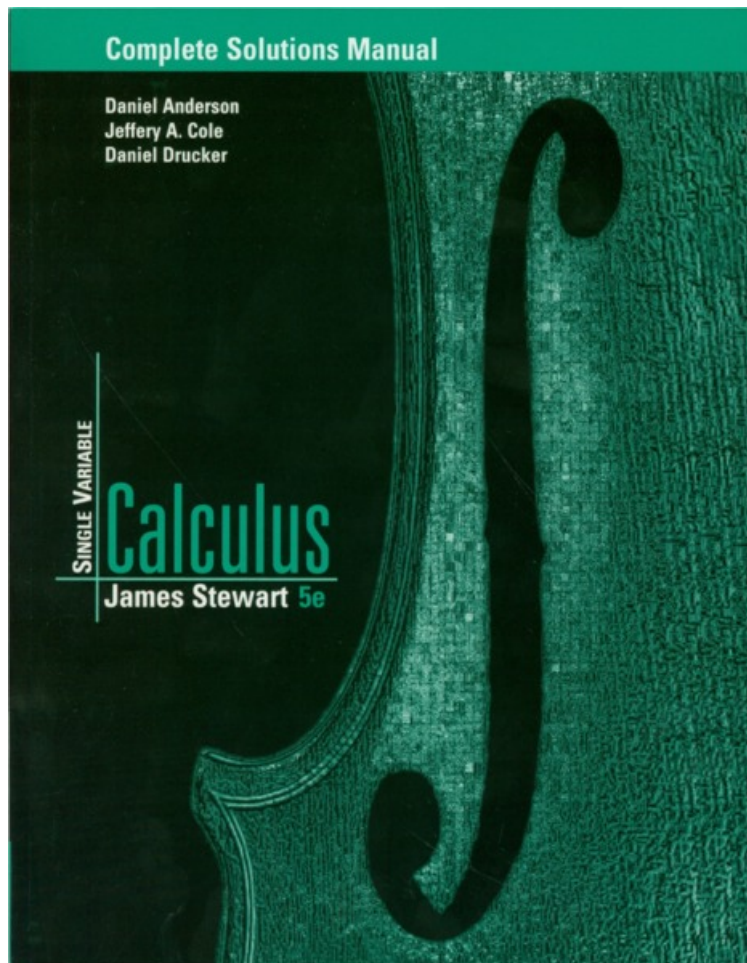
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6 □ APPLICATIONS OF INTEGRATION

6.1 Areas between Curves

$$1. A = \int_{x=0}^{x=4} (y_T - y_B) dx = \int_0^4 [(5x - x^2) - x] dx = \int_0^4 (4x - x^2) dx$$

$$= [2x^2 - \frac{1}{3}x^3]_0^4 = (32 - \frac{64}{3}) - (0) = \frac{32}{3}$$

$$2. A = \int_0^6 [2x - (x^2 - 4x)] dx = \int_0^6 (6x - x^2) dx = [3x^2 - \frac{1}{3}x^3]_0^6 = 108 - 72 = 36$$

$$3. A = \int_{y=0}^{y=1} (x_R - x_L) dy = \int_0^1 [\sqrt{y} - (y^2 - 1)] dy = \int_0^1 (y^{1/2} - y^2 + 1) dy$$

$$= [\frac{2}{3}y^{3/2} - \frac{1}{3}y^3 + y]_0^1 = (\frac{2}{3} - \frac{1}{3} + 1) - (0) = \frac{4}{3}$$

$$4. A = \int_0^3 [(2y - y^2) - (y^2 - 4y)] dy = \int_0^3 (-2y^2 + 6y) dy$$

$$= [-\frac{2}{3}y^3 + 3y^2]_0^3 = (-18 + 27) - 0 = 9$$

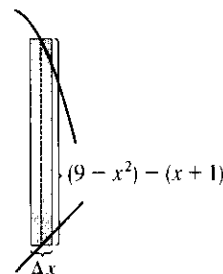
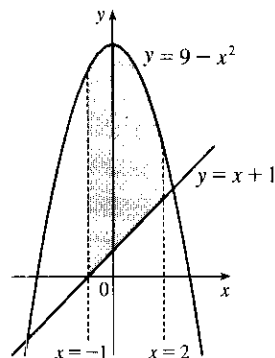
$$5. A = \int_{-1}^2 [(9 - x^2) - (x + 1)] dx$$

$$= \int_{-1}^2 (8 - x - x^2) dx$$

$$= [8x - \frac{x^2}{2} - \frac{x^3}{3}]_{-1}^2$$

$$= (16 - 2 - \frac{8}{3}) - (-8 - \frac{1}{2} + \frac{1}{3})$$

$$= 22 - 3 + \frac{1}{2} = \frac{39}{2}$$

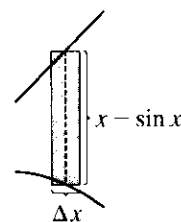
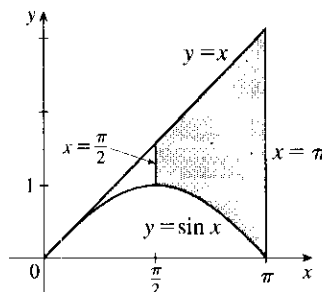


$$6. A = \int_{\pi/2}^{\pi} (x - \sin x) dx$$

$$= [\frac{x^2}{2} + \cos x]_{\pi/2}^{\pi}$$

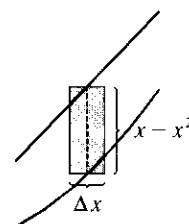
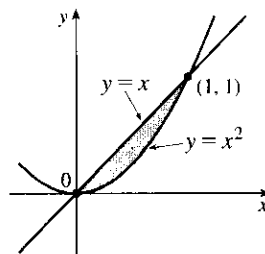
$$= (\frac{\pi^2}{2} - 1) - (\frac{\pi^2}{8} + 0)$$

$$= \frac{3\pi^2}{8} - 1$$

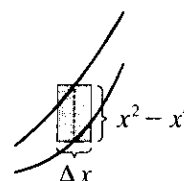
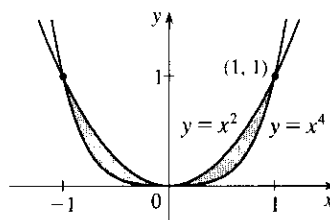


7. The curves intersect when $x = x^2 \Rightarrow x^2 - x = 0 \Leftrightarrow x(x - 1) = 0 \Leftrightarrow x = 0, 1$.

$$\begin{aligned} A &= \int_0^1 (x - x^2) dx \\ &= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

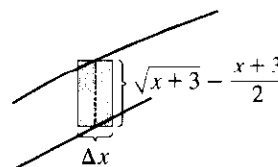
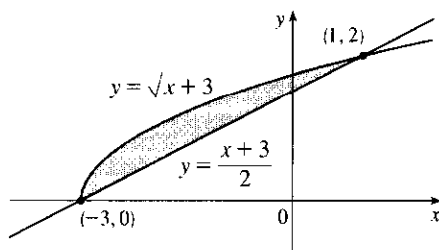


$$\begin{aligned} 8. A &= \int_{-1}^1 (x^2 - x^4) dx \\ &= 2 \int_0^1 (x^2 - x^4) dx \\ &= 2 \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 \\ &= 2 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4}{15} \end{aligned}$$



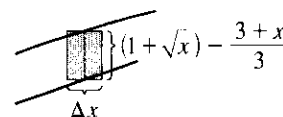
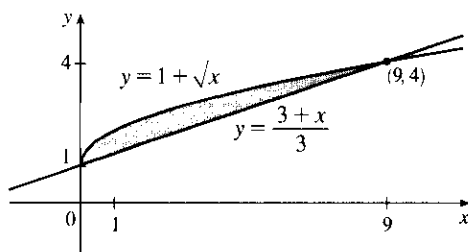
9. First find the points of intersection: $\sqrt{x+3} = \frac{x+3}{2} \Rightarrow (\sqrt{x+3})^2 = \left(\frac{x+3}{2}\right)^2 \Rightarrow x+3 = \frac{1}{4}(x+3)^2$
 $\Rightarrow 4(x+3) - (x+3)^2 = 0 \Rightarrow (x+3)[4 - (x+3)] = 0 \Rightarrow (x+3)(1-x) = 0 \Rightarrow x = -3 \text{ or } 1$. So

$$A = \int_{-3}^1 \left(\sqrt{x+3} - \frac{x+3}{2} \right) dx = \left[\frac{2}{3}(x+3)^{3/2} - \frac{(x+3)^2}{4} \right]_{-3}^1 = \left(\frac{16}{3} - 4 \right) - (0 - 0) = \frac{4}{3}$$

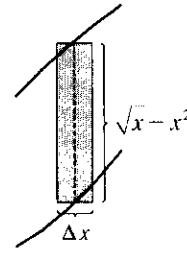
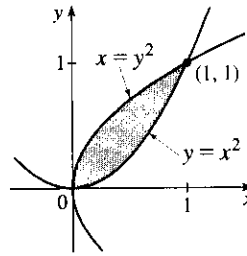


10. $1 + \sqrt{x} = \frac{3+x}{3} = 1 + \frac{x}{3} \Rightarrow \sqrt{x} = \frac{x}{3} \Rightarrow x = \frac{x^2}{9} \Rightarrow 9x - x^2 = 0 \Rightarrow x(9-x) = 0 \Rightarrow x = 0$
 or 9, so

$$\begin{aligned} A &= \int_0^9 \left[(1 + \sqrt{x}) - \left(\frac{3+x}{3} \right) \right] dx = \int_0^9 \left[(1 + \sqrt{x}) - \left(1 + \frac{x}{3} \right) \right] dx \\ &= \int_0^9 \left(\sqrt{x} - \frac{1}{3}x \right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2 \right]_0^9 = 18 - \frac{27}{2} = \frac{9}{2} \end{aligned}$$

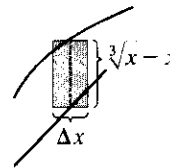
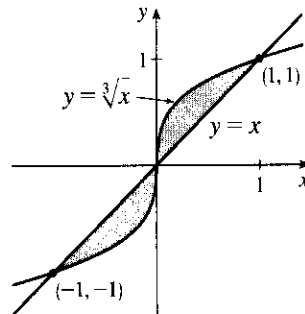


$$\begin{aligned}
 11. A &= \int_0^1 (\sqrt{x} - x^2) dx \\
 &= \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1 \\
 &= \frac{2}{3} - \frac{1}{3} \\
 &= \frac{1}{3}
 \end{aligned}$$



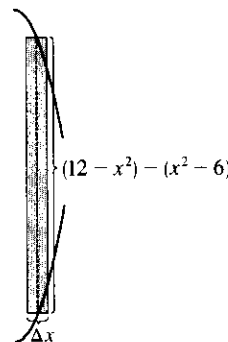
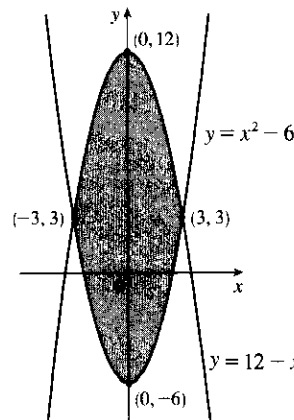
$$\begin{aligned}
 12. x = \sqrt[3]{x} &\Rightarrow x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x(x+1)(x-1) = 0 \Rightarrow \\
 &x = -1, 0, \text{ or } 1, \text{ so}
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_{-1}^1 |\sqrt[3]{x} - x| dx = \int_{-1}^0 (x - \sqrt[3]{x}) dx + \int_0^1 (\sqrt[3]{x} - x) dx = 2 \int_0^1 (x^{1/3} - x) dx \quad [\text{by symmetry}] \\
 &= 2 \left[\frac{3}{4} x^{4/3} - \frac{1}{2} x^2 \right]_0^1 = 2 \left(\frac{3}{4} - \frac{1}{2} \right) = \frac{1}{2}
 \end{aligned}$$



$$13. 12 - x^2 = x^2 - 6 \Leftrightarrow 2x^2 = 18 \Leftrightarrow x^2 = 9 \Leftrightarrow x = \pm 3, \text{ so}$$

$$\begin{aligned}
 A &= \int_{-3}^3 [(12 - x^2) - (x^2 - 6)] dx = 2 \int_0^3 (18 - 2x^2) dx \quad [\text{by symmetry}] \\
 &= 2 \left[18x - \frac{2}{3} x^3 \right]_0^3 = 2[(54 - 18) - 0] = 2(36) = 72
 \end{aligned}$$

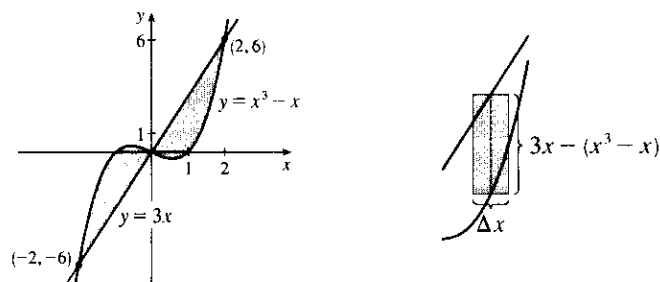


14. $x^3 - x = 3x \Rightarrow x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x(x+2)(x-2) = 0 \Rightarrow x = 0, -2, \text{ or } 2.$

By symmetry,

$$A = \int_{-2}^2 |3x - (x^3 - x)| dx = 2 \int_0^2 [3x - (x^3 - x)] dx = 2 \int_0^2 (4x - x^3) dx = 2 \left[2x^2 - \frac{1}{4}x^4 \right]_0^2$$

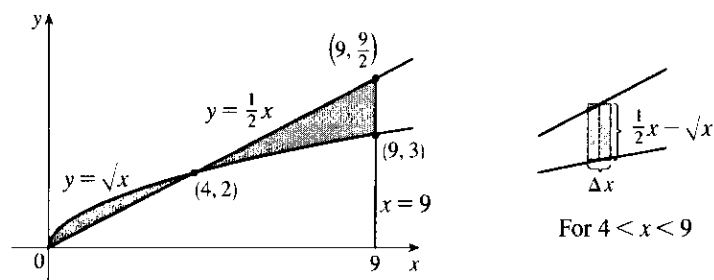
$$= 2(8 - 4) = 8$$



15. $\frac{1}{2}x = \sqrt{x} \Rightarrow \frac{1}{4}x^2 = x \Rightarrow x^2 - 4x = 0 \Rightarrow x(x - 4) = 0 \Rightarrow x = 0 \text{ or } 4, \text{ so}$

$$A = \int_0^4 (\sqrt{x} - \frac{1}{2}x) dx + \int_4^9 (\frac{1}{2}x - \sqrt{x}) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2 \right]_0^4 + \left[\frac{1}{4}x^2 - \frac{2}{3}x^{3/2} \right]_4^9$$

$$= \left[\left(\frac{16}{3} - 4 \right) - 0 \right] + \left[\left(\frac{81}{4} - 18 \right) - \left(4 - \frac{16}{3} \right) \right] = \frac{81}{4} + \frac{32}{3} - 26 = \frac{59}{12}$$

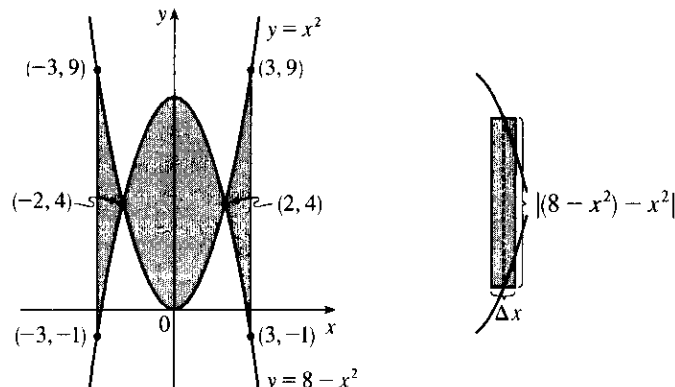


For $4 < x < 9$

16. $A = \int_{-3}^3 |(8 - x^2) - x^2| dx = 2 \int_0^3 |8 - 2x^2| dx = 2 \int_0^2 (8 - 2x^2) dx + 2 \int_2^3 (2x^2 - 8) dx$

$$= 2 \left[8x - \frac{2}{3}x^3 \right]_0^2 + 2 \left[\frac{2}{3}x^3 - 8x \right]_2^3 = 2 \left[\left(16 - \frac{16}{3} \right) - 0 \right] + 2 \left[(18 - 24) - \left(\frac{16}{3} - 16 \right) \right]$$

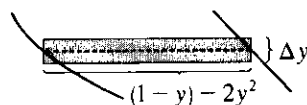
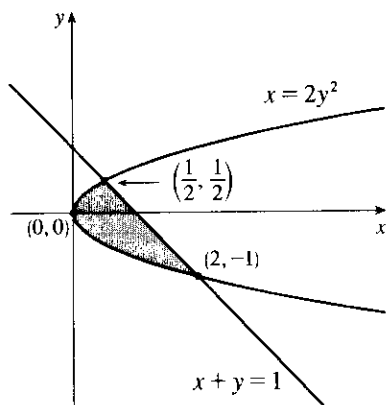
$$= 32 - \frac{32}{3} + 20 - \frac{32}{3} = 52 - \frac{64}{3} = \frac{92}{3}$$



17. $2y^2 = 1 - y \Leftrightarrow 2y^2 + y - 1 = 0 \Leftrightarrow (2y - 1)(y + 1) = 0 \Leftrightarrow y = \frac{1}{2} \text{ or } -1$, so $x = \frac{1}{2}$ or 2 and

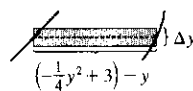
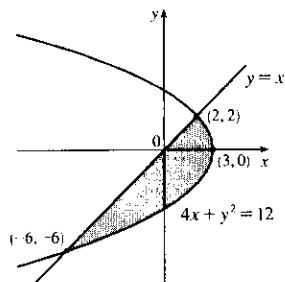
$$A = \int_{-1}^{1/2} [(1 - y) - 2y^2] dy = \int_{-1}^{1/2} (1 - y - 2y^2) dy = \left[y - \frac{1}{2}y^2 - \frac{2}{3}y^3 \right]_{-1}^{1/2}$$

$$= \left(\frac{1}{2} - \frac{1}{8} - \frac{1}{12} \right) - \left(-1 - \frac{1}{2} + \frac{2}{3} \right) = \frac{7}{24} - \left(-\frac{5}{6} \right) = \frac{7}{24} + \frac{20}{24} = \frac{27}{24} = \frac{9}{8}$$



18. $4x + x^2 = 12 \Leftrightarrow (x + 6)(x - 2) = 0 \Leftrightarrow x = -6 \text{ or } x = 2$, so $y = -6$ or $y = 2$ and

$$A = \int_{-6}^2 \left[\left(-\frac{1}{4}y^2 + 3 \right) - y \right] dy = \left[-\frac{1}{12}y^3 - \frac{1}{2}y^2 + 3y \right]_{-6}^2 = \left(-\frac{2}{3} - 2 + 6 \right) - (18 - 18 - 18) = 22 - \frac{2}{3} = \frac{64}{3}.$$



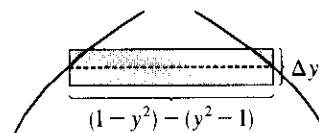
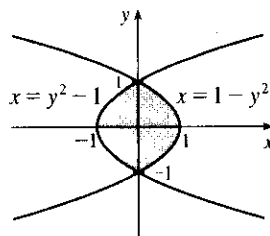
19. The curves intersect when $1 - y^2 = y^2 - 1 \Leftrightarrow 2 = 2y^2 \Leftrightarrow y^2 = 1 \Leftrightarrow y = \pm 1$.

$$A = \int_{-1}^1 [(1 - y^2) - (y^2 - 1)] dy$$

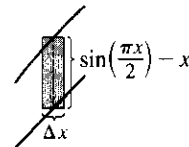
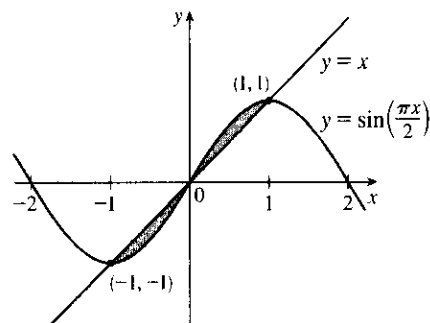
$$= \int_{-1}^1 2(1 - y^2) dy$$

$$= 2 \cdot 2 \int_0^1 (1 - y^2) dy$$

$$= 4 \left[y - \frac{1}{3}y^3 \right]_0^1 = 4 \left(1 - \frac{1}{3} \right) = \frac{8}{3}$$

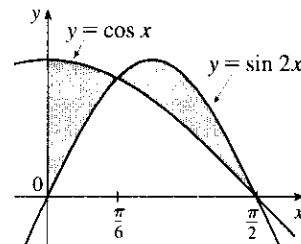


$$20. A = 2 \int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x \right] dx = 2 \left[-\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2} \right]_0^1 = 2 \left[\left(0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} - 0\right) \right] = \frac{4}{\pi} - 1$$



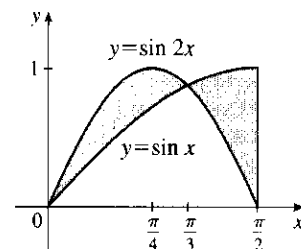
21. Notice that $\cos x = \sin 2x = 2 \sin x \cos x \Leftrightarrow$
 $2 \sin x \cos x - \cos x = 0 \Leftrightarrow \cos x (2 \sin x - 1) = 0 \Leftrightarrow$
 $2 \sin x = 1 \text{ or } \cos x = 0 \Leftrightarrow x = \frac{\pi}{6} \text{ or } \frac{\pi}{2}.$

$$\begin{aligned} A &= \int_0^{\pi/6} (\cos x - \sin 2x) dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x) dx \\ &= \left[\sin x + \frac{1}{2} \cos 2x \right]_0^{\pi/6} + \left[-\frac{1}{2} \cos 2x - \sin x \right]_{\pi/6}^{\pi/2} \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} - \left(0 + \frac{1}{2} \cdot 1\right) + \left(\frac{1}{2} - 1\right) - \left(-\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2} \end{aligned}$$



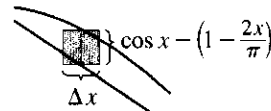
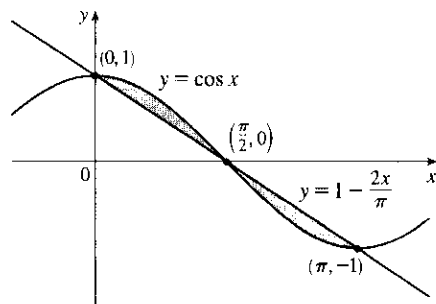
22. $\sin x = \sin 2x = 2 \sin x \cos x$ when $\sin x = 0$ and when $\cos x = \frac{1}{2}$;
 that is, when $x = 0$ or $\frac{\pi}{3}$.

$$\begin{aligned} A &= \int_0^{\pi/3} (\sin 2x - \sin x) dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x) dx \\ &= \left[-\frac{1}{2} \cos 2x + \cos x \right]_0^{\pi/3} + \left[\frac{1}{2} \cos 2x - \cos x \right]_{\pi/3}^{\pi/2} \\ &= \left[-\frac{1}{2} \left(-\frac{1}{2}\right) + \frac{1}{2} \right] - \left(-\frac{1}{2} + 1\right) \\ &\quad + \left(-\frac{1}{2} - 0\right) - \left[\frac{1}{2} \left(-\frac{1}{2}\right) - \frac{1}{2}\right] = \frac{1}{2} \end{aligned}$$



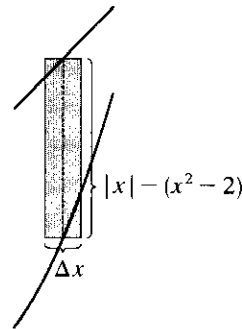
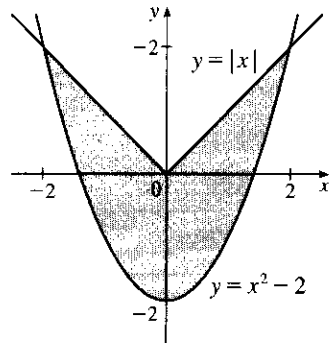
23. From the graph, we see that the curves intersect at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$. By symmetry,

$$\begin{aligned} A &= \int_0^{\pi} \left| \cos x - \left(1 - \frac{2x}{\pi}\right) \right| dx = 2 \int_0^{\pi/2} \left[\cos x - \left(1 - \frac{2x}{\pi}\right) \right] dx = 2 \int_0^{\pi/2} \left(\cos x - 1 + \frac{2x}{\pi} \right) dx \\ &= 2 \left[\sin x - x + \frac{1}{\pi} x^2 \right]_0^{\pi/2} = 2 \left[\left(1 - \frac{\pi}{2} + \frac{1}{\pi} \cdot \frac{\pi^2}{4}\right) - 0 \right] = 2 \left(1 - \frac{\pi}{2} + \frac{\pi}{4}\right) = 2 - \frac{\pi}{2} \end{aligned}$$



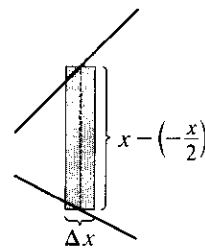
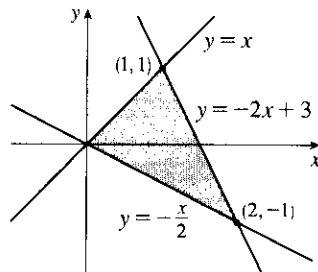
24. For $x > 0$, $x = x^2 - 2 \Rightarrow 0 = x^2 - x - 2 \Rightarrow 0 = (x - 2)(x + 1) \Rightarrow x = 2$. By symmetry,

$$\begin{aligned} \int_{-2}^2 [|x| - (x^2 - 2)] dx &= 2 \int_0^2 [x - (x^2 - 2)] dx = 2 \int_0^2 (x - x^2 + 2) dx = 2 \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x \right]_0^2 \\ &= 2 \left(2 - \frac{8}{3} + 4 \right) = \frac{20}{3} \end{aligned}$$

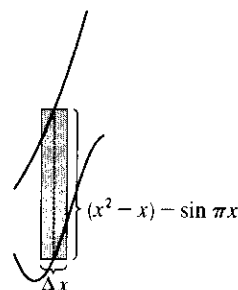
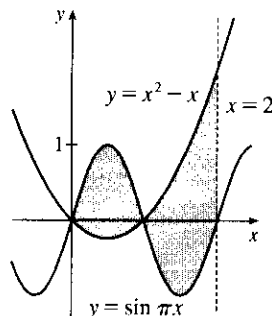


25. Graph the three functions $y = x$, $y = -\frac{1}{2}x$, and $y = -2x + 3$; then determine the points of intersection: $(0, 0)$, $(1, 1)$, and $(2, -1)$.

$$\begin{aligned} A &= \int_0^1 [x - (-\frac{1}{2}x)] dx + \int_1^2 [(3 - 2x) - (-\frac{1}{2}x)] dx = \int_0^1 \frac{3}{2}x dx + \int_1^2 (3 - \frac{3}{2}x) dx \\ &= [\frac{3}{4}x^2]_0^1 + [3x - \frac{3}{4}x^2]_1^2 = (\frac{3}{4} - 0) + (6 - 3) - (3 - \frac{3}{4}) = \frac{3}{2} \end{aligned}$$

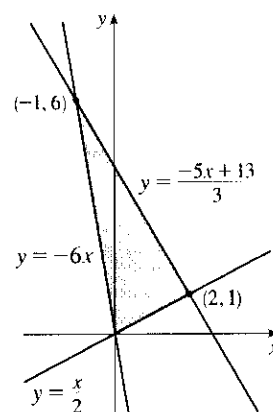


$$\begin{aligned} 26. A &= \int_0^1 [\sin \pi x - (x^2 - x)] dx + \int_1^2 [(x^2 - x) - \sin \pi x] dx \\ &= \left[-\frac{1}{\pi} \cos \pi x - \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 + \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{\pi} \cos \pi x \right]_1^2 \\ &= \left(\frac{1}{\pi} - \frac{1}{3} + \frac{1}{2} \right) - \left(-\frac{1}{\pi} \right) + \left(\frac{8}{3} - 2 + \frac{1}{\pi} \right) - \left(\frac{1}{3} - \frac{1}{2} - \frac{1}{\pi} \right) \\ &= \frac{4}{\pi} + 1 \end{aligned}$$

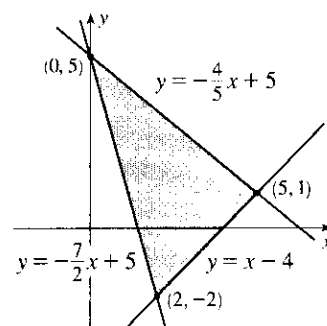


27. An equation of the line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$; through $(0, 0)$ and $(-1, 6)$ is $y = -6x$; through $(2, 1)$ and $(-1, 6)$ is $y = -\frac{5}{3}x + \frac{13}{3}$.

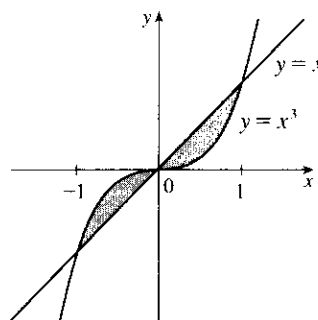
$$\begin{aligned} A &= \int_{-1}^0 \left[\left(-\frac{5}{3}x + \frac{13}{3} \right) - (-6x) \right] dx + \int_0^2 \left[\left(-\frac{5}{3}x + \frac{13}{3} \right) - \frac{1}{2}x \right] dx \\ &= \int_{-1}^0 \left(\frac{13}{3}x + \frac{13}{3} \right) dx + \int_0^2 \left(-\frac{13}{6}x + \frac{13}{3} \right) dx \\ &= \frac{13}{3} \int_{-1}^0 (x+1) dx + \frac{13}{3} \int_0^2 \left(-\frac{1}{2}x + 1 \right) dx \\ &= \frac{13}{3} \left[\frac{1}{2}x^2 + x \right]_{-1}^0 + \frac{13}{3} \left[-\frac{1}{4}x^2 + x \right]_0^2 \\ &= \frac{13}{3} \left[0 - \left(\frac{1}{2} - 1 \right) \right] + \frac{13}{3} [(-1 + 2) - 0] = \frac{13}{3} \cdot \frac{1}{2} + \frac{13}{3} \cdot 1 = \frac{13}{2} \end{aligned}$$



$$\begin{aligned} 28. A &= \int_0^2 \left[\left(-\frac{4}{5}x + 5 \right) - \left(-\frac{7}{2}x + 5 \right) \right] dx + \int_2^5 \left[\left(-\frac{4}{5}x + 5 \right) - (x - 4) \right] dx \\ &= \int_0^2 \frac{27}{10}x dx + \int_2^5 \left(-\frac{9}{5}x + 9 \right) dx \\ &= \left[\frac{27}{20}x^2 \right]_0^2 + \left[-\frac{9}{10}x^2 + 9x \right]_2^5 \\ &= \left(\frac{27}{5} - 0 \right) + \left(-\frac{45}{2} + 45 \right) - \left(-\frac{18}{5} + 18 \right) = \frac{27}{2} \end{aligned}$$

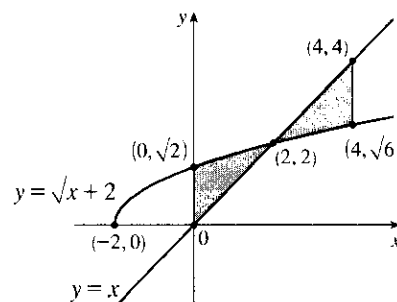


$$\begin{aligned} 29. A &= \int_{-1}^1 |x^3 - x| dx \\ &= 2 \int_0^1 (x - x^3) dx \quad [\text{by symmetry}] \\ &= 2 \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 \\ &= 2 \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2} \end{aligned}$$



30. The curves intersect when $\sqrt{x+2} = x \Rightarrow x+2 = x^2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1$ or 2 . $[-1$ is extraneous]

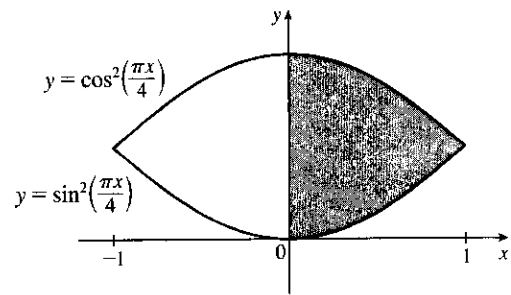
$$\begin{aligned} A &= \int_0^4 |\sqrt{x+2} - x| dx \\ &= \int_0^2 (\sqrt{x+2} - x) dx + \int_2^4 (x - \sqrt{x+2}) dx \\ &= \left[\frac{2}{3}(x+2)^{3/2} - \frac{1}{2}x^2 \right]_0^2 + \left[\frac{1}{2}x^2 - \frac{2}{3}(x+2)^{3/2} \right]_2^4 \\ &= \left(\frac{16}{3} - 2 \right) - \left(\frac{2}{3}(2\sqrt{2}) - 0 \right) + \left(8 - \frac{2}{3}(6\sqrt{6}) \right) - \left(2 - \frac{16}{3} \right) \\ &= 4 + \frac{32}{3} - \frac{4}{3}\sqrt{2} - 4\sqrt{6} = \frac{44}{3} - 4\sqrt{6} - \frac{4}{3}\sqrt{2} \end{aligned}$$



31. Let $f(x) = \cos^2\left(\frac{\pi x}{4}\right) - \sin^2\left(\frac{\pi x}{4}\right)$ and $\Delta x = \frac{1-0}{4}$.

The shaded area is given by

$$\begin{aligned} A &= \int_0^1 f(x) dx \approx M_4 \\ &= \frac{1}{4} \left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right) \right] \\ &\approx 0.6407 \end{aligned}$$



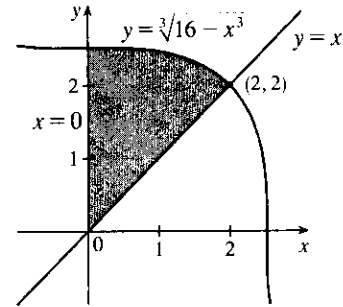
32. The curves intersect when $\sqrt[3]{16-x^3} = x \Rightarrow$

$$16 - x^3 = x^3 \Rightarrow 2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2.$$

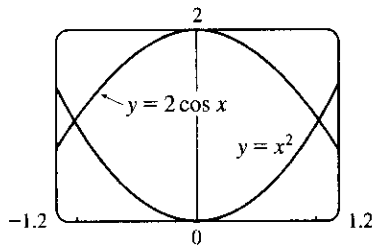
Let $f(x) = \sqrt[3]{16-x^3} - x$ and $\Delta x = \frac{2-0}{4}$.

The shaded area is given by

$$\begin{aligned} A &= \int_0^2 f(x) dx \approx M_4 \\ &= \frac{2}{4} \left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) \right] \\ &\approx 2.8144 \end{aligned}$$



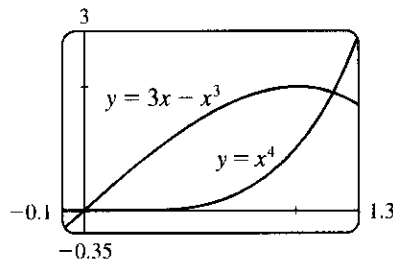
33.



From the graph, we see that the curves intersect at $x = \pm a \approx \pm 1.02$, with $2 \cos x > x^2$ on $(-a, a)$. So the area of the region bounded by the curves is

$$\begin{aligned} A &= \int_{-a}^a (2 \cos x - x^2) dx = 2 \int_0^a (2 \cos x - x^2) dx \\ &= 2 \left[2 \sin x - \frac{1}{3} x^3 \right]_0^a \approx 2.70 \end{aligned}$$

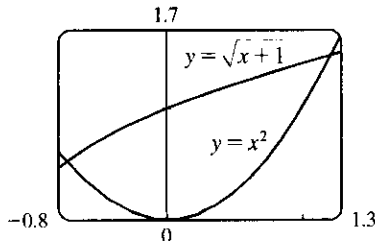
34.



From the graph, we see that the curves intersect at $x = 0$ and at $x = a \approx 1.17$, with $3x - x^3 > x^4$ on $(0, a)$. So the area of the region bounded by the curves is

$$\begin{aligned} A &= \int_0^a [(3x - x^3) - x^4] dx = \left[\frac{3}{2} x^2 - \frac{1}{4} x^4 - \frac{1}{5} x^5 \right]_0^a \\ &\approx 1.15 \end{aligned}$$

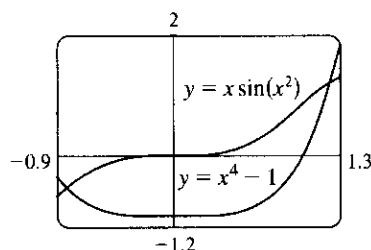
35.



From the graph, we see that the curves intersect at $x = a \approx -0.72$ and at $x = b \approx 1.22$, with $\sqrt{x+1} > x^2$ on (a, b) . So the area between the curves is

$$\begin{aligned} A &= \int_a^b (\sqrt{x+1} - x^2) dx = \left[\frac{2}{3} (x+1)^{3/2} - \frac{1}{3} x^3 \right]_a^b \\ &\approx 1.38 \end{aligned}$$

36.



From the graph, we see that the curves intersect at $x = a \approx -0.83$ and $x = b \approx 1.22$, with $x \sin(x^2) > x^4 - 1$ on (a, b) . So the area of the region bounded by the curves is

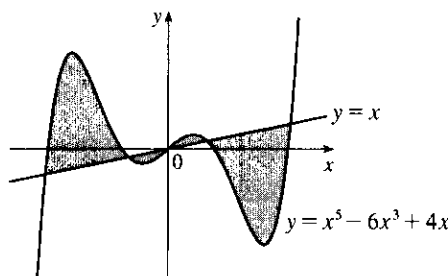
$$\begin{aligned} A &= \int_a^b [x \sin(x^2) - (x^4 - 1)] dx \\ &= \left[-\frac{1}{2} \cos(x^2) - \frac{1}{5} x^5 + x \right]_a^b \approx 1.78 \end{aligned}$$

37. As the figure illustrates, the curves $y = x$ and $y = x^5 - 6x^3 + 4x$ enclose a four-part region symmetric about the origin (since $x^5 - 6x^3 + 4x$ and x are odd functions of x). The curves intersect at values of x where $x^5 - 6x^3 + 4x = x$; that is, where $x(x^4 - 6x^2 + 3) = 0$. That happens at $x = 0$ and where

$$x^2 = \frac{6 \pm \sqrt{36 - 12}}{2} = 3 \pm \sqrt{6}; \text{ that is, at } x = -\sqrt{3 + \sqrt{6}}, -\sqrt{3 - \sqrt{6}}, 0, \sqrt{3 - \sqrt{6}}, \text{ and } \sqrt{3 + \sqrt{6}}.$$

The exact area is

$$\begin{aligned} 2 \int_0^{\sqrt{3+\sqrt{6}}} |(x^5 - 6x^3 + 4x) - x| dx &= 2 \int_0^{\sqrt{3+\sqrt{6}}} |x^5 - 6x^3 + 3x| dx \\ &= 2 \int_0^{\sqrt{3-\sqrt{6}}} (x^5 - 6x^3 + 3x) dx + 2 \int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}} (-x^5 + 6x^3 - 3x) dx \\ &\stackrel{\text{CAS}}{=} 12\sqrt{6} - 9 \end{aligned}$$



38. The inequality $x \geq 2y^2$ describes the region that lies on, or to the right of, the parabola $x = 2y^2$. The inequality $x \leq 1 - |y|$ describes the region

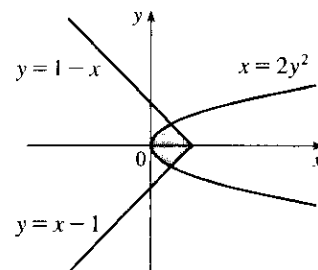
$$\text{that lies on, or to the left of, the curve } x = 1 - |y| = \begin{cases} 1 - y & \text{if } y \geq 0 \\ 1 + y & \text{if } y < 0 \end{cases}.$$

So the given region is the shaded region that lies between the curves.

$$\begin{aligned} \text{The graphs of } x = 1 - y \text{ and } x = 2y^2 \text{ intersect when } 1 - y &= 2y^2 \Leftrightarrow \\ 2y^2 + y - 1 &= 0 \Leftrightarrow (2y - 1)(y + 1) = 0 \Rightarrow y = \frac{1}{2} \text{ (for } y \geq 0). \end{aligned}$$

By symmetry,

$$A = 2 \int_0^{1/2} [(1 - y) - 2y^2] dy = 2 \left[-\frac{2}{3} y^3 - \frac{1}{2} y^2 + y \right]_0^{1/2} = 2 \left[\left(-\frac{1}{12} - \frac{1}{8} + \frac{1}{2} \right) - 0 \right] = 2 \left(\frac{7}{24} \right) = \frac{7}{12}.$$



39. 1 second = $\frac{1}{3600}$ hour, so $10 \text{ s} = \frac{1}{360}$ h. With the given data, we can take $n = 5$ to use the Midpoint Rule.

$$\Delta t = \frac{1/360 - 0}{5} = \frac{1}{1800}, \text{ so}$$

$$\begin{aligned} \text{distance}_{\text{Kelly}} - \text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K dt - \int_0^{1/360} v_C dt = \int_0^{1/360} (v_K - v_C) dt \\ &\approx M_5 = \frac{1}{1800} [(v_K - v_C)(1) + (v_K - v_C)(3) + (v_K - v_C)(5) \\ &\quad + (v_K - v_C)(7) + (v_K - v_C)(9)] \\ &= \frac{1}{1800} [(22 - 20) + (52 - 46) + (71 - 62) + (86 - 75) + (98 - 86)] \\ &= \frac{1}{1800} (2 + 6 + 9 + 11 + 12) = \frac{1}{1800} (40) = \frac{1}{45} \text{ mile, or } 117\frac{1}{3} \text{ feet} \end{aligned}$$

40. If x = distance from left end of pool and $w = w(x)$ = width at x , then the Midpoint Rule with $n = 4$ and

$$\Delta x = \frac{b - a}{n} = \frac{8 \cdot 2 - 0}{4} = 4 \text{ gives Area} = \int_0^{16} w dx \approx 4(6.2 + 6.8 + 5.0 + 4.8) = 4(22.8) = 91.2 \text{ m}^2.$$

41. We know that the area under curve A between $t = 0$ and $t = x$ is $\int_0^x v_A(t) dt = s_A(x)$, where $v_A(t)$ is the velocity of car A and s_A is its displacement. Similarly, the area under curve B between $t = 0$ and $t = x$ is $\int_0^x v_B(t) dt = s_B(x)$.

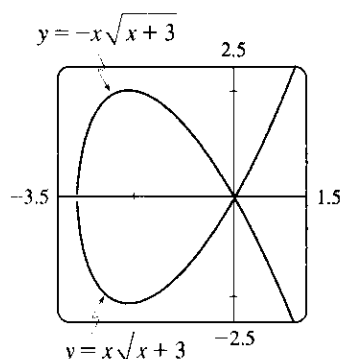
- After one minute, the area under curve A is greater than the area under curve B . So car A is ahead after one minute.
- The area of the shaded region has numerical value $s_A(1) - s_B(1)$, which is the distance by which A is ahead of B after 1 minute.
- After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve B , so car A is still ahead.
- From the graph, it appears that the area between curves A and B for $0 \leq t \leq 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \leq t \leq x$ (when car B is going faster) is the same as the area for $0 \leq t \leq 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

42. The area under $R'(x)$ from $x = 50$ to $x = 100$ represents the change in revenue, and the area under $C'(x)$ from $x = 50$ to $x = 100$ represents the change in cost. The shaded region represents the difference between these two values; that is, the increase in profit as the production level increases from 50 units to 100 units. We use the Midpoint Rule with $n = 5$ and $\Delta x = 10$:

$$\begin{aligned} M_5 &= \Delta x \{ [R'(55) - C'(55)] + [R'(65) - C'(65)] + [R'(75) - C'(75)] \\ &\quad + [R'(85) - C'(85)] + [R'(95) - C'(95)] \} \\ &\approx 10(2.40 - 0.85 + 2.20 - 0.90 + 2.00 - 1.00 + 1.80 - 1.10 + 1.70 - 1.20) \\ &= 10(5.05) = 50.5 \text{ thousand dollars} \end{aligned}$$

Using M_1 would give us $50(2 - 1) = 50$ thousand dollars.

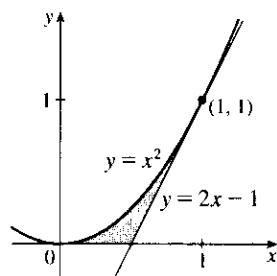
43.



To graph this function, we must first express it as a combination of explicit functions of y ; namely, $y = \pm x\sqrt{x+3}$. We can see from the graph that the loop extends from $x = -3$ to $x = 0$, and that by symmetry, the area we seek is just twice the area under the top half of the curve on this interval, the equation of the top half being $y = x\sqrt{x+3}$. So the area is $A = 2 \int_{-3}^0 (-x\sqrt{x+3}) dx$. We substitute $u = x + 3$, so $du = dx$ and the limits change to 0 and 3, and we get

$$\begin{aligned} A &= -2 \int_0^3 [(u-3)\sqrt{u}] du = -2 \int_0^3 (u^{3/2} - 3u^{1/2}) du \\ &= -2 \left[\frac{2}{5} u^{5/2} - 2u^{3/2} \right]_0^3 = -2 \left[\frac{2}{5} (3^2 \sqrt{3}) - 2(3\sqrt{3}) \right] = \frac{24}{5} \sqrt{3} \end{aligned}$$

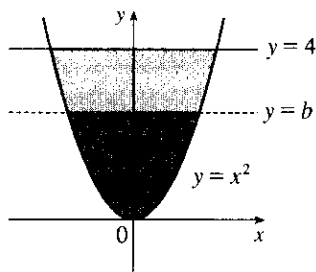
44.



We start by finding the equation of the tangent line to $y = x^2$ at the point $(1, 1)$: $y' = 2x$, so the slope of the tangent is $2(1) = 2$, and its equation is $y - 1 = 2(x - 1)$, or $y = 2x - 1$. We would need two integrals to integrate with respect to x , but only one to integrate with respect to y .

$$\begin{aligned} A &= \int_0^1 \left[\frac{1}{2}(y+1) - \sqrt{y} \right] dy = \left[\frac{1}{4}y^2 + \frac{1}{2}y - \frac{2}{3}y^{3/2} \right]_0^1 \\ &= \frac{1}{4} + \frac{1}{2} - \frac{2}{3} = \frac{1}{12} \end{aligned}$$

45.



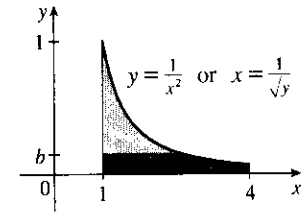
By the symmetry of the problem, we consider only the first quadrant, where $y = x^2 \Rightarrow x = \sqrt{y}$. We are looking for a number b such that $\int_0^b \sqrt{y} dy = \int_b^4 \sqrt{y} dy \Rightarrow \frac{2}{3} [y^{3/2}]_0^b = \frac{2}{3} [y^{3/2}]_b^4 \Rightarrow b^{3/2} = 4^{3/2} - b^{3/2} \Rightarrow 2b^{3/2} = 8 \Rightarrow b^{3/2} = 4 \Rightarrow b = 4^{2/3} \approx 2.52$.

$$\begin{aligned} 46. \text{ (a) We want to choose } a \text{ so that } \int_1^a \frac{1}{x^2} dx &= \int_a^4 \frac{1}{x^2} dx \Rightarrow \left[\frac{-1}{x} \right]_1^a = \left[\frac{-1}{x} \right]_a^4 \Rightarrow -\frac{1}{a} + 1 = -\frac{1}{4} + \frac{1}{a} \\ &\Rightarrow \frac{5}{4} = \frac{2}{a} \Rightarrow a = \frac{8}{5}. \end{aligned}$$

(b) The area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$ is $\frac{3}{4}$ [take $a = 4$ in the first integral in part (a)]. Now the line $y = b$ must intersect the curve $x = 1/\sqrt{y}$ and not the line $x = 4$, since the area under the line $y = 1/4^2$ from $x = 1$ to $x = 4$ is only $\frac{3}{16}$, which is less than half of $\frac{3}{4}$. We want to choose b so that the upper area in the

diagram is half of the total area under the curve $y = \frac{1}{x^2}$ from $x = 1$ to $x = 4$. This implies that

$$\begin{aligned}\int_b^1 (1/\sqrt{y} - 1) dy &= \frac{1}{2} \cdot \frac{3}{4} \Rightarrow [2\sqrt{y} - y]_b^1 = \frac{3}{8} \Rightarrow \\ 1 - 2\sqrt{b} + b &= \frac{3}{8} \Rightarrow b - 2\sqrt{b} + \frac{5}{8} = 0. \text{ Letting } c = \sqrt{b}, \text{ we get} \\ c^2 - 2c + \frac{5}{8} &= 0 \Rightarrow 8c^2 - 16c + 5 = 0. \text{ Thus,} \\ c &= \frac{16 \pm \sqrt{256 - 160}}{16} = 1 \pm \frac{\sqrt{6}}{4}. \text{ But } c = \sqrt{b} < 1 \Rightarrow c = 1 - \frac{\sqrt{6}}{4} \Rightarrow \\ b &= c^2 = 1 + \frac{3}{8} - \frac{\sqrt{6}}{2} = \frac{1}{8}(11 - 4\sqrt{6}) \approx 0.1503.\end{aligned}$$



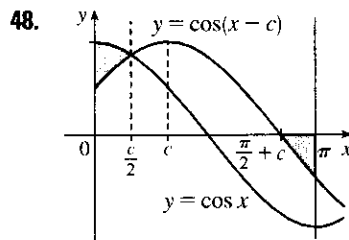
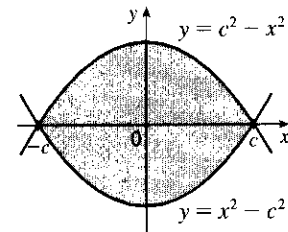
47. We first assume that $c > 0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c = 0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x = -c$ and $x = c$, and by symmetry, it is equal to four times the area in the first quadrant.

The enclosed area is

$$\begin{aligned}A &= 4 \int_0^c (c^2 - x^2) dx = 4 \left[c^2 x - \frac{1}{3} x^3 \right]_0^c \\ &= 4 \left(c^3 - \frac{1}{3} c^3 \right) = 4 \left(\frac{2}{3} c^3 \right) = \frac{8}{3} c^3\end{aligned}$$

$$\text{So } A = 576 \Leftrightarrow \frac{8}{3} c^3 = 576 \Leftrightarrow c^3 = 216 \Leftrightarrow c = \sqrt[3]{216} = 6.$$

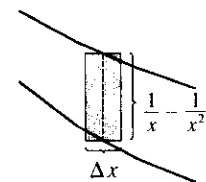
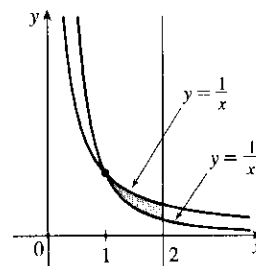
Note that $c = -6$ is another solution, since the graphs are the same.



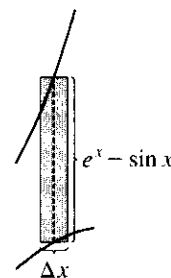
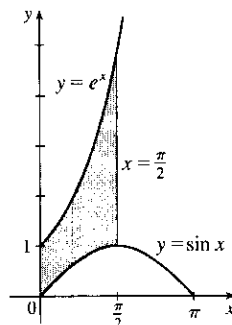
It appears from the diagram that the curves $y = \cos x$ and $y = \cos(x - c)$ intersect halfway between 0 and c , namely, when $x = c/2$. We can verify that this is indeed true by noting that $\cos(c/2 - c) = \cos(-c/2) = \cos(c/2)$. The point where $\cos(x - c)$ crosses the x -axis is $x = \frac{\pi}{2} + c$. So we require that $\int_0^{c/2} [\cos x - \cos(x - c)] dx = - \int_{\pi/2+c}^{\pi} \cos(x - c) dx$ (the negative sign on the RHS is needed since the second area is beneath the x -axis) $\Leftrightarrow$

$$\begin{aligned}[\sin x - \sin(x - c)]_0^{c/2} &= -[\sin(x - c)]_{\pi/2+c}^{\pi} \Rightarrow \\ [\sin(c/2) - \sin(-c/2)] - [-\sin(-c)] &= -\sin(\pi - c) + \sin[(\frac{\pi}{2} + c) - c] \Leftrightarrow \\ 2 \sin(c/2) - \sin c &= -\sin c + 1. \text{ [Here we have used the oddness of the sine function, and the fact that} \\ \sin(\pi - c) &= \sin c]. \text{ So } 2 \sin(c/2) = 1 \Leftrightarrow \sin(c/2) = \frac{1}{2} \Leftrightarrow c/2 = \frac{\pi}{6} \Leftrightarrow c = \frac{\pi}{3}.\end{aligned}$$

$$\begin{aligned}49. A &= \int_1^2 \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \left[\ln x + \frac{1}{x} \right]_1^2 \\ &= \left(\ln 2 + \frac{1}{2} \right) - (\ln 1 + 1) \\ &= \ln 2 - \frac{1}{2} \approx 0.19\end{aligned}$$

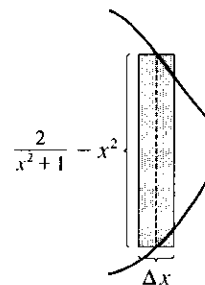
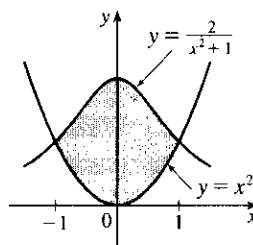


$$\begin{aligned}
 50. A &= \int_0^{\pi/2} (e^x - \sin x) dx \\
 &= [e^x + \cos x]_0^{\pi/2} \\
 &= (e^{\pi/2} + 0) - (1 + 1) \\
 &= e^{\pi/2} - 2
 \end{aligned}$$



$$\begin{aligned}
 51. \text{ The curves intersect when } x^2 = \frac{2}{x^2 + 1} &\Leftrightarrow x^4 + x^2 = 2 \Leftrightarrow x^4 + x^2 - 2 = 0 \Leftrightarrow \\
 (x^2 + 2)(x^2 - 1) = 0 &\Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1.
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_{-1}^1 \left(\frac{2}{x^2 + 1} - x^2 \right) dx \\
 &= 2 \int_0^1 \left(\frac{2}{x^2 + 1} - x^2 \right) dx \\
 &= 2 \left[2 \tan^{-1} x - \frac{1}{3} x^3 \right]_0^1 = 2 \left(2 \cdot \frac{\pi}{4} - \frac{1}{3} \right) \\
 &= \pi - \frac{2}{3} \approx 2.47
 \end{aligned}$$



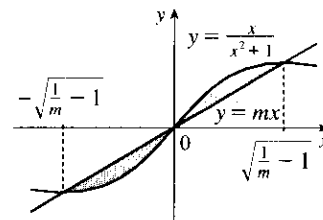
52. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation

$$\begin{aligned}
 x/(x^2 + 1) &= mx \Rightarrow x = x(mx^2 + m) \Rightarrow \\
 x(mx^2 + m) - x &= 0 \Rightarrow x(mx^2 + m - 1) = 0 \Rightarrow \\
 x = 0 \text{ or } mx^2 + m - 1 &= 0 \Rightarrow x = 0 \text{ or } x^2 = \frac{1 - m}{m} \Rightarrow
 \end{aligned}$$

$x = 0$ or $x = \pm \sqrt{\frac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if

$1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2 + 1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2 + 1)$ are both odd functions, the total area is twice the area between the curves on the interval $[0, \sqrt{1/m - 1}]$. So the total area enclosed is

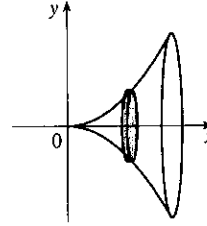
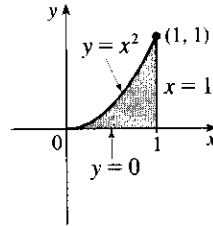
$$\begin{aligned}
 2 \int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2 + 1} - mx \right] dx &= 2 \left[\frac{1}{2} \ln(x^2 + 1) - \frac{1}{2} mx^2 \right]_0^{\sqrt{1/m-1}} \\
 &= [\ln(1/m - 1 + 1) - m(1/m - 1)] - (\ln 1 - 0) \\
 &= \ln(1/m) - 1 + m = m - \ln m - 1
 \end{aligned}$$



6.2 Volumes

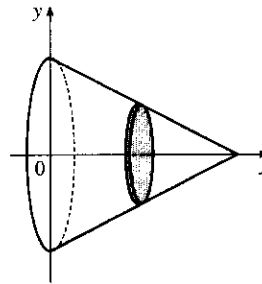
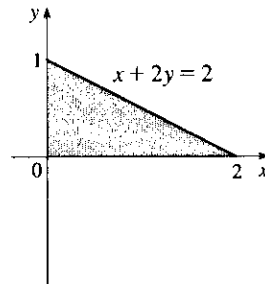
1. A cross-section is circular with radius x^2 , so its area is $A(x) = \pi(x^2)^2$.

$$V = \int_0^1 A(x) dx = \int_0^1 \pi(x^2)^2 dx = \pi \int_0^1 x^4 dx = \pi \left[\frac{1}{5} x^5 \right]_0^1 = \frac{\pi}{5}$$



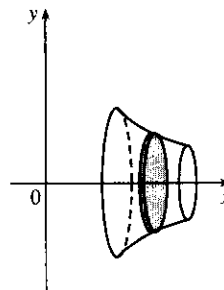
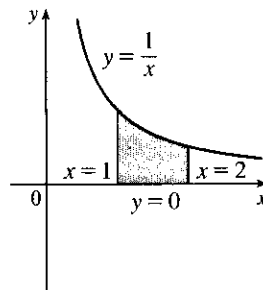
2. $x + 2y = 2 \Leftrightarrow y = 1 - \frac{1}{2}x$, so a cross-section is circular with radius $1 - \frac{1}{2}x$, and its area is $A(x) = \pi(1 - \frac{1}{2}x)^2$.

$$\begin{aligned} V &= \int_0^2 \pi y^2 dx = \pi \int_0^2 \left(1 - \frac{1}{2}x\right)^2 dx = \pi \int_0^2 \left(1 - x + \frac{1}{4}x^2\right) dx = \pi \left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 \right]_0^2 \\ &= \pi \left(2 - 2 + \frac{2}{3} \right) = \frac{2}{3}\pi \end{aligned}$$



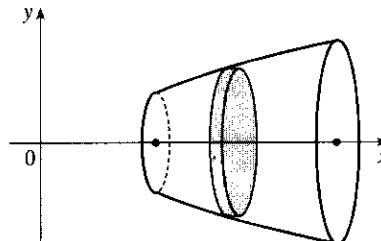
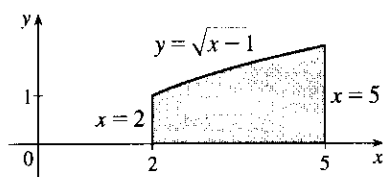
3. A cross-section is a disk with radius $1/x$, so its area is $A(x) = \pi(1/x)^2$.

$$V = \int_1^2 A(x) dx = \int_1^2 \pi \left(\frac{1}{x} \right)^2 dx = \pi \int_1^2 \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_1^2 = \pi \left[-\frac{1}{2} - (-1) \right] = \frac{\pi}{2}$$



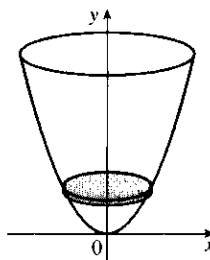
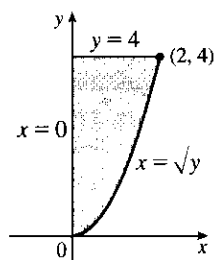
4. A cross-section is circular with radius $\sqrt{x-1}$, so its area is $A(x) = \pi(\sqrt{x-1})^2 = \pi(x-1)$.

$$V = \int_2^5 A(x) dx = \int_2^5 \pi(x-1) dx = \pi \left[\frac{1}{2}x^2 - x \right]_2^5 = \pi \left(\frac{25}{2} - 5 - \frac{4}{2} + 2 \right) = \frac{15}{2}\pi$$



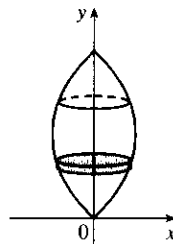
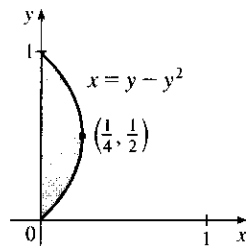
5. A cross-section is a disk with radius $\sqrt{y}$, so its area is $A(y) = \pi(\sqrt{y})^2$.

$$V = \int_0^4 A(y) dy = \int_0^4 \pi(\sqrt{y})^2 dy = \pi \int_0^4 y dy = \pi \left[\frac{1}{2}y^2 \right]_0^4 = 8\pi$$



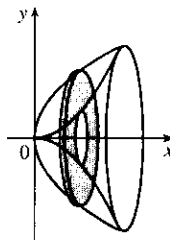
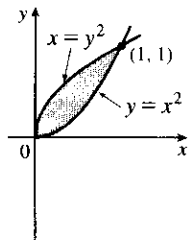
6. A cross-section is a disk with radius $y - y^2$, so its area is $A(y) = \pi(y - y^2)^2$.

$$\begin{aligned} V &= \int_0^1 A(y) dy = \int_0^1 \pi(y - y^2)^2 dy = \pi \int_0^1 (y^4 - 2y^3 + y^2) dy = \pi \left[\frac{1}{5}y^5 - \frac{1}{2}y^4 + \frac{1}{3}y^3 \right]_0^1 \\ &= \pi \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3} \right) = \frac{\pi}{30} \end{aligned}$$



7. A cross-section is a washer (annulus) with inner radius x^2 and outer radius $\sqrt{x}$, so its area is $A(x) = \pi(\sqrt{x})^2 - \pi(x^2)^2 = \pi(x - x^4)$.

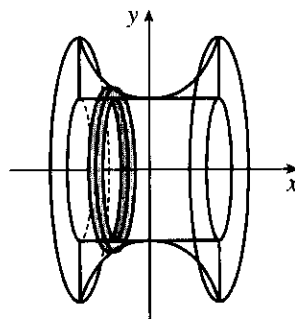
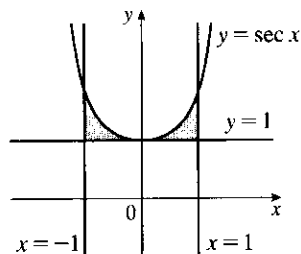
$$V = \int_0^1 A(x) dx = \pi \int_0^1 (x - x^4) dx = \pi \left[\frac{1}{2}x^2 - \frac{1}{5}x^5 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10}$$



8. A cross-section is a washer with inner radius 1 and outer radius $\sec x$, so its area is

$$A(x) = \pi(\sec x)^2 - \pi(1)^2 = \pi(\sec^2 x - 1).$$

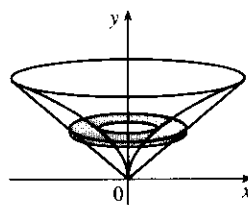
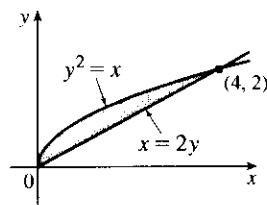
$$V = \int_{-1}^1 A(x) dx = \int_{-1}^1 \pi(\sec^2 x - 1) dx = 2\pi \int_0^1 (\sec^2 x - 1) dx = 2\pi[\tan x - x]_0^1 = 2\pi(\tan 1 - 1) \approx 3.5023$$



9. A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is

$$A(y) = \pi(2y)^2 - \pi(y^2)^2 = \pi(4y^2 - y^4).$$

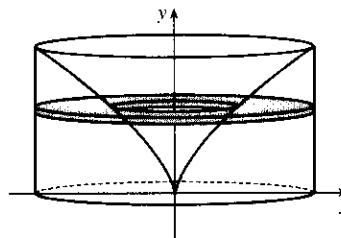
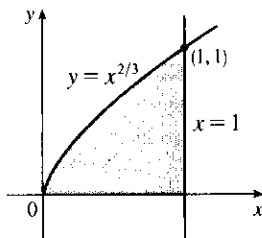
$$V = \int_0^2 A(y) dy = \pi \int_0^2 (4y^2 - y^4) dy = \pi \left[\frac{4}{3}y^3 - \frac{1}{5}y^5 \right]_0^2 = \pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{64\pi}{15}$$



10. $y = x^{2/3} \Leftrightarrow x = y^{3/2}$, so a cross-section is a washer with inner radius $y^{3/2}$ and outer radius 1, and its area is

$$A(y) = \pi(1)^2 - \pi(y^{3/2})^2 = \pi(1 - y^3).$$

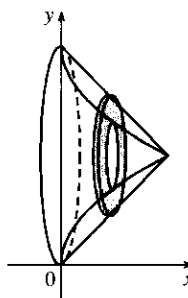
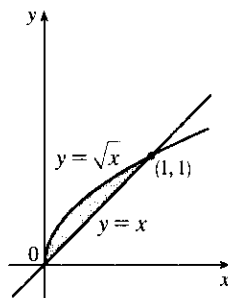
$$V = \int_0^1 A(y) dy = \pi \int_0^1 (1 - y^3) dy = \pi \left[y - \frac{1}{4}y^4 \right]_0^1 = \frac{3}{4}\pi$$



11. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is

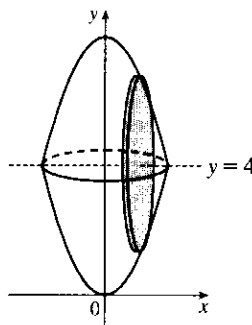
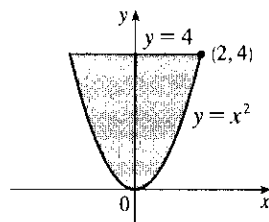
$$A(x) = \pi(1 - x)^2 - \pi(1 - \sqrt{x})^2 = \pi[(1 - 2x + x^2) - (1 - 2\sqrt{x} + x)] = \pi(-3x + x^2 + 2\sqrt{x}).$$

$$\begin{aligned} V &= \int_0^1 A(x) dx = \pi \int_0^1 (-3x + x^2 + 2\sqrt{x}) dx \\ &= \pi \left[-\frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{4}{3}x^{3/2} \right]_0^1 = \pi \left(-\frac{3}{2} + \frac{5}{3} \right) = \frac{\pi}{6} \end{aligned}$$



12. A cross-section is circular with radius $4 - x^2$, so its area is $A(x) = \pi(4 - x^2)^2 = \pi(16 - 8x^2 + x^4)$.

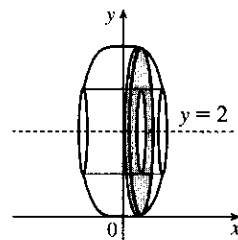
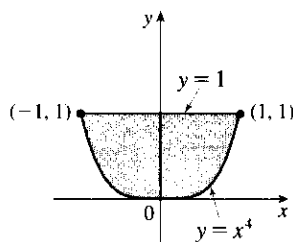
$$\begin{aligned} V &= \int_{-2}^2 A(x) dx = 2 \int_0^2 A(x) dx = 2\pi \int_0^2 (16 - 8x^2 + x^4) dx = 2\pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2 \\ &= 2\pi \left(32 - \frac{64}{3} + \frac{32}{5} \right) = 64\pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) = 64\pi \cdot \frac{8}{15} = \frac{512\pi}{15} \end{aligned}$$



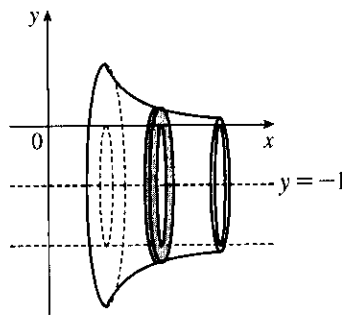
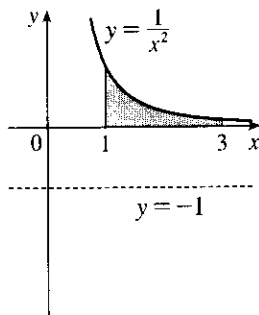
13. A cross-section is an annulus with inner radius $2 - 1$ and outer radius $2 - x^4$, so its area is

$$A(x) = \pi(2 - x^4)^2 - \pi(2 - 1)^2 = \pi(3 - 4x^4 + x^8).$$

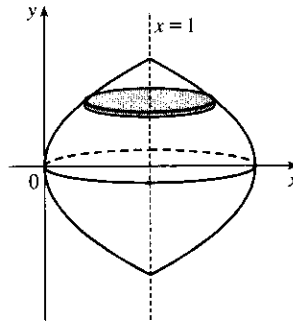
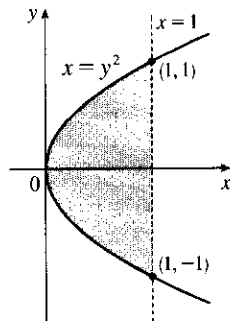
$$\begin{aligned} V &= \int_{-1}^1 A(x) dx = 2 \int_0^1 A(x) dx = 2\pi \int_0^1 (3 - 4x^4 + x^8) dx = 2\pi \left[3x - \frac{4}{5}x^5 + \frac{1}{9}x^9 \right]_0^1 \\ &= 2\pi \left(3 - \frac{4}{5} + \frac{1}{9} \right) = \frac{208}{45}\pi \end{aligned}$$



$$\begin{aligned}
 14. V &= \int_1^3 \pi \left\{ \left[\frac{1}{x^2} - (-1) \right]^2 - [0 - (-1)]^2 \right\} dx = \pi \int_1^3 \left[\left(\frac{1}{x^2} + 1 \right)^2 - 1^2 \right] dx = \pi \int_1^3 \left(\frac{1}{x^4} + \frac{2}{x^2} \right) dx \\
 &= \pi \left[-\frac{1}{3x^3} - \frac{2}{x} \right]_1^3 = \pi \left[\left(-\frac{1}{81} - \frac{2}{3} \right) - \left(-\frac{1}{3} - 2 \right) \right] = \frac{134\pi}{81}
 \end{aligned}$$

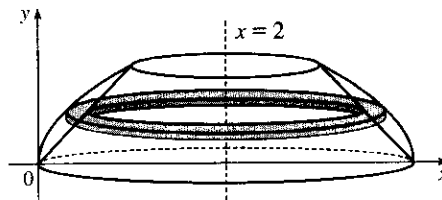
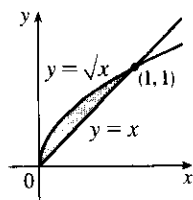


$$\begin{aligned}
 15. V &= \int_{-1}^1 \pi (1 - y^2)^2 dy = 2 \int_0^1 \pi (1 - y^2)^2 dy = 2\pi \int_0^1 (1 - 2y^2 + y^4) dy \\
 &= 2\pi \left[y - \frac{2}{3}y^3 + \frac{1}{5}y^5 \right]_0^1 = 2\pi \cdot \frac{8}{15} = \frac{16}{15}\pi
 \end{aligned}$$



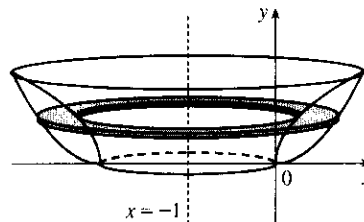
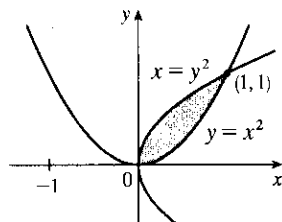
16. $y = \sqrt{x} \Rightarrow x = y^2$, so the outer radius is $2 - y^2$.

$$\begin{aligned}
 V &= \int_0^1 \pi \left[(2 - y^2)^2 - (2 - y)^2 \right] dy = \pi \int_0^1 [(4 - 4y^2 + y^4) - (4 - 4y + y^2)] dy \\
 &= \pi \int_0^1 (y^4 - 5y^2 + 4y) dy = \pi \left[\frac{1}{5}y^5 - \frac{5}{3}y^3 + 2y^2 \right]_0^1 = \pi \left(\frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8}{15}\pi
 \end{aligned}$$



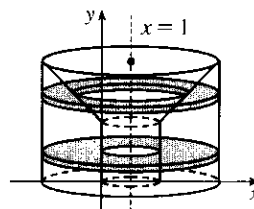
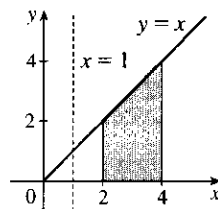
17. $y = x^2 \Rightarrow x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$$\begin{aligned} V &= \int_0^1 \pi \left\{ [\sqrt{y} - (-1)]^2 - [y^2 - (-1)]^2 \right\} dy = \pi \int_0^1 [(\sqrt{y} + 1)^2 - (y^2 + 1)^2] dy \\ &= \pi \int_0^1 (y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1) dy = \pi \int_0^1 (y + 2\sqrt{y} - y^4 - 2y^2) dy \\ &= \pi \left[\frac{1}{2}y^2 + \frac{4}{3}y^{3/2} - \frac{1}{5}y^5 - \frac{2}{3}y^3 \right]_0^1 = \pi \left(\frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3} \right) = \frac{29}{30}\pi \end{aligned}$$



18. For $0 \leq y < 2$, a cross-section is an annulus with inner radius $2 - 1$ and outer radius $4 - 1$, the area of which is $A_1(y) = \pi(4 - 1)^2 - \pi(2 - 1)^2$. For $2 \leq y \leq 4$, a cross-section is an annulus with inner radius $y - 1$ and outer radius $4 - 1$, the area of which is $A_2(y) = \pi(4 - 1)^2 - \pi(y - 1)^2$.

$$\begin{aligned} V &= \int_0^4 A(y) dy = \pi \int_0^2 [(4 - 1)^2 - (2 - 1)^2] dy + \pi \int_2^4 [(4 - 1)^2 - (y - 1)^2] dy \\ &= \pi [8y]_0^2 + \pi \int_2^4 (8 + 2y - y^2) dy = 16\pi + \pi \left[8y + y^2 - \frac{1}{3}y^3 \right]_2^4 \\ &= 16\pi + \pi \left[(32 + 16 - \frac{64}{3}) - (16 + 4 - \frac{8}{3}) \right] = \frac{76}{3}\pi \end{aligned}$$



19. $\mathcal{R}_1$ about OA (the line $y = 0$): $V = \int_0^1 A(x) dx = \int_0^1 \pi(x^3)^2 dx = \pi \int_0^1 x^6 dx = \pi \left[\frac{1}{7}x^7 \right]_0^1 = \frac{\pi}{7}$

20. $\mathcal{R}_1$ about OC (the line $x = 0$):

$$V = \int_0^1 A(y) dy = \int_0^1 [\pi(1)^2 - \pi(\sqrt[3]{y})^2] dy = \pi \int_0^1 (1 - y^{2/3}) dy = \pi \left[y - \frac{3}{5}y^{5/3} \right]_0^1 = \pi \left(1 - \frac{3}{5} \right) = \frac{2\pi}{5}$$

21. $\mathcal{R}_1$ about AB (the line $x = 1$):

$$\begin{aligned} V &= \int_0^1 A(y) dy = \int_0^1 \pi(1 - \sqrt[3]{y})^2 dy = \pi \int_0^1 (1 - 2y^{1/3} + y^{2/3}) dy \\ &= \pi \left[y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3} \right]_0^1 = \pi \left(1 - \frac{3}{2} + \frac{3}{5} \right) = \frac{\pi}{10} \end{aligned}$$

22. $\mathcal{R}_1$ about BC (the line $y = 1$):

$$\begin{aligned} V &= \int_0^1 A(x) dx = \int_0^1 [\pi(1)^2 - \pi(1 - x^3)^2] dx = \pi \int_0^1 [1 - (1 - 2x^3 + x^6)] dx \\ &= \pi \int_0^1 (2x^3 - x^6) dx = \pi \left[\frac{1}{2}x^4 - \frac{1}{7}x^7 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{7} \right) = \frac{5\pi}{14} \end{aligned}$$

23. $\mathcal{R}_2$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x) dx = \int_0^1 [\pi(1)^2 - \pi(\sqrt{x})^2] dx = \pi \int_0^1 (1 - x) dx = \pi \left[x - \frac{1}{2}x^2 \right]_0^1 = \pi \left(1 - \frac{1}{2} \right) = \frac{\pi}{2}$$

24. $\mathcal{R}_2$ about OC (the line $x = 0$): $V = \int_0^1 A(y) dy = \int_0^1 \pi(y^2)^2 dy = \pi \int_0^1 y^4 dy = \pi \left[\frac{1}{5} y^5 \right]_0^1 = \frac{\pi}{5}$

25. $\mathcal{R}_2$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y) dy = \int_0^1 [\pi(1)^2 - \pi(1 - y^2)^2] dy = \pi \int_0^1 [1 - (1 - 2y^2 + y^4)] dy$$

$$= \pi \int_0^1 (2y^2 - y^4) dy = \pi \left[\frac{2}{3} y^3 - \frac{1}{5} y^5 \right]_0^1 = \pi \left(\frac{2}{3} - \frac{1}{5} \right) = \frac{7\pi}{15}$$

26. $\mathcal{R}_2$ about BC (the line $y = 1$):

$$V = \int_0^1 A(x) dx = \int_0^1 \pi(1 - \sqrt{x})^2 dx = \pi \int_0^1 (1 - 2x^{1/2} + x) dx$$

$$= \pi \left[x - \frac{4}{3} x^{3/2} + \frac{1}{2} x^2 \right]_0^1 = \pi \left(1 - \frac{4}{3} + \frac{1}{2} \right) = \frac{\pi}{6}$$

27. $\mathcal{R}_3$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x) dx = \int_0^1 [\pi(\sqrt{x})^2 - \pi(x^3)^2] dx = \pi \int_0^1 (x - x^6) dx = \pi \left[\frac{1}{2} x^2 - \frac{1}{7} x^7 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{7} \right) = \frac{5\pi}{14}.$$

Note: Let $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$. If we rotate $\mathcal{R}$ about any of the segments OA , OC , AB , or BC , we obtain a right circular cylinder of height 1 and radius 1. Its volume is $\pi r^2 h = \pi(1)^2 \cdot 1 = \pi$. As a check for Exercises 19, 23, and 27, we can add the answers, and that sum must equal π . Thus, $\frac{\pi}{7} + \frac{\pi}{2} + \frac{5\pi}{14} = \left(\frac{2+7+5}{14} \right) \pi = \pi$.

28. $\mathcal{R}_3$ about OC (the line $x = 0$):

$$V = \int_0^1 A(y) dy = \int_0^1 [\pi(\sqrt[3]{y})^2 - \pi(y^2)^2] dy = \pi \int_0^1 (y^{2/3} - y^4) dy$$

$$= \pi \left[\frac{3}{5} y^{5/3} - \frac{1}{5} y^5 \right]_0^1 = \pi \left(\frac{3}{5} - \frac{1}{5} \right) = \frac{2\pi}{5}$$

Note: See the note in Exercise 27. For Exercises 20, 24, and 28, we have $\frac{2\pi}{5} + \frac{\pi}{5} + \frac{2\pi}{5} = \pi$.

29. $\mathcal{R}_3$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y) dy = \int_0^1 [\pi(1 - y^2)^2 - \pi(1 - \sqrt[3]{y})^2] dy = \pi \int_0^1 [(1 - 2y^2 + y^4) - (1 - 2y^{1/3} + y^{2/3})] dy$$

$$= \pi \int_0^1 (-2y^2 + y^4 + 2y^{1/3} - y^{2/3}) dy = \pi \left[-\frac{2}{3} y^3 + \frac{1}{5} y^5 + \frac{3}{2} y^{4/3} - \frac{3}{5} y^{5/3} \right]_0^1$$

$$= \pi \left(-\frac{2}{3} + \frac{1}{5} + \frac{3}{2} - \frac{3}{5} \right) = \frac{13\pi}{30}$$

Note: See the note in Exercise 27. For Exercises 21, 25, and 29, we have $\frac{\pi}{10} + \frac{7\pi}{15} + \frac{13\pi}{30} = \left(\frac{3+14+13}{30} \right) \pi = \pi$.

30. $\mathcal{R}_3$ about BC (the line $y = 1$):

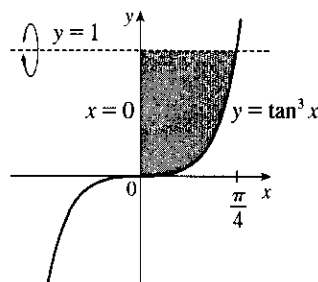
$$V = \int_0^1 A(x) dx = \int_0^1 [\pi(1 - x^3)^2 - \pi(1 - \sqrt{x})^2] dx$$

$$= \pi \int_0^1 [(1 - 2x^3 + x^6) - (1 - 2x^{1/2} + x)] dx = \pi \int_0^1 (-2x^3 + x^6 + 2x^{1/2} - x) dx$$

$$= \pi \left[-\frac{1}{2} x^4 + \frac{1}{7} x^7 + \frac{4}{3} x^{3/2} - \frac{1}{2} x^2 \right]_0^1 = \pi \left(-\frac{1}{2} + \frac{1}{7} + \frac{4}{3} - \frac{1}{2} \right) = \frac{10\pi}{21}$$

Note: See the note in Exercise 27. For Exercises 22, 26, and 30, we have $\frac{5\pi}{14} + \frac{\pi}{6} + \frac{10\pi}{21} = \left(\frac{15+7+20}{42} \right) \pi = \pi$.

31. $V = \pi \int_0^{\pi/4} (1 - \tan^3 x)^2 dx$



- 32.
- $y = (x - 2)^4$
- and
- $8x - y = 16$
- intersect when

$$(x - 2)^4 = 8x - 16 = 8(x - 2) \Leftrightarrow$$

$$(x - 2)^4 - 8(x - 2) = 0 \Leftrightarrow (x - 2)[(x - 2)^3 - 8] = 0$$

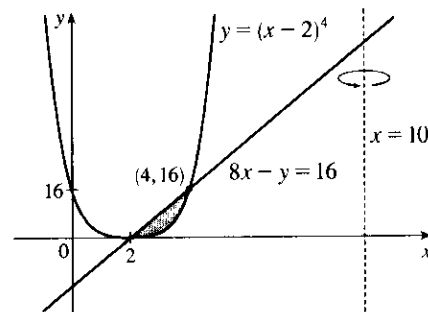
$$\Leftrightarrow x - 2 = 0 \text{ or } x - 2 = 2 \Leftrightarrow x = 2 \text{ or } 4.$$

$$y = (x - 2)^4 \Rightarrow x - 2 = \pm \sqrt[4]{y} \Rightarrow x = 2 + \sqrt[4]{y}$$

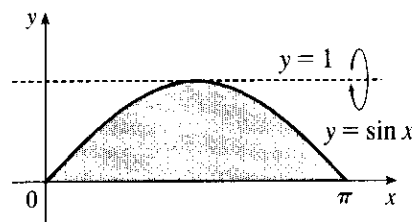
$$[\text{since } x \geq 2, 8x - y = 16 \Rightarrow 8x = y + 16 \Rightarrow$$

$$x = \frac{1}{8}y + 2.$$

$$V = \pi \int_0^{16} \left\{ \left[10 - \left(\frac{1}{8}y + 2 \right) \right]^2 - \left[10 - (2 + \sqrt[4]{y}) \right]^2 \right\} dy$$

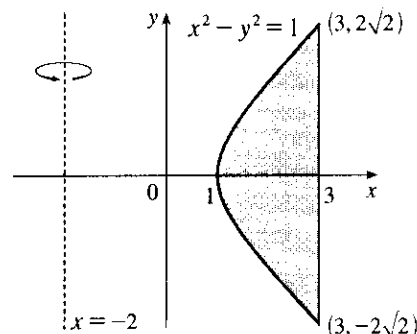


$$\begin{aligned} 33. V &= \pi \int_0^\pi [(1 - 0)^2 - (1 - \sin x)^2] dx \\ &= \pi \int_0^\pi [1^2 - (1 - \sin x)^2] dx \end{aligned}$$



$$34. V = \pi \int_0^\pi [(\sin x + 2)^2 - 2^2] dx$$

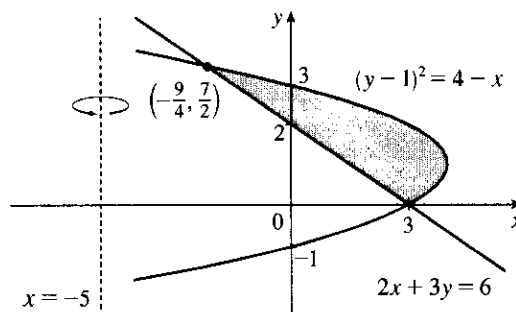
$$\begin{aligned} 35. V &= \pi \int_{-\sqrt{8}}^{\sqrt{8}} \left\{ [3 - (-2)]^2 - [\sqrt{y^2 + 1} - (-2)]^2 \right\} dy \\ &= \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[5^2 - (\sqrt{1 + y^2} + 2)^2 \right] dy \end{aligned}$$



36. Solve the equations for
- x
- :
- $(y - 1)^2 = 4 - x \Leftrightarrow x = 4 - (y - 1)^2$
- and
- $2x + 3y = 6 \Leftrightarrow x = 3 - \frac{3}{2}y$
- .

The points of intersection of the two curves are $(3, 0)$ and $(-\frac{9}{4}, \frac{7}{2})$. Therefore,

$$\begin{aligned} V &= \pi \int_0^{7/2} \left\{ [4 - (y - 1)^2 - (-5)]^2 - [3 - \frac{3}{2}y - (-5)]^2 \right\} dy \\ &= \pi \int_0^{7/2} \left\{ [9 - (y - 1)^2]^2 - (8 - \frac{3}{2}y)^2 \right\} dy \end{aligned}$$



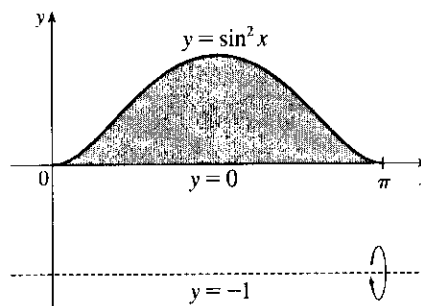
37. We see from the graph in Exercise 6.1.35 that the x -coordinates of the points of intersection are $x = a \approx -0.72$ and $x = b \approx 1.22$, with $\sqrt{x+1} > x^2$ on the interval (a, b) , so the volume of revolution is

$$\pi \int_a^b [(\sqrt{x+1})^2 - (x^2)^2] dx = \pi \int_a^b (x+1-x^4) dx = \pi \left[\frac{1}{2}x^2 + x - \frac{1}{5}x^5 \right]_a^b \approx 5.80.$$

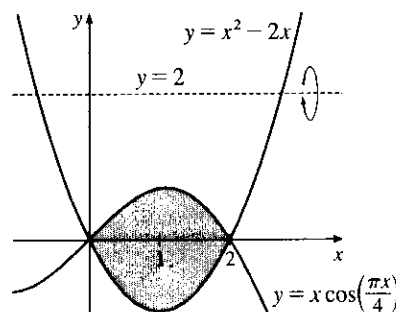
38. We see from the graph in Exercise 6.1.34 that the x -coordinates of the points of intersection are $x = 0$ and $x = a \approx 1.17$, with $3x - x^3 > x^4$ on the interval $(0, a)$, so the volume of revolution is

$$\begin{aligned} \pi \int_0^a [(3x - x^3)^2 - (x^4)^2] dx &= \pi \int_0^a (9x^2 - 6x^4 + x^6 - x^8) dx \\ &= \pi \left[3x^3 - \frac{6}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{9}x^9 \right]_0^a \approx 6.74 \end{aligned}$$

39. $V = \pi \int_0^\pi \{ [\sin^2 x - (-1)]^2 - [0 - (-1)]^2 \} dx$
 $\stackrel{\text{CAS}}{=} \frac{11}{8}\pi^2$



40. $V = \pi \int_0^2 \{ [2 - (x^2 - 2x)]^2 - [2 - x \cos(\pi x/4)]^2 \} dx$
 $\stackrel{\text{CAS}}{=} \frac{4(19\pi^2 + 120\pi - 210)}{15\pi}$



41. $\pi \int_0^{\pi/2} \cos^2 x dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x\}$ of the xy -plane about the x -axis.
42. $\pi \int_2^5 y dy = \pi \int_2^5 (\sqrt{y})^2 dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 2 \leq y \leq 5, 0 \leq x \leq \sqrt{y}\}$ of the xy -plane about the y -axis.
43. $\pi \int_0^1 (y^4 - y^8) dy = \pi \int_0^1 [(y^2)^2 - (y^4)^2] dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq y \leq 1, y^4 \leq x \leq y^2\}$ of the xy -plane about the y -axis.
44. $\pi \int_0^{\pi/2} [(1 + \cos x)^2 - 1^2] dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 1 \leq y \leq 1 + \cos x\}$ of the xy -plane about the x -axis.
 Or: The solid could be obtained by rotating the region $\mathcal{R}' = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x\}$ about the line $y = -1$.
45. There are 10 subintervals over the 15-cm length, so we'll use $n = 10/2 = 5$ for the Midpoint Rule.
 $V = \int_0^{15} A(x) dx \approx M_5 = \frac{15-0}{5} [A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(13.5)]$
 $= 3(18 + 79 + 106 + 128 + 39) = 3 \cdot 370 = 1110 \text{ cm}^3$

$$\begin{aligned}
 46. V &= \int_0^{10} A(x) dx \approx M_5 = \frac{10-0}{5} [A(1) + A(3) + A(5) + A(7) + A(9)] \\
 &= 2(0.65 + 0.61 + 0.59 + 0.55 + 0.50) = 2(2.90) = 5.80 \text{ m}^3
 \end{aligned}$$

47. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h}x$ about the x -axis.

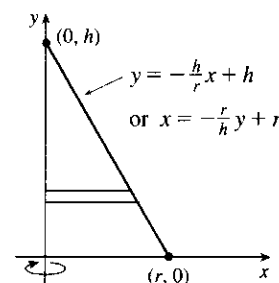
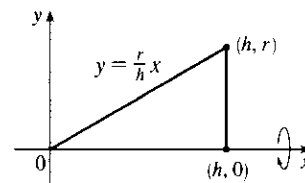
$$\begin{aligned}
 V &= \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \left[\frac{1}{3}x^3\right]_0^h \\
 &= \pi \frac{r^2}{h^2} \left(\frac{1}{3}h^3\right) = \frac{1}{3}\pi r^2 h
 \end{aligned}$$

Another solution: Revolve $x = -\frac{r}{h}y + r$ about the y -axis.

$$\begin{aligned}
 V &= \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy = \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy \\
 &= \pi \left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2y\right]_0^h = \pi \left(\frac{1}{3}r^2h - r^2h + r^2h\right) = \frac{1}{3}\pi r^2 h
 \end{aligned}$$

* Or use substitution with $u = r - \frac{r}{h}y$ and $du = -\frac{r}{h}dy$ to get

$$\pi \int_r^0 u^2 \left(-\frac{h}{r} du\right) = -\pi \frac{h}{r} \left[\frac{1}{3}u^3\right]_r^0 = -\pi \frac{h}{r} \left(-\frac{1}{3}r^3\right) = \frac{1}{3}\pi r^2 h.$$



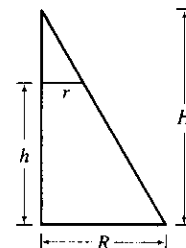
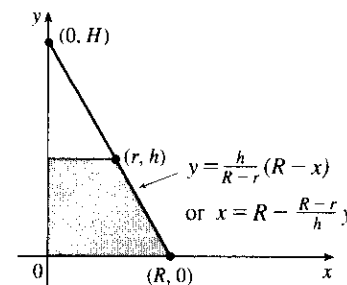
$$\begin{aligned}
 48. V &= \pi \int_0^h \left(R - \frac{R-r}{h}y\right)^2 dy \\
 &= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h}y + \left(\frac{R-r}{h}\right)^2 y^2\right] dy \\
 &= \pi \left[R^2y - \frac{R(R-r)}{h}y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3\right]_0^h \\
 &= \pi \left[R^2h - R(R-r)h + \frac{1}{3}(R-r)^2h\right] \\
 &= \frac{1}{3}\pi h [3Rr + (R^2 - 2Rr + r^2)] = \frac{1}{3}\pi h (R^2 + Rr + r^2)
 \end{aligned}$$

Another solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore,

$$Hr = HR - hR \Rightarrow hR = H(R-r) \Rightarrow H = \frac{hR}{R-r}. \text{ Now}$$

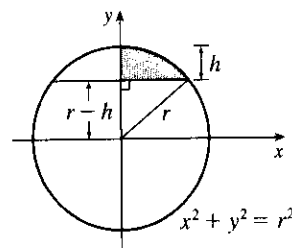
$$\begin{aligned}
 V &= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 (H-h) \quad [\text{by Exercise 47}] \\
 &= \frac{1}{3}\pi R^2 \frac{hR}{R-r} - \frac{1}{3}\pi r^2 \frac{r h}{R-r} \quad \left[H-h = \frac{rH}{R} = \frac{r h R}{R(R-r)}\right] \\
 &= \frac{1}{3}\pi h \frac{R^3 - r^3}{R-r} = \frac{1}{3}\pi h (R^2 + Rr + r^2) \\
 &= \frac{1}{3} \left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right] h = \frac{1}{3} (A_1 + A_2 + \sqrt{A_1 A_2}) h
 \end{aligned}$$

where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 50 for a related result.)



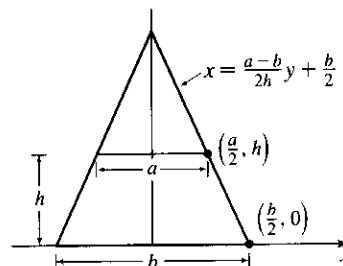
49. $x^2 + y^2 = r^2 \Leftrightarrow x^2 = r^2 - y^2$

$$\begin{aligned}
 V &= \pi \int_{r-h}^r (r^2 - y^2) dy = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^r \\
 &= \pi \left\{ \left[r^3 - \frac{r^3}{3} \right] - \left[r^2(r-h) - \frac{(r-h)^3}{3} \right] \right\} \\
 &= \pi \left\{ \frac{2}{3} r^3 - \frac{1}{3} (r-h) [3r^2 - (r-h)^2] \right\} \\
 &= \frac{1}{3} \pi \{ 2r^3 - (r-h) [3r^2 - (r^2 - 2rh + h^2)] \} \\
 &= \frac{1}{3} \pi \{ 2r^3 - (r-h) [2r^2 + 2rh - h^2] \} \\
 &= \frac{1}{3} \pi (2r^3 - 2r^3 - 2r^2h + rh^2 + 2r^2h + 2rh^2 - h^3) \\
 &= \frac{1}{3} \pi (3rh^2 - h^3) = \frac{1}{3} \pi h^2 (3r - h), \text{ or, equivalently, } \pi h^2 \left(r - \frac{h}{3} \right)
 \end{aligned}$$



50. An equation of the line is $x = \frac{\Delta x}{\Delta y} y + (x\text{-intercept}) = \frac{a/2 - b/2}{h - 0} y + \frac{b}{2} = \frac{a-b}{2h} y + \frac{b}{2}$.

$$\begin{aligned}
 V &= \int_0^h A(y) dy = \int_0^h (2x)^2 dy \\
 &= \int_0^h \left[2 \left(\frac{a-b}{2h} y + \frac{b}{2} \right) \right]^2 dy = \int_0^h \left[\frac{a-b}{h} y + b \right]^2 dy \\
 &= \int_0^h \left[\frac{(a-b)^2}{h^2} y^2 + \frac{2b(a-b)}{h} y + b^2 \right] dy \\
 &= \left[\frac{(a-b)^2}{3h^2} y^3 + \frac{b(a-b)}{h} y^2 + b^2 y \right]_0^h \\
 &= \frac{1}{3} (a-b)^2 h + b(a-b)h + b^2 h = \frac{1}{3} (a^2 - 2ab + b^2 + 3ab) h \\
 &= \frac{1}{3} (a^2 + ab + b^2) h
 \end{aligned}$$



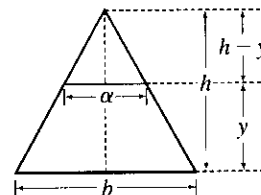
[Note that this can be written as $\frac{1}{3} (A_1 + A_2 + \sqrt{A_1 A_2}) h$, as in Exercise 48.]

If $a = b$, we get a rectangular solid with volume $b^2 h$. If $a = 0$, we get a square pyramid with volume $\frac{1}{3} b^2 h$.

51. For a cross-section at height y , we see from similar triangles that $\frac{\alpha/2}{b/2} = \frac{h-y}{h}$, so $\alpha = b \left(1 - \frac{y}{h} \right)$.

Similarly, for cross-sections having $2b$ as their base and β replacing α , $\beta = 2b \left(1 - \frac{y}{h} \right)$. So

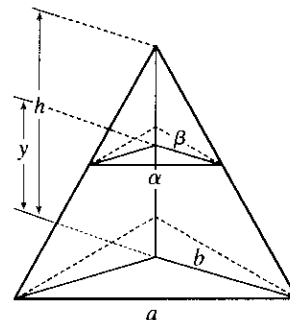
$$\begin{aligned}
 V &= \int_0^h A(y) dy = \int_0^h \left[b \left(1 - \frac{y}{h} \right) \right] \left[2b \left(1 - \frac{y}{h} \right) \right] dy \\
 &= \int_0^h 2b^2 \left(1 - \frac{y}{h} \right)^2 dy = 2b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2} \right) dy \\
 &= 2b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2} \right]_0^h = 2b^2 \left[h - h + \frac{1}{3} h \right] \\
 &= \frac{2}{3} b^2 h \quad \left[= \frac{1}{3} B h \text{ where } B \text{ is the area of the base, as with any pyramid.} \right]
 \end{aligned}$$



52. Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height y , so $a/b = \alpha/\beta \Rightarrow \alpha = a\beta/b$. Also by similar triangles, $b/h = \beta/(h - y) \Rightarrow \beta = b(h - y)/h$. These two equations imply that $\alpha = a(1 - y/h)$, and since the cross-section is an equilateral triangle, it has area

$$A(y) = \frac{1}{2} \cdot \alpha \cdot \frac{\sqrt{3}}{2} \alpha = \frac{a^2(1 - y/h)^2}{4} \sqrt{3}, \text{ so}$$

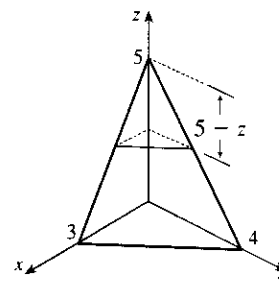
$$\begin{aligned} V &= \int_0^h A(y) dy = \frac{a^2 \sqrt{3}}{4} \int_0^h \left(1 - \frac{y}{h}\right)^2 dy \\ &= \frac{a^2 \sqrt{3}}{4} \left[-\frac{h}{3} \left(1 - \frac{y}{h}\right)^3 \right]_0^h = -\frac{\sqrt{3}}{12} a^2 h(-1) = \frac{\sqrt{3}}{12} a^2 h \end{aligned}$$



53. A cross-section at height z is a triangle similar to the base, so we'll multiply the legs of the base triangle, 3 and 4, by a proportionality factor of $(5 - z)/5$. Thus, the triangle at height z has area

$$A(z) = \frac{1}{2} \cdot 3 \left(\frac{5 - z}{5}\right) \cdot 4 \left(\frac{5 - z}{5}\right) = 6 \left(1 - \frac{z}{5}\right)^2, \text{ so}$$

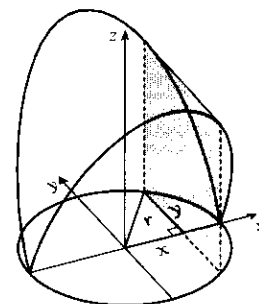
$$\begin{aligned} V &= \int_0^5 A(z) dz = 6 \int_0^5 \left(1 - z/5\right)^2 dz \\ &= 6 \int_1^0 u^2 (-5 du) \quad [u = 1 - z/5, du = -\frac{1}{5} dz] \\ &= -30 \left[\frac{1}{3} u^3 \right]_1^0 = -30 \left(-\frac{1}{3}\right) = 10 \text{ cm}^3 \end{aligned}$$



54. A cross-section is shaded in the diagram.

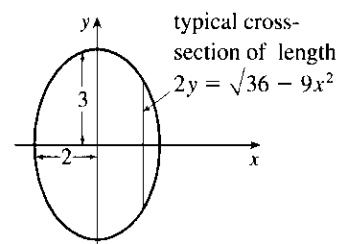
$$A(x) = (2y)^2 = (2\sqrt{r^2 - x^2})^2, \text{ so}$$

$$\begin{aligned} V &= \int_{-r}^r A(x) dx = 2 \int_0^r 4(r^2 - x^2) dx \\ &= 8 \left[r^2 x - \frac{1}{3} x^3 \right]_0^r \\ &= 8 \left(\frac{2}{3} r^3 \right) = \frac{16}{3} r^3 \end{aligned}$$

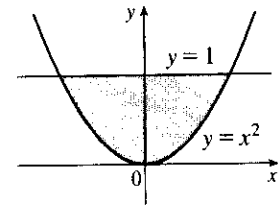


55. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse, then $l^2 + l^2 = (2y)^2 \Rightarrow 2l^2 = 4y^2 \Rightarrow l^2 = 2y^2$.

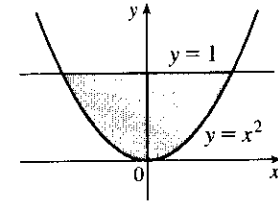
$$\begin{aligned} V &= \int_{-2}^2 A(x) dx = 2 \int_0^2 A(x) dx = 2 \int_0^2 \frac{1}{2} (l)(l) dx = 2 \int_0^2 y^2 dx \\ &= 2 \int_0^2 \frac{1}{4} (36 - 9x^2) dx = \frac{9}{2} \int_0^2 (4 - x^2) dx \\ &= \frac{9}{2} \left[4x - \frac{1}{3} x^3 \right]_0^2 = \frac{9}{2} \left(8 - \frac{8}{3} \right) = 24 \end{aligned}$$



56. The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{y}$. The corresponding equilateral triangle with side s has area $A(y) = s^2 \left(\frac{\sqrt{3}}{4} \right) = (2x)^2 \left(\frac{\sqrt{3}}{4} \right) = (2\sqrt{y})^2 \left(\frac{\sqrt{3}}{4} \right) = y\sqrt{3}$. Therefore, $V = \int_0^1 A(y) dy = \int_0^1 y\sqrt{3} dy = \sqrt{3} \left[\frac{1}{2}y^2 \right]_0^1 = \frac{\sqrt{3}}{2}$.

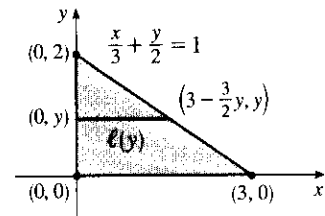


57. The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{y}$. The square has area $A(y) = (2\sqrt{y})^2 = 4y$, so $V = \int_0^1 A(y) dy = \int_0^1 4y dy = [2y^2]_0^1 = 2$.



58. A typical cross-section perpendicular to the y -axis in the base has length $\ell(y) = 3 - \frac{3}{2}y$. This length is the diameter of a cross-sectional semicircle in S , so

$$\begin{aligned} V &= \int_0^2 A(y) dy = \int_0^2 \frac{\pi}{2} \left[\frac{\ell(y)}{2} \right]^2 dy = \frac{\pi}{8} \int_0^2 \left(3 - \frac{3}{2}y \right)^2 dy \\ &= \frac{\pi}{8} \int_3^0 u^2 \left(-\frac{2}{3} du \right) \quad \left[u = 3 - \frac{3}{2}y, du = -\frac{3}{2} dy \right] \\ &= -\frac{\pi}{12} \left[\frac{1}{3} u^3 \right]_3^0 = -\frac{\pi}{12} (-9) = \frac{3\pi}{4} \end{aligned}$$

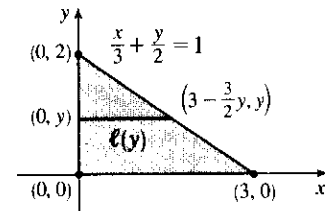


59. A typical cross-section perpendicular to the y -axis in the base has length $\ell(y) = 3 - \frac{3}{2}y$. This length is the leg of an isosceles right triangle, so

$$\begin{aligned} A(y) &= \frac{1}{2} [\ell(y)]^2 \quad \left[\frac{1}{2}bh \text{ with base} = \text{height} \right] \\ &= \frac{1}{2} \left[3 \left(1 - \frac{1}{2}y \right) \right]^2 = \frac{9}{2} \left(1 - \frac{1}{2}y \right)^2 \end{aligned}$$

Thus,

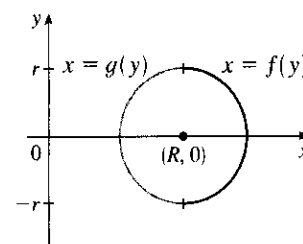
$$\begin{aligned} V &= \int_0^2 A(y) dy = \frac{9}{2} \int_1^0 u^2 (-2 du) \quad \left[u = 1 - \frac{1}{2}y, du = -\frac{1}{2} dy \right] \\ &= -9 \left[\frac{1}{3} u^3 \right]_1^0 = -9 \left(-\frac{1}{3} \right) = 3 \end{aligned}$$



60. (a) $V = \int_{-r}^r A(x) dx = 2 \int_0^r A(x) dx = 2 \int_0^r \frac{1}{2} h (2\sqrt{r^2 - x^2}) dx = 2h \int_0^r \sqrt{r^2 - x^2} dx$

(b) Observe that the integral represents one quarter of the area of a circle of radius r , so $V = 2h \cdot \frac{1}{4} \pi r^2 = \frac{1}{2} \pi h r^2$.

61. (a) The torus is obtained by rotating the circle $(x - R)^2 + y^2 = r^2$ about the y -axis. Solving for x , we see that the right half of the circle is given by $x = R + \sqrt{r^2 - y^2} = f(y)$ and the left half by $x = R - \sqrt{r^2 - y^2} = g(y)$. So



$$\begin{aligned} V &= \pi \int_{-r}^r \{ [f(y)]^2 - [g(y)]^2 \} dy \\ &= 2\pi \int_0^r \left[\left(R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2 \right) - \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2 \right) \right] dy \\ &= 2\pi \int_0^r 4R\sqrt{r^2 - y^2} dy = 8\pi R \int_0^r \sqrt{r^2 - y^2} dy \end{aligned}$$

- (b) Observe that the integral represents a quarter of the area of a circle with radius r , so

$$8\pi R \int_0^r \sqrt{r^2 - y^2} dy = 8\pi R \cdot \frac{1}{4}\pi r^2 = 2\pi^2 r^2 R.$$

62. The cross-sections perpendicular to the y -axis in Figure 17 are rectangles. The rectangle corresponding to the coordinate y has a base of length $2\sqrt{16 - y^2}$ in the xy -plane and a height of $\frac{1}{\sqrt{3}}y$, since $\angle BAC = 30^\circ$ and $|BC| = \frac{1}{\sqrt{3}}|AB|$. Thus, $A(y) = \frac{2}{\sqrt{3}}y\sqrt{16 - y^2}$ and

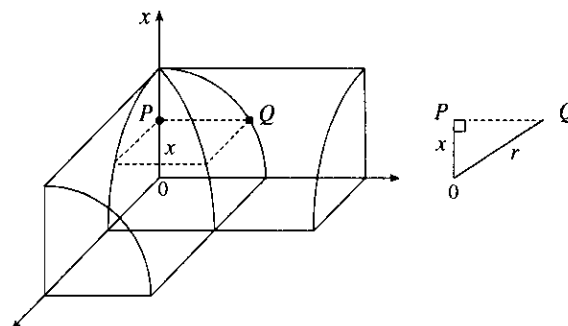
$$\begin{aligned} V &= \int_0^4 A(y) dy = \frac{2}{\sqrt{3}} \int_0^4 \sqrt{16 - y^2} y dy \\ &= \frac{2}{\sqrt{3}} \int_{16}^0 u^{1/2} \left(-\frac{1}{2} du\right) \quad [\text{Put } u = 16 - y^2, \text{ so } du = -2y dy] \\ &= \frac{1}{\sqrt{3}} \int_0^{16} u^{1/2} du = \frac{1}{\sqrt{3}} \frac{2}{3} \left[u^{3/2} \right]_0^{16} = \frac{2}{3\sqrt{3}} (64) = \frac{128}{3\sqrt{3}} \end{aligned}$$

63. (a) $\text{Volume}(S_1) = \int_0^h A(z) dz = \text{Volume}(S_2)$ since the cross-sectional area $A(z)$ at height z is the same for both solids.
- (b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h , that is, $\pi r^2 h$.

64. Each cross-section of the solid S in a plane perpendicular to the x -axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of S are shown. The area of this quarter-square is $|PQ|^2 = r^2 - x^2$. Therefore,

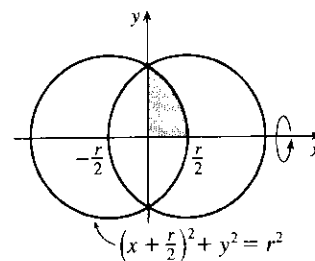
$$A(x) = 4(r^2 - x^2) \text{ and the volume of } S$$

$$\begin{aligned} V &= \int_{-r}^r A(x) dx = 4 \int_{-r}^r (r^2 - x^2) dx \\ &= 8 \int_0^r (r^2 - x^2) dx = 8 \left[r^2 x - \frac{1}{3} x^3 \right]_0^r = \frac{16}{3} r^3 \end{aligned}$$



65. The volume is obtained by rotating the area common to two circles of radius r , as shown. The volume of the right half is

$$\begin{aligned} V_{\text{right}} &= \pi \int_0^{r/2} y^2 dx = \pi \int_0^{r/2} \left[r^2 - \left(\frac{1}{2}r + x \right)^2 \right] dx \\ &= \pi \left[r^2 x - \frac{1}{3} \left(\frac{1}{2}r + x \right)^3 \right]_0^{r/2} = \pi \left[\left(\frac{1}{2}r^3 - \frac{1}{3}r^3 \right) - \left(0 - \frac{1}{24}r^3 \right) \right] = \frac{5}{24}\pi r^3 \end{aligned}$$



So by symmetry, the total volume is twice this, or $\frac{5}{12}\pi r^3$.

Another solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 49 with $h = \frac{1}{2}r$: $V = 2 \cdot \frac{1}{3}\pi h^2(3r - h) = \frac{2}{3}\pi \left(\frac{1}{2}r\right)^2 \left(3r - \frac{1}{2}r\right) = \frac{5}{12}\pi r^3$.

66. We consider two cases: one in which the ball is not completely submerged and the other in which it is.

Case 1: $0 \leq h \leq 10$ The ball will not be completely submerged, and so a cross-section of the water parallel to the surface will be the shaded area shown in the first diagram. We can find the area of the cross-section at height x above the bottom of the bowl by using the Pythagorean Theorem: $R^2 = 15^2 - (15 - x)^2$ and $r^2 = 5^2 - (x - 5)^2$, so $A(x) = \pi(R^2 - r^2) = 20\pi x$. The volume of water when it has depth h is then

$$V(h) = \int_0^h A(x) dx = \int_0^h 20\pi x dx = [10\pi x^2]_0^h = 10\pi h^2 \text{ cm}^3, 0 \leq h \leq 10.$$

Case 2: $10 < h \leq 15$ In this case we can find the volume by simply subtracting the volume displaced by the ball from the total volume inside the bowl underneath the surface of the water. The total volume underneath the surface is just the volume of a cap of the bowl,

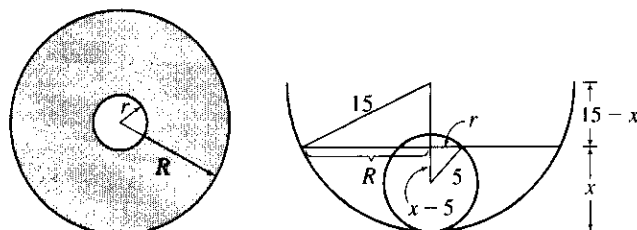
so we use the formula from Exercise 49:

$$V_{\text{cap}}(h) = \frac{1}{3}\pi h^2(45 - h). \text{ The volume of}$$

$$\text{the small sphere is } V_{\text{ball}} = \frac{4}{3}\pi(5)^3 = \frac{500}{3}\pi,$$

so the total volume is

$$V_{\text{cap}} - V_{\text{ball}} = \frac{1}{3}\pi(45h^2 - h^3 - 500) \text{ cm}^3.$$



67. Take the x -axis to be the axis of the cylindrical hole of radius r .

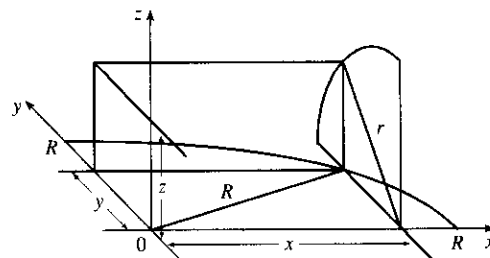
A quarter of the cross-section through y , perpendicular to the y -axis, is the rectangle shown. Using the Pythagorean Theorem twice, we see that the dimensions of this rectangle are

$$x = \sqrt{R^2 - y^2} \text{ and } z = \sqrt{r^2 - y^2}, \text{ so}$$

$$\frac{1}{4}A(y) = xz = \sqrt{r^2 - y^2} \sqrt{R^2 - y^2}, \text{ and}$$

$$V = \int_{-r}^r A(y) dy = \int_{-r}^r 4 \sqrt{r^2 - y^2} \sqrt{R^2 - y^2} dy$$

$$= 8 \int_0^r \sqrt{r^2 - y^2} \sqrt{R^2 - y^2} dy$$



68. The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2} \Rightarrow r^2 = R^2 - x^2 \Rightarrow x^2 = R^2 - r^2 \Rightarrow x = \pm\sqrt{R^2 - r^2}$. Rotating the shaded region about the x -axis gives us

$$\begin{aligned}
 V &= \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi \left[\left(\sqrt{R^2-x^2} \right)^2 - r^2 \right] dx \\
 &= 2\pi \int_0^{\sqrt{R^2-r^2}} (R^2 - x^2 - r^2) dx \quad [\text{by symmetry}] \\
 &= 2\pi \int_0^{\sqrt{R^2-r^2}} [(R^2 - r^2) - x^2] dx = 2\pi \left[(R^2 - r^2)x - \frac{1}{3}x^3 \right]_0^{\sqrt{R^2-r^2}} \\
 &= 2\pi \left[(R^2 - r^2)^{3/2} - \frac{1}{3}(R^2 - r^2)^{3/2} \right] \\
 &= 2\pi \cdot \frac{2}{3}(R^2 - r^2)^{3/2} = \frac{4\pi}{3}(R^2 - r^2)^{3/2}
 \end{aligned}$$

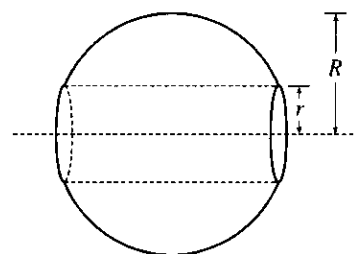
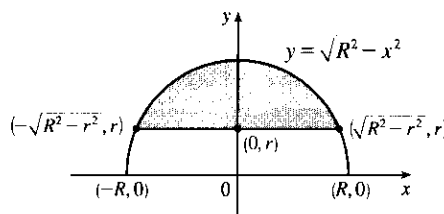
Our answer makes sense in limiting cases. As $r \rightarrow 0$,

$V \rightarrow \frac{4}{3}\pi R^3$, which is the

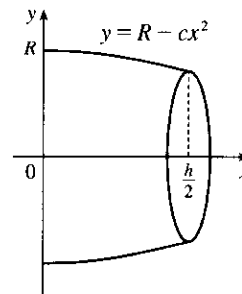
volume of the full sphere. As

$r \rightarrow R$, $V \rightarrow 0$, which makes

sense because the hole's radius is approaching that of the sphere.



69. (a) The radius of the barrel is the same at each end by symmetry, since the function $y = R - cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the x -axis, this radius is equal to the value of y at $x = \frac{1}{2}h$, which is $R - c(\frac{1}{2}h)^2 = R - d = r$.



- (b) The barrel is symmetric about the y -axis, so its volume is twice the volume of that part of the barrel for $x > 0$.

Also, the barrel is a volume of rotation, so

$$\begin{aligned}
 V &= 2 \int_0^{h/2} \pi y^2 dx = 2\pi \int_0^{h/2} (R - cx^2)^2 dx = 2\pi \left[R^2x - \frac{2}{3}Rcx^3 + \frac{1}{5}c^2x^5 \right]_0^{h/2} \\
 &= 2\pi \left(\frac{1}{2}R^2h - \frac{1}{12}Rch^3 + \frac{1}{160}c^2h^5 \right)
 \end{aligned}$$

Trying to make this look more like the expression we want, we rewrite it as

$$V = \frac{1}{3}\pi h \left[2R^2 + \left(R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4 \right) \right]. \text{ But}$$

$$R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4 = \left(R - \frac{1}{4}ch^2 \right)^2 - \frac{1}{40}c^2h^4 = (R - d)^2 - \frac{2}{5} \left(\frac{1}{4}ch^2 \right)^2 = r^2 - \frac{2}{5}d^2.$$

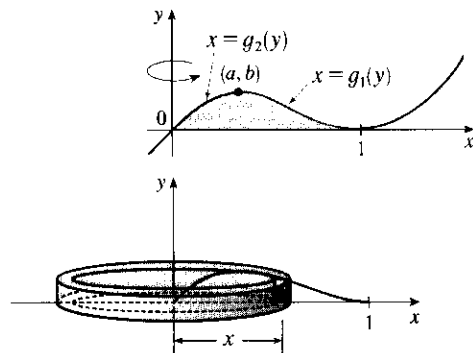
Substituting this back into V , we see that $V = \frac{1}{3}\pi h \left(2R^2 + r^2 - \frac{2}{5}d^2 \right)$, as required.

70. It suffices to consider the case where $\mathcal{R}$ is bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$, where $g(x) \leq f(x)$ for all x in $[a, b]$, since other regions can be decomposed into subregions of this type. We are concerned with the volume obtained when $\mathcal{R}$ is rotated about the line $y = -k$, which is equal to

$$\begin{aligned} V_2 &= \pi \int_a^b ([f(x) + k]^2 - [g(x) + k]^2) dx = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx + 2\pi k \int_a^b [f(x) - g(x)] dx \\ &= V_1 + 2\pi k A \end{aligned}$$

6.3 Volumes by Cylindrical Shells

1.



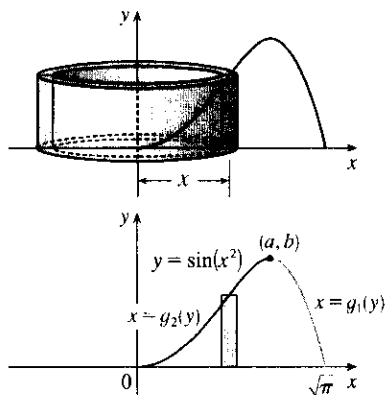
If we were to use the “washer” method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using

$$V = \pi \int_0^b \{ [g_1(y)]^2 - [g_2(y)]^2 \} dy.$$

Using shells, we find that a typical approximating shell has radius x , so its circumference is $2\pi x$. Its height is y , that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x [x(x-1)^2] dx = 2\pi \int_0^1 (x^4 - 2x^3 + x^2) dx = 2\pi \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3} \right]_0^1 = \frac{\pi}{15}$$

2.



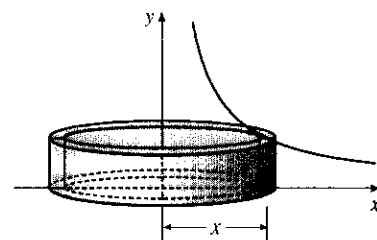
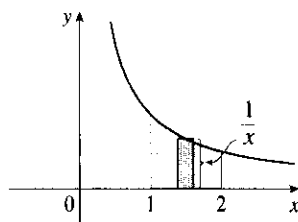
A typical cylindrical shell has circumference $2\pi x$ and height

$\sin(x^2)$. $V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2) dx$. Let $u = x^2$. Then $du = 2x dx$, so $V = \pi \int_0^{\pi} \sin u du = \pi [-\cos u]_0^{\pi} = \pi [1 - (-1)] = 2\pi$.

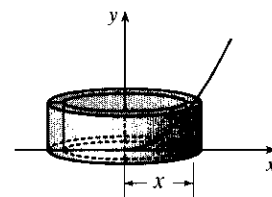
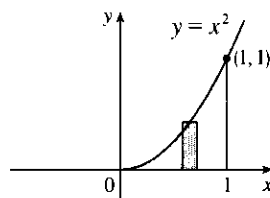
For slicing, we would first have to locate the local maximum point (a, b) of $y = \sin(x^2)$ using the methods of Chapter 4. Then we would have to solve the equation $y = \sin(x^2)$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the second figure. Finally we would find the volume using

$V = \pi \int_0^b \{ [g_1(y)]^2 - [g_2(y)]^2 \} dy$. Using shells is definitely preferable to slicing.

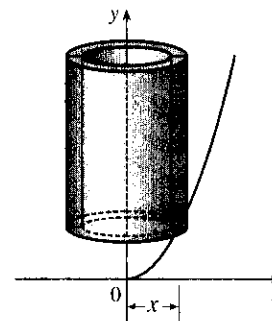
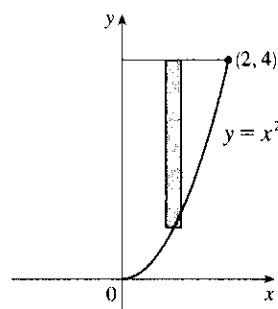
$$\begin{aligned} 3. V &= \int_1^2 2\pi x \cdot \frac{1}{x} dx = 2\pi \int_1^2 1 dx \\ &= 2\pi [x]_1^2 = 2\pi(2-1) = 2\pi \end{aligned}$$



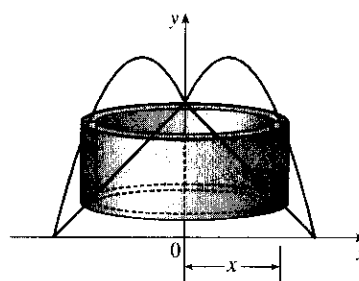
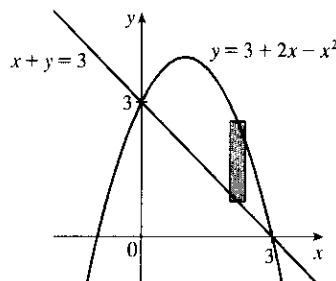
$$\begin{aligned}
 4. \quad V &= \int_0^1 2\pi x \cdot x^2 dx = 2\pi \int_0^1 x^3 dx \\
 &= 2\pi \left[\frac{1}{4} x^4 \right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}
 \end{aligned}$$



$$\begin{aligned}
 5. \quad V &= \int_0^2 2\pi x (4 - x^2) dx = 2\pi \int_0^2 (4x - x^3) dx \\
 &= 2\pi \left[2x^2 - \frac{1}{4} x^4 \right]_0^2 = 2\pi (8 - 4) \\
 &= 8\pi
 \end{aligned}$$

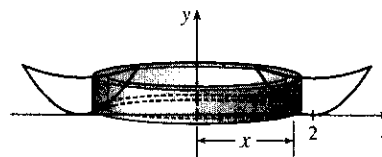
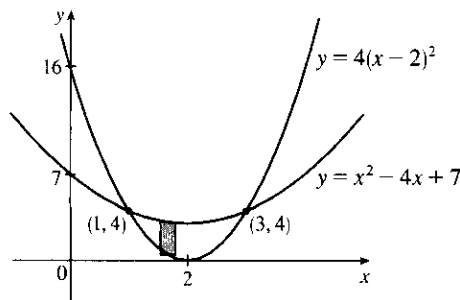


$$\begin{aligned}
 6. \quad V &= 2\pi \int_0^3 \{x[(3 + 2x - x^2) - (3 - x)]\} dx = 2\pi \int_0^3 [x(3x - x^2)] dx \\
 &= 2\pi \int_0^3 (3x^2 - x^3) dx = 2\pi \left[x^3 - \frac{1}{4} x^4 \right]_0^3 = 2\pi \left(27 - \frac{81}{4} \right) = 2\pi \left(\frac{27}{4} \right) = \frac{27\pi}{2}
 \end{aligned}$$



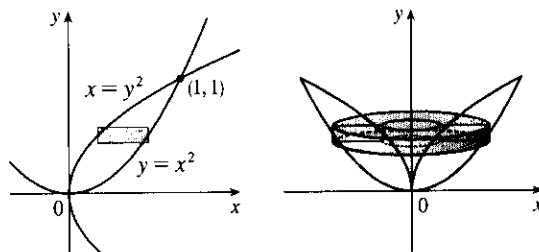
7. The curves intersect when $4(x - 2)^2 = x^2 - 4x + 7 \Leftrightarrow 4x^2 - 16x + 16 = x^2 - 4x + 7 \Leftrightarrow 3x^2 - 12x + 9 = 0 \Leftrightarrow 3(x^2 - 4x + 3) = 0 \Leftrightarrow 3(x - 1)(x - 3) = 0$, so $x = 1$ or 3 .

$$\begin{aligned}
 V &= 2\pi \int_1^3 \{x[(x^2 - 4x + 7) - 4(x - 2)^2]\} dx = 2\pi \int_1^3 [x(x^2 - 4x + 7 - 4x^2 + 16x - 16)] dx \\
 &= 2\pi \int_1^3 [x(-3x^2 + 12x - 9)] dx = 2\pi(-3) \int_1^3 (x^3 - 4x^2 + 3x) dx = -6\pi \left[\frac{1}{4} x^4 - \frac{4}{3} x^3 + \frac{3}{2} x^2 \right]_1^3 \\
 &= -6\pi \left[\left(\frac{81}{4} - 36 + \frac{27}{2} \right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2} \right) \right] = -6\pi \left(20 - 36 + 12 + \frac{4}{3} \right) = -6\pi \left(-\frac{8}{3} \right) = 16\pi
 \end{aligned}$$



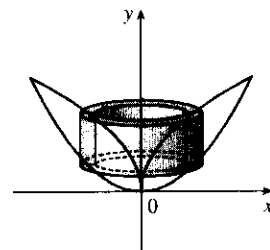
8. By slicing:

$$V = \int_0^1 \pi \left[(\sqrt{y})^2 - (y^2)^2 \right] dy = \pi \int_0^1 (y - y^4) dy = \pi \left[\frac{1}{2} y^2 - \frac{1}{5} y^5 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10}$$

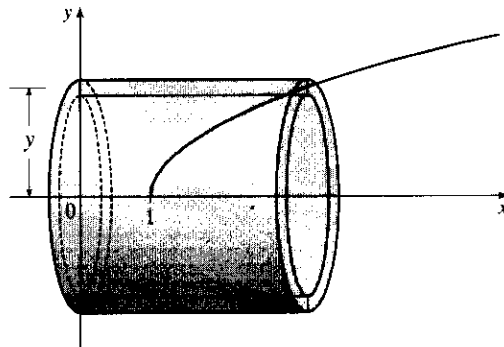
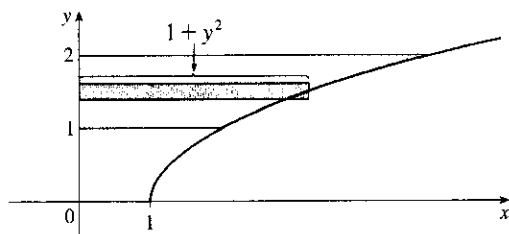


By cylindrical shells:

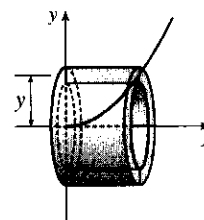
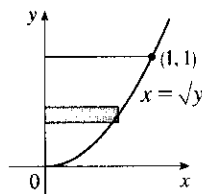
$$\begin{aligned} V &= \int_0^1 2\pi x (\sqrt{x} - x^2) dx = 2\pi \int_0^1 (x^{3/2} - x^3) dx \\ &= 2\pi \left[\frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right]_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{4} \right) \\ &= 2\pi \left(\frac{3}{20} \right) = \frac{3\pi}{10} \end{aligned}$$



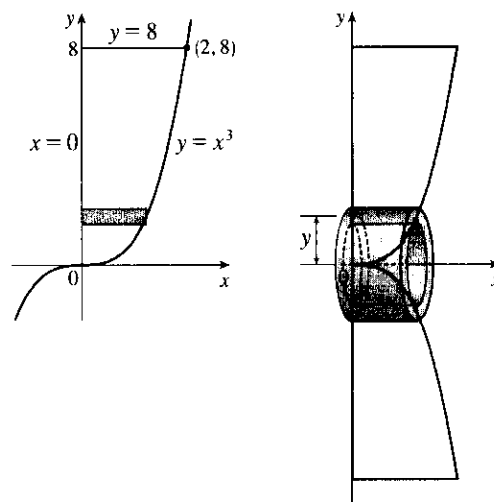
$$\begin{aligned} 9. V &= \int_1^2 2\pi y (1 + y^2) dy = 2\pi \int_1^2 (y + y^3) dy = 2\pi \left[\frac{1}{2} y^2 + \frac{1}{4} y^4 \right]_1^2 \\ &= 2\pi \left[(2 + 4) - \left(\frac{1}{2} + \frac{1}{4} \right) \right] = 2\pi \left(\frac{21}{4} \right) = \frac{21\pi}{2} \end{aligned}$$



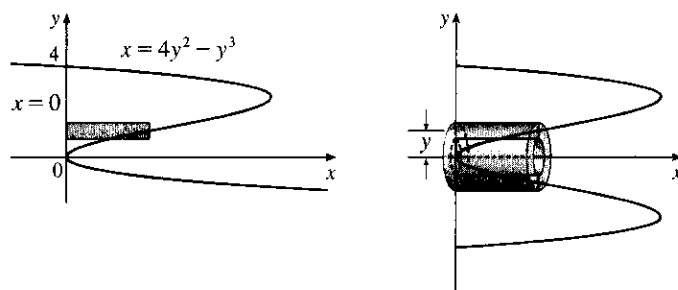
$$\begin{aligned} 10. V &= \int_0^1 2\pi y \sqrt{y} dy = 2\pi \int_0^1 y^{3/2} dy \\ &= 2\pi \left[\frac{2}{5} y^{5/2} \right]_0^1 = \frac{4\pi}{5} \end{aligned}$$



$$\begin{aligned}
 11. V &= 2\pi \int_0^8 [y(\sqrt[3]{y} - 0)] dy \\
 &= 2\pi \int_0^8 y^{4/3} dy = 2\pi \left[\frac{3}{7} y^{7/3} \right]_0^8 \\
 &= \frac{6\pi}{7} (8^{7/3}) = \frac{6\pi}{7} (2^7) = \frac{768\pi}{7}
 \end{aligned}$$



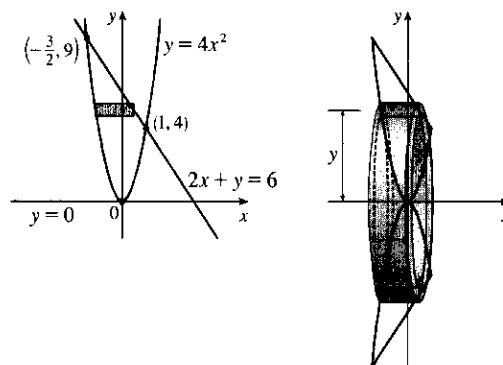
$$\begin{aligned}
 12. V &= 2\pi \int_0^4 [y(4y^2 - y^3)] dy \\
 &= 2\pi \int_0^4 (4y^3 - y^4) dy \\
 &= 2\pi \left[y^4 - \frac{1}{5} y^5 \right]_0^4 = 2\pi \left(256 - \frac{1024}{5} \right) \\
 &= 2\pi \left(\frac{256}{5} \right) = \frac{512\pi}{5}
 \end{aligned}$$



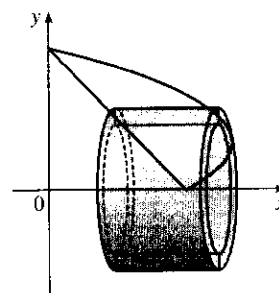
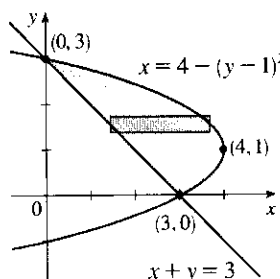
13. The curves intersect when $4x^2 = 6 - 2x \Leftrightarrow 2x^2 + x - 3 = 0 \Leftrightarrow (2x + 3)(x - 1) = 0 \Leftrightarrow x = -\frac{3}{2}$ or 1 .

Solving the equations for x gives us $y = 4x^2 \Rightarrow x = \pm \frac{1}{2}\sqrt{y}$ and $2x + y = 6 \Rightarrow x = -\frac{1}{2}y + 3$.

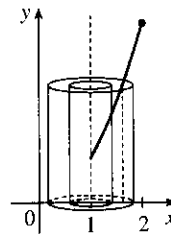
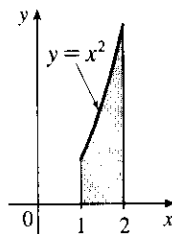
$$\begin{aligned}
 V &= 2\pi \int_0^4 \left\{ y \left[\left(\frac{1}{2}\sqrt{y} \right) - \left(-\frac{1}{2}\sqrt{y} \right) \right] \right\} dy + 2\pi \int_4^9 \left\{ y \left[\left(-\frac{1}{2}y + 3 \right) - \left(-\frac{1}{2}\sqrt{y} \right) \right] \right\} dy \\
 &= 2\pi \int_0^4 (y\sqrt{y}) dy + 2\pi \int_4^9 \left(-\frac{1}{2}y^2 + 3y + \frac{1}{2}y^{3/2} \right) dy = 2\pi \left[\frac{2}{5}y^{5/2} \right]_0^4 + 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{2}y^2 + \frac{1}{5}y^{5/2} \right]_4^9 \\
 &= 2\pi \left(\frac{2}{5} \cdot 32 \right) + 2\pi \left[\left(-\frac{243}{2} + \frac{243}{2} + \frac{243}{5} \right) - \left(-\frac{32}{3} + 24 + \frac{32}{5} \right) \right] \\
 &= \frac{128}{5}\pi + 2\pi \left(\frac{433}{15} \right) = \frac{1250}{15}\pi = \frac{250}{3}\pi
 \end{aligned}$$



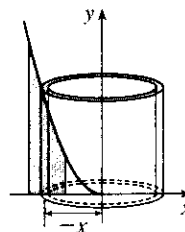
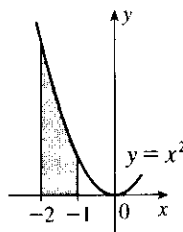
$$\begin{aligned}
 14. V &= \int_0^3 2\pi y [4 - (y-1)^2 - (3-y)] dy \\
 &= 2\pi \int_0^3 y(-y^2 + 3y) dy \\
 &= 2\pi \int_0^3 (-y^3 + 3y^2) dy = 2\pi \left[-\frac{1}{4}y^4 + y^3 \right]_0^3 \\
 &= 2\pi \left(-\frac{81}{4} + 27 \right) = 2\pi \left(\frac{27}{4} \right) = \frac{27\pi}{2}
 \end{aligned}$$



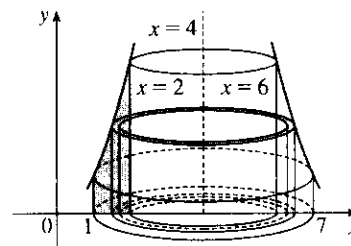
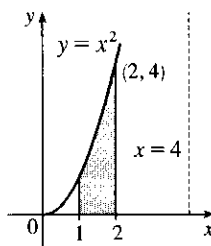
$$\begin{aligned}
 15. V &= \int_1^2 2\pi(x-1)x^2 dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 \right]_1^2 \\
 &= 2\pi \left[\left(4 - \frac{8}{3} \right) - \left(\frac{1}{4} - \frac{1}{3} \right) \right] = \frac{17}{6}\pi
 \end{aligned}$$



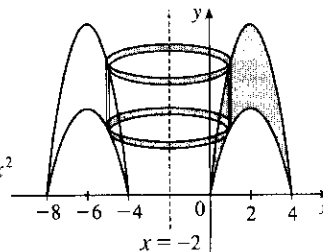
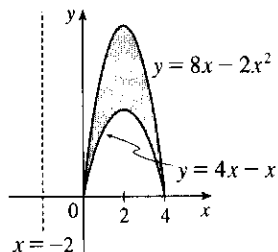
$$\begin{aligned}
 16. V &= \int_{-2}^{-1} 2\pi(-x) \cdot x^2 dx = 2\pi \left[-\frac{1}{4}x^4 \right]_{-2}^{-1} \\
 &= 2\pi \left[\left(-\frac{1}{4} \right) - (-4) \right] = \frac{15}{2}\pi
 \end{aligned}$$



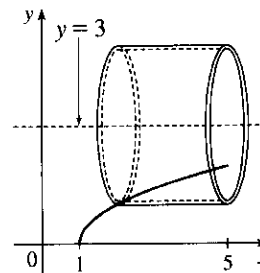
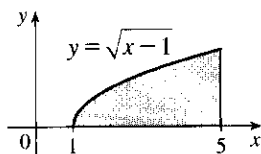
$$\begin{aligned}
 17. V &= \int_1^2 2\pi(4-x)x^2 dx = 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 \right]_1^2 \\
 &= 2\pi \left[\left(\frac{32}{3} - 4 \right) - \left(\frac{4}{3} - \frac{1}{4} \right) \right] = \frac{67}{6}\pi
 \end{aligned}$$



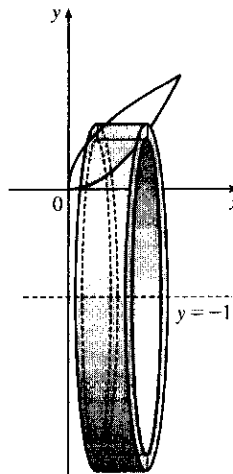
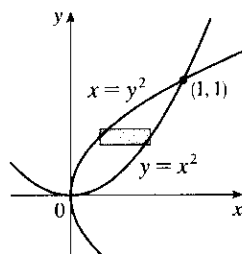
$$\begin{aligned}
 18. V &= \int_0^4 2\pi[x - (-2)][(8x - 2x^2) - (4x - x^2)] dx \\
 &= \int_0^4 2\pi(2+x)(4x - x^2) dx \\
 &= 2\pi \int_0^4 (8x + 2x^2 - x^3) dx \\
 &= 2\pi \left[4x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^4 \\
 &= 2\pi \left(64 + \frac{128}{3} - 64 \right) = \frac{256}{3}\pi
 \end{aligned}$$



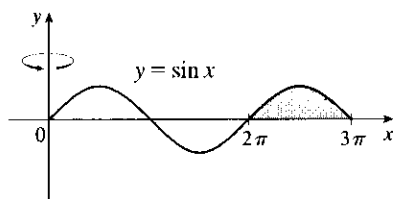
$$\begin{aligned}
 19. V &= \int_0^2 2\pi(3-y)(5-x) dy \\
 &= \int_0^2 2\pi(3-y)(5-y^2-1) dy \\
 &= \int_0^2 2\pi(12-4y-3y^2+y^3) dy \\
 &= 2\pi \left[12y - 2y^2 - y^3 + \frac{1}{4}y^4 \right]_0^2 \\
 &= 2\pi(24 - 8 - 8 + 4) = 24\pi
 \end{aligned}$$



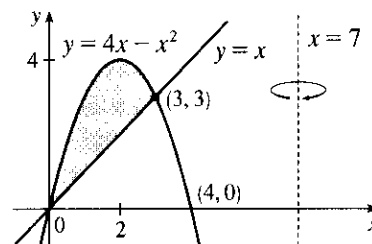
$$\begin{aligned}
 20. V &= \int_0^1 2\pi(y+1)(\sqrt{y}-y^2) dy = 2\pi \int_0^1 (y^{3/2} + y^{1/2} - y^3 - y^2) dy \\
 &= 2\pi \left[\frac{2}{5}y^{5/2} + \frac{2}{3}y^{3/2} - \frac{1}{4}y^4 - \frac{1}{3}y^3 \right]_0^1 = 2\pi \left(\frac{2}{5} + \frac{2}{3} - \frac{1}{4} - \frac{1}{3} \right) = 2\pi \left(\frac{29}{60} \right) = \frac{29\pi}{30}
 \end{aligned}$$



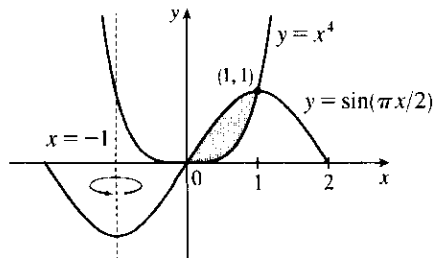
$$21. V = \int_{2\pi}^{3\pi} 2\pi x \sin x \, dx$$



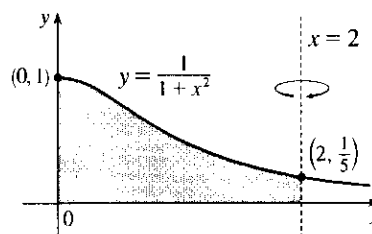
$$22. V = \int_0^3 2\pi(7-x)[(4x-x^2)-x] \, dx$$



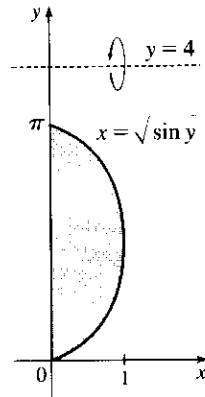
$$23. V = \int_0^1 2\pi[x - (-1)](\sin \frac{\pi}{2}x - x^4) \, dx$$



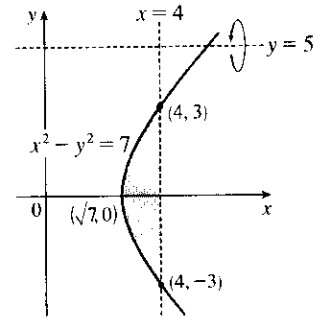
$$24. V = \int_0^2 2\pi(2-x) \left(\frac{1}{1+x^2} \right) \, dx$$



25. $V = \int_0^\pi 2\pi(4-y)\sqrt{\sin y} dy$



26. $V = \int_{-3}^3 2\pi(5-y)(4 - \sqrt{y^2 + 7}) dy$



27. $\Delta x = \frac{\pi/4 - 0}{4} = \frac{\pi}{16}$

$V = \int_0^{\pi/4} 2\pi x \tan x dx \approx 2\pi \cdot \frac{\pi}{16} \left(\frac{\pi}{32} \tan \frac{\pi}{32} + \frac{3\pi}{32} \tan \frac{3\pi}{32} + \frac{5\pi}{32} \tan \frac{5\pi}{32} + \frac{7\pi}{32} \tan \frac{7\pi}{32} \right) \approx 1.142$

28. $\Delta x = \frac{12-2}{5} = 2$, $n = 5$ and $x_i^* = 2 + (2i + 1)$, where $i = 0, 1, 2, 3, 4$. The values of $f(x)$ are taken directly from the diagram.

$$V = \int_2^{12} 2\pi x f(x) dx \approx 2\pi[3f(3) + 5f(5) + 7f(7) + 9f(9) + 11f(11)] \cdot 2$$

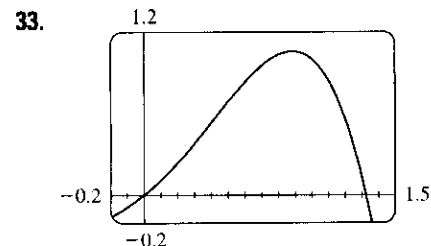
$$\approx 2\pi[3(2) + 5(4) + 7(4) + 9(2) + 11(1)]2 = 332\pi$$

29. $\int_0^3 2\pi x^5 dx = 2\pi \int_0^3 x(x^4) dx$. The solid is obtained by rotating the region $0 \leq y \leq x^4$, $0 \leq x \leq 3$ about the y-axis using cylindrical shells.

30. $2\pi \int_0^2 \frac{y}{1+y^2} dy = 2\pi \int_0^2 y \left(\frac{1}{1+y^2} \right) dy$. The solid is obtained by rotating the region $0 \leq x \leq \frac{1}{1+y^2}$, $0 \leq y \leq 2$ about the x-axis using cylindrical shells.

31. $\int_0^1 2\pi(3-y)(1-y^2) dy$. The solid is obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

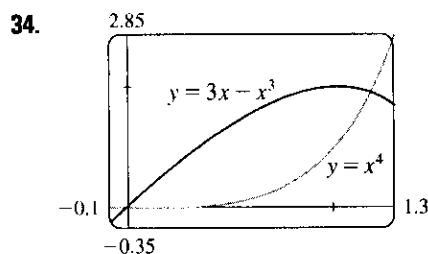
32. $\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x) dx$. The solid is obtained by rotating the region bounded by (i) $0 \leq y \leq \cos x - \sin x$, $0 \leq x \leq \frac{\pi}{4}$ or (ii) $\sin x \leq y \leq \cos x$, $0 \leq x \leq \frac{\pi}{4}$ about the line $x = \pi$ using cylindrical shells.



From the graph, the curves intersect at $x = 0$ and at $x = a \approx 1.32$, with $x + x^2 - x^4 > 0$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi \int_0^a [x(x + x^2 - x^4)] dx = 2\pi \int_0^a (x^2 + x^3 - x^5) dx$$

$$= 2\pi \left[\frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{6}x^6 \right]_0^a \approx 4.05$$

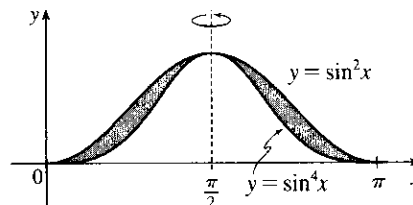


From the graph, the curves intersect at $x = 0$ and at $x = a \approx 1.17$, with $3x - x^3 > x^4$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y -axis is

$$\begin{aligned} V &= 2\pi \int_0^a \{x[(3x - x^3) - x^4]\} dx = 2\pi \int_0^a (3x^2 - x^4 - x^5) dx \\ &= 2\pi \left[x^3 - \frac{1}{5}x^5 - \frac{1}{6}x^6 \right]_0^a \approx 4.62 \end{aligned}$$

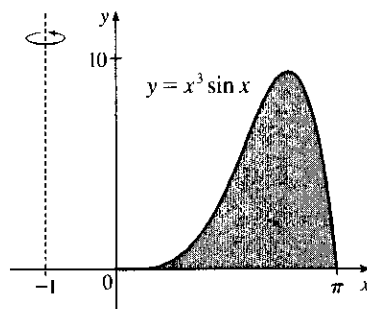
35. $V = 2\pi \int_0^{\pi/2} \left[\left(\frac{\pi}{2} - x \right) (\sin^2 x - \sin^4 x) \right] dx$

$\stackrel{\text{CAS}}{=} \frac{1}{32}\pi^3$



36. $V = 2\pi \int_0^{\pi} \{ [x - (-1)](x^3 \sin x) \} dx \stackrel{\text{CAS}}{=} 2\pi(\pi^4 + \pi^3 - 12\pi^2 - 6\pi + 48)$

$$= 2\pi^5 + 2\pi^4 - 24\pi^3 - 12\pi^2 + 96\pi$$



37. Use disks:

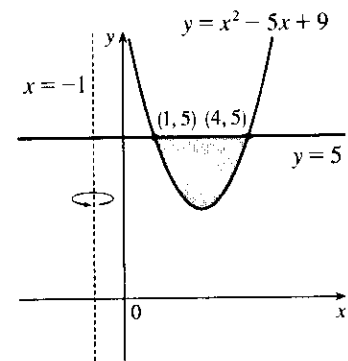
$$\begin{aligned} V &= \int_{-2}^1 \pi(x^2 + x - 2)^2 dx = \pi \int_{-2}^1 (x^4 + 2x^3 - 3x^2 - 4x + 4) dx \\ &= \pi \left[\frac{1}{5}x^5 + \frac{1}{2}x^4 - x^3 - 2x^2 + 4x \right]_{-2}^1 = \pi \left[\left(\frac{1}{5} + \frac{1}{2} - 1 - 2 + 4 \right) - \left(-\frac{32}{5} + 8 + 8 - 8 - 8 \right) \right] \\ &= \pi \left(\frac{33}{5} + \frac{3}{2} \right) = \frac{81}{10}\pi \end{aligned}$$

38. Use shells:

$$\begin{aligned} V &= \int_1^2 2\pi x(-x^2 + 3x - 2) dx = 2\pi \int_1^2 (-x^3 + 3x^2 - 2x) dx \\ &= 2\pi \left[-\frac{1}{4}x^4 + x^3 - x^2 \right]_1^2 = 2\pi \left[(-4 + 8 - 4) - \left(-\frac{1}{4} + 1 - 1 \right) \right] = \frac{\pi}{2} \end{aligned}$$

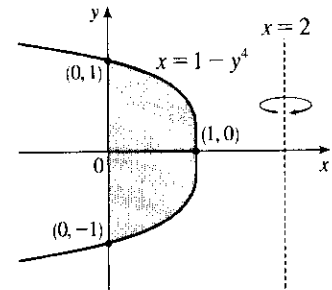
39. Use shells:

$$\begin{aligned}
 V &= \int_1^4 2\pi [x - (-1)] [5 - (x^2 - 5x + 9)] dx \\
 &= 2\pi \int_1^4 (x+1)(-x^2 + 5x - 4) dx \\
 &= 2\pi \int_1^4 (-x^3 + 4x^2 + x - 4) dx = 2\pi \left[-\frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 - 4x \right]_1^4 \\
 &= 2\pi \left[\left(-64 + \frac{256}{3} + 8 - 16 \right) - \left(-\frac{1}{4} + \frac{4}{3} + \frac{1}{2} - 4 \right) \right] \\
 &= 2\pi \left(\frac{63}{4} \right) = \frac{63\pi}{2}
 \end{aligned}$$



40. Use washers:

$$\begin{aligned}
 V &= \int_{-1}^1 \pi \left\{ [2 - 0]^2 - [2 - (1 - y^4)]^2 \right\} dy \\
 &= 2\pi \int_0^1 [4 - (1 + y^4)^2] dy \quad [\text{by symmetry}] \\
 &= 2\pi \int_0^1 [4 - (1 + 2y^4 + y^8)] dy = 2\pi \int_0^1 (3 - 2y^4 - y^8) dy \\
 &= 2\pi \left[3y - \frac{2}{5}y^5 - \frac{1}{9}y^9 \right]_0^1 = 2\pi \left(3 - \frac{2}{5} - \frac{1}{9} \right) = 2\pi \left(\frac{112}{45} \right) = \frac{224\pi}{45}
 \end{aligned}$$



41. Use disks: $V = \pi \int_0^2 \left[\sqrt{1 - (y-1)^2} \right]^2 dy = \pi \int_0^2 (2y - y^2) dy = \pi \left[y^2 - \frac{1}{3}y^3 \right]_0^2 = \pi \left(4 - \frac{8}{3} \right) = \frac{4}{3}\pi$

42. Using shells, we have

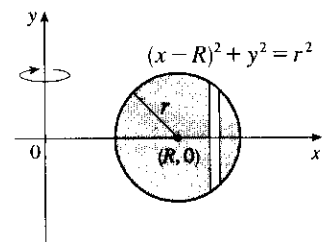
$$\begin{aligned}
 V &= \int_0^2 2\pi y \left[\sqrt{1 - (y-1)^2} - \left(-\sqrt{1 - (y-1)^2} \right) \right] dy \\
 &= 2\pi \int_0^2 y \cdot 2\sqrt{1 - (y-1)^2} dy = 4\pi \int_{-1}^1 (u+1)\sqrt{1-u^2} du \quad [\text{let } u = y-1] \\
 &= 4\pi \int_{-1}^1 u\sqrt{1-u^2} du + 4\pi \int_{-1}^1 \sqrt{1-u^2} du
 \end{aligned}$$

The first definite integral equals zero because its integrand is an odd function. The second is the area of a semicircle of radius 1, that is, $\frac{\pi}{2}$. Thus, $V = 4\pi \cdot 0 + 4\pi \cdot \frac{\pi}{2} = 2\pi^2$.

43. $V = 2 \int_0^r 2\pi x \sqrt{r^2 - x^2} dx = -2\pi \int_0^r (r^2 - x^2)^{1/2} (-2x) dx = \left[-2\pi \cdot \frac{2}{3} (r^2 - x^2)^{3/2} \right]_0^r$
 $= -\frac{4}{3}\pi(0 - r^3) = \frac{4}{3}\pi r^3$

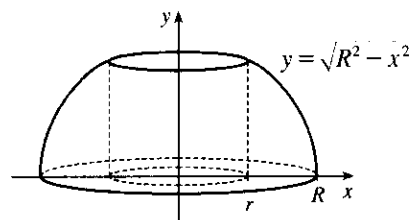
44. $V = \int_{-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x-R)^2} dx$
 $= \int_{-r}^r 4\pi(u+R)\sqrt{r^2 - u^2} du \quad [\text{let } u = x-R]$
 $= 4\pi R \int_{-r}^r \sqrt{r^2 - u^2} du + 4\pi \int_{-r}^r u\sqrt{r^2 - u^2} du$

The first integral is the area of a semicircle of radius r , that is, $\frac{1}{2}\pi r^2$, and the second is zero since the integrand is an odd function. Thus,
 $V = 4\pi R \left(\frac{1}{2}\pi r^2 \right) + 4\pi \cdot 0 = 2\pi R r^2$.



$$45. V = 2\pi \int_0^r x \left(-\frac{h}{r}x + h \right) dx = 2\pi h \int_0^r \left(-\frac{x^2}{r} + x \right) dx = 2\pi h \left[-\frac{x^3}{3r} + \frac{x^2}{2} \right]_0^r = 2\pi h \frac{r^2}{6} = \frac{\pi r^2 h}{3}$$

46. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x -axis and below the curve $y = \sqrt{R^2 - x^2}$ (the equation of the top half of the cross-section of the sphere), between $x = r$ and $x = R$, about the y -axis.



This volume is equal to

$$2 \int_{\text{inner radius}}^{\text{outer radius}} 2\pi r h \, dx = 2 \cdot 2\pi \int_r^R x \sqrt{R^2 - x^2} \, dx = 4\pi \left[-\frac{1}{3} (R^2 - x^2)^{3/2} \right]_r^R = \frac{4}{3}\pi (R^2 - r^2)^{3/2}$$

But by the Pythagorean Theorem, $R^2 - r^2 = (\frac{1}{2}h)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi (\frac{1}{2}h)^3 = \frac{1}{6}\pi h^3$, which is independent of both R and r ; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.68.

Another solution: The height of the missing cap is the radius of the sphere minus half the height of the cut-out cylinder, that is, $R - \frac{1}{2}h$. Using Exercise 6.2.49,

$$V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} = \frac{4}{3}\pi R^3 - \pi r^2 h - 2 \cdot \frac{\pi}{3} (R - \frac{1}{2}h)^2 [3R - (R - \frac{1}{2}h)] = \frac{1}{6}\pi h^3$$

6.4 Work

- By Equation 2, $W = Fd = (900)(8) = 7200$ J.
- $F = mg = (60)(9.8) = 588$ N; $W = Fd = 588 \cdot 2 = 1176$ J
- $W = \int_a^b f(x) \, dx = \int_0^9 \frac{10}{(1+x)^2} \, dx = 10 \int_1^{10} \frac{1}{u^2} \, du \quad [u = 1+x, \, du = dx]$
 $= 10 \left[-\frac{1}{u} \right]_1^{10} = 10 \left(-\frac{1}{10} + 1 \right) = 9$ ft-lb
- $W = \int_1^2 \cos\left(\frac{1}{3}\pi x\right) \, dx = \frac{3}{\pi} \left[\sin\left(\frac{1}{3}\pi x\right) \right]_1^2 = \frac{3}{\pi} \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) = 0$ N-m = 0 J.

Interpretation: From $x = 1$ to $x = \frac{3}{2}$, the force does work equal to $\int_1^{3/2} \cos\left(\frac{1}{3}\pi x\right) \, dx = \frac{3}{\pi} \left(1 - \frac{\sqrt{3}}{2} \right)$ J in accelerating the particle and increasing its kinetic energy. From $x = \frac{3}{2}$ to $x = 2$, the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from $x = 1$ to $x = \frac{3}{2}$.

- The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by $\int_0^8 F(x) \, dx = \int_0^4 F(x) \, dx + \int_4^8 F(x) \, dx = \frac{1}{2}(4)(30) + (4)(30) = 180$ J.
- $W = \int_4^{20} f(x) \, dx \approx M_4 = \Delta x [f(6) + f(10) + f(14) + f(18)] = \frac{20-4}{4} [5.8 + 8.8 + 8.2 + 5.2] = 4(28) = 112$ J
- $10 = f(x) = kx = \frac{1}{3}k$ [4 inches = $\frac{1}{3}$ foot], so $k = 30$ lb/ft and $f(x) = 30x$. Now 6 inches = $\frac{1}{2}$ foot, so $W = \int_0^{1/2} 30x \, dx = \left[15x^2 \right]_0^{1/2} = \frac{15}{4}$ ft-lb.

8. $25 = f(x) = kx = k(0.1)$ [10 cm = 0.1 m], so $k = 250$ N/m and $f(x) = 250x$. Now 5 cm = 0.05 m, so

$$W = \int_0^{0.05} 250x \, dx = [125x^2]_0^{0.05} = 125(0.0025) = 0.3125 \approx 0.31 \text{ J.}$$

9. If $\int_0^{0.12} kx \, dx = 2$ J, then $2 = [\frac{1}{2}kx^2]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N/m.

Thus, the work needed to stretch the spring from 35 cm to 40 cm is

$$\int_{0.05}^{0.10} \frac{2500}{9} x \, dx = [\frac{1250}{9} x^2]_{1/20}^{1/10} = \frac{1250}{9} (\frac{1}{100} - \frac{1}{400}) = \frac{25}{24} \approx 1.04 \text{ J.}$$

10. If $12 = \int_0^1 kx \, dx = [\frac{1}{2}kx^2]_0^1 = \frac{1}{2}k$, then $k = 24$ lb/ft and the work required is

$$\int_0^{3/4} 24x \, dx = [12x^2]_0^{3/4} = 12 \cdot \frac{9}{16} = \frac{27}{4} = 6.75 \text{ ft-lb.}$$

11. $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m = 10.8 cm

12. Let L be the natural length of the spring in meters. Then

$$6 = \int_{0.10-L}^{0.12-L} kx \, dx = [\frac{1}{2}kx^2]_{0.10-L}^{0.12-L} = \frac{1}{2}k[(0.12-L)^2 - (0.10-L)^2] \text{ and}$$

$$10 = \int_{0.12-L}^{0.14-L} kx \, dx = [\frac{1}{2}kx^2]_{0.12-L}^{0.14-L} = \frac{1}{2}k[(0.14-L)^2 - (0.12-L)^2]. \text{ Simplifying gives us}$$

$$12 = k(0.0044 - 0.04L) \text{ and } 20 = k(0.0052 - 0.04L). \text{ Subtracting the first equation from the second gives}$$

$$8 = 0.0008k, \text{ so } k = 10,000. \text{ Now the second equation becomes } 20 = 52 - 400L, \text{ so } L = \frac{32}{400} \text{ m} = 8 \text{ cm.}$$

In Exercises 13–20, n is the number of subintervals of length Δx , and x_i^* is a sample point in the i th subinterval $[x_{i-1}, x_i]$.

13. (a) The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2} \Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2} x_i^* \Delta x$ ft-lb. The total work is

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} x_i^* \Delta x = \int_0^{50} \frac{1}{2} x \, dx = [\frac{1}{4} x^2]_0^{50} = \frac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is

$$W_1 = \int_0^{25} \frac{1}{2} x \, dx = [\frac{1}{4} x^2]_0^{25} = \frac{625}{4} \text{ ft-lb. The bottom half of the rope is lifted 25 ft and the work needed to}$$

$$\text{accomplish that is } W_2 = \int_{25}^{50} \frac{1}{2} \cdot 25 \, dx = \frac{25}{2} [x]_{25}^{50} = \frac{625}{2} \text{ ft-lb. The total work done in pulling half the rope to the top of the building is } W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4} \cdot 625 = \frac{1875}{4} \text{ ft-lb.}$$

14. Assumptions: 1. After lifting, the chain is L-shaped, with 4 m of the chain lying along the ground.

2. The chain slides effortlessly and without friction along the ground while its end is lifted.

3. The weight density of the chain is constant throughout its length and therefore equals

$$(8 \text{ kg/m})(9.8 \text{ m/s}^2) = 78.4 \text{ N/m.}$$

The part of the chain x m from the lifted end is raised $6 - x$ m if $0 \leq x \leq 6$ m, and it is lifted 0 m if $x > 6$ m.

Thus, the work needed is

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n (6 - x_i^*) \cdot 78.4 \Delta x = \int_0^6 (6 - x) 78.4 \, dx = 78.4 [6x - \frac{1}{2} x^2]_0^6 = (78.4)(18) = 1411.2 \text{ J.}$$

15. The work needed to lift the cable is $\lim_{n \rightarrow \infty} \sum_{i=1}^n 2x_i^* \Delta x = \int_0^{500} 2x \, dx = [x^2]_0^{500} = 250,000$ ft-lb. The work needed to lift the coal is $800 \text{ lb} \cdot 500 \text{ ft} = 400,000$ ft-lb. Thus, the total work required is $250,000 + 400,000 = 650,000$ ft-lb.

16. The work needed to lift the bucket itself is $4 \text{ lb} \cdot 80 \text{ ft} = 320 \text{ ft-lb}$. At time t (in seconds) the bucket is $x_i^* = 2t$ ft above its original 80 ft depth, but it now holds only $(40 - 0.2t)$ lb of water. In terms of distance, the bucket holds $[40 - 0.2(\frac{1}{2}x_i^*)]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $(40 - \frac{1}{10}x_i^*) \Delta x$ ft-lb of work. Thus, the work needed to lift the water is

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n (40 - \frac{1}{10}x_i^*) \Delta x = \int_0^{80} (40 - \frac{1}{10}x) dx = [40x - \frac{1}{20}x^2]_0^{80} = (3200 - 320) \text{ ft-lb}$$

Adding the work of lifting the bucket gives a total of 3200 ft-lb of work.

17. At a height of x meters ($0 \leq x \leq 12$), the mass of the rope is $(0.8 \text{ kg/m})(12 - x \text{ m}) = (9.6 - 0.8x) \text{ kg}$ and the mass of the water is $(\frac{36}{12} \text{ kg/m})(12 - x \text{ m}) = (36 - 3x) \text{ kg}$. The mass of the bucket is 10 kg, so the total mass is $(9.6 - 0.8x) + (36 - 3x) + 10 = (55.6 - 3.8x) \text{ kg}$, and hence, the total force is $9.8(55.6 - 3.8x) \text{ N}$.

The work needed to lift the bucket Δx m through the i th subinterval of $[0, 12]$ is $9.8(55.6 - 3.8x_i^*) \Delta x$, so the total work is

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 9.8(55.6 - 3.8x_i^*) \Delta x = \int_0^{12} (9.8)(55.6 - 3.8x) dx = 9.8 \left[55.6x - 1.9x^2 \right]_0^{12} \\ &= 9.8(393.6) \approx 3857 \text{ J} \end{aligned}$$

18. The chain's weight density is $\frac{25 \text{ lb}}{10 \text{ ft}} = 2.5 \text{ lb/ft}$. The part of the chain x ft below the ceiling (for $5 \leq x \leq 10$) has to be lifted $2(x - 5)$ ft, so the work needed to lift the i th subinterval of the chain is $2(x_i^* - 5)(2.5 \Delta x)$. The total work needed is

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2(x_i^* - 5)(2.5) \Delta x = \int_5^{10} [2(x - 5)(2.5)] dx = 5 \int_5^{10} (x - 5) dx \\ &= 5 \left[\frac{1}{2}x^2 - 5x \right]_5^{10} = 5 \left[(50 - 50) - \left(\frac{25}{2} - 25 \right) \right] = 5 \left(\frac{25}{2} \right) = 62.5 \text{ ft-lb} \end{aligned}$$

19. A "slice" of water Δx m thick and lying at a depth of x_i^* m (where $0 \leq x_i^* \leq \frac{1}{2}$) has volume $(2 \times 1 \times \Delta x) \text{ m}^3$, a mass of $2000 \Delta x$ kg, weighs about $(9.8)(2000 \Delta x) = 19,600 \Delta x \text{ N}$, and thus requires about $19,600x_i^* \Delta x$ J of work for its removal. So $W = \lim_{n \rightarrow \infty} \sum_{i=1}^n 19,600x_i^* \Delta x = \int_0^{1/2} 19,600x dx = [9800x^2]_0^{1/2} = 2450 \text{ J}$.
20. A horizontal cylindrical slice of water Δx ft thick has a volume of $\pi r^2 h = \pi \cdot 12^2 \cdot \Delta x \text{ ft}^3$ and weighs about $(62.5 \text{ lb/ft}^3)(144\pi \Delta x \text{ ft}^3) = 9000\pi \Delta x \text{ lb}$. If the slice lies x_i^* ft below the edge of the pool (where $1 \leq x_i^* \leq 5$), then the work needed to pump it out is about $9000\pi x_i^* \Delta x$. Thus,

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n 9000\pi x_i^* \Delta x = \int_1^5 9000\pi x dx = [4500\pi x^2]_1^5 = 4500\pi(25 - 1) = 108,000\pi \text{ ft-lb}$$

21. A rectangular "slice" of water Δx m thick and lying x ft above the bottom has width x ft and volume $8x \Delta x \text{ m}^3$. It weighs about $(9.8 \times 1000)(8x \Delta x) \text{ N}$, and must be lifted $(5 - x) \text{ m}$ by the pump, so the work needed is about $(9.8 \times 10^3)(5 - x)(8x \Delta x) \text{ J}$. The total work required is

$$\begin{aligned} W &\approx \int_0^3 (9.8 \times 10^3)(5 - x)8x dx = (9.8 \times 10^3) \int_0^3 (40x - 8x^2) dx = (9.8 \times 10^3) \left[20x^2 - \frac{8}{3}x^3 \right]_0^3 \\ &= (9.8 \times 10^3)(180 - 72) = (9.8 \times 10^3)(108) = 1058.4 \times 10^3 \approx 1.06 \times 10^6 \text{ J} \end{aligned}$$

22. For convenience, measure depth x from the middle of the tank, so that $-1.5 \leq x \leq 1.5$ m.

Lifting a slice of water of thickness Δx at depth x requires a work contribution of

$$\Delta W \approx (9.8 \times 10^3) \left(2\sqrt{(1.5)^2 - x^2} \right) (6 \Delta x)(2.5 + x), \text{ so}$$

$$\begin{aligned} W &\approx \int_{-1.5}^{1.5} (9.8 \times 10^3) 12 \sqrt{2.25 - x^2} (2.5 + x) dx \\ &= (9.8 \times 10^3) \left[60 \int_0^{3/2} \sqrt{\frac{9}{4} - x^2} dx + 12 \int_{-3/2}^{3/2} x \sqrt{\frac{9}{4} - x^2} dx \right] \end{aligned}$$

The second integral is 0 because its integrand is an odd function, and the first integral represents the area of a quarter-circle of radius $\frac{3}{2}$. Therefore,

$$W \approx (9.8 \times 10^3) 60 \int_0^{3/2} \sqrt{\frac{9}{4} - x^2} dx = (9.8 \times 10^3) (60) \left(\frac{1}{4} \pi \right) \left(\frac{3}{2} \right)^2 = 330,750\pi \approx 1.04 \times 10^6 \text{ J}$$

23. Measure depth x downward from the flat top of the tank, so that $0 \leq x \leq 2$ ft. Then

$$\Delta W = (62.5)(2\sqrt{4 - x^2})(8 \Delta x)(x + 1) \text{ ft-lb, so}$$

$$\begin{aligned} W &\approx (62.5)(16) \int_0^2 (x + 1) \sqrt{4 - x^2} dx = 1000 \left(\int_0^2 x \sqrt{4 - x^2} dx + \int_0^2 \sqrt{4 - x^2} dx \right) \\ &= 1000 \left[\int_0^4 u^{1/2} \left(\frac{1}{2} du \right) + \frac{1}{4} \pi (2^2) \right] \quad [\text{Put } u = 4 - x^2, \text{ so } du = -2x dx] \\ &= 1000 \left(\left[\frac{1}{2} \cdot \frac{2}{3} u^{3/2} \right]_0^4 + \pi \right) = 1000 \left(\frac{8}{3} + \pi \right) \approx 5.8 \times 10^3 \text{ ft-lb} \end{aligned}$$

Note: The second integral represents the area of a quarter-circle of radius 2.

24. Let x be depth in feet, so that $0 \leq x \leq 5$. Then $\Delta W = (62.5)\pi(\sqrt{5^2 - x^2})^2 \Delta x \cdot x$ ft-lb and

$$\begin{aligned} W &\approx 62.5\pi \int_0^5 x(25 - x^2) dx = 62.5\pi \left[\frac{25}{2} x^2 - \frac{1}{4} x^4 \right]_0^5 = 62.5\pi \left(\frac{625}{2} - \frac{625}{4} \right) = 62.5\pi \left(\frac{625}{4} \right) \\ &\approx 3.07 \times 10^4 \text{ ft-lb} \end{aligned}$$

25. If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in Exercise 21, except that the work is fixed, and we are trying to find the lower limit of

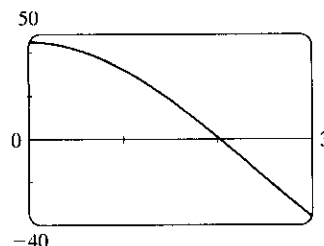
$$\text{integration: } 4.7 \times 10^5 \approx \int_h^3 (9.8 \times 10^3)(5 - x)8x dx = (9.8 \times 10^3) \left[20x^2 - \frac{8}{3}x^3 \right]_h^3 \Leftrightarrow$$

$$\frac{4.7}{9.8} \times 10^2 \approx 48 = \left(20 \cdot 3^2 - \frac{8}{3} \cdot 3^3 \right) - \left(20h^2 - \frac{8}{3}h^3 \right) \Leftrightarrow$$

$$2h^3 - 15h^2 + 45 = 0. \text{ To find the solution of this equation, we plot}$$

$$2h^3 - 15h^2 + 45 \text{ between } h = 0 \text{ and } h = 3. \text{ We see that the equation}$$

is satisfied for $h \approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.



$$26. W \approx (9.8 \times 920) \int_0^{3/2} 12 \sqrt{\frac{9}{4} - x^2} \left(\frac{5}{2} + x\right) dx = 9016 \left[30 \int_0^{3/2} \sqrt{\frac{9}{4} - x^2} dx + 12 \int_0^{3/2} x \sqrt{\frac{9}{4} - x^2} dx \right].$$

$$\text{Here } \int_0^{3/2} \sqrt{\frac{9}{4} - x^2} dx = \frac{1}{4} \pi \left(\frac{3}{2}\right)^2 = \frac{9\pi}{16} \text{ and } \int_0^{3/2} x \sqrt{\frac{9}{4} - x^2} dx = \int_0^{9/4} \frac{1}{2} u^{1/2} du$$

$$[\text{where } u = \frac{9}{4} - x^2, \text{ so } du = -2x dx] = \left[-\frac{1}{3} u^{3/2} \right]_0^{9/4} = -\frac{1}{3} \left(\frac{27}{8} \right) = -\frac{9}{8}, \text{ so}$$

$$W \approx 9016 \left[30 \cdot \frac{9\pi}{16} + 12 \cdot \frac{9}{8} \right] = 9016 \left(\frac{135}{8} \pi + \frac{27}{2} \right) \approx 6.00 \times 10^5 \text{ J.}$$

27. $V = \pi r^2 x$, so V is a function of x and P can also be regarded as a function of x . If $V_1 = \pi r^2 x_1$ and $V_2 = \pi r^2 x_2$, then

$$\begin{aligned} W &= \int_{x_1}^{x_2} F(x) dx = \int_{x_1}^{x_2} \pi r^2 P(V(x)) dx \\ &= \int_{x_1}^{x_2} P(V(x)) dV(x) \quad [\text{Let } V(x) = \pi r^2 x, \text{ so } dV(x) = \pi r^2 dx.] \\ &= \int_{V_1}^{V_2} P(V) dV \quad \text{by the Substitution Rule.} \end{aligned}$$

$$28. 160 \text{ lb/in}^2 = 160 \cdot 144 \text{ lb/ft}^2, 100 \text{ in}^3 = \frac{100}{1728} \text{ ft}^3, \text{ and } 800 \text{ in}^3 = \frac{800}{1728} \text{ ft}^3.$$

$$k = PV^{1.4} = (160 \cdot 144) \left(\frac{100}{1728} \right)^{1.4} = 23,040 \left(\frac{25}{432} \right)^{1.4} \approx 426.5. \text{ Therefore, } P \approx 426.5 V^{-1.4} \text{ and}$$

$$\begin{aligned} W &= \int_{100/1728}^{800/1728} 426.5 V^{-1.4} dV = 426.5 \left[\frac{-1}{-0.4} V^{-0.4} \right]_{25/432}^{25/54} \\ &= (426.5)(2.5) \left[\left(\frac{432}{25} \right)^{0.4} - \left(\frac{54}{25} \right)^{0.4} \right] \\ &\approx 1.88 \times 10^3 \text{ ft-lb} \end{aligned}$$

$$29. W = \int_a^b F(r) dr = \int_a^b G \frac{m_1 m_2}{r^2} dr = G m_1 m_2 \left[\frac{-1}{r} \right]_a^b = G m_1 m_2 \left(\frac{1}{a} - \frac{1}{b} \right)$$

30. By Exercise 29, $W = GMm \left(\frac{1}{R} - \frac{1}{R + 1,000,000} \right)$ where M = mass of Earth in kg, R = radius of Earth in m, and m = mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,

$$W = (6.67 \times 10^{-11}) (5.98 \times 10^{24}) (1000) \times \left(\frac{1}{6.37 \times 10^6} - \frac{1}{7.37 \times 10^6} \right) \approx 8.50 \times 10^9 \text{ J}$$

6.5 Average Value of a Function

$$1. f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{1-(-1)} \int_{-1}^1 x^2 dx = \frac{1}{2} \cdot 2 \int_0^1 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1}{3}$$

$$2. f_{\text{ave}} = \frac{1}{2-0} \int_0^2 (x-x^2) dx = \frac{1}{2} \left[\frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_0^2 = \frac{1}{2} \left(2 - \frac{8}{3} \right) = -\frac{1}{3}$$

$$3. g_{\text{ave}} = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} \cos x \, dx = \frac{2}{\pi} [\sin x]_0^{\pi/2} = \frac{2}{\pi} (1 - 0) = \frac{2}{\pi}$$

$$4. g_{\text{ave}} = \frac{1}{2-0} \int_0^2 x^2 \sqrt{1+x^3} \, dx = \frac{1}{2} \int_1^9 \sqrt{u} \cdot \frac{1}{3} \, du \quad [u = 1+x^3, du = 3x^2 \, dx]$$

$$= \frac{1}{6} \left[\frac{2}{3} u^{3/2} \right]_1^9 = \frac{1}{9} (27 - 1) = \frac{26}{9}$$

$$5. f_{\text{ave}} = \frac{1}{5-0} \int_0^5 t \sqrt{1+t^2} \, dt = \frac{1}{5} \int_1^{26} \sqrt{u} \left(\frac{1}{2} \, du \right) \quad [u = 1+t^2, du = 2t \, dt]$$

$$= \frac{1}{10} \int_1^{26} u^{1/2} \, du = \frac{1}{10} \cdot \frac{2}{3} \left[u^{3/2} \right]_1^{26} = \frac{1}{15} (26^{3/2} - 1)$$

$$6. f_{\text{ave}} = \frac{1}{\frac{\pi}{4} - 0} \int_0^{\pi/4} \sec \theta \tan \theta \, d\theta = \frac{4}{\pi} [\sec \theta]_0^{\pi/4} = \frac{4}{\pi} (\sqrt{2} - 1)$$

$$7. h_{\text{ave}} = \frac{1}{\pi - 0} \int_0^{\pi} \cos^4 x \sin x \, dx = \frac{1}{\pi} \int_1^{-1} u^4 (-du) \quad [u = \cos x, du = -\sin x \, dx]$$

$$= \frac{1}{\pi} \int_{-1}^1 u^4 \, du = \frac{1}{\pi} \cdot 2 \int_0^1 u^4 \, du = \frac{2}{\pi} \left[\frac{1}{5} u^5 \right]_0^1 = \frac{2}{5\pi}$$

$$8. h_{\text{ave}} = \frac{1}{6-1} \int_1^6 \frac{3}{(1+r)^2} \, dr = \frac{1}{5} \int_2^7 3u^{-2} \, du \quad [u = 1+r, du = dr]$$

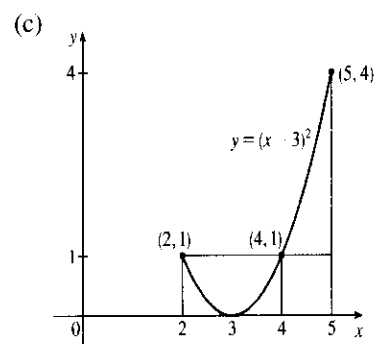
$$= -\frac{3}{5} [u^{-1}]_2^7 = -\frac{3}{5} \left(\frac{1}{7} - \frac{1}{2} \right) = \frac{3}{5} \left(\frac{1}{2} - \frac{1}{7} \right) = \frac{3}{5} \cdot \frac{5}{14} = \frac{3}{14}$$

$$9. (a) f_{\text{ave}} = \frac{1}{5-2} \int_2^5 (x-3)^2 \, dx = \frac{1}{3} \left[\frac{1}{3} (x-3)^3 \right]_2^5$$

$$= \frac{1}{9} [2^3 - (-1)^3] = \frac{1}{9} (8 + 1) = 1$$

$$(b) f(c) = f_{\text{ave}} \Leftrightarrow (c-3)^2 = 1 \Leftrightarrow c-3 = \pm 1$$

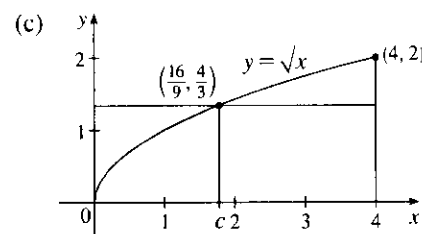
$$\Leftrightarrow c = 2 \text{ or } 4$$



$$10. (a) f_{\text{ave}} = \frac{1}{4-0} \int_0^4 \sqrt{x} \, dx = \frac{1}{4} \left[\frac{2}{3} x^{3/2} \right]_0^4$$

$$= \frac{1}{6} [x^{3/2}]_0^4 = \frac{1}{6} [8 - 0] = \frac{4}{3}$$

$$(b) f(c) = f_{\text{ave}} \Leftrightarrow \sqrt{c} = \frac{4}{3} \Leftrightarrow c = \frac{16}{9}$$



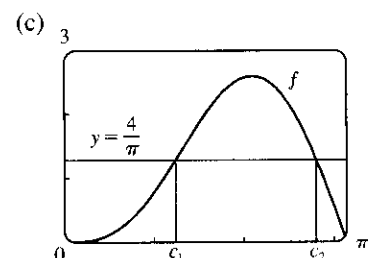
$$11. (a) f_{\text{ave}} = \frac{1}{\pi - 0} \int_0^{\pi} (2 \sin x - \sin 2x) \, dx$$

$$= \frac{1}{\pi} \left[-2 \cos x + \frac{1}{2} \cos 2x \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\left(2 + \frac{1}{2} \right) - \left(-2 + \frac{1}{2} \right) \right] = \frac{4}{\pi}$$

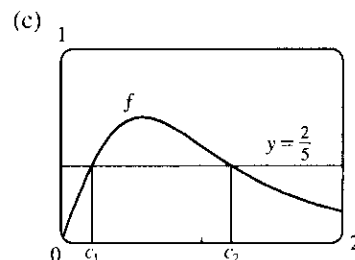
$$(b) f(c) = f_{\text{ave}} \Leftrightarrow 2 \sin c - \sin 2c = \frac{4}{\pi} \Leftrightarrow$$

$$c_1 \approx 1.238 \text{ or } c_2 \approx 2.808$$



$$\begin{aligned}
 12. (a) f_{\text{ave}} &= \frac{1}{2-0} \int_0^2 \frac{2x}{(1+x^2)^2} dx \\
 &= \frac{1}{2} \int_1^5 \frac{1}{u^2} du \quad [u = 1+x^2, du = 2x dx] \\
 &= \frac{1}{2} \left[-\frac{1}{u} \right]_1^5 = -\frac{1}{2} \left(\frac{1}{5} - 1 \right) = \frac{2}{5}
 \end{aligned}$$

$$\begin{aligned}
 (b) f(c) = f_{\text{ave}} &\Leftrightarrow \frac{2c}{(1+c^2)^2} = \frac{2}{5} \Leftrightarrow 5c = (1+c^2)^2 \\
 &\Leftrightarrow c_1 \approx 0.220 \text{ or } c_2 \approx 1.207
 \end{aligned}$$



13. f is continuous on $[1, 3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1, 3]$ such that $\int_1^3 f(x) dx = f(c)(3-1) \Rightarrow 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

14. The requirement is that $\frac{1}{b-0} \int_0^b f(x) dx = 3$. The LHS of this equation is equal to

$$\begin{aligned}
 \frac{1}{b} \int_0^b (2+6x-3x^2) dx &= \frac{1}{b} [2x+3x^2-x^3]_0^b = 2+3b-b^2, \text{ so we solve the equation } 2+3b-b^2=3 \Leftrightarrow \\
 b^2-3b+1 &= 0 \Leftrightarrow b = \frac{3 \pm \sqrt{(-3)^2-4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{3 \pm \sqrt{5}}{2}. \text{ Both roots are valid since they are positive.}
 \end{aligned}$$

$$\begin{aligned}
 15. f_{\text{ave}} &= \frac{1}{50-20} \int_{20}^{50} f(x) dx \approx \frac{1}{30} M_3 = \frac{1}{30} \cdot \frac{50-20}{3} [f(25) + f(35) + f(45)] \\
 &= \frac{1}{3} (38 + 29 + 48) = \frac{115}{3} = 38\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 16. (a) v_{\text{ave}} &= \frac{1}{12-0} \int_0^{12} v(t) dt = \frac{1}{12} I. \text{ Use the Midpoint Rule with } n=3 \text{ and } \Delta t = \frac{12-0}{3} = 4 \text{ to estimate } I. \\
 I &\approx M_3 = 4[v(2) + v(6) + v(10)] = 4[21 + 50 + 66] = 4(137) = 548. \text{ Thus, } v_{\text{ave}} \approx \frac{1}{12}(548) = 45\frac{2}{3} \text{ km/h.}
 \end{aligned}$$

$$(b) \text{ Estimating from the graph, } v(t) = 45\frac{2}{3} \text{ when } t \approx 5.2 \text{ s.}$$

17. Let $t = 0$ and $t = 12$ correspond to 9 A.M. and 9 P.M., respectively.

$$\begin{aligned}
 T_{\text{ave}} &= \frac{1}{12-0} \int_0^{12} \left[50 + 14 \sin \frac{1}{12} \pi t \right] dt = \frac{1}{12} \left[50t - 14 \cdot \frac{12}{\pi} \cos \frac{1}{12} \pi t \right]_0^{12} \\
 &= \frac{1}{12} \left[50 \cdot 12 + 14 \cdot \frac{12}{\pi} + 14 \cdot \frac{12}{\pi} \right] = \left(50 + \frac{28}{\pi} \right) ^\circ \text{F} \approx 59^\circ \text{F}
 \end{aligned}$$

$$18. T_{\text{ave}} = \frac{1}{5} \int_0^5 4x dx = \frac{1}{5} [2x^2]_0^5 = 10^\circ \text{C}$$

$$19. \rho_{\text{ave}} = \frac{1}{8} \int_0^8 \frac{12}{\sqrt{x+1}} dx = \frac{3}{2} \int_0^8 (x+1)^{-1/2} dx = [3\sqrt{x+1}]_0^8 = 9 - 3 = 6 \text{ kg/m}$$

$$\begin{aligned}
 20. s &= \frac{1}{2}gt^2 \Rightarrow t = \sqrt{2s/g} \quad [\text{since } t \geq 0]. \text{ Now } v = \frac{ds}{dt} = gt = g\sqrt{2s/g} = \sqrt{2gs} \Rightarrow v^2 = 2gs \Rightarrow \\
 s &= \frac{v^2}{2g}. \text{ We see that } v \text{ can be regarded as a function of } t \text{ or of } s: v = F(t) = gt \text{ and } v = G(s) = \sqrt{2gs}. \text{ Note that} \\
 v_T &= F(T) = gT. \text{ Displacement can be viewed as a function of } t: s = s(t) = \frac{1}{2}gt^2; \text{ also } s(t) = \frac{v^2}{2g} = \frac{[F(t)]^2}{2g}.
 \end{aligned}$$

When $t = T$, these two formulas for $s(t)$ imply that

$$\sqrt{2gs(T)} = F(T) = v_T = gT = 2\left(\frac{1}{2}gT^2\right)/T = 2s(T)/T \quad (*)$$

The average of the velocities with respect to time t during the interval $[0, T]$ is

$$\begin{aligned} v_{t\text{-ave}} = F_{\text{ave}} &= \frac{1}{T-0} \int_0^T F(t) dt = \frac{1}{T} [s(T) - s(0)] \quad [\text{by FTC}] \\ &= \frac{s(T)}{T} \quad [\text{since } s(0) = 0] = \frac{1}{2}v_T \quad [\text{by } (*)] \end{aligned}$$

But the average of the velocities with respect to displacement s during the corresponding displacement interval $[s(0), s(T)] = [0, s(T)]$ is

$$\begin{aligned} v_{s\text{-ave}} = G_{\text{ave}} &= \frac{1}{s(T)-0} \int_0^{s(T)} G(s) ds = \frac{1}{s(T)} \int_0^{s(T)} \sqrt{2gs} ds = \frac{\sqrt{2g}}{s(T)} \int_0^{s(T)} s^{1/2} ds \\ &= \frac{\sqrt{2g}}{s(T)} \cdot \frac{2}{3} [s^{3/2}]_0^{s(T)} = \frac{2}{3} \cdot \frac{\sqrt{2g}}{s(T)} \cdot [s(T)]^{3/2} = \frac{2}{3} \sqrt{2gs(T)} = \frac{2}{3}v_T \quad [\text{by } (*)] \end{aligned}$$

$$\begin{aligned} 21. V_{\text{ave}} &= \frac{1}{5} \int_0^5 V(t) dt = \frac{1}{5} \int_0^5 \frac{5}{4\pi} [1 - \cos(\frac{2}{5}\pi t)] dt = \frac{1}{4\pi} \int_0^5 [1 - \cos(\frac{2}{5}\pi t)] dt \\ &= \frac{1}{4\pi} [t - \frac{5}{2\pi} \sin(\frac{2}{5}\pi t)]_0^5 = \frac{1}{4\pi} [(5-0) - 0] = \frac{5}{4\pi} \approx 0.4 \text{ L} \end{aligned}$$

$$22. v_{\text{ave}} = \frac{1}{R-0} \int_0^R v(r) dr = \frac{1}{R} \int_0^R \frac{P}{4\eta l} (R^2 - r^2) dr = \frac{P}{4\eta l R} [R^2 r - \frac{1}{3}r^3]_0^R = \frac{P}{4\eta l R} (\frac{2}{3})R^3 = \frac{PR^2}{6\eta l}.$$

Since $v(r)$ is decreasing on $(0, R]$, $v_{\text{max}} = v(0) = \frac{PR^2}{4\eta l}$. Thus, $v_{\text{ave}} = \frac{2}{3}v_{\text{max}}$.

23. Let $F(x) = \int_a^x f(t) dt$ for x in $[a, b]$. Then F is continuous on $[a, b]$ and differentiable on (a, b) , so by the Mean Value Theorem there is a number c in (a, b) such that $F(b) - F(a) = F'(c)(b-a)$. But $F'(x) = f(x)$ by the Fundamental Theorem of Calculus. Therefore, $\int_a^b f(t) dt - 0 = f(c)(b-a)$.

$$\begin{aligned} 24. f_{\text{ave}} [a, b] &= \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{b-a} \int_a^c f(x) dx + \frac{1}{b-a} \int_c^b f(x) dx \\ &= \frac{c-a}{b-a} \left[\frac{1}{c-a} \int_a^c f(x) dx \right] + \frac{b-c}{b-a} \left[\frac{1}{b-c} \int_c^b f(x) dx \right] = \frac{c-a}{b-a} f_{\text{ave}} [a, c] + \frac{b-c}{b-a} f_{\text{ave}} [c, b] \end{aligned}$$

6 Review

CONCEPT CHECK

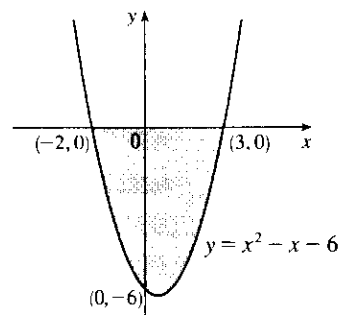
- (a) See Section 6.1, Figure 2 and Equations 6.1.1 and 6.1.2.
(b) Instead of using “top minus bottom” and integrating from left to right, we use “right minus left” and integrate from bottom to top. See Figures 11 and 12 in Section 6.1.
- The numerical value of the area represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) See the discussion in Section 6.2, near Figures 2 and 3, ending in the Definition of Volume.
- (b) See the discussion between Examples 5 and 6 in Section 6.2. If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi(r_{\text{out}}^2) - \pi(r_{\text{in}}^2)$.
4. (a) $V = 2\pi rh \Delta r = (\text{circumference})(\text{height})(\text{thickness})$
- (b) For a typical shell, find the circumference and height in terms of x or y and calculate $V = \int_a^b (\text{circumference})(\text{height})(dx \text{ or } dy)$, where a and b are the limits on x or y .
- (c) Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.
5. $\int_0^6 f(x) dx$ represents the amount of work done. Its units are newton-meters, or joules.
6. (a) The average value of a function f on an interval $[a, b]$ is $f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx$.
- (b) The Mean Value Theorem for Integrals says that there is a number c at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. For a geometric interpretation of the Mean Value Theorem for Integrals, see Figure 2 in Section 6.5 and the discussion that accompanies it.

EXERCISES

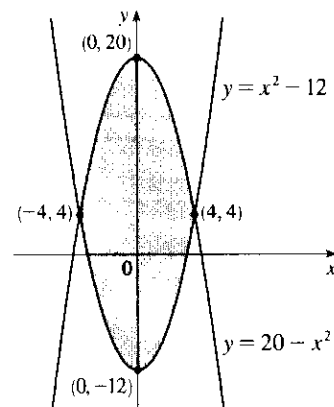
1. $0 = x^2 - x - 6 = (x-3)(x+2) \Leftrightarrow x = 3 \text{ or } -2$. So

$$\begin{aligned} A &= \int_{-2}^3 [0 - (x^2 - x - 6)] dx = \int_{-2}^3 (-x^2 + x + 6) dx \\ &= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right]_{-2}^3 \\ &= \left(-9 + \frac{9}{2} + 18\right) - \left(\frac{8}{3} + 2 - 12\right) \\ &= \frac{125}{6} \end{aligned}$$



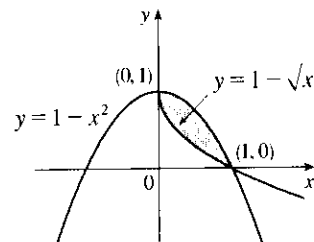
2. $20 - x^2 = x^2 - 12 \Leftrightarrow 32 = 2x^2 \Leftrightarrow x^2 = 16 \Leftrightarrow x = \pm 4$.
So

$$\begin{aligned} A &= \int_{-4}^4 [(20 - x^2) - (x^2 - 12)] dx = \int_{-4}^4 (32 - 2x^2) dx \\ &= 2 \int_0^4 (32 - 2x^2) dx \quad [\text{even function}] \\ &= 2 \left[32x - \frac{2}{3}x^3 \right]_0^4 \\ &= 2 \left(128 - \frac{128}{3} \right) = \frac{512}{3} \end{aligned}$$



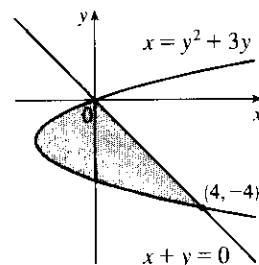
$$\begin{aligned}
 3. \quad 1 - x^2 &= 1 - \sqrt{x} \Leftrightarrow -x^2 = -\sqrt{x} \Leftrightarrow x^2 = \sqrt{x} \Rightarrow x^4 = x \\
 &\Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow \\
 x(x-1)(x^2+x+1) &= 0 \Rightarrow x = 0 \text{ or } 1. \text{ So}
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_0^1 [(1-x^2) - (1-\sqrt{x})] dx = \int_0^1 (\sqrt{x} - x^2) dx \\
 &= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
 \end{aligned}$$

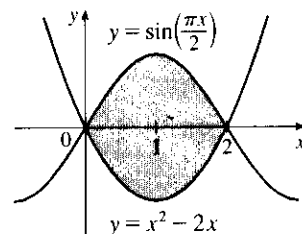


$$\begin{aligned}
 4. \quad y^2 + 3y &= -y \Leftrightarrow y^2 + 4y = 0 \Leftrightarrow y(y+4) = 0 \Leftrightarrow \\
 y &= 0 \text{ or } -4.
 \end{aligned}$$

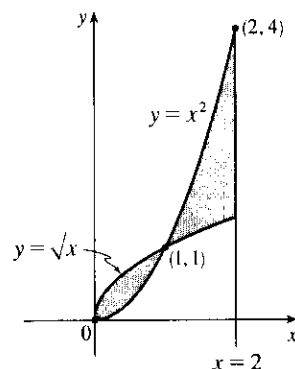
$$\begin{aligned}
 A &= \int_{-4}^0 [-y - (y^2 + 3y)] dy = \int_{-4}^0 (-y^2 - 4y) dy \\
 &= \left[-\frac{1}{3}y^3 - 2y^2 \right]_{-4}^0 = 0 - \left(\frac{64}{3} - 32 \right) = \frac{32}{3}
 \end{aligned}$$



$$\begin{aligned}
 5. \quad A &= \int_0^2 \left[\sin\left(\frac{\pi x}{2}\right) - (x^2 - 2x) \right] dx \\
 &= \left[-\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}x^3 + x^2 \right]_0^2 \\
 &= \left(\frac{2}{\pi} - \frac{8}{3} + 4 \right) - \left(-\frac{2}{\pi} - 0 + 0 \right) = \frac{4}{3} + \frac{4}{\pi}
 \end{aligned}$$

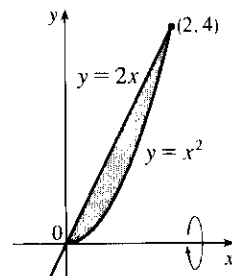


$$\begin{aligned}
 6. \quad A &= \int_0^1 (\sqrt{x} - x^2) dx + \int_1^2 (x^2 - \sqrt{x}) dx \\
 &= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 + \left[\frac{1}{3}x^3 - \frac{2}{3}x^{3/2} \right]_1^2 \\
 &= \left[\left(\frac{2}{3} - \frac{1}{3} \right) - 0 \right] + \left[\left(\frac{8}{3} - \frac{4}{3}\sqrt{2} \right) - \left(\frac{1}{3} - \frac{2}{3} \right) \right] \\
 &= \frac{10}{3} - \frac{4}{3}\sqrt{2}
 \end{aligned}$$



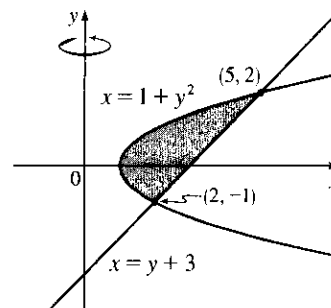
7. Using washers with inner radius x^2 and outer radius $2x$, we have

$$\begin{aligned}
 V &= \pi \int_0^2 [(2x)^2 - (x^2)^2] dx = \pi \int_0^2 (4x^2 - x^4) dx \\
 &= \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \pi \left(\frac{32}{3} - \frac{32}{5} \right) = 32\pi \cdot \frac{2}{15} \\
 &= \frac{64\pi}{15}
 \end{aligned}$$

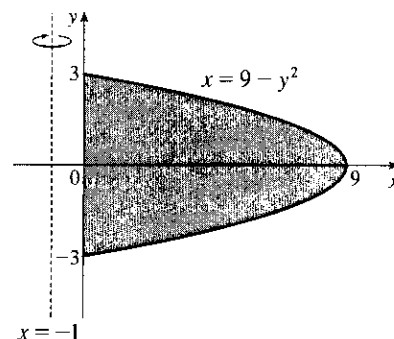


$$8. 1 + y^2 = y + 3 \Leftrightarrow y^2 - y - 2 = 0 \Leftrightarrow (y - 2)(y + 1) = 0 \Leftrightarrow y = 2 \text{ or } -1.$$

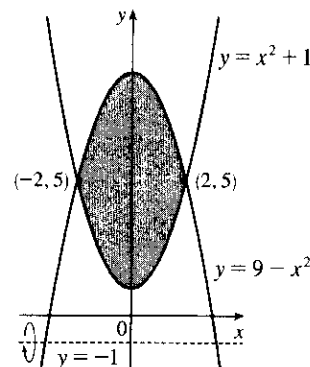
$$\begin{aligned} V &= \pi \int_{-1}^2 [(y + 3)^2 - (1 + y^2)^2] dy \\ &= \pi \int_{-1}^2 (y^2 + 6y + 9 - 1 - 2y^2 - y^4) dy \\ &= \pi \int_{-1}^2 (8 + 6y - y^2 - y^4) dy = \pi \left[8y + 3y^2 - \frac{1}{3}y^3 - \frac{1}{5}y^5 \right]_{-1}^2 \\ &= \pi \left[\left(16 + 12 - \frac{8}{3} - \frac{32}{5} \right) - \left(-8 + 3 + \frac{1}{3} + \frac{1}{5} \right) \right] \\ &= \pi \left(33 - \frac{9}{3} - \frac{33}{5} \right) = \frac{117\pi}{5} \end{aligned}$$



$$\begin{aligned} 9. V &= \pi \int_{-3}^3 \left\{ [(9 - y^2) - (-1)]^2 - [0 - (-1)]^2 \right\} dy \\ &= 2\pi \int_0^3 [(10 - y^2)^2 - 1] dy \\ &= 2\pi \int_0^3 (100 - 20y^2 + y^4 - 1) dy \\ &= 2\pi \int_0^3 (99 - 20y^2 + y^4) dy = 2\pi \left[99y - \frac{20}{3}y^3 + \frac{1}{5}y^5 \right]_0^3 \\ &= 2\pi \left(297 - 180 + \frac{243}{5} \right) = \frac{1656\pi}{5} \end{aligned}$$



$$\begin{aligned} 10. V &= \pi \int_{-2}^2 \left\{ [(9 - x^2) - (-1)]^2 - [(x^2 + 1) - (-1)]^2 \right\} dx \\ &= \pi \int_{-2}^2 [(10 - x^2)^2 - (x^2 + 2)^2] dx \\ &= 2\pi \int_0^2 (96 - 24x^2) dx = 48\pi \int_0^2 (4 - x^2) dx \\ &= 48\pi \left[4x - \frac{1}{3}x^3 \right]_0^2 = 48\pi \left(8 - \frac{8}{3} \right) = 256\pi \end{aligned}$$



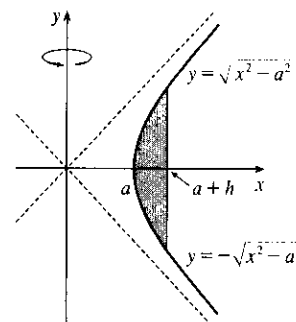
11. The graph of $x^2 - y^2 = a^2$ is a hyperbola with right and left branches. Solving for y gives us $y^2 = x^2 - a^2 \Rightarrow y = \pm\sqrt{x^2 - a^2}$. We'll use shells and the height of each shell is $\sqrt{x^2 - a^2} - (-\sqrt{x^2 - a^2}) = 2\sqrt{x^2 - a^2}$.

The volume is $V = \int_a^{a+h} 2\pi x \cdot 2\sqrt{x^2 - a^2} dx$. To evaluate, let $u = x^2 - a^2$, so $du = 2x dx$ and $x dx = \frac{1}{2} du$.

When $x = a$, $u = 0$, and when $x = a + h$,

$$u = (a + h)^2 - a^2 = a^2 + 2ah + h^2 - a^2 = 2ah + h^2.$$

$$\text{Thus, } V = 4\pi \int_0^{2ah+h^2} \sqrt{u} \left(\frac{1}{2} du \right) = 2\pi \left[\frac{2}{3} u^{3/2} \right]_0^{2ah+h^2} = \frac{4}{3}\pi (2ah + h^2)^{3/2}.$$



12. $V = \int_{3\pi/2}^{5\pi/2} 2\pi x \cos x \, dx$ [by the method of cylindrical shells]

13. $V = \int_0^1 \pi \left[(1-x^3)^2 - (1-x^2)^2 \right] dx$

14. $V = \int_0^2 2\pi(8-x^3)(2-x) \, dx$

15. (a) A cross-section is a washer with inner radius x^2 and outer radius x .

$$V = \int_0^1 \pi \left[(x)^2 - (x^2)^2 \right] dx = \int_0^1 \pi (x^2 - x^4) \, dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 = \pi \left[\frac{1}{3} - \frac{1}{5} \right] = \frac{2\pi}{15}$$

- (b) A cross-section is a washer with inner radius y and outer radius $\sqrt{y}$.

$$V = \int_0^1 \pi \left[(\sqrt{y})^2 - y^2 \right] dy = \int_0^1 \pi (y - y^2) \, dy = \pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{\pi}{6}$$

- (c) A cross-section is a washer with inner radius $2-x$ and outer radius $2-x^2$.

$$V = \int_0^1 \pi \left[(2-x^2)^2 - (2-x)^2 \right] dx = \int_0^1 \pi (x^4 - 5x^2 + 4x) \, dx = \pi \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 2x^2 \right]_0^1 \\ = \pi \left[\frac{1}{5} - \frac{5}{3} + 2 \right] = \frac{8\pi}{15}$$

16. (a) $A = \int_0^1 (2x - x^2 - x^3) \, dx = \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$

- (b) A cross-section is a washer with inner radius x^3 and outer radius $2x - x^2$, so its area is $\pi(2x - x^2)^2 - \pi(x^3)^2$.

$$V = \int_0^1 A(x) \, dx = \int_0^1 \pi \left[(2x - x^2)^2 - (x^3)^2 \right] dx = \int_0^1 \pi (4x^2 - 4x^3 + x^4 - x^6) \, dx \\ = \pi \left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \right]_0^1 = \pi \left(\frac{4}{3} - 1 + \frac{1}{5} - \frac{1}{7} \right) = \frac{41\pi}{105}$$

- (c) Using the method of cylindrical shells,

$$V = \int_0^1 2\pi x (2x - x^2 - x^3) \, dx = \int_0^1 2\pi (2x^2 - x^3 - x^4) \, dx = 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 \right]_0^1 \\ = 2\pi \left(\frac{2}{3} - \frac{1}{4} - \frac{1}{5} \right) = \frac{13\pi}{30}.$$

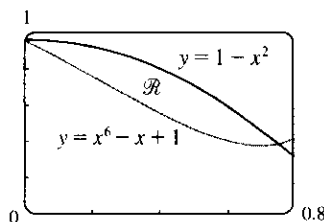
17. (a) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \tan(x^2)$ and $n = 4$, we estimate

$$A = \int_0^1 \tan(x^2) \, dx \approx \frac{1}{4} \left[\tan\left(\left(\frac{1}{8}\right)^2\right) + \tan\left(\left(\frac{3}{8}\right)^2\right) + \tan\left(\left(\frac{5}{8}\right)^2\right) + \tan\left(\left(\frac{7}{8}\right)^2\right) \right] \approx \frac{1}{4}(1.53) \approx 0.38$$

- (b) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \pi \tan^2(x^2)$ (for disks) and $n = 4$, we estimate

$$V = \int_0^1 f(x) \, dx \approx \frac{1}{4} \pi \left[\tan^2\left(\left(\frac{1}{8}\right)^2\right) + \tan^2\left(\left(\frac{3}{8}\right)^2\right) + \tan^2\left(\left(\frac{5}{8}\right)^2\right) + \tan^2\left(\left(\frac{7}{8}\right)^2\right) \right] \approx \frac{\pi}{4}(1.114) \approx 0.87$$

18. (a)



From the graph, we see that the curves intersect at $x = 0$ and at $x = a \approx 0.75$, with $1 - x^2 > x^6 - x + 1$ on $(0, a)$.

- (b) The area of $\mathcal{R}$ is

$$A = \int_0^a [(1 - x^2) - (x^6 - x + 1)] \, dx = \left[-\frac{1}{3}x^3 - \frac{1}{7}x^7 + \frac{1}{2}x^2 \right]_0^a \approx 0.12$$

- (c) Using washers, the volume generated when $\mathcal{R}$ is rotated about the x -axis is

$$V = \pi \int_0^a \left[(1 - x^2)^2 - (x^6 - x + 1)^2 \right] dx = \pi \int_0^a (-x^{12} + 2x^7 - 2x^6 + x^4 - 3x^2 + 2x) \, dx \\ = \pi \left[-\frac{1}{13}x^{13} + \frac{1}{4}x^8 - \frac{2}{7}x^7 + \frac{1}{5}x^5 - x^3 + x^2 \right]_0^a \approx 0.54$$

(d) Using shells, the volume generated when $\mathcal{R}$ is rotated about the y -axis is

$$\begin{aligned} V &= \int_0^a 2\pi x [(1-x^2) - (x^6 - x + 1)] dx = 2\pi \int_0^a (-x^3 - x^7 + x^2) dx \\ &= 2\pi \left[-\frac{1}{4}x^4 - \frac{1}{8}x^8 + \frac{1}{3}x^3 \right]_0^a \approx 0.31 \end{aligned}$$

19. The solid is obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x\}$ about the y -axis.
20. The solid is obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \sqrt{2} \cos x\}$ about the x -axis.
21. The solid is obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq y \leq 2, 0 \leq x \leq 4 - y^2\}$ about the x -axis.
22. The solid is obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq 1, 2 - \sqrt{x} \leq y \leq 2 - x^2\}$ about the x -axis.

Or: The solid is obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$ about the line $y = 2$.

23. Take the base to be the disk $x^2 + y^2 \leq 9$. Then $V = \int_{-3}^3 A(x) dx$, where $A(x_0)$ is the area of the isosceles right triangle whose hypotenuse lies along the line $x = x_0$ in the xy -plane. The length of the hypotenuse is $2\sqrt{9 - x^2}$ and the length of each leg is $\sqrt{2}\sqrt{9 - x^2}$. $A(x) = \frac{1}{2}(\sqrt{2}\sqrt{9 - x^2})^2 = 9 - x^2$, so
- $$V = 2 \int_0^3 A(x) dx = 2 \int_0^3 (9 - x^2) dx = 2 \left[9x - \frac{1}{3}x^3 \right]_0^3 = 2(27 - 9) = 36.$$

24. $V = \int_{-1}^1 A(x) dx = 2 \int_0^1 A(x) dx = 2 \int_0^1 [(2 - x^2) - x^2]^2 dx = 2 \int_0^1 [2(1 - x^2)]^2 dx$
- $$= 8 \int_0^1 (1 - 2x^2 + x^4) dx = 8 \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_0^1 = 8 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{64}{15}$$

25. Equilateral triangles with sides measuring $\frac{1}{4}x$ meters have height $\frac{1}{4}x \sin 60^\circ = \frac{\sqrt{3}}{8}x$. Therefore,

$$A(x) = \frac{1}{2} \cdot \frac{1}{4}x \cdot \frac{\sqrt{3}}{8}x = \frac{\sqrt{3}}{64}x^2. \quad V = \int_0^{20} A(x) dx = \frac{\sqrt{3}}{64} \int_0^{20} x^2 dx = \frac{\sqrt{3}}{64} \left[\frac{1}{3}x^3 \right]_0^{20} = \frac{8000\sqrt{3}}{64 \cdot 3} = \frac{125\sqrt{3}}{3} \text{ m}^3.$$

26. (a) By the symmetry of the problem, we consider only the solid to the right of the origin. The semicircular cross-sections perpendicular to the x -axis have radius $1 - x$, so $A(x) = \frac{1}{2}\pi(1 - x)^2$. Now we can calculate

$$V = 2 \int_0^1 A(x) dx = 2 \int_0^1 \frac{1}{2}\pi(1 - x)^2 dx = \int_0^1 \pi(1 - x)^2 dx = -\frac{\pi}{3}[(1 - x)^3]_0^1 = \frac{\pi}{3}.$$

- (b) Cut the solid with a plane perpendicular to the x -axis and passing through the y -axis. Fold the half of the solid in the region $x \leq 0$ under the xy -plane so that the point $(-1, 0)$ comes around and touches the point $(1, 0)$. The resulting solid is a right circular cone of radius 1 with vertex at $(x, y, z) = (1, 0, 0)$ and with its base in the yz -plane, centered at the origin. The volume of this cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 1^2 \cdot 1 = \frac{\pi}{3}$.

27. $f(x) = kx \Rightarrow 30 \text{ N} = k(15 - 12) \text{ cm} \Rightarrow k = 10 \text{ N/cm} = 1000 \text{ N/m}. \quad 20 \text{ cm} - 12 \text{ cm} = 0.08 \text{ m} \Rightarrow$

$$W = \int_0^{0.08} kx dx = 1000 \int_0^{0.08} x dx = 500 [x^2]_0^{0.08} = 500(0.08)^2 = 3.2 \text{ N}\cdot\text{m} = 3.2 \text{ J}.$$

28. The work needed to raise the elevator alone is $1600 \text{ lb} \times 30 \text{ ft} = 48,000 \text{ ft}\cdot\text{lb}$. The work needed to raise the bottom 170 ft of cable is $170 \text{ ft} \times 10 \text{ lb/ft} \times 30 \text{ ft} = 51,000 \text{ ft}\cdot\text{lb}$. The work needed to raise the top 30 ft of cable is

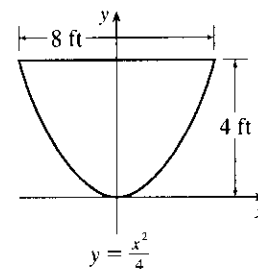
$$\int_0^{30} 10x \, dx = [5x^2]_0^{30} = 5 \cdot 900 = 4500 \text{ ft}\cdot\text{lb}. \text{ Adding these, we see that the total work needed is}$$

$$48,000 + 51,000 + 4,500 = 103,500 \text{ ft}\cdot\text{lb}.$$

29. (a) The parabola has equation $y = ax^2$ with vertex at the origin and passing through $(4, 4)$. $4 = a \cdot 4^2 \Rightarrow a = \frac{1}{4} \Rightarrow y = \frac{1}{4}x^2 \Rightarrow x^2 = 4y \Rightarrow x = 2\sqrt{y}$. Each circular disk has radius $2\sqrt{y}$ and is moved $4 - y$ ft.

$$W = \int_0^4 \pi (2\sqrt{y})^2 62.5 (4 - y) \, dy = 250\pi \int_0^4 y(4 - y) \, dy$$

$$= 250\pi \left[2y^2 - \frac{1}{3}y^3 \right]_0^4 = 250\pi \left(32 - \frac{64}{3} \right) = \frac{8000\pi}{3} \approx 8378 \text{ ft}\cdot\text{lb}$$

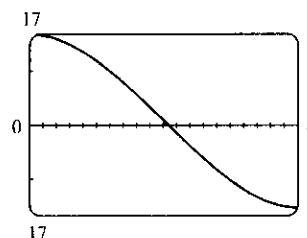


- (b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level — call it h)

$$\text{unknown: } W = 4000 \Leftrightarrow 250\pi \left[2y^2 - \frac{1}{3}y^3 \right]_h^4 = 4000 \Leftrightarrow$$

$$\frac{16}{\pi} = \left[\left(32 - \frac{64}{3} \right) - \left(2h^2 - \frac{1}{3}h^3 \right) \right] \Leftrightarrow h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0.$$

We graph the function $f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi}$ on the interval $[0, 4]$ to see where it is 0. From the graph, $f(h) = 0$ for $h \approx 2.1$. So the depth of water remaining is about 2.1 ft.



30. $f_{\text{ave}} = \frac{1}{10-0} \int_0^{10} t \sin(t^2) \, dt = \frac{1}{10} \int_0^{100} \sin u \left(\frac{1}{2} du \right) \quad [u = t^2, du = 2t \, dt]$

$$= \frac{1}{20} \left[-\cos u \right]_0^{100} = \frac{1}{20} (-\cos 100 + \cos 0) = \frac{1}{20} (1 - \cos 100) \approx 0.007$$

31. $\lim_{h \rightarrow 0} f_{\text{ave}} = \lim_{h \rightarrow 0} \frac{1}{(x+h) - x} \int_x^{x+h} f(t) \, dt = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$, where $F(x) = \int_a^x f(t) \, dt$. But we recognize this limit as being $F'(x)$ by the definition of a derivative. Therefore, $\lim_{h \rightarrow 0} f_{\text{ave}} = F'(x) = f(x)$ by FTC1.

32. (a) $\mathcal{R}_1$ is the region below the graph of $y = x^2$ and above the x -axis between $x = 0$ and $x = b$, and $\mathcal{R}_2$ is the region to the left of the graph of $x = \sqrt{y}$ and to the right of the y -axis between $y = 0$ and $y = b^2$. So the area of $\mathcal{R}_1$ is $A_1 = \int_0^b x^2 \, dx = \left[\frac{1}{3}x^3 \right]_0^b = \frac{1}{3}b^3$, and the area of $\mathcal{R}_2$ is $A_2 = \int_0^{b^2} \sqrt{y} \, dy = \left[\frac{2}{3}y^{3/2} \right]_0^{b^2} = \frac{2}{3}b^3$. So there is no solution to $A_1 = A_2$ for $b \neq 0$.

- (b) Using disks, we calculate the volume of rotation of $\mathcal{R}_1$ about the x -axis to be $V_{1,x} = \pi \int_0^b (x^2)^2 dx = \frac{1}{5} \pi b^5$.

Using cylindrical shells, we calculate the volume of rotation of $\mathcal{R}_1$ about the y -axis to be

$$V_{1,y} = 2\pi \int_0^b x(x^2) dx = 2\pi \left[\frac{1}{4} x^4 \right]_0^b = \frac{1}{2} \pi b^4. \text{ So } V_{1,x} = V_{1,y} \Leftrightarrow \frac{1}{5} \pi b^5 = \frac{1}{2} \pi b^4 \Leftrightarrow 2b = 5 \Leftrightarrow$$

$b = \frac{5}{2}$. So the volumes of rotation about the x - and y -axes are the same for $b = \frac{5}{2}$.

- (c) We use cylindrical shells to calculate the volume of rotation of $\mathcal{R}_2$ about the x -axis:

$$\mathcal{R}_{2,x} = 2\pi \int_0^{b^2} y(\sqrt{y}) dy = 2\pi \left[\frac{2}{5} y^{5/2} \right]_0^{b^2} = \frac{4}{5} \pi b^5. \text{ We already know the volume of rotation of } \mathcal{R}_1 \text{ about the}$$

x -axis from part (b), and $\mathcal{R}_{1,x} = \mathcal{R}_{2,x} \Leftrightarrow \frac{1}{5} \pi b^5 = \frac{4}{5} \pi b^5$, which has no solution for $b \neq 0$.

- (d) We use disks to calculate the volume of rotation of $\mathcal{R}_2$ about the y -axis:

$$\mathcal{R}_{2,y} = \pi \int_0^{b^2} (\sqrt{y})^2 dy = \pi \left[\frac{1}{2} y^2 \right]_0^{b^2} = \frac{1}{2} \pi b^4. \text{ We know the volume of rotation of } \mathcal{R}_1 \text{ about the } y\text{-axis from}$$

part (b), and $\mathcal{R}_{1,y} = \mathcal{R}_{2,y} \Leftrightarrow \frac{1}{2} \pi b^4 = \frac{1}{2} \pi b^4$. But this equation is true for all b , so the volumes of rotation about the y -axis are equal for all values of b .

□ PROBLEMS PLUS

1. (a) The area under the graph of f from 0 to t is equal to $\int_0^t f(x) dx$, so the requirement is that $\int_0^t f(x) dx = t^3$ for all t . We differentiate both sides of this equation with respect to t (with the help of FTC1) to get $f(t) = 3t^2$.

This function is positive and continuous, as required.

- (b) The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi[f(x)]^2 dx$. Hence, we are given that $b^2 = \int_0^b \pi[f(x)]^2 dx$ for all $b > 0$. Differentiating both sides of this equation with respect to b using the Fundamental Theorem of Calculus gives $2b = \pi[f(b)]^2 \Rightarrow f(b) = \sqrt{2b/\pi}$, since f is positive. Therefore, $f(x) = \sqrt{2x/\pi}$.

2. The total area of the region bounded by the parabola

$y = x - x^2 = x(1 - x)$ and the x -axis is

$$\int_0^1 (x - x^2) dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{6}.$$

Let the slope of the line we are looking for be m . Then the area above this

line but below the parabola is $\int_0^a [(x - x^2) - mx] dx$, where a is the x -coordinate of the point of intersection of the line and the parabola. We

find the point of intersection by solving the equation $x - x^2 = mx \Leftrightarrow$

$1 - x = m \Leftrightarrow x = 1 - m$. So the value of a is $1 - m$, and

$$\begin{aligned} \int_0^{1-m} [(x - x^2) - mx] dx &= \int_0^{1-m} [(1 - m)x - x^2] dx = \left[\frac{1}{2}(1 - m)x^2 - \frac{1}{3}x^3 \right]_0^{1-m} \\ &= \frac{1}{2}(1 - m)(1 - m)^2 - \frac{1}{3}(1 - m)^3 = \frac{1}{6}(1 - m)^3 \end{aligned}$$

We want this to be half of $\frac{1}{6}$, so $\frac{1}{6}(1 - m)^3 = \frac{1}{12} \Leftrightarrow (1 - m)^3 = \frac{6}{12} \Leftrightarrow$

$1 - m = \sqrt[3]{\frac{1}{2}} \Leftrightarrow m = 1 - \sqrt[3]{\frac{1}{2}}$. So the slope of the required line is $1 - \sqrt[3]{\frac{1}{2}} \approx 0.206$.

3. Let a and b be the x -coordinates of the points where the line intersects the

curve. From the figure, $R_1 = R_2 \Rightarrow$

$$\int_0^a [c - (8x - 27x^3)] dx = \int_a^b [(8x - 27x^3) - c] dx$$

$$\left[cx - 4x^2 + \frac{27}{4}x^4 \right]_0^a = \left[4x^2 - \frac{27}{4}x^4 - cx \right]_a^b$$

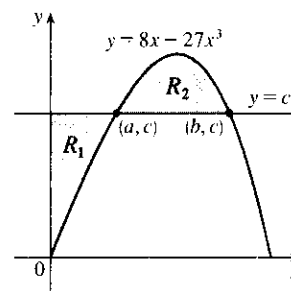
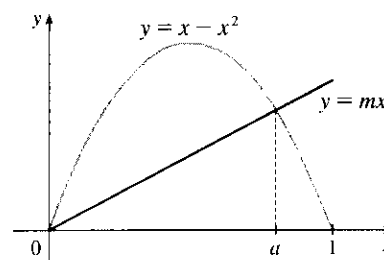
$$ac - 4a^2 + \frac{27}{4}a^4 = (4b^2 - \frac{27}{4}b^4 - bc) - (4a^2 - \frac{27}{4}a^4 - ac)$$

$$0 = 4b^2 - \frac{27}{4}b^4 - bc = 4b^2 - \frac{27}{4}b^4 - b(8b - 27b^3)$$

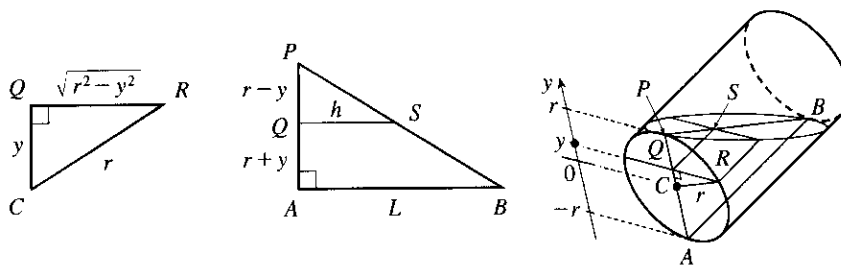
$$= 4b^2 - \frac{27}{4}b^4 - 8b^2 + 27b^4 = \frac{81}{4}b^4 - 4b^2$$

$$= b^2 \left(\frac{81}{4}b^2 - 4 \right)$$

So for $b > 0$, $b^2 = \frac{16}{81} \Rightarrow b = \frac{4}{9}$. Thus, $c = 8b - 27b^3 = 8\left(\frac{4}{9}\right) - 27\left(\frac{64}{729}\right) = \frac{32}{9} - \frac{64}{27} = \frac{32}{27}$.



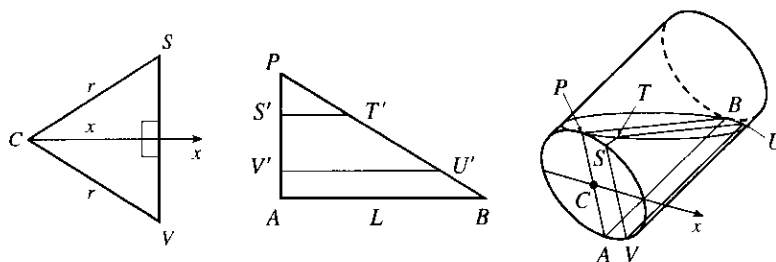
4. (a) Take slices perpendicular to the line through the center C of the bottom of the glass and the point P where the top surface of the water meets the bottom of the glass.



A typical rectangular cross-section y units above the axis of the glass has width $2|QR| = 2\sqrt{r^2 - y^2}$ and length $h = |QS| = \frac{L}{2r}(r - y)$. [Triangles PQS and PAB are similar, so $\frac{h}{L} = \frac{|PQ|}{|PA|} = \frac{r - y}{2r}$.] Thus,

$$\begin{aligned} V &= \int_{-r}^r 2\sqrt{r^2 - y^2} \cdot \frac{L}{2r}(r - y) dy = L \int_{-r}^r \left(1 - \frac{y}{r}\right) \sqrt{r^2 - y^2} dy \\ &= L \int_{-r}^r \sqrt{r^2 - y^2} dy - \frac{L}{r} \int_{-r}^r y \sqrt{r^2 - y^2} dy \\ &= L \cdot \frac{\pi r^2}{2} - \frac{L}{r} \cdot 0 \quad \left[\begin{array}{l} \text{the first integral is the area of a semicircle of radius } r, \\ \text{and the second has an odd integrand} \end{array} \right] = \frac{\pi r^2 L}{2} \end{aligned}$$

- (b) Slice parallel to the plane through the axis of the glass and the point of contact P . (This is the plane determined by P , B , and C in the figure.) $STUV$ is a typical trapezoidal slice. With respect to an x -axis with origin at C as shown, if S and V have x -coordinate x , then $|SV| = 2\sqrt{r^2 - x^2}$. Projecting the trapezoid $STUV$ onto the plane of the triangle PAB (call the projection $S'T'U'V'$), we see that $|AP| = 2r$, $|SV| = 2\sqrt{r^2 - x^2}$, and $|S'P| = |V'A| = \frac{1}{2}(|AP| - |SV|) = r - \sqrt{r^2 - x^2}$.



By similar triangles, $\frac{|ST|}{|S'P|} = \frac{|AB|}{|AP|}$, so $|ST| = (r - \sqrt{r^2 - x^2}) \cdot \frac{L}{2r}$. In the same way, we find that

$$\frac{|VU|}{|V'P|} = \frac{|AB|}{|AP|}, \text{ so } |VU| = |V'P| \cdot \frac{L}{2r} = (r + \sqrt{r^2 - x^2}) \cdot \frac{L}{2r}.$$

The area $A(x)$ of the trapezoid $STUV$ is $\frac{1}{2}|SV| \cdot (|ST| + |VU|)$; that is,

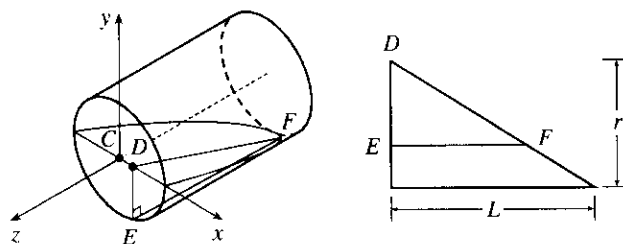
$$A(x) = \frac{1}{2} \cdot 2\sqrt{r^2 - x^2} \cdot \left[(r - \sqrt{r^2 - x^2}) \cdot \frac{L}{2r} + (r + \sqrt{r^2 - x^2}) \cdot \frac{L}{2r} \right] = L\sqrt{r^2 - x^2}. \text{ Thus,}$$

$$V = \int_{-r}^r A(x) dx = L \int_{-r}^r \sqrt{r^2 - x^2} dx = L \cdot \frac{\pi r^2}{2} = \frac{\pi r^2 L}{2}.$$

- (c) See the computation of V in part (a) or part (b).

(d) The volume of the water is exactly half the volume of the cylindrical glass, so $V = \frac{1}{2}\pi r^2 L$.

(e)



Choose x -, y -, and z -axes as shown in the figure. Then slices perpendicular to the x -axis are triangular, slices perpendicular to the y -axis are rectangular, and slices perpendicular to the z -axis are segments of circles. Using triangular slices, we find that the area $A(x)$ of a typical slice DEF , where D has x -coordinate x , is given by

$$A(x) = \frac{1}{2}|DE| \cdot |EF| = \frac{1}{2}|DE| \cdot \left(\frac{L}{r}|DE|\right) = \frac{L}{2r}|DE|^2 = \frac{L}{2r}(r^2 - x^2). \text{ Thus,}$$

$$\begin{aligned} V &= \int_{-r}^r A(x) dx = \frac{L}{2r} \int_{-r}^r (r^2 - x^2) dx = \frac{L}{r} \int_0^r (r^2 - x^2) dx = \frac{L}{r} \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= \frac{L}{r} \left(r^3 - \frac{r^3}{3} \right) = \frac{L}{r} \cdot \frac{2}{3} r^3 = \frac{2}{3} r^2 L \text{ [This is } 2/(3\pi) \approx 0.21 \text{ of the volume of the glass.]} \end{aligned}$$

5. (a) $V = \pi h^2(r - h/3) = \frac{1}{3}\pi h^2(3r - h)$. See the solution to Exercise 6.2.49.

(b) The smaller segment has height $h = 1 - x$ and so by part (a) its volume is

$$\begin{aligned} V &= \frac{1}{3}\pi(1-x)^2[3(1) - (1-x)] = \frac{1}{3}\pi(x-1)^2(x+2). \text{ This volume must be } \frac{1}{3} \text{ of the total volume of the} \\ &\text{sphere, which is } \frac{4}{3}\pi(1)^3. \text{ So } \frac{1}{3}\pi(x-1)^2(x+2) = \frac{1}{3}\left(\frac{4}{3}\pi\right) \Rightarrow (x^2 - 2x + 1)(x+2) = \frac{4}{3} \Rightarrow \\ &x^3 - 3x + 2 = \frac{4}{3} \Rightarrow 3x^3 - 9x + 2 = 0. \text{ Using Newton's method with } f(x) = 3x^3 - 9x + 2, \\ &f'(x) = 9x^2 - 9, \text{ we get } x_{n+1} = x_n - \frac{3x_n^3 - 9x_n + 2}{9x_n^2 - 9}. \text{ Taking } x_1 = 0, \text{ we get } x_2 \approx 0.2222, \text{ and} \\ &x_3 \approx 0.2261 \approx x_4, \text{ so, correct to four decimal places, } x \approx 0.2261. \end{aligned}$$

(c) With $r = 0.5$ and $s = 0.75$, the equation $x^3 - 3rx^2 + 4r^3s = 0$ becomes $x^3 - 3(0.5)x^2 + 4(0.5)^3(0.75) = 0$
 $\Rightarrow x^3 - \frac{3}{2}x^2 + 4\left(\frac{1}{8}\right)\frac{3}{4} = 0 \Rightarrow 8x^3 - 12x^2 + 3 = 0$. We use Newton's method with

$$\begin{aligned} f(x) &= 8x^3 - 12x^2 + 3, f'(x) = 24x^2 - 24x, \text{ so } x_{n+1} = x_n - \frac{8x_n^3 - 12x_n^2 + 3}{24x_n^2 - 24x_n}. \text{ Take } x_1 = 0.5. \text{ Then} \\ &x_2 \approx 0.6667, \text{ and } x_3 \approx 0.6736 \approx x_4. \text{ So to four decimal places the depth is } 0.6736 \text{ m.} \end{aligned}$$

(d) (i) From part (a) with $r = 5$ in., the volume of water in the bowl is

$$\begin{aligned} V &= \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi h^2(15 - h) = 5\pi h^2 - \frac{1}{3}\pi h^3. \text{ We are given that } \frac{dV}{dt} = 0.2 \text{ m}^3/\text{s} \text{ and we want to} \\ &\text{find } \frac{dh}{dt} \text{ when } h = 3. \text{ Now } \frac{dV}{dt} = 10\pi h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt}, \text{ so } \frac{dh}{dt} = \frac{0.2}{\pi(10h - h^2)}. \text{ When } h = 3, \text{ we have} \\ &\frac{dh}{dt} = \frac{0.2}{\pi(10 \cdot 3 - 3^2)} = \frac{1}{105\pi} \approx 0.003 \text{ in/s.} \end{aligned}$$

(ii) From part (a), the volume of water required to fill the bowl from the instant that the water is 4 in. deep is

$$V = \frac{1}{2} \cdot \frac{4}{3} \pi (5)^3 - \frac{1}{3} \pi (4)^2 (15 - 4) = \frac{2}{3} \cdot 125\pi - \frac{16}{3} \cdot 11\pi = \frac{74}{3} \pi. \text{ To find the time required to fill the bowl we divide this volume by the rate: Time} = \frac{74\pi/3}{0.2} = \frac{370\pi}{3} \approx 387 \text{ s} \approx 6.5 \text{ min}$$

6. (a) The volume above the surface is $\int_0^{L-h} A(y) dy = \int_{-h}^{L-h} A(y) dy - \int_{-h}^0 A(y) dy$. So the proportion of volume

above the surface is $\frac{\int_0^{L-h} A(y) dy}{\int_{-h}^{L-h} A(y) dy} = \frac{\int_{-h}^{L-h} A(y) dy - \int_{-h}^0 A(y) dy}{\int_{-h}^{L-h} A(y) dy}$. Now by Archimedes' Principle, we

$$\text{have } F = W \Rightarrow \rho_f g \int_{-h}^0 A(y) dy = \rho_0 g \int_{-h}^{L-h} A(y) dy, \text{ so } \int_{-h}^0 A(y) dy = (\rho_0/\rho_f) \int_{-h}^{L-h} A(y) dy.$$

Therefore, $\frac{\int_0^{L-h} A(y) dy}{\int_{-h}^{L-h} A(y) dy} = \frac{\int_{-h}^{L-h} A(y) dy - (\rho_0/\rho_f) \int_{-h}^{L-h} A(y) dy}{\int_{-h}^{L-h} A(y) dy} = \frac{\rho_f - \rho_0}{\rho_f}$, so the percentage of

volume above the surface is $100 \left(\frac{\rho_f - \rho_0}{\rho_f} \right) \%$.

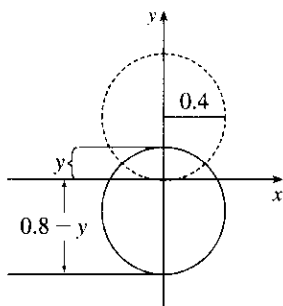
- (b) For an iceberg, the percentage of volume above the surface is $100 \left(\frac{1030 - 917}{1030} \right) \% \approx 11\%$.

- (c) No, the water does not overflow. Let V_i be the volume of the ice cube, and let V_w be the volume of the water which results from the melting. Then by the formula derived in part (a), the volume of ice above the surface of the water is $[(\rho_f - \rho_0)/\rho_f] V_i$, so the volume below the surface is $V_i - [(\rho_f - \rho_0)/\rho_f] V_i = (\rho_0/\rho_f) V_i$.

Now the mass of the ice cube is the same as the mass of the water which is created when it melts, namely

$$m = \rho_0 V_i = \rho_f V_w \Rightarrow V_w = (\rho_0/\rho_f) V_i. \text{ So when the ice cube melts, the volume of the resulting water is the same as the underwater volume of the ice cube, and so the water does not overflow.}$$

- (d)



The figure shows the instant when the height of the exposed part of the ball is y . Using the formula in Problem 5(a) with $r = 0.4$ and

$h = 0.8 - y$, we see that the volume of the submerged part of the

sphere is $\frac{1}{3} \pi (0.8 - y)^2 [1.2 - (0.8 - y)]$, so its weight is

$1000g \cdot \frac{1}{3} \pi s^2 (1.2 - s)$, where $s = 0.8 - y$. Then the work done to submerge the sphere is

$$\begin{aligned} W &= \int_0^{0.8} g \frac{1000}{3} \pi s^2 (1.2 - s) ds = g \frac{1000}{3} \pi \int_0^{0.8} (1.2s^2 - s^3) ds \\ &= g \frac{1000}{3} \pi \left[0.4s^3 - \frac{1}{4}s^4 \right]_0^{0.8} = g \frac{1000}{3} \pi (0.2048 - 0.1024) \\ &= 9.8 \frac{1000}{3} \pi (0.1024) \approx 1.05 \times 10^3 \text{ joules} \end{aligned}$$

7. We are given that the rate of change of the volume of water is $\frac{dV}{dt} = -kA(x)$, where k is some positive constant

and $A(x)$ is the area of the surface when the water has depth x . Now we are concerned with the rate of change of

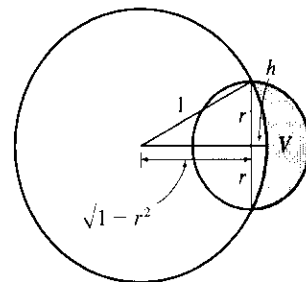
the depth of the water with respect to time, that is, $\frac{dx}{dt}$. But by the Chain Rule, $\frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt}$, so the first equation

can be written $\frac{dV}{dx} \frac{dx}{dt} = -kA(x)$ (*). Also, we know that the total volume of water up to a depth x is

$V(x) = \int_0^x A(s) ds$, where $A(s)$ is the area of a cross-section of the water at a depth s . Differentiating this equation with respect to x , we get $dV/dx = A(x)$. Substituting this into equation $\star$, we get

$$A(x)(dx/dt) = -kA(x) \Rightarrow dx/dt = -k, \text{ a constant.}$$

8. A typical sphere of radius r is shown in the figure. We wish to maximize the shaded volume V , which can be thought of as the volume of a hemisphere of radius r minus the volume of the spherical cap with height $h = 1 - \sqrt{1 - r^2}$ and radius 1.



$$\begin{aligned} V &= \frac{1}{2} \cdot \frac{4}{3} \pi r^3 - \frac{1}{3} \pi (1 - \sqrt{1 - r^2})^2 [3(1) - (1 - \sqrt{1 - r^2})] \quad [\text{by Problem 5(a)}] \\ &= \frac{1}{3} \pi [2r^3 - (2 - 2\sqrt{1 - r^2} - r^2)(2 + \sqrt{1 - r^2})] \\ &= \frac{1}{3} \pi [2r^3 - 2 + (r^2 + 2)\sqrt{1 - r^2}] \end{aligned}$$

$$\begin{aligned} V' &= \frac{1}{3} \pi \left[6r^2 + \frac{(r^2 + 2)(-r)}{\sqrt{1 - r^2}} + \sqrt{1 - r^2}(2r) \right] = \frac{1}{3} \pi \left[\frac{6r^2 \sqrt{1 - r^2} - r(r^2 + 2) + 2r(1 - r^2)}{\sqrt{1 - r^2}} \right] \\ &= \frac{1}{3} \pi \left(\frac{6r^2 \sqrt{1 - r^2} - 3r^3}{\sqrt{1 - r^2}} \right) = \frac{\pi r^2 (2\sqrt{1 - r^2} - r)}{\sqrt{1 - r^2}} \end{aligned}$$

$V'(r) = 0 \Leftrightarrow 2\sqrt{1 - r^2} = r \Leftrightarrow 4 - 4r^2 = r^2 \Leftrightarrow r^2 = \frac{4}{5} \Leftrightarrow r = \frac{2}{\sqrt{5}} \approx 0.89$. Since $V'(r) > 0$ for $0 < r < \frac{2}{\sqrt{5}}$ and $V'(r) < 0$ for $\frac{2}{\sqrt{5}} < r < 1$, we know that V attains a maximum at $r = \frac{2}{\sqrt{5}}$.

9. We must find expressions for the areas A and B , and then set them equal and see what this says about the curve C .

If $P = (a, 2a^2)$, then area A is just $\int_0^a (2x^2 - x^2) dx = \int_0^a x^2 dx = \frac{1}{3}a^3$. To find area B , we use y as the variable of integration. So we find the equation of the middle curve as a function of y : $y = 2x^2 \Leftrightarrow x = \sqrt{y/2}$,

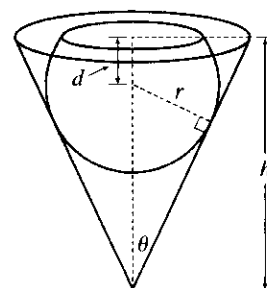
since we are concerned with the first quadrant only. We can express area B as

$$\int_0^{2a^2} [\sqrt{y/2} - C(y)] dy = \left[\frac{4}{3}(y/2)^{3/2} \right]_0^{2a^2} - \int_0^{2a^2} C(y) dy = \frac{4}{3}a^3 - \int_0^{2a^2} C(y) dy, \text{ where } C(y) \text{ is the function}$$

with graph C . Setting $A = B$, we get $\frac{1}{3}a^3 = \frac{4}{3}a^3 - \int_0^{2a^2} C(y) dy \Leftrightarrow \int_0^{2a^2} C(y) dy = a^3$. Now we differentiate this equation with respect to a using the Chain Rule and the Fundamental Theorem:

$$\begin{aligned} C(2a^2)(4a) &= 3a^2 \Rightarrow C(y) = \frac{3}{4}\sqrt{y/2}, \text{ where } y = 2a^2. \text{ Now we can solve for } y: x = \frac{3}{4}\sqrt{y/2} \Rightarrow \\ x^2 &= \frac{9}{16}(y/2) \Rightarrow y = \frac{32}{9}x^2. \end{aligned}$$

10. We want to find the volume of that part of the sphere which is below the surface of the water. As we can see from the diagram, this region is a cap of a sphere with radius r and height $r + d$. If we can find an expression for d in terms of h , r and θ , then we can determine the volume of the region [see Problem 5(a)], and then differentiate with respect to r to find the maximum. We see that



$$\sin \theta = \frac{r}{h - d} \Leftrightarrow h - d = \frac{r}{\sin \theta} \Leftrightarrow d = h - r \csc \theta.$$

Now we can use the formula from Problem 5(a) to find the volume of water displaced:

$$\begin{aligned} V &= \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi(r + d)^2[3r - (r + d)] = \frac{1}{3}\pi(r + h - r \csc \theta)^2(2r - h + r \csc \theta) \\ &= \frac{\pi}{3}[r(1 - \csc \theta) + h]^2[r(2 + \csc \theta) - h] \end{aligned}$$

Now we differentiate with respect to r :

$$\begin{aligned} dV/dr &= \frac{\pi}{3}([r(1 - \csc \theta) + h]^2(2 + \csc \theta) + 2[r(1 - \csc \theta) + h](1 - \csc \theta)[r(2 + \csc \theta) - h]) \\ &= \frac{\pi}{3}[r(1 - \csc \theta) + h]([r(1 - \csc \theta) + h](2 + \csc \theta) + 2(1 - \csc \theta)[r(2 + \csc \theta) - h]) \\ &= \frac{\pi}{3}[r(1 - \csc \theta) + h](3(2 + \csc \theta)(1 - \csc \theta)r + [(2 + \csc \theta) - 2(1 - \csc \theta)]h) \\ &= \frac{\pi}{3}[r(1 - \csc \theta) + h][3(2 + \csc \theta)(1 - \csc \theta)r + 3h \csc \theta] \end{aligned}$$

This is 0 when $r = \frac{h}{\csc \theta - 1}$ and when $r = \frac{h \csc \theta}{(\csc \theta + 2)(\csc \theta - 1)}$. Now since $V\left(\frac{h}{\csc \theta - 1}\right) = 0$

(the first factor vanishes; this corresponds to $d = -r$), the maximum volume of water is displaced when

$r = \frac{h \csc \theta}{(\csc \theta - 1)(\csc \theta + 2)}$. (Our intuition tells us that a maximum value does exist, and it must occur at a critical

number.) Multiplying numerator and denominator by $\sin^2 \theta$, we get an alternative form of the answer:

$$r = \frac{h \sin \theta}{\sin \theta + \cos 2\theta}.$$

11. (a) Stacking disks along the y -axis gives us $V = \int_0^h \pi [f(y)]^2 dy$.

(b) Using the Chain Rule, $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \pi [f(h)]^2 \frac{dh}{dt}$.

(c) $kA \sqrt{h} = \pi [f(h)]^2 \frac{dh}{dt}$. Set $\frac{dh}{dt} = C$: $\pi [f(h)]^2 C = kA \sqrt{h} \Rightarrow [f(h)]^2 = \frac{kA}{\pi C} \sqrt{h} \Rightarrow$

$f(h) = \sqrt{\frac{kA}{\pi C}} h^{1/4}$; that is, $f(y) = \sqrt{\frac{kA}{\pi C}} y^{1/4}$. The advantage of having $\frac{dh}{dt} = C$ is that the markings on the container are equally spaced.

12. (a) We first use the cylindrical shell method to express the volume V in terms of h , r , and ω :

$$\begin{aligned} V &= \int_0^r 2\pi xy \, dx = \int_0^r 2\pi x \left[h + \frac{\omega^2 x^2}{2g} \right] dx = 2\pi \int_0^r \left(hx + \frac{\omega^2 x^3}{2g} \right) dx \\ &= 2\pi \left[\frac{hx^2}{2} + \frac{\omega^2 x^4}{8g} \right]_0^r = 2\pi \left[\frac{hr^2}{2} + \frac{\omega^2 r^4}{8g} \right] = \pi hr^2 + \frac{\pi \omega^2 r^4}{4g} \Rightarrow \end{aligned}$$

$$h = \frac{V - (\pi \omega^2 r^4)/(4g)}{\pi r^2} = \frac{4gV - \pi \omega^2 r^4}{4\pi g r^2}.$$

(b) The surface touches the bottom when $h = 0 \Rightarrow 4gV - \pi\omega^2 r^4 = 0 \Rightarrow \omega^2 = \frac{4gV}{\pi r^4} \Rightarrow \omega = \frac{2\sqrt{gV}}{\sqrt{\pi r^2}}$.

To spill over the top, $y(r) > L \Leftrightarrow$

$$L < h + \frac{\omega^2 r^2}{2g} = \frac{4gV - \pi\omega^2 r^4}{4\pi g r^2} + \frac{\omega^2 r^2}{2g} = \frac{4gV}{4\pi g r^2} - \frac{\pi\omega^2 r^2}{4\pi g r^2} + \frac{\omega^2 r^2}{2g}$$

$$= \frac{V}{\pi r^2} - \frac{\omega^2 r^2}{4g} + \frac{\omega^2 r^2}{2g} = \frac{V}{\pi r^2} + \frac{\omega^2 r^2}{4g} \Leftrightarrow$$

$$\frac{\omega^2 r^2}{4g} > L - \frac{V}{\pi r^2} = \frac{\pi r^2 L - V}{\pi r^2} \Leftrightarrow \omega^2 > \frac{4g(\pi r^2 L - V)}{\pi r^4}. \text{ So for spillage, the angular speed should}$$

$$\text{be } \omega > \frac{2\sqrt{g(\pi r^2 L - V)}}{r^2 \sqrt{\pi}}.$$

(c) (i) Here we have $r = 2$, $L = 7$, $h = 7 - 5 = 2$. When $x = 1$, $y = 7 - 4 = 3$. Therefore, $3 = 2 + \frac{\omega^2 \cdot 1^2}{2 \cdot 32}$
 $\Rightarrow 1 = \frac{\omega^2}{2 \cdot 32} \Rightarrow \omega^2 = 64 \Rightarrow \omega = 8 \text{ rad/s. } V = \pi(2)(2)^2 + \frac{\pi \cdot 8^2 \cdot 2^4}{4g} = 8\pi + 8\pi = 16\pi \text{ ft}^2.$

(ii) At the wall, $x = 2$, so $y = 2 + \frac{8^2 \cdot 2^2}{2 \cdot 32} = 6$ and the surface is $7 - 6 = 1$ ft below the top of the tank.

13. We assume that P lies in the region of positive x . Since $y = x^3$ is an odd

function, this assumption will not affect the result of the calculation. Let

$P = (a, a^3)$. The slope of the tangent to the curve $y = x^3$ at P is $3a^2$, and so

the equation of the tangent is $y - a^3 = 3a^2(x - a) \Leftrightarrow y = 3a^2x - 2a^3$.

We solve this simultaneously with $y = x^3$ to find the other point of intersection:

$$x^3 = 3a^2x - 2a^3 \Leftrightarrow (x - a)^2(x + 2a) = 0. \text{ So } Q = (-2a, -8a^3) \text{ is}$$

the other point of intersection. The equation of the tangent at Q is

$$y - (-8a^3) = 12a^2[x - (-2a)] \Leftrightarrow y = 12a^2x + 16a^3. \text{ By symmetry,}$$

this tangent will intersect the curve again at $x = -2(-2a) = 4a$. The curve lies above the first tangent, and below

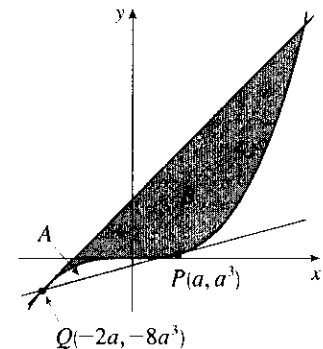
the second, so we are looking for a relationship between $A = \int_{-2a}^a [x^3 - (3a^2x - 2a^3)] dx$ and

$$B = \int_{-2a}^{4a} [(12a^2x + 16a^3) - x^3] dx. \text{ We calculate } A = \left[\frac{1}{4}x^4 - \frac{3}{2}a^2x^2 + 2a^3x \right]_{-2a}^a = \frac{3}{4}a^4 - (-6a^4) = \frac{27}{4}a^4,$$

$$\text{and } B = \left[6a^2x^2 + 16a^3x - \frac{1}{4}x^4 \right]_{-2a}^{4a} = 96a^4 - (-12a^4) = 108a^4. \text{ We see that } B = 16A = 2^4A. \text{ This is}$$

because our calculation of area B was essentially the same as that of area A , with a replaced by $-2a$, so if we

$$\text{replace } a \text{ with } -2a \text{ in our expression for } A, \text{ we get } \frac{27}{4}(-2a)^4 = 108a^4 = B.$$

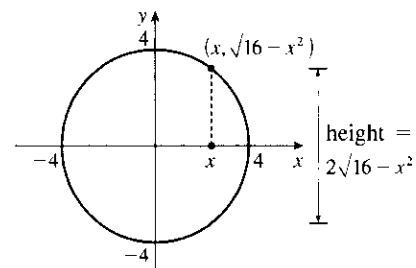


14. (a) Place the round flat tortilla on an xy -coordinate system as

shown in the first figure. An equation of the circle is

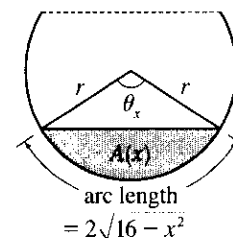
$$x^2 + y^2 = 4^2 \text{ and the height of a cross-section is}$$

$$2\sqrt{16 - x^2}.$$



Now look at a cross-section with central angle θ_x as shown in the second figure (r is the radius of the circular cylinder).

The filled area $A(x)$ is equal to the area $A_1(x)$ of the sector minus the area $A_2(x)$ of the triangle.



$$\begin{aligned}
 A(x) &= A_1(x) - A_2(x) \\
 &= \frac{1}{2}r^2\theta_x - \frac{1}{2}r^2\sin\theta_x \quad [\text{area formulas from trigonometry}] \\
 &= \frac{1}{2}r(r\theta_x) - \frac{1}{2}r^2\sin\left(\frac{s}{r}\right) \quad [\text{arc length } s = r\theta_x \Rightarrow \theta_x = s/r] \\
 &= \frac{1}{2}r \cdot 2\sqrt{16-x^2} - \frac{1}{2}r^2\sin\left(\frac{2\sqrt{16-x^2}}{r}\right) \quad [s = 2\sqrt{16-x^2}] \\
 &= r\sqrt{16-x^2} - \frac{1}{2}r^2\sin\left(\frac{2}{r}\sqrt{16-x^2}\right) \quad (*)
 \end{aligned}$$

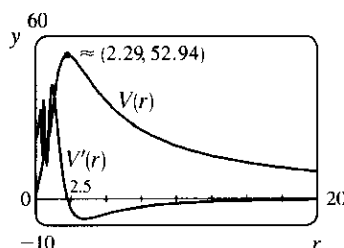
Note that the central angle θ_x will be small near the ends of the tortilla; that is, when $|x| \approx 4$. But near the center of the tortilla (when $|x| \approx 0$), the central angle θ_x may exceed 180° . Thus, the sine of θ_x will be negative and the second term in $(*)$ will be positive (actually adding area to the area of the sector). The volume of the taco can be found by integrating the cross-sectional areas from $x = -4$ to $x = 4$. Thus,

$$V(x) = \int_{-4}^4 A(x) dx = \int_{-4}^4 \left[r\sqrt{16-x^2} - \frac{1}{2}r^2\sin\left(\frac{2}{r}\sqrt{16-x^2}\right) \right] dx$$

(b) To find the value of r that maximizes the volume of the taco, we can define the function

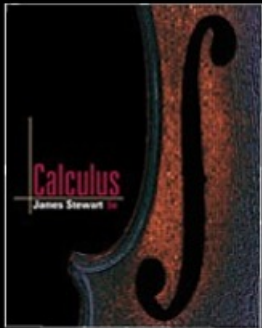
$$V(r) = \int_{-4}^4 \left[r\sqrt{16-x^2} - \frac{1}{2}r^2\sin\left(\frac{2}{r}\sqrt{16-x^2}\right) \right] dx$$

The third figure shows a graph of $y = V(r)$ and $y = V'(r)$. The maximum volume of about 52.94 occurs when $r \approx 2.2912$.

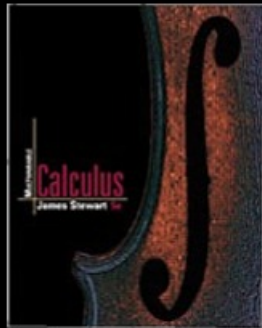


Chapter 7

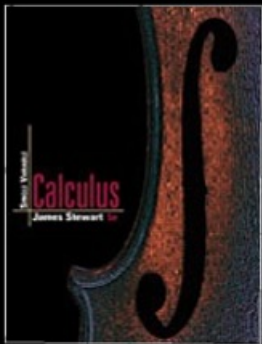
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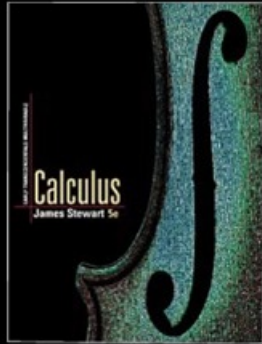
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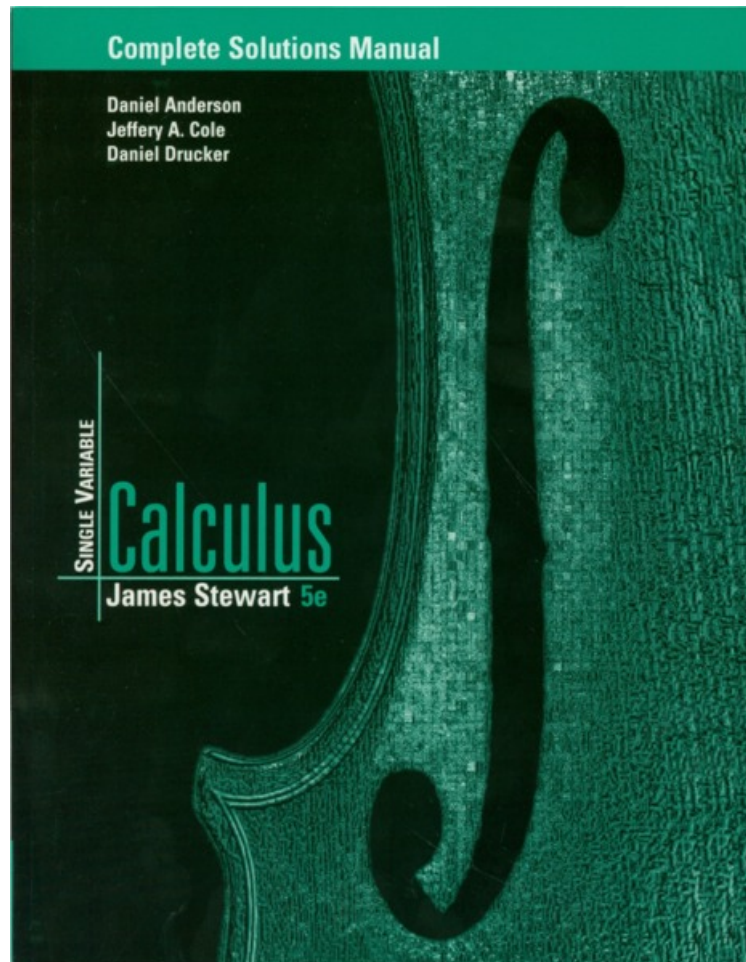
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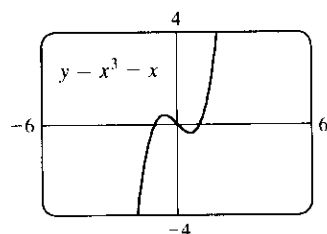


7 INVERSE FUNCTIONS: Exponential, Logarithmic, and Inverse Trigonometric Functions

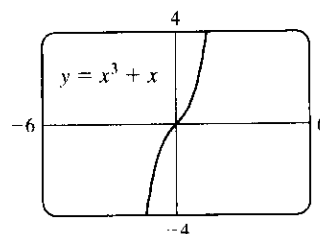
7.1 Inverse Functions

- (a) See Definition 1.
(b) It must pass the Horizontal Line Test.
- (a) $f^{-1}(y) = x \Leftrightarrow f(x) = y$ for any y in B . The domain of f^{-1} is B and the range of f^{-1} is A .
(b) See the steps in (5).
(c) Reflect the graph of f about the line $y = x$.
- f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.
- f is one-to-one since for any two different domain values, there are different range values.
- No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.
- The horizontal line $y = 0$ (the x -axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.
- The horizontal line $y = 0$ (the x -axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.
- No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.
- The graph of $f(x) = \frac{1}{2}(x + 5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one-to-one.
Algebraic solution: If $x_1 \neq x_2$, then $x_1 + 5 \neq x_2 + 5 \Rightarrow \frac{1}{2}(x_1 + 5) \neq \frac{1}{2}(x_2 + 5) \Rightarrow f(x_1) \neq f(x_2)$, so f is one-to-one.
- The graph of $f(x) = 1 + 4x - x^2$ is a parabola with axis of symmetry $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$. Pick any x -values equidistant from 2 to find two equal function values. For example, $f(1) = 4$ and $f(3) = 4$, so f is not 1-1.
- $x_1 \neq x_2 \Rightarrow \sqrt{x_1} \neq \sqrt{x_2} \Rightarrow g(x_1) \neq g(x_2)$, so g is 1-1.
- $g(x) = |x| \Rightarrow g(-1) = 1 = g(1)$, so g is not one-to-one.
- $h(x) = x^4 + 5 \Rightarrow h(1) = 6 = h(-1)$, so h is not 1-1.
- $x_1 \neq x_2 \Rightarrow x_1^4 \neq x_2^4$ [since $x \geq 0$] $\Rightarrow x_1^4 + 5 \neq x_2^4 + 5 \Rightarrow h(x_1) \neq h(x_2)$, so h is 1-1.
- A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2 , $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.
- f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

17. f does not pass the Horizontal Line Test, so f is not 1-1.



18. f passes the Horizontal Line Test, so f is 1-1.



19. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f , then the point $(9, 2)$ is on the graph of f^{-1} .
20. $f(x) = x + \cos x \Rightarrow f'(x) = 1 - \sin x \geq 0$, with equality only if $x = \frac{\pi}{2} + 2n\pi$. So f is increasing on $\mathbb{R}$, and hence, 1-1. By inspection, $f(0) = 0 + \cos 0 = 1$, so $f^{-1}(1) = 0$.
21. $h(x) = x + \sqrt{x} \Rightarrow h'(x) = 1 + 1/(2\sqrt{x}) > 0$ on $(0, \infty)$. So h is increasing and hence, 1-1. By inspection, $h(4) = 4 + \sqrt{4} = 6$, so $h^{-1}(6) = 4$.
22. (a) f is 1-1 because it passes the Horizontal Line Test.
 (b) Domain of $f = [-3, 3] = \text{Range of } f^{-1}$. Range of $f = [-2, 2] = \text{Domain of } f^{-1}$.
 (c) Since $f(-2) = 1$, $f^{-1}(1) = -2$.
23. We solve $C = \frac{5}{9}(F - 32)$ for F : $\frac{9}{5}C = F - 32 \Rightarrow F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C . $F \geq -459.67 \Rightarrow \frac{9}{5}C + 32 \geq -459.67 \Rightarrow \frac{9}{5}C \geq -491.67 \Rightarrow C \geq -273.15$, the domain of the inverse function.
24. $m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{m_0^2}{m^2} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{m_0^2}{m^2} \Rightarrow v^2 = c^2 \left(1 - \frac{m_0^2}{m^2}\right) \Rightarrow v = c \sqrt{1 - \frac{m_0^2}{m^2}}$. This formula gives us the speed v of the particle in terms of its mass m , that is, $v = f^{-1}(m)$.
25. $y = f(x) = 3 - 2x \Rightarrow 2x = 3 - y \Rightarrow x = \frac{3 - y}{2}$. Interchange x and y : $y = \frac{3 - x}{2}$. So $f^{-1}(x) = \frac{3 - x}{2}$.
26. $f(x) = \frac{4x - 1}{2x + 3} \Rightarrow y = \frac{4x - 1}{2x + 3} \Rightarrow y(2x + 3) = 4x - 1 \Rightarrow 2xy + 3y = 4x - 1 \Rightarrow 3y + 1 = 4x - 2xy \Rightarrow 3y + 1 = (4 - 2y)x \Rightarrow x = \frac{3y + 1}{4 - 2y}$. Interchange x and y : $y = \frac{3x + 1}{4 - 2x}$. So $f^{-1}(x) = \frac{3x + 1}{4 - 2x}$.
27. $f(x) = \sqrt{10 - 3x} \Rightarrow y = \sqrt{10 - 3x} \ (y \geq 0) \Rightarrow y^2 = 10 - 3x \Rightarrow 3x = 10 - y^2 \Rightarrow x = -\frac{1}{3}y^2 + \frac{10}{3}$. Interchange x and y : $y = -\frac{1}{3}x^2 + \frac{10}{3}$. So $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.
28. $y = f(x) = 2x^3 + 3 \Rightarrow y - 3 = 2x^3 \Rightarrow \frac{y - 3}{2} = x^3 \Rightarrow x = \sqrt[3]{\frac{y - 3}{2}}$. Interchange x and y : $y = \sqrt[3]{\frac{x - 3}{2}}$. So $f^{-1}(x) = \sqrt[3]{\frac{x - 3}{2}}$.

29. For $f(x) = \frac{1 - \sqrt{x}}{1 + \sqrt{x}}$, the domain is $x \geq 0$. $f(0) = 1$ and as x increases, y decreases. As $x \rightarrow \infty$,

$$\frac{1 - \sqrt{x}}{1 + \sqrt{x}} \cdot \frac{1/\sqrt{x}}{1/\sqrt{x}} = \frac{1/\sqrt{x} - 1}{1/\sqrt{x} + 1} \rightarrow \frac{-1}{1} = -1, \text{ so the range of } f \text{ is } -1 < y \leq 1. \text{ Thus, the domain of } f^{-1} \text{ is } -1 < x \leq 1.$$

$$y = \frac{1 - \sqrt{x}}{1 + \sqrt{x}} \Rightarrow y(1 + \sqrt{x}) = 1 - \sqrt{x} \Rightarrow y + y\sqrt{x} = 1 - \sqrt{x} \Rightarrow \sqrt{x} + y\sqrt{x} = 1 - y \Rightarrow$$

$$\sqrt{x}(1 + y) = 1 - y \Rightarrow \sqrt{x} = \frac{1 - y}{1 + y} \Rightarrow x = \left(\frac{1 - y}{1 + y}\right)^2. \text{ Interchange } x \text{ and } y: y = \left(\frac{1 - x}{1 + x}\right)^2. \text{ So}$$

$$f^{-1}(x) = \left(\frac{1 - x}{1 + x}\right)^2 \text{ with } -1 < x \leq 1.$$

30. $y = f(x) = 2x^2 - 8x, x \geq 2 \Rightarrow 2x^2 - 8x - y = 0, x \geq 2 \Rightarrow$

$$x = \frac{8 + \sqrt{64 + 8y}}{4} \left[\begin{array}{l} \text{quadratic formula with} \\ a = 2, b = -8, \text{ and } c = -y \end{array} \right] = \frac{8 + 2\sqrt{16 + 2y}}{4} = 2 + \frac{1}{2}\sqrt{16 + 2y}. \text{ Interchange } x \text{ and}$$

$$y: y = 2 + \frac{1}{2}\sqrt{16 + 2x}. \text{ So } f^{-1}(x) = 2 + \frac{1}{2}\sqrt{16 + 2x}.$$

Alternate solution (by completing the square): $y = 2x^2 - 8x, x \geq 2 \Rightarrow x^2 - 4x = y/2, x \geq 2 \Rightarrow$

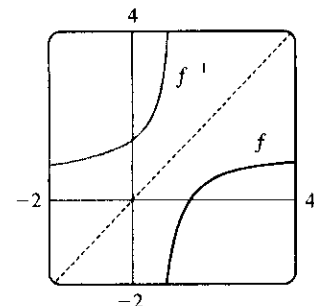
$$(x - 2)^2 = x^2 - 4x + 4 = \frac{y}{2} + 4 = \frac{y + 8}{2} = \frac{2y + 16}{4}, x \geq 2 \Rightarrow x - 2 = +\sqrt{\frac{2y + 16}{4}} \Rightarrow$$

$$x = 2 + \frac{1}{2}\sqrt{2y + 16}. \text{ Interchange } x \text{ and } y: y = 2 + \frac{1}{2}\sqrt{2x + 16}. \text{ So } f^{-1}(x) = 2 + \frac{1}{2}\sqrt{2x + 16}.$$

31. $y = f(x) = 1 - \frac{2}{x^2} \Rightarrow 1 - y = \frac{2}{x^2} \Rightarrow x^2 = \frac{2}{1 - y} \Rightarrow$

$$x = \sqrt{\frac{2}{1 - y}}, \text{ since } x > 0. \text{ Interchange } x \text{ and } y: y = \sqrt{\frac{2}{1 - x}}.$$

$$\text{So } f^{-1}(x) = \sqrt{\frac{2}{1 - x}}.$$

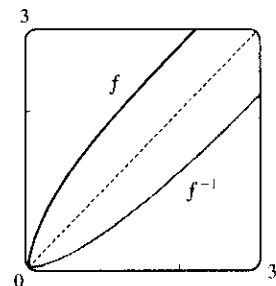


32. $y = f(x) = \sqrt{x^2 + 2x}, x > 0 \Rightarrow y > 0$ and $y^2 = x^2 + 2x \Rightarrow x^2 + 2x - y^2 = 0$. Now we use the quadratic formula:

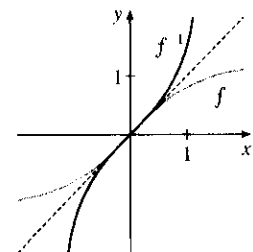
$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-y^2)}}{2 \cdot 1} = -1 \pm \sqrt{1 + y^2}. \text{ But } x > 0, \text{ so the}$$

$$\text{negative root is inadmissible. Interchange } x \text{ and } y: y = -1 + \sqrt{1 + x^2}.$$

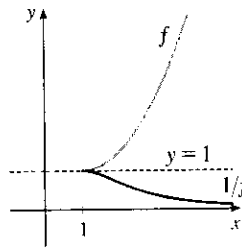
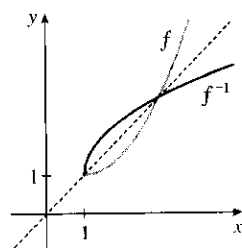
$$\text{So } f^{-1}(x) = -1 + \sqrt{1 + x^2}, x > 0.$$



33. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$.



34. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$. For the graph of $1/f$, the y -coordinates are simply the reciprocals of f . For example, if $f(5) = 9$, then $1/f(5) = \frac{1}{9}$. If we draw the horizontal line $y = 1$, we see that the only place where the graphs intersect is on that line.



35. (a) $x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$, so f is one-to-one.

(b) $f'(x) = 3x^2$ and $f(2) = 8 \Rightarrow g(8) = 2$, so $g'(8) = 1/f'(g(8)) = 1/f'(2) = \frac{1}{12}$.

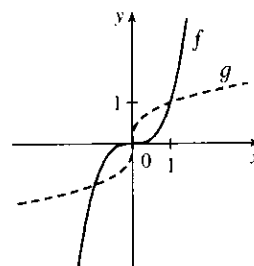
(c) $y = x^3 \Rightarrow x = y^{1/3}$. Interchanging x and y gives $y = x^{1/3}$, (e)

so $f^{-1}(x) = x^{1/3}$. Domain(g) = range(f) = $\mathbb{R}$.

Range(g) = domain(f) = $\mathbb{R}$.

(d) $g(x) = x^{1/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(8) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$

as in part (b).



36. (a) $x_1 \neq x_2 \Rightarrow x_1 - 2 \neq x_2 - 2 \Rightarrow \sqrt{x_1 - 2} \neq \sqrt{x_2 - 2} \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

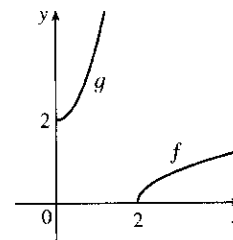
(b) $f(6) = 2$, so $g(2) = 6$. Also $f'(x) = \frac{1}{2\sqrt{x-2}}$, so $g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(6)} = \frac{1}{1/4} = 4$.

(c) $y = \sqrt{x-2} \Rightarrow y^2 = x-2 \Rightarrow x = y^2 + 2$. (e)

Interchange x and y : $y = x^2 + 2$. So $g(x) = x^2 + 2$.

Domain = $[0, \infty)$, range = $[2, \infty)$.

(d) $g(x) = x^2 + 2 \Rightarrow g'(x) = 2x \Rightarrow g'(2) = 4$.



37. (a) Since $x \geq 0$, $x_1 \neq x_2 \Rightarrow x_1^2 \neq x_2^2 \Rightarrow 9 - x_1^2 \neq 9 - x_2^2 \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f'(x) = -2x$ and $f(1) = 8 \Rightarrow g(8) = 1$, so $g'(8) = \frac{1}{f'(g(8))} = \frac{1}{f'(1)} = \frac{1}{-2} = -\frac{1}{2}$.

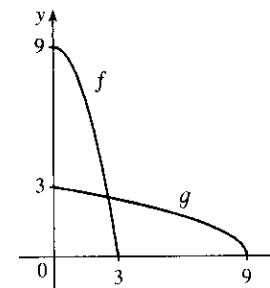
(c) $y = 9 - x^2 \Rightarrow x^2 = 9 - y \Rightarrow x = \sqrt{9 - y}$. Interchange x (e)

and y : $y = \sqrt{9 - x}$, so $f^{-1}(x) = \sqrt{9 - x}$.

Domain(g) = range(f) = $[0, 9]$.

Range(g) = domain(f) = $[0, 3]$.

(d) $g'(x) = -1/(2\sqrt{9-x}) \Rightarrow g'(8) = -\frac{1}{2}$ as in part (b).



38. (a) $x_1 \neq x_2 \Rightarrow x_1 - 1 \neq x_2 - 1 \Rightarrow \frac{1}{x_1 - 1} \neq \frac{1}{x_2 - 1} \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

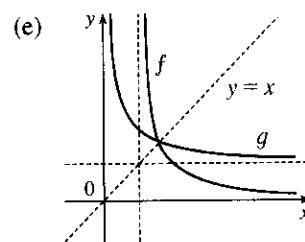
(b) $g(2) = \frac{3}{2}$ since $f(\frac{3}{2}) = 2$. Also $f'(x) = -1/(x-1)^2$, so $g'(2) = 1/f'(\frac{3}{2}) = \frac{1}{-4} = -\frac{1}{4}$.

(c) $y = 1/(x-1) \Rightarrow x-1 = 1/y \Rightarrow x = 1 + 1/y$. Interchange

x and y : $y = 1 + 1/x$. So $g(x) = 1 + 1/x$, $x > 0$ (since $y > 1$).

Domain = $(0, \infty)$, range = $(1, \infty)$.

(d) $g'(x) = -1/x^2$, so $g'(2) = -\frac{1}{4}$.



39. $f(0) = 1 \Rightarrow f^{-1}(1) = 0$, and $f(x) = x^3 + x + 1 \Rightarrow f'(x) = 3x^2 + 1$ and $f'(0) = 1$. Thus,

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{1} = 1.$$

40. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = x^5 - x^3 + 2x \Rightarrow f'(x) = 5x^4 - 3x^2 + 2$ and $f'(1) = 4$. Thus,

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{4}.$$

41. $f(0) = 3 \Rightarrow f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \Rightarrow f'(x) = 2x + \frac{\pi}{2} \sec^2(\pi x/2)$ and $f'(0) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$. Thus, $(f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(0) = 2/\pi$.

42. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = \sqrt{x^3 + x^2 + x + 1} \Rightarrow f'(x) = \frac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$ and $f'(1) = \frac{3 + 2 + 1}{2\sqrt{1 + 1 + 1 + 1}} = \frac{3}{2}$. Thus, $(f^{-1})'(2) = 1/f'(f^{-1}(2)) = 1/f'(1) = \frac{2}{3}$.

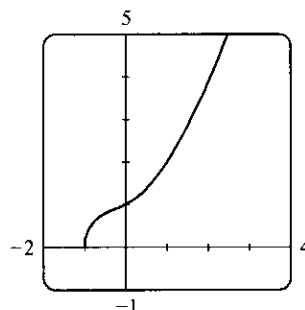
43. $f(4) = 5 \Rightarrow g(5) = 4$. Thus, $g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(4)} = \frac{1}{2/3} = \frac{3}{2}$.

44. $f(3) = 2 \Rightarrow g(2) = 3$. Thus, $g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(3)} = 9$. Hence, $G(x) = \frac{1}{g(x)} \Rightarrow G'(x) = -\frac{g'(x)}{[g(x)]^2} \Rightarrow G'(2) = -\frac{g'(2)}{[g(2)]^2} = -\frac{9}{(3)^2} = -1$.

45. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y . Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:

$$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6} (\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2})$$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\frac{1}{6} \frac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.



46. Since $\sin(2n\pi) = 0$, $h(x) = \sin x$ is not one-to-one. $h'(x) = \cos x > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, so h is increasing and hence 1-1 on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let $y = f^{-1}(x) = \sin^{-1} x$ so that $\sin y = x$. Differentiating $\sin y = x$ implicitly with respect to

x gives us $\cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$. Now $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \pm \sqrt{1 - \sin^2 y}$, but since

$\cos y > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$.

47. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1} . Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.
- (b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c) f^{-1}(x)$.

48. (a) We know that $g'(x) = \frac{1}{f'(g(x))}$. Thus,

$$g''(x) = -\frac{f''(g(x)) \cdot g'(x)}{[f'(g(x))]^2} = -\frac{f''(g(x)) \cdot [1/f'(g(x))]}{[f'(g(x))]^2} = -\frac{f''(g(x))}{f'(g(x))[f'(g(x))]^2} = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

- (b) f is increasing $\Rightarrow f'(g(x)) > 0 \Rightarrow [f'(g(x))]^3 > 0$. f is concave upward $\Rightarrow f''(g(x)) > 0$. So

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3} < 0, \text{ which implies that } g \text{ (} f \text{'s inverse) is concave downward.}$$

7.2 Exponential Functions and Their Derivatives

- (a) $f(x) = a^x$, $a > 0$

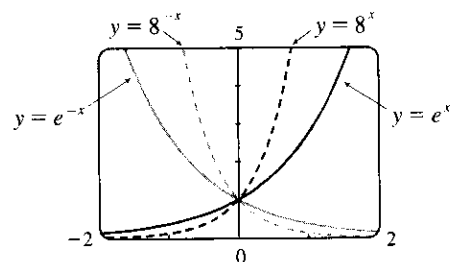
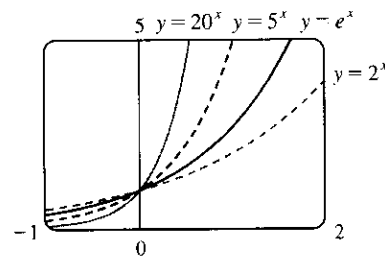
(b) $\mathbb{R}$

(c) $(0, \infty)$

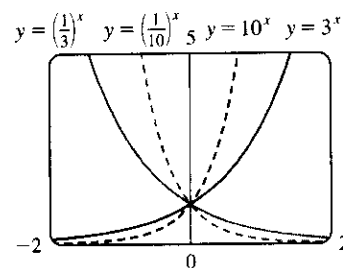
(d) See Figures 6(c), 6(b), and 6(a), respectively.
- (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

(b) $e \approx 2.71828$

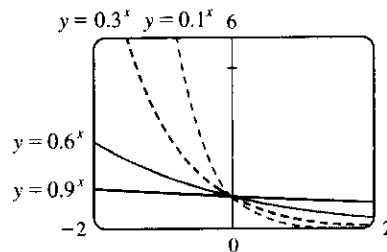
(c) $f(x) = e^x$
- All of these graphs approach 0 as $x \rightarrow -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \rightarrow \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \rightarrow -\infty$.
- The graph of e^{-x} is the reflection of the graph of e^x about the y -axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y -axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \rightarrow -\infty$.



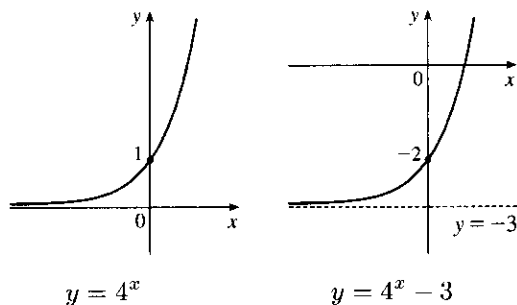
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 ($(\frac{1}{3})^x$ and $(\frac{1}{10})^x$) are decreasing. The graph of $(\frac{1}{3})^x$ is the reflection of that of 3^x about the y -axis, and the graph of $(\frac{1}{10})^x$ is the reflection of that of 10^x about the y -axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \rightarrow -\infty$.



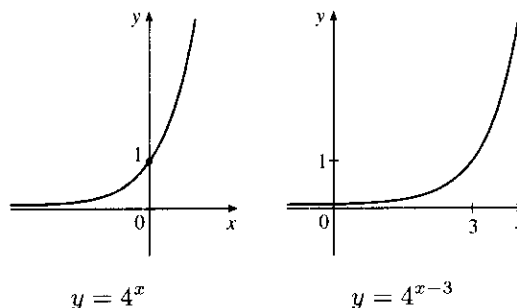
6. Each of the graphs approaches ∞ as $x \rightarrow -\infty$, and each approaches 0 as $x \rightarrow \infty$. The smaller the base, the faster the function grows as $x \rightarrow -\infty$, and the faster it approaches 0 as $x \rightarrow \infty$.



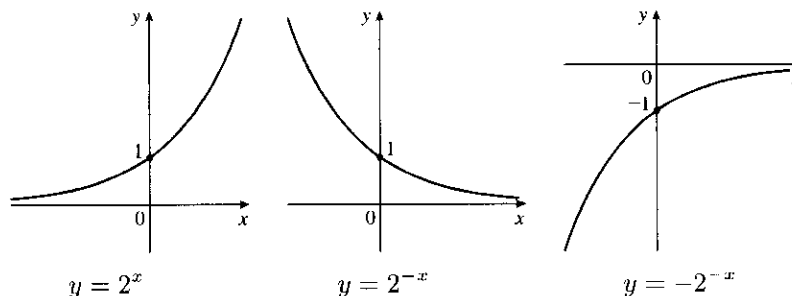
7. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y = 4^x - 3$ is $(-3, \infty)$. There is a horizontal asymptote of $y = -3$.



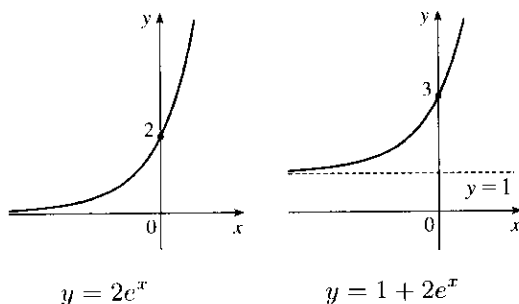
8. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units to the right. There is a horizontal asymptote of $y = 0$.



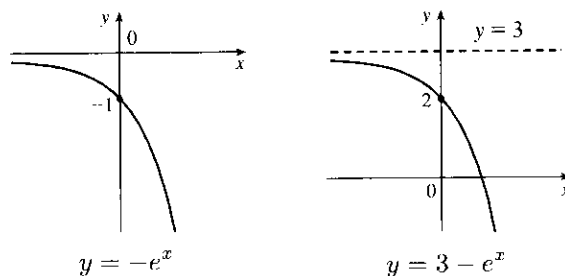
9. We start with the graph of $y = 2^x$ (Figure 3), reflect it about the y -axis, and then about the x -axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



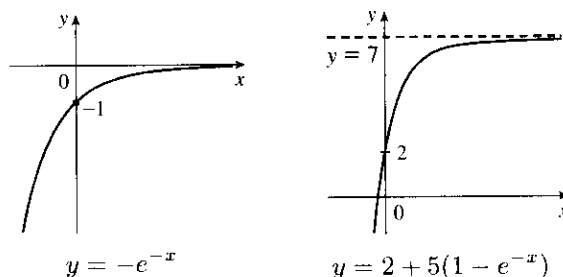
10. We start with the graph of $y = e^x$ (Figure 12), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y = 1$.



11. We start with the graph of $y = e^x$ (Figure 12), reflect it about the x -axis, and then shift 3 units upward. Note the horizontal asymptote of $y = 3$.

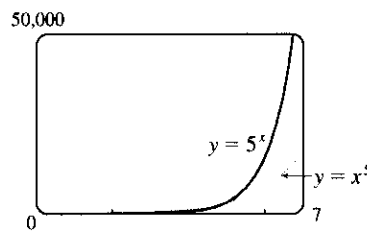
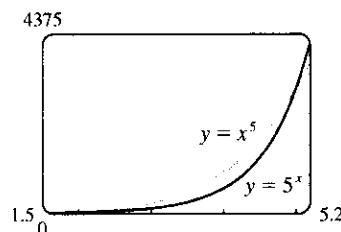
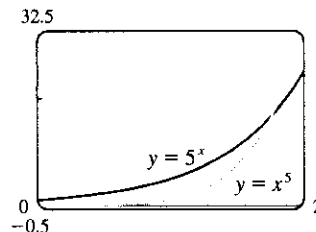
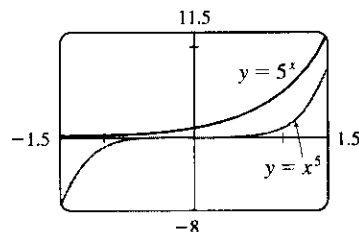


12. We start with the graph of $y = e^x$ (Figure 12), reflect it about the y -axis, and then about the x -axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.

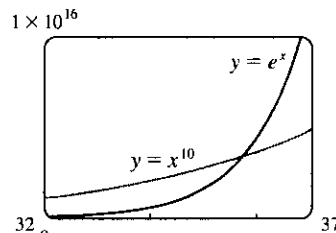
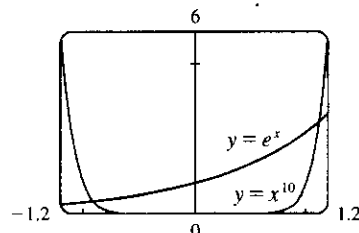


13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.
- (b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.
- (c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x -axis, we multiply the original function by -1 to get $y = -e^x$.
- (d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y -axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.
- (e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x -axis and then about the y -axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.
14. (a) This reflection consists of first reflecting the graph about the x -axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.
- (b) This reflection consists of first reflecting the graph about the y -axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.
15. (a) The denominator $1 + e^x$ is never equal to zero because $e^x > 0$, so the domain of $f(x) = 1/(1 + e^x)$ is $\mathbb{R}$.
- (b) $1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$, so the domain of $f(x) = 1/(1 - e^x)$ is $(-\infty, 0) \cup (0, \infty)$.
16. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.
- (b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.
17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1$ [$C = \frac{6}{a}$] and $24 = Ca^3 \Rightarrow 24 = \left(\frac{6}{a}\right)a^3 \Rightarrow 4 = a^2 \Rightarrow a = 2$ [since $a > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

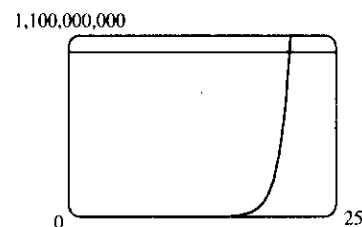
18. Given the y -intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $(2, \frac{2}{9})$ gives us $\frac{2}{9} = 2a^2 \Rightarrow \frac{1}{9} = a^2 \Rightarrow a = \frac{1}{3}$ [since $a > 0$]. The function is $f(x) = 2(\frac{1}{3})^x$ or $f(x) = 2(3)^{-x}$.
19. $2 \text{ ft} = 24 \text{ in}$, $f(24) = 24^2 \text{ in} = 576 \text{ in} = 48 \text{ ft}$. $g(24) = 2^{24} \text{ in} = 2^{24}/(12 \cdot 5280) \text{ mi} \approx 265 \text{ mi}$
20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



21. The graph of g finally surpasses that of f at $x \approx 35.8$.



22. We graph $y = e^x$ and $y = 1,000,000,000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.



23. $\lim_{x \rightarrow \infty} (1.001)^x = \infty$ by (3), since $1.001 > 1$.
24. Let $t = -x^2$. As $x \rightarrow \infty$, $t \rightarrow -\infty$. So $\lim_{x \rightarrow \infty} e^{-x^2} = \lim_{t \rightarrow -\infty} e^t = 0$ by (11).
25. Divide numerator and denominator by e^{3x} : $\lim_{x \rightarrow \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \rightarrow \infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} = \frac{1 - 0}{1 + 0} = 1$
26. If we let $t = \tan x$, then as $x \rightarrow (\pi/2)^+$, $t \rightarrow -\infty$. Thus, $\lim_{x \rightarrow (\pi/2)^+} e^{\tan x} = \lim_{t \rightarrow -\infty} e^t = 0$.
27. Let $t = 3/(2-x)$. As $x \rightarrow 2^+$, $t \rightarrow -\infty$. So $\lim_{x \rightarrow 2^+} e^{3/(2-x)} = \lim_{t \rightarrow -\infty} e^t = 0$ by (11).

28. Let $t = 3/(2 - x)$. As $x \rightarrow 2^-$, $t \rightarrow \infty$. So $\lim_{x \rightarrow 2^-} e^{3/(2-x)} = \lim_{t \rightarrow \infty} e^t = \infty$ by (11).

29. By the Product Rule, $f(x) = x^2 e^x \Rightarrow f'(x) = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2) = x^2 e^x + e^x(2x) = xe^x(x + 2)$.

30. By the Quotient Rule, $y = \frac{e^x}{1+x} \Rightarrow y' = \frac{(1+x)e^x - e^x(1)}{(1+x)^2} = \frac{e^x + xe^x - e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$.

31. By (10), $y = e^{ax^3} \Rightarrow y' = e^{ax^3} \frac{d}{dx}(ax^3) = 3ax^2 e^{ax^3}$.

32. $y = e^u(\cos u + cu) \Rightarrow y' = e^u(-\sin u + c) + (\cos u + cu)e^u = e^u(\cos u - \sin u + cu + c)$

33. $f(u) = e^{1/u} \Rightarrow f'(u) = e^{1/u} \cdot \frac{d}{du}\left(\frac{1}{u}\right) = e^{1/u} \left(\frac{-1}{u^2}\right) = \left(\frac{-1}{u^2}\right) e^{1/u}$

34. By the Product Rule, $g(x) = \sqrt{x} e^x = x^{1/2} e^x \Rightarrow g'(x) = x^{1/2}(e^x) + e^x\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}x^{-1/2}e^x(2x + 1)$.

35. By (10), $F(t) = e^{t \sin 2t} \Rightarrow$
 $F'(t) = e^{t \sin 2t}(t \sin 2t)' = e^{t \sin 2t}(t \cdot 2 \cos 2t + \sin 2t \cdot 1) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$

36. $y = e^{k \tan \sqrt{x}} \Rightarrow y' = e^{k \tan \sqrt{x}} \cdot \frac{d}{dx}(k \tan \sqrt{x}) = e^{k \tan \sqrt{x}} \left(k \sec^2 \sqrt{x} \cdot \frac{1}{2}x^{-1/2}\right) = \frac{k \sec^2 \sqrt{x}}{2\sqrt{x}} e^{k \tan \sqrt{x}}$

37. $y = \sqrt{1 + 2e^{3x}} \Rightarrow y' = \frac{1}{2}(1 + 2e^{3x})^{-1/2} \frac{d}{dx}(1 + 2e^{3x}) = \frac{1}{2\sqrt{1 + 2e^{3x}}}(2e^{3x} \cdot 3) = \frac{3e^{3x}}{\sqrt{1 + 2e^{3x}}}$

38. $y = \cos(e^{\pi x}) \Rightarrow y' = -\sin(e^{\pi x}) \cdot e^{\pi x} \cdot \pi = -\pi e^{\pi x} \sin(e^{\pi x})$

39. $y = e^{e^x} \Rightarrow y' = e^{e^x} \cdot \frac{d}{dx}(e^x) = e^{e^x} \cdot e^x$ or e^{e^x+x}

40. $y = \sqrt{1 + xe^{-2x}} \Rightarrow y' = \frac{1}{2}(1 + xe^{-2x})^{-1/2} [x(-2e^{-2x}) + e^{-2x}] = \frac{e^{-2x}(-2x + 1)}{2\sqrt{1 + xe^{-2x}}}$

41. By the Quotient Rule, $y = \frac{ae^x + b}{ce^x + d} \Rightarrow$
 $y' = \frac{(ce^x + d)(ae^x) - (ae^x + b)(ce^x)}{(ce^x + d)^2} = \frac{(ace^x + ad - ace^x - bc)e^x}{(ce^x + d)^2} = \frac{(ad - bc)e^x}{(ce^x + d)^2}$

42. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}} \Rightarrow y' = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$
 $= \frac{(e^{2x} - 2 + e^{-2x}) - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} = -\frac{4}{(e^x - e^{-x})^2}$

43. $y = e^{2x} \cos \pi x \Rightarrow y' = e^{2x}(-\pi \sin \pi x) + (\cos \pi x)(2e^{2x}) = e^{2x}(2 \cos \pi x - \pi \sin \pi x)$.

At $(0, 1)$, $y' = 1(2 - 0) = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

44. $y = \frac{e^x}{x} \Rightarrow y' = \frac{x \cdot e^x - e^x \cdot 1}{x^2} = \frac{e^x(x - 1)}{x^2}$. At $(1, e)$, $y' = 0$, and an equation of the tangent line is
 $y - e = 0(x - 1)$, or $y = e$.

45. $\frac{d}{dx}(e^{x^2 y}) = \frac{d}{dx}(x + y) \Rightarrow e^{x^2 y}(x^2 y' + y \cdot 2x) = 1 + y' \Rightarrow x^2 e^{x^2 y} y' + 2xy e^{x^2 y} = 1 + y' \Rightarrow$
 $x^2 e^{x^2 y} y' - y' = 1 - 2xy e^{x^2 y} \Rightarrow y'(x^2 e^{x^2 y} - 1) = 1 - 2xy e^{x^2 y} \Rightarrow y' = \frac{1 - 2xy e^{x^2 y}}{x^2 e^{x^2 y} - 1}$

$$46. xe^y + ye^x = 1 \Rightarrow xe^y y' + e^y \cdot 1 + ye^x + e^x y' = 0 \Rightarrow y'(xe^y + e^x) = -e^y - ye^x \Rightarrow$$

$$y' = -\frac{e^y + ye^x}{xe^y + e^x}. \text{ At } (0, 1), y' = -\frac{e + 1 \cdot 1}{0 + 1} = -(e + 1), \text{ so an equation for the tangent line is}$$

$$y - 1 = -(e + 1)(x - 0), \text{ or } y = -(e + 1)x + 1.$$

$$47. y = e^x + e^{-x/2} \Rightarrow y' = e^x - \frac{1}{2}e^{-x/2} \Rightarrow y'' = e^x + \frac{1}{4}e^{-x/2}, \text{ so}$$

$$2y'' - y' - y = 2\left(e^x + \frac{1}{4}e^{-x/2}\right) - \left(e^x - \frac{1}{2}e^{-x/2}\right) - \left(e^x + e^{-x/2}\right) = 0.$$

$$48. y = Ae^{-x} + Bxe^{-x} \Rightarrow y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \Rightarrow$$

$$y'' = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}, \text{ so}$$

$$y'' + 2y' + y = (A - 2B)e^{-x} + Bxe^{-x} + 2[(B - A)e^{-x} - Bxe^{-x}] + Ae^{-x} + Bxe^{-x} = 0.$$

$$49. y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}, \text{ so if } y = e^{rx} \text{ satisfies the differential equation } y'' + 6y' + 8y = 0,$$

then $r^2 e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x , we must have

$$r^2 + 6r + 8 = 0, \text{ or } (r + 2)(r + 4) = 0, \text{ so } r = -2 \text{ or } -4.$$

$$50. y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}. \text{ Thus, } y + y' = y'' \Leftrightarrow e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \Leftrightarrow$$

$$e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \Leftrightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}, \text{ since } e^{\lambda x} \neq 0.$$

$$51. f(x) = e^{2x} \Rightarrow f'(x) = 2e^{2x} \Rightarrow f''(x) = 2 \cdot 2e^{2x} = 2^2 e^{2x} \Rightarrow$$

$$f'''(x) = 2^2 \cdot 2e^{2x} = 2^3 e^{2x} \Rightarrow \dots \Rightarrow f^{(n)}(x) = 2^n e^{2x}$$

$$52. f(x) = xe^{-x} \Rightarrow f'(x) = x(-e^{-x}) + e^{-x} = (1 - x)e^{-x} \Rightarrow$$

$$f''(x) = (1 - x)(-e^{-x}) + e^{-x}(-1) = (x - 2)e^{-x} \Rightarrow f'''(x) = (x - 2)(-e^{-x}) + e^{-x} = (3 - x)e^{-x} \Rightarrow$$

$$f^{(4)}(x) = (3 - x)(-e^{-x}) + e^{-x}(-1) = (x - 4)e^{-x} \Rightarrow \dots \Rightarrow f^{(n)}(x) = (-1)^n(x - n)e^{-x}.$$

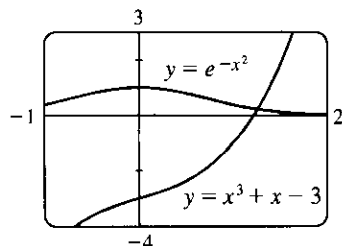
$$\text{So } D^{1000}xe^{-x} = (x - 1000)e^{-x}.$$

53. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1, 0)$.

$$(b) f(x) = e^x + x \Rightarrow f'(x) = e^x + 1, \text{ so } x_{n+1} = x_n - \frac{e^{x_n} + x_n}{e^{x_n} + 1}. \text{ Using } x_1 = -0.5, \text{ we get}$$

$$x_2 \approx -0.566311, x_3 \approx -0.567143 \approx x_4, \text{ so the root is } -0.567143 \text{ to six decimal places.}$$

54.



From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3 .

We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so

$$f'(x) = 3x^2 + 1 + 2xe^{-x^2}, \text{ and the formula is}$$

$$x_{n+1} = x_n - f(x_n)/f'(x_n). \text{ We take } x_1 = 1.2, \text{ and the formula gives}$$

$x_2 \approx 1.252462, x_3 \approx 1.251045, \text{ and } x_4 \approx x_5 \approx 1.251044. \text{ So the root of the equation, correct to six decimal places, is } x = 1.251044.$

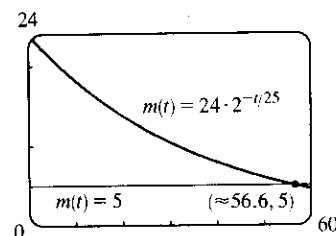
55. (a) $m(t) = 24 \cdot 2^{-t/25} \Rightarrow$

$$m(40) = 24 \cdot 2^{-40/25} \approx 7.92 \text{ mg}$$

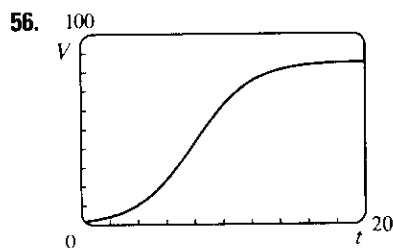
$$\begin{aligned} \text{(b) } m'(t) &= 24 \frac{d}{dt} \left(2^{-t/25} \right) \\ &\approx 24(0.69) 2^{-t/25} \frac{d}{dt} \left(-\frac{t}{25} \right) \quad [(7) \text{ and } (10)] \\ &= 24(0.69) \left(-\frac{1}{25} \right) 2^{-t/25} \end{aligned}$$

$$\text{so } m'(40) \approx -\frac{24}{25}(0.69) 2^{-40/25} \approx -0.22 \text{ mg/yr.}$$

(c)



From the graph, we can determine that $m(t) = 5 \Rightarrow t \approx 56.6$ yr.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use

$V(t) = \frac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

$$V'(t) = -\frac{a(bce^{ct})}{(1 + be^{ct})^2} \quad [\text{by the Reciprocal Rule}] \quad \text{and}$$

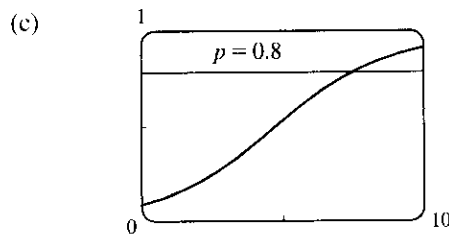
$$\begin{aligned} V''(t) &= -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{[(1 + be^{ct})^2]^2} \\ &= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2 e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3} \end{aligned}$$

So $V''(t) = 0 \Leftrightarrow 1 = be^{ct} \Leftrightarrow e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows:

$$ct = \ln(1/b) \Leftrightarrow t = (1/c) \ln(1/b) = -2 \ln \frac{1}{53} \approx 7.94 \text{ years.}$$

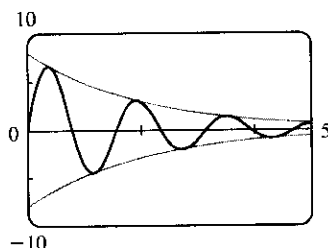
57. (a) $\lim_{t \rightarrow \infty} p(t) = \lim_{t \rightarrow \infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a \cdot 0} = 1$, since $k > 0 \Rightarrow -kt \rightarrow -\infty \Rightarrow e^{-kt} \rightarrow 0$.

(b) $p(t) = (1 + ae^{-kt})^{-1} \Rightarrow \frac{dp}{dt} = -(1 + ae^{-kt})^{-2} (-kae^{-kt}) = \frac{kae^{-kt}}{(1 + ae^{-kt})^2}$



From the graph of $p(t) = (1 + 10e^{-0.5t})^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

58. (a)



The displacement function is squeezed between the other two functions. This is because $-1 \leq \sin 4t \leq 1 \Rightarrow -8e^{-t/2} \leq 8e^{-t/2} \sin 4t \leq 8e^{-t/2}$.

(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

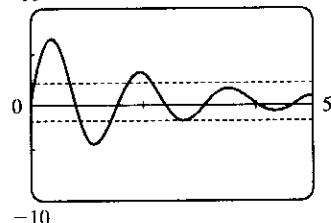
(c) The velocity of the object is the derivative of its displacement function, that is,

$$\frac{d}{dt} (8e^{-t/2} \sin 4t) = 8 \left[e^{-t/2} \cos 4t(4) + \sin 4t \left(-\frac{1}{2}\right) e^{-t/2} \right].$$

If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we

$$\text{get } v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8} \cos\left(4 \cdot \frac{\pi}{4}\right) \cdot 4 = -32e^{-\pi/8} \approx -21.6 \text{ cm/s.}$$

(d) 10

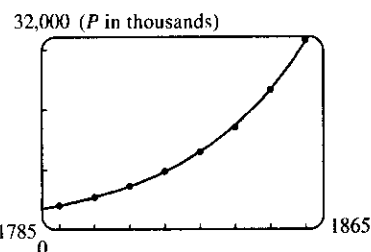


The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

59. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^t = e^{t \ln b}$ from precalculus mathematics or from Section 7.3]: $Q = ae^{t \ln b} = 100.012437e^{-10.005531t}$.

(b) Use $Q'(t) = ab^t \ln b$ or the calculator command `nDeriv(Y1, X, .04)` with $Y1 = ab^x$ to get $Q'(0.04) \approx -670.63 \mu\text{A}$. The result of Example 2 in Section 2.1 was $-670 \mu\text{A}$.

60. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) For 1800: $m_1 = \frac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \frac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \frac{23,192 - 17,063}{1850 - 1840} = 612.9$, $m_2 = \frac{31,443 - 23,192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Use the calculator command `nDeriv(Y1, X, year)` with $Y1 = ab^x$ to get

$P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41,946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

61. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

62. $g(x) = \frac{e^x}{x} \Rightarrow g'(x) = \frac{xe^x - e^x}{x^2} = 0 \Leftrightarrow e^x(x - 1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \Leftrightarrow \frac{xe^x - e^x}{x^2} > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1$ and $g'(x) < 0 \Leftrightarrow \frac{xe^x - e^x}{x^2} < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

63. (a) $f(x) = xe^x \Rightarrow f'(x) = e^x + xe^x = e^x(1 + x) > 0 \Leftrightarrow 1 + x > 0 \Leftrightarrow x > -1$, so f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

(b) $f''(x) = e^x(1 + x) + e^x = e^x(2 + x) > 0 \Leftrightarrow 2 + x > 0 \Leftrightarrow x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$.

(c) f has an inflection point at $(-2, -2e^{-2})$.

64. (a) $f(x) = x^2e^x \Rightarrow f'(x) = 2xe^x + x^2e^x = (x^2 + 2x)e^x$. $f'(x) > 0 \Leftrightarrow x(x + 2) > 0 \Leftrightarrow x < -2$ or $x > 0$, $f'(x) < 0 \Leftrightarrow -2 < x < 0$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$ and decreasing on $(-2, 0)$.

(b) $f''(x) = (2x + 2)e^x + (x^2 + 2x)e^x = (x^2 + 4x + 2)e^x = 0 \Leftrightarrow x^2 + 4x + 2 = 0 \Leftrightarrow x = -2 \pm \sqrt{2}$
 $f''(x) > 0$ when $x > -2 + \sqrt{2}$ or $x < -2 - \sqrt{2}$, so f is CU on $(-\infty, -2 - \sqrt{2})$ and $(-2 + \sqrt{2}, \infty)$ and CD on $(-2 - \sqrt{2}, -2 + \sqrt{2})$.

(c) f has inflection points at $(-2 + \sqrt{2}, (6 - 4\sqrt{2})e^{-2+\sqrt{2}})$ and $(-2 - \sqrt{2}, (6 + 4\sqrt{2})e^{-2-\sqrt{2}})$.

65. $y = f(x) = e^{-1/(x+1)}$ A. $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ B. No x -intercept;

y -intercept $= f(0) = e^{-1}$ C. No symmetry D. $\lim_{x \rightarrow \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \rightarrow 0$, so $y = 1$ is

a HA. $\lim_{x \rightarrow -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \rightarrow -\infty$, $\lim_{x \rightarrow -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \rightarrow \infty$, so

$x = -1$ is a VA. E. $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except -1 , so

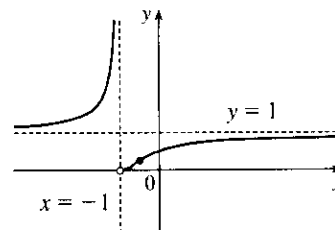
f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. F. No extreme values H.

$$\text{G. } f''(x) = \frac{e^{-1/(x+1)}}{(x+1)^4} + \frac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\frac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$$

$$\Rightarrow f''(x) > 0 \Leftrightarrow 2x + 1 < 0 \Leftrightarrow x < -\frac{1}{2}, \text{ so } f \text{ is CU on}$$

$(-\infty, -1)$ and $(-1, -\frac{1}{2})$, and CD on $(-\frac{1}{2}, \infty)$. f has an IP

at $(-\frac{1}{2}, e^{-2})$.



66. $y = f(x) = e^{2x} - e^x$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$;

x -intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$.

C. No symmetry D. $\lim_{x \rightarrow -\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA.

E. $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2}$ (*)

$\Leftrightarrow x > \ln \frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln \frac{1}{2}$, so

f is decreasing on $(-\infty, \ln \frac{1}{2})$ and increasing on $(\ln \frac{1}{2}, \infty)$. F. Local

minimum value $f(\ln \frac{1}{2}) = e^{2 \ln(1/2)} - e^{\ln(1/2)} = (\frac{1}{2})^2 - \frac{1}{2} = -\frac{1}{4}$

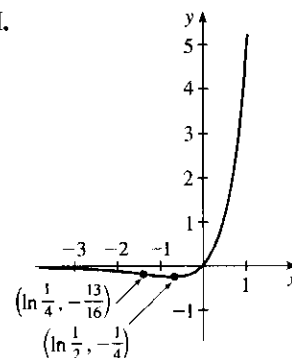
G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow$

$e^x > \frac{1}{4} \Leftrightarrow x > \ln \frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln \frac{1}{4}$.

Thus, f is CD on $(-\infty, \ln \frac{1}{4})$ and CU on $(\ln \frac{1}{4}, \infty)$. f has an IP at $(\ln \frac{1}{4}, (\frac{1}{4})^2 - \frac{1}{4}) = (\ln \frac{1}{4}, -\frac{3}{16})$.

(*) If you have not yet learned about logarithms, graph $y = e^x$ and $y = \frac{1}{2}$, find the point of intersection, and use decimal approximations for the rest of the solution.

H.



67. $y = f(x) = e^{3x} + e^{-2x}$ A. $D = \mathbb{R}$ B. y -intercept = $f(0) = 2$;

no x -intercept C. No symmetry D. No asymptotes

E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \Leftrightarrow 3e^{3x} > 2e^{-2x}$

[multiply by e^{2x}] $\Leftrightarrow e^{5x} > \frac{2}{3}$ (*) $\Leftrightarrow 5x > \ln \frac{2}{3} \Leftrightarrow$

$x > \frac{1}{5} \ln \frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \Leftrightarrow x < \frac{1}{5} \ln \frac{2}{3}$.

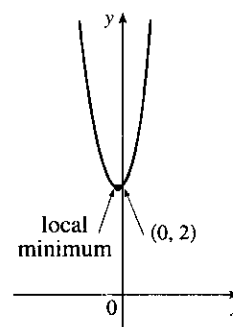
f is decreasing on $(-\infty, \frac{1}{5} \ln \frac{2}{3})$ and increasing on $(\frac{1}{5} \ln \frac{2}{3}, \infty)$.

F. Local minimum value $f(\frac{1}{5} \ln \frac{2}{3}) = (\frac{2}{3})^{3/5} + (\frac{2}{3})^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x , and f is CU on $(-\infty, \infty)$. No IP

(*) If you have not yet learned about logarithms, graph $y = e^{5x}$ and $y = \frac{2}{3}$, find the point of intersection, and use decimal approximations for the rest of the solution.

H.



68. The function $f(x) = e^{\cos x}$ is periodic

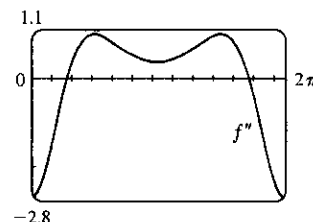
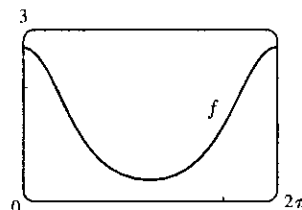
with period 2π , so we consider it only on the interval $[0, 2\pi]$. We see that it has

local maxima of about $f(0) \approx 2.72$ and

$f(2\pi) \approx 2.72$, and a local minimum of

about $f(3.14) \approx 0.37$. To find the exact

values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \Leftrightarrow x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow 0 < x < \pi$. So $f(0) = f(2\pi) = e$

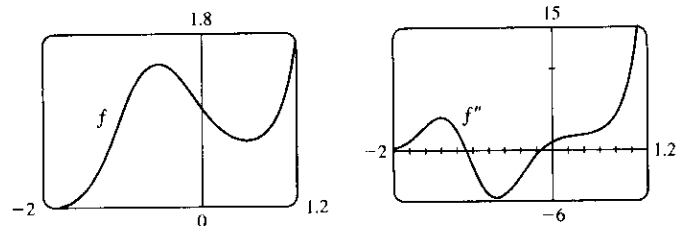


(both maxima) and $f(\pi) = e^{\cos \pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph

$$f''(x) = \frac{d}{dx}(-\sin x e^{\cos x}) = -\cos x e^{\cos x} - \sin x(e^{\cos x})(-\sin x) = e^{\cos x}(\sin^2 x - \cos x).$$

From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x -coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

69. $f(x) = e^{x^3-x} \rightarrow 0$ as $x \rightarrow -\infty$, and $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$.



To find the exact values, we calculate

$$f'(x) = (3x^2 - 1)e^{x^3-x}, \text{ which is 0 when } 3x^2 - 1 = 0 \Leftrightarrow x = \pm \frac{1}{\sqrt{3}}. \text{ The negative root corresponds to the}$$

local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local

minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph

$$f''(x) = \frac{d}{dx}[(3x^2 - 1)e^{x^3-x}] = (3x^2 - 1)e^{x^3-x}(3x^2 - 1) + e^{x^3-x}(6x) = e^{x^3-x}(9x^4 - 6x^2 + 6x + 1).$$

From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and

$x \approx -1.09$. From the graph of f , we see that these x -values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.

70. (a) As $|x| \rightarrow \infty$, $t = -x^2/(2\sigma^2) \rightarrow -\infty$, and $e^t \rightarrow 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t . Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow$

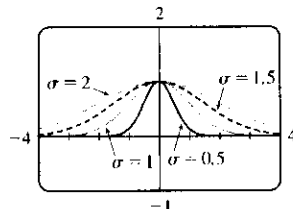
$$f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2). \quad f'(x) = 0 \Leftrightarrow x = 0. \text{ Because } f' \text{ changes from positive to negative at } x = 0, \quad f(0) = 1 \text{ is a local maximum. For inflection points, we find}$$

$$f''(x) = -\frac{1}{\sigma^2} \left[e^{-x^2/(2\sigma^2)} \cdot 1 + x e^{-x^2/(2\sigma^2)}(-x/\sigma^2) \right] = \frac{-1}{\sigma^2} e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2).$$

$$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma. \quad f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma. \text{ So } f \text{ is CD on } (-\sigma, \sigma) \text{ and CU on } (-\infty, -\sigma) \text{ and } (\sigma, \infty). \text{ IP at } (\pm\sigma, e^{-1/2}).$$

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y -axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x -axis.

71. Let $u = -3x$. Then $du = -3dx$, so

$$\int_0^5 e^{-3x} dx = -\frac{1}{3} \int_0^{-15} e^u du = -\frac{1}{3} [e^u]_0^{-15} = -\frac{1}{3} (e^{-15} - e^0) = \frac{1}{3} (1 - e^{-15}).$$

72. Let $u = -x^2$, so $du = -2x dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 x e^{-x^2} dx = \int_0^{-1} e^u \left(-\frac{1}{2} du\right) = -\frac{1}{2} [e^u]_0^{-1} = -\frac{1}{2} (e^{-1} - e^0) = \frac{1}{2} (1 - 1/e).$$

73. Let $u = 1 + e^x$. Then $du = e^x dx$, so $\int e^x \sqrt{1 + e^x} dx = \int \sqrt{u} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + e^x)^{3/2} + C$.

74. Let $u = \tan x$. Then $du = \sec^2 x dx$, so $\int \sec^2 x e^{\tan x} dx = \int e^u du = e^u + C = e^{\tan x} + C$.

$$75. \int \frac{e^x + 1}{e^x} dx = \int (1 + e^{-x}) dx = x - e^{-x} + C$$

76. Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2} dx$, so $\int \frac{e^{1/x}}{x^2} dx = -\int e^u du = -e^u + C = -e^{1/x} + C$.

77. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$, so $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$.

78. Let $u = e^x$. Then $du = e^x dx$, so $\int e^x \sin(e^x) dx = \int \sin u du = -\cos u + C = -\cos(e^x) + C$.

79. Area $= \int_0^1 (e^{3x} - e^x) dx = [\frac{1}{3}e^{3x} - e^x]_0^1 = (\frac{1}{3}e^3 - e) - (\frac{1}{3} - 1) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

80. $f''(x) = 3e^x + 5 \sin x \Rightarrow f'(x) = 3e^x - 5 \cos x + C \Rightarrow 2 = f'(0) = 3 - 5 + C \Rightarrow C = 4$, so
 $f'(x) = 3e^x - 5 \cos x + 4 \Rightarrow f(x) = 3e^x - 5 \sin x + 4x + D \Rightarrow 1 = f(0) = 3 + D \Rightarrow D = -2$,
 so $f(x) = 3e^x - 5 \sin x + 4x - 2$.

81. $V = \int_0^1 \pi (e^x)^2 dx = \pi \int_0^1 e^{2x} dx = \frac{1}{2} \pi [e^{2x}]_0^1 = \frac{\pi}{2} (e^2 - 1)$

82. $V = \int_0^1 2\pi x e^{-x^2} dx$. Let $u = x^2$. Thus $du = 2x dx$, so $V = \pi \int_0^1 e^{-u} du = \pi [-e^{-u}]_0^1 = \pi(1 - 1/e)$.

83. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}.$$

84. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\lim_{x \rightarrow \pi} \frac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos \pi) e^{\sin \pi} = -1 \cdot e^0 = -1$.

85. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 - 0 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0 \Rightarrow e^x - 1 - x \geq 0 \Rightarrow e^x \geq 1 + x$.

(b) For $0 \leq x \leq 1$, $x^2 \leq x$, so $e^{x^2} \leq e^x$ [since e^x is increasing]. Hence [from (a)] $1 + x^2 \leq e^{x^2} \leq e^x$.

$$\text{So } \frac{4}{3} = \int_0^1 (1 + x^2) dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e^x dx = e - 1 < e \Rightarrow \frac{4}{3} \leq \int_0^1 e^{x^2} dx \leq e.$$

86. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by Exercise 85(a).

Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \Rightarrow e^x \geq 1 + x + \frac{1}{2}x^2$.

(b) Using the same argument as in Exercise 85(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \leq e^{x^2} \leq e^x$

$$[\text{for } 0 \leq x \leq 1] \Rightarrow \int_0^1 (1 + x^2 + \frac{1}{2}x^4) dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e^x dx \Rightarrow \frac{43}{30} \leq \int_0^1 e^{x^2} dx \leq e - 1.$$

87. (a) By Exercise 85(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} \geq 0$

by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that

$$0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}, \text{ and hence } e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$$

for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for every positive integer n , by mathematical induction.

- (b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\bar{3} > 2.7$.

$$(c) e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} \Rightarrow \frac{e^x}{x^k} \geq \frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + \frac{1}{k!} + \frac{x}{(k+1)!} \geq \frac{x}{(k+1)!}.$$

$$\text{But } \lim_{x \rightarrow \infty} \frac{x}{(k+1)!} = \infty, \text{ so } \lim_{x \rightarrow \infty} \frac{e^x}{x^k} = \infty.$$

7.3 Logarithmic Functions

1. (a) It is defined as the inverse of the exponential function with base a , that is, $\log_a x = y \Leftrightarrow a^y = x$.
 (b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.
2. (a) The natural logarithm is the logarithm with base e , denoted $\ln x$.
 (b) The common logarithm is the logarithm with base 10, denoted $\log x$.
 (c) See Figure 3.
3. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$. Or: $\log_{10} 1000 = \log_{10} 10^3 = 3$ by (2).
 (b) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$. Or: $\log_{16} 4 = \log_{16} 16^{1/2} = \frac{1}{2}$ by (2).
4. (a) By (6), $\ln e^{-100} = -100$. (b) $\log_3 81 = 4$ since $3^4 = 81$.
5. (a) $\log_5 \frac{1}{25} = \log_5 5^{-2} = -2$ by (2). (b) $e^{\ln 15} = 15$ by (6).
6. (a) $\log_{10} 0.1 = -1$ since $10^{-1} = 0.1$.
 (b) $\log_8 320 - \log_8 5 = \log_8 \frac{320}{5} = \log_8 64 = 2$ since $8^2 = 64$.
7. (a) $\log_{12} 3 + \log_{12} 48 = \log_{12} (3 \cdot 48) = \log_{12} 144 = 2$ since $12^2 = 144$.
 (b) $\log_2 5 - \log_2 90 + 2 \log_2 3 = \log_2 5 + \log_2 3^2 - \log_2 90 = \log_2 (5 \cdot 9) - \log_2 90$
 $= \log_2 \left(\frac{45}{90} \right) = \log_2 \left(\frac{1}{2} \right) = -1$ since $2^{-1} = \frac{1}{2}$.
8. (a) $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 15} = 15$ [Or: $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3 \cdot 5 = 15$]
 (b) $e^{3 \ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$ [Or: $e^{3 \ln 2} = (e^{\ln 2})^3 = 2^3 = 8$]
9. $\log_2 \left(\frac{x^3 y}{z^2} \right) = \log_2 (x^3 y) - \log_2 z^2 = \log_2 x^3 + \log_2 y - \log_2 z^2 = 3 \log_2 x + \log_2 y - 2 \log_2 z$
 (assuming that the variables are positive)
10. $\ln \sqrt{a(b^2 + c^2)} = \ln(a(b^2 + c^2))^{1/2} = \frac{1}{2} \ln(a(b^2 + c^2)) = \frac{1}{2} [\ln a + \ln(b^2 + c^2)]$
 $= \frac{1}{2} \ln a + \frac{1}{2} \ln(b^2 + c^2)$

$$11. \ln(uv)^{10} = 10 \ln(uv) = 10(\ln u + \ln v) = 10 \ln u + 10 \ln v$$

$$12. \ln \frac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5 \ln(x+1) = \ln 3 + 2 \ln x - 5 \ln(x+1)$$

$$13. \log_{10} a - \log_{10} b + \log_{10} c = \log_{10} \frac{a}{b} + \log_{10} c = \log_{10} \left(\frac{a}{b} \cdot c \right) = \log_{10} \frac{ac}{b}$$

$$14. \ln(x+y) + \ln(x-y) - 2 \ln z = \ln((x+y)(x-y)) - \ln z^2 = \ln(x^2 - y^2) - \ln z^2 = \ln \frac{x^2 - y^2}{z^2}$$

$$15. 2 \ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$$

$$16. \ln 3 + \frac{1}{3} \ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3 \cdot 2) = \ln 6$$

$$17. \frac{1}{2} \ln x - 5 \ln(x^2 + 1) = \ln x^{1/2} - \ln(x^2 + 1)^5 = \ln \frac{\sqrt{x}}{(x^2 + 1)^5}$$

$$18. \ln x + a \ln y - b \ln z = \ln x + \ln y^a - \ln z^b = \ln(x \cdot y^a) - \ln z^b = \ln(xy^a/z^b)$$

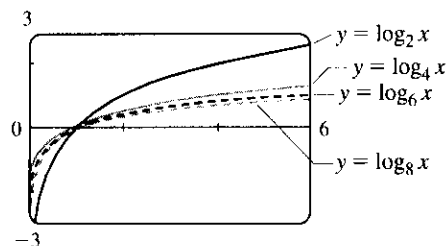
$$19. (a) \log_{12} e = \frac{\ln e}{\ln 12} = \frac{1}{\ln 12} \approx 0.402430$$

$$(b) \log_6 13.54 = \frac{\ln 13.54}{\ln 6} \approx 1.454240$$

$$(c) \log_2 \pi = \frac{\ln \pi}{\ln 2} \approx 1.651496$$

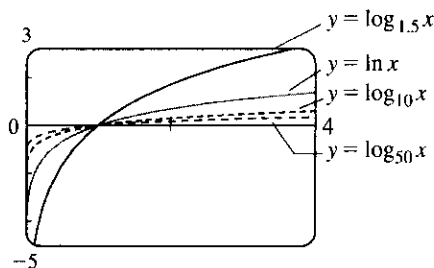
$$20. \text{To graph the functions, we use } \log_2 x = \frac{\ln x}{\ln 2}, \log_4 x = \frac{\ln x}{\ln 4}, \text{ etc.}$$

These graphs all approach $-\infty$ as $x \rightarrow 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \rightarrow \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y -axis (as $x \rightarrow 0^+$).

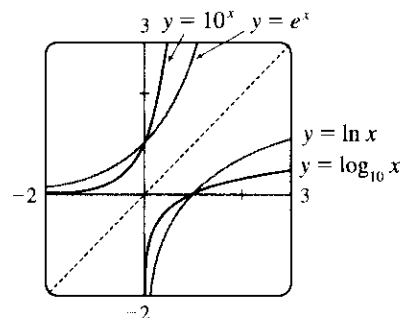


$$21. \text{To graph these functions, we use } \log_{1.5} x = \frac{\ln x}{\ln 1.5} \text{ and}$$

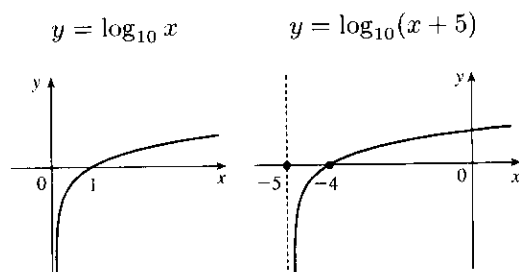
$\log_{50} x = \frac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \rightarrow 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \rightarrow \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y -axis more closely as $x \rightarrow 0^+$.



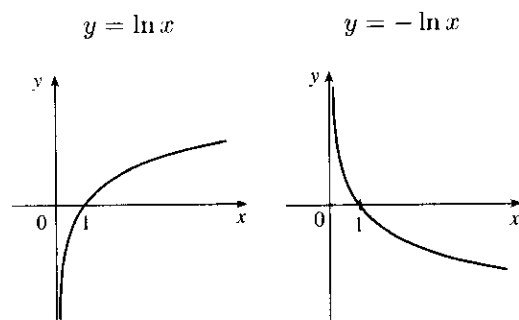
22. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x . Also note that $\log_{10} x \rightarrow \infty$ as $x \rightarrow \infty$ more slowly than $\ln x$.



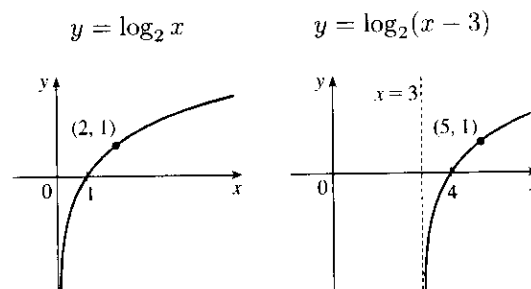
23. Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$.
Note the vertical asymptote of $x = -5$.



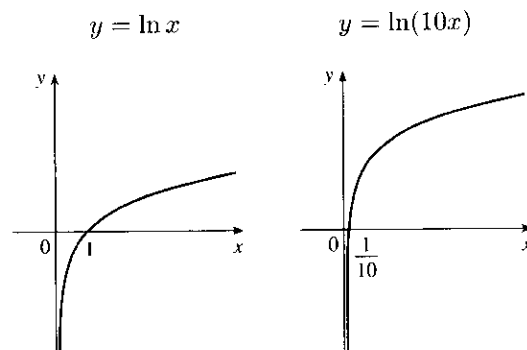
25. Reflect the graph of $y = \ln x$ about the x -axis to obtain the graph of $y = -\ln x$.



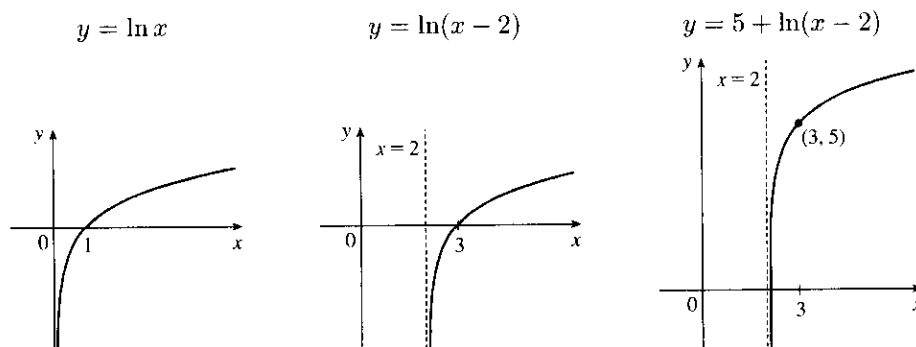
24. $\log_2(x - 3)$: Start with the graph of $y = \log_2 x$ and shift 3 units to the right.



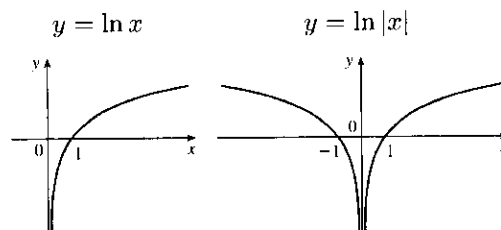
26. $y = \ln(10x)$: Start with the graph of $y = \ln x$ and compress horizontally by a factor of 10.
Or: $y = \ln(10x) = \ln 10 + \ln x$, so we could start with $y = \ln x$ and shift $\ln 10$ units upward.



27. $y = 5 + \ln(x - 2)$: Start with the graph of $y = \ln x$, shift 2 units to the right and then shift 5 units upward.



28. Reflect the portion of the graph of $y = \ln x$ to the right of the y -axis about the y -axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.



29. (a) $2 \ln x = 1 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{1/2} = \sqrt{e}$
 (b) $e^{-x} = 5 \Rightarrow -x = \ln 5 \Rightarrow x = -\ln 5$
30. (a) $e^{2x+3} - 7 = 0 \Rightarrow e^{2x+3} = 7 \Rightarrow 2x+3 = \ln 7 \Rightarrow 2x = \ln 7 - 3 \Rightarrow x = \frac{1}{2}(\ln 7 - 3)$
 (b) $\ln(5-2x) = -3 \Rightarrow 5-2x = e^{-3} \Rightarrow 2x = 5 - e^{-3} \Rightarrow x = \frac{1}{2}(5 - e^{-3})$
31. (a) $5^{x-3} = 10 \Leftrightarrow \log_{10} 5^{x-3} = \log_{10} 10 \Leftrightarrow (x-3) \log_{10} 5 = 1 \Leftrightarrow x-3 = 1/\log_{10} 5 \Leftrightarrow x = 3 + 1/\log_{10} 5$
 (b) $\log_{10}(x+1) = 4 \Leftrightarrow x+1 = 10^4 \Leftrightarrow x = 10,000 - 1 = 9999$
32. (a) $e^{3x+1} = k \Leftrightarrow 3x+1 = \ln k \Leftrightarrow x = \frac{1}{3}(\ln k - 1)$
 (b) $\log_2(mx) = c \Leftrightarrow mx = 2^c \Leftrightarrow x = 2^c/m$
33. $\ln(\ln x) = 1 \Leftrightarrow e^{\ln(\ln x)} = e^1 \Leftrightarrow \ln x = e^1 = e \Leftrightarrow e^{\ln x} = e^e \Leftrightarrow x = e^e$
34. $e^{e^x} = 10 \Leftrightarrow \ln(e^{e^x}) = \ln 10 \Leftrightarrow e^x \ln e = e^x = \ln 10 \Leftrightarrow \ln e^x = \ln(\ln 10) \Leftrightarrow x = \ln(\ln 10)$
35. $2 \ln x = \ln 2 + \ln(3x-4) \Rightarrow \ln x^2 = \ln[2(3x-4)] \Rightarrow \ln x^2 = \ln(6x-8) \Rightarrow x^2 = 6x-8 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x-2)(x-4) = 0 \Rightarrow x = 2 \text{ or } x = 4, \text{ both are valid solutions.}$
36. $\ln(2x+1) = 2 - \ln x \Rightarrow \ln x + \ln(2x+1) = \ln e^2 \Rightarrow \ln[x(2x+1)] = \ln e^2 \Rightarrow 2x^2 + x = e^2 \Rightarrow 2x^2 + x - e^2 = 0 \Rightarrow x = \frac{-1 + \sqrt{1+8e^2}}{4} \text{ [since } x > 0\text{].}$
37. $e^{ax} = Ce^{bx} \Leftrightarrow \ln e^{ax} = \ln[C(e^{bx})] \Leftrightarrow ax = \ln C + bx + \ln e^{bx} \Leftrightarrow ax = \ln C + bx \Leftrightarrow ax - bx = \ln C \Leftrightarrow (a-b)x = \ln C \Leftrightarrow x = \frac{\ln C}{a-b}$
38. $7e^x - e^{2x} = 12 \Leftrightarrow (e^x)^2 - 7e^x + 12 = 0 \Leftrightarrow (e^x - 3)(e^x - 4) = 0$, so we have either $e^x = 3 \Leftrightarrow x = \ln 3$, or $e^x = 4 \Leftrightarrow x = \ln 4$.
39. $e^{2+5x} = 100 \Rightarrow \ln(e^{2+5x}) = \ln 100 \Rightarrow 2+5x = \ln 100 \Rightarrow 5x = \ln 100 - 2 \Rightarrow x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$
40. $\ln(1+\sqrt{x}) = 2 \Rightarrow 1+\sqrt{x} = e^2 \Rightarrow \sqrt{x} = e^2 - 1 \Rightarrow x = (e^2 - 1)^2 \approx 40.8200$
41. $\ln(e^x - 2) = 3 \Rightarrow e^x - 2 = e^3 \Rightarrow e^x = e^3 + 2 \Rightarrow x = \ln(e^3 + 2) \approx 3.0949$
42. $3^{1/(x-4)} = 7 \Rightarrow \ln 3^{1/(x-4)} = \ln 7 \Rightarrow \frac{1}{x-4} \ln 3 = \ln 7 \Rightarrow x-4 = \frac{\ln 3}{\ln 7} \Rightarrow x = 4 + \frac{\ln 3}{\ln 7} \approx 4.5646$
43. (a) $e^x < 10 \Rightarrow \ln e^x < \ln 10 \Rightarrow x < \ln 10 \Rightarrow x \in (-\infty, \ln 10)$
 (b) $\ln x > -1 \Rightarrow e^{\ln x} > e^{-1} \Rightarrow x > e^{-1} \Rightarrow x \in (1/e, \infty)$
44. (a) $2 < \ln x < 9 \Rightarrow e^2 < e^{\ln x} < e^9 \Rightarrow e^2 < x < e^9 \Rightarrow x \in (e^2, e^9)$
 (b) $e^{2-3x} > 4 \Rightarrow \ln e^{2-3x} > \ln 4 \Rightarrow 2-3x > \ln 4 \Rightarrow -3x > \ln 4 - 2 \Rightarrow x < -\frac{1}{3}(\ln 4 - 2) \Rightarrow x \in (-\infty, \frac{1}{3}(2 - \ln 4))$
45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \Leftrightarrow x = 2^{36} = 68,719,476,736$. In miles, this is $68,719,476,736 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 1,084,587.7 \text{ mi}.$

46. (a) $v(t) = ce^{-kt} \Rightarrow a(t) = v'(t) = -kce^{-kt} = -kv(t)$

(b) $v(0) = ce^0 = c$, so c is the initial velocity.

(c) $v(t) = ce^{-kt} = c/2 \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln \frac{1}{2} = -\ln 2 \Rightarrow t = (\ln 2)/k$

47. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \Rightarrow$

$\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3.$

48. Let I_1 and I_2 be the intensities of the music and the mower. Then $10 \log_{10}\left(\frac{I_1}{I_0}\right) = 120$ and $10 \log_{10}\left(\frac{I_2}{I_0}\right) = 106$,

so $\log_{10}\left(\frac{I_1}{I_2}\right) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \Rightarrow \frac{I_1}{I_2} = 10^{1.4} \approx 25.$

49. (a) $n = 100 \cdot 2^{t/3} \Rightarrow \frac{n}{100} = 2^{t/3} \Rightarrow \log_2\left(\frac{n}{100}\right) = \frac{t}{3} \Rightarrow t = 3 \log_2\left(\frac{n}{100}\right)$. Using formula (7), we can write this as $t = 3 \cdot \frac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50,000 \Rightarrow t = 3 \log_2 \frac{50,000}{100} = 3 \log_2 500 = 3 \left(\frac{\ln 500}{\ln 2} \right) \approx 26.9$ hours

50. (a) $Q = Q_0(1 - e^{-t/a}) \Rightarrow \frac{Q}{Q_0} = 1 - e^{-t/a} \Rightarrow e^{-t/a} = 1 - \frac{Q}{Q_0} \Rightarrow -\frac{t}{a} = \ln\left(1 - \frac{Q}{Q_0}\right) \Rightarrow$
 $t = -a \ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q .

(b) $Q = 0.9Q_0$ and $a = 2 \Rightarrow t = -2 \ln(1 - 0.9(Q_0/Q_0)) = -2 \ln 0.1 \approx 4.6$ seconds.

51. Let $t = 2 - x$. As $x \rightarrow 2^-$, $t \rightarrow 0^+$. $\lim_{x \rightarrow 2^-} \ln(2 - x) = \lim_{t \rightarrow 0^+} \ln t = -\infty$ by (8).

52. Let $t = x^2 - 5x + 6$. As $x \rightarrow 3^+$, $t = (x - 2)(x - 3) \rightarrow 0^+$. $\lim_{x \rightarrow 3^+} \log_{10}(x^2 - 5x + 6) = \lim_{t \rightarrow 0^+} \log_{10} t = -\infty$ by (4).

53. $\lim_{x \rightarrow 0} \ln(\cos x) = \ln 1 = 0$. [$\ln(\cos x)$ is continuous at $x = 0$ since it is the composite of two continuous functions.]

54. $\lim_{x \rightarrow 0^+} \ln(\sin x) = -\infty$ since $\sin x \rightarrow 0^+$ as $x \rightarrow 0^+$.

55. $\lim_{x \rightarrow \infty} [\ln(1 + x^2) - \ln(1 + x)] = \lim_{x \rightarrow \infty} \ln \frac{1 + x^2}{1 + x} = \ln \left(\lim_{x \rightarrow \infty} \frac{1 + x^2}{1 + x} \right) = \ln \left(\lim_{x \rightarrow \infty} \frac{\frac{1}{x} + x}{\frac{1}{x} + 1} \right) = \infty$, since the limit in parentheses is ∞ .

56. $\lim_{x \rightarrow \infty} [\ln(2 + x) - \ln(1 + x)] = \lim_{x \rightarrow \infty} \ln \left(\frac{2 + x}{1 + x} \right) = \lim_{x \rightarrow \infty} \ln \left(\frac{2/x + 1}{1/x + 1} \right) = \ln \frac{1}{1} = \ln 1 = 0$

57. The domain of $f(x) = \log_2(5x - 3)$ is $\{x \mid 5x - 3 > 0\} = \{x \mid x > \frac{3}{5}\} = (\frac{3}{5}, \infty)$. Since $5x - 3$ takes on all positive values for x in $(\frac{3}{5}, \infty)$, the range of f is $\mathbb{R}$.

58. The domain of $G(t) = \ln(e^t - 2)$ is $\{t \mid e^t - 2 > 0\} = \{t \mid e^t > 2\} = \{t \mid t > \ln 2\} = (\ln 2, \infty)$. Since $e^t - 2$ takes on all positive values for t in $(\ln 2, \infty)$, the range of G is $\mathbb{R}$.

59. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \Rightarrow e^{2x} \leq 3 \Rightarrow 2x \leq \ln 3 \Rightarrow x \leq \frac{1}{2} \ln 3$.
 Thus, the domain of f is $(-\infty, \frac{1}{2} \ln 3]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow y^2 = 3 - e^{2x} \Rightarrow e^{2x} = 3 - y^2 \Rightarrow 2x = \ln(3 - y^2)$
 $\Rightarrow x = \frac{1}{2} \ln(3 - y^2)$. Interchange x and y : $y = \frac{1}{2} \ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2} \ln(3 - x^2)$. For the domain of f^{-1} , we must have $3 - x^2 > 0 \Rightarrow x^2 < 3 \Rightarrow |x| < \sqrt{3} \Rightarrow -\sqrt{3} < x < \sqrt{3} \Rightarrow 0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1} , $[0, \sqrt{3})$, equals the range of f .

60. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0 \Rightarrow \ln x > -2 \Rightarrow x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞) .

(b) $y = f(x) = \ln(2 + \ln x) \Rightarrow e^y = 2 + \ln x \Rightarrow \ln x = e^y - 2 \Rightarrow x = e^{e^y - 2}$. Interchange x and y : $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1} , as well as the range of f , is $\mathbb{R}$.

61. $y = \ln(x + 3) \Rightarrow e^y = e^{\ln(x+3)} = x + 3 \Rightarrow x = e^y - 3$.

Interchange x and y : the inverse function is $y = e^x - 3$.

62. $y = 2^{10^x} \Rightarrow \log_2 y = 10^x \Rightarrow \log_{10}(\log_2 y) = x$.

Interchange x and y : $y = \log_{10}(\log_2 x)$ is the inverse function.

63. $f(x) = e^{x^3} \Rightarrow y = e^{x^3} \Rightarrow \ln y = x^3 \Rightarrow x = \sqrt[3]{\ln y}$. Interchange x and y : $y = \sqrt[3]{\ln x}$.

So $f^{-1}(x) = \sqrt[3]{\ln x}$.

64. $y = (\ln x)^2, x \geq 1, \ln x = \sqrt{y} \Rightarrow x = e^{\sqrt{y}}$. Interchange x and y : $y = e^{\sqrt{x}}$ is the inverse function.

65. $y = \frac{10^x}{10^x + 1} \Rightarrow 10^x y + y = 10^x \Rightarrow 10^x(1 - y) = y \Rightarrow 10^x = \frac{y}{1 - y} \Rightarrow x = \log_{10} \left(\frac{y}{1 - y} \right)$.

Interchange x and y : $y = \log_{10} \left(\frac{x}{1 - x} \right)$ is the inverse function.

66. $y = \frac{1 + e^x}{1 - e^x} \Rightarrow y - ye^x = 1 + e^x \Rightarrow e^x(y + 1) = y - 1 \Rightarrow e^x = \frac{y - 1}{y + 1} \Rightarrow x = \ln \left(\frac{y - 1}{y + 1} \right)$.

Interchange x and y : $y = \ln \left(\frac{x - 1}{x + 1} \right)$ is the inverse function.

67. $f(x) = e^{3x} - e^x \Rightarrow f'(x) = 3e^{3x} - e^x$. Thus, $f'(x) > 0 \Leftrightarrow 3e^{3x} > e^x \Leftrightarrow \frac{3e^{3x}}{e^x} > \frac{e^x}{e^x} \Leftrightarrow 3e^{2x} > 1 \Leftrightarrow e^{2x} > \frac{1}{3} \Leftrightarrow 2x > \ln \left(\frac{1}{3} \right) = -\ln 3 \Leftrightarrow x > -\frac{1}{2} \ln 3$, so f is increasing on $(-\frac{1}{2} \ln 3, \infty)$.

68. $y = 2e^x - e^{-3x} \Rightarrow y' = 2e^x + 3e^{-3x} \Rightarrow y'' = 2e^x - 9e^{-3x}$. Thus, $y'' < 0 \Leftrightarrow 2e^x < 9e^{-3x} \Leftrightarrow e^{4x} < \frac{9}{2} \Leftrightarrow 4x < \ln \frac{9}{2} \Leftrightarrow x < \frac{1}{4} \ln \frac{9}{2}$, so f is concave downward on $(-\infty, \frac{1}{4} \ln \frac{9}{2})$.

69. (a) We have to show that $-f(x) = f(-x)$.

$$\begin{aligned} -f(x) &= -\ln(x + \sqrt{x^2 + 1}) = \ln \left((x + \sqrt{x^2 + 1})^{-1} \right) = \ln \frac{1}{x + \sqrt{x^2 + 1}} \\ &= \ln \left(\frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x - \sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}} \right) = \ln \frac{x - \sqrt{x^2 + 1}}{x^2 - x^2 - 1} \\ &= \ln(\sqrt{x^2 + 1} - x) = f(-x) \end{aligned}$$

Thus, f is an odd function.

(b) Let $y = \ln(x + \sqrt{x^2 + 1})$. Then $e^y = x + \sqrt{x^2 + 1} \Leftrightarrow (e^y - x)^2 = x^2 + 1 \Leftrightarrow$

$$e^{2y} - 2xe^y + x^2 = x^2 + 1 \Leftrightarrow 2xe^y = e^{2y} - 1 \Leftrightarrow x = \frac{e^{2y} - 1}{2e^y} = \frac{1}{2}(e^y - e^{-y}). \text{ Thus, the inverse}$$

function is $f^{-1}(x) = \frac{1}{2}(e^x - e^{-x})$.

70. Let (a, e^{-a}) be the point where the tangent meets the curve. The tangent has slope $-e^{-a}$ and is perpendicular to the line $2x - y = 8$, which has slope 2. So $-e^{-a} = -\frac{1}{2} \Rightarrow e^{-a} = \frac{1}{2} \Rightarrow e^a = 2 \Rightarrow a = \ln(e^a) = \ln 2$. Thus, the point on the curve is $(\ln 2, \frac{1}{2})$ and the equation of the tangent is $y - \frac{1}{2} = -\frac{1}{2}(x - \ln 2)$ or $x + 2y = 1 + \ln 2$.

71. $x^{1/\ln x} = 2 \Rightarrow \ln(x^{1/\ln x}) = \ln(2) \Rightarrow \frac{1}{\ln x} \cdot \ln x = \ln 2 \Rightarrow 1 = \ln 2$, a contradiction, so the given equation has no solution. The function $f(x) = x^{1/\ln x} = (e^{\ln x})^{1/\ln x} = e^1 = e$ for all $x > 0$, so the function $f(x) = x^{1/\ln x}$ is the constant function $f(x) = e$.

72. (a) $\lim_{x \rightarrow \infty} x^{\ln x} = \lim_{x \rightarrow \infty} (e^{\ln x})^{\ln x} = \lim_{x \rightarrow \infty} e^{(\ln x)^2} = \infty$ since $(\ln x)^2 \rightarrow \infty$ as $x \rightarrow \infty$.

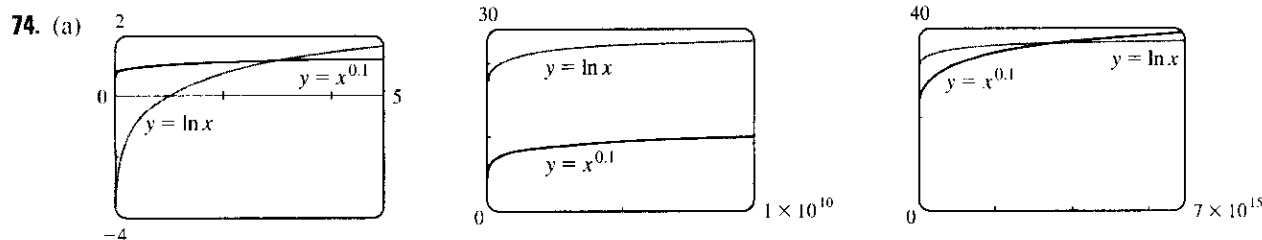
(b) $\lim_{x \rightarrow 0^+} x^{-\ln x} = \lim_{x \rightarrow 0^+} (e^{\ln x})^{-\ln x} = \lim_{x \rightarrow 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \rightarrow -\infty$ as $x \rightarrow 0^+$.

(c) $\lim_{x \rightarrow 0^+} x^{1/x} = \lim_{x \rightarrow 0^+} (e^{\ln x})^{1/x} = \lim_{x \rightarrow 0^+} e^{(\ln x)/x} = 0$ since $\frac{\ln x}{x} \rightarrow -\infty$ as $x \rightarrow 0^+$. Note that as $x \rightarrow 0^+$, $\ln x$ is a large negative number and x is a small positive number, so $(\ln x)/x \rightarrow -\infty$.

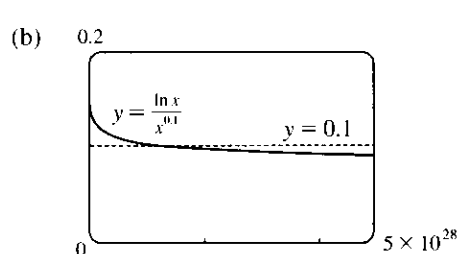
(d) $\lim_{x \rightarrow \infty} (\ln 2x)^{-\ln x} = \lim_{x \rightarrow \infty} [e^{\ln(\ln 2x)}]^{-\ln x} = \lim_{x \rightarrow \infty} e^{-\ln x \ln(\ln 2x)} = 0$ since $-\ln x \ln(\ln 2x) \rightarrow -\infty$ as $x \rightarrow \infty$.

73. (a) Let $\varepsilon > 0$ be given. We need N such that $|a^x - 0| < \varepsilon$ when $x < N$. But $a^x < \varepsilon \Leftrightarrow x < \log_a \varepsilon$. Let $N = \log_a \varepsilon$. Then $x < N \Rightarrow x < \log_a \varepsilon \Rightarrow |a^x - 0| = a^x < \varepsilon$, so $\lim_{x \rightarrow -\infty} a^x = 0$.

(b) Let $M > 0$ be given. We need N such that $a^x > M$ when $x > N$. But $a^x > M \Leftrightarrow x > \log_a M$. Let $N = \log_a M$. Then $x > N \Rightarrow x > \log_a M \Rightarrow a^x > M$, so $\lim_{x \rightarrow \infty} a^x = \infty$.



From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.



(c) From the graph at left, it seems that $\frac{\ln x}{x^{0.1}} < 0.1$ whenever $x > 1.3 \times 10^{28}$ (approximately). So we can take $N = 1.3 \times 10^{28}$, or any larger number.

75. $\ln(x^2 - 2x - 2) \leq 0 \Rightarrow 0 < x^2 - 2x - 2 \leq 1$. Now $x^2 - 2x - 2 \leq 1$ gives $x^2 - 2x - 3 \leq 0$ and hence $(x - 3)(x + 1) \leq 0$. So $-1 \leq x \leq 3$. Now $0 < x^2 - 2x - 2 \Rightarrow x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$. Therefore, $\ln(x^2 - 2x - 2) \leq 0 \Leftrightarrow -1 \leq x < 1 - \sqrt{3}$ or $1 + \sqrt{3} < x \leq 3$.

76. (a) The primes less than 25 are 2, 3, 5, 7, 11, 13, 17, 19, and 23. There are 9 of them, so $\pi(25) = 9$. We use the sieve of Eratosthenes, and arrive at the figure at right. There are 25 numbers left over, so $\pi(100) = 25$.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

- (b) Let $f(n) = \frac{\pi(n)}{n/\ln n}$. We compute $f(100) = \frac{25}{100/\ln 100} \approx 1.15$,
 $f(1000) \approx 1.16$, $f(10^4) \approx 1.13$, $f(10^5) \approx 1.10$, $f(10^6) \approx 1.08$,
 and $f(10^7) \approx 1.07$.
- (c) By the Prime Number Theorem, the number of primes less than a billion, that is, $\pi(10^9)$, should be close to $10^9/\ln 10^9 \approx 48,254,942$. In fact, $\pi(10^9) = 50,847,543$, so our estimate is off by about 5.1%. Do not attempt this calculation at home.

7.4 Derivatives of Logarithmic Functions

- The differentiation formula for logarithmic functions, $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.
- $f(x) = \ln(x^2 + 10) \Rightarrow f'(x) = \frac{1}{x^2 + 10} \cdot \frac{d}{dx}(x^2 + 10) = \frac{2x}{x^2 + 10}$
- $f(\theta) = \ln(\cos \theta) \Rightarrow f'(\theta) = \frac{1}{\cos \theta} \cdot \frac{d}{d\theta}(\cos \theta) = \frac{-\sin \theta}{\cos \theta} = -\tan \theta$
- $f(x) = \cos(\ln x) \Rightarrow f'(x) = -\sin(\ln x) \cdot \frac{1}{x} = \frac{-\sin(\ln x)}{x}$
- $f(x) = \log_2(1 - 3x) \Rightarrow f'(x) = \frac{1}{(1 - 3x) \ln 2} \cdot \frac{d}{dx}(1 - 3x) = \frac{-3}{(1 - 3x) \ln 2}$ or $\frac{3}{(3x - 1) \ln 2}$
- $f(x) = \log_{10}\left(\frac{x}{x - 1}\right) = \log_{10} x - \log_{10}(x - 1) \Rightarrow f'(x) = \frac{1}{x \ln 10} - \frac{1}{(x - 1) \ln 10}$ or $-\frac{1}{x(x - 1) \ln 10}$
- $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \Rightarrow f'(x) = \frac{1}{5}(\ln x)^{-4/5} \cdot \frac{d}{dx}(\ln x) = \frac{1}{5(\ln x)^{4/5}} \cdot \frac{1}{x} = \frac{1}{5x \sqrt[5]{(\ln x)^4}}$
- $f(x) = \ln \sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5} \ln x \Rightarrow f'(x) = \frac{1}{5} \cdot \frac{1}{x} = \frac{1}{5x}$
- $f(x) = \sqrt{x} \ln x \Rightarrow f'(x) = \sqrt{x} \left(\frac{1}{x}\right) + (\ln x) \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} = \frac{2 + \ln x}{2\sqrt{x}}$
- $f(t) = \frac{1 + \ln t}{1 - \ln t} \Rightarrow$
 $f'(t) = \frac{(1 - \ln t)(1/t) - (1 + \ln t)(-1/t)}{(1 - \ln t)^2} = \frac{(1/t)[(1 - \ln t) + (1 + \ln t)]}{(1 - \ln t)^2} = \frac{2}{t(1 - \ln t)^2}$
- $F(t) = \ln \frac{(2t + 1)^3}{(3t - 1)^4} = \ln(2t + 1)^3 - \ln(3t - 1)^4 = 3 \ln(2t + 1) - 4 \ln(3t - 1) \Rightarrow$
 $F'(t) = 3 \cdot \frac{1}{2t + 1} \cdot 2 - 4 \cdot \frac{1}{3t - 1} \cdot 3 = \frac{6}{2t + 1} - \frac{12}{3t - 1}$, or combined, $\frac{-6(t + 3)}{(2t + 1)(3t - 1)}$.

$$12. h(x) = \ln(x + \sqrt{x^2 - 1}) \Rightarrow$$

$$h'(x) = \frac{1}{x + \sqrt{x^2 - 1}} \left(1 + \frac{x}{\sqrt{x^2 - 1}} \right) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$13. g(x) = \ln \frac{a-x}{a+x} = \ln(a-x) - \ln(a+x) \Rightarrow$$

$$g'(x) = \frac{1}{a-x}(-1) - \frac{1}{a+x} = \frac{-(a+x) - (a-x)}{(a-x)(a+x)} = \frac{-2a}{a^2 - x^2}$$

$$14. F(y) = y \ln(1 + e^y) \Rightarrow F'(y) = y \cdot \frac{1}{1 + e^y} \cdot e^y + \ln(1 + e^y) \cdot 1 = \frac{ye^y}{1 + e^y} + \ln(1 + e^y)$$

$$15. f(u) = \frac{\ln u}{1 + \ln(2u)} \Rightarrow$$

$$\begin{aligned} f'(u) &= \frac{[1 + \ln(2u)] \cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u} \cdot 2}{[1 + \ln(2u)]^2} = \frac{\frac{1}{u} [1 + \ln(2u) - \ln u]}{[1 + \ln(2u)]^2} \\ &= \frac{1 + (\ln 2 + \ln u) - \ln u}{u [1 + \ln(2u)]^2} = \frac{1 + \ln 2}{u [1 + \ln(2u)]^2} \end{aligned}$$

$$16. y = \ln(x^4 \sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4 \ln x + 2 \ln \sin x \Rightarrow y' = 4 \cdot \frac{1}{x} + 2 \cdot \frac{1}{\sin x} \cdot \cos x = \frac{4}{x} + 2 \cot x$$

$$17. h(t) = t^3 - 3^t \Rightarrow h'(t) = 3t^2 - 3^t \ln 3$$

$$18. y = 10^{\tan \theta} \Rightarrow y' = 10^{\tan \theta} (\ln 10)(\sec^2 \theta)$$

$$19. y = \ln |2 - x - 5x^2| \Rightarrow y' = \frac{1}{2 - x - 5x^2} \cdot (-1 - 10x) = \frac{-10x - 1}{2 - x - 5x^2} \text{ or } \frac{10x + 1}{5x^2 + x - 2}$$

$$20. G(u) = \ln \sqrt{\frac{3u+2}{3u-2}} = \frac{1}{2} [\ln(3u+2) - \ln(3u-2)] \Rightarrow G'(u) = \frac{1}{2} \left(\frac{3}{3u+2} - \frac{3}{3u-2} \right) = \frac{-6}{9u^2 - 4}$$

$$21. y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \Rightarrow$$

$$y' = -1 + \frac{1}{1+x} = \frac{-1-x+1}{1+x} = -\frac{x}{1+x}$$

$$22. y = [\ln(1 + e^x)]^2 \Rightarrow y' = 2[\ln(1 + e^x)] \cdot \frac{1}{1 + e^x} \cdot e^x = \frac{2e^x \ln(1 + e^x)}{1 + e^x}$$

$$23. \text{Using Formula 7 and the Chain Rule, } y = 5^{-1/x} \Rightarrow y' = 5^{-1/x} (\ln 5) [-1 \cdot (-x^{-2})] = 5^{-1/x} (\ln 5) / x^2$$

$$24. y = 2^{3x^2} \Rightarrow y' = 2^{3x^2} (\ln 2) \frac{d}{dx} (3x^2) = 2^{3x^2} (\ln 2) 3x^2 (\ln 3) (2x)$$

$$25. y = x \ln x \Rightarrow y' = x(1/x) + (\ln x) \cdot 1 = 1 + \ln x \Rightarrow y'' = 1/x$$

$$26. y = \frac{\ln x}{x^2} \Rightarrow y' = \frac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \frac{x(1 - 2 \ln x)}{x^4} = \frac{1 - 2 \ln x}{x^3} \Rightarrow$$

$$y'' = \frac{x^3(-2/x) - (1 - 2 \ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2 - 3 + 6 \ln x)}{x^6} = \frac{6 \ln x - 5}{x^4}$$

$$27. y = \log_{10} x \Rightarrow y' = \frac{1}{x \ln 10} = \frac{1}{\ln 10} \left(\frac{1}{x} \right) \Rightarrow y'' = \frac{1}{\ln 10} \left(-\frac{1}{x^2} \right) = -\frac{1}{x^2 \ln 10}$$

$$28. y = \ln(\sec x + \tan x) \Rightarrow y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x \Rightarrow y'' = \sec x \tan x$$

$$29. f(x) = \frac{x}{1 - \ln(x-1)} \Rightarrow$$

$$\begin{aligned} f'(x) &= \frac{[1 - \ln(x-1)] \cdot 1 - x \cdot \frac{-1}{x-1}}{[1 - \ln(x-1)]^2} = \frac{(x-1)[1 - \ln(x-1)] + x}{[1 - \ln(x-1)]^2} = \frac{x-1 - (x-1)\ln(x-1) + x}{(x-1)[1 - \ln(x-1)]^2} \\ &= \frac{2x-1 - (x-1)\ln(x-1)}{(x-1)[1 - \ln(x-1)]^2} \end{aligned}$$

$$\begin{aligned} \text{Dom}(f) &= \{x \mid x-1 > 0 \text{ and } 1 - \ln(x-1) \neq 0\} = \{x \mid x > 1 \text{ and } \ln(x-1) \neq 1\} \\ &= \{x \mid x > 1 \text{ and } x-1 \neq e^1\} = \{x \mid x > 1 \text{ and } x \neq 1+e\} = (1, 1+e) \cup (1+e, \infty) \end{aligned}$$

$$30. f(x) = \frac{1}{1 + \ln x} \Rightarrow f'(x) = -\frac{1/x}{(1 + \ln x)^2} \quad [\text{Reciprocal Rule}] = -\frac{1}{x(1 + \ln x)^2}.$$

$$\text{Dom}(f) = \{x \mid x > 0 \text{ and } \ln x \neq -1\} = \{x \mid x > 0 \text{ and } x \neq 1/e\} = (0, 1/e) \cup (1/e, \infty).$$

$$31. f(x) = x^2 \ln(1-x^2) \Rightarrow f'(x) = 2x \ln(1-x^2) + \frac{x^2(-2x)}{1-x^2} = 2x \ln(1-x^2) - \frac{2x^3}{1-x^2}.$$

$$\text{Dom}(f) = \{x \mid 1-x^2 > 0\} = \{x \mid |x| < 1\} = (-1, 1).$$

$$32. f(x) = \ln \ln \ln x \Rightarrow f'(x) = \frac{1}{\ln \ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}.$$

$$\text{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty).$$

$$33. f(x) = \frac{x}{\ln x} \Rightarrow f'(x) = \frac{\ln x - x(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \frac{1-1}{1^2} = 0$$

$$34. f(x) = x^2 \ln x \Rightarrow f'(x) = 2x \ln x + x^2 \left(\frac{1}{x} \right) = 2x \ln x + x \Rightarrow f'(1) = 2 \ln 1 + 1 = 1$$

$$35. y = f(x) = \ln \ln x \Rightarrow f'(x) = \frac{1}{\ln x} \left(\frac{1}{x} \right) \Rightarrow f'(e) = \frac{1}{e}, \text{ so an equation of the tangent line at } (e, 0) \text{ is}$$

$$y - 0 = \frac{1}{e}(x - e), \text{ or } y = \frac{1}{e}x - 1, \text{ or } x - ey = e.$$

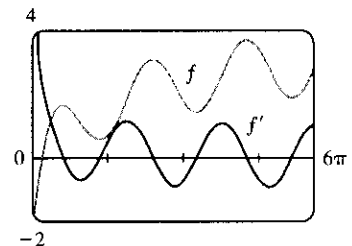
$$36. y = \ln(x^3 - 7) \Rightarrow y' = \frac{1}{x^3 - 7} \cdot 3x^2 \Rightarrow y'(2) = \frac{12}{8-7} = 12, \text{ so an equation of a tangent line at } (2, 0) \text{ is}$$

$$y - 0 = 12(x - 2) \text{ or } y = 12x - 24.$$

$$37. f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x. \text{ This is reasonable,}$$

because the graph shows that f increases when f' is positive, and

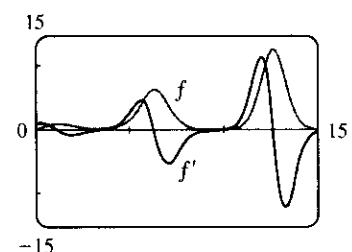
$f'(x) = 0$ when f has a horizontal tangent.



$$38. f(x) = x^{\cos x} = e^{\ln x \cos x} \Rightarrow$$

$$\begin{aligned} f'(x) &= e^{\ln x \cos x} \left[\ln x (-\sin x) + \cos x \left(\frac{1}{x} \right) \right] \\ &= x^{\cos x} \left[\frac{\cos x}{x} - \sin x \ln x \right] \end{aligned}$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



$$39. y = (2x + 1)^5 (x^4 - 3)^6 \Rightarrow \ln y = \ln((2x + 1)^5 (x^4 - 3)^6) \Rightarrow$$

$$\ln y = 5 \ln(2x + 1) + 6 \ln(x^4 - 3) \Rightarrow \frac{1}{y} y' = 5 \cdot \frac{1}{2x + 1} \cdot 2 + 6 \cdot \frac{1}{x^4 - 3} \cdot 4x^3 \Rightarrow$$

$$y' = y \left(\frac{10}{2x + 1} + \frac{24x^3}{x^4 - 3} \right) = (2x + 1)^5 (x^4 - 3)^6 \left(\frac{10}{2x + 1} + \frac{24x^3}{x^4 - 3} \right).$$

[The answer could be simplified to $y' = 2(2x + 1)^4 (x^4 - 3)^5 (29x^4 + 12x^3 - 15)$, but this is unnecessary.]

$$40. y = \sqrt{x} e^{x^2} (x^2 + 1)^{10} \Rightarrow \ln y = \ln \sqrt{x} + \ln e^{x^2} + \ln(x^2 + 1)^{10} \Rightarrow \ln y = \frac{1}{2} \ln x + x^2 + 10 \ln(x^2 + 1)$$

$$\Rightarrow \frac{1}{y} y' = \frac{1}{2} \cdot \frac{1}{x} + 2x + 10 \cdot \frac{1}{x^2 + 1} \cdot 2x \Rightarrow y' = \sqrt{x} e^{x^2} (x^2 + 1)^{10} \left(\frac{1}{2x} + 2x + \frac{20x}{x^2 + 1} \right)$$

$$41. y = \frac{\sin^2 x \tan^4 x}{(x^2 + 1)^2} \Rightarrow \ln y = \ln(\sin^2 x \tan^4 x) - \ln(x^2 + 1)^2 \Rightarrow$$

$$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2 + 1)^2 \Rightarrow \ln y = 2 \ln |\sin x| + 4 \ln |\tan x| - 2 \ln(x^2 + 1) \Rightarrow$$

$$\frac{1}{y} y' = 2 \cdot \frac{1}{\sin x} \cdot \cos x + 4 \cdot \frac{1}{\tan x} \cdot \sec^2 x - 2 \cdot \frac{1}{x^2 + 1} \cdot 2x \Rightarrow$$

$$y' = \frac{\sin^2 x \tan^4 x}{(x^2 + 1)^2} \left(2 \cot x + \frac{4 \sec^2 x}{\tan x} - \frac{4x}{x^2 + 1} \right)$$

$$42. y = \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}} \Rightarrow \ln y = \frac{1}{4} \ln(x^2 + 1) - \frac{1}{4} \ln(x^2 - 1) \Rightarrow \frac{1}{y} y' = \frac{1}{4} \cdot \frac{1}{x^2 + 1} \cdot 2x - \frac{1}{4} \cdot \frac{1}{x^2 - 1} \cdot 2x \Rightarrow$$

$$y' = \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}} \cdot \frac{1}{2} \left(\frac{x}{x^2 + 1} - \frac{x}{x^2 - 1} \right) = \frac{1}{2} \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}} \left(\frac{-2x}{x^4 - 1} \right) = \frac{x}{1 - x^4} \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}}$$

$$43. y = x^x \Rightarrow \ln y = \ln x^x \Rightarrow \ln y = x \ln x \Rightarrow y'/y = x(1/x) + (\ln x) \cdot 1 \Rightarrow$$

$$y' = y(1 + \ln x) \Rightarrow y' = x^x(1 + \ln x)$$

$$44. y = x^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln x \Rightarrow \frac{y'}{y} = \frac{1}{x} \left(\frac{1}{x} \right) + (\ln x) \left(-\frac{1}{x^2} \right) \Rightarrow y' = x^{1/x} \frac{1 - \ln x}{x^2}$$

$$45. y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} \Rightarrow \ln y = \sin x \ln x \Rightarrow \frac{y'}{y} = (\sin x) \cdot \frac{1}{x} + (\ln x)(\cos x) \Rightarrow$$

$$y' = y \left(\frac{\sin x}{x} + \ln x \cos x \right) \Rightarrow y' = x^{\sin x} \left(\frac{\sin x}{x} + \ln x \cos x \right)$$

$$46. y = (\sin x)^x \Rightarrow \ln y = x \ln(\sin x) \Rightarrow \frac{y'}{y} = x \cdot \frac{1}{\sin x} \cdot \cos x + [\ln(\sin x)] \cdot 1 \Rightarrow$$

$$y' = (\sin x)^x [x \cot x + \ln(\sin x)]$$

$$47. y = (\ln x)^x \Rightarrow \ln y = \ln(\ln x)^x \Rightarrow \ln y = x \ln \ln x \Rightarrow \frac{y'}{y} = x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} + (\ln \ln x) \cdot 1 \Rightarrow$$

$$y' = y \left(\frac{x}{x \ln x} + \ln \ln x \right) \Rightarrow y' = (\ln x)^x \left(\frac{1}{\ln x} + \ln \ln x \right)$$

$$48. y = x^{\ln x} \Rightarrow \ln y = \ln x \ln x = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2 \ln x \left(\frac{1}{x} \right) \Rightarrow y' = x^{\ln x} \left(\frac{2 \ln x}{x} \right)$$

$$49. y = x^{e^x} \Rightarrow \ln y = e^x \ln x \Rightarrow \frac{y'}{y} = e^x \cdot \frac{1}{x} + (\ln x) \cdot e^x \Rightarrow y' = x^{e^x} e^x \left(\ln x + \frac{1}{x} \right)$$

50. $y = (\ln x)^{\cos x} \Rightarrow \ln y = \cos x \ln(\ln x) \Rightarrow \frac{y'}{y} = \cos x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} + (\ln \ln x)(-\sin x) \Rightarrow$

$$y' = (\ln x)^{\cos x} \left(\frac{\cos x}{x \ln x} - \sin x \ln \ln x \right)$$

$$\mathbf{51.} \quad y = \ln(x^2 + y^2) \Rightarrow y' = \frac{1}{x^2 + y^2} \frac{d}{dx}(x^2 + y^2) \Rightarrow y' = \frac{2x + 2yy'}{x^2 + y^2} \Rightarrow x^2y' + y^2y' = 2x + 2yy'$$

$$\Rightarrow x^2 y' + y^2 y' - 2y y' = 2x \quad \Rightarrow (x^2 + y^2 - 2y) y' = 2x \quad \Rightarrow y' = \frac{2x}{x^2 + y^2 - 2y}$$

$$\mathbf{52.} \quad x^y = y^x \quad \Rightarrow \quad y \ln x = x \ln y \quad \Rightarrow \quad y \cdot \frac{1}{x} + (\ln x) \cdot y' = x \cdot \frac{1}{y} \cdot y' + \ln y \quad \Rightarrow \quad y' \ln x - \frac{x}{y} y' = \ln y - \frac{y}{x} \quad \Rightarrow$$

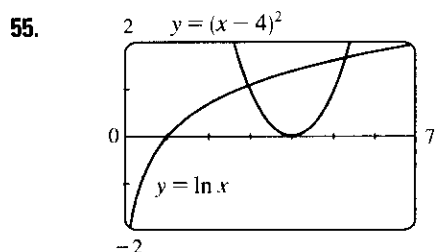
$$y' = \frac{\ln y - y/x}{\ln x - x/y}$$

53. $f(x) = \ln(x-1) \Rightarrow f'(x) = 1/(x-1) = (x-1)^{-1} \Rightarrow f''(x) = -(x-1)^{-2} \Rightarrow$
 $f'''(x) = 2(x-1)^{-3} \Rightarrow f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \Rightarrow \cdots \Rightarrow$

$$f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1} \frac{(n-1)!}{(x-1)^n}$$

54. $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8 (8x^7 \ln x + x^7)$. But the eighth derivative of x^7 is 0, so we now have

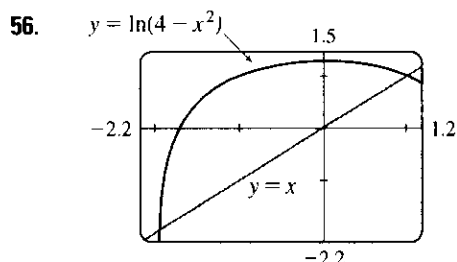
$$\begin{aligned} D^8(8x^7 \ln x) &= D^7(8 \cdot 7x^6 \ln x + 8x^6) = D^7(8 \cdot 7x^6 \ln x) \\ &= D^6(8 \cdot 7 \cdot 6x^5 \ln x) = \cdots = D(8!x^0 \ln x) = 8!/x. \end{aligned}$$



From the graph, it appears that the curves $y = (x - 4)^2$ and $y = \ln x$ intersect just to the left of $x = 3$ and to the right of $x = 5$, at about $x = 5.3$. Let $f(x) = \ln x - (x - 4)^2$. Then $f'(x) = 1/x - 2(x - 4)$ so Newton's Method says that

$$x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \frac{\ln x_n - (x_n - 4)^2}{1/x_n - 2(x_n - 4)}.$$
 Taking

$x_0 = 3$, we get $x_1 \approx 2.957738$, $x_2 \approx 2.958516 \approx x_3$, so the first root is 2.958516, to six decimal places. Taking $x_0 = 5$, we get $x_1 \approx 5.290755$, $x_2 \approx 5.290718 \approx x_3$, so the second (and final) root is 5.290718, to six decimal places.



We use Newton's Method with $f(x) = \ln(4 - x^2) - x$ and

$$f'(x) = \frac{1}{4-x^2}(-2x) - 1 = -1 - \frac{2x}{4-x^2}.$$
 The formula is

$x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the roots occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use

$x_1 = -1.9$ as an initial approximation to the first root, we get

$x_2 \approx -2.009611$, and $f(x) = \ln(x - 2)^2 - x$ is undefined at this point,

making it impossible to calculate x_3 . We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second root gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two roots of the equation $\ln(4 - x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

$$57. f(x) = \frac{\ln x}{\sqrt{x}} \Rightarrow f'(x) = \frac{\sqrt{x}(1/x) - (\ln x)[1/(2\sqrt{x})]}{x} = \frac{2 - \ln x}{2x^{3/2}} \Rightarrow$$

$$f''(x) = \frac{2x^{3/2}(-1/x) - (2 - \ln x)(3x^{1/2})}{4x^3} = \frac{3\ln x - 8}{4x^{5/2}} > 0 \Leftrightarrow \ln x > \frac{8}{3} \Leftrightarrow x > e^{8/3}, \text{ so } f \text{ is CU}$$

on $(e^{8/3}, \infty)$ and CD on $(0, e^{8/3})$. The inflection point is $(e^{8/3}, \frac{8}{3}e^{-4/3})$.

$$58. f(x) = x \ln x, f'(x) = \ln x + 1 = 0 \text{ when } \ln x = -1 \Leftrightarrow x = e^{-1}. f'(x) > 0 \Leftrightarrow \ln x + 1 > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > 1/e. f'(x) < 0 \Leftrightarrow \ln x + 1 < 0 \Leftrightarrow x < 1/e. \text{ Therefore, there is an absolute minimum value of } f(1/e) = (1/e) \ln(1/e) = -1/e.$$

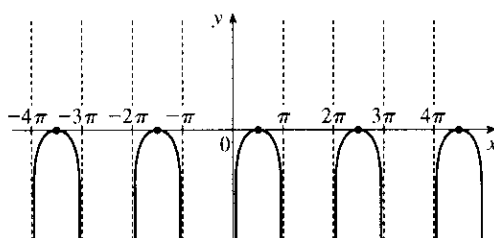
$$59. y = f(x) = \ln(\sin x)$$

$$\text{A. } D = \{x \in \mathbb{R} \mid \sin x > 0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi) \\ = \dots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \dots$$

B. No y -intercept; x -intercepts: $f(x) = 0 \Leftrightarrow \ln(\sin x) = 0 \Leftrightarrow \sin x = e^0 = 1 \Leftrightarrow x = 2n\pi + \frac{\pi}{2}$ for each integer n . C. f is periodic with period 2π . D. $\lim_{x \rightarrow (2n\pi)^+} f(x) = -\infty$ and $\lim_{x \rightarrow [(2n+1)\pi]^-} f(x) = -\infty$, so

the lines $x = n\pi$ are VAs for all integers n . E. $f'(x) = \frac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n , and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $(2n\pi, 2n\pi + \frac{\pi}{2})$ and decreasing on $(2n\pi + \frac{\pi}{2}, (2n+1)\pi)$ for each integer n . F. Local maximum values $f(2n\pi + \frac{\pi}{2}) = 0$, no local minimum. G. $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n . No IP

H.



$$60. y = \ln(\tan^2 x) \quad \text{A. } D = \{x \mid x \neq n\pi/2\} \quad \text{B. } x\text{-intercepts } n\pi + \frac{\pi}{4}, \text{ no } y\text{-intercept.} \quad \text{C. } f(-x) = f(x), \text{ so the curve is symmetric about the } y\text{-axis. Also } f(x + \pi) = f(x), \text{ so } f \text{ is periodic with period } \pi, \text{ and we consider parts D-G only for } -\frac{\pi}{2} < x < \frac{\pi}{2}. \quad \text{D. } \lim_{x \rightarrow 0} \ln(\tan^2 x) = -\infty \text{ and } \lim_{x \rightarrow (\pi/2)^-} \ln(\tan^2 x) = \infty.$$

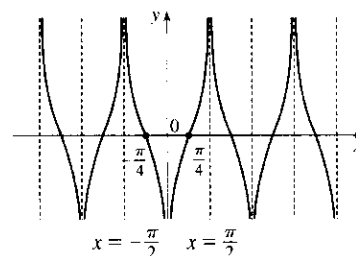
$$\lim_{x \rightarrow (-\pi/2)^+} \ln(\tan^2 x) = \infty, \text{ so } x = 0, x = \pm \frac{\pi}{2} \text{ are VA.} \quad \text{E. } f'(x) = \frac{2 \tan x \sec^2 x}{\tan^2 x} = 2 \frac{\sec^2 x}{\tan x} > 0 \Leftrightarrow$$

$\tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f is increasing on $(0, \frac{\pi}{2})$ and decreasing on $(-\frac{\pi}{2}, 0)$. F. No maximum or minimum

$$\text{G. } f'(x) = \frac{2}{\sin x \cos x} = \frac{4}{\sin 2x} \Rightarrow f''(x) = \frac{-8 \cos 2x}{\sin^2 2x} < 0$$

$\Leftrightarrow \cos 2x > 0 \Leftrightarrow -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $(-\frac{\pi}{4}, 0)$ and $(0, \frac{\pi}{4})$ and CU on $(-\frac{\pi}{2}, -\frac{\pi}{4})$ and $(\frac{\pi}{4}, \frac{\pi}{2})$. IP are $(\pm \frac{\pi}{4}, 0)$.

H.



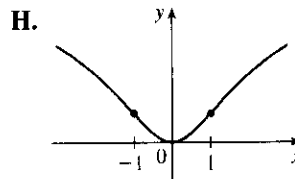
61. $y = f(x) = \ln(1 + x^2)$ A. $D = \mathbb{R}$ B. Both intercepts are 0. C. $f(-x) = f(x)$, so the curve is symmetric about the y -axis. D. $\lim_{x \rightarrow \pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. E. $f'(x) = \frac{2x}{1 + x^2} > 0 \Leftrightarrow$

$x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

F. $f(0) = 0$ is a local and absolute minimum.

G. $f''(x) = \frac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2} > 0 \Leftrightarrow$

$|x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.



62. $y = f(x) = \ln(x^2 - 3x + 2) = \ln[(x - 1)(x - 2)]$

A. $D = \{x \text{ in } \mathbb{R}: x^2 - 3x + 2 > 0\} = (-\infty, 1) \cup (2, \infty)$.

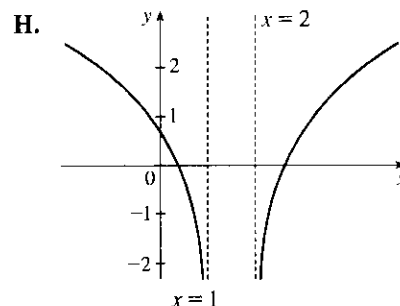
B. y -intercept: $f(0) = \ln 2$; x -intercepts: $f(x) = 0 \Leftrightarrow x^2 - 3x + 2 = e^0 \Leftrightarrow x^2 - 3x + 1 = 0 \Leftrightarrow$

$x = \frac{3 \pm \sqrt{5}}{2} \Rightarrow x \approx 0.38, 2.62$ C. No symmetry D. $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = -\infty$, so $x = 1$ and

$x = 2$ are VAs. No HA. E. $f'(x) = \frac{2x - 3}{x^2 - 3x + 2} = \frac{2(x - 3/2)}{(x - 1)(x - 2)}$, so $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 2$. Thus, f is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$. F. No extreme values

G. $f''(x) = \frac{(x^2 - 3x + 2) \cdot 2 - (2x - 3)^2}{(x^2 - 3x + 2)^2}$
 $= \frac{2x^2 - 6x + 4 - 4x^2 + 12x - 9}{(x^2 - 3x + 2)^2}$
 $= \frac{-2x^2 + 6x - 5}{(x^2 - 3x + 2)^2}$

The numerator is negative for all x and the denominator is positive, so $f''(x) < 0$ for all x in the domain of f . Thus, f is CD on $(-\infty, 1)$ and $(2, \infty)$. No IP

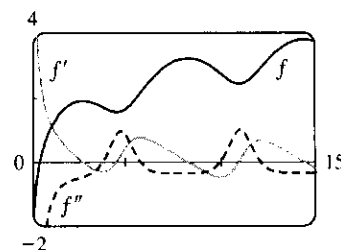


63. We use the CAS to calculate $f'(x) = \frac{2 + \sin x + x \cos x}{2x + x \sin x}$ and

$f''(x) = \frac{2x^2 \sin x + 4 \sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4 \sin x - 5)}$. From the graphs, it

seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8, 5.7, 10.0$ and 12.0 .

Looking back at the graph of $f(x) = \ln(2x + x \sin x)$, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.



64. We see that if $c \leq 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \Rightarrow |x| > \sqrt{-c}$, and

$\lim_{x \rightarrow \sqrt{-c}^+} f(x) = \lim_{x \rightarrow -\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \rightarrow -\infty$ as $y \rightarrow 0$. Thus, for $c < 0$, there are vertical

asymptotes at $x = \pm\sqrt{-c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$,

$\lim_{x \rightarrow 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima,

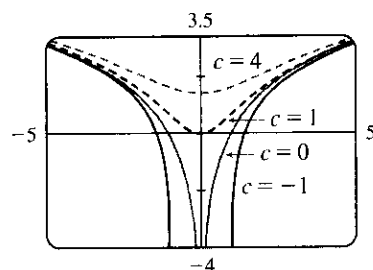
minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \Rightarrow f'(x) = \frac{1}{x^2 + c} (2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get

$$f''(x) = \frac{1}{x^2 + c} (2) + 2x \left[- (x^2 + c)^{-2} (2x) \right] = \frac{2(c - x^2)}{(x^2 + c)^2}. \text{ Now}$$

if $c \leq 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when

$$c = x^2 \Leftrightarrow x = \pm\sqrt{c}. \text{ So for } c > 0 \text{ there are inflection points at}$$

$\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



$$65. \int_2^4 \frac{3}{x} dx = 3 \int_2^4 \frac{1}{x} dx = 3 \left[\ln |x| \right]_2^4 = 3(\ln 4 - \ln 2) = 3 \ln \frac{4}{2} = 3 \ln 2$$

$$66. \int_1^2 \frac{4 + u^2}{u^3} du = \int_1^2 (4u^{-3} + u^{-1}) du = \left[\frac{4}{-2} u^{-2} + \ln |u| \right]_1^2 = \left[\frac{-2}{u^2} + \ln u \right]_1^2 \\ = \left(-\frac{1}{2} + \ln 2 \right) - \left(-2 + \ln 1 \right) = \frac{3}{2} + \ln 2$$

$$67. \int_1^2 \frac{dt}{8 - 3t} = \left[-\frac{1}{3} \ln |8 - 3t| \right]_1^2 = -\frac{1}{3} \ln 2 - \left(-\frac{1}{3} \ln 5 \right) = \frac{1}{3} (\ln 5 - \ln 2) = \frac{1}{3} \ln \frac{5}{2}$$

Or: Let $u = 8 - 3t$. Then $du = -3 dt$, so

$$\int_1^2 \frac{dt}{8 - 3t} = \int_5^2 \frac{-\frac{1}{3} du}{u} = \left[-\frac{1}{3} \ln |u| \right]_5^2 = -\frac{1}{3} \ln 2 - \left(-\frac{1}{3} \ln 5 \right) = \frac{1}{3} (\ln 5 - \ln 2) = \frac{1}{3} \ln \frac{5}{2}.$$

$$68. \int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x} \right) dx = \left[\frac{1}{2} x^2 + 2x + \ln x \right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) \\ = \frac{85}{2} + \ln \frac{9}{4}$$

$$69. \int_1^e \frac{x^2 + x + 1}{x} dx = \int_1^e \left(x + 1 + \frac{1}{x} \right) dx = \left[\frac{1}{2} x^2 + x + \ln x \right]_1^e = \left(\frac{1}{2} e^2 + e + 1 \right) - \left(\frac{1}{2} + 1 + 0 \right) \\ = \frac{1}{2} e^2 + e - \frac{1}{2}$$

$$70. \text{ Let } u = \ln x. \text{ Then } du = \frac{1}{x} dx, \text{ so } \int_e^6 \frac{dx}{x \ln x} = \int_1^{\ln 6} \frac{1}{u} du = \left[\ln |u| \right]_1^{\ln 6} = \ln \ln 6 - \ln 1 = \ln \ln 6$$

$$71. \text{ Let } u = 6x - x^3. \text{ Then } du = (6 - 3x^2) dx = 3(2 - x^2) dx, \text{ so}$$

$$\int \frac{2 - x^2}{6x - x^3} dx = \int \frac{\frac{1}{3} du}{u} = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |6x - x^3| + C.$$

$$72. \text{ Let } u = 2 + \sin x. \text{ Then } du = \cos x dx, \text{ so}$$

$$\int \frac{\cos x}{2 + \sin x} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |2 + \sin x| + C = \ln(2 + \sin x) + C \quad [\text{since } 2 + \sin x > 0].$$

73. Let $u = \ln x$. Then $du = \frac{dx}{x} \Rightarrow \int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

74. Let $u = e^x + 1$. Then $du = e^x dx$, so $\int \frac{e^x}{e^x + 1} dx = \int \frac{du}{u} = \ln |u| + C = \ln(e^x + 1) + C$.

75. $\int_1^{10} 10^t dt = \left[\frac{10^t}{\ln 10} \right]_1^{10} = \frac{10^{10}}{\ln 10} - \frac{10^1}{\ln 10} = \frac{10^{10} - 10}{\ln 10} = \frac{9999999990}{\ln 10}$

76. Let $u = x^2$. Then $du = 2x dx$, so $\int x 2x^2 dx = \frac{1}{2} \int 2^u du = \frac{1}{2} \frac{2^u}{\ln 2} + C = \frac{1}{2 \ln 2} 2^{x^2} + C$.

77. (a) $\frac{d}{dx} (\ln |\sin x| + C) = \frac{1}{\sin x} \cos x = \cot x$

(b) Let $u = \sin x$. Then $du = \cos x dx$, so $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln |u| + C = \ln |\sin x| + C$.

78. Let $u = x - 2$. Then the area is

$$A = - \int_{-4}^{-1} \frac{2}{x-2} dx = -2 \int_{-6}^{-3} \frac{du}{u} = \left[-2 \ln |u| \right]_{-6}^{-3} = -2 \ln 3 + 2 \ln 6 = 2 \ln 2 \approx 1.386.$$

79. The cross-sectional area is $\pi(1/\sqrt{x+1})^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1} dx = \pi [\ln(x+1)]_0^1 = \pi(\ln 2 - \ln 1) = \pi \ln 2.$$

80. Using cylindrical shells, we get $V = \int_0^3 \frac{2\pi x}{x^2+1} dx = \pi [\ln(1+x^2)]_0^3 = \pi \ln 10$.

81. $W = \int_{V_1}^{V_2} P dV = \int_{600}^{1000} \frac{C}{V} dV = C \int_{600}^{1000} \frac{1}{V} dV = C [\ln |V|]_{600}^{1000}$
 $= C(\ln 1000 - \ln 600) = C \ln \frac{1000}{600} = C \ln \frac{5}{3}$

Initially, $PV = C$, where $P = 150$ kPa and $V = 600$ cm³, so $C = (150)(600) = 90,000$. Thus,

$$W = 90,000 \ln \frac{5}{3} \approx 45,974 \text{ kPa} \cdot \text{cm}^3 = 45,974(10^3 \text{ Pa})(10^{-6} \text{ m}^3) = 45,974 \text{ Pa} \cdot \text{m}^3 = 45,974 \text{ N} \cdot \text{m}$$

$$[\text{Pa} = \text{N/m}^2] \quad = 45,974 \text{ J}$$

82. $f''(x) = x^{-2}, x > 0 \Rightarrow f'(x) = -1/x + C \Rightarrow f(x) = -\ln x + Cx + D$. $0 = f(1) = C + D$ and

$$0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \Rightarrow C = \ln 2 \text{ and } D = -\ln 2. \text{ So}$$

$$f(x) = -\ln x + (\ln 2)x - \ln 2.$$

83. $f(x) = 2x + \ln x \Rightarrow f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \Rightarrow g(2) = 1$, so

$$g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}.$$

84. $f(x) = e^x + \ln x \Rightarrow f'(x) = e^x + 1/x$. $h = f^{-1}$ and $f(1) = e \Rightarrow h(e) = 1$, so

$$h'(e) = 1/f'(1) = 1/(e+1).$$

85. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2 + 1) = mx \Rightarrow x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow x = 0$ or

$$x = \frac{\pm \sqrt{-4(m)(m-1)}}{2m} = \pm \sqrt{\frac{1}{m} - 1}. \text{ Note that if } m = 1, \text{ this}$$

has only the solution $x = 0$, and no region is determined. But if

$$1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1, \text{ then there are two}$$

solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2 + 1)$ at the origin is

$y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and

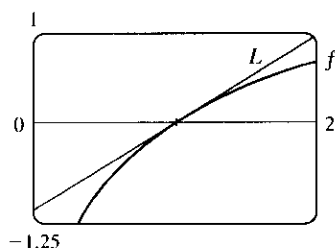
negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2 + 1)$ are both odd functions,

the total area is twice the area between the curves on the interval $[0, \sqrt{1/m - 1}]$. So the total area enclosed is

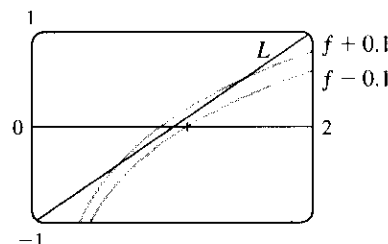
$$\begin{aligned} 2 \int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx \right] dx &= 2 \left[\frac{1}{2} \ln(x^2+1) - \frac{1}{2} mx^2 \right]_0^{\sqrt{1/m-1}} \\ &= \left[\ln \left(\frac{1}{m} - 1 + 1 \right) - m \left(\frac{1}{m} - 1 \right) \right] - (\ln 1 - 0) \\ &= \ln \left(\frac{1}{m} \right) + m - 1 = m - \ln m - 1 \end{aligned}$$

86. (a) Let $f(x) = \ln x \Rightarrow f'(x) = 1/x \Rightarrow f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x-1$.

(b)



(c)



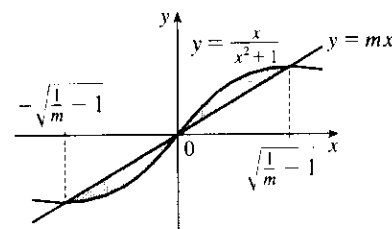
From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

87. If $f(x) = \ln(1+x)$, then $f'(x) = \frac{1}{1+x}$, so $f'(0) = 1$.

$$\text{Thus, } \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1.$$

88. Let $m = n/x$. Then $n = xm$, and as $n \rightarrow \infty$, $m \rightarrow \infty$.

$$\text{Therefore, } \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^{mx} = \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m \right]^x = e^x \text{ by Equation 9.}$$



7.2* The Natural Logarithmic Function

$$1. \ln \frac{x^3 y}{z^2} = \ln x^3 y - \ln z^2 = \ln x^3 + \ln y - \ln z^2 = 3 \ln x + \ln y - 2 \ln z$$

$$2. \ln \sqrt{a(b^2 + c^2)} = \ln(a(b^2 + c^2))^{1/2} = \frac{1}{2} \ln(a(b^2 + c^2)) = \frac{1}{2} [\ln a + \ln(b^2 + c^2)] \\ = \frac{1}{2} \ln a + \frac{1}{2} \ln(b^2 + c^2)$$

$$3. \ln(uv)^{10} = 10 \ln(uv) = 10(\ln u + \ln v) = 10 \ln u + 10 \ln v$$

$$4. \ln \frac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5 \ln(x+1) = \ln 3 + 2 \ln x - 5 \ln(x+1)$$

$$5. 2 \ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$$

$$6. \ln 3 + \frac{1}{3} \ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3 \cdot 2) = \ln 6$$

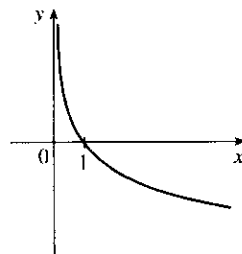
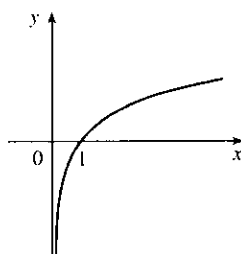
$$7. \frac{1}{2} \ln x - 5 \ln(x^2 + 1) = \ln x^{1/2} - \ln(x^2 + 1)^5 = \ln \frac{\sqrt{x}}{(x^2 + 1)^5}$$

$$8. \ln x + a \ln y - b \ln z = \ln x + \ln y^a - \ln z^b = \ln(x \cdot y^a) - \ln z^b = \ln(xy^a/z^b)$$

9. Reflect the graph of $y = \ln x$ about the x -axis to obtain the graph of $y = -\ln x$.

$$y = \ln x$$

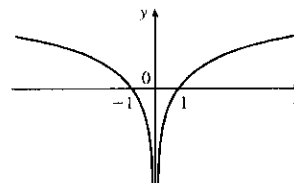
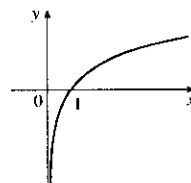
$$y = -\ln x$$



10. Reflect the portion of the graph of $y = \ln x$ to the right of the y -axis about the y -axis. The graph of $y = \ln |x|$ is that reflection in addition to the original portion.

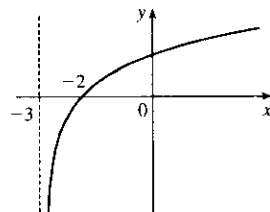
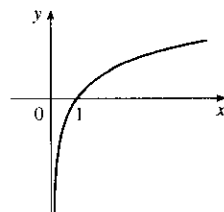
$$y = \ln x$$

$$y = \ln |x|$$



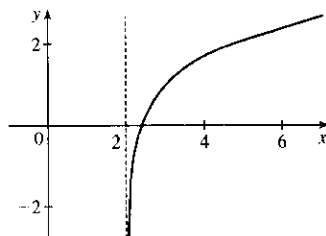
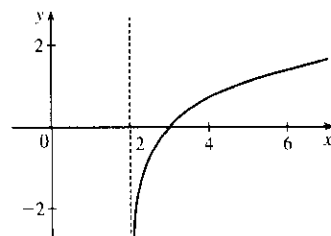
11. $y = \ln x$

$$y = \ln(x+3)$$



12. $y = \ln(x-2)$

$$y = 1 + \ln(x-2)$$



$$13. f(x) = \sqrt{x} \ln x \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \left(\frac{1}{x} \right) = \frac{\ln x + 2}{2\sqrt{x}}$$

$$14. f(x) = \ln(x^2 + 10) \Rightarrow f'(x) = \frac{1}{x^2 + 10} \frac{d}{dx} (x^2 + 10) = \frac{2x}{x^2 + 10}$$

$$15. f(\theta) = \ln(\cos \theta) \Rightarrow f'(\theta) = \frac{1}{\cos \theta} \frac{d}{d\theta} (\cos \theta) = \frac{-\sin \theta}{\cos \theta} = -\tan \theta$$

$$16. f(x) = \cos(\ln x) \Rightarrow f'(x) = -\sin(\ln x) \cdot \frac{1}{x} = \frac{-\sin(\ln x)}{x}$$

$$17. f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \Rightarrow f'(x) = \frac{1}{5}(\ln x)^{-4/5} \frac{d}{dx} (\ln x) = \frac{1}{5(\ln x)^{4/5}} \cdot \frac{1}{x} = \frac{1}{5x \sqrt[5]{(\ln x)^4}}$$

$$18. f(x) = \ln \sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5} \ln x \Rightarrow f'(x) = \frac{1}{5} \cdot \frac{1}{x} = \frac{1}{5x}$$

$$19. g(x) = \ln \frac{a-x}{a+x} = \ln(a-x) - \ln(a+x) \Rightarrow$$

$$g'(x) = \frac{1}{a-x}(-1) - \frac{1}{a+x} = \frac{-(a+x) - (a-x)}{(a-x)(a+x)} = \frac{-2a}{a^2 - x^2}$$

$$20. h(x) = \ln(x + \sqrt{x^2 - 1}) \Rightarrow$$

$$h'(x) = \frac{1}{x + \sqrt{x^2 - 1}} \left(1 + \frac{x}{\sqrt{x^2 - 1}} \right) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$21. f(u) = \frac{\ln u}{1 + \ln(2u)} \Rightarrow$$

$$\begin{aligned} f'(u) &= \frac{[1 + \ln(2u)] \cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u} \cdot 2}{[1 + \ln(2u)]^2} = \frac{\frac{1}{u} [1 + \ln(2u) - \ln u]}{[1 + \ln(2u)]^2} \\ &= \frac{1 + (\ln 2 + \ln u) - \ln u}{u [1 + \ln(2u)]^2} = \frac{1 + \ln 2}{u [1 + \ln(2u)]^2} \end{aligned}$$

$$22. f(t) = \frac{1 + \ln t}{1 - \ln t} \Rightarrow$$

$$f'(t) = \frac{(1 - \ln t)(1/t) - (1 + \ln t)(-1/t)}{(1 - \ln t)^2} = \frac{(1/t)[(1 - \ln t) + (1 + \ln t)]}{(1 - \ln t)^2} = \frac{2}{t(1 - \ln t)^2}$$

$$23. F(t) = \ln \frac{(2t+1)^3}{(3t-1)^4} = \ln(2t+1)^3 - \ln(3t-1)^4 = 3 \ln(2t+1) - 4 \ln(3t-1) \Rightarrow$$

$$F'(t) = 3 \cdot \frac{1}{2t+1} \cdot 2 - 4 \cdot \frac{1}{3t-1} \cdot 3 = \frac{6}{2t+1} - \frac{12}{3t-1}, \text{ or combined, } \frac{-6(t+3)}{(2t+1)(3t-1)}.$$

$$24. y = \ln(x^4 \sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4 \ln x + 2 \ln \sin x \Rightarrow y' = 4 \cdot \frac{1}{x} + 2 \cdot \frac{1}{\sin x} \cdot \cos x = \frac{4}{x} + 2 \cot x$$

$$25. y = \ln |2 - x - 5x^2| \Rightarrow y' = \frac{1}{2 - x - 5x^2} \cdot (-1 - 10x) = \frac{-10x - 1}{2 - x - 5x^2} \text{ or } \frac{10x + 1}{5x^2 + x - 2}$$

$$26. G(u) = \ln \sqrt{\frac{3u+2}{3u-2}} = \frac{1}{2} [\ln(3u+2) - \ln(3u-2)] \Rightarrow G'(u) = \frac{1}{2} \left(\frac{3}{3u+2} - \frac{3}{3u-2} \right) = \frac{-6}{9u^2 - 4}$$

$$27. y = \ln \left(\frac{x+1}{x-1} \right)^{3/5} = \frac{3}{5} [\ln(x+1) - \ln(x-1)] \Rightarrow y' = \frac{3}{5} \left(\frac{1}{x+1} - \frac{1}{x-1} \right) = \frac{-6}{5(x^2 - 1)}$$

$$28. y = (\ln \tan x)^2 \Rightarrow y' = 2(\ln \tan x) \cdot \frac{1}{\tan x} \cdot \sec^2 x = \frac{2(\ln \tan x) \sec^2 x}{\tan x}$$

$$29. y = \tan[\ln(ax + b)] \Rightarrow y' = \sec^2[\ln(ax + b)] \cdot \frac{1}{ax + b} \cdot a = \sec^2[\ln(ax + b)] \frac{a}{ax + b}$$

$$30. y = \ln|\tan 2x| \Rightarrow y' = \frac{2\sec^2 2x}{\tan 2x}$$

$$31. y = \ln \ln x \Rightarrow y' = \frac{1}{\ln x} \frac{d}{dx}(\ln x) = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x} \Rightarrow$$

$$y'' = -\frac{\frac{d}{dx}(x \ln x)}{(x \ln x)^2} \quad [\text{Reciprocal Rule}] = -\frac{x \cdot \frac{1}{x} + \ln x \cdot 1}{(x \ln x)^2} = -\frac{1 + \ln x}{(x \ln x)^2}$$

$$32. y = \frac{\ln x}{x^2} \Rightarrow y' = \frac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \frac{x(1 - 2 \ln x)}{x^4} = \frac{1 - 2 \ln x}{x^3} \Rightarrow$$

$$y'' = \frac{x^3(-2/x) - (1 - 2 \ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2 - 3 + 6 \ln x)}{x^6} = \frac{6 \ln x - 5}{x^4}$$

$$33. f(x) = \frac{x}{1 - \ln(x-1)} \Rightarrow$$

$$\begin{aligned} f'(x) &= \frac{[1 - \ln(x-1)] \cdot 1 - x \cdot \frac{-1}{x-1}}{[1 - \ln(x-1)]^2} = \frac{(x-1)[1 - \ln(x-1)] + x}{[1 - \ln(x-1)]^2} = \frac{x-1 - (x-1)\ln(x-1) + x}{(x-1)[1 - \ln(x-1)]^2} \\ &= \frac{2x-1 - (x-1)\ln(x-1)}{(x-1)[1 - \ln(x-1)]^2} \end{aligned}$$

$$\begin{aligned} \text{Dom}(f) &= \{x \mid x-1 > 0 \text{ and } 1 - \ln(x-1) \neq 0\} = \{x \mid x > 1 \text{ and } \ln(x-1) \neq 1\} \\ &= \{x \mid x > 1 \text{ and } x-1 \neq e^1\} = \{x \mid x > 1 \text{ and } x \neq 1+e\} = (1, 1+e) \cup (1+e, \infty) \end{aligned}$$

$$34. f(x) = \frac{1}{1 + \ln x} \Rightarrow f'(x) = -\frac{1/x}{(1 + \ln x)^2} \quad [\text{Reciprocal Rule}] = -\frac{1}{x(1 + \ln x)^2}.$$

$$\text{Dom}(f) = \{x \mid x > 0 \text{ and } \ln x \neq -1\} = \{x \mid x > 0 \text{ and } x \neq 1/e\} = (0, 1/e) \cup (1/e, \infty).$$

$$35. f(x) = \sqrt{1 - \ln x} \text{ is defined} \Leftrightarrow x > 0 \text{ [so that } \ln x \text{ is defined]} \text{ and } 1 - \ln x \geq 0$$

$$\Leftrightarrow x > 0 \text{ and } \ln x \leq 1 \Leftrightarrow 0 < x \leq e, \text{ so the domain of } f \text{ is } (0, e]. \text{ Now}$$

$$f'(x) = \frac{1}{2}(1 - \ln x)^{-1/2} \cdot \frac{d}{dx}(1 - \ln x) = \frac{1}{2\sqrt{1 - \ln x}} \cdot \left(-\frac{1}{x}\right) = \frac{-1}{2x\sqrt{1 - \ln x}}.$$

$$36. f(x) = \ln \ln \ln x \Rightarrow f'(x) = \frac{1}{\ln \ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}.$$

$$\text{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty).$$

$$37. f(x) = \frac{x}{\ln x} \Rightarrow f'(x) = \frac{\ln x - x(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \frac{1-1}{1^2} = 0$$

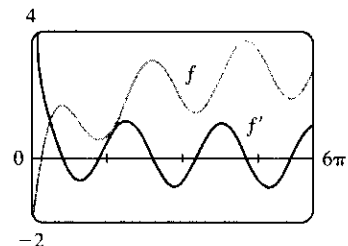
$$38. f(t) = t \ln(4 + 3t) \Rightarrow f'(t) = t \cdot \frac{1}{4 + 3t} \cdot 3 + \ln(4 + 3t) = \frac{3t}{4 + 3t} + \ln(4 + 3t),$$

$$\text{so } f'(-1) = \frac{-3}{1} + \ln 1 = -3 + 0 = -3.$$

$$39. f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x. \text{ This is reasonable,}$$

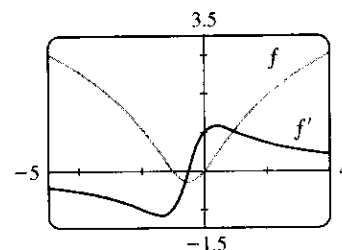
because the graph shows that f increases when f' is positive, and

$f'(x) = 0$ when f has a horizontal tangent.



$$40. f(x) = \ln(x^2 + x + 1) \Rightarrow f'(x) = \frac{1}{x^2 + x + 1} (2x + 1).$$

Notice from the graph that f is increasing when $f'(x)$ is positive.



$$41. y = \sin(2 \ln x) \Rightarrow y' = \cos(2 \ln x) \cdot \frac{2}{x}. \text{ At } (1, 0), y' = \cos 0 \cdot \frac{2}{1} = 2, \text{ so an equation of the tangent line is } y - 0 = 2 \cdot (x - 1), \text{ or } y = 2x - 2.$$

$$42. y = \ln(x^3 - 7) \Rightarrow y' = \frac{1}{x^3 - 7} \cdot 3x^2 \Rightarrow y'(2) = \frac{12}{8 - 7} = 12, \text{ so an equation of a tangent line at } (2, 0) \text{ is } y - 0 = 12(x - 2) \text{ or } y = 12x - 24.$$

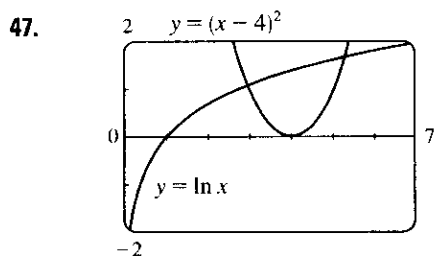
$$43. y = \ln(x^2 + y^2) \Rightarrow y' = \frac{1}{x^2 + y^2} \frac{d}{dx}(x^2 + y^2) \Rightarrow y' = \frac{2x + 2yy'}{x^2 + y^2} \Rightarrow x^2 y' + y^2 y' = 2x + 2yy' \\ \Rightarrow x^2 y' + y^2 y' - 2yy' = 2x \Rightarrow (x^2 + y^2 - 2y)y' = 2x \Rightarrow y' = \frac{2x}{x^2 + y^2 - 2y}$$

$$44. \ln xy = \ln x + \ln y = y \sin x \Rightarrow 1/x + y'/y = y \cos x + y' \sin x \Rightarrow y'(1/y - \sin x) = y \cos x - 1/x \Rightarrow \\ y' = \frac{y \cos x - 1/x}{1/y - \sin x} = \left(\frac{y}{x}\right) \frac{xy \cos x - 1}{1 - y \sin x}$$

$$45. f(x) = \ln(x - 1) \Rightarrow f'(x) = 1/(x - 1) = (x - 1)^{-1} \Rightarrow f''(x) = -(x - 1)^{-2} \Rightarrow \\ f'''(x) = 2(x - 1)^{-3} \Rightarrow f^{(4)}(x) = -2 \cdot 3(x - 1)^{-4} \Rightarrow \dots \Rightarrow \\ f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n - 1)(x - 1)^{-n} = (-1)^{n-1} \frac{(n - 1)!}{(x - 1)^n}$$

$$46. y = x^8 \ln x, \text{ so } D^9 y = D^8 y' = D^8 (8x^7 \ln x + x^7). \text{ But the eighth derivative of } x^7 \text{ is 0, so we now have}$$

$$D^8 (8x^7 \ln x) = D^7 (8 \cdot 7x^6 \ln x + 8x^6) = D^7 (8 \cdot 7x^6 \ln x) \\ = D^6 (8 \cdot 7 \cdot 6x^5 \ln x) = \dots = D (8! x^0 \ln x) = 8!/x.$$

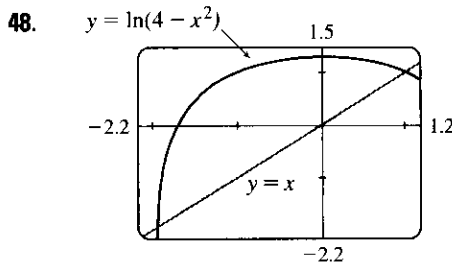


From the graph, it appears that the curves $y = (x - 4)^2$ and $y = \ln x$ intersect just to the left of $x = 3$ and to the right of $x = 5$, at about $x = 5.3$. Let $f(x) = \ln x - (x - 4)^2$. Then $f'(x) = 1/x - 2(x - 4)$, so Newton's Method says that

$$x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \frac{\ln x_n - (x_n - 4)^2}{1/x_n - 2(x_n - 4)}.$$
 Taking

$x_0 = 3$, we get $x_1 \approx 2.957738$, $x_2 \approx 2.958516 \approx x_3$, so the first root is

2.958516, to six decimal places. Taking $x_0 = 5$, we get $x_1 \approx 5.290755$, $x_2 \approx 5.290718 \approx x_3$, so the second (and final) root is 5.290718, to six decimal places.



We use Newton's Method with $f(x) = \ln(4 - x^2) - x$ and

$$f'(x) = \frac{1}{4 - x^2}(-2x) - 1 = -1 - \frac{2x}{4 - x^2}.$$

The formula is

$$x_{n+1} = x_n - f(x_n)/f'(x_n).$$

From the graphs it seems that the roots

occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use

$x_1 = -1.9$ as an initial approximation to the first root, we get

$x_2 \approx -2.009611$, and $f(x) = \ln(x - 2)^2 - x$ is undefined at this point,

making it impossible to calculate x_3 . We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second root gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two roots of the equation $\ln(4 - x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

49. $y = f(x) = \ln(\sin x)$

A. $D = \{x \in \mathbb{R} \mid \sin x > 0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi)$
 $= \dots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \dots$

B. No y -intercept; x -intercepts: $f(x) = 0 \Leftrightarrow \ln(\sin x) = 0 \Leftrightarrow \sin x = e^0 = 1 \Leftrightarrow x = 2n\pi + \frac{\pi}{2}$ for each integer n .

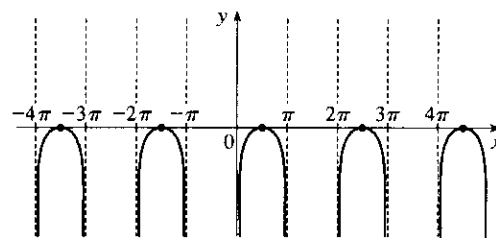
C. f is periodic with period 2π . D. $\lim_{x \rightarrow (2n\pi)^+} f(x) = -\infty$ and $\lim_{x \rightarrow [(2n+1)\pi]^-} f(x) = -\infty$, so

the lines $x = n\pi$ are VAs for all integers n . E. $f'(x) = \frac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n , and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $(2n\pi, 2n\pi + \frac{\pi}{2})$ and

decreasing on $(2n\pi + \frac{\pi}{2}, (2n+1)\pi)$ for each integer n . F. Local maximum values $f(2n\pi + \frac{\pi}{2}) = 0$, no local

minimum. G. $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n . No IP

H.



50. $y = \ln(\tan^2 x)$ A. $D = \{x \mid x \neq n\pi/2\}$ B. x -intercepts $n\pi + \frac{\pi}{4}$, no y -intercept. C. $f(-x) = f(x)$, so the curve is symmetric about the y -axis. Also $f(x + \pi) = f(x)$, so f is periodic with period π , and we consider parts

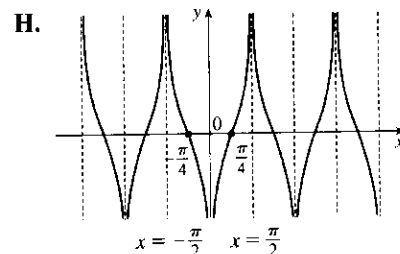
D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. D. $\lim_{x \rightarrow 0} \ln(\tan^2 x) = -\infty$ and $\lim_{x \rightarrow (\pi/2)^-} \ln(\tan^2 x) = \infty$,

$\lim_{x \rightarrow -(\pi/2)^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm \frac{\pi}{2}$ are VA. E. $f'(x) = \frac{2 \tan x \sec^2 x}{\tan^2 x} = 2 \frac{\sec^2 x}{\tan x} > 0 \Leftrightarrow$

$\tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f is increasing on $(0, \frac{\pi}{2})$ and decreasing on $(-\frac{\pi}{2}, 0)$. **F.** No maximum or minimum

$$\text{G. } f'(x) = \frac{2}{\sin x \cos x} = \frac{4}{\sin 2x} \Rightarrow f''(x) = \frac{-8 \cos 2x}{\sin^2 2x} < 0$$

$\Leftrightarrow \cos 2x > 0 \Leftrightarrow -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $(-\frac{\pi}{4}, 0)$ and $(0, \frac{\pi}{4})$ and CU on $(-\frac{\pi}{2}, -\frac{\pi}{4})$ and $(\frac{\pi}{4}, \frac{\pi}{2})$. IP are $(\pm \frac{\pi}{4}, 0)$.



51. $y = f(x) = \ln(1 + x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric

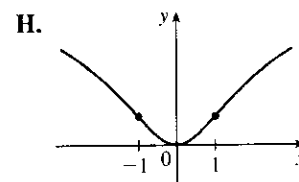
about the y -axis. **D.** $\lim_{x \rightarrow \pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. **E.** $f'(x) = \frac{2x}{1 + x^2} > 0 \Leftrightarrow$

$x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

F. $f(0) = 0$ is a local and absolute minimum.

$$\text{G. } f''(x) = \frac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2} > 0 \Leftrightarrow$$

$|x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.



52. $y = f(x) = \ln(x^2 - 3x + 2) = \ln[(x - 1)(x - 2)]$

A. $D = \{x \text{ in } \mathbb{R} : x^2 - 3x + 2 > 0\} = (-\infty, 1) \cup (2, \infty)$.

B. y -intercept: $f(0) = \ln 2$; x -intercepts: $f(x) = 0 \Leftrightarrow x^2 - 3x + 2 = e^0 \Leftrightarrow x^2 - 3x + 1 = 0 \Leftrightarrow$

$$x = \frac{3 \pm \sqrt{5}}{2} \Rightarrow x \approx 0.38, 2.62 \quad \text{C. No symmetry} \quad \text{D. } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = -\infty, \text{ so } x = 1 \text{ and}$$

$x = 2$ are VAs. No HA. **E.** $f'(x) = \frac{2x - 3}{x^2 - 3x + 2} = \frac{2(x - 3/2)}{(x - 1)(x - 2)}$, so $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$

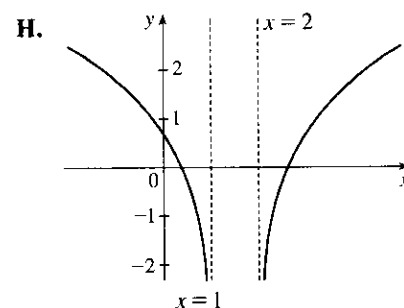
for $x > 2$. Thus, f is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$. **F.** No extreme values

$$\begin{aligned} \text{G. } f''(x) &= \frac{(x^2 - 3x + 2) \cdot 2 - (2x - 3)^2}{(x^2 - 3x + 2)^2} \\ &= \frac{2x^2 - 6x + 4 - 4x^2 + 12x - 9}{(x^2 - 3x + 2)^2} \\ &= \frac{-2x^2 + 6x - 5}{(x^2 - 3x + 2)^2} \end{aligned}$$

The numerator is negative for all x and the denominator is positive,

so $f''(x) < 0$ for all x in the domain of f . Thus, f is CD on

$(-\infty, 1)$ and $(2, \infty)$. No IP

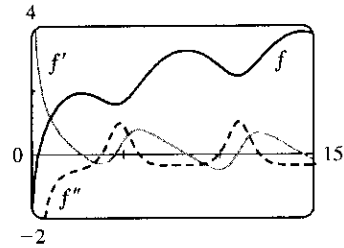


53. We use the CAS to calculate $f'(x) = \frac{2 + \sin x + x \cos x}{2x + x \sin x}$ and

$$f''(x) = \frac{2x^2 \sin x + 4 \sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4 \sin x - 5)}.$$

From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8, 5.7, 10.0$ and 12.0 .

Looking back at the graph of $f(x) = \ln(2x + x \sin x)$, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.



54. We see that if $c \leq 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \Rightarrow |x| > \sqrt{-c}$, and

$$\lim_{x \rightarrow \sqrt{-c}^+} f(x) = \lim_{x \rightarrow -\sqrt{-c}^-} f(x) = -\infty, \text{ since } \ln y \rightarrow -\infty \text{ as } y \rightarrow 0. \text{ Thus, for } c < 0, \text{ there are vertical}$$

asymptotes at $x = \pm\sqrt{-c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$,

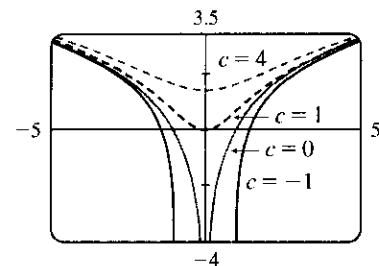
$\lim_{x \rightarrow 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima,

minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \Rightarrow f'(x) = \frac{1}{x^2 + c} (2x)$, so by the First

Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get

$$f''(x) = \frac{1}{x^2 + c} (2) + 2x \left[-(x^2 + c)^{-2} (2x) \right] = \frac{2(c - x^2)}{(x^2 + c)^2}.$$

Now if $c \leq 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \Leftrightarrow x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



55. $y = (2x + 1)^5 (x^4 - 3)^6 \Rightarrow \ln y = \ln((2x + 1)^5 (x^4 - 3)^6) \Rightarrow$

$$\ln y = 5 \ln(2x + 1) + 6 \ln(x^4 - 3) \Rightarrow \frac{1}{y} y' = 5 \cdot \frac{1}{2x + 1} \cdot 2 + 6 \cdot \frac{1}{x^4 - 3} \cdot 4x^3 \Rightarrow$$

$$y' = y \left(\frac{10}{2x + 1} + \frac{24x^3}{x^4 - 3} \right) = (2x + 1)^5 (x^4 - 3)^6 \left(\frac{10}{2x + 1} + \frac{24x^3}{x^4 - 3} \right).$$

[The answer could be simplified to $y' = 2(2x + 1)^4 (x^4 - 3)^5 (29x^4 + 12x^3 - 15)$, but this is unnecessary.]

56. $y = \frac{(x^3 + 1)^4 \sin^2 x}{x^{1/3}} \Rightarrow \ln |y| = 4 \ln |x^3 + 1| + 2 \ln |\sin x| - \frac{1}{3} \ln |x|.$

$$\text{So } \frac{y'}{y} = 4 \frac{3x^2}{x^3 + 1} + 2 \frac{\cos x}{\sin x} - \frac{1}{3x} \Rightarrow y' = \frac{(x^3 + 1)^4 \sin^2 x}{x^{1/3}} \left(\frac{12x^2}{x^3 + 1} + 2 \cot x - \frac{1}{3x} \right).$$

$$57. y = \frac{\sin^2 x \tan^4 x}{(x^2 + 1)^2} \Rightarrow \ln y = \ln(\sin^2 x \tan^4 x) - \ln(x^2 + 1)^2 \Rightarrow$$

$$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2 + 1)^2 \Rightarrow \ln y = 2 \ln |\sin x| + 4 \ln |\tan x| - 2 \ln(x^2 + 1) \Rightarrow$$

$$\frac{1}{y} y' = 2 \cdot \frac{1}{\sin x} \cdot \cos x + 4 \cdot \frac{1}{\tan x} \cdot \sec^2 x - 2 \cdot \frac{1}{x^2 + 1} \cdot 2x \Rightarrow$$

$$y' = \frac{\sin^2 x \tan^4 x}{(x^2 + 1)^2} \left(2 \cot x + \frac{4 \sec^2 x}{\tan x} - \frac{4x}{x^2 + 1} \right)$$

$$58. y = \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}} \Rightarrow \ln y = \frac{1}{4} \ln(x^2 + 1) - \frac{1}{4} \ln(x^2 - 1) \Rightarrow \frac{1}{y} y' = \frac{1}{4} \cdot \frac{1}{x^2 + 1} \cdot 2x - \frac{1}{4} \cdot \frac{1}{x^2 - 1} \cdot 2x \Rightarrow$$

$$y' = \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}} \cdot \frac{1}{2} \left(\frac{x}{x^2 + 1} - \frac{x}{x^2 - 1} \right) = \frac{1}{2} \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}} \left(\frac{-2x}{x^4 - 1} \right) = \frac{x}{1 - x^4} \sqrt[4]{\frac{x^2 + 1}{x^2 - 1}}$$

$$59. \int_2^4 \frac{3}{x} dx = 3 \int_2^4 \frac{1}{x} dx = 3 \left[\ln |x| \right]_2^4 = 3(\ln 4 - \ln 2) = 3 \ln \frac{4}{2} = 3 \ln 2$$

$$60. \int_1^2 \frac{4 + u^2}{u^3} du = \int_1^2 (4u^{-3} + u^{-1}) du = \left[-\frac{4}{2} u^{-2} + \ln |u| \right]_1^2 = \left[\frac{-2}{u^2} + \ln u \right]_1^2$$

$$= \left(-\frac{1}{2} + \ln 2 \right) - (-2 + \ln 1) = \frac{3}{2} + \ln 2$$

$$61. \int_1^2 \frac{dt}{8 - 3t} = \left[-\frac{1}{3} \ln |8 - 3t| \right]_1^2 = -\frac{1}{3} \ln 2 - \left(-\frac{1}{3} \ln 5 \right) = \frac{1}{3} (\ln 5 - \ln 2) = \frac{1}{3} \ln \frac{5}{2}$$

Or: Let $u = 8 - 3t$. Then $du = -3 dt$, so

$$\int_1^2 \frac{dt}{8 - 3t} = \int_5^2 \frac{-\frac{1}{3} du}{u} = \left[-\frac{1}{3} \ln |u| \right]_5^2 = -\frac{1}{3} \ln 2 - \left(-\frac{1}{3} \ln 5 \right) = \frac{1}{3} (\ln 5 - \ln 2) = \frac{1}{3} \ln \frac{5}{2}.$$

$$62. \int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x} \right) dx = \left[\frac{1}{2} x^2 + 2x + \ln x \right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4)$$

$$= \frac{85}{2} + \ln \frac{9}{4}$$

$$63. \int_1^e \frac{x^2 + x + 1}{x} dx = \int_1^e \left(x + 1 + \frac{1}{x} \right) dx = \left[\frac{1}{2} x^2 + x + \ln x \right]_1^e = \left(\frac{1}{2} e^2 + e + 1 \right) - \left(\frac{1}{2} + 1 + 0 \right)$$

$$= \frac{1}{2} e^2 + e - \frac{1}{2}$$

$$64. \text{ Let } u = \ln x. \text{ Then } du = \frac{1}{x} dx, \text{ so } \int_e^6 \frac{dx}{x \ln x} = \int_1^{\ln 6} \frac{1}{u} du = \left[\ln |u| \right]_1^{\ln 6} = \ln \ln 6 - \ln 1 = \ln \ln 6$$

$$65. \text{ Let } u = 6x - x^3. \text{ Then } du = (6 - 3x^2) dx = 3(2 - x^2) dx, \text{ so}$$

$$\int \frac{2 - x^2}{6x - x^3} dx = \int \frac{\frac{1}{3} du}{u} = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |6x - x^3| + C.$$

66. Let $u = 2 + \sin x$. Then $du = \cos x \, dx$, so

$$\int \frac{\cos x}{2 + \sin x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |2 + \sin x| + C = \ln(2 + \sin x) + C \quad [\text{since } 2 + \sin x > 0].$$

67. Let $u = \ln x$. Then $du = \frac{dx}{x} \Rightarrow \int \frac{(\ln x)^2}{x} \, dx = \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

68. Let $u = 1 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} \, dx$, so $\int \frac{(1 + \sqrt{x})^4}{\sqrt{x}} \, dx = 2 \int u^4 \, du = \frac{2}{5}u^5 + C = \frac{2}{5}(1 + \sqrt{x})^5 + C$.

69. (a) $\frac{d}{dx} (\ln |\sin x| + C) = \frac{1}{\sin x} \cos x = \cot x$

(b) Let $u = \sin x$. Then $du = \cos x \, dx$, so $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln |\sin x| + C$.

70. Let $u = x - 2$. Then the area is

$$A = - \int_{-4}^{-1} \frac{2}{x-2} \, dx = -2 \int_{-6}^{-3} \frac{du}{u} = [-2 \ln |u|]_{-6}^{-3} = -2 \ln 3 + 2 \ln 6 = 2 \ln 2 \approx 1.386.$$

71. The cross-sectional area is $\pi(1/\sqrt{x+1})^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1} \, dx = \pi [\ln(x+1)]_0^1 = \pi(\ln 2 - \ln 1) = \pi \ln 2.$$

72. Using cylindrical shells, we get $V = \int_0^3 \frac{2\pi x}{x^2+1} \, dx = \pi [\ln(1+x^2)]_0^3 = \pi \ln 10$.

73. $W = \int_{V_1}^{V_2} P \, dV = \int_{600}^{1000} \frac{C}{V} \, dV = C \int_{600}^{1000} \frac{1}{V} \, dV = C [\ln |V|]_{600}^{1000}$
 $= C(\ln 1000 - \ln 600) = C \ln \frac{1000}{600} = C \ln \frac{5}{3}$

Initially, $PV = C$, where $P = 150$ kPa and $V = 600$ cm³, so $C = (150)(600) = 90,000$. Thus,

$$W = 90,000 \ln \frac{5}{3} \approx 45,974 \text{ kPa} \cdot \text{cm}^3 = 45,974(10^3 \text{ Pa})(10^{-6} \text{ m}^3) = 45,974 \text{ Pa} \cdot \text{m}^3 = 45,974 \text{ N} \cdot \text{m}$$

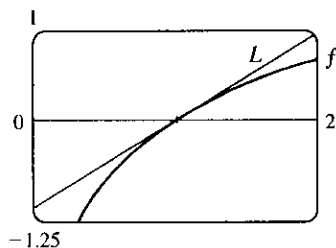
$$[\text{Pa} = \text{N/m}^2] = 45,974 \text{ J}$$

74. $f''(x) = x^{-2}, x > 0 \Rightarrow f'(x) = -1/x + C \Rightarrow f(x) = -\ln x + Cx + D$. $0 = f(1) = C + D$ and
 $0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \Rightarrow C = \ln 2$ and $D = -\ln 2$. So
 $f(x) = -\ln x + (\ln 2)x - \ln 2$.

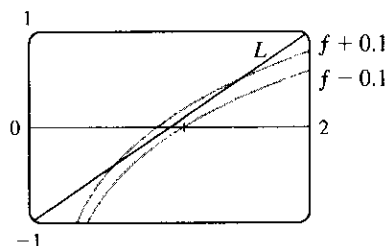
75. $f(x) = 2x + \ln x \Rightarrow f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \Rightarrow g(2) = 1$, so
 $g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}$.

76. (a) Let $f(x) = \ln x \Rightarrow f'(x) = 1/x \Rightarrow f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x-1$.

(b)

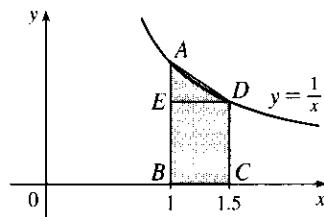


(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

77. (a)



We interpret $\ln 1.5$ as the area under the curve $y = 1/x$ from $x = 1$ to $x = 1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot \frac{1}{2} \left(1 + \frac{2}{3}\right) = \frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3} < \ln 1.5 < \frac{5}{12}$.

(b) With $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.05$, we have

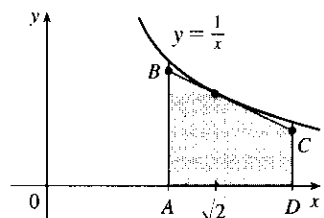
$$\begin{aligned} \ln 1.5 &= \int_1^{1.5} (1/t) dt \approx (0.05)[f(1.025) + f(1.075) + \cdots + f(1.475)] \\ &= (0.05) \left[\frac{1}{1.025} + \frac{1}{1.075} + \cdots + \frac{1}{1.475} \right] \approx 0.4054 \end{aligned}$$

78. (a) $y = \frac{1}{t}$, $y' = -\frac{1}{t^2}$. The slope of AD is $\frac{1/2 - 1}{2 - 1} = -\frac{1}{2}$. Let c be the t -coordinate of the point on $y = \frac{1}{t}$ with

slope $-\frac{1}{2}$. Then $-\frac{1}{c^2} = -\frac{1}{2} \Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}$ since $c > 0$. Therefore the tangent line is given by

$$y - \frac{1}{\sqrt{2}} = -\frac{1}{2}(t - \sqrt{2}) \Rightarrow y = -\frac{1}{2}t + \sqrt{2}.$$

(b)



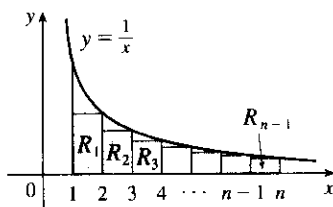
Since the graph of $y = 1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC . Now

$|AB| = -\frac{1}{2} + \sqrt{2}$ and $|CD| = -1 + \sqrt{2}$. So the area of the trapezoid $ABCD$ is

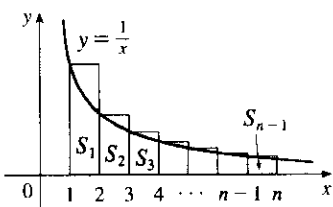
$$\frac{1}{2} \left[\left(-\frac{1}{2} + \sqrt{2}\right) + \left(-1 + \sqrt{2}\right) \right] 1 = -\frac{3}{4} + \sqrt{2} \approx 0.6642. \text{ So}$$

$\ln 2 > \text{area of trapezoid } ABCD > 0.66$.

79.



The area of R_i is $\frac{1}{i+1}$ and so $\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < \int_1^n \frac{1}{t} dt = \ln n$.



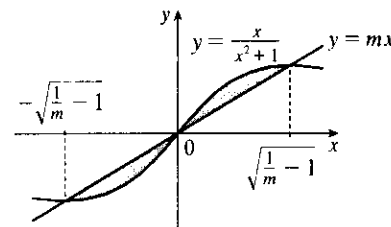
The area of S_i is $\frac{1}{i}$ and so $1 + \frac{1}{2} + \dots + \frac{1}{n-1} > \int_1^n \frac{1}{t} dt = \ln n$.

80. If $f(x) = \ln(x^r)$, then $f'(x) = (1/x^r)(rx^{r-1}) = r/x$. But if $g(x) = r \ln x$, then $g'(x) = r/x$. So f and g must differ by a constant: $\ln(x^r) = r \ln x + C$. Put $x = 1$: $\ln(1^r) = r \ln 1 + C \Rightarrow C = 0$, so $\ln(x^r) = r \ln x$.

81. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2 + 1) = mx \Rightarrow$

$$x = 0 \text{ or } mx^2 + m - 1 = 0 \Rightarrow x = 0 \text{ or}$$

$$x = \frac{\pm \sqrt{-4(m)(m-1)}}{2m} = \pm \sqrt{\frac{1}{m} - 1}. \text{ Note that if } m = 1, \text{ this}$$



has only the solution $x = 0$, and no region is determined. But if

$$1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1, \text{ then there are two}$$

solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2 + 1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2 + 1)$ are both odd functions,

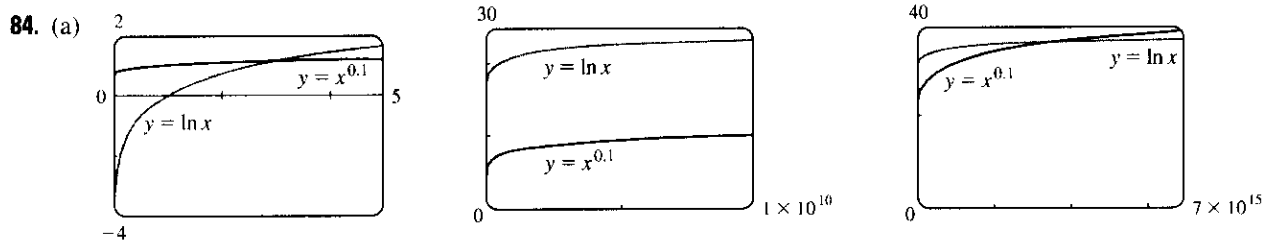
the total area is twice the area between the curves on the interval $[0, \sqrt{1/m - 1}]$. So the total area enclosed is

$$\begin{aligned} 2 \int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx \right] dx &= 2 \left[\frac{1}{2} \ln(x^2+1) - \frac{1}{2} mx^2 \right]_0^{\sqrt{1/m-1}} \\ &= \left[\ln \left(\frac{1}{m} - 1 + 1 \right) - m \left(\frac{1}{m} - 1 \right) \right] - (\ln 1 - 0) \\ &= \ln \left(\frac{1}{m} \right) + m - 1 = m - \ln m - 1 \end{aligned}$$

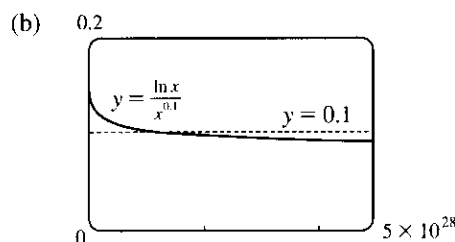
$$82. \lim_{x \rightarrow \infty} [\ln(2+x) - \ln(1+x)] = \lim_{x \rightarrow \infty} \ln \left(\frac{2+x}{1+x} \right) = \lim_{x \rightarrow \infty} \ln \left(\frac{2/x+1}{1/x+1} \right) = \ln \frac{1}{1} = \ln 1 = 0$$

83. If $f(x) = \ln(1+x)$, then $f'(x) = \frac{1}{1+x}$, so $f'(0) = 1$.

$$\text{Thus, } \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1.$$



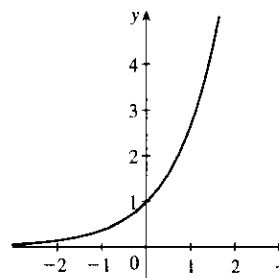
From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.



(c) From the graph at left, it seems that $\frac{\ln x}{x^{0.1}} < 0.1$ whenever $x > 1.3 \times 10^{28}$ (approximately). So we can take $N = 1.3 \times 10^{28}$, or any larger number.

7.3* The Natural Exponential Function

1. (a) e is the number such that $\ln e = 1$. (c)
(b) $e \approx 2.71828$



The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

2. (a) $e^{\ln 6} = 6$ (b) $\ln \sqrt{e} = \ln(e^{1/2}) = \frac{1}{2}$
3. (a) $\ln e^{\sqrt{2}} = \sqrt{2}$ (b) $e^{3 \ln 2} = (e^{\ln 2})^3 = 2^3 = 8$
4. (a) $\ln e^{\sin x} = \sin x$ (b) $e^{x + \ln x} = e^x e^{\ln x} = x e^x$
5. (a) $2 \ln x = 1 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{1/2} = \sqrt{e}$
(b) $e^{-x} = 5 \Rightarrow -x = \ln 5 \Rightarrow x = -\ln 5$

$$6. (a) e^{2x+3} - 7 = 0 \Rightarrow e^{2x+3} = 7 \Rightarrow 2x + 3 = \ln 7 \Rightarrow 2x = \ln 7 - 3 \Rightarrow x = \frac{1}{2}(\ln 7 - 3)$$

$$(b) \ln(5 - 2x) = -3 \Rightarrow 5 - 2x = e^{-3} \Rightarrow 2x = 5 - e^{-3} \Rightarrow x = \frac{1}{2}(5 - e^{-3})$$

$$7. \ln(\ln x) = 1 \Leftrightarrow e^{\ln(\ln x)} = e^1 \Leftrightarrow \ln x = e^1 = e \Leftrightarrow e^{\ln x} = e^e \Leftrightarrow x = e^e$$

$$8. e^{e^x} = 10 \Leftrightarrow \ln(e^{e^x}) = \ln 10 \Leftrightarrow e^x \ln e = e^x = \ln 10 \Leftrightarrow \ln e^x = \ln(\ln 10) \Leftrightarrow x = \ln(\ln 10)$$

$$9. 2 \ln x = \ln 2 + \ln(3x - 4) \Rightarrow \ln x^2 = \ln[2(3x - 4)] \Rightarrow \ln x^2 = \ln(6x - 8) \Rightarrow x^2 = 6x - 8 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x - 2)(x - 4) = 0 \Rightarrow x = 2 \text{ or } x = 4, \text{ both are valid solutions.}$$

$$10. \ln(2x + 1) = 2 - \ln x \Rightarrow \ln x + \ln(2x + 1) = \ln e^2 \Rightarrow \ln[x(2x + 1)] = \ln e^2 \Rightarrow 2x^2 + x = e^2 \Rightarrow 2x^2 + x - e^2 = 0 \Rightarrow x = \frac{-1 + \sqrt{1 + 8e^2}}{4} \quad \{\text{since } x > 0\}.$$

$$11. e^{ax} = Ce^{bx} \Leftrightarrow \ln e^{ax} = \ln[C(e^{bx})] \Leftrightarrow ax = \ln C + bx + \ln e^{bx} \Leftrightarrow ax = \ln C + bx \Leftrightarrow ax - bx = \ln C \Leftrightarrow (a - b)x = \ln C \Leftrightarrow x = \frac{\ln C}{a - b}$$

$$12. 7e^x - e^{2x} = 12 \Leftrightarrow (e^x)^2 - 7e^x + 12 = 0 \Leftrightarrow (e^x - 3)(e^x - 4) = 0, \text{ so we have either } e^x = 3 \Leftrightarrow x = \ln 3, \text{ or } e^x = 4 \Leftrightarrow x = \ln 4.$$

$$13. e^{2+5x} = 100 \Rightarrow \ln(e^{2+5x}) = \ln 100 \Rightarrow 2 + 5x = \ln 100 \Rightarrow 5x = \ln 100 - 2 \Rightarrow x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$$

$$14. \ln(1 + \sqrt{x}) = 2 \Rightarrow 1 + \sqrt{x} = e^2 \Rightarrow \sqrt{x} = e^2 - 1 \Rightarrow x = (e^2 - 1)^2 \approx 40.8200$$

$$15. \ln(e^x - 2) = 3 \Rightarrow e^x - 2 = e^3 \Rightarrow e^x = e^3 + 2 \Rightarrow x = \ln(e^3 + 2) \approx 3.0949$$

$$16. e^{1/(x-4)} = 7 \Rightarrow \ln e^{1/(x-4)} = \ln 7 \Rightarrow \frac{1}{x-4} = \ln 7 \Rightarrow \frac{1}{\ln 7} = x - 4 \Rightarrow x = 4 + \frac{1}{\ln 7} \approx 4.5139$$

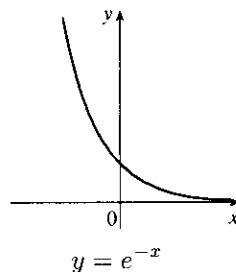
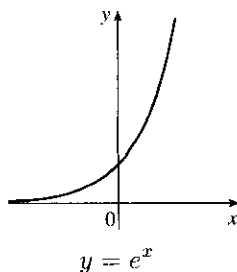
$$17. (a) e^x < 10 \Rightarrow \ln e^x < \ln 10 \Rightarrow x < \ln 10 \Rightarrow x \in (-\infty, \ln 10)$$

$$(b) \ln x > -1 \Rightarrow e^{\ln x} > e^{-1} \Rightarrow x > e^{-1} \Rightarrow x \in (1/e, \infty)$$

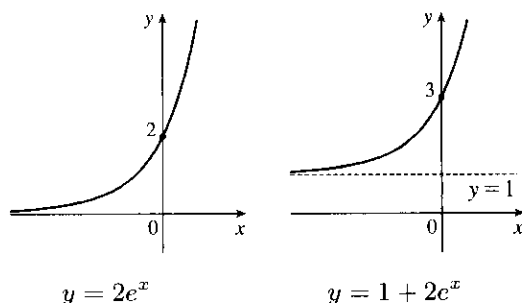
$$18. (a) 2 < \ln x < 9 \Rightarrow e^2 < e^{\ln x} < e^9 \Rightarrow e^2 < x < e^9 \Rightarrow x \in (e^2, e^9)$$

$$(b) e^{2-3x} > 4 \Rightarrow \ln e^{2-3x} > \ln 4 \Rightarrow 2 - 3x > \ln 4 \Rightarrow -3x > \ln 4 - 2 \Rightarrow x < -\frac{1}{3}(\ln 4 - 2) \Rightarrow x \in (-\infty, \frac{1}{3}(2 - \ln 4))$$

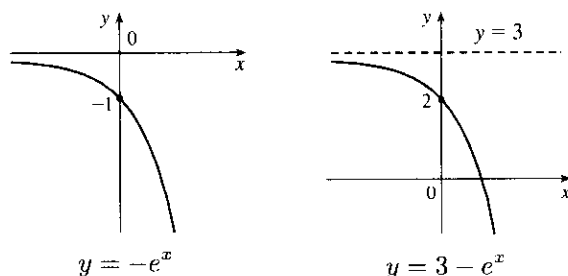
19.



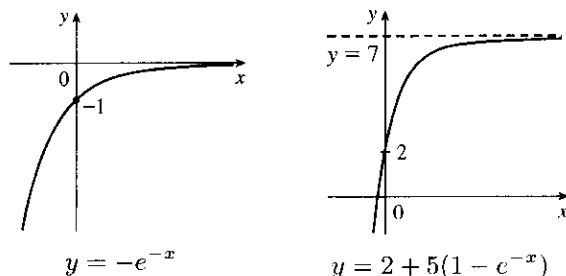
20. We start with the graph of $y = e^x$ (Figure 2), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y = 1$.



21. We start with the graph of $y = e^x$ (Figure 2), reflect it about the x -axis, and then shift 3 units upward. Note the horizontal asymptote of $y = 3$.



22. We start with the graph of $y = e^x$ (Figure 2), reflect it about the y -axis, and then about the x -axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.



23. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.
- (b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.
- (c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x -axis, we multiply the original function by -1 to get $y = -e^x$.
- (d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y -axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.
- (e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x -axis and then about the y -axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.
24. (a) This reflection consists of first reflecting the graph about the x -axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.
- (b) This reflection consists of first reflecting the graph about the y -axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

$$25. \lim_{x \rightarrow \infty} e^{1-x^3} = \lim_{x \rightarrow \infty} (e^1 \cdot e^{-x^3}) = e \lim_{x \rightarrow \infty} \frac{1}{e^{x^3}} = e \cdot 0 = 0$$

26. If we let $t = \tan x$, then as $x \rightarrow (\pi/2)^+$, $t \rightarrow -\infty$. Thus, $\lim_{x \rightarrow (\pi/2)^+} e^{\tan x} = \lim_{t \rightarrow -\infty} e^t = 0$.
27. Divide numerator and denominator by e^{3x} : $\lim_{x \rightarrow \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \rightarrow \infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} = \frac{1 - 0}{1 + 0} = 1$
28. Divide numerator and denominator by e^{-3x} : $\lim_{x \rightarrow -\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \rightarrow -\infty} \frac{e^{6x} - 1}{e^{6x} + 1} = \frac{0 - 1}{0 + 1} = -1$
29. Let $t = 3/(2 - x)$. As $x \rightarrow 2^+$, $t \rightarrow -\infty$. So $\lim_{x \rightarrow 2^+} e^{3/(2-x)} = \lim_{t \rightarrow -\infty} e^t = 0$ by (6).
30. Let $t = 3/(2 - x)$. As $x \rightarrow 2^-$, $t \rightarrow \infty$. So $\lim_{x \rightarrow 2^-} e^{3/(2-x)} = \lim_{t \rightarrow \infty} e^t = \infty$ by (6).
31. By the Product Rule, $f(x) = x^2 e^x \Rightarrow f'(x) = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2) = x^2 e^x + e^x(2x) = x e^x(x + 2)$.
32. By the Quotient Rule, $y = \frac{e^x}{1+x} \Rightarrow y' = \frac{(1+x)e^x - e^x(1)}{(1+x)^2} = \frac{e^x + x e^x - e^x}{(x+1)^2} = \frac{x e^x}{(x+1)^2}$.
33. By (9), $y = e^{ax^3} \Rightarrow y' = e^{ax^3} \frac{d}{dx}(ax^3) = 3ax^2 e^{ax^3}$.
34. $y = e^u(\cos u + cu) \Rightarrow y' = e^u(-\sin u + c) + (\cos u + cu)e^u = e^u(\cos u - \sin u + cu + c)$
35. $f(u) = e^{1/u} \Rightarrow f'(u) = e^{1/u} \cdot \frac{d}{du}\left(\frac{1}{u}\right) = e^{1/u}\left(\frac{-1}{u^2}\right) = \left(\frac{-1}{u^2}\right)e^{1/u}$
36. $y = e^x \ln x \Rightarrow y' = e^x\left(\frac{1}{x}\right) + (\ln x)(e^x) = e^x\left(\ln x + \frac{1}{x}\right)$
37. By (9), $F(t) = e^{t \sin 2t} \Rightarrow$
 $F'(t) = e^{t \sin 2t}(t \sin 2t)' = e^{t \sin 2t}(t \cdot 2 \cos 2t + \sin 2t \cdot 1) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$
38. $y = e^{k \tan \sqrt{x}} \Rightarrow y' = e^{k \tan \sqrt{x}} \cdot \frac{d}{dx}(k \tan \sqrt{x}) = e^{k \tan \sqrt{x}}\left(k \sec^2 \sqrt{x} \cdot \frac{1}{2}x^{-1/2}\right) = \frac{k \sec^2 \sqrt{x}}{2\sqrt{x}} e^{k \tan \sqrt{x}}$
39. $y = \sqrt{1 + 2e^{3x}} \Rightarrow y' = \frac{1}{2}(1 + 2e^{3x})^{-1/2} \frac{d}{dx}(1 + 2e^{3x}) = \frac{1}{2\sqrt{1 + 2e^{3x}}}(2e^{3x} \cdot 3) = \frac{3e^{3x}}{\sqrt{1 + 2e^{3x}}}$
40. $y = \cos(e^{\pi x}) \Rightarrow y' = -\sin(e^{\pi x}) \cdot e^{\pi x} \cdot \pi = -\pi e^{\pi x} \sin(e^{\pi x})$
41. $y = e^{e^x} \Rightarrow y' = e^{e^x} \cdot \frac{d}{dx}(e^x) = e^{e^x} \cdot e^x$ or e^{e^x+x}
42. $y = \sqrt{1 + x e^{-2x}} \Rightarrow y' = \frac{1}{2}(1 + x e^{-2x})^{-1/2} [x(-2e^{-2x}) + e^{-2x}] = \frac{e^{-2x}(-2x + 1)}{2\sqrt{1 + x e^{-2x}}}$
43. By the Quotient Rule, $y = \frac{ae^x + b}{ce^x + d} \Rightarrow$
 $y' = \frac{(ce^x + d)(ae^x) - (ae^x + b)(ce^x)}{(ce^x + d)^2} = \frac{(ace^x + ad - ace^x - bc)e^x}{(ce^x + d)^2} = \frac{(ad - bc)e^x}{(ce^x + d)^2}$
44. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}} \Rightarrow y' = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$
 $= \frac{(e^{2x} - 2 + e^{-2x}) - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} = -\frac{4}{(e^x - e^{-x})^2}$

45. $y = e^{2x} \cos \pi x \Rightarrow y' = e^{2x}(-\pi \sin \pi x) + (\cos \pi x)(2e^{2x}) = e^{2x}(2 \cos \pi x - \pi \sin \pi x).$

At $(0, 1)$, $y' = 1(2 - 0) = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

46. $y = \frac{e^x}{x} \Rightarrow y' = \frac{x \cdot e^x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}.$ At $(1, e)$, $y' = 0$, and an equation of the tangent line is $y - e = 0(x - 1)$, or $y = e$.

47. $\frac{d}{dx}(e^{x^2y}) = \frac{d}{dx}(x + y) \Rightarrow e^{x^2y}(x^2y' + y \cdot 2x) = 1 + y' \Rightarrow x^2e^{x^2y}y' + 2xye^{x^2y} = 1 + y' \Rightarrow$
 $x^2e^{x^2y}y' - y' = 1 - 2xye^{x^2y} \Rightarrow y'(x^2e^{x^2y} - 1) = 1 - 2xye^{x^2y} \Rightarrow y' = \frac{1 - 2xye^{x^2y}}{x^2e^{x^2y} - 1}$

48. $y = Ae^{-x} + Bxe^{-x} \Rightarrow y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \Rightarrow$
 $y'' = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}$, so
 $y'' + 2y' + y = (A - 2B)e^{-x} + Bxe^{-x} + 2[(B - A)e^{-x} - Bxe^{-x}] + Ae^{-x} + Bxe^{-x} = 0.$

49. $y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2e^{rx}$, so if $y = e^{rx}$ satisfies the differential equation $y'' + 6y' + 8y = 0$, then $r^2e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x , we must have $r^2 + 6r + 8 = 0$, or $(r + 2)(r + 4) = 0$, so $r = -2$ or -4 .

50. $y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \Leftrightarrow e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \Leftrightarrow$
 $e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \Leftrightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

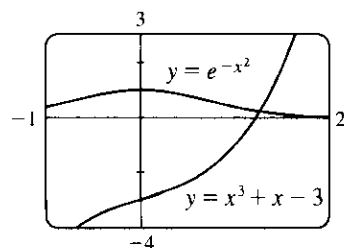
51. $f(x) = e^{2x} \Rightarrow f'(x) = 2e^{2x} \Rightarrow f''(x) = 2 \cdot 2e^{2x} = 2^2e^{2x} \Rightarrow$
 $f'''(x) = 2^2 \cdot 2e^{2x} = 2^3e^{2x} \Rightarrow \dots \Rightarrow f^{(n)}(x) = 2^n e^{2x}$

52. $f(x) = xe^{-x} \Rightarrow f'(x) = x(-e^{-x}) + e^{-x} = (1 - x)e^{-x} \Rightarrow$
 $f''(x) = (1 - x)(-e^{-x}) + e^{-x}(-1) = (x - 2)e^{-x} \Rightarrow f'''(x) = (x - 2)(-e^{-x}) + e^{-x} = (3 - x)e^{-x} \Rightarrow$
 $f^{(4)}(x) = (3 - x)(-e^{-x}) + e^{-x}(-1) = (x - 4)e^{-x} \Rightarrow \dots \Rightarrow f^{(n)}(x) = (-1)^n(x - n)e^{-x}.$
 So $D^{1000}xe^{-x} = (x - 1000)e^{-x}.$

53. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1, 0)$.

(b) $f(x) = e^x + x \Rightarrow f'(x) = e^x + 1$, so $x_{n+1} = x_n - \frac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get
 $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the root is -0.567143 to six decimal places.

54.



From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3 .

We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so

$$f'(x) = 3x^2 + 1 + 2xe^{-x^2}, \text{ and the formula is}$$

$x_{n+1} = x_n - f(x_n)/f'(x_n)$. We take $x_1 = 1.2$, and the formula gives
 $x_2 \approx 1.252462$, $x_3 \approx 1.251045$, and $x_4 \approx x_5 \approx 1.251044$. So the root
 of the equation, correct to six decimal places, is $x = 1.251044$.

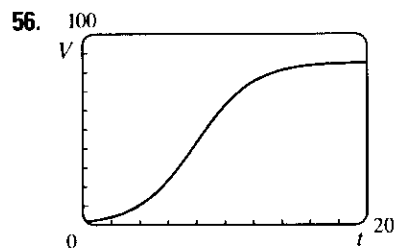
55. (a) $m(t) = 24 \cdot e^{-(\ln 2)t/25} = 24 \cdot 2^{-t/25} \Rightarrow m(40) = 24 \cdot 2^{-40/25} \approx 7.92$ mg.

(b) $m'(t) = 24 \frac{d}{dt} \left[e^{-(\ln 2)t/25} \right] = 24 \cdot e^{-(\ln 2)t/25} \left(-\frac{\ln 2}{25} \right)$, so

$m'(40) = 24e^{-(\ln 2)(40)/25} \left(-\frac{\ln 2}{25} \right) \approx -0.22$ mg/yr

(c) $m(t) = 5 \Rightarrow 24e^{-(\ln 2)t/25} = 5 \Rightarrow e^{-(\ln 2)t/25} = \frac{5}{24} \Rightarrow -(\ln 2)t/25 = \ln \frac{5}{24} \Rightarrow$

$t = -25 \frac{\ln \frac{5}{24}}{\ln 2} \approx 56.6$ yr



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use

$V(t) = \frac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

$V'(t) = -\frac{a(bce^{ct})}{(1 + be^{ct})^2}$ [by the Reciprocal Rule] and

$$V''(t) = -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{[(1 + be^{ct})^2]^2}$$

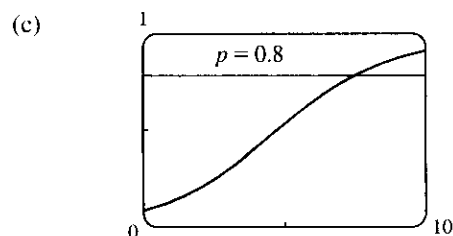
$$= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2 e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3}$$

So $V''(t) = 0 \Leftrightarrow 1 = be^{ct} \Leftrightarrow e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows:

$$ct = \ln(1/b) \Leftrightarrow t = (1/c) \ln(1/b) = -2 \ln \frac{1}{53} \approx 7.94 \text{ years.}$$

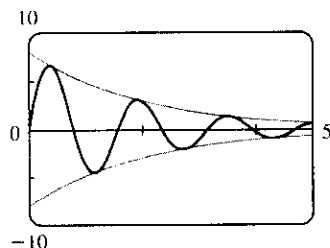
57. (a) $\lim_{t \rightarrow \infty} p(t) = \lim_{t \rightarrow \infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a \cdot 0} = 1$, since $k > 0 \Rightarrow -kt \rightarrow -\infty \Rightarrow e^{-kt} \rightarrow 0$.

(b) $p(t) = (1 + ae^{-kt})^{-1} \Rightarrow \frac{dp}{dt} = -(1 + ae^{-kt})^{-2} (-kae^{-kt}) = \frac{kae^{-kt}}{(1 + ae^{-kt})^2}$



From the graph of $p(t) = (1 + 10e^{-0.5t})^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

58. (a)



The displacement function is squeezed between the other two functions. This is because $-1 \leq \sin 4t \leq 1 \Rightarrow$
 $-8e^{-t/2} \leq 8e^{-t/2} \sin 4t \leq 8e^{-t/2}.$

(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

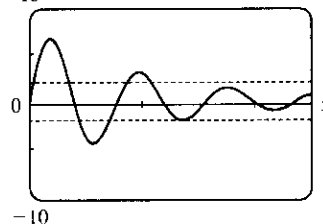
(c) The velocity of the object is the derivative of its displacement function, that is,

$$\frac{d}{dt} (8e^{-t/2} \sin 4t) = 8 \left[e^{-t/2} \cos 4t(4) + \sin 4t \left(-\frac{1}{2}\right) e^{-t/2} \right].$$

If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we

$$\text{get } v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8} \cos\left(4 \cdot \frac{\pi}{4}\right) \cdot 4 = -32e^{-\pi/8} \approx -21.6 \text{ cm/s.}$$

(d)



The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

59. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

60. $g(x) = \frac{e^x}{x} \Rightarrow g'(x) = \frac{xe^x - e^x}{x^2} = 0 \Leftrightarrow e^x(x - 1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \Leftrightarrow \frac{xe^x - e^x}{x^2} > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1$ and $g'(x) < 0 \Leftrightarrow \frac{xe^x - e^x}{x^2} < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

61. $y = xe^{3x} \Rightarrow y' = xe^{3x} \cdot 3 + e^{3x} \cdot 1 = (3x + 1)e^{3x} \Rightarrow y'' = (3x + 1)e^{3x} \cdot 3 + e^{3x} \cdot 3 = (9x + 6)e^{3x}$. The curve is concave upward at $x \Leftrightarrow y'' > 0$ at $x \Leftrightarrow 9x + 6 > 0 \Leftrightarrow x > -\frac{2}{3}$. Thus, the curve is concave upward on $(-\frac{2}{3}, \infty)$.

62. $f(x) = x^2e^{-x} \Rightarrow f'(x) = x^2(-e^{-x}) + e^{-x} \cdot 2x = (2x - x^2)e^{-x}$, so $f'(x) > 0 \Leftrightarrow 2x - x^2 > 0 \Leftrightarrow x(2 - x) > 0 \Leftrightarrow 0 < x < 2$, so f is increasing on $(0, 2)$.

63. $y = f(x) = e^{-1/(x+1)}$ A. $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ B. No x -intercept;

y -intercept $= f(0) = e^{-1}$ C. No symmetry D. $\lim_{x \rightarrow \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \rightarrow 0$, so $y = 1$ is

a HA. $\lim_{x \rightarrow -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \rightarrow -\infty$, $\lim_{x \rightarrow -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \rightarrow \infty$, so

$x = -1$ is a VA. E. $f'(x) = e^{-1/(x+1)} / (x+1)^2 \Rightarrow f'(x) > 0$ for all x except $x = -1$, so

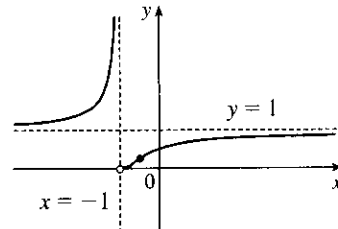
f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. F. No extreme values H.

$$G. f''(x) = \frac{e^{-1/(x+1)}}{(x+1)^4} + \frac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\frac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$$

$$\Rightarrow f''(x) > 0 \Leftrightarrow 2x+1 < 0 \Leftrightarrow x < -\frac{1}{2}, \text{ so } f \text{ is CU on}$$

$(-\infty, -1)$ and $(-1, -\frac{1}{2})$, and CD on $(-\frac{1}{2}, \infty)$. f has an IP

at $(-\frac{1}{2}, e^{-2})$.



64. $y = f(x) = e^{2x} - e^x$ A. $D = \mathbb{R}$ B. y -intercept: $f(0) = 0$;

x -intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$.

C. No symmetry D. $\lim_{x \rightarrow -\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA.

E. $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2} \Leftrightarrow$

$x > \ln \frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln \frac{1}{2}$, so f is

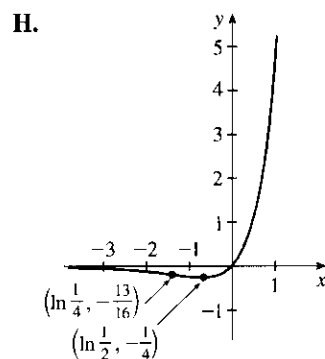
decreasing on $(-\infty, \ln \frac{1}{2})$ and increasing on $(\ln \frac{1}{2}, \infty)$. F. Local

minimum value $f(\ln \frac{1}{2}) = e^{2 \ln(1/2)} - e^{\ln(1/2)} = (\frac{1}{2})^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow$

$e^x > \frac{1}{4} \Leftrightarrow x > \ln \frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln \frac{1}{4}$.

Thus, f is CD on $(-\infty, \ln \frac{1}{4})$ and CU on $(\ln \frac{1}{4}, \infty)$. f has an IP at $(\ln \frac{1}{4}, (\frac{1}{4})^2 - \frac{1}{4}) = (\ln \frac{1}{4}, -\frac{3}{16})$.



65. $y = f(x) = e^{3x} + e^{-2x}$ A. $D = \mathbb{R}$ B. y -intercept $= f(0) = 2$;

no x -intercept C. No symmetry D. No asymptotes

E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \Leftrightarrow 3e^{3x} > 2e^{-2x}$

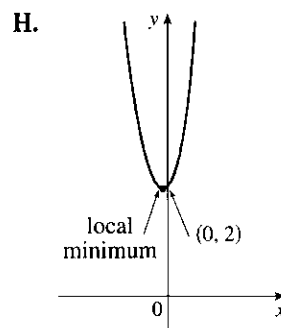
[multiply by e^{2x}] $\Leftrightarrow e^{5x} > \frac{2}{3} \Leftrightarrow 5x > \ln \frac{2}{3} \Leftrightarrow$

$x > \frac{1}{5} \ln \frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \Leftrightarrow x < \frac{1}{5} \ln \frac{2}{3}$.

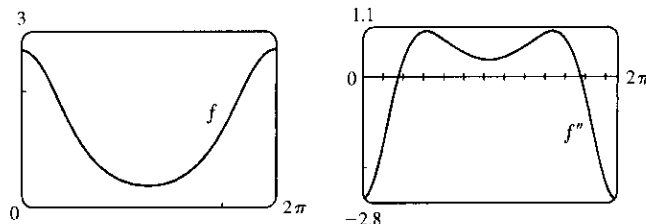
f is decreasing on $(-\infty, \frac{1}{5} \ln \frac{2}{3})$ and increasing on $(\frac{1}{5} \ln \frac{2}{3}, \infty)$.

F. Local minimum value $f(\frac{1}{5} \ln \frac{2}{3}) = (\frac{2}{3})^{3/5} + (\frac{2}{3})^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x , and f is CU on $(-\infty, \infty)$. No IP



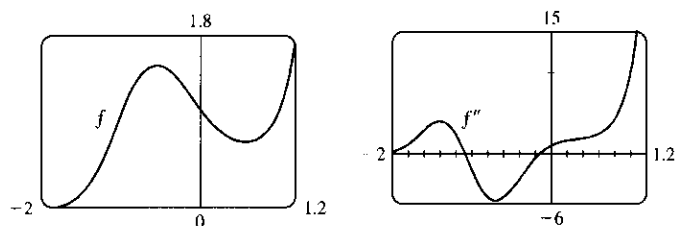
66. The function $f(x) = e^{\cos x}$ is periodic with period 2π , so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact



values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \Leftrightarrow x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow 0 < x < \pi$. So $f(0) = f(2\pi) = e$ (both maxima) and $f(\pi) = e^{\cos \pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph

$f''(x) = \frac{d}{dx}(-\sin x e^{\cos x}) = -\cos x e^{\cos x} - \sin x(e^{\cos x})(-\sin x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x -coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

67. $f(x) = e^{x^3-x} \rightarrow 0$ as $x \rightarrow -\infty$, and $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$.



To find the exact values, we calculate

$f'(x) = (3x^2 - 1)e^{x^3-x}$, which is 0 when $3x^2 - 1 = 0 \Leftrightarrow x = \pm \frac{1}{\sqrt{3}}$. The negative root corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph

$$f''(x) = \frac{d}{dx}[(3x^2 - 1)e^{x^3-x}] = (3x^2 - 1)e^{x^3-x}(3x^2 - 1) + e^{x^3-x}(6x) = e^{x^3-x}(9x^4 - 6x^2 + 6x + 1).$$

From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f , we see that these x -values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.

68. (a) As $|x| \rightarrow \infty$, $t = -x^2/(2\sigma^2) \rightarrow -\infty$, and $e^t \rightarrow 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t . Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow$

$$f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2). \quad f'(x) = 0 \Leftrightarrow x = 0. \quad \text{Because } f' \text{ changes from positive to negative at } x = 0,$$

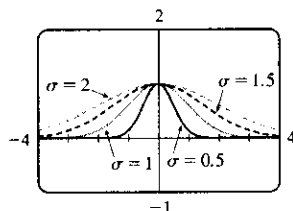
$f(0) = 1$ is a local maximum. For inflection points, we find

$$f''(x) = -\frac{1}{\sigma^2} \left[e^{-x^2/(2\sigma^2)} \cdot 1 + x e^{-x^2/(2\sigma^2)} (-x/\sigma^2) \right] = \frac{-1}{\sigma^2} e^{-x^2/(2\sigma^2)} (1 - x^2/\sigma^2).$$

$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma$. $f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞) . IP at $(\pm\sigma, e^{-1/2})$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y -axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x -axis.

69. Let $u = -3x$. Then $du = -3 dx$, so

$$\int_0^5 e^{-3x} dx = -\frac{1}{3} \int_0^{-15} e^u du = -\frac{1}{3} [e^u]_0^{-15} = -\frac{1}{3} (e^{-15} - e^0) = \frac{1}{3} (1 - e^{-15}).$$

70. Let $u = -x^2$, so $du = -2x dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 x e^{-x^2} dx = \int_0^{-1} e^u (-\frac{1}{2} du) = -\frac{1}{2} [e^u]_0^{-1} = -\frac{1}{2} (e^{-1} - e^0) = \frac{1}{2} (1 - 1/e).$$

71. Let $u = 1 + e^x$. Then $du = e^x dx$, so $\int e^x \sqrt{1 + e^x} dx = \int \sqrt{u} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + e^x)^{3/2} + C$.

72. Let $u = \tan x$. Then $du = \sec^2 x dx$, so $\int \sec^2 x e^{\tan x} dx = \int e^u du = e^u + C = e^{\tan x} + C$.

$$73. \int \frac{e^x + 1}{e^x} dx = \int (1 + e^{-x}) dx = x - e^{-x} + C$$

74. Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2} dx$, so $\int \frac{e^{1/x}}{x^2} dx = -\int e^u du = -e^u + C = -e^{1/x} + C$.

75. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$, so $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$.

76. Let $u = e^x$. Then $du = e^x dx$, so $\int e^x \sin(e^x) dx = \int \sin u du = -\cos u + C = -\cos(e^x) + C$.

77. Area = $\int_0^1 (e^{3x} - e^x) dx = [\frac{1}{3}e^{3x} - e^x]_0^1 = (\frac{1}{3}e^3 - e) - (\frac{1}{3} - 1) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

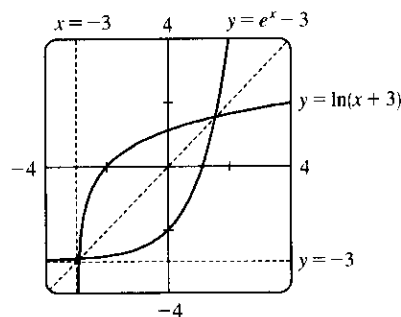
78. $f''(x) = 3e^x + 5 \sin x \Rightarrow f'(x) = 3e^x - 5 \cos x + C \Rightarrow 2 = f'(0) = 3 - 5 + C \Rightarrow C = 4$, so

$$f'(x) = 3e^x - 5 \cos x + 4 \Rightarrow f(x) = 3e^x - 5 \sin x + 4x + D \Rightarrow 1 = f(0) = 3 + D \Rightarrow D = -2, \\ \text{so } f(x) = 3e^x - 5 \sin x + 4x - 2.$$

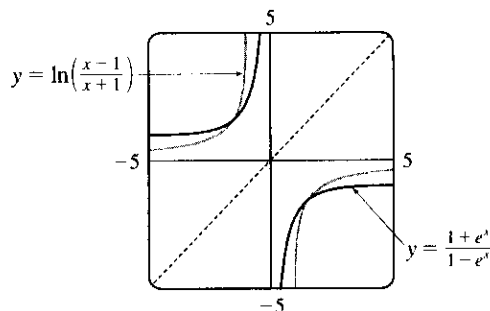
79. $V = \int_0^1 \pi (e^x)^2 dx = \pi \int_0^1 e^{2x} dx = \frac{1}{2} \pi [e^{2x}]_0^1 = \frac{\pi}{2} (e^2 - 1)$

80. $V = \int_0^1 2\pi x e^{-x^2} dx$. Let $u = x^2$. Thus $du = 2x dx$, so $V = \pi \int_0^1 e^{-u} du = \pi [-e^{-u}]_0^1 = \pi (1 - 1/e)$.

81. $y = \ln(x+3) \Rightarrow e^y = x+3 \Rightarrow$
 $x = e^y - 3$. Interchanging x and y , we get
 $y = e^x - 3$, so $f^{-1}(x) = e^x - 3$.



82. $y = \frac{1+e^x}{1-e^x} \Rightarrow y - ye^x = 1 + e^x \Rightarrow$
 $e^x(y+1) = y-1 \Rightarrow e^x = \frac{y-1}{y+1} \Rightarrow$
 $x = \ln\left(\frac{y-1}{y+1}\right)$. Interchange x and y :
 $y = \ln\left(\frac{x-1}{x+1}\right)$ is the inverse function.



83. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}.$$

84. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\lim_{x \rightarrow \pi} \frac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos \pi)e^{\sin \pi} = -1 \cdot e^0 = -1$.

85. Using the second law of logarithms and Equation 5, we have $\ln(e^x/e^y) = \ln e^x - \ln e^y = x - y = \ln(e^{x-y})$.

Since $\ln$ is a one-to-one function, it follows that $e^x/e^y = e^{x-y}$.

86. Using the third law of logarithms and Equation 5, we have $\ln e^{rx} = rx = r \ln e^x = \ln(e^x)^r$. Since $\ln$ is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

87. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0 \Rightarrow e^x - 1 - x \geq 0 \Rightarrow e^x \geq 1 + x$.

- (b) For $0 \leq x \leq 1$, $x^2 \leq x$, so $e^{x^2} \leq e^x$ [since e^x is increasing]. Hence [from (a)] $1 + x^2 \leq e^{x^2} \leq e^x$.

$$\text{So } \frac{4}{3} = \int_0^1 (1 + x^2) dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e^x dx = e - 1 < e \Rightarrow \frac{4}{3} \leq \int_0^1 e^{x^2} dx \leq e.$$

88. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by Exercise 87(a).

Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \Rightarrow e^x \geq 1 + x + \frac{1}{2}x^2$.

- (b) Using the same argument as in Exercise 87(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \leq e^{x^2} \leq e^x$

$$[\text{for } 0 \leq x \leq 1] \Rightarrow \int_0^1 (1 + x^2 + \frac{1}{2}x^4) dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e^x dx \Rightarrow \frac{43}{30} \leq \int_0^1 e^{x^2} dx \leq e - 1.$$

89. (a) By Exercise 87(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} \geq 0$

by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that

$$0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}, \text{ and hence } e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$$

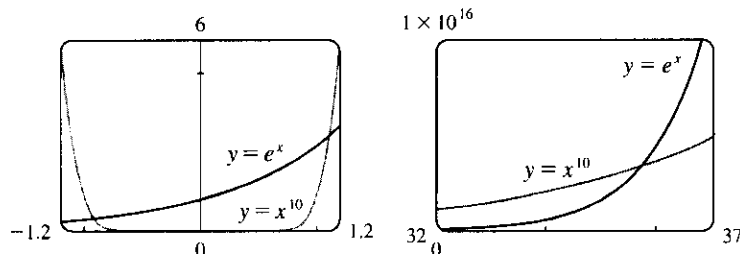
for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for every positive integer n , by mathematical induction.

- (b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\bar{3} > 2.7$.

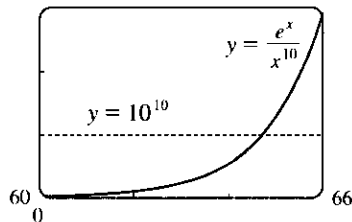
$$(c) e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} \Rightarrow \frac{e^x}{x^k} \geq \frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + \frac{1}{k!} + \frac{x}{(k+1)!} \geq \frac{x}{(k+1)!}.$$

But $\lim_{x \rightarrow \infty} \frac{x}{(k+1)!} = \infty$, so $\lim_{x \rightarrow \infty} \frac{e^x}{x^k} = \infty$.

90. (a) The graph of g finally surpasses that of f at $x \approx 35.8$.



- (b) 3×10^{10}



- (c) From the graph in part (b), it seems that $e^x/x^{10} > 10^{10}$ whenever $x > 65$, approximately. So we can take $N \geq 65$.

7.4* General Logarithmic and Exponential Functions

1. (a) $a^x = e^{x \ln a}$

(b) The domain of $f(x) = a^x$ is $\mathbb{R}$.

(c) The range of $f(x) = a^x$ ($a \neq 1$) is $(0, \infty)$.

(d) (i) See Figure 1. (ii) See Figure 3. (iii) See Figure 2.

2. (a) $\log_a x$ is the number y such that $a^y = x$.

(b) The domain of $f(x) = \log_a x$ is $(0, \infty)$.

(c) The range of $f(x) = \log_a x$ is $\mathbb{R}$.

(d) See Figure 9.

3. $5^{\sqrt{7}} = (e^{\ln 5})^{\sqrt{7}} = e^{\sqrt{7} \ln 5}$

4. $10^{x^2} = (e^{\ln 10})^{x^2} = e^{x^2 \ln 10}$

5. $(\cos x)^x = (e^{\ln \cos x})^x = e^{x \ln(\cos x)}$.

6. $x^{\cos x} = (e^{\ln x})^{\cos x} = e^{(\cos x)(\ln x)}$

7. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$.

(b) $\log_2 \frac{1}{16} = -4$ since $2^{-4} = \frac{1}{16}$. [Or: $\log_2 \frac{1}{16} = \log_2 2^{-4} = -4$]

8. (a) $\log_{10} 0.1 = -1$ since $10^{-1} = 0.1$.

(b) $\log_8 320 - \log_8 5 = \log_8 \frac{320}{5} = \log_8 64 = 2$ since $8^2 = 64$.

9. (a) $\log_{12} 3 + \log_{12} 48 = \log_{12}(3 \cdot 48) = \log_{12} 144 = 2$ since $12^2 = 144$.

(b) $\log_5 5^{\sqrt{2}} = \sqrt{2}$ by the cancellation property $\log_a a^x = x$.

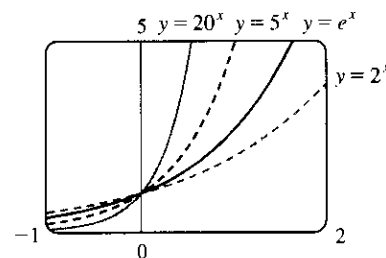
[Or: $\log_5 5^{\sqrt{2}} = \sqrt{2} \log_5 5 = \sqrt{2} \cdot 1 = \sqrt{2}$]

10. (a) $\log_a \frac{1}{a} = -1$ since $a^{-1} = \frac{1}{a}$. [Or: $\log_a \frac{1}{a} = \log_a a^{-1} = -1$]

(b) $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10} 4} \cdot 10^{\log_{10} 7} = 4 \cdot 7 = 28$

[Or: $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10}(4 \cdot 7)} = 10^{\log_{10} 28} = 28$]

11. All of these graphs approach 0 as $x \rightarrow -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \rightarrow \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \rightarrow -\infty$.



12. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1

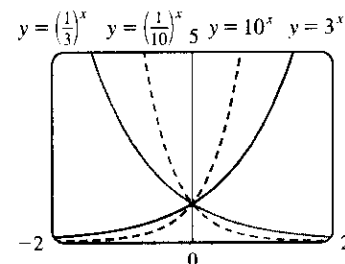
$\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the

reflection of that of 3^x about the y -axis, and the graph of

$\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y -axis. The

graph of 10^x increases more quickly than that of 3^x for

$x > 0$, and approaches 0 faster as $x \rightarrow -\infty$.



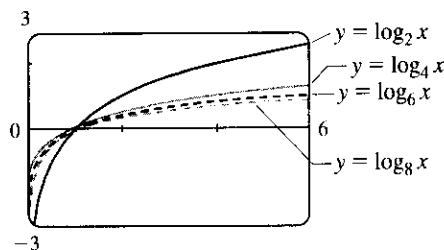
13. (a) $\log_{12} e = \frac{\ln e}{\ln 12} = \frac{1}{\ln 12} \approx 0.402430$

(b) $\log_6 13.54 = \frac{\ln 13.54}{\ln 6} \approx 1.454240$

(c) $\log_2 \pi = \frac{\ln \pi}{\ln 2} \approx 1.651496$

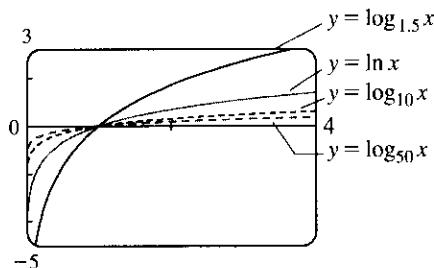
14. To graph the functions, we use $\log_2 x = \frac{\ln x}{\ln 2}$, $\log_4 x = \frac{\ln x}{\ln 4}$, etc.

These graphs all approach $-\infty$ as $x \rightarrow 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \rightarrow \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y -axis (as $x \rightarrow 0^+$).

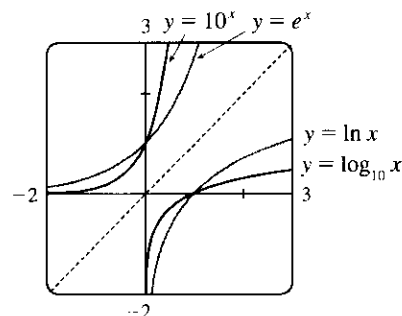


15. To graph these functions, we use $\log_{1.5} x = \frac{\ln x}{\ln 1.5}$ and

$\log_{50} x = \frac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \rightarrow 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \rightarrow \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y -axis more closely as $x \rightarrow 0^+$.



16. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x . Also note that $\log_{10} x \rightarrow \infty$ as $x \rightarrow \infty$ more slowly than $\ln x$.



17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1$ $[C = \frac{6}{a}]$ and $24 = Ca^3 \Rightarrow 24 = (\frac{6}{a})a^3 \Rightarrow 4 = a^2 \Rightarrow a = 2$ [since $a > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

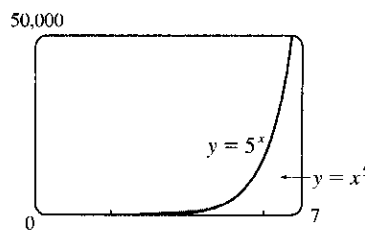
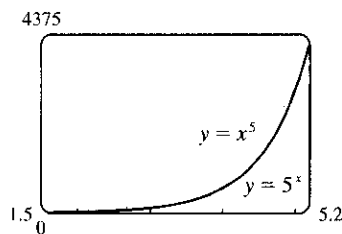
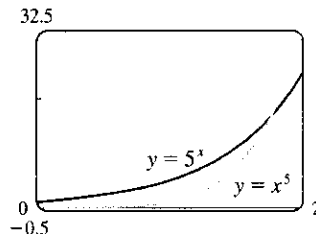
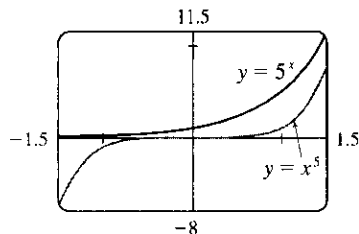
18. Given the y -intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $(2, \frac{2}{9})$ gives us $\frac{2}{9} = 2a^2 \Rightarrow \frac{1}{9} = a^2 \Rightarrow a = \frac{1}{3}$ [since $a > 0$]. The function is $f(x) = 2(\frac{1}{3})^x$ or $f(x) = 2(3)^{-x}$.

19. (a) $2 \text{ ft} = 24 \text{ in}$, $f(24) = 24^2 \text{ in} = 576 \text{ in} = 48 \text{ ft}$. $g(24) = 2^{24} \text{ in} = 2^{24}/(12 \cdot 5280) \text{ mi} \approx 265 \text{ mi}$

(b) $3 \text{ ft} = 36 \text{ in}$, so we need x such that $\log_2 x = 36 \Leftrightarrow x = 2^{36} = 68,719,476,736$. In miles, this is

$$68,719,476,736 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 1,084,587.7 \text{ mi}.$$

20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point (1.8, 17.1) the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At (5, 3125) there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



21. $\lim_{t \rightarrow \infty} 2^{-t^2} = \lim_{u \rightarrow -\infty} 2^u$ [where $u = -t^2$] $= 0$
22. Let $t = x^2 - 5x + 6$. As $x \rightarrow 3^+$, $t = (x-2)(x-3) \rightarrow 0^+$. $\lim_{x \rightarrow 3^+} \log_{10}(x^2 - 5x + 6) = \lim_{t \rightarrow 0^+} \log_{10} t = -\infty$ by (4) in Section 7.3.
23. $h(t) = t^3 - 3^t \Rightarrow h'(t) = 3t^2 - 3^t \ln 3$
24. $g(x) = x^4 4^x \Rightarrow g'(x) = x^4 4^x \ln 4 + 4^x \cdot 4x^3 = x^3 4^x (x \ln 4 + 4)$
25. Using Formula 4 and the Chain Rule, $y = 5^{-1/x} \Rightarrow y' = 5^{-1/x} (\ln 5) [-1 \cdot (-x^{-2})] = 5^{-1/x} (\ln 5) / x^2$
26. $y = 10^{\tan \theta} \Rightarrow y' = 10^{\tan \theta} (\ln 10) (\sec^2 \theta)$
27. $f(u) = (2^u + 2^{-u})^{10} \Rightarrow$
 $f'(u) = 10(2^u + 2^{-u})^9 \frac{d}{du} (2^u + 2^{-u}) = 10(2^u + 2^{-u})^9 [2^u \ln 2 + 2^{-u} \ln 2 \cdot (-1)]$
 $= 10 \ln 2 (2^u + 2^{-u})^9 (2^u - 2^{-u})$
28. $y = 2^{3x^2} \Rightarrow y' = 2^{3x^2} (\ln 2) \frac{d}{dx} (3x^2) = 2^{3x^2} (\ln 2) 3x^2 (\ln 3) (2x)$
29. $f(x) = \log_3(x^2 - 4) \Rightarrow f'(x) = \frac{1}{(x^2 - 4) \ln 3} (2x) = \frac{2x}{(x^2 - 4) \ln 3}$
30. $f(x) = \log_{10} \left(\frac{x}{x-1} \right) = \log_{10} x - \log_{10}(x-1) \Rightarrow f'(x) = \frac{1}{x \ln 10} - \frac{1}{(x-1) \ln 10}$ or $-\frac{1}{x(x-1) \ln 10}$
31. $y = x^x \Rightarrow \ln y = x \ln x \Rightarrow y'/y = \ln x + x(1/x) \Rightarrow y' = x^x (\ln x + 1)$
32. $y = x^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln x \Rightarrow \frac{y'}{y} = -\frac{1}{x^2} \ln x + \frac{1}{x} \left(\frac{1}{x} \right) \Rightarrow y' = x^{1/x} \frac{1 - \ln x}{x^2}$
33. $y = x^{\sin x} \Rightarrow \ln y = \sin x \ln x \Rightarrow \frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x} \Rightarrow y' = x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right)$

$$34. y = (\sin x)^x \Rightarrow \ln y = x \ln(\sin x) \Rightarrow y'/y = \ln(\sin x) + x(\cos x)/(\sin x) \Rightarrow y' = (\sin x)^x [\ln(\sin x) + x \cot x]$$

$$35. y = (\ln x)^x \Rightarrow \ln y = x \ln \ln x \Rightarrow \frac{y'}{y} = \ln \ln x + x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \Rightarrow y' = (\ln x)^x \left(\ln \ln x + \frac{1}{\ln x} \right)$$

$$36. y = x^{\ln x} \Rightarrow \ln y = \ln x \ln x = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2 \ln x \left(\frac{1}{x} \right) \Rightarrow y' = x^{\ln x} \left(\frac{2 \ln x}{x} \right)$$

$$37. y = x^{e^x} \Rightarrow \ln y = e^x \ln x \Rightarrow \frac{y'}{y} = e^x \ln x + \frac{e^x}{x} \Rightarrow y' = x^{e^x} e^x \left(\ln x + \frac{1}{x} \right)$$

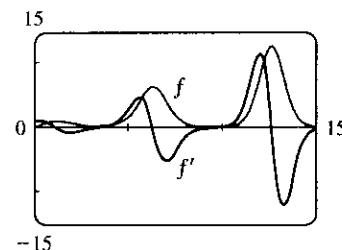
$$38. y = (\ln x)^{\cos x} \Rightarrow \ln y = \cos x \ln(\ln x) \Rightarrow \frac{y'}{y} = \cos x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} + (\ln \ln x)(-\sin x) \Rightarrow y' = (\ln x)^{\cos x} \left(\frac{\cos x}{x \ln x} - \sin x \ln \ln x \right)$$

$$39. y = 10^x \Rightarrow y' = 10^x \ln 10, \text{ so at } (1, 10), \text{ the slope of the tangent line is } 10^1 \ln 10 = 10 \ln 10, \text{ and its equation is } y - 10 = 10 \ln 10(x - 1), \text{ or } y = (10 \ln 10)x + 10(1 - \ln 10).$$

$$40. f(x) = x^{\cos x} = e^{\ln x \cos x} \Rightarrow$$

$$f'(x) = e^{\ln x \cos x} \left[\ln x (-\sin x) + \cos x \left(\frac{1}{x} \right) \right] = x^{\cos x} \left[\frac{\cos x}{x} - \sin x \ln x \right]$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



$$41. \int_1^{10} 10^t dt = \left[\frac{10^t}{\ln 10} \right]_1^{10} = \frac{10^{10}}{\ln 10} - \frac{10^1}{\ln 10} = \frac{100 - 10}{\ln 10} = \frac{90}{\ln 10}$$

$$42. \text{ Let } v = -2u. \text{ Then } dv = -2 du \text{ and}$$

$$\int_0^1 4^{-2u} du = \int_0^{-2} 4^v \left(-\frac{1}{2} \right) dv = -\frac{1}{2} \left[\frac{4^v}{\ln 4} \right]_0^{-2} = -\frac{1}{2 \ln 4} (4^{-2} - 4^0) = -\frac{1}{2 \ln 2^2} \left(\frac{1}{16} - 1 \right) = -\frac{1}{4 \ln 2} \left(-\frac{15}{16} \right) = \frac{15}{64 \ln 2}$$

$$43. \int \frac{\log_{10} x}{x} dx = \int \frac{(\ln x)/(\ln 10)}{x} dx = \frac{1}{\ln 10} \int \frac{\ln x}{x} dx. \text{ Now put } u = \ln x, \text{ so } du = \frac{1}{x} dx, \text{ and the expression becomes } \frac{1}{\ln 10} \int u du = \frac{1}{\ln 10} \left(\frac{1}{2} u^2 + C_1 \right) = \frac{1}{2 \ln 10} (\ln x)^2 + C.$$

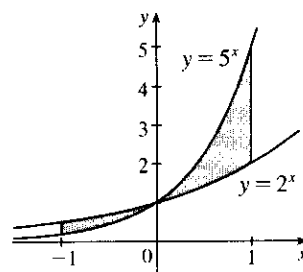
$$\text{Or: The substitution } u = \log_{10} x \text{ gives } du = \frac{dx}{x \ln 10} \text{ and we get } \int \frac{\log_{10} x}{x} dx = \frac{1}{2} \ln 10 (\log_{10} x)^2 + C.$$

$$44. \int (x^5 + 5^x) dx = \frac{1}{6} x^6 + \frac{1}{\ln 5} 5^x + C$$

$$45. \text{ Let } u = \sin \theta. \text{ Then } du = \cos \theta d\theta \text{ and } \int 3^{\sin \theta} \cos \theta d\theta = \int 3^u du = \frac{3^u}{\ln 3} + C = \frac{1}{\ln 3} 3^{\sin \theta} + C.$$

$$46. \text{ Let } u = 2^x + 1. \text{ Then } du = 2^x \ln 2 dx, \text{ so } \int \frac{2^x}{2^x + 1} dx = \int \frac{1}{u} \frac{du}{\ln 2} = \frac{1}{\ln 2} \ln |u| + C = \frac{1}{\ln 2} \ln(2^x + 1) + C.$$

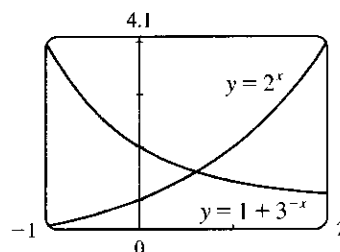
$$\begin{aligned}
 47. A &= \int_{-1}^0 (2^x - 5^x) dx + \int_0^1 (5^x - 2^x) dx \\
 &= \left[\frac{2^x}{\ln 2} - \frac{5^x}{\ln 5} \right]_{-1}^0 + \left[\frac{5^x}{\ln 5} - \frac{2^x}{\ln 2} \right]_0^1 \\
 &= \left(\frac{1}{\ln 2} - \frac{1}{\ln 5} \right) - \left(\frac{1/2}{\ln 2} - \frac{1/5}{\ln 5} \right) + \left(\frac{5}{\ln 5} - \frac{2}{\ln 2} \right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 2} \right) \\
 &= \frac{16}{5 \ln 5} - \frac{1}{2 \ln 2}
 \end{aligned}$$



48. Using disks, the volume is $V = \int_0^1 \pi [10^{-x}]^2 dx = \pi \int_0^1 10^{-2x} dx$. To evaluate the integral, we let $u = -2x \Rightarrow du = -2 dx$, $x = 0 \Rightarrow u = 0$, and $x = 1 \Rightarrow u = -2$, so we have

$$V = -\frac{\pi}{2} \int_0^{-2} 10^u du = -\frac{\pi}{2} \left[\frac{1}{\ln 10} 10^u \right]_0^{-2} = -\frac{\pi}{2 \ln 10} (10^{-2} - 1) = \frac{99\pi}{200 \ln 10}$$

49. We see that the graphs of $y = 2^x$ and $y = 1 + 3^{-x}$ intersect at $x \approx 0.6$. We let $f(x) = 2^x - 1 - 3^{-x}$ and calculate $f'(x) = 2^x \ln 2 + 3^{-x} \ln 3$, and using the formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ (Newton's Method), we get $x_1 = 0.6$, $x_2 \approx x_3 \approx 0.600967$. So, correct to six decimal places, the root occurs at $x = 0.600967$.



50. $x^y = y^x \Rightarrow y \ln x = x \ln y \Rightarrow y \cdot \frac{1}{x} + (\ln x) \cdot y' = x \cdot \frac{1}{y} \cdot y' + \ln y \Rightarrow y' \ln x - \frac{x}{y} y' = \ln y - \frac{y}{x} \Rightarrow$
 $y' = \frac{\ln y - y/x}{\ln x - x/y}$

51. $y = \frac{10^x}{10^x + 1} \Leftrightarrow (10^x + 1)y = 10^x \Leftrightarrow 10^x \cdot y + y = 10^x \Leftrightarrow y = 10^x - 10^x y \Leftrightarrow$
 $y = 10^x(1 - y) \Leftrightarrow 10^x = \frac{y}{1 - y} \Leftrightarrow \log_{10} 10^x = \log_{10} \left(\frac{y}{1 - y} \right) \Leftrightarrow x = \log_{10} y - \log_{10}(1 - y).$
 Interchange x and y : $y = \log_{10} x - \log_{10}(1 - x)$ is the inverse function.

52. $\lim_{x \rightarrow 0^+} x^{-\ln x} = \lim_{x \rightarrow 0^+} (e^{\ln x})^{-\ln x} = \lim_{x \rightarrow 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \rightarrow -\infty$ as $x \rightarrow 0^+$.

53. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \Rightarrow$
 $\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3.$

54. Let I_1 and I_2 be the intensities of the music and the mower. Then $10 \log_{10} \left(\frac{I_1}{I_0} \right) = 120$ and $10 \log_{10} \left(\frac{I_2}{I_0} \right) = 106$,
 so $\log_{10} \left(\frac{I_1}{I_2} \right) = \log_{10} \left(\frac{I_1/I_0}{I_2/I_0} \right) = \log_{10} \left(\frac{I_1}{I_0} \right) - \log_{10} \left(\frac{I_2}{I_0} \right) = 12 - 10.6 = 1.4 \Rightarrow \frac{I_1}{I_2} = 10^{1.4} \approx 25.$

55. We find I with the loudness formula from Exercise 55, substituting $I_0 = 10^{-12}$ and $L = 50$:

$$50 = 10 \log_{10} \frac{I}{10^{-12}} \Leftrightarrow 5 = \log_{10} \frac{I}{10^{-12}} \Leftrightarrow 10^5 = \frac{I}{10^{-12}} \Leftrightarrow I = 10^{-7} \text{ watt/m}^2. \text{ Now we}$$

$$\text{differentiate } L \text{ with respect to } I: L = 10 \log_{10} \frac{I}{I_0} \Rightarrow \frac{dL}{dI} = 10 \frac{1}{(I/I_0) \ln 10} \left(\frac{1}{I_0} \right) = \frac{10}{\ln 10} \left(\frac{1}{I} \right).$$

$$\text{Substituting } I = 10^{-7}, \text{ we get } L'(50) = \frac{10}{\ln 10} \left(\frac{1}{10^{-7}} \right) = \frac{10^8}{\ln 10} \approx 4.34 \times 10^7 \frac{\text{dB}}{\text{watt/m}^2}.$$

56. (a) $I(x) = I_0 a^x \Rightarrow I'(x) = I_0 (\ln a) a^x = (I_0 a^x) \ln a = I(x) \ln a$

(b) We substitute $I_0 = 8$, $a = 0.38$ and $x = 20$ into the first expression for $I'(x)$ above:

$$I'(20) = 8(\ln 0.38)(0.38)^{20} \approx -3.05 \times 10^{-8}.$$

(c) The average value of the function $I(x)$ between $x = 0$ and $x = 20$ is

$$\frac{\int_0^{20} I(x) dx}{20 - 0} = \frac{1}{20} \int_0^{20} 8(0.38)^x dx = \frac{2}{5} \left[\frac{(0.38)^x}{\ln 0.38} \right]_0^{20} = \frac{2(0.38^{20} - 1)}{5 \ln 0.38} \approx 0.41.$$

57. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$.

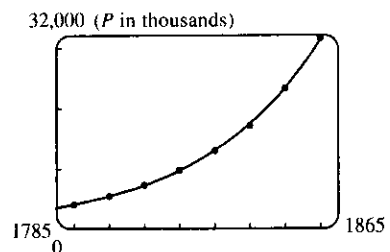
We can change this model to one with base e and exponent $\ln b$ [$b^t = e^{t \ln b}$ from precalculus mathematics or from Section 7.3]: $Q = ae^{t \ln b} = 100.012437e^{-10.005531t}$.

(b) Use $Q'(t) = ab^t \ln b$ or the calculator command `nDeriv(Y1, X, .04)` with $Y_1 = ab^x$ to get

$$Q'(0.04) \approx -670.63 \mu\text{A}. \text{ The result of Example 2 in Section 2.1 was } -670 \mu\text{A}.$$

58. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$,

where P is measured in thousands of people. The fit appears to be very good.



(b) For 1800: $m_1 = \frac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \frac{7240 - 5308}{1810 - 1800} = 193.2$.

$$\text{So } P'(1800) \approx (m_1 + m_2)/2 = 165.55 \text{ thousand people/year.}$$

For 1850: $m_1 = \frac{23,192 - 17,063}{1850 - 1840} = 612.9$, $m_2 = \frac{31,443 - 23,192}{1860 - 1850} = 825.1$.

$$\text{So } P'(1850) \approx (m_1 + m_2)/2 = 719 \text{ thousand people/year.}$$

(c) Use the calculator command `nDeriv(Y1, X, year)` with $Y_1 = ab^x$ to get

$$P'(1800) \approx 156.85 \text{ and } P'(1850) \approx 686.07. \text{ These estimates are somewhat less than the ones in part (b).}$$

(d) $P(1870) \approx 41,946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

59. Using Definition 1 and the second law of exponents for e^x , we have

$$a^{x-y} = e^{(x-y) \ln a} = e^{x \ln a - y \ln a} = \frac{e^{x \ln a}}{e^{y \ln a}} = \frac{a^x}{a^y}.$$

60. Using Definition 1, the first law of logarithms, and the first law of exponents for e^x , we have

$$(ab)^x = e^{x \ln(ab)} = e^{x(\ln a + \ln b)} = e^{x \ln a + x \ln b} = e^{x \ln a} e^{x \ln b} = a^x b^x.$$

61. Let $\log_a x = r$ and $\log_a y = s$. Then $a^r = x$ and $a^s = y$.

(a) $xy = a^r a^s = a^{r+s} \Rightarrow \log_a(xy) = r + s = \log_a x + \log_a y$

(b) $\frac{x}{y} = \frac{a^r}{a^s} = a^{r-s} \Rightarrow \log_a \frac{x}{y} = r - s = \log_a x - \log_a y$

(c) $x^y = (a^r)^y = a^{ry} \Rightarrow \log_a(x^y) = ry = y \log_a x$

62. Let $m = n/x$. Then $n = xm$, and as $n \rightarrow \infty$, $m \rightarrow \infty$.

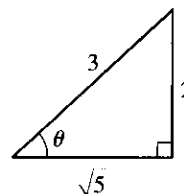
$$\text{Therefore, } \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^{mx} = \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m \right]^x = e^x \text{ by Equation 9.}$$

7.5 Inverse Trigonometric Functions

1. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
 (b) $\cos^{-1}(-1) = \pi$ since $\cos \pi = -1$ and π is in $[0, \pi]$.
2. (a) $\arctan(-1) = -\frac{\pi}{4}$ since $\tan(-\frac{\pi}{4}) = -1$ and $-\frac{\pi}{4}$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
 (b) $\csc^{-1} 2 = \frac{\pi}{6}$ since $\csc \frac{\pi}{6} = 2$ and $\frac{\pi}{6}$ is in $(0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$.
3. (a) $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$ since $\tan \frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
 (b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
4. (a) $\sec^{-1} \sqrt{2} = \frac{\pi}{4}$ since $\sec \frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$.
 (b) $\arcsin 1 = \frac{\pi}{2}$ since $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
5. (a) $\arccos(\cos 2\pi) = \arccos(1) = 0$
 (b) $\tan(\tan^{-1} 5) = 5$
6. (a) $\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$
 (b) $\cos\left(\arcsin \frac{1}{2}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

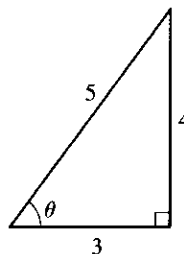
7. Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$.

Then $\tan(\sin^{-1}(\frac{2}{3})) = \tan \theta = \frac{2}{\sqrt{5}}$.



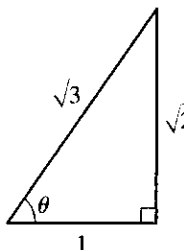
8. Let $\theta = \arccos \frac{3}{5}$.

Then $\csc(\arccos(\frac{3}{5})) = \csc \theta = \frac{5}{4}$.



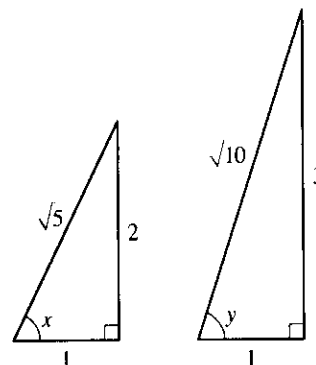
9. Let $\theta = \tan^{-1} \sqrt{2}$. Then

$$\begin{aligned} \sin(2 \tan^{-1} \sqrt{2}) &= \sin(2\theta) = 2 \sin \theta \cos \theta \\ &= 2 \left(\frac{\sqrt{2}}{\sqrt{3}} \right) \left(\frac{1}{\sqrt{3}} \right) = \frac{2\sqrt{2}}{3} \end{aligned}$$



10. Let $x = \tan^{-1} 2$ and $y = \tan^{-1} 3$. Then

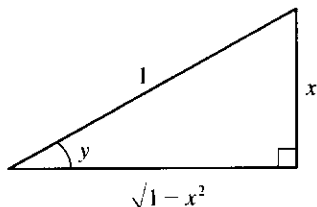
$$\begin{aligned}\cos(\tan^{-1} 2 + \tan^{-1} 3) &= \cos(x + y) = \cos x \cos y - \sin x \sin y \\&= \frac{1}{\sqrt{5}} \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}} \frac{3}{\sqrt{10}} \\&= \frac{-5}{\sqrt{50}} = \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}}\end{aligned}$$



11. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \Rightarrow \cos y \geq 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

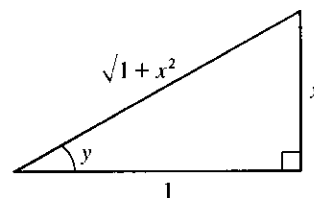
12. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle we see that

$$\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1-x^2}}.$$



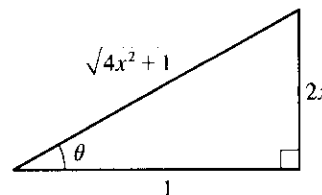
13. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}.$$

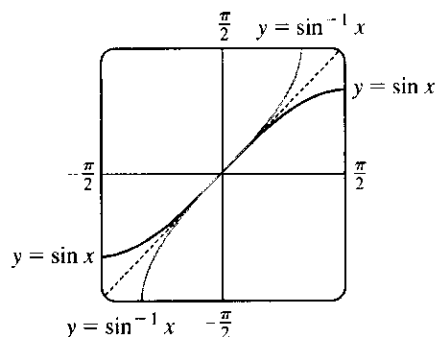


14. Let $\theta = \arctan 2x$. Then $\tan \theta = 2x$, so from the diagram we see that

$$\csc(\arctan 2x) = \csc \theta = \frac{\sqrt{4x^2 + 1}}{2x}.$$

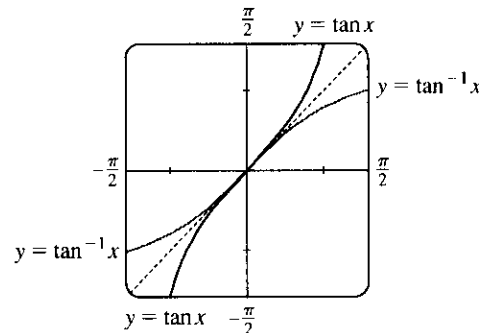


15.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

16.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

17. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \leq y \leq \pi \Rightarrow -\sin y \frac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}. \quad [\text{Note that } \sin y \geq 0 \text{ for } 0 \leq y \leq \pi.]$$

18. (a) Let $a = \sin^{-1} x$ and $b = \cos^{-1} x$. Then $\cos a = \sqrt{1 - \sin^2 a} = \sqrt{1 - x^2}$ since $\cos a \geq 0$ for $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$. Similarly, $\sin b = \sqrt{1 - x^2}$. So

$$\begin{aligned}\sin(\sin^{-1} x + \cos^{-1} x) &= \sin(a + b) = \sin a \cos b + \cos a \sin b = x \cdot x + \sqrt{1 - x^2} \sqrt{1 - x^2} \\ &= x^2 + (1 - x^2) = 1\end{aligned}$$

But $-\frac{\pi}{2} \leq \sin^{-1} x + \cos^{-1} x \leq \frac{3\pi}{2}$, and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

- (b) We differentiate $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ with respect to x , and get

$$\frac{1}{\sqrt{1-x^2}} + \frac{d}{dx} \cos^{-1} x = 0 \Rightarrow \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

19. Let $y = \cot^{-1} x$. Then $\cot y = x \Rightarrow -\csc^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\frac{1}{1 + \cot^2 y} = -\frac{1}{1 + x^2}$.

20. Let $y = \sec^{-1} x$. Then $\sec y = x$ and $y \in (0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2})$. Differentiate with respect to x :

$$\sec y \tan y \left(\frac{dy}{dx} \right) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}} = \frac{1}{x \sqrt{x^2 - 1}}. \text{ Note that}$$

$$\tan^2 y = \sec^2 y - 1 \Rightarrow \tan y = \sqrt{\sec^2 y - 1} \text{ since } \tan y > 0 \text{ when } 0 < y < \frac{\pi}{2} \text{ or } \pi < y < \frac{3\pi}{2}.$$

21. Let $y = \csc^{-1} x$. Then $\csc y = x \Rightarrow -\csc y \cot y \frac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\csc y \cot y} = -\frac{1}{\csc y \sqrt{\csc^2 y - 1}} = -\frac{1}{x \sqrt{x^2 - 1}}. \text{ Note that } \cot y \geq 0 \text{ on the domain of } \csc^{-1} x.$$

22. $y = \sqrt{\tan^{-1} x} = (\tan^{-1} x)^{1/2} \Rightarrow$

$$y' = \frac{1}{2} (\tan^{-1} x)^{-1/2} \cdot \frac{d}{dx} (\tan^{-1} x) = \frac{1}{2 \sqrt{\tan^{-1} x}} \cdot \frac{1}{1 + x^2} = \frac{1}{2 \sqrt{\tan^{-1} x} (1 + x^2)}$$

23. $y = \tan^{-1} \sqrt{x} \Rightarrow y' = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx} (\sqrt{x}) = \frac{1}{1 + x} \left(\frac{1}{2} x^{-1/2} \right) = \frac{1}{2 \sqrt{x} (1 + x)}$

24. $h(x) = \sqrt{1 - x^2} \arcsin x \Rightarrow$

$$h'(x) = \sqrt{1 - x^2} \cdot \frac{1}{\sqrt{1 - x^2}} + \arcsin x \left[\frac{1}{2} (1 - x^2)^{-1/2} (-2x) \right] = 1 - \frac{x \arcsin x}{\sqrt{1 - x^2}}$$

25. $y = \sin^{-1}(2x + 1) \Rightarrow$

$$y' = \frac{1}{\sqrt{1 - (2x + 1)^2}} \cdot \frac{d}{dx} (2x + 1) = \frac{1}{\sqrt{1 - (4x^2 + 4x + 1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2 - 4x}} = \frac{1}{\sqrt{-x^2 - x}}$$

26. $f(x) = x \ln(\arctan x) \Rightarrow$

$$f'(x) = x \cdot \frac{1}{\arctan x} \cdot \frac{1}{1 + x^2} + \ln(\arctan x) \cdot 1 = \frac{x}{(1 + x^2) \arctan x} + \ln(\arctan x)$$

27. $H(x) = (1 + x^2) \arctan x \Rightarrow H'(x) = (1 + x^2) \frac{1}{1 + x^2} + (\arctan x)(2x) = 1 + 2x \arctan x$

28. $h(t) = e^{\sec^{-1} t} \Rightarrow h'(t) = e^{\sec^{-1} t} \frac{d}{dt} (\sec^{-1} t) = \frac{e^{\sec^{-1} t}}{t \sqrt{t^2 - 1}}$

29. $y = \cos^{-1}(e^{2x}) \Rightarrow y' = -\frac{1}{\sqrt{1 - (e^{2x})^2}} \cdot \frac{d}{dx} (e^{2x}) = -\frac{2e^{2x}}{\sqrt{1 - e^{4x}}}$

30. $y = x \cos^{-1} x - \sqrt{1 - x^2} \Rightarrow y' = \cos^{-1} x - \frac{x}{\sqrt{1 - x^2}} + \frac{x}{\sqrt{1 - x^2}} = \cos^{-1} x$

$$31. y = \arctan(\cos \theta) \Rightarrow y' = \frac{1}{1 + (\cos \theta)^2} (-\sin \theta) = -\frac{\sin \theta}{1 + \cos^2 \theta}$$

$$\begin{aligned}
 32. y = \tan^{-1}(x - \sqrt{x^2 + 1}) &\Rightarrow \\
 y' &= \frac{1}{1 + (x - \sqrt{x^2 + 1})^2} \left(1 - \frac{x}{\sqrt{x^2 + 1}}\right) = \frac{1}{1 + x^2 - 2x\sqrt{x^2 + 1} + x^2 + 1} \left(\frac{\sqrt{x^2 + 1} - x}{\sqrt{x^2 + 1}}\right) \\
 &= \frac{\sqrt{x^2 + 1} - x}{2(1 + x^2 - x\sqrt{x^2 + 1})\sqrt{x^2 + 1}} = \frac{\sqrt{x^2 + 1} - x}{2[\sqrt{x^2 + 1}(1 + x^2) - x(x^2 + 1)]} \\
 &= \frac{\sqrt{x^2 + 1} - x}{2[(1 + x^2)(\sqrt{x^2 + 1} - x)]} = \frac{1}{2(1 + x^2)}
 \end{aligned}$$

$$\begin{aligned}
 33. h(t) &= \cot^{-1}(t) + \cot^{-1}(1/t) \Rightarrow \\
 h'(t) &= -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2} \cdot \frac{d}{dt} \frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1} \cdot \left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.
 \end{aligned}$$

Note that this makes sense because $h(t) = \frac{\pi}{2}$ for $t > 0$ and $h(t) = -\frac{\pi}{2}$ for $t < 0$.

$$\begin{aligned}
 34. y &= \tan^{-1}\left(\frac{x}{a}\right) + \ln \sqrt{\frac{x-a}{x+a}} = \tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} \ln(x-a) - \frac{1}{2} \ln(x+a) \Rightarrow \\
 y' &= \frac{a}{x^2+a^2} + \frac{1/2}{x-a} - \frac{1/2}{x+a} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2} = \frac{2ax^2}{x^4-a^4}
 \end{aligned}$$

$$\begin{aligned}
 35. y &= \arccos\left(\frac{b+a\cos x}{a+b\cos x}\right) \Rightarrow \\
 y' &= -\frac{1}{\sqrt{1-\left(\frac{b+a\cos x}{a+b\cos x}\right)^2}} \frac{(a+b\cos x)(-a\sin x) - (b+a\cos x)(-b\sin x)}{(a+b\cos x)^2} \\
 &= \frac{1}{\sqrt{a^2+b^2\cos^2 x - b^2 - a^2\cos^2 x}} \frac{(a^2-b^2)\sin x}{|a+b\cos x|} \\
 &= \frac{1}{\sqrt{a^2-b^2}\sqrt{1-\cos^2 x}} \frac{(a^2-b^2)\sin x}{|a+b\cos x|} = \frac{\sqrt{a^2-b^2}\sin x}{|a+b\cos x||\sin x|}
 \end{aligned}$$

But $0 \leq x \leq \pi$, so $|\sin x| = \sin x$. Also $a > b > 0 \Rightarrow b\cos x \geq -b > -a$, so $a+b\cos x > 0$.

Thus $y' = \frac{\sqrt{a^2-b^2}}{a+b\cos x}$.

$$\begin{aligned}
 36. f(x) &= \arcsin(e^x) \Rightarrow f'(x) = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}. \\
 \text{Domain}(f) &= \{x \mid -1 \leq e^x \leq 1\} = \{x \mid 0 < e^x \leq 1\} = (-\infty, 0]. \\
 \text{Domain}(f') &= \{x \mid 1 - e^{2x} > 0\} = \{x \mid e^{2x} < 1\} = \{x \mid 2x < 0\} = (-\infty, 0).
 \end{aligned}$$

$$\begin{aligned}
 37. g(x) &= \cos^{-1}(3-2x) \Rightarrow g'(x) = -\frac{1}{\sqrt{1-(3-2x)^2}} (-2) = \frac{2}{\sqrt{1-(3-2x)^2}}. \\
 \text{Domain}(g) &= \{x \mid -1 \leq 3-2x \leq 1\} = \{x \mid -4 \leq -2x \leq -2\} = \{x \mid 2 \geq x \geq 1\} = [1, 2]. \\
 \text{Domain}(g') &= \{x \mid 1 - (3-2x)^2 > 0\} = \{x \mid (3-2x)^2 < 1\} = \{x \mid |3-2x| < 1\} \\
 &= \{x \mid -1 < 3-2x < 1\} = \{x \mid -4 < -2x < -2\} = \{x \mid 2 > x > 1\} = (1, 2)
 \end{aligned}$$

$$38. \tan^{-1}(xy) = 1 + x^2y \Rightarrow \frac{1}{1+x^2y^2}(xy' + y \cdot 1) = 0 + x^2y' + 2xy \Rightarrow$$

$$y' \left(\frac{x}{1+x^2y^2} - x^2 \right) = 2xy - \frac{y}{1+x^2y^2} \Rightarrow$$

$$y' = \frac{2xy - \frac{y}{1+x^2y^2}}{\frac{x}{1+x^2y^2} - x^2} = \frac{2xy(1+x^2y^2) - y}{x - x^2(1+x^2y^2)} = \frac{y(-1-2x-2x^3y^2)}{x(1-x-x^3y^2)}.$$

$$39. g(x) = x \sin^{-1}\left(\frac{x}{4}\right) + \sqrt{16-x^2} \Rightarrow g'(x) = \sin^{-1}\left(\frac{x}{4}\right) + \frac{x}{4\sqrt{1-(x/4)^2}} - \frac{x}{\sqrt{16-x^2}} = \sin^{-1}\left(\frac{x}{4}\right) \Rightarrow$$

$$g'(2) = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

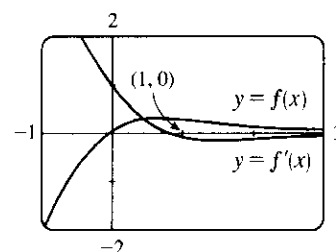
$$40. y = 3 \arccos \frac{x}{2} \Rightarrow y' = 3 \left[-\frac{1}{\sqrt{1-(x/2)^2}} \right] \left(\frac{1}{2} \right), \text{ so at } (1, \pi), y' = -\frac{3}{2\sqrt{1-\frac{1}{4}}} = -\sqrt{3}. \text{ An equation of}$$

the tangent line is $y - \pi = -\sqrt{3}(x - 1)$, or $y = -\sqrt{3}x + \pi + \sqrt{3}$.

$$41. f(x) = e^{-x} \arctan x \Rightarrow$$

$$f'(x) = \frac{e^{-x}}{1+x^2} - e^{-x} \arctan x. \text{ The answer is reasonable}$$

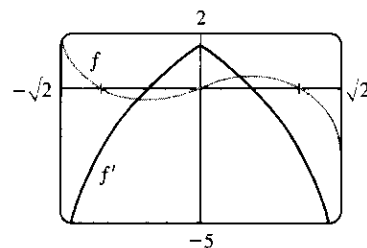
since f' is positive where f is increasing and f' is negative where f is decreasing.



$$42. f(x) = x \arcsin(1-x^2) \Rightarrow$$

$$f'(x) = x \left[\frac{-2x}{\sqrt{1-(1-x^2)^2}} \right] + \arcsin(1-x^2) \cdot 1$$

$$= \arcsin(1-x^2) - \frac{2x^2}{\sqrt{2x^2-x^4}}$$



This is reasonable because the graphs show that f is increasing when f' is positive, and that f has an inflection point when f' changes from increasing to decreasing.

$$43. \lim_{x \rightarrow -1^+} \sin^{-1} x = \sin^{-1}(-1) = -\frac{\pi}{2}$$

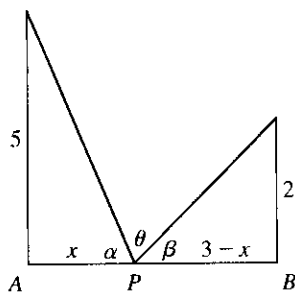
$$44. \text{ Let } t = \frac{1+x^2}{1+2x^2}. \text{ As } x \rightarrow \infty, t = \frac{1+x^2}{1+2x^2} = \frac{1/x^2+1}{1/x^2+2} \rightarrow \frac{1}{2}.$$

$$\lim_{x \rightarrow \infty} \arccos\left(\frac{1+x^2}{1+2x^2}\right) = \lim_{t \rightarrow 1/2} \arccos t = \arccos \frac{1}{2} = \frac{\pi}{3}.$$

$$45. \text{ Let } t = e^x. \text{ As } x \rightarrow \infty, t \rightarrow \infty. \lim_{x \rightarrow \infty} \arctan(e^x) = \lim_{t \rightarrow \infty} \arctan t = \frac{\pi}{2} \text{ by (8).}$$

$$46. \text{ Let } t = \ln x. \text{ As } x \rightarrow 0^+, t \rightarrow -\infty. \lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) = \lim_{t \rightarrow -\infty} \tan^{-1} t = -\frac{\pi}{2} \text{ by (8).}$$

47.



From the figure, $\tan \alpha = \frac{5}{x}$ and $\tan \beta = \frac{2}{3-x}$. Since

$$\alpha + \beta + \theta = 180^\circ = \pi, \theta = \pi - \tan^{-1}\left(\frac{5}{x}\right) - \tan^{-1}\left(\frac{2}{3-x}\right) \Rightarrow$$

$$\begin{aligned} \frac{d\theta}{dx} &= -\frac{1}{1 + \left(\frac{5}{x}\right)^2} \left(-\frac{5}{x^2}\right) - \frac{1}{1 + \left(\frac{2}{3-x}\right)^2} \left[\frac{2}{(3-x)^2}\right] \\ &= \frac{x^2}{x^2 + 25} \cdot \frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2 + 4} \cdot \frac{2}{(3-x)^2}. \end{aligned}$$

$$\text{Now } \frac{d\theta}{dx} = 0 \Rightarrow \frac{5}{x^2 + 25} = \frac{2}{x^2 - 6x + 13} \Rightarrow 2x^2 + 50 = 5x^2 - 30x + 65 \Rightarrow$$

$3x^2 - 30x + 15 = 0 \Rightarrow x^2 - 10x + 5 = 0 \Rightarrow x = 5 \pm 2\sqrt{5}$. We reject the root with the + sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

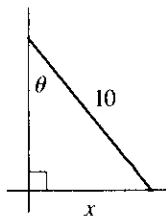
48. Let x be the distance from the observer to the wall. Then, from the given figure,

$$\theta = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right), x > 0 \Rightarrow$$

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{1}{1 + [(h+d)/x]^2} \left[-\frac{h+d}{x^2}\right] - \frac{1}{1 + (d/x)^2} \left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2 + (h+d)^2} + \frac{d}{x^2 + d^2} \\ &= \frac{d[x^2 + (h+d)^2] - (h+d)(x^2 + d^2)}{[x^2 + (h+d)^2](x^2 + d^2)} = \frac{h^2d + hd^2 - hx^2}{[x^2 + (h+d)^2](x^2 + d^2)} = 0 \Leftrightarrow \end{aligned}$$

$hx^2 = h^2d + hd^2 \Leftrightarrow x^2 = hd + d^2 \Leftrightarrow x = \sqrt{d(h+d)}$. Since $d\theta/dx > 0$ for all $x < \sqrt{d(h+d)}$ and $d\theta/dx < 0$ for all $x > \sqrt{d(h+d)}$, the absolute maximum occurs when $x = \sqrt{d(h+d)}$.

49.

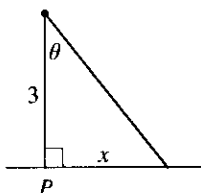


$$\frac{dx}{dt} = 2 \text{ ft/s}, \sin \theta = \frac{x}{10} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{10}\right), \frac{d\theta}{dx} = \frac{1/10}{\sqrt{1 - (x/10)^2}},$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \frac{dx}{dt} = \frac{1/10}{\sqrt{1 - (x/10)^2}} (2) \text{ rad/s},$$

$$\left. \frac{d\theta}{dt} \right|_{x=6} = \frac{2/10}{\sqrt{1 - (6/10)^2}} \text{ rad/s} = \frac{1}{4} \text{ rad/s}$$

50.



$$\frac{d\theta}{dt} = 4 \text{ rev/min} = 8\pi \cdot 60 \text{ rad/h. From the diagram, we see that } \tan \theta = \frac{x}{3}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{x}{3}\right). \text{ Thus, } 8\pi \cdot 60 = \frac{d\theta}{dt} = \frac{d\theta}{dx} \frac{dx}{dt} = \frac{1/3}{1 + (x/3)^2} \frac{dx}{dt}.$$

$$\text{So } \frac{dx}{dt} = 8\pi \cdot 60 \cdot 3 \left[1 + \left(\frac{x}{3}\right)^2\right] \text{ km/h, and at } x = 1,$$

$$\frac{dx}{dt} = 8\pi \cdot 60 \cdot 3 \left[1 + \frac{1}{9}\right] \text{ km/h} = 1600\pi \text{ km/h.}$$

51. $y = f(x) = \sin^{-1}(x/(x+1))$ A. $D = \{x \mid -1 \leq x/(x+1) \leq 1\}$. For $x > -1$ we have

$$-x-1 \leq x \leq x+1 \Leftrightarrow 2x \geq -1 \Leftrightarrow x \geq -\frac{1}{2}, \text{ so } D = [-\frac{1}{2}, \infty).$$

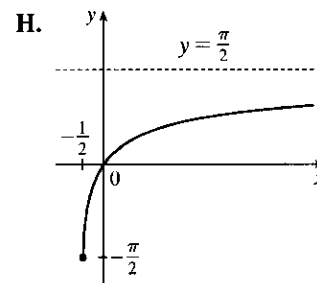
B. Intercepts are 0 C. No symmetry D. $\lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{x}{x+1}\right) = \lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{1}{1+1/x}\right) = \sin^{-1} 1 = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a HA.

$$\text{E. } f'(x) = \frac{1}{\sqrt{1 - [x/(x+1)]^2}} \cdot \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)\sqrt{2x+1}} > 0,$$

so f is increasing on $(-\frac{1}{2}, \infty)$. F. No local maximum or minimum,

$$f(-\frac{1}{2}) = \sin^{-1}(-1) = -\frac{\pi}{2} \text{ is an absolute minimum}$$

$$\begin{aligned} \text{G. } f''(x) &= -\frac{\sqrt{2x+1} + (x+1)/\sqrt{2x+1}}{(x+1)^2(2x+1)} \\ &= -\frac{3x+2}{(x+1)^2(2x+1)^{3/2}} < 0 \text{ on } D, \text{ so } f \text{ is CD on } (-\frac{1}{2}, \infty). \end{aligned}$$



52. $y = f(x) = \tan^{-1}\left(\frac{x-1}{x+1}\right)$ A. $D = \{x \mid x \neq -1\}$

B. x -intercept = 1, y -intercept = $f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$ C. No symmetry

D. $\lim_{x \rightarrow \pm\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \lim_{x \rightarrow \pm\infty} \tan^{-1}\left(\frac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA. Also

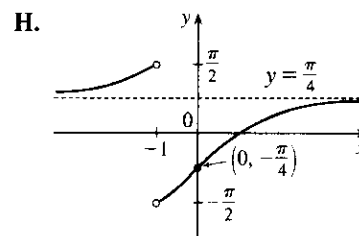
$$\lim_{x \rightarrow -1^+} \tan^{-1}\left(\frac{x-1}{x+1}\right) = -\frac{\pi}{2} \text{ and } \lim_{x \rightarrow -1^-} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}.$$

$$\begin{aligned} \text{E. } f'(x) &= \frac{1}{1 + [(x-1)/(x+1)]^2} \cdot \frac{(x+1) - (x-1)}{(x+1)^2} \\ &= \frac{2}{(x+1)^2 + (x-1)^2} = \frac{1}{x^2 + 1} > 0 \end{aligned}$$

so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. F. No extreme values

G. $f''(x) = -2x/(x^2 + 1)^2 > 0 \Leftrightarrow x < 0$, so f is CU on $(-\infty, -1)$ and $(-1, 0)$, and CD on $(0, \infty)$.

IP at $(0, -\frac{\pi}{4})$



53. $y = f(x) = x - \tan^{-1} x$ A. $D = \mathbb{R}$ B. Intercepts are 0 C. $f(-x) = -f(x)$, so the curve is symmetric about the origin. D. $\lim_{x \rightarrow \infty} (x - \tan^{-1} x) = \infty$ and $\lim_{x \rightarrow -\infty} (x - \tan^{-1} x) = -\infty$, no HA.

But $f(x) - (x - \frac{\pi}{2}) = -\tan^{-1} x + \frac{\pi}{2} \rightarrow 0$ as $x \rightarrow \infty$, and

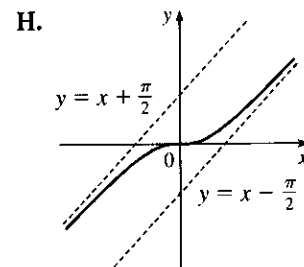
$f(x) - (x + \frac{\pi}{2}) = -\tan^{-1} x - \frac{\pi}{2} \rightarrow 0$ as $x \rightarrow -\infty$, so $y = x \pm \frac{\pi}{2}$ are

slant asymptotes. E. $f'(x) = 1 - \frac{1}{x^2 + 1} = \frac{x^2}{x^2 + 1} > 0$, so f is

increasing on $\mathbb{R}$. F. No extrema

$$\text{G. } f''(x) = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2} > 0 \Leftrightarrow x > 0, \text{ so}$$

f is CU on $(0, \infty)$, CD on $(-\infty, 0)$. IP at $(0, 0)$.



54. $y = \tan^{-1}(\ln x)$ A. $D = (0, \infty)$ B. No y -intercept, x -intercept when $\tan^{-1}(\ln x) = 0 \Leftrightarrow \ln x = 0$

$$\Leftrightarrow x = 1. \text{ C. No symmetry D. } \lim_{x \rightarrow \infty} \tan^{-1}(\ln x) = \frac{\pi}{2},$$

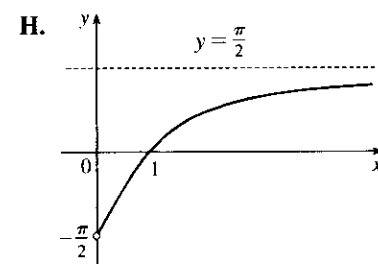
so $y = \frac{\pi}{2}$ is a HA. Also $\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}$.

$$\text{E. } f'(x) = \frac{1}{x[1 + (\ln x)^2]} > 0, \text{ so } f \text{ is increasing on } (0, \infty).$$

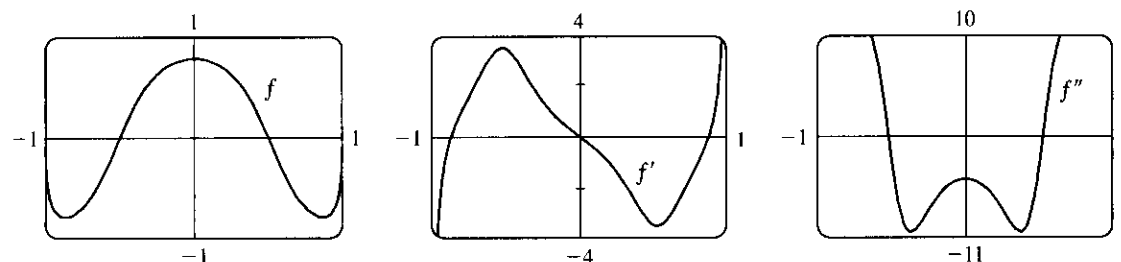
F. No maximum or minimum

$$\text{G. } f''(x) = \frac{-[1 + (\ln x)^2 + x(2 \ln x/x)]}{x^2[1 + (\ln x)^2]^2} = -\frac{(1 + \ln x)^2}{x^2[1 + (\ln x)^2]^2} < 0,$$

so f is CD on $(0, \infty)$.



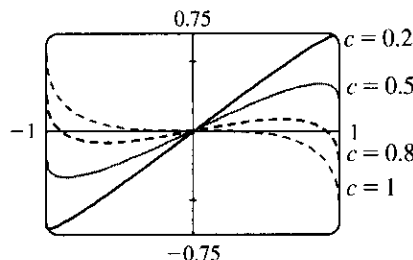
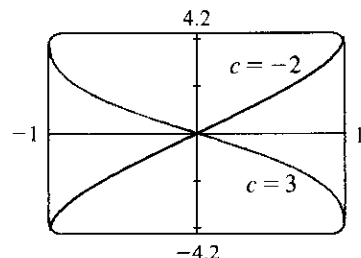
55. $f(x) = \arctan(\cos(3 \arcsin x))$. We use a CAS to compute f' and f'' , and to graph f , f' , and f'' :



From the graph of f' , it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$. From the graph of f'' , it appears that there are inflection points at $x = \pm 0.52$.

56. First note that the function $f(x) = x - c \sin^{-1} x$ is only defined on the interval $[-1, 1]$, since $\sin^{-1}$ is only defined on that interval. We differentiate to get $f'(x) = 1 - c/\sqrt{1-x^2}$. Now if $c \leq 0$, then $f'(x) \geq 1$, so there is no

extremum and f is increasing on its domain. If $c > 1$, then $f'(x) < 0$, so there is no local extremum and f is decreasing on its domain, and if $c = 1$, then there is still no extremum, since $f'(x)$ does not change sign at $x = 0$. So we can only have local extrema if $0 < c < 1$. In this case, f is increasing where $f'(x) > 0 \Leftrightarrow \sqrt{1-x^2} > c \Leftrightarrow |x| < \sqrt{1-c^2}$, and decreasing where $\sqrt{1-c^2} < |x| \leq 1$. f has a maximum at $x = \sqrt{1-c^2}$ and a minimum at $x = -\sqrt{1-c^2}$.



57. $f(x) = 2x + 5(1-x^2)^{-1/2} = 2x + 5/\sqrt{1-x^2} \Rightarrow F(x) = x^2 + 5 \sin^{-1} x + C$

58. $f'(x) = 4 - 3(1+x^2)^{-1} \Rightarrow f(x) = 4x - 3 \tan^{-1} x + C \Rightarrow f(\frac{\pi}{4}) = \pi - 3 + C = 0 \Rightarrow C = 3 - \pi$,
so $f(x) = 4x - 3 \tan^{-1} x + 3 - \pi$.

59. $\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}} dt = 6 \int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}} dt = 6 [\sin^{-1} t]_{1/2}^{\sqrt{3}/2} = 6 \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) - \sin^{-1} \left(\frac{1}{2} \right) \right]$
 $= 6 \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = 6 \left(\frac{\pi}{6} \right) = \pi$

60. $\int_0^1 \frac{4}{t^2+1} dt = 4 \int_0^1 \frac{1}{1+t^2} dt = 4 [\tan^{-1} t]_0^1 = 4 (\tan^{-1} 1 - \tan^{-1} 0) = 4 \left(\frac{\pi}{4} - 0 \right) = \pi$

61. Let $u = 4x$. Then $du = 4 dx$, so

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{4} \int_0^{\sqrt{3}} \frac{1}{1+u^2} du = \frac{1}{4} [\tan^{-1} u]_0^{\sqrt{3}} = \frac{1}{4} (\tan^{-1} \sqrt{3} - \tan^{-1} 0) = \frac{1}{4} \left(\frac{\pi}{3} - 0 \right) = \frac{\pi}{12}.$$

62. Let $u = 2t$. Then $\sqrt{1-4t^2} = \sqrt{1-u^2}$ and $du = 2 dt$, so

$$\int \frac{dt}{\sqrt{1-4t^2}} = \int \frac{\frac{1}{2} du}{\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} (2t) + C.$$

63. Let $u = \sin^{-1} x$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$, so $\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} u du = \left[\frac{u^2}{2} \right]_0^{\pi/6} = \frac{1}{2} \left(\frac{\pi}{6} \right)^2 = \frac{\pi^2}{72}$.

64. Let $u = -\cos x$. Then $du = \sin x dx$, so

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx = \int_{-1}^0 \frac{1}{1+u^2} du = [\tan^{-1} u]_{-1}^0 = \tan^{-1} 0 - \tan^{-1} (-1) = 0 - \left(-\frac{\pi}{4} \right) = \frac{\pi}{4}.$$

65. $\int \frac{x+9}{x^2+9} dx = \int \frac{x}{x^2+9} dx + 9 \int \frac{1}{x^2+9} dx = \frac{1}{2} \ln(x^2+9) + 3 \tan^{-1} \frac{x}{3} + C$

(Let $u = x^2 + 9$ in the first integral; use Equation 14 in the second.)

66. Let $u = \tan^{-1} x$. Then $du = dx/(1+x^2)$, so $\int \frac{\tan^{-1} x}{1+x^2} dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(\tan^{-1} x)^2 + C$.

67. Let $u = t^3$. Then $du = 3t^2 dt$ and $\int \frac{t^2}{\sqrt{1-t^6}} dt = \int \frac{\frac{1}{3} du}{\sqrt{1-u^2}} = \frac{1}{3} \sin^{-1} u + C = \frac{1}{3} \sin^{-1}(t^3) + C$.

68. Let $u = \frac{1}{2}x$. Then $du = \frac{1}{2} dx \Rightarrow$

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^2-4}} &= \int \frac{dx}{2x\sqrt{(x/2)^2-1}} = \int \frac{2 du}{4u\sqrt{u^2-1}} = \frac{1}{2} \int \frac{du}{u\sqrt{u^2-1}} \\ &= \frac{1}{2} \sec^{-1} u + C = \frac{1}{2} \sec^{-1}\left(\frac{1}{2}x\right) + C \end{aligned}$$

69. Let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$ and $\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2 du}{1+u^2} = 2 \tan^{-1} u + C = 2 \tan^{-1} \sqrt{x} + C$.

70. Let $u = e^{2x}$. Then $du = 2e^{2x} dx \Rightarrow \int \frac{e^{2x} dx}{\sqrt{1-e^{4x}}} = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1}(e^{2x}) + C$

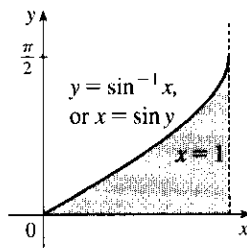
71. Let $u = x/a$. Then $du = dx/a$, so

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{a\sqrt{1-(x/a)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} \frac{x}{a} + C.$$

72. We use the disk method: $A = \int_0^2 \pi \left[\frac{1}{\sqrt{x^2+4}} \right]^2 dx = \pi \int_0^2 \frac{1}{x^2+4} dx$. By Formula 14, this is equal to

$$\pi \left[\frac{1}{2} \tan^{-1}(x/2) \right]_0^2 = \frac{\pi}{2} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi^2}{8}.$$

73.



The integral represents the area below the curve $y = \sin^{-1} x$ on the interval $x \in [0, 1]$. The bounding curves are $y = \sin^{-1} x \Leftrightarrow x = \sin y$, $y = 0$ and $x = 1$. We see that y ranges between $\sin^{-1} 0 = 0$ and $\sin^{-1} 1 = \frac{\pi}{2}$. So we have to integrate the function $x = 1 - \sin y$ between $y = 0$ and $y = \frac{\pi}{2}$:

$$\int_0^1 \sin^{-1} x dx = \int_0^{\pi/2} (1 - \sin y) dy = \left(\frac{\pi}{2} + \cos \frac{\pi}{2} \right) - (0 + \cos 0) = \frac{\pi}{2} - 1.$$

74. Let $a = \arctan x$ and $b = \arctan y$. Then by the addition formula for the tangent (see Reference Page 2 in the

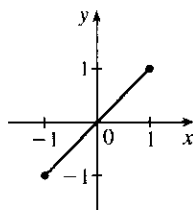
$$\text{textbook}), \tan(a+b) = \frac{\tan a + \tan b}{1 - (\tan a)(\tan b)} = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} \Rightarrow \tan(a+b) = \frac{x+y}{1-xy}$$

$$\Rightarrow \arctan x + \arctan y = a+b = \arctan\left(\frac{x+y}{1-xy}\right), \text{ since } -\frac{\pi}{2} < \arctan x + \arctan y < \frac{\pi}{2}.$$

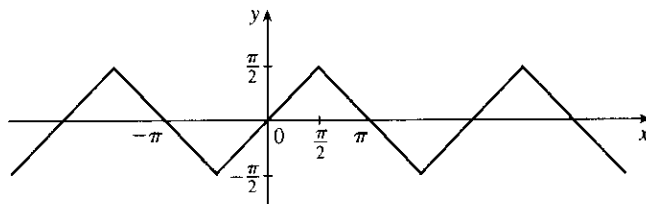
75. (a) $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan\left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}}\right) = \arctan 1 = \frac{\pi}{4}$

$$\begin{aligned} \text{(b) } 2 \arctan \frac{1}{3} + \arctan \frac{1}{7} &= \left(\arctan \frac{1}{3} + \arctan \frac{1}{3} \right) + \arctan \frac{1}{7} = \arctan\left(\frac{\frac{1}{3} + \frac{1}{3}}{1 - \frac{1}{3} \cdot \frac{1}{3}}\right) + \arctan \frac{1}{7} \\ &= \arctan \frac{3}{4} + \arctan \frac{1}{7} = \arctan\left(\frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \cdot \frac{1}{7}}\right) = \arctan 1 = \frac{\pi}{4} \end{aligned}$$

76. (a) $f(x) = \sin(\sin^{-1} x)$

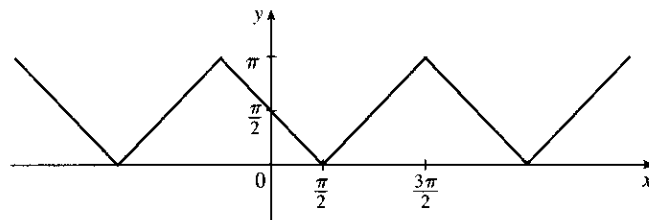


(b) $g(x) = \sin^{-1}(\sin x)$



(c) $g'(x) = \frac{d}{dx} \sin^{-1}(\sin x) = \frac{1}{\sqrt{1 - \sin^2 x}} \cos x = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|}$

(d) $h(x) = \cos^{-1}(\sin x)$, so $h'(x) = -\frac{\cos x}{\sqrt{1 - \sin^2 x}} = -\frac{\cos x}{|\cos x|}$



77. Let $f(x) = 2 \sin^{-1} x - \cos^{-1}(1 - 2x^2)$. Then

$$f'(x) = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{1-(1-2x^2)^2}} = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{2x\sqrt{1-x^2}} = 0 \quad [\text{since } x \geq 0]$$

Thus $f'(x) = 0$ for all $x \in [0, 1]$. Thus $f(x) = C$. To find C let $x = 0$. Thus $2 \sin^{-1}(0) - \cos^{-1}(1) = 0 = C$.

Therefore we see that $f(x) = 2 \sin^{-1} x - \cos^{-1}(1 - 2x^2) = 0 \Rightarrow 2 \sin^{-1} x = \cos^{-1}(1 - 2x^2)$.

78. Let $f(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) - 2 \tan^{-1} \sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0, \infty)$. Thus

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1) - (x-1)}{(x+1)^2} - \frac{2}{1+x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}(x+1)} - \frac{1}{\sqrt{x}(x+1)} = 0$$

Then $f(x) = C$. To find C , we let $x = 0 \Rightarrow \sin^{-1}(-1) - 2 \tan^{-1}(0) + \frac{\pi}{2} = C \Rightarrow -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$.

Thus, $f(x) = 0 \Rightarrow \sin^{-1}\left(\frac{x-1}{x+1}\right) = 2 \tan^{-1} \sqrt{x} - \frac{\pi}{2}$.

79. $y = \sec^{-1} x \Rightarrow \sec y = x \Rightarrow \sec y \tan y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}$. Now

$\tan^2 y = \sec^2 y - 1 = x^2 - 1$, so $\tan y = \pm \sqrt{x^2 - 1}$. For $y \in [0, \frac{\pi}{2})$, $x \geq 1$, so $\sec y = x = |x|$ and $\tan y \geq 0$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$. For $y \in (\frac{\pi}{2}, \pi]$, $x \leq -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2-1} \Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x(-\sqrt{x^2-1})} = \frac{1}{(-x)\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

- 80.** (a) Since $|\arctan(1/x)| < \frac{\pi}{2}$, we have $0 \leq |x \arctan(1/x)| \leq \frac{\pi}{2} |x| \rightarrow 0$ as $x \rightarrow 0$. So, by the Squeeze Theorem,

$\lim_{x \rightarrow 0} f(x) = 0 = f(0)$, so f is continuous at 0.

- (b) Here $\frac{f(x) - f(0)}{x - 0} = \frac{x \arctan(1/x) - 0}{x} = \arctan\left(\frac{1}{x}\right)$. So (see Exercise 44 in

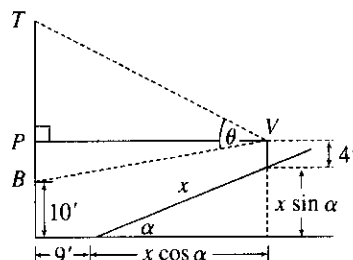
Section 3.2 for a discussion of left- and right-hand derivatives)

$$f'_-(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \arctan\left(\frac{1}{x}\right) = \lim_{y \rightarrow -\infty} \arctan y = -\frac{\pi}{2}, \text{ while}$$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \arctan\left(\frac{1}{x}\right) = \lim_{y \rightarrow \infty} \arctan y = \frac{\pi}{2}. \text{ So } f'(0) \text{ does not exist.}$$

APPLIED PROJECT Where to Sit at the Movies

1.



$|VP| = 9 + x \cos \alpha$, $|PT| = 35 - (4 + x \sin \alpha) = 31 - x \sin \alpha$, and

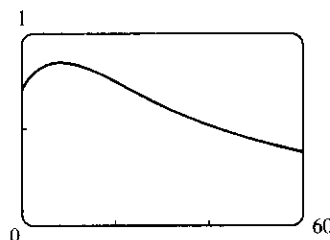
$|PB| = (4 + x \sin \alpha) - 10 = x \sin \alpha - 6$. So using the Pythagorean Theorem, we have

$$|VT| = \sqrt{|VP|^2 + |PT|^2} = \sqrt{(9 + x \cos \alpha)^2 + (31 - x \sin \alpha)^2} = a, \text{ and}$$

$|VB| = \sqrt{|VP|^2 + |PB|^2} = \sqrt{(9 + x \cos \alpha)^2 + (x \sin \alpha - 6)^2} = b$. Using the Law of Cosines on $\triangle VBT$, we

$$\text{get } 25^2 = a^2 + b^2 - 2ab \cos \theta \Leftrightarrow \cos \theta = \frac{a^2 + b^2 - 625}{2ab} \Leftrightarrow \theta = \arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right), \text{ as required.}$$

2. From the graph of θ , it appears that the value of x which maximizes θ is $x \approx 8.25$ ft. Assuming that the first row is at $x = 0$, the row closest to this value of x is the fourth row, at $x = 9$ ft, and from the graph, the viewing angle in this row seems to be about 0.85 radians, or about 49° .



3. With a CAS, we type in the definition of θ , substitute in the proper values of a and b in terms of x and $\alpha = 20^\circ = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical root finder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 8.253062$, as approximated in Problem 2.
4. From the graph in Problem 2, it seems that the average value of the function on the interval $[0, 60]$ is about 0.6. We can use a CAS to approximate $\frac{1}{60} \int_0^{60} \theta(x) dx \approx 0.625 \approx 36^\circ$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(60) \approx 0.38$ and, from Problem 2, the maximum value is about 0.85.

7.6 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^0) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1 + 1) = 1$
2. (a) $\tanh 0 = \frac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \frac{e^1 - e^{-1}}{e^1 + e^{-1}} = \frac{e^2 - 1}{e^2 + 1} \approx 0.76159$
3. (a) $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{e^{\ln 2} - (e^{\ln 2})^{-1}}{2} = \frac{2 - 2^{-1}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$
(b) $\sinh 2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$
4. (a) $\cosh 3 = \frac{1}{2}(e^3 + e^{-3}) \approx 10.06766$ (b) $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$
5. (a) $\operatorname{sech} 0 = \frac{1}{\cosh 0} = \frac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.
6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$
(b) Using Equation 3, we have $\sinh^{-1} 1 = \ln(1 + \sqrt{1^2 + 1}) = \ln(1 + \sqrt{2}) \approx 0.88137$.
7. $\sinh(-x) = \frac{1}{2}[e^{-x} - e^{-(-x)}] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$
8. $\cosh(-x) = \frac{1}{2}[e^{-x} + e^{-(-x)}] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$
9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$
10. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$
11. $\sinh x \cosh y + \cosh x \sinh y = \left[\frac{1}{2}(e^x - e^{-x})\right]\left[\frac{1}{2}(e^y + e^{-y})\right] + \left[\frac{1}{2}(e^x + e^{-x})\right]\left[\frac{1}{2}(e^y - e^{-y})\right]$
 $= \frac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})]$
 $= \frac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \frac{1}{2}[e^{x+y} - e^{-(x+y)}] = \sinh(x+y)$

$$\begin{aligned} 12. \cosh x \cosh y + \sinh x \sinh y &= \left[\frac{1}{2}(e^x + e^{-x}) \right] \left[\frac{1}{2}(e^y + e^{-y}) \right] + \left[\frac{1}{2}(e^x - e^{-x}) \right] \left[\frac{1}{2}(e^y - e^{-y}) \right] \\ &= \frac{1}{4} [(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})] \\ &= \frac{1}{4} (2e^{x+y} + 2e^{-x-y}) = \frac{1}{2} [e^{x+y} + e^{-(x+y)}] = \cosh(x+y) \end{aligned}$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \Leftrightarrow \coth^2 x - 1 = \operatorname{csch}^2 x.$$

$$\begin{aligned} 14. \tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \frac{\frac{\sinh x \cosh y}{\cosh x \cosh y} + \frac{\cosh x \sinh y}{\cosh x \cosh y}}{\frac{\cosh x \cosh y}{\cosh x \cosh y} + \frac{\sinh x \sinh y}{\cosh x \cosh y}} \\ &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \end{aligned}$$

15. Putting $y = x$ in the result from Exercise 11, we have

$$\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x.$$

16. Putting $y = x$ in the result from Exercise 12, we have

$$\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$$

$$\begin{aligned} 17. \tanh(\ln x) &= \frac{\sinh(\ln x)}{\cosh(\ln x)} = \frac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \frac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \frac{x - x^{-1}}{x + x^{-1}} \\ &= \frac{x - 1/x}{x + 1/x} = \frac{(x^2 - 1)/x}{(x^2 + 1)/x} = \frac{x^2 - 1}{x^2 + 1} \end{aligned}$$

$$\begin{aligned} 18. \frac{1 + \tanh x}{1 - \tanh x} &= \frac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \frac{\cosh x + \sinh x}{\cosh x - \sinh x} = \frac{\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})} \\ &= \frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x} - e^x + e^{-x}} = \frac{2e^x}{2e^{-x}} = e^{2x} \end{aligned}$$

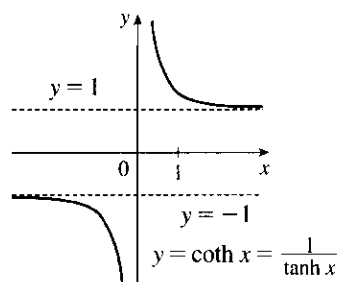
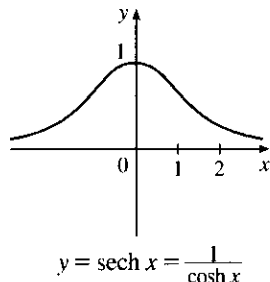
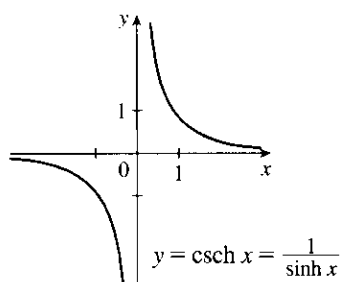
$$\text{Or: Using the results of Exercises 9 and 10, } \frac{\cosh x + \sinh x}{\cosh x - \sinh x} = \frac{e^x}{e^{-x}} = e^{2x}$$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

20. $\sinh x = \frac{3}{4} \Rightarrow \operatorname{csch} x = 1/\sinh x = \frac{4}{3}$, $\cosh^2 x = \sinh^2 x + 1 = \frac{9}{16} + 1 = \frac{25}{16} \Rightarrow \cosh x = \frac{5}{4}$ (since $\cosh x > 0$), $\operatorname{sech} x = 1/\cosh x = \frac{4}{5}$, $\tanh x = \sinh x/\cosh x = \frac{3/4}{5/4} = \frac{3}{5}$, and $\coth x = 1/\tanh x = \frac{5}{3}$.

21. $\tanh x = \frac{4}{5} > 0$, so $x > 0$. $\coth x = 1/\tanh x = \frac{5}{4}$, $\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \Rightarrow \operatorname{sech} x = \frac{3}{5}$ (since $\operatorname{sech} x > 0$), $\cosh x = 1/\operatorname{sech} x = \frac{5}{3}$, $\sinh x = \tanh x \cosh x = \frac{4}{5} \cdot \frac{5}{3} = \frac{4}{3}$, and $\operatorname{csch} x = 1/\sinh x = \frac{3}{4}$.

22.



$$23. (a) \lim_{x \rightarrow \infty} \tanh x = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1 - 0}{1 + 0} = 1$$

$$(b) \lim_{x \rightarrow -\infty} \tanh x = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x \rightarrow -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0 - 1}{0 + 1} = -1$$

$$(c) \lim_{x \rightarrow \infty} \sinh x = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{2} = \infty$$

$$(d) \lim_{x \rightarrow -\infty} \sinh x = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{2} = -\infty$$

$$(e) \lim_{x \rightarrow \infty} \operatorname{sech} x = \lim_{x \rightarrow \infty} \frac{2}{e^x + e^{-x}} = 0$$

$$(f) \lim_{x \rightarrow \infty} \operatorname{coth} x = \lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \rightarrow \infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1 + 0}{1 - 0} = 1 \quad [\text{Or: Use part (a)}]$$

$$(g) \lim_{x \rightarrow 0^+} \operatorname{coth} x = \lim_{x \rightarrow 0^+} \frac{\cosh x}{\sinh x} = \infty, \text{ since } \sinh x \rightarrow 0 \text{ through positive values and } \cosh x \rightarrow 1.$$

$$(h) \lim_{x \rightarrow 0^-} \operatorname{coth} x = \lim_{x \rightarrow 0^-} \frac{\cosh x}{\sinh x} = -\infty, \text{ since } \sinh x \rightarrow 0 \text{ through negative values and } \cosh x \rightarrow 1.$$

$$(i) \lim_{x \rightarrow -\infty} \operatorname{csch} x = \lim_{x \rightarrow -\infty} \frac{2}{e^x - e^{-x}} = 0$$

$$24. (a) \frac{d}{dx} \cosh x = \frac{d}{dx} \left[\frac{1}{2}(e^x + e^{-x}) \right] = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

$$(b) \frac{d}{dx} \tanh x = \frac{d}{dx} \left[\frac{\sinh x}{\cosh x} \right] = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$(c) \frac{d}{dx} \operatorname{csch} x = \frac{d}{dx} \left[\frac{1}{\sinh x} \right] = -\frac{\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x} \cdot \frac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$$

$$(d) \frac{d}{dx} \operatorname{sech} x = \frac{d}{dx} \left[\frac{1}{\cosh x} \right] = -\frac{\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x} \cdot \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$$

$$(e) \frac{d}{dx} \operatorname{coth} x = \frac{d}{dx} \left[\frac{\cosh x}{\sinh x} \right] = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x} \\ = -\operatorname{csch}^2 x$$

$$25. \text{ Let } y = \sinh^{-1} x. \text{ Then } \sinh y = x \text{ and, by Example 1(a), } \cosh^2 y - \sinh^2 y = 1 \Rightarrow [\text{with } \cosh y > 0]$$

$$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}. \text{ So by Exercise 9, } e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \Rightarrow$$

$$y = \ln(x + \sqrt{1 + x^2}).$$

26. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 9,
 $e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1} \Rightarrow y = \ln(x + \sqrt{x^2 - 1})$.

Another method: Write $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$ and solve a quadratic, as in Example 3.

27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \frac{e^y}{e^y} = \frac{e^{2y} - 1}{e^{2y} + 1} \Rightarrow$
 $xe^{2y} + x = e^{2y} - 1 \Rightarrow 1 + x = e^{2y} - xe^{2y} \Rightarrow 1 + x = e^{2y}(1 - x) \Rightarrow$
 $e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$.

- (b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$$e^{2y} = \frac{1 + \tanh y}{1 - \tanh y} = \frac{1+x}{1-x} \Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

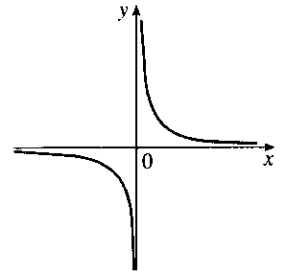
28. (a) (i) $y = \operatorname{csch}^{-1} x \Leftrightarrow \operatorname{csch} y = x \quad (x \neq 0)$

- (ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \frac{2}{e^y - e^{-y}} \Rightarrow xe^y - xe^{-y} = 2$
 $\Rightarrow x(e^y)^2 - 2e^y - x = 0 \Rightarrow e^y = \frac{1 \pm \sqrt{x^2 + 1}}{x}$.

But $e^y > 0$, so for $x > 0$, $e^y = \frac{1 + \sqrt{x^2 + 1}}{x}$ and for $x < 0$, $e^y = \frac{1 - \sqrt{x^2 + 1}}{x}$.

Thus, $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2 + 1}}{|x|}\right)$.

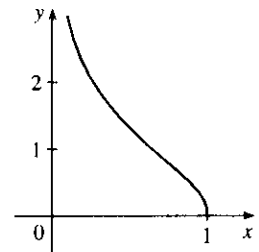


- (b) (i) $y = \operatorname{sech}^{-1} x \Leftrightarrow \operatorname{sech} y = x$ and $y > 0$.

- (ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \frac{2}{e^y + e^{-y}} \Rightarrow xe^y + xe^{-y} = 2$
 $\Rightarrow x(e^y)^2 - 2e^y + x = 0 \Leftrightarrow e^y = \frac{1 \pm \sqrt{1 - x^2}}{x}$.

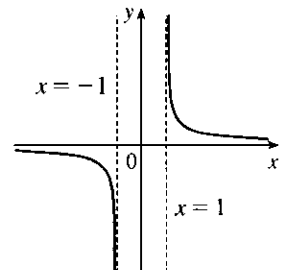
But $y > 0 \Rightarrow e^y > 1$. This rules out the minus sign because $\frac{1 - \sqrt{1 - x^2}}{x} > 1 \Leftrightarrow 1 - \sqrt{1 - x^2} > x$
 $\Leftrightarrow 1 - x > \sqrt{1 - x^2} \Leftrightarrow 1 - 2x + x^2 > 1 - x^2 \Leftrightarrow x^2 > x \Leftrightarrow x > 1$, but $x = \operatorname{sech} y \leq 1$. Thus,
 $e^y = \frac{1 + \sqrt{1 - x^2}}{x} \Rightarrow \operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$.



- (c) (i) $y = \operatorname{coth}^{-1} x \Leftrightarrow \operatorname{coth} y = x$

- (ii) We sketch the graph of coth^{-1} by reflecting the graph of coth (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{coth}^{-1} x$. Then $x = \operatorname{coth} y = \frac{e^y + e^{-y}}{e^y - e^{-y}} \Rightarrow$
 $xe^y - xe^{-y} = e^y + e^{-y} \Rightarrow (x-1)e^y = (x+1)e^{-y} \Rightarrow$
 $e^{2y} = \frac{x+1}{x-1} \Rightarrow 2y = \ln\left(\frac{x+1}{x-1}\right) \Rightarrow \operatorname{coth}^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$



29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \Rightarrow \sinh y \frac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}} \quad (\text{since } \sinh y \geq 0 \text{ for } y \geq 0). \text{ Or: Use Formula 4.}$$

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \Rightarrow -\operatorname{csch} y \coth y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{csch} y \coth y}$.

By Exercise 13, $\coth y = \pm \sqrt{\operatorname{csch}^2 y + 1} = \pm \sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2 + 1}$.

If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2 + 1}$. In either case we have

$$\frac{dy}{dx} = -\frac{1}{\operatorname{csch} y \coth y} = -\frac{1}{|x| \sqrt{x^2 + 1}}.$$

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \Rightarrow -\operatorname{sech} y \tanh y \frac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \tanh y} = -\frac{1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} = -\frac{1}{x \sqrt{1 - x^2}}. \quad (\text{Note that } y > 0 \text{ and so } \tanh y > 0.)$$

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \Rightarrow -\operatorname{csch}^2 y \frac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{csch}^2 y} = \frac{1}{1 - \coth^2 y} = \frac{1}{1 - x^2} \text{ by Exercise 13.}$$

30. $f(x) = \tanh 4x \Rightarrow f'(x) = 4 \operatorname{sech}^2 4x$

31. $f(x) = x \cosh x \Rightarrow f'(x) = x (\cosh x)' + (\cosh x)(x)' = x \sinh x + \cosh x$

32. $g(x) = \sinh^2 x \Rightarrow g'(x) = 2 \sinh x \cosh x$

33. $h(x) = \sinh(x^2) \Rightarrow h'(x) = \cosh(x^2) \cdot 2x = 2x \cosh(x^2)$

34. $F(x) = \sinh x \tanh x \Rightarrow F'(x) = \sinh x \operatorname{sech}^2 x + \tanh x \cosh x$

35. $G(x) = \frac{1 - \cosh x}{1 + \cosh x} \Rightarrow$

$$\begin{aligned} G'(x) &= \frac{(1 + \cosh x)(-\sinh x) - (1 - \cosh x)(\sinh x)}{(1 + \cosh x)^2} \\ &= \frac{-\sinh x - \sinh x \cosh x - \sinh x + \sinh x \cosh x}{(1 + \cosh x)^2} = \frac{-2 \sinh x}{(1 + \cosh x)^2} \end{aligned}$$

36. $f(t) = e^t \operatorname{sech} t \Rightarrow f'(t) = e^t (-\operatorname{sech} t \tanh t) + (\operatorname{sech} t) e^t = e^t \operatorname{sech} t (1 - \tanh t)$

37. $h(t) = \coth \sqrt{1+t^2} \Rightarrow h'(t) = -\operatorname{csch}^2 \sqrt{1+t^2} \cdot \frac{1}{2}(1+t^2)^{-1/2} (2t) = -\frac{t \operatorname{csch}^2 \sqrt{1+t^2}}{\sqrt{1+t^2}}$

38. $f(t) = \ln(\sinh t) \Rightarrow f'(t) = \frac{1}{\sinh t} \cosh t = \coth t$

39. $H(t) = \tanh(e^t) \Rightarrow H'(t) = \operatorname{sech}^2(e^t) \cdot e^t = e^t \operatorname{sech}^2(e^t)$

40. $y = \sinh(\cosh x) \Rightarrow y' = \cosh(\cosh x) \cdot \sinh x$

41. $y = e^{\cosh 3x} \Rightarrow y' = e^{\cosh 3x} \cdot \sinh 3x \cdot 3 = 3e^{\cosh 3x} \sinh 3x$

$$42. y = x^2 \sinh^{-1}(2x) \Rightarrow y' = x^2 \cdot \frac{1}{\sqrt{1+(2x)^2}} \cdot 2 + \sinh^{-1}(2x) \cdot 2x = 2x \left[\frac{x}{\sqrt{1+4x^2}} + \sinh^{-1}(2x) \right]$$

$$43. y = \tanh^{-1} \sqrt{x} \Rightarrow y' = \frac{1}{1-(\sqrt{x})^2} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}(1-x)}$$

$$44. y = x \tanh^{-1} x + \ln \sqrt{1-x^2} = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) \Rightarrow$$

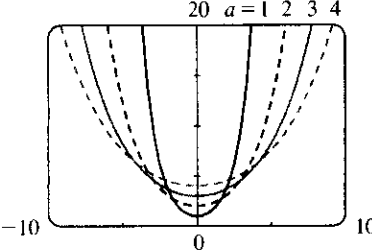
$$y' = \tanh^{-1} x + \frac{x}{1-x^2} + \frac{1}{2} \left(\frac{1}{1-x^2} \right) (-2x) = \tanh^{-1} x$$

$$45. y = x \sinh^{-1}(x/3) - \sqrt{9+x^2} \Rightarrow$$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x \frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

$$46. y = \operatorname{sech}^{-1} \sqrt{1-x^2} \Rightarrow y' = -\frac{1}{\sqrt{1-x^2} \sqrt{1-(1-x^2)}} \cdot \frac{-2x}{2\sqrt{1-x^2}} = \frac{x}{(1-x^2)|x|}$$

$$47. y = \coth^{-1} \sqrt{x^2+1} \Rightarrow y' = \frac{1}{1-(x^2+1)} \cdot \frac{2x}{2\sqrt{x^2+1}} = -\frac{1}{x\sqrt{x^2+1}}$$

48.  For $y = a \cosh(x/a)$ with $a > 0$, we have the y -intercept equal to a . As a increases, the graph flattens.

49. (a) $y = 20 \cosh(x/20) - 15 \Rightarrow y' = 20 \sinh(x/20) \cdot \frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh \frac{7}{20} \approx 0.3572$.

- (b) If α is the angle between the tangent line and the x -axis, then $\tan \alpha = \text{slope of the line} = \sinh \frac{7}{20}$, so $\alpha = \tan^{-1}(\sinh \frac{7}{20}) \approx 0.343 \text{ rad} \approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

50. We differentiate the function twice, then substitute into the differential equation: $y = \frac{T}{\rho g} \cosh \frac{\rho g x}{T} \Rightarrow$

$$\frac{dy}{dx} = \frac{T}{\rho g} \sinh\left(\frac{\rho g x}{T}\right) \frac{\rho g}{T} = \sinh \frac{\rho g x}{T} \Rightarrow \frac{d^2 y}{dx^2} = \cosh\left(\frac{\rho g x}{T}\right) \frac{\rho g}{T} = \frac{\rho g}{T} \cosh \frac{\rho g x}{T}.$$

We evaluate the two sides separately: $\text{LHS} = \frac{d^2 y}{dx^2} = \frac{\rho g}{T} \cosh \frac{\rho g x}{T},$

$$\text{RHS} = \frac{\rho g}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T} \sqrt{1 + \sinh^2 \frac{\rho g x}{T}} = \frac{\rho g}{T} \cosh \frac{\rho g x}{T}, \text{ by the identity proved in Example 1(a).}$$

51. (a) $y = A \sinh mx + B \cosh mx \Rightarrow y' = mA \cosh mx + mB \sinh mx \Rightarrow$
 $y'' = m^2 A \sinh mx + m^2 B \cosh mx = m^2 (A \sinh mx + B \cosh mx) = m^2 y$

- (b) From part (a), a solution of $y'' = 9y$ is $y(x) = A \sinh 3x + B \cosh 3x$. So
 $-4 = y(0) = A \sinh 0 + B \cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A \cosh 3x - 12 \sinh 3x \Rightarrow$
 $6 = y'(0) = 3A \Rightarrow A = 2$, so $y = 2 \sinh 3x - 4 \cosh 3x$.

$$52. \lim_{x \rightarrow \infty} \frac{\sinh x}{e^x} = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{2e^x} = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{2} = \frac{1 - 0}{2} = \frac{1}{2}$$

53. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \Rightarrow x = \sinh^{-1} 1 = \ln(1 + \sqrt{2})$, by Equation 3.

Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1 + \sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $(\ln(1 + \sqrt{2}), \sqrt{2})$.

$$\begin{aligned} 54. \cosh x &= \cosh[\ln(\sec \theta + \tan \theta)] = \frac{1}{2} \left[e^{\ln(\sec \theta + \tan \theta)} + e^{-\ln(\sec \theta + \tan \theta)} \right] \\ &= \frac{1}{2} \left[\sec \theta + \tan \theta + \frac{1}{\sec \theta + \tan \theta} \right] = \frac{1}{2} \left[\sec \theta + \tan \theta + \frac{\sec \theta - \tan \theta}{(\sec \theta + \tan \theta)(\sec \theta - \tan \theta)} \right] \\ &= \frac{1}{2} \left[\sec \theta + \tan \theta + \frac{\sec \theta - \tan \theta}{\sec^2 \theta - \tan^2 \theta} \right] = \frac{1}{2} (\sec \theta + \tan \theta + \sec \theta - \tan \theta) = \sec \theta \end{aligned}$$

55. Let $u = \cosh x$. Then $du = \sinh x \, dx$, so $\int \sinh x \cosh^2 x \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \cosh^3 x + C$.

56. Let $u = 1 + 4x$. Then $du = 4 \, dx$, so $\int \sinh(1 + 4x) \, dx = \frac{1}{4} \int \sinh u \, du = \frac{1}{4} \cosh u + C = \frac{1}{4} \cosh(1 + 4x) + C$.

57. Let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$ and $\int \frac{\sinh \sqrt{x}}{\sqrt{x}} \, dx = \int \sinh u \cdot 2 \, du = 2 \cosh u + C = 2 \cosh \sqrt{x} + C$.

58. Let $u = \cosh x$. Then $du = \sinh x \, dx$, and

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln(\cosh x) + C.$$

$$59. \int \frac{\cosh x}{\cosh^2 x - 1} \, dx = \int \frac{\cosh x}{\sinh^2 x} \, dx = \int \frac{\cosh x}{\sinh x} \cdot \frac{1}{\sinh x} \, dx = \int \coth x \operatorname{csch} x \, dx = -\operatorname{csch} x + C$$

60. Let $u = 2 + \tanh x$. Then $du = \operatorname{sech}^2 x \, dx$, so

$$\int \frac{\operatorname{sech}^2 x}{2 + \tanh x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |2 + \tanh x| + C = \ln(2 + \tanh x) + C$$

(since $2 + \tanh x > 1$).

61. Let $t = 3u$. Then $dt = 3 \, du$ and

$$\begin{aligned} \int_4^6 \frac{1}{\sqrt{t^2 - 9}} \, dt &= \int_{4/3}^2 \frac{1}{\sqrt{9u^2 - 9}} 3 \, du = \int_{4/3}^2 \frac{du}{\sqrt{u^2 - 1}} = \left[\cosh^{-1} u \right]_{4/3}^2 = \cosh^{-1} 2 - \cosh^{-1} \left(\frac{4}{3} \right) \quad \text{or} \\ &= \left[\cosh^{-1} u \right]_{4/3}^2 = \left[\ln(u + \sqrt{u^2 - 1}) \right]_{4/3}^2 \\ &= \ln(2 + \sqrt{3}) - \ln\left(\frac{4 + \sqrt{7}}{3}\right) = \ln\left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right) \end{aligned}$$

62. Let $u = 4t$. Then $du = 4 \, dt$ and

$$\begin{aligned} \int_0^1 \frac{dt}{\sqrt{16t^2 + 1}} &= \int_0^4 \frac{\frac{1}{4} du}{\sqrt{u^2 + 1}} = \frac{1}{4} \left[\sinh^{-1} u \right]_0^4 = \frac{1}{4} \left[\ln(u + \sqrt{u^2 + 1}) \right]_0^4 \\ &= \frac{1}{4} [\ln(4 + \sqrt{17}) - \ln 1] = \frac{1}{4} \ln(4 + \sqrt{17}) \end{aligned}$$

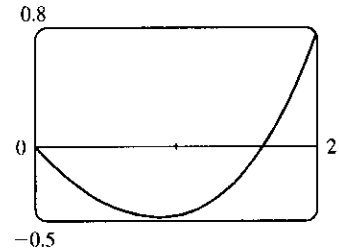
63. Let $u = e^x$. Then $du = e^x \, dx$ and $\int \frac{e^x}{1 - e^{2x}} \, dx = \int \frac{du}{1 - u^2} = \tanh^{-1} u + C = \tanh^{-1}(e^x) + C$

$$\left[\text{or } \frac{1}{2} \ln \left(\frac{1 + e^x}{1 - e^x} \right) + C \right].$$

64. We want $\int_0^1 \sinh cx \, dx = 1$. To calculate the integral, we put $u = cx$, so $du = c \, dx$, the upper limit becomes c , and the equation becomes

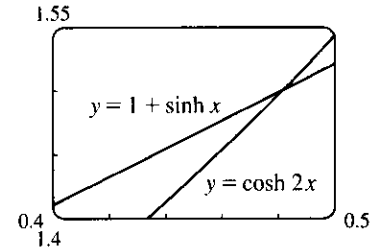
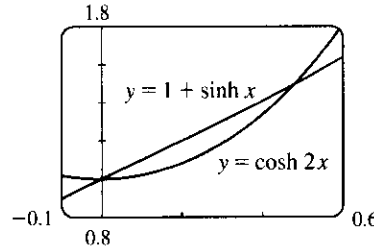
$$\frac{1}{c} \int_0^c \sinh u \, du = 1 \Leftrightarrow \frac{1}{c} [\cosh c - 1] = 1 \Leftrightarrow \cosh c - 1 = c.$$

We plot the function $f(c) = \cosh c - c - 1$, and see that its positive root lies at approximately $c = 1.62$. So the equation $\int_0^1 \sinh cx \, dx = 1$ holds for $c \approx 1.62$.



65. (a) From the graphs, we estimate

that the two curves $y = \cosh 2x$ and $y = 1 + \sinh x$ intersect at $x = 0$ and at $x = a \approx 0.481$.



- (b) We have found the two roots of the equation $\cosh 2x = 1 + \sinh x$ to be $x = 0$ and $x = a \approx 0.481$. Note from the first graph that $1 + \sinh x > \cosh 2x$ on the interval $(0, a)$, so the area between the two curves is

$$\begin{aligned} A &= \int_0^a (1 + \sinh x - \cosh 2x) \, dx = \left[x + \cosh x - \frac{1}{2} \sinh 2x \right]_0^a \\ &= [a + \cosh a - \frac{1}{2} \sinh 2a] - [0 + \cosh 0 - \frac{1}{2} \sinh 0] \approx 0.0402 \end{aligned}$$

66. The area of the triangle with vertices O , P , and $(\cosh t, 0)$ is $\frac{1}{2} \sinh t \cosh t$, and the area under the curve $x^2 - y^2 = 1$, from $x = 1$ to $x = \cosh t$, is $\int_1^{\cosh t} \sqrt{x^2 - 1} \, dx$. Therefore, the area of the shaded region is $A(t) = \frac{1}{2} \sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1} \, dx$. So, by FTC1,

$$\begin{aligned} A'(t) &= \frac{1}{2} (\cosh^2 t + \sinh^2 t) - \sqrt{\cosh^2 t - 1} \sinh t = \frac{1}{2} (\cosh^2 t + \sinh^2 t) - \sqrt{\sinh^2 t} \sinh t \\ &= \frac{1}{2} (\cosh^2 t + \sinh^2 t) - \sinh^2 t = \frac{1}{2} (\cosh^2 t - \sinh^2 t) = \frac{1}{2} (1) = \frac{1}{2} \end{aligned}$$

Thus $A(t) = \frac{1}{2}t + C$, since $A'(t) = \frac{1}{2}$. To calculate C , we let $t = 0$. Thus,

$$A(0) = \frac{1}{2} \sinh 0 \cosh 0 - \int_1^{\cosh 0} \sqrt{x^2 - 1} \, dx = \frac{1}{2} (0) + C \Rightarrow C = 0. \text{ Thus } A(t) = \frac{1}{2}t.$$

67. If $ae^x + be^{-x} = \alpha \cosh(x + \beta)$ [or $\alpha \sinh(x + \beta)$], then

$ae^x + be^{-x} = \frac{\alpha}{2} (e^{x+\beta} \pm e^{-x-\beta}) = \frac{\alpha}{2} (e^x e^\beta \pm e^{-x} e^{-\beta}) = (\frac{\alpha}{2} e^\beta) e^x \pm (\frac{\alpha}{2} e^{-\beta}) e^{-x}$. Comparing coefficients of e^x and e^{-x} , we have $a = \frac{\alpha}{2} e^\beta$ (1) and $b = \pm \frac{\alpha}{2} e^{-\beta}$ (2). We need to find α and β . Dividing equation (1) by equation (2) gives us $\frac{a}{b} = \pm e^{2\beta} \Rightarrow (*) \quad 2\beta = \ln(\pm \frac{a}{b}) \Rightarrow \beta = \frac{1}{2} \ln(\pm \frac{a}{b})$. Solving equations (1) and (2) for e^β gives us $e^\beta = \frac{2a}{\alpha}$ and $e^\beta = \pm \frac{\alpha}{2b}$, so $\frac{2a}{\alpha} = \pm \frac{\alpha}{2b} \Rightarrow \alpha^2 = \pm 4ab \Rightarrow \alpha = 2\sqrt{\pm ab}$.

(*) If $\frac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab} \cosh(x + \frac{1}{2} \ln \frac{a}{b})$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab} \sinh(x + \frac{1}{2} \ln(-\frac{a}{b}))$.

7.7 Indeterminate Forms and L'Hospital's Rule

The use of L'Hospital's Rule is indicated by an **H** above the equal sign: $\stackrel{\text{H}}{=}$.

1. (a) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.
 (b) $\lim_{x \rightarrow a} \frac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.
 (c) $\lim_{x \rightarrow a} \frac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.
 (d) If $\lim_{x \rightarrow a} p(x) = \infty$ and $f(x) \rightarrow 0$ through positive values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \rightarrow 0$ through negative values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \rightarrow 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]
 (e) $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.
 2. (a) $\lim_{x \rightarrow a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.
 (b) When x is near a , $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x \rightarrow a} [h(x)p(x)] = \infty$.
 (c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x \rightarrow a} [p(x)q(x)] = \infty$.
 3. (a) When x is near a , $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim_{x \rightarrow a} [f(x) - p(x)] = -\infty$.
 (b) $\lim_{x \rightarrow a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.
 (c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim_{x \rightarrow a} [p(x) + q(x)] = \infty$.
 4. (a) $\lim_{x \rightarrow a} [f(x)]^{q(x)}$ is an indeterminate form of type 0^0 .
 (b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a , $p(x) \rightarrow \infty$ and $\ln f(x) \rightarrow -\infty$, so $\ln y \rightarrow -\infty$. Therefore, $\lim_{x \rightarrow a} [f(x)]^{p(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = 0$, provided f^p is defined.
 (c) $\lim_{x \rightarrow a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞ .
 (d) $\lim_{x \rightarrow a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0 .
 (e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a , $q(x) \rightarrow \infty$ and $\ln p(x) \rightarrow \infty$, so $\ln y \rightarrow \infty$. Therefore, $\lim_{x \rightarrow a} [p(x)]^{q(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = \infty$.
 (f) $\lim_{x \rightarrow a} \sqrt[q(x)]{p(x)} = \lim_{x \rightarrow a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0 .
 5. This limit has the form $\frac{0}{0}$. We can simply factor the numerator to evaluate this limit.

$$\lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} \frac{(x + 1)(x - 1)}{x + 1} = \lim_{x \rightarrow -1} (x - 1) = -2$$

6. $\lim_{x \rightarrow -2} \frac{x+2}{x^2+3x+2} = \lim_{x \rightarrow -2} \frac{x+2}{(x+1)(x+2)} = \lim_{x \rightarrow -2} \frac{1}{x+1} = -1$
7. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow 1} \frac{x^9-1}{x^5-1} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{9x^8}{5x^4} = \frac{9}{5} \lim_{x \rightarrow 1} x^4 = \frac{9}{5}(1) = \frac{9}{5}$
8. $\lim_{x \rightarrow 1} \frac{x^a-1}{x^b-1} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$
9. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow (\pi/2)^+} \frac{\cos x}{1-\sin x} \stackrel{H}{=} \lim_{x \rightarrow (\pi/2)^+} \frac{-\sin x}{-\cos x} = \lim_{x \rightarrow (\pi/2)^+} \tan x = -\infty$.
10. $\lim_{x \rightarrow 0} \frac{x+\tan x}{\sin x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1+\sec^2 x}{\cos x} = \frac{1+1^2}{1} = 2$
11. This limit has the form $\frac{0}{0}$. $\lim_{t \rightarrow 0} \frac{e^t-1}{t^3} \stackrel{H}{=} \lim_{t \rightarrow 0} \frac{e^t}{3t^2} = \infty$ since $e^t \rightarrow 1$ and $3t^2 \rightarrow 0^+$ as $t \rightarrow 0$.
12. $\lim_{t \rightarrow 0} \frac{e^{3t}-1}{t} \stackrel{H}{=} \lim_{t \rightarrow 0} \frac{3e^{3t}}{1} = 3$
13. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow 0} \frac{\tan px}{\tan qx} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{p \sec^2 px}{q \sec^2 qx} = \frac{p(1)^2}{q(1)^2} = \frac{p}{q}$
14. $\lim_{\theta \rightarrow \pi/2} \frac{1-\sin \theta}{\csc \theta} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.
15. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$
16. $\lim_{x \rightarrow \infty} \frac{e^x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{1} = \lim_{x \rightarrow \infty} e^x = \infty$
17. $\lim_{x \rightarrow 0^+} [(\ln x)/x] = -\infty$ since $\ln x \rightarrow -\infty$ as $x \rightarrow 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.
18. $\lim_{x \rightarrow \infty} \frac{\ln \ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x \ln x} = 0$
19. This limit has the form $\frac{0}{0}$. $\lim_{t \rightarrow 0} \frac{5^t-3^t}{t} \stackrel{H}{=} \lim_{t \rightarrow 0} \frac{5^t \ln 5 - 3^t \ln 3}{1} = \ln 5 - \ln 3 = \ln \frac{5}{3}$
20. $\lim_{x \rightarrow 1} \frac{\ln x}{\sin \pi x} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{1/x}{\pi \cos \pi x} = \frac{1}{\pi(-1)} = -\frac{1}{\pi}$
21. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow 0} \frac{e^x-1-x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x-1}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}$
22. $\lim_{x \rightarrow 0} \frac{e^x-1-x-x^2/2}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x-1-x}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x-1}{6x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x}{6} = \frac{1}{6}$
23. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x \rightarrow \infty} \frac{e^x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty$
24. $\lim_{x \rightarrow 0} \frac{\sin x}{\sinh x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x}{\cosh x} = \frac{1}{1} = 1$
25. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1/\sqrt{1-x^2}}{1} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{1-x^2}} = \frac{1}{1} = 1$
26. $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{6x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\cos x}{6} = -\frac{1}{6}$
27. This limit has the form $\frac{0}{0}$. $\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin x}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x}{2} = \frac{1}{2}$

$$28. \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2(\ln x)(1/x)}{1} = 2 \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} 2 \lim_{x \rightarrow \infty} \frac{1/x}{1} = 2(0) = 0$$

$$29. \lim_{x \rightarrow 0} \frac{x + \sin x}{x + \cos x} = \frac{0 + 0}{0 + 1} = \frac{0}{1} = 0. \text{ L'Hospital's Rule does not apply.}$$

$$30. \lim_{x \rightarrow 0} \frac{\cos mx - \cos nx}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-m \sin mx + n \sin nx}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-m^2 \cos mx + n^2 \cos nx}{2} = \frac{1}{2}(n^2 - m^2)$$

$$31. \text{ This limit has the form } \frac{\infty}{\infty}. \lim_{x \rightarrow \infty} \frac{x}{\ln(1 + 2e^x)} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{1 + 2e^x} \cdot 2e^x} = \lim_{x \rightarrow \infty} \frac{1 + 2e^x}{2e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2e^x}{2e^x} = 1$$

$$32. \lim_{x \rightarrow 0} \frac{x}{\tan^{-1}(4x)} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1}{\frac{1}{1 + (4x)^2} \cdot 4} = \lim_{x \rightarrow 0} \frac{1 + 16x^2}{4} = \frac{1}{4}$$

$$33. \text{ This limit has the form } \frac{0}{0}. \lim_{x \rightarrow 1} \frac{1 - x + \ln x}{1 + \cos \pi x} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{-1 + 1/x}{-\pi \sin \pi x} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{-1/x^2}{-\pi^2 \cos \pi x} = \frac{-1}{-\pi^2(-1)} = -\frac{1}{\pi^2}$$

$$34. \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2}}{\sqrt{2x^2 + 1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{x^2 + 2}{2x^2 + 1}} = \sqrt{\lim_{x \rightarrow \infty} \frac{x^2 + 2}{2x^2 + 1}} = \sqrt{\lim_{x \rightarrow \infty} \frac{1 + 2/x^2}{2 + 1/x^2}} = \sqrt{\frac{1}{2}}$$

$$35. \text{ This limit has the form } \frac{0}{0}. \lim_{x \rightarrow 1} \frac{x^a - ax + a - 1}{(x - 1)^2} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{ax^{a-1} - a}{2(x - 1)} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{a(a - 1)x^{a-2}}{2} = \frac{a(a - 1)}{2}$$

$$36. \lim_{x \rightarrow 0} \frac{1 - e^{-2x}}{\sec x} = \frac{1 - 1}{1} = 0. \text{ L'Hospital's Rule does not apply.}$$

37. This limit has the form $0 \cdot (-\infty)$. We need to write this product as a quotient, but keep in mind that we will have to differentiate both the numerator and the denominator. If we differentiate $\frac{1}{\ln x}$, we get a complicated expression that results in a more difficult limit. Instead we write the quotient as $\frac{\ln x}{x^{-1/2}}$.

$$\lim_{x \rightarrow 0^+} \sqrt{x} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1/2}} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} \cdot \frac{-2x^{3/2}}{-2x^{3/2}} = \lim_{x \rightarrow 0^+} (-2\sqrt{x}) = 0$$

$$38. \lim_{x \rightarrow -\infty} x^2 e^x = \lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}} \stackrel{H}{=} \lim_{x \rightarrow -\infty} \frac{2x}{-e^{-x}} \stackrel{H}{=} \lim_{x \rightarrow -\infty} \frac{2}{e^{-x}} = \lim_{x \rightarrow -\infty} 2e^x = 0$$

39. This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \cot 2x \sin 6x = \lim_{x \rightarrow 0} \frac{\sin 6x}{\tan 2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{6 \cos 6x}{2 \sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

$$40. \lim_{x \rightarrow 0^+} \sin x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-\csc x \cot x} = - \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \cdot \tan x \right) \\ = - \left(\lim_{x \rightarrow 0^+} \frac{\sin x}{x} \right) \left(\lim_{x \rightarrow 0^+} \tan x \right) = -1 \cdot 0 = 0$$

$$41. \text{ This limit has the form } \infty \cdot 0. \lim_{x \rightarrow \infty} x^3 e^{-x^2} = \lim_{x \rightarrow \infty} \frac{x^3}{e^{x^2}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x \rightarrow \infty} \frac{3x}{2e^{x^2}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{3}{4xe^{x^2}} = 0$$

$$42. \lim_{x \rightarrow \pi/4} (1 - \tan x) \sec x = (1 - 1)\sqrt{2} = 0. \text{ L'Hospital's Rule does not apply.}$$

43. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x \rightarrow 1^+} \ln x \tan(\pi x/2) = \lim_{x \rightarrow 1^+} \frac{\ln x}{\cot(\pi x/2)} \stackrel{H}{=} \lim_{x \rightarrow 1^+} \frac{1/x}{(-\pi/2) \csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

$$44. \lim_{x \rightarrow \infty} x \tan(1/x) = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\sec^2(1/x)(-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \sec^2(1/x) = 1^2 = 1$$

$$45. \lim_{x \rightarrow 0} \left(\frac{1}{x} - \csc x \right) = \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0} \frac{\sin x - x}{x \sin x}$$

$$\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x \cos x + \sin x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0$$

$$46. \lim_{x \rightarrow 0} (\csc x - \cot x) = \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = 0$$

47. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x) = \lim_{x \rightarrow \infty} \left(\frac{\sqrt{x^2 + x} - x}{1} \cdot \frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x} \right) = \lim_{x \rightarrow \infty} \frac{(x^2 + x) - x^2}{\sqrt{x^2 + x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + x} + x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 1/x} + 1} = \frac{1}{\sqrt{1 + 1} + 1} = \frac{1}{2}.$$

As an alternate solution, write $\sqrt{x^2 + x} - x$ as $\sqrt{x^2 + x} - \sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $(\sqrt{1 + 1/x} - 1)/(1/x)$, and apply l'Hospital's Rule.

$$48. \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \rightarrow 1} \frac{x-1 - \ln x}{(x-1) \ln x} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{1 - 1/x}{(x-1)(1/x) + \ln x} \cdot \frac{x}{x}$$

$$= \lim_{x \rightarrow 1} \frac{x-1}{x-1+x \ln x} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{1}{1+1+\ln x} = \frac{1}{2+0} = \frac{1}{2}$$

49. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x .

$$\lim_{x \rightarrow \infty} (x - \ln x) = \lim_{x \rightarrow \infty} x \left(1 - \frac{\ln x}{x} \right) = \infty \text{ since } \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

50. As $x \rightarrow \infty$, $1/x \rightarrow 0$, and $e^{1/x} \rightarrow 1$. So the limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x .

$$\lim_{x \rightarrow \infty} (xe^{1/x} - x) = \lim_{x \rightarrow \infty} x(e^{1/x} - 1) = \lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} e^{1/x} = e^0 = 1$$

51. $y = x^{x^2} \Rightarrow \ln y = x^2 \ln x$, so

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x^2} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0^+} \left(-\frac{1}{2} x^2 \right) = 0 \Rightarrow$$

$$\lim_{x \rightarrow 0^+} x^{x^2} = \lim_{x \rightarrow 0^+} e^{\ln y} = e^0 = 1.$$

52. $y = (\tan 2x)^x \Rightarrow \ln y = x \cdot \ln \tan 2x$, so

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} x \cdot \ln \tan 2x = \lim_{x \rightarrow 0^+} \frac{\ln \tan 2x}{1/x}$$

$$\stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{(1/\tan 2x)(2 \sec^2 2x)}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{-2x^2 \cos 2x}{\sin 2x \cos^2 2x} = \lim_{x \rightarrow 0^+} \frac{2x}{\sin 2x} \cdot \lim_{x \rightarrow 0^+} \frac{-x}{\cos 2x} = 1 \cdot 0 = 0 =$$

$$\lim_{x \rightarrow 0^+} (\tan 2x)^x = \lim_{x \rightarrow 0^+} e^{\ln y} = e^0 = 1.$$

53. $y = (1 - 2x)^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(1 - 2x)$, so $\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln(1 - 2x)}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-2/(1 - 2x)}{1} = -2 \Rightarrow$

$$\lim_{x \rightarrow 0} (1 - 2x)^{1/x} = \lim_{x \rightarrow 0} e^{\ln y} = e^{-2}.$$

$$54. y = \left(1 + \frac{a}{x}\right)^{bx} \Rightarrow \ln y = bx \ln\left(1 + \frac{a}{x}\right), \text{ so}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{b \ln(1 + a/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{b \left(\frac{1}{1 + a/x}\right) \left(-\frac{a}{x^2}\right)}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{ab}{1 + a/x} = ab \Rightarrow$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx} = \lim_{x \rightarrow \infty} e^{\ln y} = e^{ab}.$$

$$55. y = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x \Rightarrow \ln y = x \ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right) \Rightarrow$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\left(-\frac{3}{x^2} - \frac{10}{x^3}\right) / \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{3 + \frac{10}{x}}{1 + \frac{3}{x} + \frac{5}{x^2}} = 3,$$

$$\text{so } \lim_{x \rightarrow \infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = \lim_{x \rightarrow \infty} e^{\ln y} = e^3.$$

$$56. y = x^{(\ln 2)/(1 + \ln x)} \Rightarrow \ln y = \frac{\ln 2}{1 + \ln x} \ln x \Rightarrow$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{(\ln 2)(\ln x)}{1 + \ln x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x \rightarrow \infty} \ln 2 = \ln 2,$$

$$\text{so } \lim_{x \rightarrow \infty} x^{(\ln 2)/(1 + \ln x)} = \lim_{x \rightarrow \infty} e^{\ln y} = e^{\ln 2} = 2.$$

$$57. y = x^{1/x} \Rightarrow \ln y = (1/x) \ln x \Rightarrow \lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0 \Rightarrow$$

$$\lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} e^{\ln y} = e^0 = 1$$

$$58. y = (e^x + x)^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(e^x + x), \text{ so}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(e^x + x)}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{e^x + 1} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{e^x} = 1 \Rightarrow$$

$$\lim_{x \rightarrow \infty} (e^x + x)^{1/x} = \lim_{x \rightarrow \infty} e^{\ln y} = e^1 = e.$$

$$59. y = \left(\frac{x}{x+1}\right)^x \Rightarrow \ln y = x \ln\left(\frac{x}{x+1}\right) \Rightarrow$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln\left(\frac{x}{x+1}\right) = \lim_{x \rightarrow \infty} \frac{\ln x - \ln(x+1)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x - 1/(x+1)}{-1/x^2}$$

$$= \lim_{x \rightarrow \infty} \left(-x + \frac{x^2}{x+1}\right) = \lim_{x \rightarrow \infty} \frac{-x}{x+1} = -1$$

$$\text{so } \lim_{x \rightarrow \infty} \left(\frac{x}{x+1}\right)^x = \lim_{x \rightarrow \infty} e^{\ln y} = e^{-1}$$

$$\text{Or: } \lim_{x \rightarrow \infty} \left(\frac{x}{x+1}\right)^x = \lim_{x \rightarrow \infty} \left[\left(\frac{x+1}{x}\right)^{-1}\right]^x = \left[\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x\right]^{-1} = e^{-1}$$

$$60. y = (\cos 3x)^{5/x} \Rightarrow \ln y = \frac{5}{x} \ln(\cos 3x) \Rightarrow \lim_{x \rightarrow 0} \ln y = 5 \lim_{x \rightarrow 0} \frac{\ln(\cos 3x)}{x} \stackrel{H}{=} 5 \lim_{x \rightarrow 0} \frac{-3 \tan 3x}{1} = 0,$$

$$\text{so } \lim_{x \rightarrow 0} (\cos 3x)^{5/x} = e^0 = 1.$$

$$61. y = (\cos x)^{1/x^2} \Rightarrow \ln y = \frac{1}{x^2} \ln \cos x \Rightarrow$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln \cos x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{-\tan x}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{-\sec^2 x}{2} = -\frac{1}{2} \Rightarrow$$

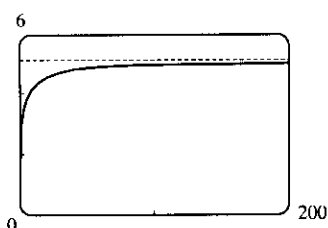
$$\lim_{x \rightarrow 0^+} (\cos x)^{1/x^2} = \lim_{x \rightarrow 0^+} e^{\ln y} = e^{-1/2} = 1/\sqrt{e}$$

$$62. y = \left(\frac{2x-3}{2x+5} \right)^{2x+1} \Rightarrow \ln y = (2x+1) \ln \left(\frac{2x-3}{2x+5} \right) \Rightarrow$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(2x-3) - \ln(2x+5)}{1/(2x+1)} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2/(2x-3) - 2/(2x+5)}{-2/(2x+1)^2} = \lim_{x \rightarrow \infty} \frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x \rightarrow \infty} \frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8 \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{2x-3}{2x+5} \right)^{2x+1} = e^{-8}$$

63.



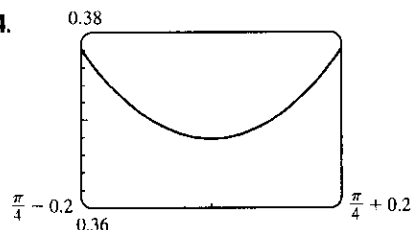
From the graph, it appears that $\lim_{x \rightarrow \infty} x [\ln(x+5) - \ln x] = 5$.

To prove this, we first note that

$$\ln(x+5) - \ln x = \ln \frac{x+5}{x} = \ln \left(1 + \frac{5}{x} \right) \rightarrow \ln 1 = 0 \text{ as } x \rightarrow \infty. \text{ Thus,}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} x [\ln(x+5) - \ln x] &= \lim_{x \rightarrow \infty} \frac{\ln(x+5) - \ln x}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x+5} - \frac{1}{x}}{-1/x^2} \\ &= \lim_{x \rightarrow \infty} \left[\frac{x - (x+5)}{x(x+5)} \cdot \frac{-x^2}{1} \right] = \lim_{x \rightarrow \infty} \frac{5x^2}{x^2 + 5x} = 5 \end{aligned}$$

64.



From the graph, it appears that $\lim_{x \rightarrow \pi/4} (\tan x)^{\tan 2x} \approx 0.368$.

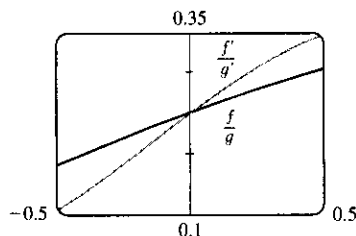
The limit has the form 1^∞ . Now $y = (\tan x)^{\tan 2x} \Rightarrow$

$\ln y = \tan 2x \ln(\tan x)$, so

$$\lim_{x \rightarrow \pi/4} \ln y = \lim_{x \rightarrow \pi/4} \frac{\ln(\tan x)}{\cot 2x} \stackrel{H}{=} \lim_{x \rightarrow \pi/4} \frac{\sec^2 x / \tan x}{-2 \csc^2 2x} = \frac{2/1}{-2(1)} = -1$$

$$\Rightarrow \lim_{x \rightarrow \pi/4} (\tan x)^{\tan 2x} = \lim_{x \rightarrow \pi/4} e^{\ln y} = e^{-1} = 1/e \approx 0.3679.$$

65.

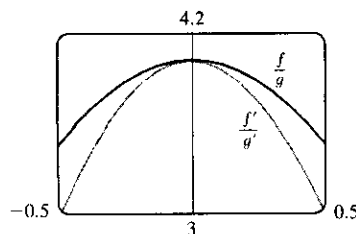


From the graph, it appears that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = 0.25. \text{ We calculate}$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x^3 + 4x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x}{3x^2 + 4} = \frac{1}{4}.$$

66.



From the graph, it appears that $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = 4$.

We calculate

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{2x \sin x}{\sec x - 1} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2(x \cos x + \sin x)}{\sec x \tan x} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2(-x \sin x + \cos x + \cos x)}{\sec x(\sec^2 x) + \tan x(\sec x \tan x)} = \frac{4}{1} = 4 \end{aligned}$$

- 67.
- $y = f(x) = xe^{-x}$
- A.
- $D = \mathbb{R}$
- B. Intercepts are 0 C. No symmetry

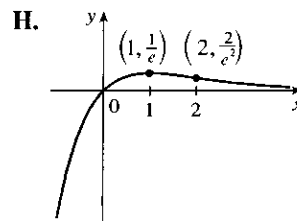
$$\text{D. } \lim_{x \rightarrow \infty} xe^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} \stackrel{\text{H}}{=} \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0, \text{ so } y = 0 \text{ is a HA.}$$

$$\lim_{x \rightarrow -\infty} xe^{-x} = -\infty \quad \text{E. } f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) > 0 \Leftrightarrow$$

$x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

F. Absolute and local maximum value $f(1) = 1/e$.

G. $f''(x) = e^{-x}(x-2) > 0 \Leftrightarrow x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. IP at $(2, 2/e^2)$



- 68.
- $y = f(x) = x(\ln x)^2$
- A.
- $D = (0, \infty)$
- B.
- x
- intercept = 1, no
- y
- intercept C. No symmetry

$$\text{D. } \lim_{x \rightarrow \infty} x(\ln x)^2 = \infty, \lim_{x \rightarrow 0^+} x(\ln x)^2 = \lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{1/x} \stackrel{\text{H}}{=} \lim_{x \rightarrow 0^+} \frac{2(\ln x)(1/x)}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{2 \ln x}{-1/x} \stackrel{\text{H}}{=}$$

$$\lim_{x \rightarrow 0^+} \frac{2/x}{1/x^2} = \lim_{x \rightarrow 0^+} 2x = 0, \text{ no asymptote} \quad \text{E. } f'(x) = (\ln x)^2 + 2 \ln x = (\ln x)(\ln x + 2) = 0 \text{ when } \ln x = 0$$

$\Leftrightarrow x = 1$ and when $\ln x = -2 \Leftrightarrow x = e^{-2}$. $f'(x) > 0$ when $0 < x < e^{-2}$ and when $x > 1$, so

f is increasing on $(0, e^{-2})$ and $(1, \infty)$ and decreasing on $(e^{-2}, 1)$.

F. Local maximum value $f(e^{-2}) = 4e^{-2}$,

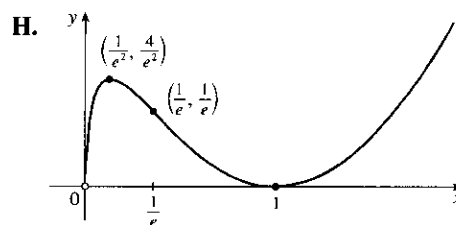
local minimum value $f(1) = 0$

$$\text{G. } f''(x) = 2(\ln x)(1/x) + 2/x = (2/x)(\ln x + 1) = 0$$

when $\ln x = -1 \Leftrightarrow x = e^{-1}$. $f''(x) > 0 \Leftrightarrow$

$x > 1/e$, so f is CU on $(1/e, \infty)$, CD on $(0, 1/e)$.

IP at $(1/e, 1/e)$



- 69.
- $y = f(x) = xe^{-x^2}$
- A.
- $D = \mathbb{R}$
- B. Intercepts are 0 C.
- $f(-x) = -f(x)$
- , so the curve is symmetric

about the origin. D. $\lim_{x \rightarrow \pm\infty} xe^{-x^2} = \lim_{x \rightarrow \pm\infty} \frac{x}{e^{x^2}} \stackrel{\text{H}}{=} \lim_{x \rightarrow \pm\infty} \frac{1}{2xe^{x^2}} = 0$, so $y = 0$ is a HA.

E. $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) > 0 \Leftrightarrow x^2 < \frac{1}{2} \Leftrightarrow |x| < \frac{1}{\sqrt{2}}$, so f is increasing on

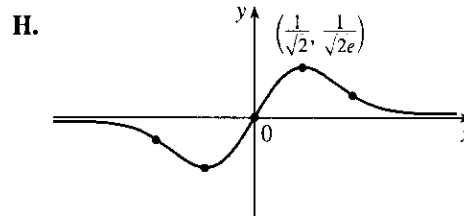
$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and decreasing on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \infty)$. F. Local maximum value $f(\frac{1}{\sqrt{2}}) = 1/\sqrt{2}e$, local

minimum value $f(-\frac{1}{\sqrt{2}}) = -1/\sqrt{2}e$ G. $f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2} = 2xe^{-x^2}(2x^2 - 3) > 0$

$\Leftrightarrow x > \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}} < x < 0$, so f is CU on $(\sqrt{\frac{3}{2}}, \infty)$

and $(-\sqrt{\frac{3}{2}}, 0)$ and CD on $(-\infty, -\sqrt{\frac{3}{2}})$ and $(0, \sqrt{\frac{3}{2}})$.

IP are $(0, 0)$ and $(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}e^{-3/2})$.



70. $y = f(x) = e^x/x$ A. $D = \{x \mid x \neq 0\}$ B. No intercept C. No symmetry D. $\lim_{x \rightarrow \infty} \frac{e^x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty$,

$$\lim_{x \rightarrow -\infty} \frac{e^x}{x} = 0, \text{ so } y = 0 \text{ is a HA. } \lim_{x \rightarrow 0^+} \frac{e^x}{x} = \infty, \lim_{x \rightarrow 0^-} \frac{e^x}{x} = -\infty, \text{ so } x = 0 \text{ is a VA.}$$

E. $f'(x) = \frac{xe^x - e^x}{x^2} > 0 \Leftrightarrow (x-1)e^x > 0 \Leftrightarrow x > 1$,

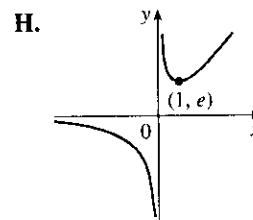
so f is increasing on $(1, \infty)$, and decreasing on $(-\infty, 0)$ and $(0, 1)$.

F. $f(1) = e$ is a local minimum value.

G. $f''(x) = \frac{x^2(xe^x) - 2x(xe^x - e^x)}{x^4} = \frac{e^x(x^2 - 2x + 2)}{x^3} > 0$

$\Leftrightarrow x > 0$ since $x^2 - 2x + 2 > 0$ for all x . So f is CU on $(0, \infty)$ and CD

on $(-\infty, 0)$. No IP



71. $y = f(x) = x - \ln(1+x)$ A. $D = \{x \mid x > -1\} = (-1, \infty)$ B. Intercepts are 0 C. No symmetry

D. $\lim_{x \rightarrow -1^+} [x - \ln(1+x)] = \infty$, so $x = -1$ is a VA. $\lim_{x \rightarrow \infty} [x - \ln(1+x)] = \lim_{x \rightarrow \infty} x \left[1 - \frac{\ln(1+x)}{x} \right] = \infty$,

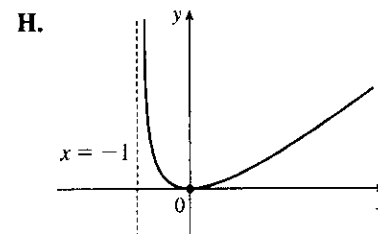
since $\lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/(1+x)}{1} = 0$.

E. $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \Leftrightarrow x > 0$ since $x+1 > 0$.

So f is increasing on $(0, \infty)$ and decreasing on $(-1, 0)$.

F. $f(0) = 0$ is an absolute minimum.

G. $f''(x) = 1/(1+x)^2 > 0$, so f is CU on $(-1, \infty)$.



72. $y = f(x) = e^x - 3e^{-x} - 4x$ A. $D = \mathbb{R}$ B. y -intercept = -2; x -intercept ≈ 2.22 C. No symmetry

D. $\lim_{x \rightarrow \infty} (e^x - 3e^{-x} - 4x) = \lim_{x \rightarrow \infty} x \left(\frac{e^x}{x} - 3\frac{e^{-x}}{x} - 4 \right) = \infty$, since $\lim_{x \rightarrow \infty} \frac{e^x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty$.

Similarly, $\lim_{x \rightarrow -\infty} (e^x - 3e^{-x} - 4x) = -\infty$. No HA; no VA

E. $f'(x) = e^x + 3e^{-x} - 4 = e^{-x}(e^{2x} - 4e^x + 3) = e^{-x}(e^x - 3)(e^x - 1) > 0 \Leftrightarrow e^x > 3 \text{ or } e^x < 1 \Leftrightarrow$

$x > \ln 3 \text{ or } x < 0$. So f is increasing on $(-\infty, 0)$ and $(\ln 3, \infty)$ and

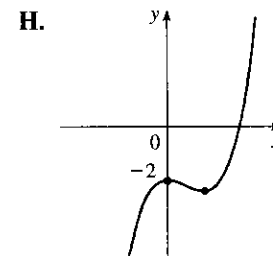
decreasing on $(0, \ln 3)$. F. Local maximum value $f(0) = -2$,

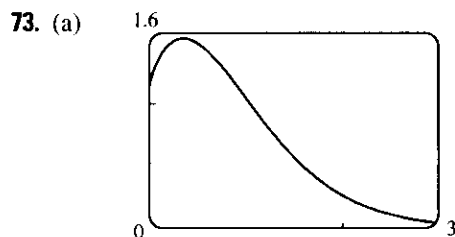
local minimum value $f(\ln 3) = 2 - 4 \ln 3$

G. $f''(x) = e^x - 3e^{-x} = e^{-x}(e^{2x} - 3) > 0 \Leftrightarrow e^{2x} > 3 \Leftrightarrow$

$x > \frac{1}{2} \ln 3$, so f is CU on $(\frac{1}{2} \ln 3, \infty)$ and CD on $(-\infty, \frac{1}{2} \ln 3)$.

IP at $(\frac{1}{2} \ln 3, -2 \ln 3)$.





(b) $y = f(x) = x^{-x}$. We note that

$$\ln f(x) = \ln x^{-x} = -x \ln x = -\frac{\ln x}{1/x}, \text{ so}$$

$$\lim_{x \rightarrow 0^+} \ln f(x) \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{-1/x}{-x^{-2}} = \lim_{x \rightarrow 0^+} x = 0. \text{ Thus}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^0 = 1.$$

(c) From the graph, it appears that there is a local and absolute maximum of about

$$f(0.37) \approx 1.44. \text{ To find the exact value, we differentiate: } f(x) = x^{-x} = e^{-x \ln x} \Rightarrow$$

$$f'(x) = e^{-x \ln x} \left[-x \left(\frac{1}{x} \right) + \ln x (-1) \right] = -x^{-x} (1 + \ln x). \text{ This is 0 only when } 1 + \ln x = 0 \Leftrightarrow$$

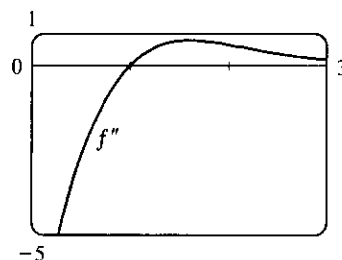
$x = e^{-1}$. Also $f'(x)$ changes from positive to negative at e^{-1} . So the maximum value is

$$f(1/e) = (1/e)^{-1/e} = e^{1/e}.$$

(d) We differentiate again to get

$$\begin{aligned} f''(x) &= -x^{-x}(1/x) + (1 + \ln x)^2(x^{-x}) \\ &= x^{-x} [(1 + \ln x)^2 - 1/x] \end{aligned}$$

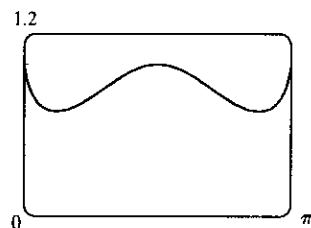
From the graph of $f''(x)$, it seems that $f''(x)$ changes from negative to positive at $x = 1$, so we estimate that f has an IP at $x = 1$.



74. (a) $f(x) = (\sin x)^{\sin x}$ is continuous where $\sin x > 0$, that is,

on intervals of the form $(2n\pi, (2n+1)\pi)$, so we have

graphed f on $(0, \pi)$.



(b) $y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x \ln \sin x$, so

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \sin x \ln \sin x = \lim_{x \rightarrow 0^+} \frac{\ln \sin x}{\csc x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{\cot x}{-\csc x \cot x} = \lim_{x \rightarrow 0^+} (-\sin x) = 0$$

$$\Rightarrow \lim_{x \rightarrow 0^+} y = e^0 = 1.$$

(c) It appears that we have a local maximum at $(1.57, 1)$ and local minima at $(0.38, 0.69)$ and

$$(2.76, 0.69). \quad y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x \ln \sin x \Rightarrow$$

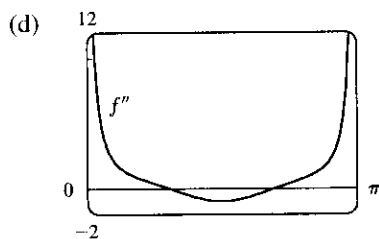
$$\frac{y'}{y} = (\sin x) \left(\frac{\cos x}{\sin x} \right) + (\ln \sin x) \cos x = \cos x (1 + \ln \sin x) \Rightarrow y' = (\sin x)^{\sin x} (\cos x) (1 + \ln \sin x).$$

$$y' = 0 \Rightarrow \cos x = 0 \text{ or } \ln \sin x = -1 \Rightarrow x_2 = \frac{\pi}{2} \text{ or } \sin x = e^{-1}. \text{ On } (0, \pi), \sin x = e^{-1} \Rightarrow$$

$$x_1 = \sin^{-1}(e^{-1}) \text{ and } x_3 = \pi - \sin^{-1}(e^{-1}). \text{ Approximating these points gives us}$$

$$(x_1, f(x_1)) \approx (0.3767, 0.6922), (x_2, f(x_2)) \approx (1.5708, 1), \text{ and } (x_3, f(x_3)) \approx (2.7649, 0.6922). \text{ The}$$

approximations confirm our estimates.



From the graph, we see that $f''(x) = 0$ at $x \approx 0.94$ and $x \approx 2.20$. Since f'' changes sign at these values, they are x -coordinates of inflection points.

75. (a) $f(x) = x^{1/x}$



(b) Recall that $a^b = e^{b \ln a}$. $\lim_{x \rightarrow 0^+} x^{1/x} = \lim_{x \rightarrow 0^+} e^{(1/x) \ln x}$. As $x \rightarrow 0^+$,

$\frac{\ln x}{x} \rightarrow -\infty$, so $x^{1/x} = e^{(1/x) \ln x} \rightarrow 0$. This indicates that there is a hole at $(0, 0)$. As $x \rightarrow \infty$, we have the indeterminate form ∞^0 .

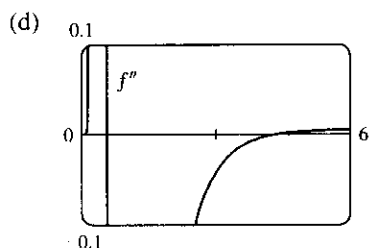
$\lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} e^{(1/x) \ln x}$, but $\lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$, so $\lim_{x \rightarrow \infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any

critical numbers. $y = x^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln x \Rightarrow \frac{y'}{y} = \frac{1}{x} \cdot \frac{1}{x} + (\ln x) \left(-\frac{1}{x^2} \right) \Rightarrow$

$y' = x^{1/x} \left(\frac{1 - \ln x}{x^2} \right) = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so

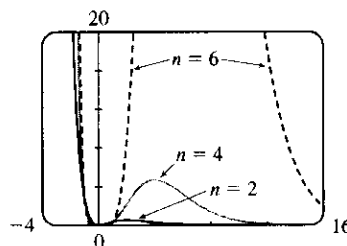
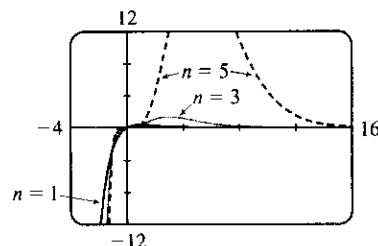
$f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.



From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$.

Since f'' changes sign at these values, they are x -coordinates of inflection points.

76.



The first figure shows representative examples of $f(x) = x^n e^{-x}$ with n odd. n is even in the second figure. All curves pass through the origin and approach $y = 0$ as $x \rightarrow \infty$. $f'(x) = \frac{x^n(n-x)}{x e^x} = 0 \Leftrightarrow x = n$ or $x = 0$ (the latter for $n > 1$). At $x = 0$, we have a local minimum for n even. At $x = n$, we have a local maximum for

all n . As n increases, $(n, f(n))$ gets farther away from the origin. $f''(x) = \frac{x^n(x^2 - 2nx + n^2 - n)}{x^2 e^x} = 0 \Leftrightarrow x = n \pm \sqrt{n}$ or $x = 0$ (the latter for $n > 2$). As n increases, the IP move farther away from the origin—they are symmetric about the line $x = n$.

77. If $c < 0$, then $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x}{e^{cx}} \stackrel{H}{=} \lim_{x \rightarrow -\infty} \frac{1}{ce^{cx}} = 0$, and $\lim_{x \rightarrow \infty} f(x) = \infty$.

If $c > 0$, then $\lim_{x \rightarrow -\infty} f(x) = -\infty$, and $\lim_{x \rightarrow \infty} f(x) \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{ce^{cx}} = 0$.

If $c = 0$, then $f(x) = x$, so $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$ respectively.

So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f , we differentiate: $f(x) = xe^{-cx} \Rightarrow$

$f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \Leftrightarrow x = 1/c$. If $c < 0$ then this

represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$;

and if $c > 0$, it represents a maximum value. As $|c|$ increases, the

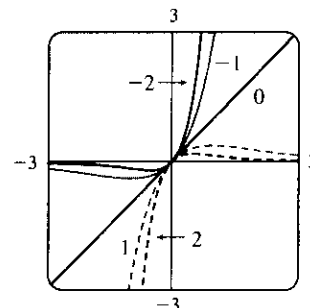
maximum or minimum point gets closer to the origin. To find the inflection

points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \Rightarrow$

$f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This

changes sign when $cx - 2 = 0 \Leftrightarrow x = 2/c$. So as $|c|$ increases, the

points of inflection get closer to the origin.



78. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\begin{aligned} \lim_{x \rightarrow a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} &\stackrel{H}{=} \lim_{x \rightarrow a} \frac{\frac{1}{2}(2a^3x - x^4)^{-1/2}(2a^3 - 4x^3) - a(\frac{1}{3})(aax)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)} \\ &= \frac{\frac{1}{2}(2a^3a - a^4)^{-1/2}(2a^3 - 4a^3) - \frac{1}{3}a^3(a^2a)^{-2/3}}{-\frac{1}{4}(aa^3)^{-3/4}(3aa^2)} \\ &= \frac{(a^4)^{-1/2}(-a^3) - \frac{1}{3}a^3(a^3)^{-2/3}}{-\frac{3}{4}a^3(a^4)^{-3/4}} = \frac{-a - \frac{1}{3}a}{-\frac{3}{4}} = \frac{4}{3}\left(\frac{4}{3}a\right) = \frac{16}{9}a \end{aligned}$$

79. First we will find $\lim_{n \rightarrow \infty} \left(1 + \frac{i}{n}\right)^{nt}$, which is of the form 1^∞ . $y = \left(1 + \frac{i}{n}\right)^{nt} \Rightarrow \ln y = nt \ln\left(1 + \frac{i}{n}\right)$, so

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} nt \ln\left(1 + \frac{i}{n}\right) = t \lim_{n \rightarrow \infty} \frac{\ln(1 + i/n)}{1/n} \stackrel{H}{=} t \lim_{n \rightarrow \infty} \frac{(-i/n^2)}{(1 + i/n)(-1/n^2)} = t \lim_{n \rightarrow \infty} \frac{i}{1 + i/n} = ti$$

$$\Rightarrow \lim_{n \rightarrow \infty} y = e^{it}. \text{ Thus, as } n \rightarrow \infty, A = A_0 \left(1 + \frac{i}{n}\right)^{nt} \rightarrow A_0 e^{it}.$$

$$\begin{aligned} 80. (a) \lim_{t \rightarrow \infty} v &= \lim_{t \rightarrow \infty} \frac{mg}{c} (1 - e^{-ct/m}) = \frac{mg}{c} \lim_{t \rightarrow \infty} (1 - e^{-ct/m}) \\ &= \frac{mg}{c} (1 - 0) \quad [\text{because } -ct/m \rightarrow -\infty \text{ as } t \rightarrow \infty] = \frac{mg}{c}, \end{aligned}$$

which is the speed the object approaches as time goes on, the so-called limiting velocity.

$$\begin{aligned} (b) \lim_{m \rightarrow \infty} v &= \lim_{m \rightarrow \infty} \frac{mg}{c} (1 - e^{-ct/m}) = \frac{g}{c} \lim_{m \rightarrow \infty} \frac{1 - e^{-ct/m}}{1/m} \stackrel{H}{=} \frac{g}{c} \lim_{m \rightarrow \infty} \frac{-e^{-ct/m}(ct/m^2)}{-1/m^2} \\ &= \frac{g}{c} (ct) \lim_{m \rightarrow \infty} e^{-ct/m} = gt(1) \quad [\text{because } -ct/m \rightarrow 0 \text{ as } m \rightarrow \infty] = gt. \end{aligned}$$

The speed of a very heavy falling object is approximately proportional to the elapsed time t , provided it can fall

for time t in an environment where the given model continues to hold. [If t is too large, the object may hit the ground in less than time t , or it may have to start falling too high above the earth, where there is almost no air.]

81. Both numerator and denominator approach 0 as $x \rightarrow 0$, so we use l'Hospital's Rule (and FTC1):

$$\lim_{x \rightarrow 0} \frac{S(x)}{x^3} = \lim_{x \rightarrow 0} \frac{\int_0^x \sin(\pi t^2/2) dt}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin(\pi x^2/2)}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\pi x \cos(\pi x^2/2)}{6x} = \frac{\pi}{6} \cdot \cos 0 = \frac{\pi}{6}$$

82. Both numerator and denominator approach 0 as $a \rightarrow 0$, so we use l'Hospital's Rule. (Note that we are differentiating with respect to a , since that is the quantity which is changing.) We also use the Fundamental Theorem of Calculus, Part 1:

$$\lim_{a \rightarrow 0} T(x, t) = \lim_{a \rightarrow 0} \frac{C \int_0^a e^{-(x-u)^2/(4kt)} du}{a \sqrt{4\pi kt}} \stackrel{H}{=} \lim_{a \rightarrow 0} \frac{C e^{-(x-a)^2/(4kt)}}{\sqrt{4\pi kt}} = \frac{C e^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$$

83. Since $f(2) = 0$, the given limit has the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{f(2+3x) + f(2+5x)}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{f'(2+3x) \cdot 3 + f'(2+5x) \cdot 5}{1} = f'(2) \cdot 3 + f'(2) \cdot 5 = 8f'(2) = 8 \cdot 7 = 56$$

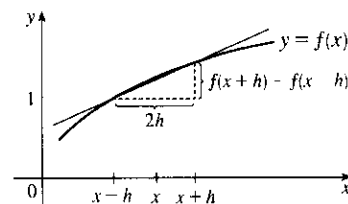
84. $L = \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2} \right) = \lim_{x \rightarrow 0} \frac{\sin 2x + ax^3 + bx}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2 \cos 2x + 3ax^2 + b}{3x^2}$. As $x \rightarrow 0$, $3x^2 \rightarrow 0$, and $(2 \cos 2x + 3ax^2 + b) \rightarrow b + 2$, so the last limit exists only if $b + 2 = 0$, that is, $b = -2$. Thus,

$$\lim_{x \rightarrow 0} \frac{2 \cos 2x + 3ax^2 - 2}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-4 \sin 2x + 6ax}{6x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-8 \cos 2x + 6a}{6} = \frac{6a - 8}{6}, \text{ which is equal to 0 if and only if } a = \frac{4}{3}. \text{ Hence, } L = 0 \text{ if and only if } b = -2 \text{ and } a = \frac{4}{3}.$$

85. Since $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = f(x) - f(x) = 0$ (f is differentiable and hence continuous) and $\lim_{h \rightarrow 0} 2h = 0$, we use l'Hospital's Rule:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} \stackrel{H}{=} \lim_{h \rightarrow 0} \frac{f'(x+h)(1) - f'(x-h)(-1)}{2} = \frac{f'(x) + f'(x)}{2} = \frac{2f'(x)}{2} = f'(x)$$

$\frac{f(x+h) - f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h \rightarrow 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



86. Since $\lim_{h \rightarrow 0} [f(x+h) - 2f(x) + f(x-h)] = f(x) - 2f(x) + f(x) = 0$ (f is differentiable and hence continuous) and $\lim_{h \rightarrow 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \stackrel{H}{=} \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 85 to $f'(x)$.

$$87. \lim_{x \rightarrow \infty} \frac{e^x}{x^n} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{nx^{n-1}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{n(n-1)x^{n-2}} \stackrel{H}{=} \dots \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{n!} = \infty$$

$$88. \lim_{x \rightarrow \infty} \frac{\ln x}{x^p} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{px^{p-1}} = \lim_{x \rightarrow \infty} \frac{1}{px^p} = 0 \text{ since } p > 0.$$

$$89. \lim_{x \rightarrow 0^+} x^\alpha \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-\alpha}} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-\alpha x^{-\alpha-1}} = \lim_{x \rightarrow 0^+} \frac{x^\alpha}{-\alpha} = 0 \text{ since } \alpha > 0.$$

90. Using l'Hospital's Rule and FTC1, we have

$$\lim_{x \rightarrow 0} \frac{\int_0^x \sin(t^2) dt}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^x \sin(t^2) dt}{3x^2} = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{3x^2} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} = \frac{1}{3}$$

91. Let the radius of the circle be r . We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac{1}{2}r|PQ| = \frac{1}{2}r(r \sin \theta) = \frac{1}{2}r^2 \sin \theta$. So we have

$$A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = \frac{1}{2}r^2(\theta - \sin \theta). \text{ Now by elementary trigonometry,}$$

$$B(\theta) = \frac{1}{2}|QR||PQ| = \frac{1}{2}(r - |OQ|)|PQ| = \frac{1}{2}(r - r \cos \theta)(r \sin \theta) = \frac{1}{2}r^2(1 - \cos \theta) \sin \theta.$$

So the limit we want is

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)} &= \lim_{\theta \rightarrow 0^+} \frac{\frac{1}{2}r^2(\theta - \sin \theta)}{\frac{1}{2}r^2(1 - \cos \theta) \sin \theta} \stackrel{H}{=} \lim_{\theta \rightarrow 0^+} \frac{1 - \cos \theta}{(1 - \cos \theta) \cos \theta + \sin \theta (\sin \theta)} \\ &= \lim_{\theta \rightarrow 0^+} \frac{1 - \cos \theta}{\cos \theta - \cos^2 \theta + \sin^2 \theta} \stackrel{H}{=} \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{-\sin \theta - 2 \cos \theta (-\sin \theta) + 2 \sin \theta (\cos \theta)} \\ &= \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{-\sin \theta + 4 \sin \theta \cos \theta} = \lim_{\theta \rightarrow 0^+} \frac{1}{-1 + 4 \cos \theta} = \frac{1}{-1 + 4 \cos 0} = \frac{1}{3} \end{aligned}$$

92. The area $A(t) = \int_0^t \sin(x^2) dx$, and the area $B(t) = \frac{1}{2}t \sin(t^2)$. Since $\lim_{t \rightarrow 0^+} A(t) = 0 = \lim_{t \rightarrow 0^+} B(t)$, we can use l'Hospital's Rule:

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{A(t)}{B(t)} &\stackrel{H}{=} \lim_{t \rightarrow 0^+} \frac{\sin(t^2)}{\frac{1}{2} \sin(t^2) + \frac{1}{2}t[2t \cos(t^2)]} \quad [\text{by FTC1 and the Product Rule}] \\ &\stackrel{H}{=} \lim_{t \rightarrow 0^+} \frac{2t \cos(t^2)}{t \cos(t^2) - 2t^3 \sin(t^2) + 2t \cos(t^2)} = \lim_{t \rightarrow 0^+} \frac{2 \cos(t^2)}{3 \cos(t^2) - 2t^2 \sin(t^2)} \\ &= \frac{2}{3 - 0} = \frac{2}{3} \end{aligned}$$

93. (a) We show that $\lim_{x \rightarrow 0} \frac{f(x)}{x^n} = 0$ for every integer $n \geq 0$. Let $y = \frac{1}{x^2}$. Then

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^{2n}} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{(x^2)^n} = \lim_{y \rightarrow \infty} \frac{y^n}{e^y} \stackrel{H}{=} \lim_{y \rightarrow \infty} \frac{ny^{n-1}}{e^y} \stackrel{H}{=} \dots \stackrel{H}{=} \lim_{y \rightarrow \infty} \frac{n!}{e^y} = 0 \Rightarrow$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^n} = \lim_{x \rightarrow 0} x^n \frac{f(x)}{x^{2n}} = \lim_{x \rightarrow 0} x^n \lim_{x \rightarrow 0} \frac{f(x)}{x^{2n}} = 0. \text{ Thus, } f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = 0.$$

- (b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the n th derivative. Then $f'(x) = f(x)(2/x^3)$, so

$$\begin{aligned} f^{(n+1)}(x) &= \left[x^{k_n} [p'_n(x)f(x) + p_n(x)f'(x)] - k_n x^{k_n-1} p_n(x)f(x) \right] x^{-2k_n} \\ &= \left[x^{k_n} p'_n(x) + p_n(x)(2/x^3) - k_n x^{k_n-1} p_n(x) \right] f(x) x^{-2k_n} \\ &= \left[x^{k_n+3} p'_n(x) + 2p_n(x) - k_n x^{k_n+2} p_n(x) \right] f(x) x^{-(2k_n+3)} \end{aligned}$$

which has the desired form.

Now we show by induction that $f^{(n)}(0) = 0$ for all n . By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

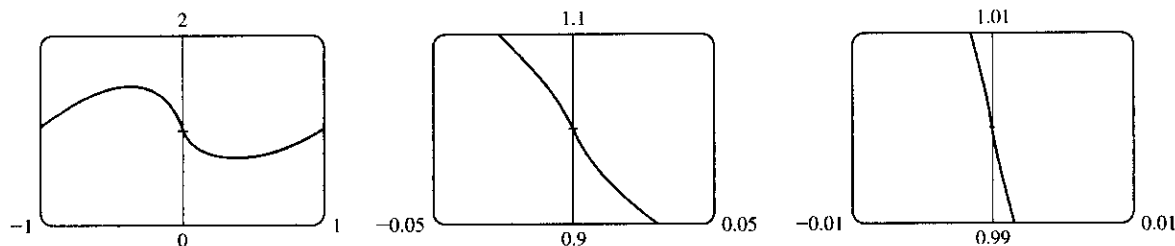
$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x \rightarrow 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f^{(n)}(x)}{x} = \lim_{x \rightarrow 0} \frac{p_n(x)f(x)/x^{k_n}}{x} = \lim_{x \rightarrow 0} \frac{p_n(x)f(x)}{x^{k_n+1}} \\ &= \lim_{x \rightarrow 0} p_n(x) \lim_{x \rightarrow 0} \frac{f(x)}{x^{k_n+1}} = p_n(0) \cdot 0 = 0 \end{aligned}$$

94. (a) For f to be continuous, we need $\lim_{x \rightarrow 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln |x|^x = x \ln |x|$.

So $\lim_{x \rightarrow 0} \ln f(x) = \lim_{x \rightarrow 0} x \ln |x| = \lim_{x \rightarrow 0} \frac{\ln |x|}{1/x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = 0$. Therefore,

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0.

- (b) From the graphs, it appears that f is differentiable at 0.



- (c) To find f' , we use logarithmic differentiation: $\ln f(x) = x \ln |x| \Rightarrow \frac{f'(x)}{f(x)} = x \left(\frac{1}{x} \right) + \ln |x| \Rightarrow$

$f'(x) = f(x)(1 + \ln |x|) = |x|^x(1 + \ln |x|)$, $x \neq 0$. Now $f'(x) \rightarrow -\infty$ as $x \rightarrow 0$ [since $|x|^x \rightarrow 1$ and $(1 + \ln |x|) \rightarrow -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln |x| \rightarrow -\infty$ very slowly as $x \rightarrow 0$.

7 Review

CONCEPT CHECK

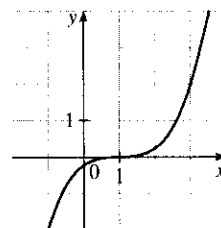
1. (a) See Definition 1 in Section 7.1. It must pass the Horizontal Line Test.
 (b) See Definition 2 in Section 7.1. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.
 (c) $g'(a) = (f^{-1})'(a) = \frac{1}{f'(g(a))}$
2. (a) The function $f(x) = e^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.
 (b) The function $f(x) = \ln x$ has domain $(0, \infty)$ and range $\mathbb{R}$.
 (c) The graphs are reflections of one another about the line $y = x$. See Figure 7.3.3 or Figure 7.3*.1.
 (d) $\log_a x = \frac{\ln x}{\ln a}$
3. (a) See Definition 7.5.1. Domain = $[-1, 1]$, Range = $[-\frac{\pi}{2}, \frac{\pi}{2}]$
 (b) See Definition 7.5.4. Domain = $[-1, 1]$, Range = $[0, \pi]$
 (c) See Definition 7.5.7. Domain = $\mathbb{R}$, Range = $(-\frac{\pi}{2}, \frac{\pi}{2})$. See Figure 10 in Section 7.5.
4. $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
5. (a) $y = e^x \Rightarrow y' = e^x$
 (b) $y = a^x \Rightarrow y' = a^x \ln a$
 (c) $y = \ln x \Rightarrow y' = 1/x$
 (d) $y = \log_a x \Rightarrow y' = 1/(x \ln a)$
 (e) $y = \sin^{-1} x \Rightarrow y' = 1/\sqrt{1-x^2}$
 (f) $y = \cos^{-1} x \Rightarrow y' = -1/\sqrt{1-x^2}$
 (g) $y = \tan^{-1} x \Rightarrow y' = 1/(1+x^2)$
 (h) $y = \sinh x \Rightarrow y' = \cosh x$
 (i) $y = \cosh x \Rightarrow y' = \sinh x$
 (j) $y = \tanh x \Rightarrow y' = \operatorname{sech}^2 x$
 (k) $y = \sinh^{-1} x \Rightarrow y' = 1/\sqrt{1+x^2}$
 (l) $y = \cosh^{-1} x \Rightarrow y' = 1/\sqrt{x^2-1}$
 (m) $y = \tanh^{-1} x \Rightarrow y' = 1/(1-x^2)$
6. (a) e is the number such that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.
 (b) $e = \lim_{x \rightarrow 0} (1+x)^{1/x}$
 (c) The differentiation formula for $y = a^x$ [$y' = a^x \ln a$] is simplest when $a = e$ because $\ln e = 1$.
 (d) The differentiation formula for $y = \log_a x$ [$y' = 1/(x \ln a)$] is simplest when $a = e$ because $\ln e = 1$.
7. (a) See l'Hospital's Rule and the three notes that follow it in Section 7.7.
 (b) Write fg as $\frac{f}{1/g}$ or $\frac{g}{1/f}$.
 (c) Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method.
 (d) Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ or by writing f^g as $e^{g \ln f}$.

TRUE-FALSE QUIZ

- True. If f is one-to-one, with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$ by the first cancellation equation [see (4) in Section 7.1].
- False. By Theorem 7 in Section 7.1, $(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))}$, *not* $\frac{1}{f'(6)}$ unless $f^{-1}(6) = 6$.
- False. For example, $\cos \frac{\pi}{2} = \cos(-\frac{\pi}{2})$, so $\cos x$ is not 1-1.
- False. It is true that $\tan \frac{3\pi}{4} = -1$, but since the range of $\tan^{-1}$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, we must have $\tan^{-1}(-1) = -\frac{\pi}{4}$.
- True, since $\ln x$ is an increasing function on $(0, \infty)$.
- True, by Equation 7.4*.1.
- True. We can divide by e^x since $e^x \neq 0$ for every x .
- False. For example, $\ln(1+1) = \ln 2$, but $\ln 1 + \ln 1 = 0$. In fact $\ln a + \ln b = \ln(ab)$.
- False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6 \ln x = 6 \ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$.
- False. $\frac{d}{dx} 10^x = 10^x \ln 10$
- False. $\ln 10$ is a constant, so its derivative is 0.
- True. $y = e^{3x} \Rightarrow \ln y = 3x \Rightarrow x = \frac{1}{3} \ln y \Rightarrow$ the inverse function is $y = \frac{1}{3} \ln x$.
- False. The “-1” is not an exponent; it is an indication of an inverse function.
- False. For example, $\tan^{-1} 20$ is defined; $\sin^{-1} 20$ and $\cos^{-1} 20$ are not.
- True. See Figure 2 in Section 7.6.
- True. $\ln \frac{1}{10} = -\ln 10 = -\int_1^{10} (1/x) dx$, by Equation 7.4.4 or by Definition 7.2*.1.
- True. $\int_2^{16} (1/x) dx = \ln x \Big|_2^{16} = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8 = \ln 2^3 = 3 \ln 2$
- False. L'Hospital's Rule does not apply since $\lim_{x \rightarrow \pi^-} \frac{\tan x}{1 - \cos x} = \frac{0}{2} = 0$.

EXERCISES

- No. f is not 1-1 because the graph of f fails the Horizontal Line Test.
- g is one-to-one because it passes the Horizontal Line Test.
 - When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.
 - The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1} .
 - We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1} .

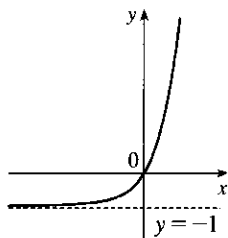


3. (a) $f^{-1}(3) = 7$ since $f(7) = 3$.

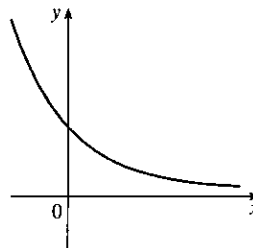
(b) $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(7)} = \frac{1}{8}$

4. $y = \frac{x+1}{2x+1}$. Interchanging x and y gives us $x = \frac{y+1}{2y+1} \Rightarrow 2xy + x = y + 1 \Rightarrow 2xy - y = 1 - x \Rightarrow$
 $y(2x - 1) = 1 - x \Rightarrow y = \frac{1-x}{2x-1} = f^{-1}(x).$

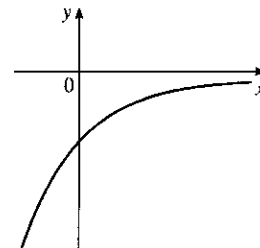
5. $y = 5^x - 1$



6. $y = e^{-x}$

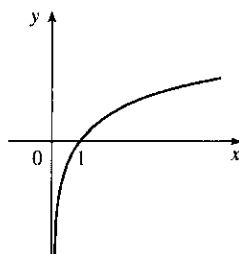


$y = -e^{-x}$

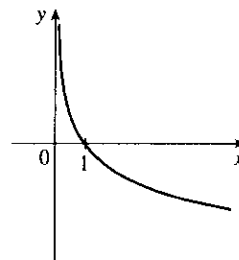


7. Reflect the graph of $y = \ln x$ about the x -axis to obtain the graph of $y = -\ln x$.

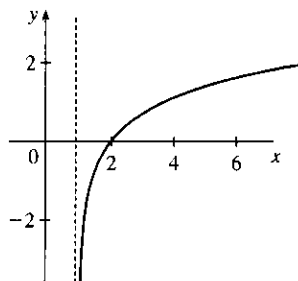
$y = \ln x$



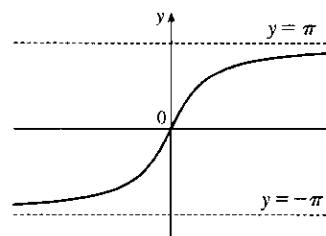
$y = -\ln x$



8. $y = \ln(x-1)$



9. $y = 2 \arctan x$



10. We have seen that if $a > 1$, then $a^x > x^a$ for sufficiently large x . (See Exercise 7.2.20.) In general, we could show that $\lim_{x \rightarrow \infty} (a^x/x^a) = \infty$ by using l'Hospital's Rule repeatedly. Also, $\log_a x$ increases much more slowly than either x^a or a^x . [Compare the graph of $\log_a x$ with those of x^a and a^x , or use l'Hospital's Rule to show that $\lim_{x \rightarrow \infty} [(\log_a x)/x^a] = 0$.] So for large x , $\log_a x < x^a < a^x$.

11. (a) $e^{2 \ln 3} = (e^{\ln 3})^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

12. (a) $\ln e^\pi = \pi$

(b) $\tan(\arcsin \frac{1}{2}) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

13. $\ln x = \frac{1}{3} \Leftrightarrow \log_e x = \frac{1}{3} \Rightarrow x = e^{1/3}$
14. $e^x = \frac{1}{3} \Rightarrow x = \ln \frac{1}{3} = \ln 1 - \ln 3 = -\ln 3$
15. $e^{e^x} = 17 \Rightarrow \ln e^{e^x} = \ln 17 \Rightarrow e^x = \ln 17 \Rightarrow \ln e^x = \ln(\ln 17) \Rightarrow x = \ln \ln 17$
16. $\ln(1 + e^{-x}) = 3 \Rightarrow 1 + e^{-x} = e^3 \Rightarrow e^{-x} = e^3 - 1 \Rightarrow \ln e^{-x} = \ln(e^3 - 1) \Rightarrow -x = \ln(e^3 - 1) \Rightarrow x = -\ln(e^3 - 1)$
17. $\ln(x+1) + \ln(x-1) = 1 \Rightarrow \ln[(x+1)(x-1)] = 1 \Rightarrow \ln(x^2 - 1) = \ln e \Rightarrow x^2 - 1 = e \Rightarrow x^2 = e + 1 \Rightarrow x = \sqrt{e+1}$ since $\ln(x-1)$ is defined only when $x > 1$.
18. $\log_5(c^x) = d \Rightarrow x \log_5 c = d \Rightarrow x = \frac{d}{\log_5 c}$.
 Or: $\log_5(c^x) = d \Rightarrow 5^d = c^x \Rightarrow \ln 5^d = \ln c^x \Rightarrow d \ln 5 = x \ln c \Rightarrow x = \frac{d \ln 5}{\ln c}$.
19. $\tan^{-1} x = 1 \Rightarrow \tan \tan^{-1} x = \tan 1 \Rightarrow x = \tan 1 (\approx 1.5574)$
20. $\sin x = 0.3 \Rightarrow x = \sin^{-1} 0.3 = \alpha$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The reference angle for α is $\pi - \alpha$, so all solutions are $x = \alpha + 2n\pi$ and $x = \pi - \alpha + 2n\pi$ [or $(2n+1)\pi - \alpha$]
21. $f(t) = t^2 \ln t \Rightarrow f'(t) = t^2 \cdot \frac{1}{t} + (\ln t)(2t) = t + 2t \ln t$ or $t(1 + 2 \ln t)$
22. $g(t) = \frac{e^t}{1 + e^t} \Rightarrow g'(t) = \frac{(1 + e^t)e^t - e^t(e^t)}{(1 + e^t)^2} = \frac{e^t}{(1 + e^t)^2}$
23. $h(\theta) = e^{\tan 2\theta} \Rightarrow h'(\theta) = e^{\tan 2\theta} \cdot \sec^2 2\theta \cdot 2 = 2 \sec^2 2\theta e^{\tan 2\theta}$
24. $h(u) = 10^{\sqrt{u}} \Rightarrow h'(u) = 10^{\sqrt{u}} \cdot \ln 10 \cdot \frac{1}{2\sqrt{u}} = \frac{(\ln 10)10^{\sqrt{u}}}{2\sqrt{u}}$
25. $y = \ln |\sec 5x + \tan 5x| \Rightarrow$
 $y' = \frac{1}{\sec 5x + \tan 5x} (\sec 5x \tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \frac{5 \sec 5x (\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5 \sec 5x$
26. $y = e^{-t}(t^2 - 2t + 2) \Rightarrow$
 $y' = e^{-t}(2t - 2) + (t^2 - 2t + 2)(-e^{-t}) = e^{-t}(2t - 2 - t^2 + 2t - 2) = e^{-t}(-t^2 + 4t - 4)$
27. $y = e^{cx}(c \sin x - \cos x) \Rightarrow y' = ce^{cx}(c \sin x - \cos x) + e^{cx}(c \cos x + \sin x) = (c^2 + 1)e^{cx} \sin x$
28. $y = \sin^{-1}(e^x) \Rightarrow y' = 1/\sqrt{1 - (e^x)^2} \cdot e^x = e^x/\sqrt{1 - e^{2x}}$
29. $y = \ln(\sec^2 x) = 2 \ln |\sec x| \Rightarrow y' = (2/\sec x)(\sec x \tan x) = 2 \tan x$
30. $y = \ln(x^2 e^x) = 2 \ln |x| + x \Rightarrow y' = 2/x + 1$
31. $y = xe^{-1/x} \Rightarrow y' = e^{-1/x} + xe^{-1/x}(1/x^2) = e^{-1/x}(1 + 1/x)$
32. $y = x^r e^{sx} \Rightarrow y' = rx^{r-1} e^{sx} + sx^r e^{sx}$
33. $y = 2^{-t^2} \Rightarrow y' = 2^{-t^2}(\ln 2)(-2t) = (-2 \ln 2)t 2^{-t^2}$
34. $y = e^{\cos x} + \cos(e^x) \Rightarrow y' = -\sin x e^{\cos x} - e^x \sin(e^x)$
35. $H(v) = v \tan^{-1} v \Rightarrow H'(v) = v \cdot \frac{1}{1+v^2} + \tan^{-1} v \cdot 1 = \frac{v}{1+v^2} + \tan^{-1} v$
36. $F(z) = \log_{10}(1+z^2) \Rightarrow F'(z) = \frac{1}{(\ln 10)(1+z^2)} \cdot 2z = \frac{2z}{(\ln 10)(1+z^2)}$
37. $y = x \sinh(x^2) \Rightarrow y' = x \cosh(x^2) \cdot 2x + \sinh(x^2) \cdot 1 = 2x^2 \cosh(x^2) + \sinh(x^2)$

$$38. y = (\cos x)^x \Rightarrow \ln y = \ln(\cos x)^x = x \ln \cos x \Rightarrow \frac{y'}{y} = x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \Rightarrow$$

$$y' = (\cos x)^x (\ln \cos x - x \tan x)$$

$$39. y = \ln \sin x - \frac{1}{2} \sin^2 x \Rightarrow y' = \frac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2 \sin x \cdot \cos x = \cot x - \sin x \cos x$$

$$40. y = \arctan(\arcsin \sqrt{x}) \Rightarrow y' = \frac{1}{1 + (\arcsin \sqrt{x})^2} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$41. y = \ln \frac{1}{x} + \frac{1}{\ln x} = \ln x^{-1} + (\ln x)^{-1} = -\ln x + (\ln x)^{-1} \Rightarrow$$

$$y' = -1 \cdot \frac{1}{x} + (-1)(\ln x)^{-2} \cdot \frac{1}{x} = -\frac{1}{x} - \frac{1}{x(\ln x)^2}$$

$$42. xe^y = y - 1 \Rightarrow e^y + xe^y y' = y' \Rightarrow y' = e^y / (1 - xe^y)$$

$$43. y = \ln(\cosh 3x) \Rightarrow y' = (1/\cosh 3x)(\sinh 3x)(3) = 3 \tanh 3x$$

$$44. y = \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} \Rightarrow$$

$$\ln y = \ln \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} = \ln(x^2 + 1)^4 - \ln[(2x + 1)^3(3x - 1)^5]$$

$$= 4 \ln(x^2 + 1) - [\ln(2x + 1)^3 + \ln(3x - 1)^5] = 4 \ln(x^2 + 1) - 3 \ln(2x + 1) - 5 \ln(3x - 1) \Rightarrow$$

$$\frac{y'}{y} = 4 \cdot \frac{1}{x^2 + 1} \cdot 2x - 3 \cdot \frac{1}{2x + 1} \cdot 2 - 5 \cdot \frac{1}{3x - 1} \cdot 3 \Rightarrow$$

$$y' = \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} \left(\frac{8x}{x^2 + 1} - \frac{6}{2x + 1} - \frac{15}{3x - 1} \right).$$

[The answer could be simplified to $y' = -\frac{(x^2 + 56x + 9)(x^2 + 1)^3}{(2x + 1)^4(3x - 1)^6}$, but this is unnecessary.]

$$45. y = \cosh^{-1}(\sinh x) \Rightarrow y' = (\cosh x) / \sqrt{\sinh^2 x - 1}$$

$$46. y = x \tanh^{-1} \sqrt{x} \Rightarrow y' = \tanh^{-1} \sqrt{x} + x \frac{1}{1 - (\sqrt{x})^2} \frac{1}{2\sqrt{x}} = \tanh^{-1} \sqrt{x} + \frac{\sqrt{x}}{2(1-x)}$$

$$47. f(x) = e^{\sin^3(\ln(x^2+1))} \Rightarrow$$

$$f'(x) = e^{\sin^3(\ln(x^2+1))} \cdot 3 \sin^2(\ln(x^2+1)) \cdot \cos(\ln(x^2+1)) \cdot \frac{1}{x^2+1} \cdot 2x$$

$$= \frac{6x}{x^2+1} \sin^2(\ln(x^2+1)) \cdot \cos(\ln(x^2+1)) \cdot e^{\sin^3(\ln(x^2+1))}$$

$$48. \frac{d}{dx} \left(\frac{1}{2} \tan^{-1} x + \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} \right) = \frac{d}{dx} \left(\frac{1}{2} \tan^{-1} x + \frac{1}{2} \ln |x+1| - \frac{1}{4} \ln(x^2+1) \right)$$

$$= \frac{1}{2} \frac{1}{x^2+1} + \frac{1}{2} \frac{1}{x+1} - \frac{1}{4} \frac{2x}{x^2+1} = \frac{1}{2} \left(\frac{1}{x^2+1} - \frac{x}{x^2+1} + \frac{1}{x+1} \right)$$

$$= \frac{1}{2} \left(\frac{1-x}{x^2+1} + \frac{1}{x+1} \right) = \frac{1}{2} \left(\frac{1-x^2}{(x^2+1)(1+x)} + \frac{x^2+1}{(x^2+1)(1+x)} \right)$$

$$= \frac{1}{2} \frac{2}{(x^2+1)(1+x)} = \frac{1}{(1+x)(x^2+1)}$$

$$49. f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)} g'(x)$$

$$50. f(x) = g(e^x) \Rightarrow f'(x) = g'(e^x) e^x$$

$$51. f(x) = \ln |g(x)| \Rightarrow f'(x) = \frac{1}{g(x)} g'(x) = \frac{g'(x)}{g(x)}$$

$$52. f(x) = g(\ln x) \Rightarrow f'(x) = g'(\ln x) \cdot \frac{1}{x} = \frac{g'(\ln x)}{x}$$

$$53. f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2 \Rightarrow f''(x) = 2^x (\ln 2)^2 \Rightarrow \dots \Rightarrow f^{(n)}(x) = 2^x (\ln 2)^n$$

$$54. f(x) = \ln(2x) = \ln 2 + \ln x \Rightarrow f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = 2x^{-3}, f^{(4)}(x) = -2 \cdot 3x^{-4}, \dots, \\ f^{(n)}(x) = (-1)^{n-1} (n-1)! x^{-n}$$

55. We first show it is true for $n = 1$: $f'(x) = e^x + xe^x = (x+1)e^x$. We now assume it is true for $n = k$:

$f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$$f^{(k+1)}(x) = \frac{d}{dx} [f^{(k)}(x)] = \frac{d}{dx} [(x+k)e^x] = e^x + (x+k)e^x = [x+(k+1)]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

$$56. \text{ Using implicit differentiation, } y = x + \arctan y \Rightarrow y' = 1 + \frac{1}{1+y^2} y' \Rightarrow y' \left(1 - \frac{1}{1+y^2} \right) = 1 \Rightarrow$$

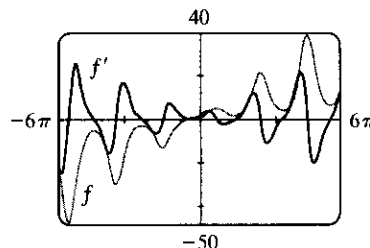
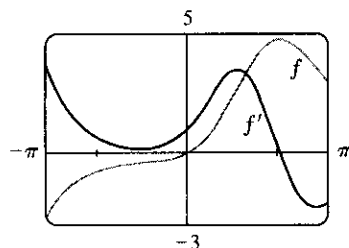
$$y' \left(\frac{y^2}{1+y^2} \right) = 1 \Rightarrow y' = \frac{1+y^2}{y^2} = \frac{1}{y^2} + 1.$$

$$57. y = (2+x)e^{-x} \Rightarrow y' = (2+x)(-e^{-x}) + e^{-x} \cdot 1 = e^{-x}[-(2+x)+1] = e^{-x}(-x-1). \text{ At } (0, 2), \\ y' = 1(-1) = -1, \text{ so an equation of the tangent line is } y - 2 = -1(x - 0), \text{ or } y = -x + 2.$$

$$58. y = f(x) = x \ln x \Rightarrow f'(x) = \ln x + 1, \text{ so the slope of the tangent at } (e, e) \text{ is } f'(e) = 2 \text{ and an equation is} \\ y - e = 2(x - e) \text{ or } y = 2x - e.$$

$$59. y = [\ln(x+4)]^2 \Rightarrow y' = 2 \frac{\ln(x+4)}{x+4} = 0 \Leftrightarrow \ln(x+4) = 0 \Leftrightarrow x+4 = 1 \Leftrightarrow x = -3, \text{ so the} \\ \text{tangent is horizontal at } (-3, 0).$$

$$60. f(x) = xe^{\sin x} \Rightarrow f'(x) = x[e^{\sin x}(\cos x)] + e^{\sin x}(1) = e^{\sin x}(x \cos x + 1). \text{ As a check on our work, we} \\ \text{notice from the graphs that } f'(x) > 0 \text{ when } f \text{ is increasing. Also, we see in the larger viewing rectangle a certain} \\ \text{similarity in the graphs of } f \text{ and } f': \text{ the sizes of the oscillations of } f \text{ and } f' \text{ are linked.}$$



$$61. (a) \text{ The line } x - 4y = 1 \text{ has slope } \frac{1}{4}. \text{ The tangent to } y = e^x \text{ has slope } \frac{1}{4} \text{ when } y' = e^x = \frac{1}{4} \Rightarrow \\ x = \ln \frac{1}{4} = -\ln 4, \text{ so an equation is } y - \frac{1}{4} = \frac{1}{4}(x + \ln 4) \text{ or } y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1).$$

$$(b) \text{ The slope of the tangent at the point } (a, e^a) \text{ is } \left. \frac{d}{dx} e^x \right|_{x=a} = e^a. \text{ An equation of the tangent line is thus}$$

$$y - e^a = e^a(x - a). \text{ We substitute } x = 0, y = 0 \text{ into this equation, since we want the line to pass through the} \\ \text{origin: } 0 - e^a = e^a(0 - a) \Leftrightarrow -e^a = e^a(-a) \Leftrightarrow a = 1. \text{ So an equation of the tangent is} \\ y - e = e(x - 1), \text{ or } y = ex.$$

$$62. (a) \lim_{t \rightarrow \infty} C(t) = \lim_{t \rightarrow \infty} [K(e^{-at} - e^{-bt})] = K \lim_{t \rightarrow \infty} (e^{-at} - e^{-bt}) = K(0 - 0) = 0 \text{ because } -at \rightarrow -\infty \text{ and } -bt \rightarrow -\infty \text{ as } t \rightarrow \infty.$$

$$(b) C(t) = K(e^{-at} - e^{-bt}) \Rightarrow C'(t) = K(e^{-at}(-a) - e^{-bt}(-b)) = K(-ae^{-at} + be^{-bt})$$

$$(c) C'(t) = 0 \Rightarrow be^{-bt} = ae^{-at} \Rightarrow \frac{b}{a} = e^{(-a+b)t} \Rightarrow \ln \frac{b}{a} = (b-a)t \Rightarrow t = \frac{\ln(b/a)}{b-a}$$

$$63. \lim_{x \rightarrow \infty} e^{-3x} = 0 \text{ since } -3x \rightarrow -\infty \text{ as } x \rightarrow \infty \text{ and } \lim_{t \rightarrow -\infty} e^t = 0.$$

$$64. \lim_{x \rightarrow 10^-} \ln(100 - x^2) = -\infty \text{ since as } x \rightarrow 10^-, (100 - x^2) \rightarrow 0^+.$$

$$65. \text{ Let } t = 2/(x-3). \text{ As } x \rightarrow 3^-, t \rightarrow -\infty. \lim_{x \rightarrow 3^-} e^{2/(x-3)} = \lim_{t \rightarrow -\infty} e^t = 0$$

$$66. \text{ If } y = x^3 - x = x(x^2 - 1), \text{ then as } x \rightarrow \infty, y \rightarrow \infty. \lim_{x \rightarrow \infty} \arctan(x^3 - x) = \lim_{y \rightarrow \infty} \arctan y = \frac{\pi}{2} \text{ by (7.5.8).}$$

$$67. \text{ Let } t = \sinh x. \text{ As } x \rightarrow 0^+, t \rightarrow 0^+. \lim_{x \rightarrow 0^+} \ln(\sinh x) = \lim_{t \rightarrow 0^+} \ln t = -\infty$$

$$68. -1 \leq \sin x \leq 1 \Rightarrow -e^{-x} \leq e^{-x} \sin x \leq e^{-x}. \text{ Now } \lim_{x \rightarrow \infty} (\pm e^{-x}) = 0, \text{ so by the Squeeze Theorem,}$$

$$\lim_{x \rightarrow \infty} e^{-x} \sin x = 0.$$

$$69. \lim_{x \rightarrow \infty} \frac{(1 + 2^x)/2^x}{(1 - 2^x)/2^x} = \lim_{x \rightarrow \infty} \frac{1/2^x + 1}{1/2^x - 1} = \frac{0 + 1}{0 - 1} = -1$$

$$70. \text{ Let } t = x/4, \text{ so } x = 4t. \text{ As } x \rightarrow \infty, t \rightarrow \infty. \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^x = \lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^{4t} = \left[\lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^t \right]^4 = e^4$$

$$71. \lim_{x \rightarrow 0} \frac{\tan \pi x}{\ln(1+x)} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\pi \sec^2 \pi x}{1/(1+x)} = \frac{\pi \cdot 1^2}{1/1} = \pi$$

$$72. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 + x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin x}{2x + 1} = \frac{0}{1} = 0$$

$$73. \lim_{x \rightarrow 0} \frac{e^{4x} - 1 - 4x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{4e^{4x} - 4}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{16e^{4x}}{2} = \lim_{x \rightarrow 0} 8e^{4x} = 8 \cdot 1 = 8$$

$$74. \lim_{x \rightarrow \infty} \frac{e^{4x} - 1 - 4x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{4e^{4x} - 4}{2x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{16e^{4x}}{2} = \lim_{x \rightarrow \infty} 8e^{4x} = \infty$$

$$75. \lim_{x \rightarrow \infty} x^3 e^{-x} = \lim_{x \rightarrow \infty} \frac{x^3}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{6x}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0$$

$$76. \lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x^2} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0^+} \left(-\frac{1}{2}x^2\right) = 0$$

$$\begin{aligned} 77. \lim_{x \rightarrow 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) &= \lim_{x \rightarrow 1^+} \left(\frac{x \ln x - x + 1}{(x-1) \ln x} \right) \stackrel{H}{=} \lim_{x \rightarrow 1^+} \frac{x \cdot (1/x) + \ln x - 1}{(x-1) \cdot (1/x) + \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{\ln x}{1 - 1/x + \ln x} \stackrel{H}{=} \lim_{x \rightarrow 1^+} \frac{1/x}{1/x^2 + 1/x} = \frac{1}{1+1} = \frac{1}{2} \end{aligned}$$

78. $y = (\tan x)^{\cos x} \Rightarrow \ln y = \cos x \ln \tan x$, so

$$\begin{aligned}\lim_{x \rightarrow (\pi/2)^-} \ln y &= \lim_{x \rightarrow (\pi/2)^-} \frac{\ln \tan x}{\sec x} \stackrel{H}{=} \lim_{x \rightarrow (\pi/2)^-} \frac{(1/\tan x) \sec^2 x}{\sec x \tan x} = \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{\tan^2 x} \\ &= \lim_{x \rightarrow (\pi/2)^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0, \text{ so}\end{aligned}$$

$$\lim_{x \rightarrow (\pi/2)^-} (\tan x)^{\cos x} = \lim_{x \rightarrow (\pi/2)^-} e^{\ln y} = e^0 = 1.$$

79. $y = f(x) = \tan^{-1}(1/x)$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x \rightarrow \pm\infty} \tan^{-1}(1/x) = \tan^{-1} 0 = 0$, so $y = 0$ is a HA. $\lim_{x \rightarrow 0^+} \tan^{-1}(1/x) = \frac{\pi}{2}$

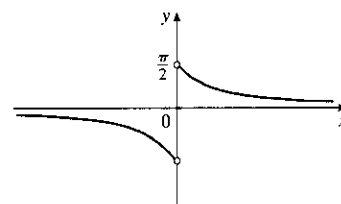
and $\lim_{x \rightarrow 0^-} \tan^{-1}(1/x) = -\frac{\pi}{2}$ since $\frac{1}{x} \rightarrow \pm\infty$ as $x \rightarrow 0^\pm$.

E. $f'(x) = \frac{1}{1 + (1/x)^2} (-1/x^2) = \frac{-1}{x^2 + 1} \Rightarrow f'(x) < 0$,

so f is decreasing on $(-\infty, 0)$ and $(0, \infty)$. **F.** No maximum nor minimum

G. $f''(x) = \frac{2x}{(x^2 + 1)^2} > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$.

H.



80. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \leq 1/x \leq 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept

C. $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim_{x \rightarrow \pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \frac{1}{\sqrt{1 - (1/x)^2}} \left(-\frac{1}{x^2}\right) = \frac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

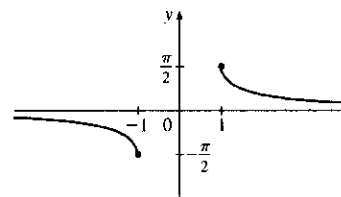
F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

G. $f''(x) = \frac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \frac{x(2x^2 - 1)}{(x^4 - x^2)^{3/2}} > 0$ for $x > 1$ and

$f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$.

No IP

H.



81. $y = f(x) = x \ln x$ **A.** $D = (0, \infty)$ **B.** No y -intercept; x -intercept 1. **C.** No symmetry **D.** No asymptote

[Note that the graph approaches the point $(0, 0)$ as $x \rightarrow 0^+$.]

E. $f'(x) = x(1/x) + (\ln x)(1) = 1 + \ln x$, so $f'(x) \rightarrow -\infty$ as $x \rightarrow 0^+$

and $f'(x) \rightarrow \infty$ as $x \rightarrow \infty$. $f'(x) = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow$

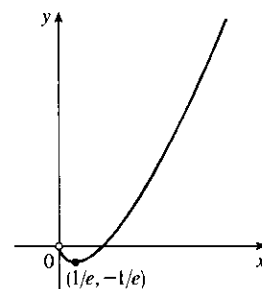
$x = e^{-1} = 1/e$. $f'(x) > 0$ for $x > 1/e$, so f is decreasing on $(0, 1/e)$

and increasing on $(1/e, \infty)$. **F.** Local minimum: $f(1/e) = -1/e$.

No local maximum. **G.** $f''(x) = 1/x$, so $f''(x) > 0$ for $x > 0$. The

graph is CU on $(0, \infty)$ and there is no IP.

H.



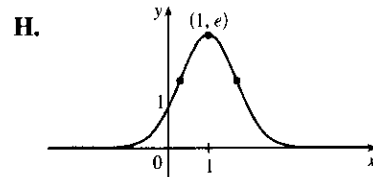
82. $y = f(x) = e^{2x-x^2}$ A. $D = \mathbb{R}$ B. y -intercept 1; no x -intercept C. No symmetry D. $\lim_{x \rightarrow \pm\infty} e^{2x-x^2} = 0$,

so $y = 0$ is a HA. E. $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1-x)e^{2x-x^2} > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. F. $f(1) = e$ is a local and absolute maximum value.

G. $f''(x) = 2(2x^2 - 4x + 1)e^{2x-x^2} = 0 \Leftrightarrow x = 1 \pm \frac{\sqrt{2}}{2}$.

$f''(x) > 0 \Leftrightarrow x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $(-\infty, 1 - \frac{\sqrt{2}}{2})$ and $(1 + \frac{\sqrt{2}}{2}, \infty)$, and CD on $(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2})$.

IP at $(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e})$



83. $y = f(x) = e^x + e^{-3x}$ A. $D = \mathbb{R}$ B. y -intercept 2; no x -intercept C. No symmetry

D. $\lim_{x \rightarrow \pm\infty} (e^x + e^{-3x}) = \infty$, no asymptote E. $y = f(x) = e^x + e^{-3x} \Rightarrow$

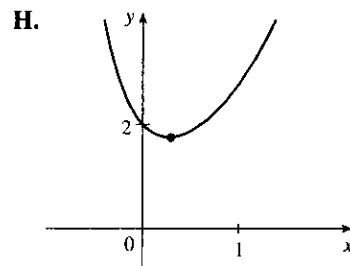
$f'(x) = e^x - 3e^{-3x} = e^{-3x}(e^{4x} - 3) > 0 \Leftrightarrow e^{4x} > 3 \Leftrightarrow$

$4x > \ln 3 \Leftrightarrow x > \frac{1}{4} \ln 3 \approx 0.27$, so f is increasing on $(\frac{1}{4} \ln 3, \infty)$

and decreasing on $(-\infty, \frac{1}{4} \ln 3)$.

F. Absolute minimum value $f(\frac{1}{4} \ln 3) = 3^{1/4} + 3^{-3/4} \approx 1.75$.

G. $f''(x) = e^x + 9e^{-3x} > 0$, so f is CU on $(-\infty, \infty)$. No IP



84. $y = f(x) = \ln(x^2 - 1)$ A. $D = (-\infty, -1) \cup (1, \infty)$ B. No y -intercept; x -intercepts $\pm\sqrt{2}$ C. Symmetric about the y -axis D. $\lim_{x \rightarrow \pm\infty} \ln(x^2 - 1) = \infty$, $\lim_{x \rightarrow 1^+} \ln(x^2 - 1) = -\infty$, $\lim_{x \rightarrow -1^-} \ln(x^2 - 1) = -\infty$, so $x = 1$

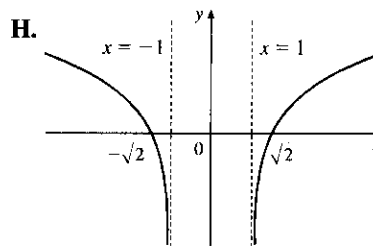
and $x = -1$ are VA. E. $y = f(x) = \ln(x^2 - 1) \Rightarrow f'(x) = \frac{2x}{x^2 - 1} > 0$ for $x > 1$ and $f'(x) < 0$

for $x < -1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$.

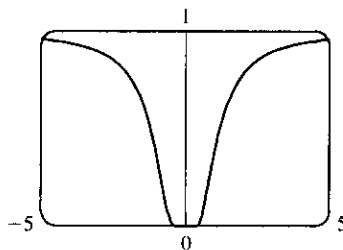
Note that the domain of f is $|x| > 1$. F. No extreme value

G. $f''(x) = -2 \frac{x^2 + 1}{(x^2 - 1)^2} < 0$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$.

No IP



85.



From the graph, we estimate the points of inflection to be about

$(\pm 0.82, 0.22)$. $f(x) = e^{-1/x^2} \Rightarrow f'(x) = 2x^{-3}e^{-1/x^2} \Rightarrow$

$$f''(x) = 2 \left[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4}) \right]$$

$$= 2x^{-6}e^{-1/x^2}(2 - 3x^2).$$

This is 0 when $2 - 3x^2 = 0 \Leftrightarrow x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $(\pm\sqrt{\frac{2}{3}}, e^{-3/2})$.

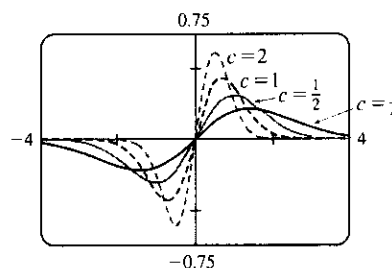
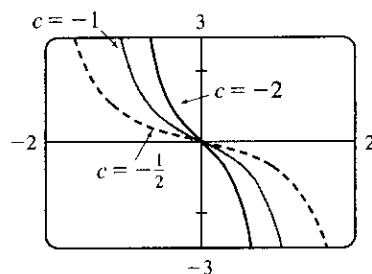
86. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x . To find the maxima and minima, we differentiate:

$$f(x) = cxe^{-cx^2} \Rightarrow f'(x) = c[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)] = ce^{-cx^2}(-2cx^2 + 1). \text{ This is 0 where}$$

$-2cx^2 + 1 = 0 \Leftrightarrow x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x -coordinates approach 0 as c increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f(\pm 1/\sqrt{2c}) = c(\pm 1/\sqrt{2c})e^{-c(\pm 1/\sqrt{2c})^2} = \pm \sqrt{c/2e}$.

So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim_{x \rightarrow \pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim_{x \rightarrow \pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}(-2cx^2 + 1) \Rightarrow f''(x) = c[e^{-cx^2}(-4cx) + (-2cx^2 + 1)(-2cxe^{-cx^2})] = -2c^2xe^{-cx^2}(3 - 2cx^2)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \Leftrightarrow x = \pm\sqrt{3/(2c)} \Rightarrow$ IP at $(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}e^{-3/2})$. If $c > 0$ there are three inflection points, and as c increases, the x -coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.



87. $s(t) = Ae^{-ct} \cos(\omega t + \delta) \Rightarrow$

$$v(t) = s'(t) = A\{e^{-ct}[-\omega \sin(\omega t + \delta)] + \cos(\omega t + \delta)(-ce^{-ct})\} \\ = -Ae^{-ct}[\omega \sin(\omega t + \delta) + c \cos(\omega t + \delta)] \Rightarrow$$

$$a(t) = v'(t) = -A\{e^{-ct}[\omega^2 \cos(\omega t + \delta) - c\omega \sin(\omega t + \delta)] + [\omega \sin(\omega t + \delta) + c \cos(\omega t + \delta)](-ce^{-ct})\} \\ = -Ae^{-ct}[\omega^2 \cos(\omega t + \delta) - c\omega \sin(\omega t + \delta) - c\omega \sin(\omega t + \delta) - c^2 \cos(\omega t + \delta)] \\ = -Ae^{-ct}[(\omega^2 - c^2) \cos(\omega t + \delta) - 2c\omega \sin(\omega t + \delta)] \\ = Ae^{-ct}[(c^2 - \omega^2) \cos(\omega t + \delta) + 2c\omega \sin(\omega t + \delta)]$$

88. (a) Let $f(x) = \ln x + x - 3$. Then $f'(x) = 1/x + 1 > 0$ (for $x > 0$) and $f(2) \approx -0.307$ and $f(e) \approx 0.718$. f is differentiable on $(2, e)$, continuous on $[2, e]$ and $f(2) < 0$, $f(e) > 0$. Therefore, by the Intermediate Value Theorem there exists a number c in $(2, e)$ such that $f(c) = 0$. Thus, there is one root. But $f'(x) > 0$ for $x \in (2, e)$, so f is increasing on $(2, e)$, which means that there is exactly one root.

(b) We use Newton's Method with $f(x) = \ln x + x - 3$, $f'(x) = 1/x + 1$, and $x_1 = 2$.

$$x_2 = x_1 - \frac{\ln x_1 + x_1 - 3}{1/x_1 + 1} = 2 - \frac{\ln 2 + 2 - 3}{1/2 + 1} \approx 2.20457. \text{ Similarly, } x_3 \approx 2.20794, x_4 = 2.20794. \text{ Thus, the root of the equation, correct to four decimal places, is } 2.2079.$$

89. Let $P(t) = \frac{64}{1 + 31e^{-0.7944t}} = \frac{A}{1 + Be^{ct}} = A(1 + Be^{ct})^{-1}$, where $A = 64$, $B = 31$, and $c = -0.7944$.

$$P'(t) = -A(1 + Be^{ct})^{-2} (Bce^{ct}) = -ABce^{ct}(1 + Be^{ct})^{-2}$$

$$\begin{aligned} P''(t) &= -ABce^{ct} \left[-2(1 + Be^{ct})^{-3} (Bce^{ct}) \right] + (1 + Be^{ct})^{-2} (-ABc^2e^{ct}) \\ &= -ABc^2e^{ct}(1 + Be^{ct})^{-3} [-2Be^{ct} + (1 + Be^{ct})] = -\frac{ABc^2e^{ct}(1 - Be^{ct})}{(1 + Be^{ct})^3} \end{aligned}$$

The population is increasing most rapidly when its graph changes from CU to CD; that is,

when $P''(t) = 0$ in this case. $P''(t) = 0 \Rightarrow Be^{ct} = 1 \Rightarrow e^{ct} = \frac{1}{B} \Rightarrow$

$$ct = \ln \frac{1}{B} \Rightarrow t = \frac{\ln(1/B)}{c} = \frac{\ln(1/31)}{-0.7944} \approx 4.32 \text{ days. Note that}$$

$$P\left(\frac{1}{c} \ln \frac{1}{B}\right) = \frac{A}{1 + Be^{c(1/c) \ln(1/B)}} = \frac{A}{1 + Be^{\ln(1/B)}} = \frac{A}{1 + B(1/B)} = \frac{A}{1 + 1} = \frac{A}{2}, \text{ one-half the limit of } P \text{ as } t \rightarrow \infty.$$

90. Let $t = 4u$. Then $dt = 4 du$ and

$$\begin{aligned} \int_0^4 \frac{1}{16 + t^2} dt &= \int_0^1 \frac{1}{16 + 16u^2} \cdot 4 du = \frac{1}{4} \int_0^1 \frac{du}{1 + u^2} = \frac{1}{4} [\tan^{-1} u]_0^1 \\ &= \frac{1}{4} (\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{4} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{16} \end{aligned}$$

91. Let $u = -2y^2$. Then $du = -4y dy$ and

$$\int_0^1 ye^{-2y^2} dy = \int_0^{-2} e^u \left(-\frac{1}{4} du\right) = -\frac{1}{4} [e^u]_0^{-2} = -\frac{1}{4} (e^{-2} - 1) = \frac{1}{4} (1 - e^{-2}).$$

92. $\int_2^5 \frac{dr}{1 + 2r} = \frac{1}{2} [\ln |1 + 2r|]_2^5 = \frac{1}{2} (\ln 11 - \ln 5) = \frac{1}{2} \ln \frac{11}{5}$

93. $\int_2^4 \frac{1 + x - x^2}{x^2} dx = \int_2^4 \left(x^{-2} + \frac{1}{x} - 1 \right) dx = \left[-\frac{1}{x} + \ln x - x \right]_2^4$
 $= \left(-\frac{1}{4} + \ln 4 - 4 \right) - \left(-\frac{1}{2} + \ln 2 - 2 \right) = \ln 2 - \frac{7}{4}$

94. Let $u = \sin x$. Then $du = \cos x dx$, so

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx = \int_0^1 \frac{1}{1 + u^2} du = [\tan^{-1} u]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

95. Let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}} \Rightarrow \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C.$

96. Let $u = \ln x$. Then $du = \frac{dx}{x} \Rightarrow \int \frac{\cos(\ln x)}{x} dx = \int \cos u du = \sin u + C = \sin(\ln x) + C.$

97. Let $u = x^2 + 2x$. Then $du = (2x + 2) dx = 2(x + 1) dx$ and

$$\int \frac{x + 1}{x^2 + 2x} dx = \int \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 2x| + C.$$

98. Let $u = e^{-x}$. Then $du = -e^{-x} dx$ and

$$\int \frac{e^{-x}}{1 + e^{-2x}} dx = \int \frac{-du}{1 + u^2} = -\tan^{-1} u + C = -\tan^{-1}(e^{-x}) + C.$$

99. Let $u = \ln(\cos x)$. Then $du = \frac{-\sin x}{\cos x} dx = -\tan x dx \Rightarrow$

$$\int \tan x \ln(\cos x) dx = -\int u du = -\frac{1}{2}u^2 + C = -\frac{1}{2}[\ln(\cos x)]^2 + C.$$

100. Let $u = x^2$. Then $du = 2x dx \Rightarrow \int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1}(x^2) + C.$

101. Let $u = \tan \theta$. Then $du = \sec^2 \theta d\theta$ and $\int 2^{\tan \theta} \sec^2 \theta d\theta = \int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{\tan \theta}}{\ln 2} + C.$

102. $\int \sinh au du = \frac{1}{a} \cosh au + C$

103. Let $u = 1 + \sec \theta$, so $du = \sec \theta \tan \theta d\theta \Rightarrow \int \frac{\sec \theta \tan \theta}{1 + \sec \theta} d\theta = \int \frac{1}{u} du = \ln |u| + C = \ln |1 + \sec \theta| + C.$

104. $1 + e^{2x} > e^{2x} \Rightarrow \sqrt{1 + e^{2x}} > \sqrt{e^{2x}} = e^x \Rightarrow \int_0^1 \sqrt{1 + e^{2x}} dx \geq \int_0^1 e^x dx = e^x \Big|_0^1 = e - 1$

105. $\cos x \leq 1 \Rightarrow e^x \cos x \leq e^x \Rightarrow \int_0^1 e^x \cos x dx \leq \int_0^1 e^x dx = e^x \Big|_0^1 = e - 1$

106. For $0 \leq x \leq 1$, $0 \leq \sin^{-1} x \leq \frac{\pi}{2}$, so $\int_0^1 x \sin^{-1} x dx \leq \int_0^1 x \left(\frac{\pi}{2}\right) dx = \frac{\pi}{4} x^2 \Big|_0^1 = \frac{\pi}{4}.$

107. $f(x) = \int_1^{\sqrt{x}} \frac{e^s}{s} ds \Rightarrow f'(x) = \frac{d}{dx} \int_1^{\sqrt{x}} \frac{e^s}{s} ds = \frac{e^{\sqrt{x}}}{\sqrt{x}} \frac{d}{dx} \sqrt{x} = \frac{e^{\sqrt{x}}}{\sqrt{x}} \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x}$

108. $f(x) = \int_{\ln x}^{2x} e^{-t^2} dt \Rightarrow f'(x) = \frac{d}{dx} \int_{\ln x}^{2x} e^{-t^2} dt = -\frac{d}{dx} \int_0^{\ln x} e^{-t^2} dt + \frac{d}{dx} \int_0^{2x} e^{-t^2} dt$
 $= -e^{-(\ln x)^2} \left(\frac{1}{x}\right) + e^{-(2x)^2} (2) = -\frac{e^{-(\ln x)^2}}{x} + 2e^{-4x^2}$

109. $f_{\text{ave}} = \frac{1}{4-1} \int_1^4 \frac{1}{x} dx = \frac{1}{3} [\ln |x|]_1^4 = \frac{1}{3} [\ln 4 - \ln 1] = \frac{1}{3} \ln 4$

110. $A = \int_{-2}^0 (e^{-x} - e^x) dx + \int_0^1 (e^x - e^{-x}) dx = [-e^{-x} - e^x]_{-2}^0 + [e^x + e^{-x}]_0^1$
 $= [(-1 - 1) - (-e^2 - e^{-2})] + [(e + e^{-1}) - (1 + 1)] = e^2 + e + e^{-1} + e^{-2} - 4$

111. $V = \int_0^1 \frac{2\pi x}{1+x^4} dx$ by cylindrical shells. Let $u = x^2 \Rightarrow du = 2x dx$. Then

$$V = \int_0^1 \frac{\pi}{1+u^2} du = \pi [\tan^{-1} u]_0^1 = \pi (\tan^{-1} 1 - \tan^{-1} 0) = \pi \left(\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

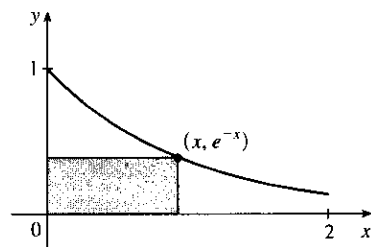
112. $f(x) = x + x^2 + e^x \Rightarrow f'(x) = 1 + 2x + e^x$ and $f(0) = 1 \Rightarrow g(1) = 0$, so

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

113. $f(x) = \ln x + \tan^{-1} x \Rightarrow f(1) = \ln 1 + \tan^{-1} 1 = \frac{\pi}{4} \Rightarrow g\left(\frac{\pi}{4}\right) = 1.$

$$f'(x) = \frac{1}{x} + \frac{1}{1+x^2}, \text{ so } g'\left(\frac{\pi}{4}\right) = \frac{1}{f'(1)} = \frac{1}{3/2} = \frac{2}{3}.$$

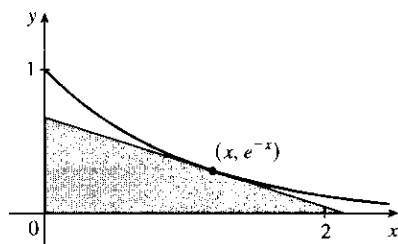
114.



The area of such a rectangle is just the product of its sides, that is,

$A(x) = x \cdot e^{-x}$. We want to find the maximum of this function, so we differentiate: $A'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1 - x)$. This is 0 only at $x = 1$, and changes from positive to negative there, so by the First Derivative Test this gives a local maximum. So the largest area is $A(1) = 1/e$.

115.



We find the equation of a tangent to the curve $y = e^{-x}$, so that we can find the x - and y -intercepts of this tangent, and then we can find the area of the triangle. The slope of the tangent at the point (a, e^{-a}) is given by $\left. \frac{d}{dx} e^{-x} \right|_{x=a} = -e^{-a}$, and so the equation of the tangent is $y - e^{-a} = -e^{-a}(x - a) \Leftrightarrow y = e^{-a}(a - x + 1)$.

The y -intercept of this line is $y = e^{-a}(a - 0 + 1) = e^{-a}(a + 1)$. To find the x -intercept we set

$y = 0 \Rightarrow e^{-a}(a - x + 1) = 0 \Rightarrow x = a + 1$. So the area of the triangle is

$A(a) = \frac{1}{2} [e^{-a}(a + 1)](a + 1) = \frac{1}{2} e^{-a}(a + 1)^2$. We differentiate this with respect to a :

$A'(a) = \frac{1}{2} [e^{-a}(2)(a + 1) + (a + 1)^2 e^{-a}(-1)] = \frac{1}{2} e^{-a}(1 - a^2)$. This is 0 at $a = \pm 1$, and the root $a = 1$ gives a maximum, by the First Derivative Test. So the maximum area of the triangle is

$$A(1) = \frac{1}{2} e^{-1}(1 + 1)^2 = 2e^{-1} = 2/e.$$

116. Using Formula 5.2.3 with $a = 0$ and $b = 1$, we have $\int_0^1 e^x dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n e^{i/n}$. This series is a geometric

series with $a = r = e^{1/n}$, so $\sum_{i=1}^n e^{i/n} = e^{1/n} \frac{e^{n/n} - 1}{e^{1/n} - 1} = e^{1/n} \frac{e - 1}{e^{1/n} - 1} \Rightarrow$

$$\int_0^1 e^x dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n e^{i/n} = \lim_{n \rightarrow \infty} (e - 1) e^{1/n} \frac{1/n}{e^{1/n} - 1}. \text{ As } n \rightarrow \infty, 1/n \rightarrow 0^+, \text{ so } e^{1/n} \rightarrow e^0 = 1.$$

Let $t = 1/n$. Then $e^{1/n} - 1 = e^t - 1 \rightarrow 0^+$, so l'Hospital's Rule gives $\lim_{t \rightarrow 0} \frac{t}{e^t - 1} = \lim_{t \rightarrow 0} \frac{1}{e^t} = 1$ and we have

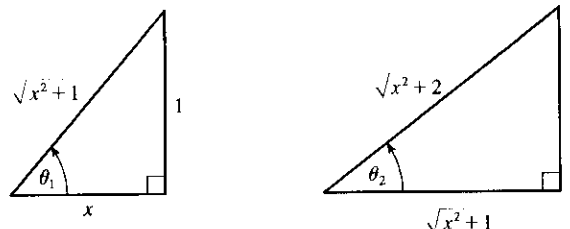
$$\int_0^1 e^x dx = \left[\lim_{t \rightarrow 0^+} (e - 1) e^t \right] \left[\lim_{t \rightarrow 0^+} \frac{t}{e^t - 1} \right] = e - 1.$$

117. $\lim_{x \rightarrow -1} F(x) = \lim_{x \rightarrow -1} \frac{b^{x+1} - a^{x+1}}{x + 1} \stackrel{H}{=} \lim_{x \rightarrow -1} \frac{b^{x+1} \ln b - a^{x+1} \ln a}{1} = \ln b - \ln a = F(-1)$, so F is continuous at -1 .

118. Let $\theta_1 = \operatorname{arccot} x$, so $\cot \theta_1 = x = x/1$. So $\sin(\operatorname{arccot} x) = \sin \theta_1 = \frac{1}{\sqrt{x^2 + 1}}$.

Let $\theta_2 = \arctan \left[\frac{1}{\sqrt{x^2 + 1}} \right]$, so $\tan \theta_2 = \frac{1}{\sqrt{x^2 + 1}}$.

Hence, $\cos(\arctan(\sin(\operatorname{arccot} x))) = \cos \theta_2 = \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 2}} = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$.



119. Differentiating both sides of the given equation, using the Fundamental Theorem for each side, gives

$$f(x) = e^{2x} + 2xe^{2x} + e^{-x}f(x). \text{ So } f(x)(1 - e^{-x}) = e^{2x} + 2xe^{2x}. \text{ Hence } f(x) = \frac{e^{2x}(1 + 2x)}{1 - e^{-x}}.$$

120. (a) Let $f(x) = x - \ln x - 1$, so $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. Since $x > 0$, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. So there is an absolute minimum at $x = 1$ with $f(1) = 0$.

So for $x > 0$, $x \neq 1$, $x - \ln x - 1 = f(x) > f(1) = 0$, and hence $\ln x < x - 1$.

(b) Here let $f(x) = \ln x - \frac{x-1}{x} = \ln x - 1 + \frac{1}{x}$. So $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$. As in (a), we see that there is an absolute minimum value at $x = 1$ and that $f(1) = 0$. So for $x > 0$, $x \neq 1$, $\ln x - \frac{x-1}{x} = f(x) > f(1) = 0$ and hence $\frac{x-1}{x} < \ln x$.

(c) Let $b > a > 0$, so $b/a > 1$. Letting $x = b/a$ in the inequalities in (a) and (b) gives

$\frac{b-a}{b} = \frac{b/a - 1}{b/a} < \ln \frac{b}{a} < \frac{b}{a} - 1 = \frac{b-a}{a}$. Noting that $\ln \frac{b}{a} = \ln b - \ln a$, the result follows after dividing through by $b - a$.

(d) Let $f(x) = \ln x$. From the given diagram, we see that

(slope of tangent at $x = b$) < (slope of secant line) < (slope of tangent at $x = a$). Since $f'(x) = \frac{1}{x}$, we

therefore have $\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$. To make this geometric argument more rigorous, we could use the Mean Value Theorem: For any a and b with $0 < a < b$, there exists some $c \in (a, b)$ for which

$$f'(c) = \frac{1}{c} = \frac{\ln b - \ln a}{b - a}. \text{ But } \frac{1}{x} \text{ is a decreasing function on } (0, \infty), \text{ so } \frac{1}{b} < \frac{1}{c} = \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

(e) Since $\frac{1}{b} < \frac{1}{x} < \frac{1}{a}$ for $a < x < b$, Property 8 says that $\frac{1}{b}(b-a) < \int_a^b \frac{1}{x} dx < \frac{1}{a}(b-a) \Rightarrow$

$\frac{1}{b}(b-a) < \ln b - \ln a < \frac{1}{a}(b-a) \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$. (Note from the proof of Property 8 that we are justified in making all of the inequalities strict.)

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□ PROBLEMS PLUS

- Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm \frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.
- We use proof by contradiction. Suppose that $\log_2 5$ is a rational number. Then $\log_2 5 = m/n$ where m and n are positive integers $\Rightarrow 2^{m/n} = 5 \Rightarrow 2^m = 5^n$. But this is impossible since 2^m is even and 5^n is odd. So $\log_2 5$ is irrational.

- Consider the statement that $\frac{d^n}{dx^n}(e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$. For $n = 1$,

$$\frac{d}{dx}(e^{ax} \sin bx) = ae^{ax} \sin bx + be^{ax} \cos bx, \text{ and}$$

$$\begin{aligned} re^{ax} \sin(bx + \theta) &= re^{ax} [\sin bx \cos \theta + \cos bx \sin \theta] = re^{ax} \left(\frac{a}{r} \sin bx + \frac{b}{r} \cos bx \right) \\ &= ae^{ax} \sin bx + be^{ax} \cos bx \end{aligned}$$

$$\text{since } \tan \theta = \frac{b}{a} \Rightarrow \sin \theta = \frac{b}{r} \text{ and } \cos \theta = \frac{a}{r}.$$

So the statement is true for $n = 1$. Assume it is true for $n = k$. Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax} \sin bx) &= \frac{d}{dx} \left[r^k e^{ax} \sin(bx + k\theta) \right] = r^k ae^{ax} \sin(bx + k\theta) + r^k e^{ax} b \cos(bx + k\theta) \\ &= r^k e^{ax} [a \sin(bx + k\theta) + b \cos(bx + k\theta)] \end{aligned}$$

But

$$\begin{aligned} \sin[bx + (k+1)\theta] &= \sin[(bx + k\theta) + \theta] = \sin(bx + k\theta) \cos \theta + \sin \theta \cos(bx + k\theta) \\ &= \frac{a}{r} \sin(bx + k\theta) + \frac{b}{r} \cos(bx + k\theta) \end{aligned}$$

Hence, $a \sin(bx + k\theta) + b \cos(bx + k\theta) = r \sin[bx + (k+1)\theta]$. So

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax} \sin bx) &= r^k e^{ax} [a \sin(bx + k\theta) + b \cos(bx + k\theta)] = r^k e^{ax} [r \sin(bx + (k+1)\theta)] \\ &= r^{k+1} e^{ax} \sin(bx + (k+1)\theta) \end{aligned}$$

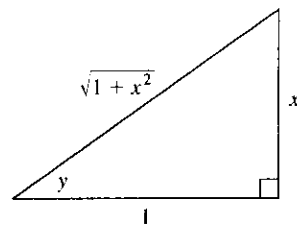
Therefore, the statement is true for all n by mathematical induction.

4. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}. \text{ Using this fact we have that}$$

$$\sin(\tan^{-1}(\sinh x)) = \frac{\sinh x}{\sqrt{1+\sinh^2 x}} = \frac{\sinh x}{\cosh x} = \tanh x. \text{ Hence,}$$

$$\sin^{-1}(\tanh x) = \sin^{-1}(\sin(\tan^{-1}(\sinh x))) = \tan^{-1}(\sinh x).$$



5. We first show that $\frac{x}{1+x^2} < \tan^{-1} x$ for $x > 0$. Let $f(x) = \tan^{-1} x - \frac{x}{1+x^2}$. Then

$$f'(x) = \frac{1}{1+x^2} - \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{(1+x^2) - (1-x^2)}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \text{ for } x > 0. \text{ So } f(x) \text{ is}$$

increasing on $(0, \infty)$. Hence, $0 < x \Rightarrow 0 = f(0) < f(x) = \tan^{-1} x - \frac{x}{1+x^2}$. So $\frac{x}{1+x^2} < \tan^{-1} x$

for $0 < x$. We next show that $\tan^{-1} x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1} x$. Then

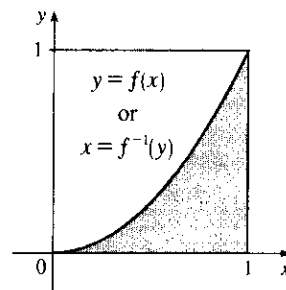
$$h'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0. \text{ Hence, } h(x) \text{ is increasing on } (0, \infty). \text{ So for } 0 < x,$$

$0 = h(0) < h(x) = x - \tan^{-1} x$. Hence, $\tan^{-1} x < x$ for $x > 0$, and we conclude that $\frac{x}{1+x^2} < \tan^{-1} x < x$ for $x > 0$.

6. The shaded region has area $\int_0^1 f(x) dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y) dy$

gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$.

$$\text{So } \int_0^1 f^{-1}(y) dy = \frac{2}{3}.$$



7. By the Fundamental Theorem of Calculus, $f(x) = \int_1^x \sqrt{1+t^3} dt \Rightarrow f'(x) = \sqrt{1+x^3} > 0$ for $x > -1$.

So f is increasing on $(-1, \infty)$ and hence is one-to-one. Note that $f(1) = 0$, so $f^{-1}(1) = 0 \Rightarrow$

$$(f^{-1})'(0) = 1/f'(1) = \frac{1}{\sqrt{2}}.$$

8. $y = \frac{x}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \arctan \frac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$\begin{aligned} y' &= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \sin^2 x / (k + \cos x)^2} \cdot \frac{\cos x(k + \cos x) + \sin^2 x}{(k + \cos x)^2} \\ &= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k \cos x + \cos^2 x + \sin^2 x}{(k + \cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k \cos x + 1}{k^2 + 2k \cos x + 1} \\ &= \frac{k^2 + 2k \cos x + 1 - 2k \cos x - 2}{\sqrt{a^2-1}(k^2 + 2k \cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2-1}(k^2 + 2k \cos x + 1)} \end{aligned}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a(a + \sqrt{a^2-1}) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and

$$k^2 - 1 = 2(ak - 1). \text{ So } y' = \frac{2(ak - 1)}{\sqrt{a^2 - 1}(2ak + 2k \cos x)} = \frac{ak - 1}{\sqrt{a^2 - 1}k(a + \cos x)}. \text{ But}$$

$$ak - 1 = a^2 + a\sqrt{a^2 - 1} - 1 = k\sqrt{a^2 - 1}, \text{ so } y' = 1/(a + \cos x).$$

9. If $L = \lim_{x \rightarrow \infty} \left(\frac{x+a}{x-a} \right)^x$, then L has the indeterminate form 1^∞ , so

$$\begin{aligned} \ln L &= \lim_{x \rightarrow \infty} \ln \left(\frac{x+a}{x-a} \right)^x = \lim_{x \rightarrow \infty} x \ln \left(\frac{x+a}{x-a} \right) = \lim_{x \rightarrow \infty} \frac{\ln(x+a) - \ln(x-a)}{1/x} \\ &\stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x+a} - \frac{1}{x-a}}{-1/x^2} = \lim_{x \rightarrow \infty} \left[\frac{(x-a) - (x+a)}{(x+a)(x-a)} \cdot \frac{-x^2}{1} \right] \\ &= \lim_{x \rightarrow \infty} \frac{2ax^2}{x^2 - a^2} = \lim_{x \rightarrow \infty} \frac{2a}{1 - a^2/x^2} = 2a. \end{aligned}$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$.

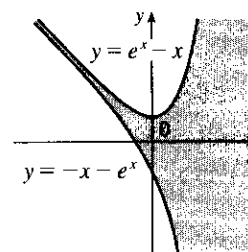
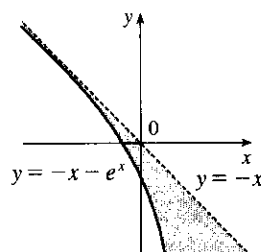
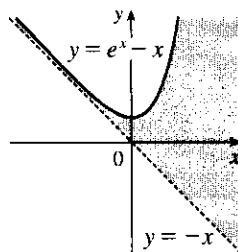
10. Case (i) (first graph): For $x + y \geq 0$, that is, $y \geq -x$, $|x + y| = x + y \leq e^x \Rightarrow y \leq e^x - x$.

Note that $y = e^x - x$ is always above the line $y = -x$ and that $y = -x$ is a slant asymptote.

Case (ii) (second graph): For $x + y < 0$, that is, $y < -x$, $|x + y| = -x - y \leq e^x \Rightarrow y \geq -x - e^x$.

Note that $-x - e^x$ is always below the line $y = -x$ and $y = -x$ is a slant asymptote.

Putting the two pieces together gives the third graph.



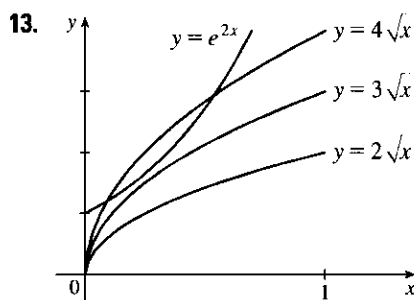
11. Both sides of the inequality are positive, so $\cosh(\sinh x) < \sinh(\cosh x) \Leftrightarrow \cosh^2(\sinh x) < \sinh^2(\cosh x)$
- $$\Leftrightarrow \sinh^2(\sinh x) + 1 < \sinh^2(\cosh x) \Leftrightarrow 1 < [\sinh(\cosh x) - \sinh(\sinh x)][\sinh(\cosh x) + \sinh(\sinh x)]$$
- $$\Leftrightarrow 1 < \left[\sinh\left(\frac{e^x + e^{-x}}{2}\right) - \sinh\left(\frac{e^x - e^{-x}}{2}\right) \right] \left[\sinh\left(\frac{e^x + e^{-x}}{2}\right) + \sinh\left(\frac{e^x - e^{-x}}{2}\right) \right]$$
- $$\Leftrightarrow 1 < [2 \cosh(e^x/2) \sinh(e^x/2)][2 \sinh(e^x/2) \cosh(e^x/2)] \quad [\text{use the addition formulas and cancel}]$$
- $$\Leftrightarrow 1 < [2 \sinh(e^x/2) \cosh(e^x/2)][2 \sinh(e^x/2) \cosh(e^x/2)] \Leftrightarrow 1 < \sinh e^x \sinh e^{-x},$$

by the half-angle formula. Now both e^x and e^{-x} are positive, and $\sinh y > y$ for $y > 0$, since $\sinh 0 = 0$ and

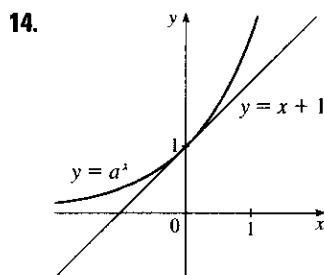
$(\sinh y - y)' = \cosh y - 1 > 0$ for $x > 0$, so $1 = e^x e^{-x} < \sinh e^x \sinh e^{-x}$. So, following this chain of

reasoning backward, we arrive at the desired result.

12. First, we recognize some symmetry in the inequality: $\frac{e^{x+y}}{xy} \geq e^2 \Leftrightarrow \frac{e^x}{x} \cdot \frac{e^y}{y} \geq e \cdot e$. This suggests that we need to show that $\frac{e^x}{x} \geq e$ for $x > 0$. If we can do this, then the inequality $\frac{e^y}{y} \geq e$ is true, and the given inequality follows. $f(x) = \frac{e^x}{x} \Rightarrow f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2} = 0 \Rightarrow x = 1$. By the First Derivative Test, we have a minimum of $f(1) = e$, so $e^x/x \geq e$ for all x .



Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ ($k > 0$). From the graphs of f and g , we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$. $f(a) = g(a) \Rightarrow e^{2a} = k\sqrt{a}$ (1) and $f'(a) = g'(a) \Rightarrow 2e^{2a} = \frac{k}{2\sqrt{a}} \Rightarrow e^{2a} = \frac{k}{4\sqrt{a}}$. So we must have $k\sqrt{a} = \frac{k}{4\sqrt{a}} \Rightarrow (\sqrt{a})^2 = \frac{k}{4k} \Rightarrow a = \frac{1}{4}$. From (1), $e^{2(1/4)} = k\sqrt{1/4} \Rightarrow k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.



We see that at $x = 0$, $f(x) = a^x = 1 + x = 1$, so if $y = a^x$ is to lie above $y = 1 + x$, the two curves must just touch at $(0, 1)$, that is, we must have $f'(0) = 1$.

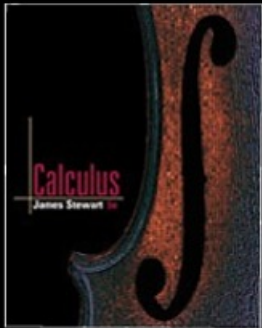
[To see this analytically, note that $a^x \geq 1 + x \Rightarrow a^x - 1 \geq x \Rightarrow \frac{a^x - 1}{x} \geq 1$ for $x > 0$, so $f'(0) = \lim_{x \rightarrow 0^+} \frac{a^x - 1}{x} \geq 1$. Similarly, for $x < 0$, $a^x - 1 \geq x \Rightarrow \frac{a^x - 1}{x} \leq 1$, so $f'(0) = \lim_{x \rightarrow 0^-} \frac{a^x - 1}{x} \leq 1$. Since $1 \leq f'(0) \leq 1$, we must have $f'(0) = 1$.] But $f'(x) = a^x \ln a \Rightarrow f'(0) = \ln a$, so we have $\ln a = 1 \Leftrightarrow a = e$.

Another method: The inequality certainly holds for $x \leq -1$, so consider $x > -1$, $x \neq 0$. Then $a^x \geq 1 + x \Rightarrow a \geq (1+x)^{1/x}$ for $x > 0 \Rightarrow a \geq \lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$, by Equation 7.4.8. Also, $a^x \geq 1 + x \Rightarrow a \leq (1+x)^{1/x}$ for $x < 0 \Rightarrow a \leq \lim_{x \rightarrow 0^-} (1+x)^{1/x} = e$. So since $e \leq a \leq e$, we must have $a = e$.

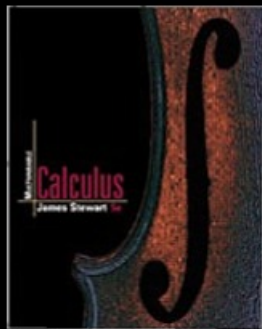
15. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, $x > 0$, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$. $g'(x) = e^{(1/x) \ln x} \left(-\frac{1}{x^2} \ln x + \frac{1}{x} \cdot \frac{1}{x} \right) = x^{1/x} \left(\frac{1}{x^2} \right) (1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \leq e^{1/e}$. Conversely, suppose that $0 < a \leq e^{1/e}$. Then $a^e \leq e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

Chapter 8

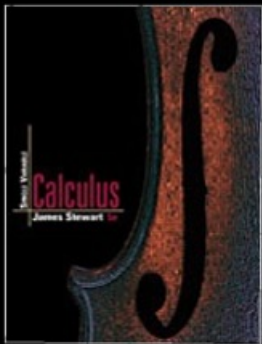
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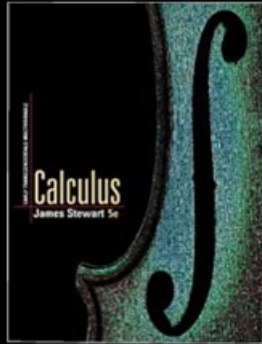
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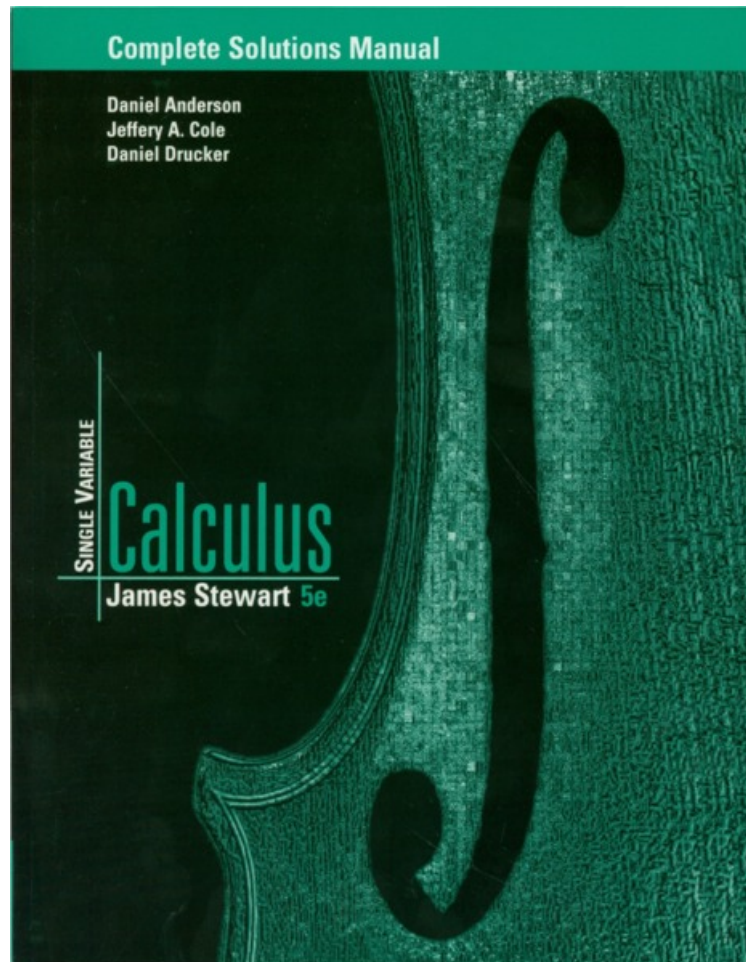
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8 TECHNIQUES OF INTEGRATION

8.1 Integration by Parts

1. Let $u = \ln x$, $dv = x dx \Rightarrow du = dx/x$, $v = \frac{1}{2}x^2$. Then by Equation 2, $\int u dv = uv - \int v du$,

$$\begin{aligned}\int x \ln x dx &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2(dx/x) = \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{1}{2}x^2 + C \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\end{aligned}$$

2. Let $u = \theta$, $dv = \sec^2 \theta d\theta \Rightarrow du = d\theta$, $v = \tan \theta$. Then

$$\int \theta \sec^2 \theta d\theta = \theta \tan \theta - \int \tan \theta d\theta = \theta \tan \theta - \ln |\sec \theta| + C.$$

3. Let $u = x$, $dv = \cos 5x dx \Rightarrow du = dx$, $v = \frac{1}{5} \sin 5x$. Then by Equation 2,

$$\int x \cos 5x dx = \frac{1}{5}x \sin 5x - \int \frac{1}{5} \sin 5x dx = \frac{1}{5}x \sin 5x + \frac{1}{25} \cos 5x + C.$$

4. Let $u = x$, $dv = e^{-x} dx \Rightarrow du = dx$, $v = -e^{-x}$. Then

$$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C.$$

5. Let $u = r$, $dv = e^{r/2} dr \Rightarrow du = dr$, $v = 2e^{r/2}$. Then

$$\int r e^{r/2} dr = 2r e^{r/2} - \int 2e^{r/2} dr = 2r e^{r/2} - 4e^{r/2} + C.$$

6. Let $u = t$, $dv = \sin 2t dt \Rightarrow du = dt$, $v = -\frac{1}{2} \cos 2t$. Then

$$\int t \sin 2t dt = -\frac{1}{2}t \cos 2t + \frac{1}{2} \int \cos 2t dt = -\frac{1}{2}t \cos 2t + \frac{1}{4} \sin 2t + C.$$

7. Let $u = x^2$, $dv = \sin \pi x dx \Rightarrow du = 2x dx$ and $v = -\frac{1}{\pi} \cos \pi x$. Then

$$I = \int x^2 \sin \pi x dx = -\frac{1}{\pi} x^2 \cos \pi x + \frac{2}{\pi} \int x \cos \pi x dx (*).$$

Next let $U = x$, $dV = \cos \pi x dx \Rightarrow dU = dx$, $V = \frac{1}{\pi} \sin \pi x$, so

$$\int x \cos \pi x dx = \frac{1}{\pi} x \sin \pi x - \frac{1}{\pi} \int \sin \pi x dx = \frac{1}{\pi} x \sin \pi x + \frac{1}{\pi^2} \cos \pi x + C_1. \text{ Substituting for } \int x \cos \pi x dx \text{ in } (*),$$

$$\text{we get } I = -\frac{1}{\pi} x^2 \cos \pi x + \frac{2}{\pi} \left(\frac{1}{\pi} x \sin \pi x + \frac{1}{\pi^2} \cos \pi x + C_1 \right) = -\frac{1}{\pi} x^2 \cos \pi x + \frac{2}{\pi^2} x \sin \pi x + \frac{2}{\pi^3} \cos \pi x + C,$$

$$\text{where } C = \frac{2}{\pi} C_1.$$

8. Let $u = x^2$, $dv = \cos mx dx \Rightarrow du = 2x dx$, $v = \frac{1}{m} \sin mx$.

$$\text{Then } I = \int x^2 \cos mx dx = \frac{1}{m} x^2 \sin mx - \frac{2}{m} \int x \sin mx dx (*). \text{ Next let}$$

$$U = x, dV = \sin mx dx \Rightarrow dU = dx, V = -\frac{1}{m} \cos mx, \text{ so}$$

$$\int x \sin mx dx = -\frac{1}{m} x \cos mx + \frac{1}{m} \int \cos mx dx = -\frac{1}{m} x \cos mx + \frac{1}{m^2} \sin mx + C_1.$$

Substituting for $\int x \sin mx dx$ in (*), we get

$$I = \frac{1}{m} x^2 \sin mx - \frac{2}{m} \left(-\frac{1}{m} x \cos mx + \frac{1}{m^2} \sin mx + C_1 \right) = \frac{1}{m} x^2 \sin mx + \frac{2}{m^2} x \cos mx - \frac{2}{m^3} \sin mx + C,$$

$$\text{where } C = -\frac{2}{m} C_1.$$

9. Let $u = \ln(2x + 1)$, $dv = dx \Rightarrow du = \frac{2}{2x+1} dx$, $v = x$. Then

$$\begin{aligned}\int \ln(2x+1) dx &= x \ln(2x+1) - \int \frac{2x}{2x+1} dx = x \ln(2x+1) - \int \frac{(2x+1)-1}{2x+1} dx \\ &= x \ln(2x+1) - \int \left(1 - \frac{1}{2x+1}\right) dx = x \ln(2x+1) - x + \frac{1}{2} \ln(2x+1) + C \\ &= \frac{1}{2}(2x+1) \ln(2x+1) - x + C\end{aligned}$$

10. Let $u = \sin^{-1} x$, $dv = dx \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$\begin{aligned}\int \sin^{-1} x dx &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx. \text{ Setting } t = 1-x^2, \text{ we get } dt = -2x dx, \text{ so} \\ -\int \frac{x dx}{\sqrt{1-x^2}} &= -\int t^{-1/2} \left(-\frac{1}{2} dt\right) = \frac{1}{2} \left(2t^{1/2}\right) + C = t^{1/2} + C = \sqrt{1-x^2} + C. \text{ Hence,} \\ \int \sin^{-1} x dx &= x \sin^{-1} x + \sqrt{1-x^2} + C.\end{aligned}$$

11. Let $u = \arctan 4t$, $dv = dt \Rightarrow du = \frac{4}{1+(4t)^2} dt = \frac{4}{1+16t^2} dt$, $v = t$. Then

$$\begin{aligned}\int \arctan 4t dt &= t \arctan 4t - \int \frac{4t}{1+16t^2} dt = t \arctan 4t - \frac{1}{8} \int \frac{32t}{1+16t^2} dt \\ &= t \arctan 4t - \frac{1}{8} \ln(1+16t^2) + C\end{aligned}$$

12. Let $u = \ln p$, $dv = p^5 dp \Rightarrow du = \frac{1}{p} dp$, $v = \frac{1}{6} p^6$. Then

$$\int p^5 \ln p dp = \frac{1}{6} p^6 \ln p - \frac{1}{6} \int p^5 dp = \frac{1}{6} p^6 \ln p - \frac{1}{36} p^6 + C.$$

13. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2 \ln x \cdot \frac{1}{x} dx$, $v = x$. Then by Equation 2,

$$\begin{aligned}I = \int (\ln x)^2 dx &= x(\ln x)^2 - 2 \int x \ln x \cdot \frac{1}{x} dx = x(\ln x)^2 - 2 \int \ln x dx. \text{ Next let } U = \ln x, dV = dx \Rightarrow \\ dU &= 1/x dx, V = x \text{ to get } \int \ln x dx = x \ln x - \int x \cdot (1/x) dx = x \ln x - \int dx = x \ln x - x + C_1. \text{ Thus,} \\ I &= x(\ln x)^2 - 2(x \ln x - x + C_1) = x(\ln x)^2 - 2x \ln x + 2x + C, \text{ where } C = -2C_1.\end{aligned}$$

14. Let $u = t^3$, $dv = e^t dt \Rightarrow du = 3t^2 dt$, $v = e^t$. Then $I = \int t^3 e^t dt = t^3 e^t - \int 3t^2 e^t dt$. Integrate by parts twice more with $dv = e^t dt$.

$$\begin{aligned}I &= t^3 e^t - (3t^2 e^t - \int 6te^t dt) = t^3 e^t - 3t^2 e^t + 6te^t - \int 6e^t dt \\ &= t^3 e^t - 3t^2 e^t + 6te^t - 6e^t + C = (t^3 - 3t^2 + 6t - 6)e^t + C\end{aligned}$$

More generally, if $p(t)$ is a polynomial of degree n in t , then repeated integration by parts shows that

$$\int p(t) e^t dt = \left[p(t) - p'(t) + p''(t) - p'''(t) + \cdots + (-1)^n p^{(n)}(t) \right] e^t + C.$$

15. First let $u = \sin 3\theta$, $dv = e^{2\theta} d\theta \Rightarrow du = 3 \cos 3\theta d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then

$$I = \int e^{2\theta} \sin 3\theta d\theta = \frac{1}{2}e^{2\theta} \sin 3\theta - \frac{3}{2} \int e^{2\theta} \cos 3\theta d\theta. \text{ Next let } U = \cos 3\theta,$$

$$dV = e^{2\theta} d\theta \Rightarrow dU = -3 \sin 3\theta d\theta, V = \frac{1}{2}e^{2\theta} \text{ to get}$$

$$\int e^{2\theta} \cos 3\theta d\theta = \frac{1}{2}e^{2\theta} \cos 3\theta + \frac{3}{2} \int e^{2\theta} \sin 3\theta d\theta. \text{ Substituting in the previous formula gives}$$

$$I = \frac{1}{2}e^{2\theta} \sin 3\theta - \frac{3}{4}e^{2\theta} \cos 3\theta - \frac{9}{4} \int e^{2\theta} \sin 3\theta d\theta = \frac{1}{2}e^{2\theta} \sin 3\theta - \frac{3}{4}e^{2\theta} \cos 3\theta - \frac{9}{4}I \Rightarrow$$

$$\frac{13}{4}I = \frac{1}{2}e^{2\theta} \sin 3\theta - \frac{3}{4}e^{2\theta} \cos 3\theta + C_1. \text{ Hence, } I = \frac{1}{13}e^{2\theta}(2 \sin 3\theta - 3 \cos 3\theta) + C, \text{ where } C = \frac{4}{13}C_1.$$

16. First let $u = e^{-\theta}$, $dv = \cos 2\theta d\theta \Rightarrow du = -e^{-\theta} d\theta$, $v = \frac{1}{2} \sin 2\theta$. Then

$$I = \int e^{-\theta} \cos 2\theta d\theta = \frac{1}{2}e^{-\theta} \sin 2\theta - \int \frac{1}{2} \sin 2\theta (-e^{-\theta} d\theta) = \frac{1}{2}e^{-\theta} \sin 2\theta + \frac{1}{2} \int e^{-\theta} \sin 2\theta d\theta.$$

$$\text{Next let } U = e^{-\theta}, dV = \sin 2\theta d\theta \Rightarrow dU = -e^{-\theta} d\theta, V = -\frac{1}{2} \cos 2\theta, \text{ so}$$

$$\int e^{-\theta} \sin 2\theta d\theta = -\frac{1}{2}e^{-\theta} \cos 2\theta - \int \left(-\frac{1}{2}\right) \cos 2\theta (-e^{-\theta} d\theta) = -\frac{1}{2}e^{-\theta} \cos 2\theta - \frac{1}{2} \int e^{-\theta} \cos 2\theta d\theta.$$

$$\text{So } I = \frac{1}{2}e^{-\theta} \sin 2\theta + \frac{1}{2} \left[\left(-\frac{1}{2}e^{-\theta} \cos 2\theta\right) - \frac{1}{2}I \right] = \frac{1}{2}e^{-\theta} \sin 2\theta - \frac{1}{4}e^{-\theta} \cos 2\theta - \frac{1}{4}I \Rightarrow$$

$$\frac{5}{4}I = \frac{1}{2}e^{-\theta} \sin 2\theta - \frac{1}{4}e^{-\theta} \cos 2\theta + C_1 \Rightarrow$$

$$I = \frac{4}{5} \left(\frac{1}{2}e^{-\theta} \sin 2\theta - \frac{1}{4}e^{-\theta} \cos 2\theta + C_1 \right) = \frac{2}{5}e^{-\theta} \sin 2\theta - \frac{1}{5}e^{-\theta} \cos 2\theta + C.$$

17. Let $u = y$, $dv = \sinh y dy \Rightarrow du = dy$, $v = \cosh y$. Then

$$\int y \sinh y dy = y \cosh y - \int \cosh y dy = y \cosh y - \sinh y + C.$$

18. Let $u = y$, $dv = \cosh ay dy \Rightarrow du = dy$, $v = \frac{\sinh ay}{a}$. Then

$$\int y \cosh ay dy = \frac{y \sinh ay}{a} - \frac{1}{a} \int \sinh ay dy = \frac{y \sinh ay}{a} - \frac{\cosh ay}{a^2} + C.$$

19. Let $u = t$, $dv = \sin 3t dt \Rightarrow du = dt$, $v = -\frac{1}{3} \cos 3t$. Then

$$\int_0^{\pi} t \sin 3t dt = \left[-\frac{1}{3}t \cos 3t \right]_0^{\pi} + \frac{1}{3} \int_0^{\pi} \cos 3t dt = \left(\frac{1}{3}\pi - 0 \right) + \frac{1}{9} [\sin 3t]_0^{\pi} = \frac{\pi}{3}.$$

20. First let $u = x^2 + 1$, $dv = e^{-x} dx \Rightarrow du = 2x dx$, $v = -e^{-x}$. By (6),

$$\int_0^1 (x^2 + 1)e^{-x} dx = \left[-(x^2 + 1)e^{-x} \right]_0^1 + \int_0^1 2xe^{-x} dx = -2e^{-1} + 1 + 2 \int_0^1 xe^{-x} dx. \text{ Next let}$$

$$U = x, dV = e^{-x} dx \Rightarrow dU = dx, V = -e^{-x}. \text{ By (6) again,}$$

$$\int_0^1 xe^{-x} dx = \left[-xe^{-x} \right]_0^1 + \int_0^1 e^{-x} dx = -e^{-1} + \left[-e^{-x} \right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1. \text{ So}$$

$$\int_0^1 (x^2 + 1)e^{-x} dx = -2e^{-1} + 1 + 2(-2e^{-1} + 1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3.$$

21. Let $u = \ln x$, $dv = x^{-2} dx \Rightarrow du = \frac{1}{x} dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x} \right]_1^2 + \int_1^2 x^{-2} dx = -\frac{1}{2} \ln 2 + \ln 1 + \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} \ln 2 + 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2} \ln 2.$$

22. Let $u = \ln t$, $dv = \sqrt{t} dt \Rightarrow du = dt/t$, $v = \frac{2}{3}t^{3/2}$. By Formula 6,

$$\int_1^4 \sqrt{t} \ln t dt = \left[\frac{2}{3}t^{3/2} \ln t \right]_1^4 - \frac{2}{3} \int_1^4 \sqrt{t} dt = \frac{2}{3} \cdot 8 \cdot \ln 4 - 0 - \left[\frac{2}{3} \cdot \frac{2}{3}t^{3/2} \right]_1^4 = \frac{16}{3} \ln 4 - \frac{4}{9} (8 - 1) = \frac{16}{3} \ln 4 - \frac{28}{9}.$$

23. Let $u = y$, $dv = \frac{dy}{e^{2y}} = e^{-2y} dy \Rightarrow du = dy$, $v = -\frac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}} dy = \left[-\frac{1}{2}ye^{-2y}\right]_0^1 + \frac{1}{2} \int_0^1 e^{-2y} dy = \left(-\frac{1}{2}e^{-2} + 0\right) - \frac{1}{4}\left[e^{-2y}\right]_0^1 = -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} = \frac{1}{4} - \frac{3}{4}e^{-2}.$$

24. Let $u = x$, $dv = \csc^2 x dx \Rightarrow du = dx$, $v = -\cot x$. Then

$$\begin{aligned} \int_{\pi/4}^{\pi/2} x \csc^2 x dx &= [-x \cot x]_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2} \cot x dx = -\frac{\pi}{2} \cdot 0 + \frac{\pi}{4} \cdot 1 + \left[\ln |\sin x|\right]_{\pi/4}^{\pi/2} \quad [\text{see Exercise 5.5.75}] \\ &= \frac{\pi}{4} + \ln 1 - \ln \frac{1}{\sqrt{2}} = \frac{\pi}{4} + 0 - \ln 2^{-1/2} = \frac{\pi}{4} + \frac{1}{2} \ln 2 \end{aligned}$$

25. Let $u = \cos^{-1} x$, $dv = dx \Rightarrow du = -\frac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$\begin{aligned} I &= \int_0^{1/2} \cos^{-1} x dx = [x \cos^{-1} x]_0^{1/2} + \int_0^{1/2} \frac{x dx}{\sqrt{1-x^2}} = \frac{1}{2} \cdot \frac{\pi}{3} + \int_1^{3/4} t^{-1/2} \left[-\frac{1}{2} dt\right], \text{ where } t = 1 - x^2 \\ &\Rightarrow dt = -2x dx. \text{ Thus, } I = \frac{\pi}{6} + \frac{1}{2} \int_{3/4}^1 t^{-1/2} dt = \frac{\pi}{6} + [\sqrt{t}]_{3/4}^1 = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}(\pi + 6 - 3\sqrt{3}). \end{aligned}$$

26. Let $u = x$, $dv = 5^x dx \Rightarrow du = dx$, $v = (5^x/\ln 5)$. Then

$$\begin{aligned} \int_0^1 x 5^x dx &= \left[\frac{x 5^x}{\ln 5}\right]_0^1 - \int_0^1 \frac{5^x}{\ln 5} dx = \frac{5}{\ln 5} - 0 - \frac{1}{\ln 5} \left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{\ln 5} - \frac{5}{(\ln 5)^2} + \frac{1}{(\ln 5)^2} \\ &= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2} \end{aligned}$$

27. Let $u = \ln(\sin x)$, $dv = \cos x dx \Rightarrow du = \frac{\cos x}{\sin x} dx$, $v = \sin x$. Then

$$I = \int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \int \cos x dx = \sin x \ln(\sin x) - \sin x + C.$$

Another method: Substitute $t = \sin x$, so $dt = \cos x dx$. Then $I = \int \ln t dt = t \ln t - t + C$ (see Example 2) and so $I = \sin x (\ln \sin x - 1) + C$.

28. Let $u = \arctan(1/x)$, $dv = dx \Rightarrow du = \frac{1}{1+(1/x)^2} \cdot \frac{-1}{x^2} dx = \frac{-dx}{x^2+1}$, $v = x$. Then

$$\begin{aligned} \int_1^{\sqrt{3}} \arctan(1/x) dx &= \left[x \arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x dx}{x^2+1} = \sqrt{3} \frac{\pi}{6} - 1 \cdot \frac{\pi}{4} + \frac{1}{2} [\ln(x^2+1)]_1^{\sqrt{3}} \\ &= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{2} + \frac{1}{2} \ln \frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{2} + \frac{1}{2} \ln 2 \end{aligned}$$

29. Let $w = \ln x \Rightarrow dw = dx/x$. Then $x = e^w$ and $dx = e^w dw$, so

$$\begin{aligned} \int \cos(\ln x) dx &= \int e^w \cos w dw = \frac{1}{2} e^w (\sin w + \cos w) + C \quad [\text{by the method of Example 4}] \\ &= \frac{1}{2} x [\sin(\ln x) + \cos(\ln x)] + C \end{aligned}$$

30. Let $u = r^2$, $dv = \frac{r}{\sqrt{4+r^2}} dr \Rightarrow du = 2r dr$, $v = \sqrt{4+r^2}$. By (6),

$$\begin{aligned} \int_0^1 \frac{r^3}{\sqrt{4+r^2}} dr &= \left[r^2 \sqrt{4+r^2}\right]_0^1 - 2 \int_0^1 r \sqrt{4+r^2} dr = \sqrt{5} - \frac{2}{3} \left[(4+r^2)^{3/2}\right]_0^1 \\ &= \sqrt{5} - \frac{2}{3}(5)^{3/2} + \frac{2}{3}(8) = \sqrt{5} \left(1 - \frac{10}{3}\right) + \frac{16}{3} = \frac{16}{3} - \frac{7}{3}\sqrt{5} \end{aligned}$$

31. Let $u = (\ln x)^2$, $dv = x^4 dx \Rightarrow du = 2\frac{\ln x}{x} dx$, $v = \frac{x^5}{5}$. By (6),

$$\int_1^2 x^4 (\ln x)^2 dx = \left[\frac{x^5}{5} (\ln x)^2 \right]_1^2 - 2 \int_1^2 \frac{x^4}{5} \ln x dx = \frac{32}{5} (\ln 2)^2 - 0 - 2 \int_1^2 \frac{x^4}{5} \ln x dx.$$

$$\text{Let } U = \ln x, dV = \frac{x^4}{5} dx \Rightarrow dU = \frac{1}{x} dx, V = \frac{x^5}{25}.$$

$$\text{Then } \int_1^2 \frac{x^4}{5} \ln x dx = \left[\frac{x^5}{25} \ln x \right]_1^2 - \int_1^2 \frac{x^4}{25} dx = \frac{32}{25} \ln 2 - 0 - \left[\frac{x^5}{125} \right]_1^2 = \frac{32}{25} \ln 2 - \left(\frac{32}{125} - \frac{1}{125} \right).$$

$$\text{So } \int_1^2 x^4 (\ln x)^2 dx = \frac{32}{5} (\ln 2)^2 - 2 \left(\frac{32}{25} \ln 2 - \frac{31}{125} \right) = \frac{32}{5} (\ln 2)^2 - \frac{64}{25} \ln 2 + \frac{62}{125}.$$

32. Let $u = \sin(t-s)$, $dv = e^s ds \Rightarrow du = -\cos(t-s) ds$, $v = e^s$. Then

$$I = \int_0^t e^s \sin(t-s) ds = [e^s \sin(t-s)]_0^t + \int_0^t e^s \cos(t-s) ds = e^t \sin 0 - e^0 \sin t + I_1. \text{ For}$$

$$I_1, \text{ let } U = \cos(t-s), dV = e^s ds \Rightarrow dU = \sin(t-s) ds, V = e^s. \text{ So}$$

$$I_1 = [e^s \cos(t-s)]_0^t - \int_0^t e^s \sin(t-s) ds = e^t \cos 0 - e^0 \cos t - I. \text{ Thus, } I = -\sin t + e^t - \cos t - I \Rightarrow$$

$$2I = e^t - \cos t - \sin t \Rightarrow I = \frac{1}{2}(e^t - \cos t - \sin t).$$

33. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w dw$. Thus, $\int \sin \sqrt{x} dx = \int 2w \sin w dw$. Now use parts with $u = 2w$, $dv = \sin w dw$, $du = 2 dw$, $v = -\cos w$ to get

$$\begin{aligned} \int 2w \sin w dw &= -2w \cos w + \int 2 \cos w dw = -2w \cos w + 2 \sin w + C \\ &= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + C = 2(\sin \sqrt{x} - \sqrt{x} \cos \sqrt{x}) + C \end{aligned}$$

34. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w dw$. Thus, $\int_1^4 e^{\sqrt{x}} dx = \int_1^2 e^w 2w dw$. Now use parts with $u = 2w$, $dv = e^w dw$, $du = 2 dw$, $v = e^w$ to get $\int_1^2 e^w 2w dw = [2we^w]_1^2 - 2 \int_1^2 e^w dw = 4e^2 - 2e - 2(e^2 - e) = 2e^2$.

35. Let $x = \theta^2$, so that $dx = 2\theta d\theta$. Thus, $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2 \cos(\theta^2) \cdot \frac{1}{2}(2\theta d\theta) = \frac{1}{2} \int_{\pi/2}^{\pi} x \cos x dx$. Now use parts with $u = x$, $dv = \cos x dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned} \frac{1}{2} \int_{\pi/2}^{\pi} x \cos x dx &= \frac{1}{2} ([x \sin x]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin x dx) = \frac{1}{2} [x \sin x + \cos x]_{\pi/2}^{\pi} \\ &= \frac{1}{2} (\pi \sin \pi + \cos \pi) - \frac{1}{2} \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \right) = \frac{1}{2} (\pi \cdot 0 - 1) - \frac{1}{2} \left(\frac{\pi}{2} \cdot 1 + 0 \right) = -\frac{1}{2} - \frac{\pi}{4} \end{aligned}$$

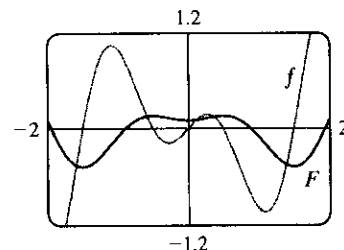
36. $\int x^5 e^{x^2} dx = \int (x^2)^2 e^{x^2} x dx = \int t^2 e^t \frac{1}{2} dt$ [where $t = x^2 \Rightarrow \frac{1}{2} dt = x dx$]
 $= \frac{1}{2} (t^2 - 2t + 2) e^t + C$ [by Example 3] $= \frac{1}{2} (x^4 - 2x^2 + 2) e^{x^2} + C$

In Exercises 37–40, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

37. Let $u = x$, $dv = \cos \pi x dx \Rightarrow du = dx$, $v = (\sin \pi x) / \pi$. Then

$$\int x \cos \pi x dx = x \cdot \frac{\sin \pi x}{\pi} - \int \frac{\sin \pi x}{\pi} dx = \frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} + C.$$

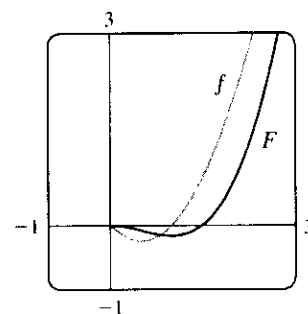
We see from the graph that this is reasonable, since F has extreme values where f is 0.



38. Let $u = \ln x$, $dv = x^{3/2} dx \Rightarrow du = \frac{1}{x} dx$, $v = \frac{2}{5}x^{5/2}$. Then

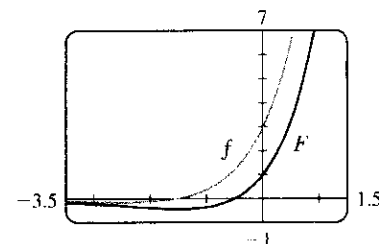
$$\begin{aligned}\int x^{3/2} \ln x \, dx &= \frac{2}{5}x^{5/2} \ln x - \frac{2}{5} \int x^{3/2} dx = \frac{2}{5}x^{5/2} \ln x - \left(\frac{2}{5}\right)^2 x^{5/2} + C \\ &= \frac{2}{5}x^{5/2} \ln x - \frac{4}{25}x^{5/2} + C.\end{aligned}$$

We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



39. Let $u = 2x + 3$, $dv = e^x dx \Rightarrow du = 2 dx$, $v = e^x$. Then

$$\begin{aligned}\int (2x + 3)e^x \, dx &= (2x + 3)e^x - 2 \int e^x dx = (2x + 3)e^x - 2e^x + C = \\ &= (2x + 1)e^x + C. \text{ We see from the graph that this is reasonable, since } F \\ &\text{has a minimum where } f \text{ changes from negative to positive.}\end{aligned}$$

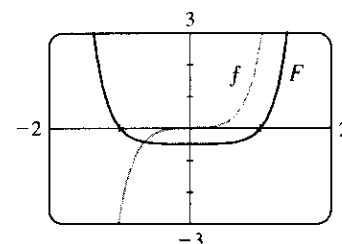


40. $\int x^3 e^{x^2} dx = \int x^2 \cdot x e^{x^2} dx = I$.

Let $u = x^2$, $dv = x e^{x^2} dx \Rightarrow du = 2x dx$, $v = \frac{1}{2}e^{x^2}$. Then

$$I = \frac{1}{2}x^2 e^{x^2} - \int x e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C = \frac{1}{2}e^{x^2}(x^2 - 1) + C.$$

We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



41. (a) Take $n = 2$ in Example 6 to get $\int \sin^2 x \, dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1 \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

$$(b) \int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{8}x - \frac{3}{16} \sin 2x + C.$$

42. (a) Let $u = \cos^{n-1} x$, $dv = \cos x \, dx \Rightarrow du = -(n-1) \cos^{n-2} x \sin x \, dx$, $v = \sin x$ in (2):

$$\begin{aligned}\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx\end{aligned}$$

Rearranging terms gives $n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$ or

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

- (b) Take $n = 2$ in part (a) to get $\int \cos^2 x \, dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1 \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$.

$$(c) \int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{8}x + \frac{3}{16} \sin 2x + C$$

43. (a) From Example 6, $\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$. Using (6),

$$\begin{aligned}\int_0^{\pi/2} \sin^n x \, dx &= \left[-\frac{\cos x \sin^{n-1} x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \\ &= (0 - 0) + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx\end{aligned}$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x \, dx = \frac{2}{3} \int_0^{\pi/2} \sin x \, dx = \left[-\frac{2}{3} \cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x \, dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x \, dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n + 1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$$\int_0^{\pi/2} \sin^{2k+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{3 \cdot 5 \cdot 7 \cdots (2k+1)}. \text{ By Example 6,}$$

$$\begin{aligned} \int_0^{\pi/2} \sin^{2k+3} x \, dx &= \frac{2k+2}{2k+3} \int_0^{\pi/2} \sin^{2k+1} x \, dx = \frac{2k+2}{2k+3} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{3 \cdot 5 \cdot 7 \cdots (2k+1)} \\ &= \frac{2 \cdot 4 \cdot 6 \cdots (2k)[2(k+1)]}{3 \cdot 5 \cdot 7 \cdots (2k+1)[2(k+1)+1]}, \end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

44. Using Exercise 43(a), we see that the formula holds for $n = 1$, because

$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \int_0^{\pi/2} 1 \, dx = \frac{1}{2} \left[x\right]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2}.$$

Now assume it holds for some $k \geq 1$. Then $\int_0^{\pi/2} \sin^{2k} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \frac{\pi}{2}$. By Exercise 43(a),

$$\begin{aligned} \int_0^{\pi/2} \sin^{2(k+1)} x \, dx &= \frac{2k+1}{2k+2} \int_0^{\pi/2} \sin^{2k} x \, dx = \frac{2k+1}{2k+2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \frac{\pi}{2} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k)(2k+2)} \cdot \frac{\pi}{2}, \end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

45. Let $u = (\ln x)^n$, $dv = dx \Rightarrow du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$$\int (\ln x)^n dx = x(\ln x)^n - \int n x (\ln x)^{n-1} (dx/x) = x(\ln x)^n - n \int (\ln x)^{n-1} dx.$$

46. Let $u = x^n$, $dv = e^x dx \Rightarrow du = nx^{n-1} dx$, $v = e^x$. By Equation 2, $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$.

47. Let $u = (x^2 + a^2)^n$, $dv = dx \Rightarrow du = n(x^2 + a^2)^{n-1} 2x dx$, $v = x$. Then

$$\begin{aligned} \int (x^2 + a^2)^n dx &= x(x^2 + a^2)^n - 2n \int x^2 (x^2 + a^2)^{n-1} dx \\ &= x(x^2 + a^2)^n - 2n \left[\int (x^2 + a^2)^n dx - a^2 \int (x^2 + a^2)^{n-1} dx \right] \quad [\text{since } x^2 = (x^2 + a^2) - a^2] \\ \Rightarrow (2n+1) \int (x^2 + a^2)^n dx &= x(x^2 + a^2)^n + 2na^2 \int (x^2 + a^2)^{n-1} dx, \text{ and} \\ \int (x^2 + a^2)^n dx &= \frac{x(x^2 + a^2)^n}{2n+1} + \frac{2na^2}{2n+1} \int (x^2 + a^2)^{n-1} dx \quad [\text{provided } 2n+1 \neq 0]. \end{aligned}$$

48. Let $u = \sec^{n-2} x$, $dv = \sec^2 x dx \Rightarrow du = (n-2) \sec^{n-3} x \sec x \tan x dx$, $v = \tan x$. Then by Equation 2,

$$\begin{aligned} \int \sec^n x dx &= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x \tan^2 x dx \\ &= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx \\ &= \tan x \sec^{n-2} x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx \end{aligned}$$

so $(n-1) \int \sec^n x dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x dx$. If $n-1 \neq 0$, then

$$\int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

49. Take
- $n = 3$
- in Exercise 45 to get

$$\int (\ln x)^3 dx = x(\ln x)^3 - 3 \int (\ln x)^2 dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C \quad [\text{by Exercise 13}].$$

Or: Instead of using Exercise 13, apply Exercise 45 again with $n = 2$.

50. Take
- $n = 4$
- in Exercise 46 to get

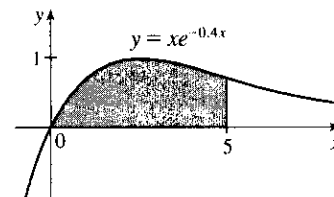
$$\begin{aligned} \int x^4 e^x dx &= x^4 e^x - 4 \int x^3 e^x dx = x^4 e^x - 4(x^3 - 3x^2 + 6x - 6)e^x + C \quad [\text{by Exercise 14}] \\ &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C \end{aligned}$$

Or: Instead of using Exercise 14, apply Exercise 46 with $n = 3$, then $n = 2$, then $n = 1$.

51. Area =
- $\int_0^5 x e^{-0.4x} dx$
- . Let
- $u = x$
- ,
- $dv = e^{-0.4x} dx \Rightarrow$

$$du = dx, v = -2.5e^{-0.4x}. \text{ Then}$$

$$\begin{aligned} \text{area} &= [-2.5x e^{-0.4x}]_0^5 + 2.5 \int_0^5 e^{-0.4x} dx \\ &= -12.5e^{-2} + 0 + 2.5[-2.5e^{-0.4x}]_0^5 \\ &= -12.5e^{-2} - 6.25(e^{-2} - 1) = 6.25 - 18.75e^{-2} \quad \text{or} \quad \frac{25}{4} - \frac{75}{4}e^{-2} \end{aligned}$$

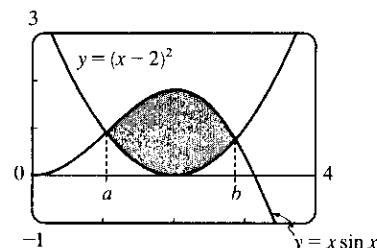


52. The curves $y = x \ln x$ and $y = 5 \ln x$ intersect when $x \ln x = 5 \ln x \Leftrightarrow x \ln x - 5 \ln x = 0 \Leftrightarrow (x - 5) \ln x = 0$; that is, when $x = 1$ or $x = 5$. For $1 < x < 5$, we have $5 \ln x > x \ln x$ since $\ln x > 0$. Thus,
- $$\text{area} = \int_1^5 (5 \ln x - x \ln x) dx = \int_1^5 [(5 - x) \ln x] dx. \text{ Let } u = \ln x, dv = (5 - x) dx \Rightarrow du = dx/x,$$
- $$v = 5x - \frac{1}{2}x^2. \text{ Then}$$

$$\begin{aligned} \text{area} &= [(\ln x)(5x - \frac{1}{2}x^2)]_1^5 - \int_1^5 [(5x - \frac{1}{2}x^2) \frac{1}{x}] dx = (\ln 5)(\frac{25}{2}) - 0 - \int_1^5 (5 - \frac{1}{2}x) dx \\ &= \frac{25}{2} \ln 5 - [5x - \frac{1}{4}x^2]_1^5 = \frac{25}{2} \ln 5 - [(25 - \frac{25}{4}) - (5 - \frac{1}{4})] = \frac{25}{2} \ln 5 - 14 \end{aligned}$$

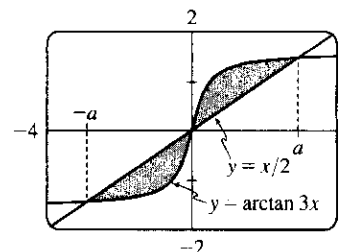
53. The curves $y = x \sin x$ and $y = (x - 2)^2$ intersect at $a \approx 1.04748$ and $b \approx 2.87307$, so

$$\begin{aligned} \text{area} &= \int_a^b [x \sin x - (x - 2)^2] dx \\ &= [-x \cos x + \sin x - \frac{1}{3}(x - 2)^3]_a^b \quad [\text{by Example 1}] \\ &\approx 2.81358 - 0.63075 = 2.18283 \end{aligned}$$



54. The curves $y = \arctan 3x$ and $y = x/2$ intersect at $x = \pm a \approx \pm 2.91379$, so

$$\begin{aligned} \text{area} &= \int_{-a}^a |\arctan 3x - \frac{1}{2}x| dx = 2 \int_0^a (\arctan 3x - \frac{1}{2}x) dx \\ &= 2[x \arctan 3x - \frac{1}{6} \ln(1 + 9x^2) - \frac{1}{4}x^2]_0^a \quad [\text{see Example 5}] \\ &\approx 2(1.39768) = 2.79536. \end{aligned}$$



55. $V = \int_0^1 2\pi x \cos(\pi x/2) dx$. Let $u = x$, $dv = \cos(\pi x/2) dx \Rightarrow du = dx, v = \frac{2}{\pi} \sin(\pi x/2)$.

$$\begin{aligned} V &= 2\pi \left[\frac{2}{\pi} x \sin\left(\frac{\pi x}{2}\right) \right]_0^1 - 2\pi \cdot \frac{2}{\pi} \int_0^1 \sin\left(\frac{\pi x}{2}\right) dx = 2\pi \left(\frac{2}{\pi} - 0 \right) - 4 \left[-\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) \right]_0^1 \\ &= 4 + \frac{8}{\pi}(0 - 1) = 4 - \frac{8}{\pi}. \end{aligned}$$

$$56. \text{Volume} = \int_0^1 2\pi x(e^x - e^{-x}) dx = 2\pi \int_0^1 (xe^x - xe^{-x}) dx$$

$$= 2\pi \left[\int_0^1 xe^x dx - \int_0^1 xe^{-x} dx \right] \quad [\text{both integrals by parts}]$$

$$= 2\pi [(xe^x - e^x) - (-xe^{-x} - e^{-x})]_0^1 = 2\pi [2/e - 0] = 4\pi/e$$

$$57. \text{Volume} = \int_{-1}^0 2\pi(1-x)e^{-x} dx. \text{ Let } u = 1-x, dv = e^{-x} dx \Rightarrow du = -dx, v = -e^{-x}.$$

$$V = 2\pi [(1-x)(-e^{-x})]_{-1}^0 - 2\pi \int_{-1}^0 e^{-x} dx = 2\pi [(x-1)(e^{-x}) + e^{-x}]_{-1}^0$$

$$= 2\pi [xe^{-x}]_{-1}^0 = 2\pi(0 + e) = 2\pi e$$

$$58. \text{Volume} = \int_1^\pi 2\pi y \cdot \ln y dy = 2\pi \left[\frac{1}{2} y^2 \ln y - \frac{1}{4} y^2 \right]_1^\pi \quad [\text{by parts with } u = \ln y \text{ and } dv = y dy]$$

$$= 2\pi \left[\frac{1}{4} y^2 (2 \ln y - 1) \right]_1^\pi = 2\pi \left[\frac{\pi^2 (2 \ln \pi - 1)}{4} - \frac{(0-1)}{4} \right] = \pi^3 \ln \pi - \frac{\pi^3}{2} + \frac{\pi}{2}$$

$$59. \text{The average value of } f(x) = x^2 \ln x \text{ on the interval } [1, 3] \text{ is } f_{\text{ave}} = \frac{1}{3-1} \int_1^3 x^2 \ln x dx = \frac{1}{2} I.$$

$$\text{Let } u = \ln x, dv = x^2 dx \Rightarrow du = (1/x) dx, v = \frac{1}{3} x^3. \text{ So}$$

$$I = \left[\frac{1}{3} x^3 \ln x \right]_1^3 - \int_1^3 \frac{1}{3} x^2 dx = (9 \ln 3 - 0) - \left[\frac{1}{9} x^3 \right]_1^3 = 9 \ln 3 - (3 - \frac{1}{9}) = 9 \ln 3 - \frac{26}{9}.$$

$$\text{Thus, } f_{\text{ave}} = \frac{1}{2} I = \frac{1}{2} (9 \ln 3 - \frac{26}{9}) = \frac{9}{2} \ln 3 - \frac{13}{9}.$$

$$60. \text{The rocket will have height } H = \int_0^{60} v(t) dt \text{ after 60 seconds.}$$

$$H = \int_0^{60} \left[-gt - v_e \ln \left(\frac{m-rt}{m} \right) \right] dt = -g \left[\frac{1}{2} t^2 \right]_0^{60} - v_e \left[\int_0^{60} \ln(m-rt) dt - \int_0^{60} \ln m dt \right]$$

$$= -g(1800) + v_e(\ln m)(60) - v_e \int_0^{60} \ln(m-rt) dt$$

$$\text{Let } u = \ln(m-rt), dv = dt \Rightarrow du = \frac{1}{m-rt}(-r) dt, v = t. \text{ Then}$$

$$\int_0^{60} \ln(m-rt) dt = [t \ln(m-rt)]_0^{60} + \int_0^{60} \frac{rt}{m-rt} dt = 60 \ln(m-60r) + \int_0^{60} \left(-1 + \frac{m}{m-rt} \right) dt$$

$$= 60 \ln(m-60r) + \left[-t - \frac{m}{r} \ln(m-rt) \right]_0^{60}$$

$$= 60 \ln(m-60r) - 60 - \frac{m}{r} \ln(m-60r) + \frac{m}{r} \ln m$$

$$\text{So } H = -1800g + 60v_e \ln m - 60v_e \ln(m-60r) + 60v_e + \frac{m}{r} v_e \ln(m-60r) - \frac{m}{r} v_e \ln m. \text{ Substituting } g = 9.8, m = 30,000, r = 160, \text{ and } v_e = 3000 \text{ gives us } H \approx 14,844 \text{ m.}$$

$$61. \text{Since } v(t) > 0 \text{ for all } t, \text{ the desired distance is } s(t) = \int_0^t v(w) dw = \int_0^t w^2 e^{-w} dw.$$

$$\text{First let } u = w^2, dv = e^{-w} dw \Rightarrow du = 2w dw, v = -e^{-w}. \text{ Then } s(t) = [-w^2 e^{-w}]_0^t + 2 \int_0^t w e^{-w} dw.$$

$$\text{Next let } U = w, dV = e^{-w} dw \Rightarrow dU = dw, V = -e^{-w}. \text{ Then}$$

$$s(t) = -t^2 e^{-t} + 2 \left([-w e^{-w}]_0^t + \int_0^t e^{-w} dw \right) = -t^2 e^{-t} + 2 \left(-t e^{-t} + 0 + [-e^{-w}]_0^t \right)$$

$$= -t^2 e^{-t} + 2(-t e^{-t} - e^{-t} + 1) = -t^2 e^{-t} - 2t e^{-t} - 2e^{-t} + 2$$

$$= 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}$$

62. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x) dx \Rightarrow du = f'(x) dx$, $v = g'(x)$.

Then $\int_0^a f(x)g''(x) dx = [f(x)g'(x)]_0^a - \int_0^a f'(x)g'(x) dx = f(a)g'(a) - \int_0^a f'(x)g'(x) dx$.

Now let $U = f'(x)$, $dV = g'(x) dx \Rightarrow dU = f''(x) dx$ and $V = g(x)$, so

$$\int_0^a f'(x)g'(x) dx = [f'(x)g(x)]_0^a - \int_0^a f''(x)g(x) dx = f'(a)g(a) - \int_0^a f''(x)g(x) dx.$$

Combining the two results, we get $\int_0^a f(x)g''(x) dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x) dx$.

63. For $I = \int_1^4 xf''(x) dx$, let $u = x$, $dv = f''(x) dx \Rightarrow du = dx$, $v = f'(x)$. Then

$$I = [xf'(x)]_1^4 - \int_1^4 f'(x) dx = 4f'(4) - 1 \cdot f'(1) - [f(4) - f(1)] = 4 \cdot 3 - 1 \cdot 5 - (7 - 2) = 12 - 5 - 5 = 2.$$

We used the fact that f'' is continuous to guarantee that I exists.

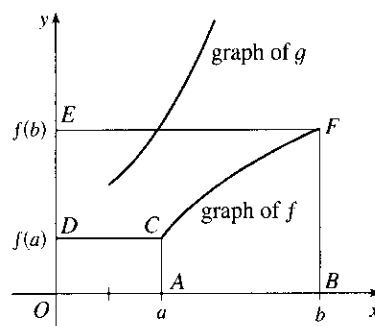
64. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x) dx = bf(b) - af(a) - \int_a^b xf'(x) dx$. Now let $y = f(x)$, so that $x = g(y)$ and

$dy = f'(x) dx$. Then $\int_a^b xf'(x) dx = \int_{f(a)}^{f(b)} g(y) dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$$\begin{aligned} &= bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y) dy \\ &= (\text{area of rectangle } OBF E) - (\text{area of rectangle } OACD) - (\text{area of region } DCFE) \end{aligned}$$



(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x dx = e \ln e - 1 \ln 1 - \int_{\ln 1}^{\ln e} e^y dy = e - \int_0^1 e^y dy = e - [e^y]_0^1 = e - (e - 1) = 1.$$

65. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2 dy - \int_0^c \pi a^2 dy - \int_c^d \pi [g(y)]^2 dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2 dy$. Let $y = f(x)$, which gives $dy = f'(x) dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x) dx$. Now integrate

by parts with $u = x^2$, and $dv = f'(x) dx \Rightarrow du = 2x dx$, $v = f(x)$, and

$$\begin{aligned} \int_a^b x^2 f'(x) dx &= [x^2 f(x)]_a^b - \int_a^b 2x f(x) dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x f(x) dx, \text{ but } f(a) = c \text{ and } f(b) = d \\ \Rightarrow V &= \pi b^2 d - \pi a^2 c - \pi [b^2 d - a^2 c - \int_a^b 2x f(x) dx] = \pi \int_a^b 2x f(x) dx. \end{aligned}$$

66. (a) We note that for $0 \leq x \leq \frac{\pi}{2}$, $0 \leq \sin x \leq 1$, so $\sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$.

(b) Substituting directly into the result from Exercise 44, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\frac{1 \cdot 3 \cdot 5 \cdots [2(n+1)-1]}{2 \cdot 4 \cdot 6 \cdots [2(n+1)]} \frac{\pi}{2}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n} . The inequalities are preserved since I_{2n} is positive:

$\frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq \frac{I_{2n}}{I_{2n}}$. Now from part (b), the left term is equal to $\frac{2n+1}{2n+2}$, so the expression becomes $\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$. Now $\lim_{n \rightarrow \infty} \frac{2n+1}{2n+2} = \lim_{n \rightarrow \infty} 1 = 1$, so by the Squeeze Theorem, $\lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

(d) We substitute the results from Exercises 43 and 44 into the result from part (c):

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n \rightarrow \infty} \frac{\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}} \\ &= \lim_{n \rightarrow \infty} \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right] \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \left(\frac{2}{\pi} \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{2}{\pi} \quad [\text{rearrange terms}] \end{aligned}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

(e) The area of the k th rectangle is k . At the $2n$ th step, the area is increased from $2n - 1$ to $2n$ by multiplying the width by $\frac{2n}{2n - 1}$, and at the $(2n + 1)$ th step, the area is increased from $2n$ to $2n + 1$ by multiplying the height by $\frac{2n + 1}{2n}$. These two steps multiply the ratio of width to height by $\frac{2n}{2n - 1}$ and $\frac{1}{(2n + 1)/(2n)} = \frac{2n}{2n + 1}$ respectively. So, by part (d), the limiting ratio is $\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots = \frac{\pi}{2}$.

8.2 Trigonometric Integrals

The symbols $\stackrel{s}{=}$ and $\stackrel{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x \, dx\}$ and $\{u = \cos x, du = -\sin x \, dx\}$, respectively.

$$\begin{aligned} 1. \int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \stackrel{c}{=} \int (1 - u^2) u^2 (-du) \\ &= \int (u^2 - 1) u^2 \, du = \int (u^4 - u^2) \, du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C \end{aligned}$$

$$\begin{aligned} 2. \int \sin^6 x \cos^3 x \, dx &= \int \sin^6 x \cos^2 x \cos x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \stackrel{s}{=} \int u^6 (1 - u^2) \, du \\ &= \int (u^6 - u^8) \, du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C \end{aligned}$$

$$\begin{aligned} \mathbf{3.} \quad \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x \, dx &= \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x \, dx = \int_{\pi/2}^{3\pi/4} \sin^5 x (1 - \sin^2 x) \cos x \, dx \\ &\stackrel{s}{=} \int_1^{\sqrt{2}/2} u^5 (1 - u^2) \, du = \int_1^{\sqrt{2}/2} (u^5 - u^7) \, du = \left[\frac{1}{6} u^6 - \frac{1}{8} u^8 \right]_1^{\sqrt{2}/2} \\ &= \left(\frac{1}{6} - \frac{1/16}{8} \right) - \left(\frac{1}{6} - \frac{1}{8} \right) = -\frac{11}{384} \end{aligned}$$

4. $\int_0^{\pi/2} \cos^5 x \, dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x \, dx = \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x \, dx \stackrel{s}{=} \int_0^1 (1 - u^2)^2 \, du$
 $= \int_0^1 (1 - 2u^2 + u^4) \, du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5} \right) - 0 = \frac{8}{15}$

5. $\int \cos^5 x \sin^4 x \, dx = \int \cos^4 x \sin^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \sin^4 x \cos x \, dx \stackrel{c}{=} \int (1 - u^2)^2 u^4 \, du$
 $= \int (1 - 2u^2 + u^4) u^4 \, du = \int (u^4 - 2u^6 + u^8) \, du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$
 $= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C$
6. $\int \sin^3(mx) \, dx = \int (1 - \cos^2 mx) \sin mx \, dx = -\frac{1}{m} \int (1 - u^2) \, du \quad [u = \cos mx, du = -m \sin mx \, dx]$
 $= -\frac{1}{m} \left(u - \frac{1}{3} u^3 \right) + C = -\frac{1}{m} \left(\cos mx - \frac{1}{3} \cos^3 mx \right) + C$
 $= \frac{1}{3m} \cos^3 mx - \frac{1}{m} \cos mx + C$
7. $\int_0^{\pi/2} \cos^2 \theta \, d\theta = \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) \, d\theta \quad [\text{half-angle identity}]$
 $= \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{1}{2} \left[\left(\frac{\pi}{2} + 0 \right) - (0 + 0) \right] = \frac{\pi}{4}$
8. $\int_0^{\pi/2} \sin^2(2\theta) \, d\theta = \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) \, d\theta = \frac{1}{2} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} = \frac{1}{2} \left[\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right] = \frac{\pi}{4}$
9. $\int_0^{\pi} \sin^4(3t) \, dt = \int_0^{\pi} [\sin^2(3t)]^2 \, dt = \int_0^{\pi} \left[\frac{1}{2} (1 - \cos 6t) \right]^2 \, dt = \frac{1}{4} \int_0^{\pi} (1 - 2 \cos 6t + \cos^2 6t) \, dt$
 $= \frac{1}{4} \int_0^{\pi} \left[1 - 2 \cos 6t + \frac{1}{2} (1 + \cos 12t) \right] \, dt = \frac{1}{4} \int_0^{\pi} \left(\frac{3}{2} - 2 \cos 6t + \frac{1}{2} \cos 12t \right) \, dt$
 $= \frac{1}{4} \left[\frac{3}{2} t - \frac{1}{3} \sin 6t + \frac{1}{24} \sin 12t \right]_0^{\pi} = \frac{1}{4} \left[\left(\frac{3\pi}{2} - 0 + 0 \right) - (0 - 0 + 0) \right] = \frac{3\pi}{8}$
10. $\int_0^{\pi} \cos^6 \theta \, d\theta = \int_0^{\pi} (\cos^2 \theta)^3 \, d\theta = \int_0^{\pi} \left[\frac{1}{2} (1 + \cos 2\theta) \right]^3 \, d\theta = \frac{1}{8} \int_0^{\pi} (1 + 3 \cos 2\theta + 3 \cos^2 2\theta + \cos^3 2\theta) \, d\theta$
 $= \frac{1}{8} \left[\theta + \frac{3}{2} \sin 2\theta \right]_0^{\pi} + \frac{1}{8} \int_0^{\pi} \left[\frac{3}{2} (1 + \cos 4\theta) \right] \, d\theta + \frac{1}{8} \int_0^{\pi} [(1 - \sin^2 2\theta) \cos 2\theta] \, d\theta$
 $= \frac{1}{8} \pi + \frac{3}{16} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi} + \frac{1}{8} \int_0^0 (1 - u^2) \left(\frac{1}{2} du \right) \quad [u = \sin 2\theta, du = 2 \cos 2\theta \, d\theta]$
 $= \frac{\pi}{8} + \frac{3\pi}{16} + 0 = \frac{5\pi}{16}$
11. $\int (1 + \cos \theta)^2 \, d\theta = \int (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta = \theta + 2 \sin \theta + \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta$
 $= \theta + 2 \sin \theta + \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + C = \frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta + C$
12. Let $u = x, dv = \cos^2 x \, dx \Rightarrow du = dx, v = \int \cos^2 x \, dx = \int \frac{1}{2} (1 + \cos 2x) \, dx = \frac{1}{2} x + \frac{1}{4} \sin 2x$, so
 $\int x \cos^2 x \, dx = x \left(\frac{1}{2} x + \frac{1}{4} \sin 2x \right) - \int \left(\frac{1}{2} x + \frac{1}{4} \sin 2x \right) \, dx = \frac{1}{2} x^2 + \frac{1}{4} x \sin 2x - \frac{1}{4} x^2 + \frac{1}{8} \cos 2x + C$
 $= \frac{1}{4} x^2 + \frac{1}{4} x \sin 2x + \frac{1}{8} \cos 2x + C$
13. $\int_0^{\pi/4} \sin^4 x \cos^2 x \, dx = \int_0^{\pi/4} \sin^2 x (\sin x \cos x)^2 \, dx = \int_0^{\pi/4} \frac{1}{2} (1 - \cos 2x) \left(\frac{1}{2} \sin 2x \right)^2 \, dx$
 $= \frac{1}{8} \int_0^{\pi/4} (1 - \cos 2x) \sin^2 2x \, dx = \frac{1}{8} \int_0^{\pi/4} \sin^2 2x \, dx - \frac{1}{8} \int_0^{\pi/4} \sin^2 2x \cos 2x \, dx$
 $= \frac{1}{16} \int_0^{\pi/4} (1 - \cos 4x) \, dx - \frac{1}{16} \left[\frac{1}{3} \sin^3 2x \right]_0^{\pi/4} = \frac{1}{16} \left[x - \frac{1}{4} \sin 4x - \frac{1}{3} \sin^3 2x \right]_0^{\pi/4}$
 $= \frac{1}{16} \left(\frac{\pi}{4} - 0 - \frac{1}{3} \right) = \frac{1}{192} (3\pi - 4)$
14. $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx = \int_0^{\pi/2} \frac{1}{4} (4 \sin^2 x \cos^2 x) \, dx = \int_0^{\pi/2} \frac{1}{4} (2 \sin x \cos x)^2 \, dx = \frac{1}{4} \int_0^{\pi/2} \sin^2 2x \, dx$
 $= \frac{1}{4} \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4x) \, dx = \frac{1}{8} \int_0^{\pi/2} (1 - \cos 4x) \, dx = \frac{1}{8} \left[x - \frac{1}{4} \sin 4x \right]_0^{\pi/2}$
 $= \frac{1}{8} \left(\frac{\pi}{2} \right) = \frac{\pi}{16}$
15. $\int \sin^3 x \sqrt{\cos x} \, dx = \int (1 - \cos^2 x) \sqrt{\cos x} \sin x \, dx \stackrel{c}{=} \int (1 - u^2) u^{1/2} (-du) = \int (u^{5/2} - u^{1/2}) \, du$
 $= \frac{2}{7} u^{7/2} - \frac{2}{3} u^{3/2} + C = \frac{2}{7} (\cos x)^{7/2} - \frac{2}{3} (\cos x)^{3/2} + C$
 $= \left(\frac{2}{7} \cos^3 x - \frac{2}{3} \cos x \right) \sqrt{\cos x} + C$

16. Let $u = \sin \theta$. Then $du = \cos \theta d\theta$ and

$$\begin{aligned}\int \cos \theta \cos^5(\sin \theta) d\theta &= \int \cos^5 u du = \int (\cos^2 u)^2 \cos u du = \int (1 - \sin^2 u)^2 \cos u du \\ &= \int (1 - 2\sin^2 u + \sin^4 u) \cos u du = I\end{aligned}$$

Now let $x = \sin u$. Then $dx = \cos u du$ and

$$\begin{aligned}I &= \int (1 - 2x^2 + x^4) dx = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C = \sin u - \frac{2}{3}\sin^3 u + \frac{1}{5}\sin^5 u + C \\ &= \sin(\sin \theta) - \frac{2}{3}\sin^3(\sin \theta) + \frac{1}{5}\sin^5(\sin \theta) + C\end{aligned}$$

$$\begin{aligned}17. \int \cos^2 x \tan^3 x dx &= \int \frac{\sin^3 x}{\cos x} dx \stackrel{c}{=} \int \frac{(1 - u^2)(-du)}{u} = \int \left[\frac{-1}{u} + u \right] du \\ &= -\ln |u| + \frac{1}{2}u^2 + C = \frac{1}{2}\cos^2 x - \ln |\cos x| + C\end{aligned}$$

$$\begin{aligned}18. \int \cot^5 \theta \sin^4 \theta d\theta &= \int \frac{\cos^5 \theta}{\sin^5 \theta} \sin^4 \theta d\theta = \int \frac{\cos^5 \theta}{\sin \theta} d\theta = \int \frac{\cos^4 \theta}{\sin \theta} \cos \theta d\theta = \int \frac{(1 - \sin^2 \theta)^2}{\sin \theta} \cos \theta d\theta \\ &\stackrel{c}{=} \int \frac{(1 - u^2)^2}{u} du = \int \frac{1 - 2u^2 + u^4}{u} du = \int \left(\frac{1}{u} - 2u + u^3 \right) du \\ &= \ln |u| - u^2 + \frac{1}{4}u^4 + C = \ln |\sin \theta| - \sin^2 \theta + \frac{1}{4}\sin^4 \theta + C\end{aligned}$$

$$\begin{aligned}19. \int \frac{1 - \sin x}{\cos x} dx &= \int (\sec x - \tan x) dx = \ln |\sec x + \tan x| - \ln |\sec x| + C \quad \left[\begin{array}{l} \text{by (1) and the boxed} \\ \text{formula above it} \end{array} \right] \\ &= \ln |(\sec x + \tan x) \cos x| + C = \ln |1 + \sin x| + C \\ &= \ln (1 + \sin x) + C \quad \text{since } 1 + \sin x \geq 0\end{aligned}$$

$$\begin{aligned}\text{Or: } \int \frac{1 - \sin x}{\cos x} dx &= \int \frac{1 - \sin x}{\cos x} \cdot \frac{1 + \sin x}{1 + \sin x} dx = \int \frac{(1 - \sin^2 x) dx}{\cos x (1 + \sin x)} = \int \frac{\cos x dx}{1 + \sin x} \\ &= \int \frac{dw}{w} \quad [\text{where } w = 1 + \sin x, dw = \cos x dx] \\ &= \ln |w| + C = \ln |1 + \sin x| + C = \ln (1 + \sin x) + C\end{aligned}$$

$$20. \int \cos^2 x \sin 2x dx = 2 \int \cos^3 x \sin x dx \stackrel{c}{=} -2 \int u^3 du = -\frac{1}{2}u^4 + C = -\frac{1}{2}\cos^4 x + C$$

$$21. \text{ Let } u = \tan x, du = \sec^2 x dx. \text{ Then } \int \sec^2 x \tan x dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

$$\text{Or: Let } v = \sec x, dv = \sec x \tan x dx. \text{ Then } \int \sec^2 x \tan x dx = \int v dv = \frac{1}{2}v^2 + C = \frac{1}{2}\sec^2 x + C.$$

$$\begin{aligned}22. \int_0^{\pi/2} \sec^4(t/2) dt &= \int_0^{\pi/4} \sec^4 x (2 dx) \quad [x = t/2, dx = \frac{1}{2} dt] = 2 \int_0^{\pi/4} \sec^2 x (1 + \tan^2 x) dx \\ &= 2 \int_0^1 (1 + u^2) du \quad [u = \tan x, du = \sec^2 x dx] = 2 \left[u + \frac{1}{3}u^3 \right]_0^1 = 2 \left(1 + \frac{1}{3} \right) = \frac{8}{3}\end{aligned}$$

$$23. \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

$$\begin{aligned}24. \int \tan^4 x dx &= \int \tan^2 x (\sec^2 x - 1) dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C \\ &(\text{Set } u = \tan x \text{ in the first integral and use Exercise 23 for the second.})\end{aligned}$$

$$\begin{aligned}25. \int \sec^6 t dt &= \int \sec^4 t \cdot \sec^2 t dt = \int (\tan^2 t + 1)^2 \sec^2 t dt = \int (u^2 + 1)^2 du \quad [u = \tan t, du = \sec^2 t dt] \\ &= \int (u^4 + 2u^2 + 1) du = \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C = \frac{1}{5}\tan^5 t + \frac{2}{3}\tan^3 t + \tan t + C\end{aligned}$$

$$\begin{aligned}26. \int_0^{\pi/4} \sec^4 \theta \tan^4 \theta d\theta &= \int_0^{\pi/4} (\tan^2 \theta + 1) \tan^4 \theta \sec^2 \theta d\theta = \int_0^1 (u^2 + 1)u^4 du \quad [u = \tan \theta, du = \sec^2 \theta d\theta] \\ &= \int_0^1 (u^6 + u^4) du = \left[\frac{1}{7}u^7 + \frac{1}{5}u^5 \right]_0^1 = \frac{1}{7} + \frac{1}{5} = \frac{12}{35}\end{aligned}$$

$$\begin{aligned}
 27. \int_0^{\pi/3} \tan^5 x \sec^4 x \, dx &= \int_0^{\pi/3} \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx \\
 &= \int_0^{\sqrt{3}} u^5 (u^2 + 1) \, du \quad [u = \tan x, du = \sec^2 x \, dx] \\
 &= \int_0^{\sqrt{3}} (u^7 + u^5) \, du = \left[\frac{1}{8} u^8 + \frac{1}{6} u^6 \right]_0^{\sqrt{3}} = \frac{81}{8} + \frac{27}{6} = \frac{81}{8} + \frac{9}{2} = \frac{81}{8} + \frac{36}{8} = \frac{117}{8}
 \end{aligned}$$

Alternate solution:

$$\begin{aligned}
 \int_0^{\pi/3} \tan^5 x \sec^4 x \, dx &= \int_0^{\pi/3} \tan^4 x \sec^3 x \sec x \tan x \, dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x \, dx \\
 &= \int_1^2 (u^2 - 1)^2 u^3 \, du \quad [u = \sec x, du = \sec x \tan x \, dx] \\
 &= \int_1^2 (u^4 - 2u^2 + 1) u^3 \, du = \int_1^2 (u^7 - 2u^5 + u^3) \, du \\
 &= \left[\frac{1}{8} u^8 - \frac{2}{6} u^6 + \frac{1}{4} u^4 \right]_1^2 = \left(32 - \frac{64}{3} + 4 \right) - \left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4} \right) = \frac{117}{8}
 \end{aligned}$$

$$\begin{aligned}
 28. \int \tan^3(2x) \sec^5(2x) \, dx &= \int \tan^2(2x) \sec^4(2x) \cdot \sec(2x) \tan(2x) \, dx \\
 &= \int (u^2 - 1) u^4 \left(\frac{1}{2} du \right) \quad [u = \sec(2x), du = 2 \sec(2x) \tan(2x) \, dx] \\
 &= \frac{1}{2} \int (u^6 - u^4) \, du = \frac{1}{14} u^7 - \frac{1}{10} u^5 + C = \frac{1}{14} \sec^7(2x) - \frac{1}{10} \sec^5(2x) + C
 \end{aligned}$$

$$\begin{aligned}
 29. \int \tan^3 x \sec x \, dx &= \int \tan^2 x \sec x \tan x \, dx = \int (\sec^2 x - 1) \sec x \tan x \, dx \\
 &= \int (u^2 - 1) \, du \quad [u = \sec x, du = \sec x \tan x \, dx] \\
 &= \frac{1}{3} u^3 - u + C = \frac{1}{3} \sec^3 x - \sec x + C
 \end{aligned}$$

$$\begin{aligned}
 30. \int_0^{\pi/3} \tan^5 x \sec^6 x \, dx &= \int_0^{\pi/3} \tan^5 x \sec^4 x \sec^2 x \, dx = \int_0^{\pi/3} \tan^5 x (1 + \tan^2 x)^2 \sec^2 x \, dx \\
 &= \int_0^{\sqrt{3}} u^5 (1 + u^2)^2 \, du \quad [u = \tan x, du = \sec^2 x \, dx] = \int_0^{\sqrt{3}} u^5 (1 + 2u^2 + u^4) \, du \\
 &= \int_0^{\sqrt{3}} (u^5 + 2u^7 + u^9) \, du = \left[\frac{1}{6} u^6 + \frac{1}{4} u^8 + \frac{1}{10} u^{10} \right]_0^{\sqrt{3}} = \frac{27}{6} + \frac{81}{4} + \frac{243}{10} = \frac{981}{20}
 \end{aligned}$$

Alternate solution:

$$\begin{aligned}
 \int_0^{\pi/3} \tan^5 x \sec^6 x \, dx &= \int_0^{\pi/3} \tan^4 x \sec^5 x \sec x \tan x \, dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^5 x \sec x \tan x \, dx \\
 &= \int_1^2 (u^2 - 1)^2 u^5 \, du \quad [u = \sec x, du = \sec x \tan x \, dx] \\
 &= \int_1^2 (u^4 - 2u^2 + 1) u^5 \, du = \int_1^2 (u^9 - 2u^7 + u^5) \, du \\
 &= \left[\frac{1}{10} u^{10} - \frac{1}{4} u^8 + \frac{1}{6} u^6 \right]_1^2 = \left(\frac{512}{5} - 64 + \frac{32}{3} \right) - \left(\frac{1}{10} - \frac{1}{4} + \frac{1}{6} \right) = \frac{981}{20}
 \end{aligned}$$

$$\begin{aligned}
 31. \int \tan^5 x \, dx &= \int (\sec^2 x - 1)^2 \tan x \, dx = \int \sec^4 x \tan x \, dx - 2 \int \sec^2 x \tan x \, dx + \int \tan x \, dx \\
 &= \int \sec^3 x \sec x \tan x \, dx - 2 \int \tan x \sec^2 x \, dx + \int \tan x \, dx \\
 &= \frac{1}{4} \sec^4 x - \tan^2 x + \ln |\sec x| + C \quad \left[\text{or } \frac{1}{4} \sec^4 x - \sec^2 x + \ln |\sec x| + C \right]
 \end{aligned}$$

$$\begin{aligned}
 32. \int \tan^6 ay \, dy &= \int \tan^4 ay (\sec^2 ay - 1) \, dy = \int \tan^4 ay \sec^2 ay \, dy - \int \tan^4 ay \, dy \\
 &= \frac{1}{5a} \tan^5 ay - \int \tan^2 ay (\sec^2 ay - 1) \, dy \\
 &= \frac{1}{5a} \tan^5 ay - \int \tan^2 ay \sec^2 ay \, dy + \int (\sec^2 ay - 1) \, dy \\
 &= \frac{1}{5a} \tan^5 ay - \frac{1}{3a} \tan^3 ay + \frac{1}{a} \tan ay - y + C
 \end{aligned}$$

$$\begin{aligned}
 33. \int \frac{\tan^3 \theta}{\cos^4 \theta} \, d\theta &= \int \tan^3 \theta \sec^4 \theta \, d\theta = \int \tan^3 \theta \cdot (\tan^2 \theta + 1) \cdot \sec^2 \theta \, d\theta \\
 &= \int u^3 (u^2 + 1) \, du \quad [u = \tan \theta, du = \sec^2 \theta \, d\theta] \\
 &= \int (u^5 + u^3) \, du = \frac{1}{6} u^6 + \frac{1}{4} u^4 + C = \frac{1}{6} \tan^6 \theta + \frac{1}{4} \tan^4 \theta + C
 \end{aligned}$$

$$\begin{aligned} 34. \int \tan^2 x \sec x \, dx &= \int (\sec^2 x - 1) \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx \\ &= \frac{1}{2}(\sec x \tan x + \ln |\sec x + \tan x|) - \ln |\sec x + \tan x| + C \quad [\text{by Example 8 and (1)}] \\ &= \frac{1}{2}(\sec x \tan x - \ln |\sec x + \tan x|) + C \end{aligned}$$

$$35. \int_{\pi/6}^{\pi/2} \cot^2 x \, dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1) \, dx = [-\cot x - x]_{\pi/6}^{\pi/2} = (0 - \frac{\pi}{2}) - (-\sqrt{3} - \frac{\pi}{6}) = \sqrt{3} - \frac{\pi}{3}$$

$$\begin{aligned} 36. \int_{\pi/4}^{\pi/2} \cot^3 x \, dx &= \int_{\pi/4}^{\pi/2} \cot x (\csc^2 x - 1) \, dx = \int_{\pi/4}^{\pi/2} \cot x \csc^2 x \, dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \, dx \\ &= \left[-\frac{1}{2} \cot^2 x - \ln |\sin x| \right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\frac{1}{2} - \ln \frac{1}{\sqrt{2}} \right] = \frac{1}{2} + \ln \frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \ln 2) \end{aligned}$$

$$\begin{aligned} 37. \int \cot^3 \alpha \csc^3 \alpha \, d\alpha &= \int \cot^2 \alpha \csc^2 \alpha \cdot \csc \alpha \cot \alpha \, d\alpha = \int (\csc^2 \alpha - 1) \csc^2 \alpha \cdot \csc \alpha \cot \alpha \, d\alpha \\ &= \int (u^2 - 1)u^2 \cdot (-du) \quad [u = \csc \alpha, du = -\csc \alpha \cot \alpha \, d\alpha] \\ &= \int (u^2 - u^4) \, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\csc^3 \alpha - \frac{1}{5}\csc^5 \alpha + C \end{aligned}$$

$$\begin{aligned} 38. \int \csc^4 x \cot^6 x \, dx &= \int \cot^6 x (\cot^2 x + 1) \csc^2 x \, dx \\ &= \int u^6(u^2 + 1) \cdot (-du) \quad [u = \cot x, du = -\csc^2 x \, dx] \\ &= \int (-u^8 - u^6) \, du = -\frac{1}{9}u^9 - \frac{1}{7}u^7 + C = -\frac{1}{9}\cot^9 x - \frac{1}{7}\cot^7 x + C \end{aligned}$$

$$\begin{aligned} 39. I = \int \csc x \, dx &= \int \frac{\csc x (\csc x - \cot x)}{\csc x - \cot x} \, dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} \, dx. \text{ Let } u = \csc x - \cot x \Rightarrow \\ du &= (-\csc x \cot x + \csc^2 x) \, dx. \text{ Then } I = \int du/u = \ln |u| = \ln |\csc x - \cot x| + C. \end{aligned}$$

$$\begin{aligned} 40. \text{ Let } u = \csc x, dv = \csc^2 x \, dx. \text{ Then } du = -\csc x \cot x \, dx, v = -\cot x \Rightarrow \\ \int \csc^3 x \, dx &= -\csc x \cot x - \int \csc x \cot^2 x \, dx = -\csc x \cot x - \int \csc x (\csc^2 x - 1) \, dx \\ &= -\csc x \cot x + \int \csc x \, dx - \int \csc^3 x \, dx \end{aligned}$$

Solving for $\int \csc^3 x \, dx$ and using Exercise 39, we get

$$\int \csc^3 x \, dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \int \csc x \, dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x| + C. \text{ Thus,}$$

$$\begin{aligned} \int_{\pi/6}^{\pi/3} \csc^3 x \, dx &= \left[-\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/3} \\ &= -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{2} \ln \left| \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right| + \frac{1}{2} \cdot 2 \cdot \sqrt{3} - \frac{1}{2} \ln |2 - \sqrt{3}| \\ &= -\frac{1}{3} + \sqrt{3} + \frac{1}{2} \ln \frac{1}{\sqrt{3}} - \frac{1}{2} \ln (2 - \sqrt{3}) \approx 1.7825 \end{aligned}$$

41. Use Equation 2(b):

$$\begin{aligned} \int \sin 5x \sin 2x \, dx &= \int \frac{1}{2} [\cos(5x - 2x) - \cos(5x + 2x)] \, dx = \frac{1}{2} \int (\cos 3x - \cos 7x) \, dx \\ &= \frac{1}{6} \sin 3x - \frac{1}{14} \sin 7x + C \end{aligned}$$

42. Use Equation 2(a):

$$\begin{aligned} \int \sin 3x \cos x \, dx &= \int \frac{1}{2} [\sin(3x + x) + \sin(3x - x)] \, dx = \frac{1}{2} \int (\sin 4x + \sin 2x) \, dx \\ &= -\frac{1}{8} \cos 4x - \frac{1}{4} \cos 2x + C \end{aligned}$$

43. Use Equation 2(c):

$$\begin{aligned} \int \cos 7\theta \cos 5\theta \, d\theta &= \int \frac{1}{2} [\cos(7\theta - 5\theta) + \cos(7\theta + 5\theta)] \, d\theta = \frac{1}{2} \int (\cos 2\theta + \cos 12\theta) \, d\theta \\ &= \frac{1}{2} \left(\frac{1}{2} \sin 2\theta + \frac{1}{12} \sin 12\theta \right) + C = \frac{1}{4} \sin 2\theta + \frac{1}{24} \sin 12\theta + C \end{aligned}$$

$$\begin{aligned}
 44. \int \frac{\cos x + \sin x}{\sin 2x} dx &= \frac{1}{2} \int \frac{\cos x + \sin x}{\sin x \cos x} dx = \frac{1}{2} \int (\csc x + \sec x) dx \\
 &= \frac{1}{2} \left(\ln |\csc x - \cot x| + \ln |\sec x + \tan x| \right) + C \quad [\text{by Exercise 39 and (1)}]
 \end{aligned}$$

$$45. \int \frac{1 - \tan^2 x}{\sec^2 x} dx = \int (\cos^2 x - \sin^2 x) dx = \int \cos 2x dx = \frac{1}{2} \sin 2x + C$$

$$\begin{aligned}
 46. \int \frac{dx}{\cos x - 1} &= \int \frac{1}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1} dx = \int \frac{\cos x + 1}{\cos^2 x - 1} dx = \int \frac{\cos x + 1}{-\sin^2 x} dx \\
 &= \int (-\cot x \csc x - \csc^2 x) dx = \csc x + \cot x + C
 \end{aligned}$$

$$\begin{aligned}
 47. \text{Let } u = \tan(t^2) \Rightarrow du &= 2t \sec^2(t^2) dt. \text{ Then} \\
 \int t \sec^2(t^2) \tan^4(t^2) dt &= \int u^4 \left(\frac{1}{2} du\right) = \frac{1}{10} u^5 + C = \frac{1}{10} \tan^5(t^2) + C.
 \end{aligned}$$

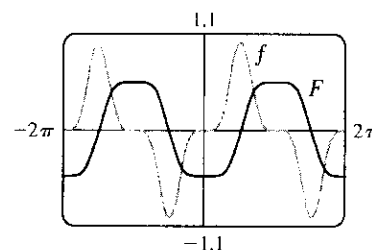
$$\begin{aligned}
 48. \text{Let } u = \tan^7 x, dv &= \sec x \tan x dx \Rightarrow du = 7 \tan^6 x \sec^2 x dx, v = \sec x. \text{ Then} \\
 \int \tan^8 x \sec x dx &= \int \tan^7 x \cdot \sec x \tan x dx = \tan^7 x \sec x - \int 7 \tan^6 x \sec^2 x \sec x dx \\
 &= \tan^7 x \sec x - 7 \int \tan^6 x (\tan^2 x + 1) \sec x dx \\
 &= \tan^7 x \sec x - 7 \int \tan^8 x \sec x dx - 7 \int \tan^6 x \sec x dx.
 \end{aligned}$$

Thus, $8 \int \tan^8 x \sec x dx = \tan^7 x \sec x - 7 \int \tan^6 x \sec x dx$ and

$$\int_0^{\pi/4} \tan^8 x \sec x dx = \frac{1}{8} [\tan^7 x \sec x]_0^{\pi/4} - \frac{7}{8} \int_0^{\pi/4} \tan^6 x \sec x dx = \frac{\sqrt{2}}{8} - \frac{7}{8} I.$$

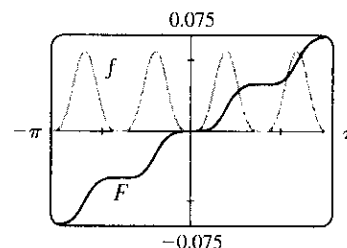
$$\begin{aligned}
 49. \text{Let } u = \cos x \Rightarrow du &= -\sin x dx. \text{ Then} \\
 \int \sin^5 x dx &= \int (1 - \cos^2 x)^2 \sin x dx = \int (1 - u^2)^2 (-du) \\
 &= \int (-1 + 2u^2 - u^4) du = -\frac{1}{5} u^5 + \frac{2}{3} u^3 - u + C \\
 &= -\frac{1}{5} \cos^5 x + \frac{2}{3} \cos^3 x - \cos x + C
 \end{aligned}$$

Notice that F is increasing when $f(x) > 0$, so the graphs serve as a check on our work.



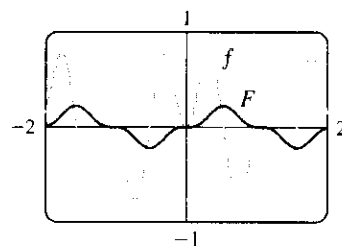
$$\begin{aligned}
 50. \int \sin^4 x \cos^4 x dx &= \int \left(\frac{1}{2} \sin 2x\right)^4 dx = \frac{1}{16} \int \sin^4 2x dx = \frac{1}{16} \int \left[\frac{1}{2} (1 - \cos 4x)\right]^2 dx \\
 &= \frac{1}{64} \int (1 - 2 \cos 4x + \cos^2 4x) dx \\
 &= \frac{1}{64} \left(x - \frac{1}{2} \sin 4x\right) + \frac{1}{128} \int (1 + \cos 8x) dx \\
 &= \frac{1}{64} \left(x - \frac{1}{2} \sin 4x\right) + \frac{1}{128} \left(x + \frac{1}{8} \sin 8x\right) + C \\
 &= \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x + C
 \end{aligned}$$

Notice that $f(x) = 0$ whenever F has a horizontal tangent.

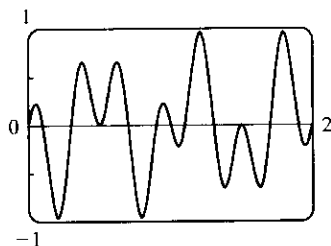


$$\begin{aligned}
 51. \int \sin 3x \sin 6x dx &= \int \frac{1}{2} [\cos(3x - 6x) - \cos(3x + 6x)] dx \\
 &= \frac{1}{2} \int (\cos 3x - \cos 9x) dx \\
 &= \frac{1}{6} \sin 3x - \frac{1}{18} \sin 9x + C
 \end{aligned}$$

Notice that $f(x) = 0$ whenever F has a horizontal tangent.



58.



It seems from the graph that $\int_0^2 \sin 2\pi x \cos 5\pi x \, dx = 0$, since each bulge above the x -axis seems to have a corresponding depression below the x -axis. To evaluate the integral, we use a trigonometric identity:

$$\begin{aligned} \int_0^1 \sin 2\pi x \cos 5\pi x \, dx &= \frac{1}{2} \int_0^2 [\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)] \, dx \\ &= \frac{1}{2} \int_0^2 [\sin(-3\pi x) + \sin 7\pi x] \, dx \\ &= \frac{1}{2} \left[\frac{1}{-3\pi} \cos(-3\pi x) - \frac{1}{7\pi} \cos 7\pi x \right]_0^2 \\ &= \frac{1}{2} \left[\frac{1}{-3\pi} (1 - 1) - \frac{1}{7\pi} (1 - 1) \right] = 0 \end{aligned}$$

$$59. V = \int_{\pi/2}^{\pi} \pi \sin^2 x \, dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2} (1 - \cos 2x) \, dx = \pi \left[\frac{1}{2} x - \frac{1}{4} \sin 2x \right]_{\pi/2}^{\pi} = \pi \left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0 \right) = \frac{\pi^2}{4}$$

$$\begin{aligned} 60. \text{Volume} &= \int_0^{\pi/4} \pi (\tan^2 x)^2 \, dx = \pi \int_0^{\pi/4} \tan^2 x (\sec^2 x - 1) \, dx = \pi \int_0^{\pi/4} \tan^2 x \sec^2 x \, dx - \pi \int_0^{\pi/4} \tan^2 x \, dx \\ &= \pi \int_0^{\pi/4} u^2 \, du - \pi \int_0^{\pi/4} (\sec^2 x - 1) \, dx \quad [\text{where } u = \tan x \text{ and } du = \sec^2 x \, dx] \\ &= \pi \left[\frac{1}{3} u^3 \right]_{x=0}^{\pi/4} - \pi [\tan x - x]_0^{\pi/4} = \pi \left[\frac{1}{3} \tan^3 x - \tan x + x \right]_0^{\pi/4} = \pi \left[\frac{1}{3} - 1 + \frac{\pi}{4} \right] = \pi \left(\frac{\pi}{4} - \frac{2}{3} \right) \end{aligned}$$

$$\begin{aligned} 61. \text{Volume} &= \pi \int_0^{\pi/2} [(1 + \cos x)^2 - 1^2] \, dx = \pi \int_0^{\pi/2} (2 \cos x + \cos^2 x) \, dx \\ &= \pi \left[2 \sin x + \frac{1}{2} x + \frac{1}{4} \sin 2x \right]_0^{\pi/2} = \pi \left(2 + \frac{\pi}{4} \right) = 2\pi + \frac{\pi^2}{4} \end{aligned}$$

$$\begin{aligned} 62. \text{Volume} &= \pi \int_0^{\pi/2} [1^2 - (1 - \cos x)^2] \, dx = \pi \int_0^{\pi/2} (2 \cos x - \cos^2 x) \, dx \\ &= \pi \left[2 \sin x - \frac{1}{2} x - \frac{1}{4} \sin 2x \right]_0^{\pi/2} = \pi \left(2 - \frac{\pi}{4} - 0 \right) = 2\pi - \frac{\pi^2}{4} \end{aligned}$$

$$\begin{aligned} 63. s &= f(t) = \int_0^t \sin \omega u \cos^2 \omega u \, du. \text{ Let } y = \cos \omega u \Rightarrow dy = -\omega \sin \omega u \, du. \text{ Then} \\ s &= -\frac{1}{\omega} \int_1^{\cos \omega t} y^2 \, dy = -\frac{1}{\omega} \left[\frac{1}{3} y^3 \right]_1^{\cos \omega t} = \frac{1}{3\omega} (1 - \cos^3 \omega t). \end{aligned}$$

$$\begin{aligned} 64. (a) \text{ We want to calculate the square root of the average value of } [E(t)]^2 &= [155 \sin(120\pi t)]^2 = 155^2 \sin^2(120\pi t). \\ \text{First, we calculate the average value itself, by integrating } [E(t)]^2 &\text{ over one cycle (between } t = 0 \text{ and } t = \frac{1}{60}, \\ \text{since there are 60 cycles per second) and dividing by } &\left(\frac{1}{60} - 0 \right): \end{aligned}$$

$$\begin{aligned} [E(t)]_{\text{ave}}^2 &= \frac{1}{1/60} \int_0^{1/60} [155^2 \sin^2(120\pi t)] \, dt = 60 \cdot 155^2 \int_0^{1/60} \frac{1}{2} [1 - \cos(240\pi t)] \, dt \\ &= 60 \cdot 155^2 \left(\frac{1}{2} \right) \left[t - \frac{1}{240\pi} \sin(240\pi t) \right]_0^{1/60} = 60 \cdot 155^2 \left(\frac{1}{2} \right) \left[\left(\frac{1}{60} - 0 \right) - (0 - 0) \right] = \frac{155^2}{2} \end{aligned}$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}} \approx 110$ V.

$$(b) 220 = \sqrt{[E(t)]_{\text{ave}}^2} \Rightarrow$$

$$\begin{aligned} 220^2 &= [E(t)]_{\text{ave}}^2 = \frac{1}{1/60} \int_0^{1/60} A^2 \sin^2(120\pi t) \, dt = 60 A^2 \int_0^{1/60} \frac{1}{2} [1 - \cos(240\pi t)] \, dt \\ &= 30 A^2 \left[t - \frac{1}{240\pi} \sin(240\pi t) \right]_0^{1/60} = 30 A^2 \left[\left(\frac{1}{60} - 0 \right) - (0 - 0) \right] = \frac{1}{2} A^2 \end{aligned}$$

$$\text{Thus, } 220^2 = \frac{1}{2} A^2 \Rightarrow A = 220 \sqrt{2} \approx 311 \text{ V.}$$

65. Just note that the integrand is odd [$f(-x) = -f(x)$].

Or: If $m \neq n$, calculate

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx &= \int_{-\pi}^{\pi} \frac{1}{2} [\sin(m-n)x + \sin(m+n)x] \, dx \\ &= \frac{1}{2} \left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0\end{aligned}$$

If $m = n$, then the first term in each set of brackets is zero.

66. $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] \, dx$. If $m \neq n$,

this is equal to $\frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$$\int_{-\pi}^{\pi} \frac{1}{2} [1 - \cos(m+n)x] \, dx = \left[\frac{1}{2}x \right]_{-\pi}^{\pi} - \left[\frac{\sin(m+n)x}{2(m+n)} \right]_{-\pi}^{\pi} = \pi - 0 = \pi.$$

67. $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x + \cos(m+n)x] \, dx$. If $m \neq n$,

this is equal to $\frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$$\int_{-\pi}^{\pi} \frac{1}{2} [1 + \cos(m+n)x] \, dx = \left[\frac{1}{2}x \right]_{-\pi}^{\pi} + \left[\frac{\sin(m+n)x}{2(m+n)} \right]_{-\pi}^{\pi} = \pi + 0 = \pi.$$

$$68. \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\left(\sum_{n=1}^m a_n \sin nx \right) \sin mx \right] \, dx = \sum_{n=1}^m \frac{a_n}{\pi} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx.$$

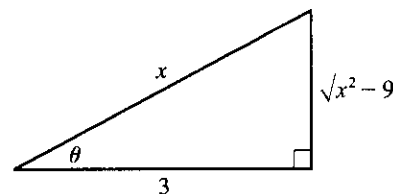
By Exercise 66, every term is zero except the m th one, and that term is $\frac{a_m}{\pi} \cdot \pi = a_m$.

8.3 Trigonometric Substitution

1. Let $x = 3 \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$. Then

$$dx = 3 \sec \theta \tan \theta \, d\theta \text{ and}$$

$$\begin{aligned}\sqrt{x^2 - 9} &= \sqrt{9 \sec^2 \theta - 9} = \sqrt{9(\sec^2 \theta - 1)} = \sqrt{9 \tan^2 \theta} \\ &= 3 |\tan \theta| = 3 \tan \theta \text{ for the relevant values of } \theta.\end{aligned}$$

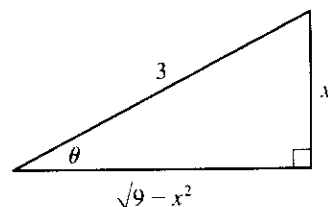


$$\int \frac{1}{x^2 \sqrt{x^2 - 9}} \, dx = \int \frac{1}{9 \sec^2 \theta \cdot 3 \tan \theta} 3 \sec \theta \tan \theta \, d\theta = \frac{1}{9} \int \cos \theta \, d\theta = \frac{1}{9} \sin \theta + C = \frac{1}{9} \frac{\sqrt{x^2 - 9}}{x} + C$$

Note that $-\sec(\theta + \pi) = \sec \theta$, so the figure is sufficient for the case $\pi \leq \theta < \frac{3\pi}{2}$.

2. Let $x = 3 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx = 3 \cos \theta d\theta$ and

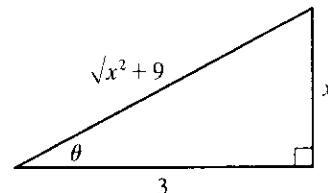
$$\begin{aligned}\sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} \\ &= 3|\cos\theta| = 3\cos\theta \text{ for the relevant values of } \theta.\end{aligned}$$



$$\begin{aligned}\int x^3 \sqrt{9-x^2} dx &= \int 3^3 \sin^3 \theta \cdot 3 \cos \theta \cdot 3 \cos \theta d\theta = 3^5 \int \sin^3 \theta \cos^2 \theta d\theta \\ &= 3^5 \int \sin^2 \theta \cos^2 \theta \sin \theta d\theta = 3^5 \int (1-\cos^2 \theta) \cos^2 \theta \sin \theta d\theta \\ &= 3^5 \int (1-u^2) u^2 (-du) \quad [u = \cos \theta, du = -\sin \theta d\theta] \\ &= 3^5 \int (u^4 - u^2) du = 3^5 \left(\frac{1}{5} u^5 - \frac{1}{3} u^3 \right) + C = 3^5 \left(\frac{1}{5} \cos^5 \theta - \frac{1}{3} \cos^3 \theta \right) + C \\ &= 3^5 \left[\frac{1}{5} \frac{(9-x^2)^{5/2}}{3^5} - \frac{1}{3} \frac{(9-x^2)^{3/2}}{3^3} \right] + C \\ &= \frac{1}{5} (9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C \text{ or } -\frac{1}{5} (x^2+6)(9-x)^{3/2} + C\end{aligned}$$

3. Let $x = 3 \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3 \sec^2 \theta d\theta$ and

$$\begin{aligned}\sqrt{x^2+9} &= \sqrt{9\tan^2\theta+9} = \sqrt{9(\tan^2\theta+1)} = \sqrt{9\sec^2\theta} \\ &= 3|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$



$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+9}} dx &= \int \frac{3^3 \tan^3 \theta}{3 \sec \theta} 3 \sec^2 \theta d\theta = 3^3 \int \tan^3 \theta \sec \theta d\theta = 3^3 \int \tan^2 \theta \tan \theta \sec \theta d\theta \\ &= 3^3 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta = 3^3 \int (u^2 - 1) du \quad [u = \sec \theta, du = \sec \theta \tan \theta d\theta] \\ &= 3^3 \left(\frac{1}{3} u^3 - u \right) + C = 3^3 \left(\frac{1}{3} \sec^3 \theta - \sec \theta \right) + C = 3^3 \left[\frac{1}{3} \frac{(x^2+9)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3} \right] + C \\ &= \frac{1}{3} (x^2+9)^{3/2} - 9\sqrt{x^2+9} + C \text{ or } \frac{1}{3} (x^2-18)\sqrt{x^2+9} + C\end{aligned}$$

4. Let $x = 4 \sin \theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $dx = 4 \cos \theta d\theta$ and

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4|\cos\theta| = 4\cos\theta. \text{ When } x=0, 4\sin\theta=0 \Rightarrow \theta=0, \text{ and}$$

$$\text{when } x=2\sqrt{3}, 4\sin\theta=2\sqrt{3} \Rightarrow \sin\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{3}. \text{ Thus, substitution gives}$$

$$\begin{aligned}\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} dx &= \int_0^{\pi/3} \frac{4^3 \sin^3 \theta}{4 \cos \theta} 4 \cos \theta d\theta = 4^3 \int_0^{\pi/3} \sin^3 \theta d\theta \\ &= 4^3 \int_0^{\pi/3} (1-\cos^2 \theta) \sin \theta d\theta \\ &\stackrel{c}{=} -4^3 \int_1^{1/2} (1-u^2) du = -64 \left[u - \frac{1}{3} u^3 \right]_1^{1/2} \\ &= -64 \left[\left(\frac{1}{2} - \frac{1}{24} \right) - \left(1 - \frac{1}{3} \right) \right] = -64 \left(-\frac{5}{24} \right) = \frac{40}{3}\end{aligned}$$

Or: Let $u = 16 - x^2$, $x^2 = 16 - u$, $du = -2x dx$.

5. Let $t = \sec \theta$, so $dt = \sec \theta \tan \theta d\theta$, $t = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$, and $t = 2 \Rightarrow \theta = \frac{\pi}{3}$. Then

$$\begin{aligned} \int_{\sqrt{2}}^2 \frac{1}{t^3 \sqrt{t^2 - 1}} dt &= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3 \theta \tan \theta} \sec \theta \tan \theta d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2 \theta} d\theta = \int_{\pi/4}^{\pi/3} \cos^2 \theta d\theta \\ &= \int_{\pi/4}^{\pi/3} \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_{\pi/4}^{\pi/3} \\ &= \frac{1}{2} \left[\left(\frac{\pi}{3} + \frac{1}{2} \frac{\sqrt{3}}{2} \right) - \left(\frac{\pi}{4} + \frac{1}{2} \cdot 1 \right) \right] = \frac{1}{2} \left(\frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2} \right) = \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4} \end{aligned}$$

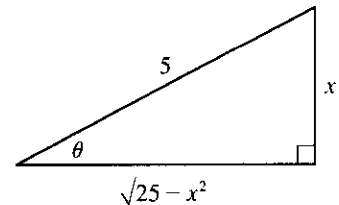
6. Let $x = 2 \tan \theta$, so $dx = 2 \sec^2 \theta d\theta$, $x = 0 \Rightarrow \theta = 0$, and $x = 2 \Rightarrow \theta = \frac{\pi}{4}$. Then

$$\begin{aligned} \int_0^2 x^3 \sqrt{x^2 + 4} dx &= \int_0^{\pi/4} 2^3 \tan^3 \theta \cdot 2 \sec \theta \cdot 2 \sec^2 \theta d\theta = 2^5 \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta \sec \theta \tan \theta d\theta \\ &= 2^5 \int_0^{\pi/4} (\sec^2 \theta - 1) \sec^2 \theta \sec \theta \tan \theta d\theta \\ &= 2^5 \int_1^{\sqrt{2}} (u^2 - 1) u^2 du \quad [u = \sec \theta, du = \sec \theta \tan \theta d\theta] \\ &= 2^5 \int_1^{\sqrt{2}} (u^4 - u^2) du = 2^5 \left[\frac{1}{5} u^5 - \frac{1}{3} u^3 \right]_1^{\sqrt{2}} = 2^5 \left[\left(\frac{1}{5} \cdot 4\sqrt{2} - \frac{1}{3} \cdot 2\sqrt{2} \right) - \left(\frac{1}{5} - \frac{1}{3} \right) \right] \\ &= 32 \left(\frac{2}{15} \sqrt{2} + \frac{2}{15} \right) = \frac{64}{15} (\sqrt{2} + 1) \end{aligned}$$

Or: Let $u = x^2 + 4$, $x^2 = u - 4$, $du = 2x dx$.

7. Let $x = 5 \sin \theta$, so $dx = 5 \cos \theta d\theta$. Then

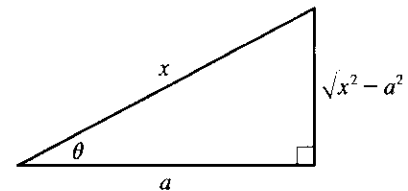
$$\begin{aligned} \int \frac{1}{x^2 \sqrt{25 - x^2}} dx &= \int \frac{1}{5^2 \sin^2 \theta \cdot 5 \cos \theta} 5 \cos \theta d\theta \\ &= \frac{1}{25} \int \csc^2 \theta d\theta = -\frac{1}{25} \cot \theta + C \\ &= -\frac{1}{25} \frac{\sqrt{25 - x^2}}{x} + C \end{aligned}$$



8. Let $x = a \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$. Then

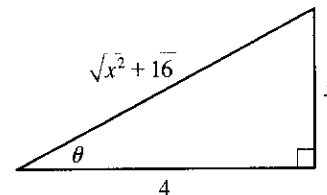
$dx = a \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - a^2} = a \tan \theta$, so

$$\begin{aligned} \int \frac{\sqrt{x^2 - a^2}}{x^4} dx &= \int \frac{a \tan \theta}{a^4 \sec^4 \theta} a \sec \theta \tan \theta d\theta \\ &= \frac{1}{a^2} \int \sin^2 \theta \cos \theta d\theta \\ &= \frac{1}{3a^2} \sin^3 \theta + C = \frac{(x^2 - a^2)^{3/2}}{3a^2 x^3} + C \end{aligned}$$



9. Let $x = 4 \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4 \sec^2 \theta d\theta$ and

$$\begin{aligned}\sqrt{x^2 + 16} &= \sqrt{16 \tan^2 \theta + 16} = \sqrt{16(\tan^2 \theta + 1)} \\ &= \sqrt{16 \sec^2 \theta} = 4 |\sec \theta| \\ &= 4 \sec \theta \text{ for the relevant values of } \theta.\end{aligned}$$

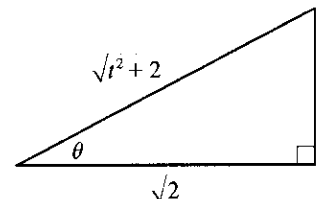


$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + 16}} &= \int \frac{4 \sec^2 \theta d\theta}{4 \sec \theta} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C_1 \\ &= \ln \left| \frac{\sqrt{x^2 + 16}}{4} + \frac{x}{4} \right| + C_1 = \ln |\sqrt{x^2 + 16} + x| - \ln |4| + C_1 \\ &= \ln(\sqrt{x^2 + 16} + x) + C, \text{ where } C = C_1 - \ln 4.\end{aligned}$$

(Since $\sqrt{x^2 + 16} + x > 0$, we don't need the absolute value.)

10. Let $t = \sqrt{2} \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dt = \sqrt{2} \sec^2 \theta d\theta$ and

$$\begin{aligned}\sqrt{t^2 + 2} &= \sqrt{2 \tan^2 \theta + 2} = \sqrt{2(\tan^2 \theta + 1)} = \sqrt{2 \sec^2 \theta} \\ &= \sqrt{2} |\sec \theta| = \sqrt{2} \sec \theta \text{ for the relevant values of } \theta.\end{aligned}$$

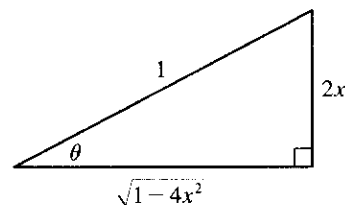


$$\begin{aligned}\int \frac{t^5}{\sqrt{t^2 + 2}} dt &= \int \frac{4 \sqrt{2} \tan^5 \theta}{\sqrt{2} \sec \theta} \sqrt{2} \sec^2 \theta d\theta = 4 \sqrt{2} \int \tan^5 \theta \sec \theta d\theta = 4 \sqrt{2} \int (\sec^2 \theta - 1)^2 \sec \theta \tan \theta d\theta \\ &= 4 \sqrt{2} \int (u^2 - 1)^2 du \quad [u = \sec \theta, du = \sec \theta \tan \theta d\theta] = 4 \sqrt{2} \int (u^4 - 2u^2 + 1) du \\ &= 4 \sqrt{2} \left(\frac{1}{5} u^5 - \frac{2}{3} u^3 + u \right) + C = \frac{4 \sqrt{2}}{15} u(3u^4 - 10u^2 + 15) + C \\ &= \frac{4 \sqrt{2}}{15} \cdot \frac{\sqrt{t^2 + 2}}{\sqrt{2}} \left[3 \cdot \frac{(t^2 + 2)^2}{2^2} - 10 \frac{t^2 + 2}{2} + 15 \right] + C \\ &= \frac{4}{15} \sqrt{t^2 + 2} \cdot \frac{1}{4} [3(t^4 + 4t^2 + 4) - 20(t^2 + 2) + 60] + C \\ &= \frac{1}{15} \sqrt{t^2 + 2} (3t^4 - 8t^2 + 32) + C\end{aligned}$$

11. Let $2x = \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $x = \frac{1}{2} \sin \theta$,

$$dx = \frac{1}{2} \cos \theta d\theta, \text{ and } \sqrt{1 - 4x^2} = \sqrt{1 - (\sin \theta)^2} = \cos \theta.$$

$$\begin{aligned}\int \sqrt{1 - 4x^2} dx &= \int \cos \theta \left(\frac{1}{2} \cos \theta \right) d\theta = \frac{1}{4} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{4} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{4} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{4} \left[\sin^{-1}(2x) + 2x \sqrt{1 - 4x^2} \right] + C\end{aligned}$$



12. $\int_0^1 x \sqrt{x^2 + 4} dx = \int_4^5 \sqrt{u} \left(\frac{1}{2} du \right) \quad [u = x^2 + 4, du = 2x dx] = \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2} \right]_4^5 = \frac{1}{3} (5\sqrt{5} - 8)$

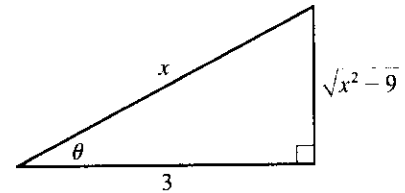
13. Let $x = 3 \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$. Then

$$dx = 3 \sec \theta \tan \theta d\theta \text{ and } \sqrt{x^2 - 9} = 3 \tan \theta, \text{ so}$$

$$\int \frac{\sqrt{x^2 - 9}}{x^3} dx = \int \frac{3 \tan \theta}{27 \sec^3 \theta} 3 \sec \theta \tan \theta d\theta = \frac{1}{3} \int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta$$

$$= \frac{1}{3} \int \sin^2 \theta d\theta = \frac{1}{3} \int \frac{1}{2} (1 - \cos 2\theta) d\theta = \frac{1}{6} \theta - \frac{1}{12} \sin 2\theta + C = \frac{1}{6} \theta - \frac{1}{6} \sin \theta \cos \theta + C$$

$$= \frac{1}{6} \sec^{-1} \left(\frac{x}{3} \right) - \frac{1}{6} \frac{\sqrt{x^2 - 9}}{x} \frac{3}{x} + C = \frac{1}{6} \sec^{-1} \left(\frac{x}{3} \right) - \frac{\sqrt{x^2 - 9}}{2x^2} + C$$



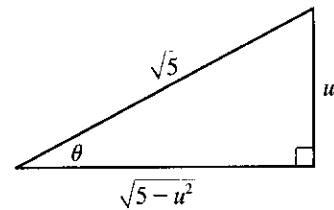
14. Let $u = \sqrt{5} \sin \theta$, so $du = \sqrt{5} \cos \theta d\theta$. Then

$$\int \frac{du}{u \sqrt{5 - u^2}} = \int \frac{1}{\sqrt{5} \sin \theta \cdot \sqrt{5} \cos \theta} \sqrt{5} \cos \theta d\theta = \frac{1}{\sqrt{5}} \int \csc \theta d\theta$$

$$= \frac{1}{\sqrt{5}} \ln |\csc \theta - \cot \theta| + C \quad [\text{by Exercise 8.2.39}]$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5}}{u} - \frac{\sqrt{5 - u^2}}{u} \right| + C$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5} - \sqrt{5 - u^2}}{u} \right| + C$$

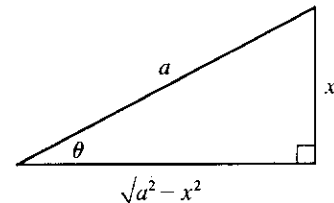


15. Let $x = a \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx = a \cos \theta d\theta$ and

$$\int \frac{x^2 dx}{(a^2 - x^2)^{3/2}} = \int \frac{a^2 \sin^2 \theta a \cos \theta d\theta}{a^3 \cos^3 \theta} = \int \tan^2 \theta d\theta$$

$$= \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C$$

$$= \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a} + C$$

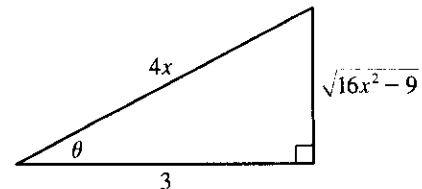


16. Let $4x = 3 \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$. Then

$$dx = \frac{3}{4} \sec \theta \tan \theta d\theta \text{ and } \sqrt{16x^2 - 9} = 3 \tan \theta, \text{ so}$$

$$\int \frac{dx}{x^2 \sqrt{16x^2 - 9}} = \int \frac{\frac{3}{4} \sec \theta \tan \theta d\theta}{\left(\frac{3}{4}\right)^2 \sec^2 \theta 3 \tan \theta}$$

$$= \frac{4}{9} \int \cos \theta d\theta = \frac{4}{9} \sin \theta + C = \frac{4}{9} \frac{\sqrt{16x^2 - 9}}{4x} + C = \frac{\sqrt{16x^2 - 9}}{9x} + C$$



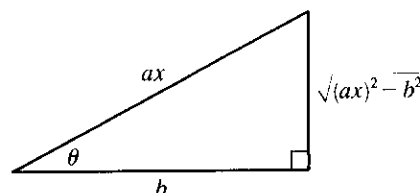
17. Let $u = x^2 - 7$, so $du = 2x dx$. Then $\int \frac{x}{\sqrt{x^2 - 7}} dx = \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} \cdot 2 \sqrt{u} + C = \sqrt{x^2 - 7} + C$.

18. Let
- $ax = b \sec \theta$
- , so
- $(ax)^2 = b^2 \sec^2 \theta \Rightarrow$

$$(ax)^2 - b^2 = b^2 \sec^2 \theta - b^2 = b^2 (\sec^2 \theta - 1) = b^2 \tan^2 \theta. \text{ So}$$

$$\sqrt{(ax)^2 - b^2} = b \tan \theta, dx = \frac{b}{a} \sec \theta \tan \theta d\theta, \text{ and}$$

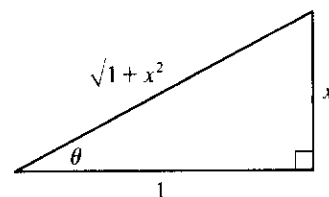
$$\begin{aligned} \int \frac{dx}{[(ax)^2 - b^2]^{3/2}} &= \int \frac{\frac{b}{a} \sec \theta \tan \theta}{b^3 \tan^3 \theta} d\theta = \frac{1}{ab^2} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{ab^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta = \frac{1}{ab^2} \int \csc \theta \cot \theta d\theta \\ &= -\frac{1}{ab^2} \csc \theta + C = -\frac{1}{ab^2} \frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2 \sqrt{(ax)^2 - b^2}} + C \end{aligned}$$



19. Let
- $x = \tan \theta$
- , where
- $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- . Then
- $dx = \sec^2 \theta d\theta$

$$\text{and } \sqrt{1+x^2} = \sec \theta, \text{ so}$$

$$\begin{aligned} \int \frac{\sqrt{1+x^2}}{x} dx &= \int \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta = \int \frac{\sec \theta}{\tan \theta} (1 + \tan^2 \theta) d\theta \\ &= \int (\csc \theta + \sec \theta \tan \theta) d\theta \\ &= \ln |\csc \theta - \cot \theta| + \sec \theta + C \quad [\text{by Exercise 8.2.39}] \\ &= \ln \left| \frac{\sqrt{1+x^2}}{x} - \frac{1}{x} \right| + \frac{\sqrt{1+x^2}}{1} + C = \ln \left| \frac{\sqrt{1+x^2} - 1}{x} \right| + \sqrt{1+x^2} + C \end{aligned}$$



20. Let
- $t = 5 \sin \theta$
- , where
- $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- . Then
- $dt = 5 \cos \theta d\theta$

$$\text{and } \sqrt{25-t^2} = 5 \cos \theta, \text{ so}$$

$$\begin{aligned} \int \frac{t}{\sqrt{25-t^2}} dt &= \int \frac{5 \sin \theta}{5 \cos \theta} 5 \cos \theta d\theta = 5 \int \sin \theta d\theta \\ &= -5 \cos \theta + C = -5 \cdot \frac{\sqrt{25-t^2}}{5} + C = -\sqrt{25-t^2} + C \end{aligned}$$

$$\text{Or: Let } u = 25 - t^2, \text{ so } du = -2t dt.$$

21. Let
- $u = 4 - 9x^2 \Rightarrow du = -18x dx$
- . Then
- $x^2 = \frac{1}{9}(4-u)$
- and

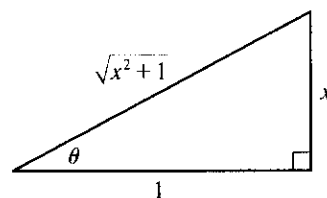
$$\begin{aligned} \int_0^{2/3} x^3 \sqrt{4-9x^2} dx &= \int_4^0 \frac{1}{9}(4-u) u^{1/2} \left(-\frac{1}{18}\right) du = \frac{1}{162} \int_0^4 (4u^{1/2} - u^{3/2}) du \\ &= \frac{1}{162} \left[\frac{8}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right]_0^4 = \frac{1}{162} \left[\frac{64}{3} - \frac{64}{5} \right] = \frac{64}{1215} \end{aligned}$$

$$\text{Or: Let } 3x = 2 \sin \theta, \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

22. Let
- $x = \tan \theta$
- , where
- $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- . Then
- $dx = \sec^2 \theta d\theta$
- ,

$$\sqrt{x^2+1} = \sec \theta \text{ and } x=0 \Rightarrow \theta=0, x=1 \Rightarrow \theta=\frac{\pi}{4}, \text{ so}$$

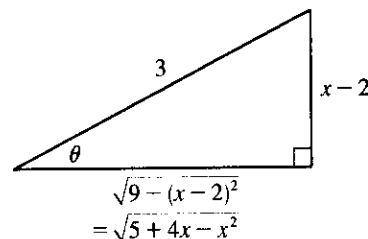
$$\begin{aligned} \int_0^1 \sqrt{x^2+1} dx &= \int_0^{\pi/4} \sec \theta \sec^2 \theta d\theta = \int_0^{\pi/4} \sec^3 \theta d\theta \\ &= \frac{1}{2} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} \quad [\text{by Example 8.2.8}] \\ &= \frac{1}{2} \left[\sqrt{2} \cdot 1 + \ln(1 + \sqrt{2}) - 0 - \ln(1+0) \right] = \frac{1}{2} \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right] \end{aligned}$$



23. $5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x - 2)^2 + 9$. Let

$x - 2 = 3 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so $dx = 3 \cos \theta d\theta$. Then

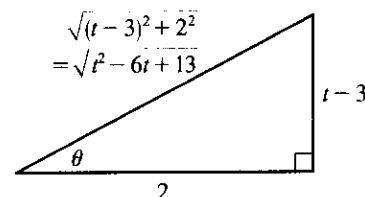
$$\begin{aligned} \int \sqrt{5 + 4x - x^2} dx &= \int \sqrt{9 - (x - 2)^2} dx = \int \sqrt{9 - 9 \sin^2 \theta} 3 \cos \theta d\theta \\ &= \int \sqrt{9 \cos^2 \theta} 3 \cos \theta d\theta = \int 9 \cos^2 \theta d\theta \\ &= \frac{9}{2} \int (1 + \cos 2\theta) d\theta = \frac{9}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta + C = \frac{9}{2} \theta + \frac{9}{4} (2 \sin \theta \cos \theta) + C \\ &= \frac{9}{2} \sin^{-1} \left(\frac{x-2}{3} \right) + \frac{9}{2} \cdot \frac{x-2}{3} \cdot \frac{\sqrt{5+4x-x^2}}{3} + C \\ &= \frac{9}{2} \sin^{-1} \left(\frac{x-2}{3} \right) + \frac{1}{2} (x-2) \sqrt{5+4x-x^2} + C \end{aligned}$$



24. $t^2 - 6t + 13 = (t^2 - 6t + 9) + 4 = (t - 3)^2 + 2^2$. Let $t - 3 = 2 \tan \theta$,

so $dt = 2 \sec^2 \theta d\theta$. Then

$$\begin{aligned} \int \frac{dt}{\sqrt{t^2 - 6t + 13}} &= \int \frac{1}{\sqrt{(2 \tan \theta)^2 + 2^2}} 2 \sec^2 \theta d\theta = \int \frac{2 \sec^2 \theta}{2 \sec \theta} d\theta \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C_1 \quad [\text{by Formula 8.2.1}] \\ &= \ln \left| \frac{\sqrt{t^2 - 6t + 13}}{2} + \frac{t-3}{2} \right| + C_1 \\ &= \ln |\sqrt{t^2 - 6t + 13} + t - 3| + C \quad \text{where } C = C_1 - \ln 2 \end{aligned}$$

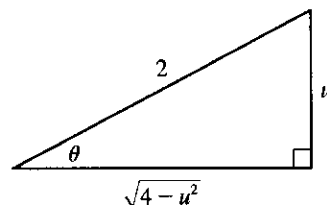


25. $9x^2 + 6x - 8 = (3x + 1)^2 - 9$, so let $u = 3x + 1$, $du = 3dx$. Then $\int \frac{dx}{\sqrt{9x^2 + 6x - 8}} = \int \frac{\frac{1}{3} du}{\sqrt{u^2 - 9}}$. Now let $u = 3 \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$. Then $du = 3 \sec \theta \tan \theta d\theta$ and $\sqrt{u^2 - 9} = 3 \tan \theta$, so

$$\begin{aligned} \int \frac{\frac{1}{3} du}{\sqrt{u^2 - 9}} &= \int \frac{\sec \theta \tan \theta d\theta}{3 \tan \theta} = \frac{1}{3} \int \sec \theta d\theta = \frac{1}{3} \ln |\sec \theta + \tan \theta| + C_1 = \frac{1}{3} \ln \left| \frac{u + \sqrt{u^2 - 9}}{3} \right| + C_1 \\ &= \frac{1}{3} \ln |u + \sqrt{u^2 - 9}| + C = \frac{1}{3} \ln |3x + 1 + \sqrt{9x^2 + 6x - 8}| + C \end{aligned}$$

26. $4x - x^2 = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2$, so let $u = x - 2$. Then $x = u + 2$ and $dx = du$, so

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{4x - x^2}} &= \int \frac{(u + 2)^2 du}{\sqrt{4 - u^2}} = \int \frac{(2 \sin \theta + 2)^2}{2 \cos \theta} 2 \cos \theta d\theta \quad [\text{Put } u = 2 \sin \theta] \\ &= 4 \int (\sin^2 \theta + 2 \sin \theta + 1) d\theta \\ &= 2 \int (1 - \cos 2\theta) d\theta + 8 \int \sin \theta d\theta + 4 \int d\theta \\ &= 2\theta - \sin 2\theta - 8 \cos \theta + 4\theta + C \\ &= 6\theta - 8 \cos \theta - 2 \sin \theta \cos \theta + C \\ &= 6 \sin^{-1} \left(\frac{1}{2} u \right) - 4 \sqrt{4 - u^2} - \frac{1}{2} u \sqrt{4 - u^2} + C \\ &= 6 \sin^{-1} \left(\frac{x-2}{2} \right) - 4 \sqrt{4x - x^2} - \left(\frac{x-2}{2} \right) \sqrt{4x - x^2} + C \end{aligned}$$



27. $x^2 + 2x + 2 = (x + 1)^2 + 1$. Let $u = x + 1$, $du = dx$. Then

$$\begin{aligned}\int \frac{dx}{(x^2 + 2x + 2)^2} &= \int \frac{du}{(u^2 + 1)^2} = \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \quad \left[\begin{array}{l} \text{where } u = \tan \theta, du = \sec^2 \theta d\theta, \\ \text{and } u^2 + 1 = \sec^2 \theta \end{array} \right] \\ &= \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta = \frac{1}{2}(\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{2} \left[\tan^{-1} u + \frac{u}{1 + u^2} \right] + C = \frac{1}{2} \left[\tan^{-1}(x + 1) + \frac{x + 1}{x^2 + 2x + 2} \right] + C\end{aligned}$$

28. $5 - 4x - x^2 = -(x^2 + 4x + 4) + 9 = 9 - (x + 2)^2$. Let $u = x + 2 \Rightarrow du = dx$. Then

$$\begin{aligned}\int \frac{dx}{(5 - 4x - x^2)^{5/2}} &= \int \frac{du}{(9 - u^2)^{5/2}} = \int \frac{3 \cos \theta d\theta}{(3 \cos \theta)^5} \quad \left[\begin{array}{l} \text{where } u = 3 \sin \theta, du = 3 \cos \theta d\theta, \\ \text{and } \sqrt{9 - u^2} = 3 \cos \theta \end{array} \right] \\ &= \frac{1}{81} \int \sec^4 \theta d\theta = \frac{1}{81} \int (\tan^2 \theta + 1) \sec^2 \theta d\theta = \frac{1}{81} \left[\frac{1}{3} \tan^3 \theta + \tan \theta \right] + C \\ &= \frac{1}{243} \left[\frac{u^3}{(9 - u^2)^{3/2}} + \frac{3u}{\sqrt{9 - u^2}} \right] + C = \frac{1}{243} \left[\frac{(x + 2)^3}{(5 - 4x - x^2)^{3/2}} + \frac{3(x + 2)}{\sqrt{5 - 4x - x^2}} \right] + C\end{aligned}$$

29. Let $u = x^2$, $du = 2x dx$. Then

$$\begin{aligned}\int x \sqrt{1 - x^4} dx &= \int \sqrt{1 - u^2} \left(\frac{1}{2} du \right) = \frac{1}{2} \int \cos \theta \cdot \cos \theta d\theta \quad \left[\begin{array}{l} \text{where } u = \sin \theta, du = \cos \theta d\theta, \\ \text{and } \sqrt{1 - u^2} = \cos \theta \end{array} \right] \\ &= \frac{1}{2} \int \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{4} \theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{4} \theta + \frac{1}{4} \sin \theta \cos \theta + C \\ &= \frac{1}{4} \sin^{-1} u + \frac{1}{4} u \sqrt{1 - u^2} + C = \frac{1}{4} \sin^{-1}(x^2) + \frac{1}{4} x^2 \sqrt{1 - x^4} + C\end{aligned}$$

30. Let $u = \sin t$, $du = \cos t dt$. Then

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos t}{\sqrt{1 + \sin^2 t}} dt &= \int_0^1 \frac{1}{\sqrt{1 + u^2}} du = \int_0^{\pi/4} \frac{1}{\sec \theta} \sec^2 \theta d\theta \quad \left[\begin{array}{l} \text{where } u = \tan \theta, du = \sec^2 \theta d\theta, \\ \text{and } \sqrt{1 + u^2} = \sec \theta \end{array} \right] \\ &= \int_0^{\pi/4} \sec \theta d\theta = \left[\ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} \quad [\text{by (1) in Section 8.2}] \\ &= \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1)\end{aligned}$$

31. (a) Let $x = a \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2 + a^2} = a \sec \theta$ and

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{a \sec^2 \theta d\theta}{a \sec \theta} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C_1 = \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C_1 \\ &= \ln \left(x + \sqrt{x^2 + a^2} \right) + C \quad \text{where } C = C_1 - \ln |a|\end{aligned}$$

(b) Let $x = a \sinh t$, so that $dx = a \cosh t dt$ and $\sqrt{x^2 + a^2} = a \cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \cosh t dt}{a \cosh t} = t + C = \sinh^{-1} \frac{x}{a} + C.$$

32. (a) Let $x = a \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\begin{aligned}I &= \int \frac{x^2}{(x^2 + a^2)^{3/2}} dx = \int \frac{a^2 \tan^2 \theta}{a^3 \sec^3 \theta} a \sec^2 \theta d\theta = \int \frac{\tan^2 \theta}{\sec \theta} d\theta = \int \frac{\sec^2 \theta - 1}{\sec \theta} d\theta \\ &= \int (\sec \theta - \cos \theta) d\theta = \ln |\sec \theta + \tan \theta| - \sin \theta + C \\ &= \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| - \frac{x}{\sqrt{x^2 + a^2}} + C = \ln \left(x + \sqrt{x^2 + a^2} \right) - \frac{x}{\sqrt{x^2 + a^2}} + C_1\end{aligned}$$

(b) Let $x = a \sinh t$. Then

$$\begin{aligned} I &= \int \frac{a^2 \sinh^2 t}{a^3 \cosh^3 t} a \cosh t dt = \int \tanh^2 t dt = \int (1 - \operatorname{sech}^2 t) dt = t - \tanh t + C \\ &= \sinh^{-1} \frac{x}{a} - \frac{x}{\sqrt{a^2 + x^2}} + C \end{aligned}$$

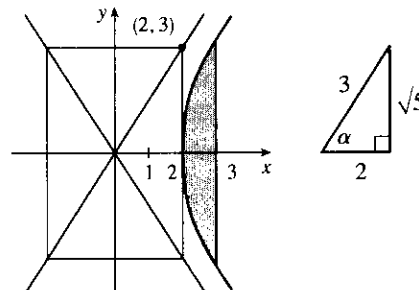
33. The average value of $f(x) = \sqrt{x^2 - 1}/x$ on the interval $[1, 7]$ is

$$\begin{aligned} \frac{1}{7-1} \int_1^7 \frac{\sqrt{x^2 - 1}}{x} dx &= \frac{1}{6} \int_0^\alpha \frac{\tan \theta}{\sec \theta} \cdot \sec \theta \tan \theta d\theta \quad \left[\begin{array}{l} \text{where } x = \sec \theta, dx = \sec \theta \tan \theta d\theta, \\ \sqrt{x^2 - 1} = \tan \theta, \text{ and } \alpha = \sec^{-1} 7 \end{array} \right] \\ &= \frac{1}{6} \int_0^\alpha \tan^2 \theta d\theta = \frac{1}{6} \int_0^\alpha (\sec^2 \theta - 1) d\theta \\ &= \frac{1}{6} [\tan \theta - \theta]_0^\alpha = \frac{1}{6} (\tan \alpha - \alpha) \\ &= \frac{1}{6} (\sqrt{48} - \sec^{-1} 7) \end{aligned}$$

34. $9x^2 - 4y^2 = 36 \Rightarrow y = \pm \frac{3}{2} \sqrt{x^2 - 4} \Rightarrow$

$$\text{area} = 2 \int_2^3 \frac{3}{2} \sqrt{x^2 - 4} dx = 3 \int_2^3 \sqrt{x^2 - 4} dx$$

$$\begin{aligned} &= 3 \int_0^\alpha 2 \tan \theta \sec \theta \tan \theta d\theta \quad \left[\begin{array}{l} \text{where } x = 2 \sec \theta, \\ dx = 2 \sec \theta \tan \theta d\theta, \\ \alpha = \sec^{-1} \frac{3}{2} \end{array} \right] \\ &= 12 \int_0^\alpha (\sec^2 \theta - 1) \sec \theta d\theta = 12 \int_0^\alpha (\sec^3 \theta - \sec \theta) d\theta \\ &= 12 \left[\frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) - \ln |\sec \theta + \tan \theta| \right]_0^\alpha \\ &= 6 \left[\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta| \right]_0^\alpha \\ &= 6 \left[\frac{3\sqrt{5}}{4} - \ln \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right) \right] = \frac{9\sqrt{5}}{2} - 6 \ln \left(\frac{3+\sqrt{5}}{2} \right) \end{aligned}$$



35. Area of $\triangle POQ = \frac{1}{2}(r \cos \theta)(r \sin \theta) = \frac{1}{2}r^2 \sin \theta \cos \theta$. Area of region $PQR = \int_{r \cos \theta}^r \sqrt{r^2 - x^2} dx$.

Let $x = r \cos u \Rightarrow dx = -r \sin u du$ for $\theta \leq u \leq \frac{\pi}{2}$. Then we obtain

$$\begin{aligned} \int \sqrt{r^2 - x^2} dx &= \int r \sin u (-r \sin u) du = -r^2 \int \sin^2 u du = -\frac{1}{2}r^2(u - \sin u \cos u) + C \\ &= -\frac{1}{2}r^2 \cos^{-1}(x/r) + \frac{1}{2}x \sqrt{r^2 - x^2} + C \end{aligned}$$

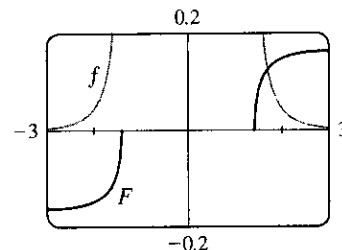
so

$$\begin{aligned} \text{area of region } PQR &= \frac{1}{2} \left[-r^2 \cos^{-1}(x/r) + x \sqrt{r^2 - x^2} \right]_{r \cos \theta}^r \\ &= \frac{1}{2} [0 - (-r^2 \theta + r \cos \theta r \sin \theta)] \\ &= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta \cos \theta \end{aligned}$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) = $\frac{1}{2}r^2\theta$.

36. Let $x = \sqrt{2} \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2} \sec \theta \tan \theta d\theta$. Then

$$\begin{aligned} \int \frac{dx}{x^4 \sqrt{x^2 - 2}} &= \int \frac{\sqrt{2} \sec \theta \tan \theta d\theta}{4 \sec^4 \theta \sqrt{2} \tan \theta} \\ &= \frac{1}{4} \int \cos^3 \theta d\theta = \frac{1}{4} \int (1 - \sin^2 \theta) \cos \theta d\theta \\ &= \frac{1}{4} \left[\sin \theta - \frac{1}{3} \sin^3 \theta \right] + C \quad [\text{substitute } u = \sin \theta] \\ &= \frac{1}{4} \left[\frac{\sqrt{x^2 - 2}}{x} - \frac{(x^2 - 2)^{3/2}}{3x^3} \right] + C \end{aligned}$$



From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

37. From the graph, it appears that the curve $y = x^2 \sqrt{4 - x^2}$ and the line $y = 2 - x$ intersect at about $x = 0.81$ and $x = 2$, with $x^2 \sqrt{4 - x^2} > 2 - x$ on $(0.81, 2)$. So the area bounded by the curve and the line is $A \approx$

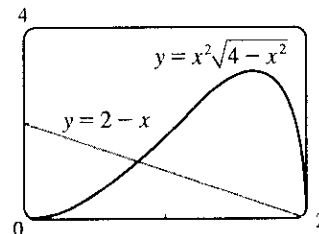
$$\int_{0.81}^2 [x^2 \sqrt{4 - x^2} - (2 - x)] dx = \int_{0.81}^2 x^2 \sqrt{4 - x^2} dx - \left[2x - \frac{1}{2}x^2 \right]_{0.81}^2.$$

To evaluate the integral, we put $x = 2 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then

$$dx = 2 \cos \theta d\theta, \quad x = 2 \Rightarrow \theta = \sin^{-1} 1 = \frac{\pi}{2}, \quad \text{and } x = 0.81 \Rightarrow \theta = \sin^{-1} 0.405 \approx 0.417. \text{ So}$$

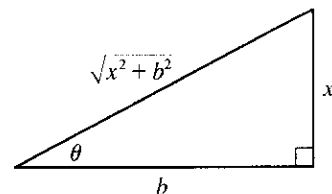
$$\begin{aligned} \int_{0.81}^2 x^2 \sqrt{4 - x^2} dx &\approx \int_{0.417}^{\pi/2} 4 \sin^2 \theta (2 \cos \theta) (2 \cos \theta d\theta) = 4 \int_{0.417}^{\pi/2} \sin^2 2\theta d\theta = 4 \int_{0.417}^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) d\theta \\ &= 2 \left[\theta - \frac{1}{4} \sin 4\theta \right]_{0.417}^{\pi/2} = 2 \left[\left(\frac{\pi}{2} - 0 \right) - \left(0.417 - \frac{1}{4} (0.995) \right) \right] \approx 2.81 \end{aligned}$$

$$\text{Thus, } A \approx 2.81 - \left[\left(2 \cdot 2 - \frac{1}{2} \cdot 2^2 \right) - \left(2 \cdot 0.81 - \frac{1}{2} \cdot 0.81^2 \right) \right] \approx 2.10.$$



38. Let $x = b \tan \theta$, so that $dx = b \sec^2 \theta d\theta$ and $\sqrt{x^2 + b^2} = b \sec \theta$.

$$\begin{aligned} E(P) &= \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0 (x^2 + b^2)^{3/2}} dx = \frac{\lambda b}{4\pi\epsilon_0} \int_{\theta_1}^{\theta_2} \frac{1}{(b \sec \theta)^3} b \sec^2 \theta d\theta \\ &= \frac{\lambda}{4\pi\epsilon_0 b} \int_{\theta_1}^{\theta_2} \frac{1}{\sec \theta} d\theta = \frac{\lambda}{4\pi\epsilon_0 b} \int_{\theta_1}^{\theta_2} \cos \theta d\theta = \frac{\lambda}{4\pi\epsilon_0 b} \left[\sin \theta \right]_{\theta_1}^{\theta_2} \\ &= \frac{\lambda}{4\pi\epsilon_0 b} \left[\frac{x}{\sqrt{x^2 + b^2}} \right]_{-a}^{L-a} = \frac{\lambda}{4\pi\epsilon_0 b} \left(\frac{L-a}{\sqrt{(L-a)^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \right) \end{aligned}$$



39. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y - b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is

$$\begin{aligned} A &= \int_{-r}^r [(b + \sqrt{r^2 - x^2}) - \sqrt{R^2 - x^2}] dx = 2 \int_0^r (b + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2}) dx \\ &= 2 \int_0^r b dx + 2 \int_0^r \sqrt{r^2 - x^2} dx - 2 \int_0^r \sqrt{R^2 - x^2} dx \end{aligned}$$

The first integral is just $2br = 2r \sqrt{R^2 - r^2}$. To evaluate the other two integrals, note that

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \int a^2 \cos^2 \theta d\theta \quad [x = a \sin \theta, dx = a \cos \theta d\theta] = \frac{1}{2} a^2 \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} a^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{2} a^2 (\theta + \sin \theta \cos \theta) + C \\ &= \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2} \left(\frac{x}{a}\right) \frac{\sqrt{a^2 - x^2}}{a} + C = \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + \frac{x}{2} \sqrt{a^2 - x^2} + C \end{aligned}$$

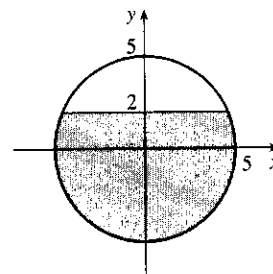
so the desired area is

$$\begin{aligned} A &= 2r \sqrt{R^2 - r^2} + \left[r^2 \arcsin(x/r) + x \sqrt{r^2 - x^2} \right]_0^r - \left[R^2 \arcsin(x/R) + x \sqrt{R^2 - x^2} \right]_0^r \\ &= 2r \sqrt{R^2 - r^2} + r^2 \left(\frac{\pi}{2} \right) - \left[R^2 \arcsin(r/R) + r \sqrt{R^2 - r^2} \right] = r \sqrt{R^2 - r^2} + \frac{\pi}{2} r^2 - R^2 \arcsin(r/R) \end{aligned}$$

40. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is

$$\begin{aligned} A &= 2 \int_{-5}^2 \sqrt{25 - y^2} dy \\ &= \left[25 \arcsin(y/5) + y \sqrt{25 - y^2} \right]_{-5}^2 \quad [\text{substitute } y = 5 \sin \theta] \\ &= 25 \arcsin \frac{2}{5} + 2 \sqrt{21} + \frac{25}{2} \pi \approx 58.72 \text{ ft}^2 \end{aligned}$$

so the fraction of the total capacity in use is $\frac{A}{\pi(5)^2} \approx \frac{58.72}{25\pi} \approx 0.748$ or 74.8%.



41. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y - R)^2 \Rightarrow x = \pm \sqrt{r^2 - (y - R)^2}$, so $g(y) = 2 \sqrt{r^2 - (y - R)^2}$ and

$$\begin{aligned} V &= \int_{R-r}^{R+r} 2\pi y \cdot 2 \sqrt{r^2 - (y - R)^2} dy = \int_{-r}^r 4\pi(u + R) \sqrt{r^2 - u^2} du \quad [\text{where } u = y - R] \\ &= 4\pi \int_{-r}^r u \sqrt{r^2 - u^2} du + 4\pi R \int_{-r}^r \sqrt{r^2 - u^2} du \quad \left[\begin{array}{l} \text{where } u = r \sin \theta, du = r \cos \theta d\theta \\ \text{in the second integral} \end{array} \right] \\ &= 4\pi \left[-\frac{1}{3} (r^2 - u^2)^{3/2} \right]_{-r}^r + 4\pi R \int_{-\pi/2}^{\pi/2} r^2 \cos^2 \theta d\theta = -\frac{4\pi}{3} (0 - 0) + 4\pi R r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta \\ &= 2\pi R r^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta = 2\pi R r^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = 2\pi^2 R r^2 \end{aligned}$$

Another method: Use washers instead of shells, so $V = 8\pi R \int_0^r \sqrt{r^2 - y^2} dy$ as in Exercise 6.2.61(a), but evaluate the integral using $y = r \sin \theta$.

8.4 Integration of Rational Functions by Partial Fractions

1. (a) $\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{3x+1}$
- (b) $\frac{1}{x^3 + 2x^2 + x} = \frac{1}{x(x^2 + 2x + 1)} = \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$
2. (a) $\frac{x-1}{x^3 + x^2} = \frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$
- (b) $\frac{x-1}{x^3 + x} = \frac{x-1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx+C}{x^2 + 1}$
3. (a) $\frac{2}{x^2 + 3x - 4} = \frac{2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$
- (b) $x^2 + x + 1$ is irreducible, so $\frac{x^2}{(x-1)(x^2 + x + 1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2 + x + 1}$.

$$4. (a) \frac{x^3}{x^2 + 4x + 3} = x - 4 + \frac{13x + 12}{x^2 + 4x + 3} = x - 4 + \frac{13x + 12}{(x+1)(x+3)} = x - 4 + \frac{A}{x+1} + \frac{B}{x+3}$$

$$(b) \frac{2x+1}{(x+1)^3(x^2+4)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

$$5. (a) \frac{x^4}{x^4-1} = \frac{(x^4-1)+1}{x^4-1} = 1 + \frac{1}{x^4-1} \quad [\text{or use long division}] = 1 + \frac{1}{(x^2-1)(x^2+1)}$$

$$= 1 + \frac{1}{(x-1)(x+1)(x^2+1)} = 1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$(b) \frac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+4} + \frac{Et+F}{(t^2+4)^2}$$

$$6. (a) \frac{x^4}{(x^3+x)(x^2-x+3)} = \frac{x^4}{x(x^2+1)(x^2-x+3)} = \frac{x^3}{(x^2+1)(x^2-x+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-x+3}$$

$$(b) \frac{1}{x^6-x^3} = \frac{1}{x^3(x^3-1)} = \frac{1}{x^3(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex+F}{x^2+x+1}$$

$$7. \int \frac{x}{x-6} dx = \int \frac{(x-6)+6}{x-6} dx = \int \left(1 + \frac{6}{x-6}\right) dx = x + 6 \ln|x-6| + C$$

$$8. \int \frac{r^2}{r+4} dr = \int \left(\frac{r^2-16}{r+4} + \frac{16}{r+4}\right) dr = \int \left(r-4 + \frac{16}{r+4}\right) dr \quad [\text{or use long division}]$$

$$= \frac{1}{2}r^2 - 4r + 16 \ln|r+4| + C$$

$$9. \frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}. \text{ Multiply both sides by } (x+5)(x-2) \text{ to get } x-9 = A(x-2) + B(x+5).$$

Substituting 2 for x gives $-7 = 7B \Leftrightarrow B = -1$. Substituting -5 for x gives $-14 = -7A \Leftrightarrow A = 2$. Thus,

$$\int \frac{x-9}{(x+5)(x-2)} dx = \int \left(\frac{2}{x+5} + \frac{-1}{x-2}\right) dx = 2 \ln|x+5| - \ln|x-2| + C$$

$$10. \frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1} \Rightarrow 1 = A(t-1) + B(t+4).$$

$$t=1 \Rightarrow 1=5B \Rightarrow B=\frac{1}{5}. \quad t=-4 \Rightarrow 1=-5A \Rightarrow A=-\frac{1}{5}. \text{ Thus,}$$

$$\int \frac{1}{(t+4)(t-1)} dt = \int \left(\frac{-1/5}{t+4} + \frac{1/5}{t-1}\right) dt = -\frac{1}{5} \ln|t+4| + \frac{1}{5} \ln|t-1| + C \quad \text{or} \quad \frac{1}{5} \ln \left| \frac{t-1}{t+4} \right| + C$$

$$11. \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}. \text{ Multiply both sides by } (x+1)(x-1) \text{ to get}$$

$$1 = A(x-1) + B(x+1). \text{ Substituting 1 for } x \text{ gives } 1 = 2B \Leftrightarrow B = \frac{1}{2}.$$

$$\text{Substituting } -1 \text{ for } x \text{ gives } 1 = -2A \Leftrightarrow A = -\frac{1}{2}. \text{ Thus,}$$

$$\int_2^3 \frac{1}{x^2-1} dx = \int_2^3 \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right) dx = \left[-\frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1|\right]_2^3$$

$$= \left(-\frac{1}{2} \ln 4 + \frac{1}{2} \ln 2\right) - \left(-\frac{1}{2} \ln 3 + \frac{1}{2} \ln 1\right) = \frac{1}{2}(\ln 2 + \ln 3 - \ln 4) \quad [\text{or } \frac{1}{2} \ln \frac{3}{2}]$$

$$12. \frac{x-1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}. \text{ Multiply both sides by } (x+1)(x+2) \text{ to get } x-1 = A(x+2) + B(x+1).$$

$$\text{Substituting } -2 \text{ for } x \text{ gives } -3 = -B \Leftrightarrow B = 3. \text{ Substituting } -1 \text{ for } x \text{ gives } -2 = A. \text{ Thus,}$$

$$\int_0^1 \frac{x-1}{x^2+3x+2} dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2}\right) dx = [-2 \ln|x+1| + 3 \ln|x+2|]_0^1$$

$$= (-2 \ln 2 + 3 \ln 3) - (-2 \ln 1 + 3 \ln 2) = 3 \ln 3 - 5 \ln 2 \quad [\text{or } \ln \frac{27}{32}]$$

$$13. \int \frac{ax}{x^2 - bx} dx = \int \frac{ax}{x(x-b)} dx = \int \frac{a}{x-b} dx = a \ln|x-b| + C$$

$$14. \text{ If } a \neq b, \frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right), \text{ so if } a \neq b, \text{ then}$$

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a} (\ln|x+a| - \ln|x+b|) + C = \frac{1}{b-a} \ln \left| \frac{x+a}{x+b} \right| + C$$

$$\text{If } a = b, \text{ then } \int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C.$$

$$15. \frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 2x+3 = A(x+1) + B. \text{ Take } x = -1 \text{ to get } B = 1, \text{ and equate}$$

coefficients of x to get $A = 2$. Now

$$\begin{aligned} \int_0^1 \frac{2x+3}{(x+1)^2} dx &= \int_0^1 \left[\frac{2}{x+1} + \frac{1}{(x+1)^2} \right] dx = \left[2 \ln(x+1) - \frac{1}{x+1} \right]_0^1 \\ &= 2 \ln 2 - \frac{1}{2} - (2 \ln 1 - 1) = 2 \ln 2 + \frac{1}{2} \end{aligned}$$

$$16. \frac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \frac{3x-4}{(x-3)(x+2)}. \text{ Write } \frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}. \text{ Then}$$

$$3x-4 = A(x+2) + B(x-3). \text{ Taking } x = 3 \text{ and } x = -2, \text{ we get } 5 = 5A \Leftrightarrow A = 1 \text{ and } -10 = -5B \Leftrightarrow B = 2, \text{ so}$$

$$\begin{aligned} \int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6} dx &= \int_0^1 \left(x + 1 + \frac{1}{x-3} + \frac{2}{x+2} \right) dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2 \ln|x+2| \right]_0^1 \\ &= \left(\frac{1}{2} + 1 + \ln 2 + 2 \ln 3 \right) - (0 + 0 + \ln 3 + 2 \ln 2) = \frac{3}{2} + \ln 3 - \ln 2 = \frac{3}{2} + \ln \frac{3}{2} \end{aligned}$$

$$17. \frac{4y^2 - 7y - 12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3} \Rightarrow 4y^2 - 7y - 12 = A(y+2)(y-3) + By(y-3) + Cy(y+2).$$

Setting $y = 0$ gives $-12 = -6A$, so $A = 2$. Setting $y = -2$ gives $18 = 10B$, so $B = \frac{9}{5}$. Setting $y = 3$ gives

$$3 = 15C, \text{ so } C = \frac{1}{5}. \text{ Now}$$

$$\begin{aligned} \int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)} dy &= \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3} \right) dy = \left[2 \ln|y| + \frac{9}{5} \ln|y+2| + \frac{1}{5} \ln|y-3| \right]_1^2 \\ &= 2 \ln 2 + \frac{9}{5} \ln 4 + \frac{1}{5} \ln 1 - 2 \ln 1 - \frac{9}{5} \ln 3 - \frac{1}{5} \ln 2 \\ &= 2 \ln 2 + \frac{18}{5} \ln 2 - \frac{1}{5} \ln 2 - \frac{9}{5} \ln 3 = \frac{27}{5} \ln 2 - \frac{9}{5} \ln 3 = \frac{9}{5} (3 \ln 2 - \ln 3) = \frac{9}{5} \ln \frac{8}{3} \end{aligned}$$

$$18. \frac{x^2 + 2x - 1}{x^3 - x} = \frac{x^2 + 2x - 1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}. \text{ Multiply both sides by } x(x+1)(x-1) \text{ to get}$$

$$x^2 + 2x - 1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1). \text{ Substituting } 0 \text{ for } x \text{ gives } -1 = -A \Leftrightarrow A = 1.$$

$$\text{Substituting } -1 \text{ for } x \text{ gives } -2 = 2B \Leftrightarrow B = -1. \text{ Substituting } 1 \text{ for } x \text{ gives } 2 = 2C \Leftrightarrow C = 1. \text{ Thus,}$$

$$\int \frac{x^2 + 2x - 1}{x^3 - x} dx = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1} \right) dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln \left| \frac{x(x-1)}{x+1} \right| + C.$$

19. $\frac{1}{(x+5)^2(x-1)} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-1} \Rightarrow 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$. Setting $x = -5$ gives $1 = -6B$, so $B = -\frac{1}{6}$. Setting $x = 1$ gives $1 = 36C$, so $C = \frac{1}{36}$. Setting $x = -2$ gives $1 = A(3)(-3) + B(-3) + C(3^2) = -9A - 3B + 9C = -9A + \frac{1}{2} + \frac{1}{4} = -9A + \frac{3}{4}$, so $9A = -\frac{1}{4}$ and $A = -\frac{1}{36}$. Now

$$\begin{aligned}\int \frac{1}{(x+5)^2(x-1)} dx &= \int \left[\frac{-1/36}{x+5} - \frac{1/6}{(x+5)^2} + \frac{1/36}{x-1} \right] dx \\ &= -\frac{1}{36} \ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36} \ln|x-1| + C\end{aligned}$$

20. $\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \Rightarrow x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$. Setting $x = 3$ gives $A = \frac{9}{25}$. Take $x = -2$ to get $C = -\frac{4}{5}$, and equate the coefficients of x^2 to get $1 = A + B \Rightarrow B = \frac{16}{25}$. Then

$$\begin{aligned}\int \frac{x^2}{(x-3)(x+2)^2} dx &= \int \left[\frac{9/25}{x-3} + \frac{16/25}{x+2} - \frac{4/5}{(x+2)^2} \right] dx \\ &= \frac{9}{25} \ln|x-3| + \frac{16}{25} \ln|x+2| + \frac{4}{5(x+2)} + C\end{aligned}$$

21. $\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x = -2$ to get $C = 3$, and take $x = 0$ to get $B = -1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2} dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2} \right) dx = 2 \ln|x| + \frac{1}{x} + 3 \ln|x+2| + C.$$

22. $\frac{1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} \Rightarrow 1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2$. Set $s = 0$, giving $B = 1$. Then set $s = 1$ to get $D = 1$. Equate the coefficients of s^3 to get $0 = A + C$ or $A = -C$, and finally set $s = 2$ to get $1 = 2A + 1 - 4A + 4$ or $A = 2$. Now

$$\int \frac{ds}{s^2(s-1)^2} = \int \left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2} \right] ds = 2 \ln|s| - \frac{1}{s} - 2 \ln|s-1| - \frac{1}{s-1} + C.$$

23. $\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$. Multiply by $(x+1)^3$ to get $x^2 = A(x+1)^2 + B(x+1) + C$. Setting $x = -1$ gives $C = 1$. Equating the coefficients of x^2 gives $A = 1$, and setting $x = 0$ gives $B = -2$.

$$\text{Now } \int \frac{x^2 dx}{(x+1)^3} = \int \left[\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3} \right] dx = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$$

24. $\frac{x}{x+1} = \frac{(x+1)-1}{x+1} = 1 - \frac{1}{x+1}$, so $\frac{x^3}{(x+1)^3} = \left[1 - \frac{1}{x+1} \right]^3 = 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}$. Thus,
- $$\int \frac{x^3}{(x+1)^3} dx = \int \left[1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3} \right] dx = x - 3 \ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C.$$

25. $\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get

$10 = A(x^2+9) + (Bx+C)(x-1)$ (*). Substituting 1 for x gives $10 = 10A \Leftrightarrow A = 1$. Substituting 0 for x gives $10 = 9A - C \Rightarrow C = 9(1) - 10 = -1$. The coefficients of the x^2 -terms in (*) must be equal, so $0 = A + B \Rightarrow B = -1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)} dx = \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+9} \right) dx = \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9} \right) dx$$

$$= \ln|x-1| - \frac{1}{2} \ln(x^2+9) \quad [\text{let } u = x^2+9] - \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) \quad [\text{Formula 10}] + C$$

26. $\frac{x^2-x+6}{x^3+3x} = \frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get

$x^2-x+6 = A(x^2+3) + (Bx+C)x$. Substituting 0 for x gives $6 = 3A \Leftrightarrow A = 2$. The coefficients of the x^2 -terms must be equal, so $1 = A + B \Rightarrow B = 1 - 2 = -1$. The coefficients of the x -terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x} dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3} \right) dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3} \right) dx$$

$$= 2 \ln|x| - \frac{1}{2} \ln(x^2+3) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$$

27. $\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$$x^3+x^2+2x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \Leftrightarrow$$

$$x^3+x^2+2x+1 = (Ax^3+Bx^2+2Ax+2B) + (Cx^3+Dx^2+Cx+D) \Leftrightarrow$$

$x^3+x^2+2x+1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$A+C=1 \quad (1) \qquad B+D=1 \quad (2)$$

$$2A+C=2 \quad (3) \qquad 2B+D=1 \quad (4)$$

Subtracting equation (1) from equation (3) gives us $A = 1$, so $C = 0$. Subtracting equation (2) from equation (4)

gives us $B = 0$, so $D = 1$. Thus, $I = \int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+2} \right) dx$. For $\int \frac{x}{x^2+1} dx$,

let $u = x^2+1$ so $du = 2x dx$ and then $\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln(x^2+1) + C$. For

$\int \frac{1}{x^2+2} dx$, use Formula 10 with $a = \sqrt{2}$. So $\int \frac{1}{x^2+2} dx = \int \frac{1}{x^2+(\sqrt{2})^2} dx = \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + C$.

Thus, $I = \frac{1}{2} \ln(x^2+1) + \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + C$.

28. $\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} \Rightarrow$

$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Setting $x = 1$ gives $B = -1$.

Equating the coefficients of x^3 gives $A = -C$. Equating the constant terms gives $-1 = -A - 1 + D$, so $D = A$.

and setting $x = 2$ gives $-1 = 5A - 5 - 2A + A$ or $A = 1$. We have

$$\begin{aligned}\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} dx &= \int \left[\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x-1}{x^2+1} \right] dx \\ &= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2} \ln(x^2+1) + \tan^{-1} x + C\end{aligned}$$

$$\begin{aligned}29. \int \frac{x+4}{x^2+2x+5} dx &= \int \frac{x+1}{x^2+2x+5} dx + \int \frac{3}{x^2+2x+5} dx = \frac{1}{2} \int \frac{(2x+2) dx}{x^2+2x+5} + \int \frac{3 dx}{(x+1)^2+4} \\ &= \frac{1}{2} \ln|x^2+2x+5| + 3 \int \frac{2 du}{4(u^2+1)} \quad \left[\begin{array}{l} \text{where } x+1=2u, \\ \text{and } dx=2 du \end{array} \right] \\ &= \frac{1}{2} \ln(x^2+2x+5) + \frac{3}{2} \tan^{-1} u + C = \frac{1}{2} \ln(x^2+2x+5) + \frac{3}{2} \tan^{-1} \left(\frac{x+1}{2} \right) + C\end{aligned}$$

$$\begin{aligned}30. \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} &= \frac{x^3 - 2x^2 + x + 1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \Rightarrow \\ x^3 - 2x^2 + x + 1 &= (Ax+B)(x^2+4) + (Cx+D)(x^2+1). \text{ Equating coefficients gives } A+C=1, \\ B+D &= -2, 4A+C=1, 4B+D=1 \Rightarrow A=0, C=1, B=1, D=-3. \text{ Now} \\ \int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx &= \int \frac{dx}{x^2+1} + \int \frac{x-3}{x^2+4} dx = \tan^{-1} x + \frac{1}{2} \ln(x^2+4) - \frac{3}{2} \tan^{-1}(x/2) + C.\end{aligned}$$

$$\begin{aligned}31. \frac{1}{x^3-1} &= \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow 1 = A(x^2+x+1) + (Bx+C)(x-1). \\ \text{Take } x &= 1 \text{ to get } A = \frac{1}{3}. \text{ Equating coefficients of } x^2 \text{ and then comparing the constant terms, we get } 0 = \frac{1}{3} + B, \\ 1 &= \frac{1}{3} - C, \text{ so } B = -\frac{1}{3}, C = -\frac{2}{3} \Rightarrow\end{aligned}$$

$$\begin{aligned}\int \frac{1}{x^3-1} dx &= \int \frac{\frac{1}{3}}{x-1} dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1} dx = \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{x+2}{x^2+x+1} dx \\ &= \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{x+1/2}{x^2+x+1} dx - \frac{1}{3} \int \frac{(3/2) dx}{(x+1/2)^2+3/4} \\ &= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln(x^2+x+1) - \frac{1}{2} \left(\frac{2}{\sqrt{3}} \right) \tan^{-1} \left(\frac{x+\frac{1}{2}}{\sqrt{3}/2} \right) + K \\ &= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln(x^2+x+1) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}}(2x+1) \right) + K\end{aligned}$$

$$\begin{aligned}32. \int_0^1 \frac{x}{x^2+4x+13} dx &= \int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13} dx - 2 \int_0^1 \frac{dx}{(x+2)^2+9} \\ &= \frac{1}{2} \int_{13}^{18} \frac{dy}{y} - 2 \int_{2/3}^1 \frac{3 du}{9u^2+9} \quad \left[\begin{array}{l} \text{where } y = x^2+4x+13, dy = (2x+4) dx, \\ x+2 = 3u, \text{ and } dx = 3 du \end{array} \right] \\ &= \frac{1}{2} [\ln y]_{13}^{18} - \frac{2}{3} [\tan^{-1} u]_{2/3}^1 = \frac{1}{2} \ln \frac{18}{13} - \frac{2}{3} \left(\frac{\pi}{4} - \tan^{-1} \left(\frac{2}{3} \right) \right) \\ &= \frac{1}{2} \ln \frac{18}{13} - \frac{\pi}{6} + \frac{2}{3} \tan^{-1} \left(\frac{2}{3} \right)\end{aligned}$$

33. Let $u = x^3 + 3x^2 + 4$. Then $du = 3(x^2 + 2x) dx \Rightarrow$

$$\int_2^5 \frac{x^2 + 2x}{x^3 + 3x^2 + 4} dx = \frac{1}{3} \int_{24}^{204} \frac{du}{u} = \frac{1}{3} [\ln u]_{24}^{204} = \frac{1}{3} (\ln 204 - \ln 24) = \frac{1}{3} \ln \frac{204}{24} = \frac{1}{3} \ln \frac{17}{2}.$$

34. $\frac{x^3}{x^3 + 1} = \frac{(x^3 + 1) - 1}{x^3 + 1} = 1 - \frac{1}{x^3 + 1} = 1 - \left(\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \right) \Rightarrow$

$1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$. Equate the terms of degree 2, 1 and 0 to get $0 = A + B$,

$0 = -A + B + C$, $1 = A + C$. Solve the three equations to get $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$. So

$$\begin{aligned} \int \frac{x^3}{x^3 + 1} dx &= \int \left[1 - \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{2}{3}}{x^2 - x + 1} \right] dx \\ &= x - \frac{1}{3} \ln|x+1| + \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx - \frac{1}{2} \int \frac{dx}{(x-\frac{1}{2})^2 + \frac{3}{4}} \\ &= x - \frac{1}{3} \ln|x+1| + \frac{1}{6} \ln(x^2 - x + 1) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} (2x-1) \right) + K \end{aligned}$$

35. $\frac{1}{x^4 - x^2} = \frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get

$1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$. Setting $x = 1$ gives $C = \frac{1}{2}$, taking

$x = -1$ gives $D = -\frac{1}{2}$. Equating the coefficients of x^3 gives $0 = A + C + D = A$. Finally, setting $x = 0$ yields

$$B = -1. \text{ Now } \int \frac{dx}{x^4 - x^2} = \int \left[\frac{-1}{x^2} + \frac{1/2}{x-1} - \frac{1/2}{x+1} \right] dx = \frac{1}{x} + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C.$$

36. Let $u = x^4 + 5x^2 + 4 \Rightarrow du = (4x^3 + 10x) dx = 2(2x^3 + 5x) dx$, so

$$\int_0^1 \frac{2x^3 + 5x}{x^4 + 5x^2 + 4} dx = \frac{1}{2} \int_4^{10} \frac{du}{u} = \frac{1}{2} [\ln|u|]_4^{10} = \frac{1}{2} (\ln 10 - \ln 4) = \frac{1}{2} \ln \frac{5}{2}.$$

37. $\int \frac{x-3}{(x^2+2x+4)^2} dx = \int \frac{x-3}{[(x+1)^2+3]^2} dx = \int \frac{u-4}{(u^2+3)^2} du$ [with $u = x+1$]

$$\begin{aligned} &= \int \frac{u du}{(u^2+3)^2} - 4 \int \frac{du}{(u^2+3)^2} = \frac{1}{2} \int \frac{dv}{v^2} - 4 \int \frac{\sqrt{3} \sec^2 \theta d\theta}{9 \sec^4 \theta} \quad \left[\begin{array}{l} v = u^2 + 3 \text{ in the first integral;} \\ u = \sqrt{3} \tan \theta \text{ in the second} \end{array} \right] \\ &= \frac{-1}{(2v)} - \frac{4\sqrt{3}}{9} \int \cos^2 \theta d\theta = \frac{-1}{2(u^2+3)} - \frac{2\sqrt{3}}{9} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9} \left[\tan^{-1} \left(\frac{x+1}{\sqrt{3}} \right) + \frac{\sqrt{3}(x+1)}{x^2+2x+4} \right] + C \\ &= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9} \tan^{-1} \left(\frac{x+1}{\sqrt{3}} \right) - \frac{2(x+1)}{3(x^2+2x+4)} + C \end{aligned}$$

38. $\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \Rightarrow x^4+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$.

Setting $x = 0$ gives $A = 1$, and equating the coefficients of x^4 gives $1 = A + B$, so $B = 0$. Now

$$\frac{C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \frac{x^4+1}{x(x^2+1)^2} - \frac{1}{x} = \frac{1}{x} \left[\frac{x^4+1-(x^4+2x^2+1)}{(x^2+1)^2} \right] = \frac{-2x}{(x^2+1)^2}, \text{ so we can take } C=0,$$

$$D=-2, \text{ and } E=0. \text{ Hence, } \int \frac{x^4+1}{x(x^2+1)^2} dx = \int \left[\frac{1}{x} - \frac{2x}{(x^2+1)^2} \right] dx = \ln|x| + \frac{1}{x^2+1} + C.$$

39. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$, $dx = 2u du \Rightarrow$

$$\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2u du}{(u^2-1)u} = 2 \int \frac{du}{u^2-1} = \ln \left| \frac{u-1}{u+1} \right| + C = \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C.$$

40. Let $u = \sqrt{x+2}$. Then $x = u^2 - 2$, $dx = 2u du \Rightarrow$

$$I = \int \frac{dx}{x-\sqrt{x+2}} = \int \frac{2u du}{u^2-2-u} = 2 \int \frac{u du}{u^2-u-2} \text{ and } \frac{u}{u^2-u-2} = \frac{A}{u-2} + \frac{B}{u+1} \Rightarrow$$

$u = A(u+1) + B(u-2)$. Substituting -1 for u gives $-1 = -3B \Leftrightarrow B = \frac{1}{3}$ and substituting 2 for u gives

$$2 = 3A \Leftrightarrow A = \frac{2}{3}. \text{ Thus,}$$

$$\begin{aligned} I &= \frac{2}{3} \int \left[\frac{2}{u-2} + \frac{1}{u+1} \right] du = \frac{2}{3} (2 \ln|u-2| + \ln|u+1|) + C \\ &= \frac{2}{3} [2 \ln|\sqrt{x+2}-2| + \ln(\sqrt{x+2}+1)] + C \end{aligned}$$

41. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u du$. Thus,

$$\begin{aligned} \int_9^{16} \frac{\sqrt{x}}{x-4} dx &= \int_3^4 \frac{u}{u^2-4} 2u du = 2 \int_3^4 \frac{u^2}{u^2-4} du = 2 \int_3^4 \left(1 + \frac{4}{u^2-4} \right) du \quad [\text{by long division}] \\ &= 2 + 8 \int_3^4 \frac{du}{(u+2)(u-2)}. (*) \end{aligned}$$

Multiply $\frac{1}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so

$$\frac{1}{(u+2)(u-2)} = \frac{-1/4}{u+2} + \frac{1/4}{u-2} \text{ and } (*) \text{ is}$$

$$\begin{aligned} 2 + 8 \int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2} \right) du &= 2 + 8 \left[-\frac{1}{4} \ln|u+2| + \frac{1}{4} \ln|u-2| \right]_3^4 \\ &= 2 + \left[2 \ln|u-2| - 2 \ln|u+2| \right]_3^4 = 2 + 2 \left[\ln \left| \frac{u-2}{u+2} \right| \right]_3^4 \\ &= 2 + 2 \left(\ln \frac{2}{6} - \ln \frac{1}{5} \right) = 2 + 2 \ln \frac{2/6}{1/5} \\ &= 2 + 2 \ln \frac{5}{3} \text{ or } 2 + \ln \left(\frac{5}{3} \right)^2 = 2 + \ln \frac{25}{9} \end{aligned}$$

42. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 du \Rightarrow$

$$\begin{aligned} \int_0^1 \frac{1}{1+\sqrt[3]{x}} dx &= \int_0^1 \frac{3u^2 du}{1+u} = \int_0^1 \left(3u - 3 + \frac{3}{1+u} \right) du = \left[\frac{3}{2}u^2 - 3u + 3 \ln(1+u) \right]_0^1 \\ &= 3 \left(\ln 2 - \frac{1}{2} \right) \end{aligned}$$

43. Let $u = \sqrt[3]{x^2 + 1}$. Then $x^2 = u^3 - 1$, $2x dx = 3u^2 du \Rightarrow$

$$\begin{aligned}\int \frac{x^3 dx}{\sqrt[3]{x^2 + 1}} &= \int \frac{(u^3 - 1) \frac{3}{2} u^2 du}{u} = \frac{3}{2} \int (u^4 - u) du = \frac{3}{10} u^5 - \frac{3}{4} u^2 + C \\ &= \frac{3}{10} (x^2 + 1)^{5/3} - \frac{3}{4} (x^2 + 1)^{2/3} + C\end{aligned}$$

44. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u du \Rightarrow$

$$\int_{1/3}^3 \frac{\sqrt{x}}{x^2 + x} dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u \cdot 2u du}{u^4 + u^2} = 2 \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2 + 1} = 2 [\tan^{-1} u]_{1/\sqrt{3}}^{\sqrt{3}} = 2 \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{3}$$

45. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5 du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\begin{aligned}\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} &= \int \frac{6u^5 du}{u^3 - u^2} = 6 \int \frac{u^5}{u^2(u-1)} du = 6 \int \frac{u^3}{u-1} du \\ &= 6 \int \left(u^2 + u + 1 + \frac{1}{u-1} \right) du \quad [\text{by long division}] \\ &= 6 \left(\frac{1}{3} u^3 + \frac{1}{2} u^2 + u + \ln |u-1| \right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6 \ln |\sqrt[6]{x} - 1| + C\end{aligned}$$

46. Let $u = \sqrt[12]{x}$. Then $x = u^{12}$, $dx = 12u^{11} du \Rightarrow$

$$\begin{aligned}\int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}} &= \int \frac{12u^{11} du}{u^4 + u^3} = 12 \int \frac{u^8 du}{u+1} = 12 \int \left(u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - 1 + \frac{1}{u+1} \right) du \\ &= \frac{3}{2} u^8 - \frac{12}{7} u^7 + 2u^6 - \frac{12}{5} u^5 + 3u^4 - 4u^3 + 6u^2 - 12u + 12 \ln |u+1| + C \\ &= \frac{3}{2} x^{2/3} - \frac{12}{7} x^{7/12} + 2\sqrt{x} - \frac{12}{5} x^{5/12} + 3\sqrt[3]{x} - 4\sqrt[4]{x} + 6\sqrt[6]{x} - 12\sqrt[12]{x} + 12 \ln (\sqrt[12]{x} + 1) + C\end{aligned}$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \frac{du}{u} \Rightarrow$

$$\begin{aligned}\int \frac{e^{2x} dx}{e^{2x} + 3e^x + 2} &= \int \frac{u^2 (du/u)}{u^2 + 3u + 2} = \int \frac{u du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2} \right] du \\ &= 2 \ln |u+2| - \ln |u+1| + C = \ln [(e^x + 2)^2 / (e^x + 1)] + C\end{aligned}$$

48. Let $u = \sin x$. Then $du = \cos x dx \Rightarrow$

$$\begin{aligned}\int \frac{\cos x dx}{\sin^2 x + \sin x} &= \int \frac{du}{u^2 + u} = \int \frac{du}{u(u+1)} = \int \left[\frac{1}{u} - \frac{1}{u+1} \right] du \\ &= \ln \left| \frac{u}{u+1} \right| + C = \ln \left| \frac{\sin x}{1 + \sin x} \right| + C\end{aligned}$$

49. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \frac{2x-1}{x^2-x+2} dx$, $v = x$, and (by integration by parts)

$$\begin{aligned} \int \ln(x^2 - x + 2) dx &= x \ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2} dx = x \ln(x^2 - x + 2) - \int \left(2 + \frac{x-4}{x^2-x+2} \right) dx \\ &= x \ln(x^2 - x + 2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2-x+2} dx + \frac{7}{2} \int \frac{dx}{(x-\frac{1}{2})^2 + \frac{7}{4}} \\ &= x \ln(x^2 - x + 2) - 2x - \frac{1}{2} \ln(x^2 - x + 2) + \frac{7}{2} \int \frac{\frac{\sqrt{7}}{2} du}{\frac{7}{4}(u^2 + 1)} \quad \left[\begin{array}{l} \text{where } x - \frac{1}{2} = \frac{\sqrt{7}}{2} u, \\ dx = \frac{\sqrt{7}}{2} du, \\ (x - \frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2 + 1) \end{array} \right] \\ &= (x - \frac{1}{2}) \ln(x^2 - x + 2) - 2x + \sqrt{7} \tan^{-1} u + C \\ &= (x - \frac{1}{2}) \ln(x^2 - x + 2) - 2x + \sqrt{7} \tan^{-1} \frac{2x-1}{\sqrt{7}} + C \end{aligned}$$

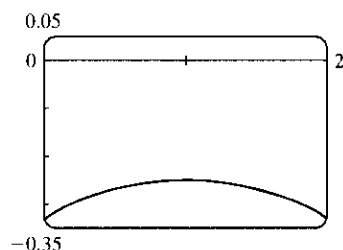
50. Let $u = \tan^{-1} x$, $dv = x dx \Rightarrow du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\int x \tan^{-1} x dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$. To evaluate the last integral, use long division or observe

$$\text{that } \int \frac{x^2}{1+x^2} dx = \int \frac{(1+x^2) - 1}{1+x^2} dx = \int 1 dx - \int \frac{1}{1+x^2} dx = x - \tan^{-1} x + C_1. \text{ So}$$

$$\int x \tan^{-1} x dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x + C_1) = \frac{1}{2}(x^2 \tan^{-1} x + \tan^{-1} x - x) + C.$$

51.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2 \cdot 0.3) = -0.6$. Now

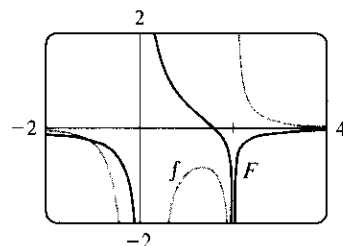
$$\begin{aligned} \frac{1}{x^2 - 2x - 3} &= \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \Leftrightarrow \\ 1 &= (A+B)x + A - 3B, \text{ so } A = -B \text{ and } A - 3B = 1 \Leftrightarrow A = \frac{1}{4} \\ \text{and } B &= -\frac{1}{4}, \text{ so the integral becomes} \end{aligned}$$

$$\begin{aligned} \int_0^2 \frac{dx}{x^2 - 2x - 3} &= \frac{1}{4} \int_0^2 \frac{dx}{x-3} - \frac{1}{4} \int_0^2 \frac{dx}{x+1} = \frac{1}{4} [\ln|x-3| - \ln|x+1|]_0^2 \\ &= \frac{1}{4} \left[\ln \left| \frac{x-3}{x+1} \right| \right]_0^2 = \frac{1}{4} (\ln \frac{1}{3} - \ln 3) = -\frac{1}{2} \ln 3 \approx -0.55 \end{aligned}$$

52. $\frac{1}{x^3 - 2x^2} = \frac{1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \Rightarrow 1 = (A+C)x^2 + (B-2A)x - 2B$, so $A+C = B-2A = 0$

and $-2B = 1 \Rightarrow B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $\frac{1}{x^3 - 2x^2}$ is

$$\begin{aligned} \int \frac{dx}{x^3 - 2x^2} &= -\frac{1}{4} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x^2} + \frac{1}{4} \int \frac{dx}{x-2} \\ &= -\frac{1}{4} \ln|x| - \frac{1}{2}(-1/x) + \frac{1}{4} \ln|x-2| + C \\ &= \frac{1}{4} \ln \left| \frac{x-2}{x} \right| + \frac{1}{2x} + C \end{aligned}$$



We plot this function with $C = 0$ on the same screen as $y = \frac{1}{x^3 - 2x^2}$.

$$\begin{aligned} 53. \int \frac{dx}{x^2 - 2x} &= \int \frac{dx}{(x-1)^2 - 1} = \int \frac{du}{u^2 - 1} \quad [\text{put } u = x - 1] \\ &= \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C \quad [\text{by Equation 6}] = \frac{1}{2} \ln \left| \frac{x-2}{x} \right| + C \end{aligned}$$

$$\begin{aligned} 54. \int \frac{(2x+1)dx}{4x^2+12x-7} &= \frac{1}{4} \int \frac{(8x+12)dx}{4x^2+12x-7} - \int \frac{2dx}{(2x+3)^2-16} \\ &= \frac{1}{4} \ln |4x^2+12x-7| - \int \frac{du}{u^2-16} \quad [\text{put } u = 2x+3] \\ &= \frac{1}{4} \ln |4x^2+12x-7| - \frac{1}{8} \ln |(u-4)/(u+4)| + C \quad [\text{by Equation 6}] \\ &= \frac{1}{4} \ln |4x^2+12x-7| - \frac{1}{8} \ln |(2x-1)/(2x+7)| + C \end{aligned}$$

55. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1} t$. The figure gives

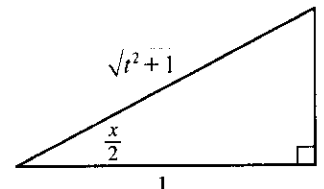
$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$

$$(b) \cos x = \cos\left(2 \cdot \frac{x}{2}\right) = 2 \cos^2\left(\frac{x}{2}\right) - 1$$

$$= 2 \left(\frac{1}{\sqrt{1+t^2}} \right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\sin x = \sin\left(2 \cdot \frac{x}{2}\right) = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = 2 \frac{t}{\sqrt{1+t^2}} \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

$$(c) \frac{x}{2} = \arctan t \Rightarrow x = 2 \arctan t \Rightarrow dx = \frac{2}{1+t^2} dt$$



56. Let $t = \tan(x/2)$. Then, using Exercise 55, $dx = \frac{2}{1+t^2} dt$, $\sin x = \frac{2t}{1+t^2} \Rightarrow$

$$\begin{aligned} \int \frac{dx}{3-5\sin x} &= \int \frac{2 dt/(1+t^2)}{3-10t/(1+t^2)} = \int \frac{2 dt}{3(1+t^2)-10t} = 2 \int \frac{dt}{3t^2-10t+3} \\ &= \frac{1}{4} \int \left[\frac{1}{t-3} - \frac{3}{3t-1} \right] dt = \frac{1}{4} (\ln |t-3| - \ln |3t-1|) + C = \frac{1}{4} \ln \left| \frac{\tan(x/2)-3}{3\tan(x/2)-1} \right| + C \end{aligned}$$

57. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 55, we have

$$\begin{aligned} \int \frac{1}{3\sin x - 4\cos x} dx &= \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)} \frac{2 dt}{1+t^2} = 2 \int \frac{dt}{3(2t) - 4(1-t^2)} = \int \frac{dt}{2t^2 + 3t - 2} \\ &= \int \frac{dt}{(2t-1)(t+2)} = \int \left[\frac{2}{5} \frac{1}{2t-1} - \frac{1}{5} \frac{1}{t+2} \right] dt \quad [\text{using partial fractions}] \\ &= \frac{1}{5} [\ln |2t-1| - \ln |t+2|] + C = \frac{1}{5} \ln \left| \frac{2t-1}{t+2} \right| + C = \frac{1}{5} \ln \left| \frac{2\tan(x/2)-1}{\tan(x/2)+2} \right| + C \end{aligned}$$

58. Let $t = \tan(x/2)$. Then, by Exercise 55,

$$\begin{aligned} \int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x} &= \int_{1/\sqrt{3}}^1 \frac{2dt/(1+t^2)}{1+2t/(1+t^2) - (1-t^2)/(1+t^2)} = \int_{1/\sqrt{3}}^1 \frac{2dt}{1+t^2+2t-1+t^2} \\ &= \int_{1/\sqrt{3}}^1 \left[\frac{1}{t} - \frac{1}{t+1} \right] dt = [\ln t - \ln(t+1)]_{1/\sqrt{3}}^1 = \ln \frac{1}{2} - \ln \frac{1}{\sqrt{3}+1} = \ln \frac{\sqrt{3}+1}{2} \end{aligned}$$

59. Let $t = \tan(x/2)$. Then, by Exercise 55,

$$\begin{aligned}\int \frac{dx}{2 \sin x + \sin 2x} &= \frac{1}{2} \int \frac{dx}{\sin x + \sin x \cos x} = \frac{1}{2} \int \frac{2 dt/(1+t^2)}{2t/(1+t^2) + 2t(1-t^2)/(1+t^2)^2} \\ &= \frac{1}{2} \int \frac{(1+t^2) dt}{t(1+t^2) + t(1-t^2)} = \frac{1}{4} \int \frac{(1+t^2) dt}{t} = \frac{1}{4} \int \left(\frac{1}{t} + t \right) dt \\ &= \frac{1}{4} \ln |t| + \frac{1}{8} t^2 + C = \frac{1}{4} \ln \left| \tan\left(\frac{1}{2}x\right) \right| + \frac{1}{8} \tan^2\left(\frac{1}{2}x\right) + C\end{aligned}$$

60. $x^2 - 6x + 8 = (x-3)^2 - 1$ is positive for $5 \leq x \leq 10$, so

$$\begin{aligned}\text{area} &= \int_5^{10} \frac{dx}{(x-3)^2 - 1} = \int_2^7 \frac{du}{u^2 - 1} \quad [\text{put } u = x-3] = \left[\frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| \right]_2^7 \\ &= \frac{1}{2} \ln \frac{3}{4} - \frac{1}{2} \ln \frac{1}{3} = \frac{1}{2} (\ln 3 - 2 \ln 2 + \ln 3) = \ln 3 - \ln 2 = \ln \frac{3}{2}\end{aligned}$$

61. $\frac{x+1}{x-1} = 1 + \frac{2}{x-1} > 0$ for $2 \leq x \leq 3$, so

$$\text{area} = \int_2^3 \left[1 + \frac{2}{x-1} \right] dx = \left[x + 2 \ln |x-1| \right]_2^3 = (3 + 2 \ln 2) - (2 + 2 \ln 1) = 1 + 2 \ln 2.$$

62. (a) We use disks, so the volume is $V = \pi \int_0^1 \left[\frac{1}{x^2 + 3x + 2} \right]^2 dx = \pi \int_0^1 \frac{dx}{(x+1)^2(x+2)^2}$. To evaluate the

integral, we use partial fractions: $\frac{1}{(x+1)^2(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} \Rightarrow$

$1 = A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2$. We set $x = -1$, giving $B = 1$, then set $x = -2$, giving $D = 1$. Now equating coefficients of x^3 gives $A = -C$, and then equating constants gives $1 = 4A + 4 + 2(-A) + 1 \Rightarrow A = -2 \Rightarrow C = 2$. So the expression becomes

$$\begin{aligned}V &= \pi \int_0^1 \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{x+2} + \frac{1}{(x+2)^2} \right] dx = \pi \left[2 \ln \left| \frac{x+2}{x+1} \right| - \frac{1}{x+1} - \frac{1}{x+2} \right]_0^1 \\ &= \pi \left[\left(2 \ln \frac{3}{2} - \frac{1}{2} - \frac{1}{3} \right) - \left(2 \ln 2 - 1 - \frac{1}{2} \right) \right] = \pi \left(2 \ln \frac{3/2}{2} + \frac{2}{3} \right) = \pi \left(\frac{2}{3} + \ln \frac{9}{16} \right)\end{aligned}$$

(b) In this case, we use cylindrical shells, so the volume is $V = 2\pi \int_0^1 \frac{x dx}{x^2 + 3x + 2} = 2\pi \int_0^1 \frac{x dx}{(x+1)(x+2)}$.

We use partial fractions to simplify the integrand: $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \Rightarrow$

$x = (A+B)x + 2A+B$. So $A+B=1$ and $2A+B=0 \Rightarrow A=-1$ and $B=2$. So the volume is

$$\begin{aligned}2\pi \int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2} \right] dx &= 2\pi \left[-\ln |x+1| + 2 \ln |x+2| \right]_0^1 \\ &= 2\pi (-\ln 2 + 2 \ln 3 + \ln 1 - 2 \ln 2) = 2\pi (2 \ln 3 - 3 \ln 2) = 2\pi \ln \frac{9}{8}\end{aligned}$$

63. $\frac{P+S}{P[(r-1)P-S]} = \frac{A}{P} + \frac{B}{(r-1)P-S} \Rightarrow P+S = A[(r-1)P-S] + BP = [(r-1)A+B]P - AS$

$\Rightarrow (r-1)A+B=1, -A=1 \Rightarrow A=-1, B=r$. Now

$$t = \int \frac{P+S}{P[(r-1)P-S]} dP = \int \left[\frac{-1}{P} + \frac{r}{(r-1)P-S} \right] dP = -\int \frac{dP}{P} + \frac{r}{r-1} \int \frac{r-1}{(r-1)P-S} dP$$

so $t = -\ln P + \frac{r}{r-1} \ln|(r-1)P - S| + C$. Here $r = 0.10$ and $S = 900$, so

$$\begin{aligned} t &= -\ln P + \frac{0.1}{-0.9} \ln|-0.9P - 900| + C = -\ln P - \frac{1}{9} \ln(|-1| |0.9P + 900|) \\ &= -\ln P - \frac{1}{9} \ln(0.9P + 900) + C \end{aligned}$$

When $t = 0$, $P = 10,000$, so $0 = -\ln 10,000 - \frac{1}{9} \ln(9900) + C$. Thus, $C = \ln 10,000 + \frac{1}{9} \ln 9900$ [≈ 10.2326], so our equation becomes

$$\begin{aligned} t &= \ln 10,000 - \ln P + \frac{1}{9} \ln 9900 - \frac{1}{9} \ln(0.9P + 900) = \ln \frac{10,000}{P} + \frac{1}{9} \ln \frac{9900}{0.9P + 900} \\ &= \ln \frac{10,000}{P} + \frac{1}{9} \ln \frac{1100}{0.1P + 100} = \ln \frac{10,000}{P} + \frac{1}{9} \ln \frac{11,000}{P + 1000} \end{aligned}$$

64. If we subtract and add $2x^2$, we get

$$\begin{aligned} x^4 + 1 &= x^4 + 2x^2 + 1 - 2x^2 = (x^2 + 1)^2 - 2x^2 = (x^2 + 1)^2 - (\sqrt{2}x)^2 \\ &= [(x^2 + 1) - \sqrt{2}x][(x^2 + 1) + \sqrt{2}x] = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1) \end{aligned}$$

So we can decompose $\frac{1}{x^4 + 1} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1} \Rightarrow$

$1 = (Ax + B)(x^2 - \sqrt{2}x + 1) + (Cx + D)(x^2 + \sqrt{2}x + 1)$. Setting the constant terms equal gives $B + D = 1$,

then from the coefficients of x^3 we get $A + C = 0$. Now from the coefficients of x we get

$A + C + (B - D)\sqrt{2} = 0 \Leftrightarrow [(1 - D) - D]\sqrt{2} = 0 \Rightarrow D = \frac{1}{2} \Rightarrow B = \frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}(C - A) + B + D = 0 \Rightarrow C - A = -\frac{1}{\sqrt{2}} \Rightarrow C = -\frac{\sqrt{2}}{4}$ and $A = \frac{\sqrt{2}}{4}$.

So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\begin{aligned} \frac{1}{x^4 + 1} &= \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2 + \sqrt{2}x + 1} + \frac{-\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2 - \sqrt{2}x + 1} = \frac{1}{4\sqrt{2}} \left[\frac{2x + 2\sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{2x - 2\sqrt{2}}{x^2 - \sqrt{2}x + 1} \right] \\ &= \frac{\sqrt{2}}{8} \left[\frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} \right] + \frac{1}{4} \left[\frac{1}{\left(x + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} + \frac{1}{\left(x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} \right] \end{aligned}$$

Now we integrate:

$$\int \frac{dx}{x^4 + 1} = \frac{\sqrt{2}}{8} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + \frac{\sqrt{2}}{4} \left[\tan^{-1}(\sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x - 1) \right] + C.$$

65. (a) In Maple, we define $f(x)$, and then use `convert(f, parfrac, x)`; to obtain

$$f(x) = \frac{24,110/4879}{5x + 2} - \frac{668/323}{2x + 1} - \frac{9438/80,155}{3x - 7} + \frac{(22,098x + 48,935)/260,015}{x^2 + x + 5}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

$$\begin{aligned}
 \text{(b)} \quad \int f(x) dx &= \frac{24,110}{4879} \cdot \frac{1}{5} \ln|5x+2| - \frac{668}{323} \cdot \frac{1}{2} \ln|2x+1| - \frac{9438}{80,155} \cdot \frac{1}{3} \ln|3x-7| \\
 &\quad + \frac{1}{260,015} \int \frac{22,098(x + \frac{1}{2}) + 37,886}{(x + \frac{1}{2})^2 + \frac{19}{4}} dx + C \\
 &= \frac{24,110}{4879} \cdot \frac{1}{5} \ln|5x+2| - \frac{668}{323} \cdot \frac{1}{2} \ln|2x+1| - \frac{9438}{80,155} \cdot \frac{1}{3} \ln|3x-7| \\
 &\quad + \frac{1}{260,015} \left[22,098 \cdot \frac{1}{2} \ln(x^2 + x + 5) + 37,886 \cdot \sqrt{\frac{4}{19}} \tan^{-1} \left(\frac{1}{\sqrt{19/4}} (x + \frac{1}{2}) \right) \right] + C \\
 &= \frac{4822}{4879} \ln|5x+2| - \frac{334}{323} \ln|2x+1| - \frac{3146}{80,155} \ln|3x-7| + \frac{11,049}{260,015} \ln(x^2 + x + 5) \\
 &\quad + \frac{75,772}{260,015\sqrt{19}} \tan^{-1} \left[\frac{1}{\sqrt{19}} (2x+1) \right] + C
 \end{aligned}$$

Using a CAS, we get

$$\begin{aligned}
 &\frac{4822 \ln(5x+2)}{4879} - \frac{334 \ln(2x+1)}{323} - \frac{3146 \ln(3x-7)}{80,155} \\
 &\quad + \frac{11,049 \ln(x^2 + x + 5)}{260,015} + \frac{3988\sqrt{19}}{260,015} \tan^{-1} \left[\frac{\sqrt{19}}{19} (2x+1) \right]
 \end{aligned}$$

The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

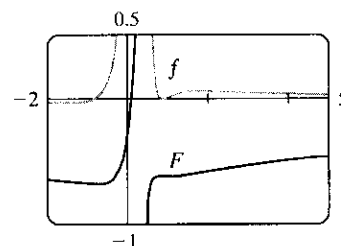
66. (a) In Maple, we define $f(x)$, and then use `convert(f, parfrac, x)`; to get

$$f(x) = \frac{5828/1815}{(5x-2)^2} - \frac{59,096/19,965}{5x-2} + \frac{2(2843x+816)/3993}{2x^2+1} + \frac{(313x-251)/363}{(2x^2+1)^2}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

- (b) As we saw in Exercise 65, computer algebra systems omit the absolute value signs in $\int (1/y) dy = \ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get

$$\begin{aligned}
 \int f(x) dx &= -\frac{5828}{9075(5x-2)} - \frac{59,096 \ln|5x-2|}{99,825} + \frac{2843 \ln(2x^2+1)}{7986} \\
 &\quad + \frac{503}{15,972} \sqrt{2} \tan^{-1}(\sqrt{2}x) - \frac{1}{2904} \frac{1004x+626}{2x^2+1} + C
 \end{aligned}$$



- (c) From the graph, we see that f goes from negative to positive at $x \approx -0.78$, then back to negative at $x \approx 0.8$, and finally back to positive at $x = 1$. Also, $\lim_{x \rightarrow 0.4} f(x) = \infty$. So we see (by the First Derivative Test) that $\int f(x) dx$ has minima at $x \approx -0.78$ and $x = 1$, and a maximum at $x \approx 0.80$, and that $\int f(x) dx$ is unbounded as $x \rightarrow 0.4$. Note also that just to the right of $x = 0.4$, f has large values, so $\int f(x) dx$ increases rapidly, but slows down as f drops toward 0. $\int f(x) dx$ decreases from about 0.8 to 1, then increases slowly since f stays small and positive.

67. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x , $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x . By continuity of F and G , the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a . Thus,

$$\begin{aligned} F(a) &= \lim_{x \rightarrow a} F(x) \quad [\text{by continuity of } F] = \lim_{x \rightarrow a} G(x) \quad [\text{whenever } Q(x) \neq 0] \\ &= G(a) \quad [\text{by continuity of } G] \end{aligned}$$

68. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\frac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\frac{f(x)}{x^2(x+1)^3} = \frac{ax^2 + bx + 1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \Rightarrow b = 3$. This is the quantity we are looking for, since $f'(0) = b$.

8.5 Strategy for Integration

- $\int \frac{\sin x + \sec x}{\tan x} dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x} \right) dx = \int (\cos x + \csc x) dx = \sin x + \ln |\csc x - \cot x| + C$
- $\begin{aligned} \int \tan^3 \theta d\theta &= \int (\sec^2 \theta - 1) \tan \theta d\theta = \int \tan \theta \sec^2 \theta d\theta - \int \frac{\sin \theta}{\cos \theta} d\theta \\ &= \int u du + \int \frac{dv}{v} \quad \left[\begin{array}{l} u = \tan \theta, \quad v = \cos \theta, \\ du = \sec^2 \theta d\theta \quad dv = -\sin \theta d\theta \end{array} \right] \\ &= \frac{1}{2} u^2 + \ln |v| + C = \frac{1}{2} \tan^2 \theta + \ln |\cos \theta| + C \end{aligned}$
- $\begin{aligned} \int_0^2 \frac{2t}{(t-3)^2} dt &= \int_{-3}^{-1} \frac{2(u+3)}{u^2} du \quad [u = t-3, du = dt] = \int_{-3}^{-1} \left(\frac{2}{u} + \frac{6}{u^2} \right) du = \left[2 \ln |u| - \frac{6}{u} \right]_{-3}^{-1} \\ &= (2 \ln 1 + 6) - (2 \ln 3 + 2) = 4 - 2 \ln 3 \text{ or } 4 - \ln 9 \end{aligned}$
- Let $u = x^2$. Then $du = 2x dx \Rightarrow \int \frac{x dx}{\sqrt{3-x^4}} = \frac{1}{2} \int \frac{du}{\sqrt{3-u^2}} = \frac{1}{2} \sin^{-1} \frac{u}{\sqrt{3}} + C = \frac{1}{2} \sin^{-1} \frac{x^2}{\sqrt{3}} + C$.
- Let $u = \arctan y$. Then $du = \frac{dy}{1+y^2} \Rightarrow \int_{-1}^1 \frac{e^{\arctan y}}{1+y^2} dy = \int_{-\pi/4}^{\pi/4} e^u du = [e^u]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.
- $\begin{aligned} \int x \csc x \cot x dx &\quad \left[\begin{array}{l} u = x, \quad dv = \csc x \cot x dx, \\ du = dx \quad v = -\csc x \end{array} \right] = -x \csc x - \int (-\csc x) dx \\ &= -x \csc x + \ln |\csc x - \cot x| + C \end{aligned}$
- $\begin{aligned} \int_1^3 r^4 \ln r dr &\quad \left[\begin{array}{l} u = \ln r, \quad dv = r^4 dr, \\ du = \frac{dr}{r} \quad v = \frac{1}{5} r^5 \end{array} \right] = \left[\frac{1}{5} r^5 \ln r \right]_1^3 - \int_1^3 \frac{1}{5} r^4 dr = \frac{243}{5} \ln 3 - 0 - \left[\frac{1}{25} r^5 \right]_1^3 \\ &= \frac{243}{5} \ln 3 - \left(\frac{243}{25} - \frac{1}{25} \right) = \frac{243}{5} \ln 3 - \frac{242}{25} \end{aligned}$

8. $\frac{x-1}{x^2-4x-5} = \frac{x-1}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1} \Rightarrow x-1 = A(x+1) + B(x-5)$. Setting $x = -1$ gives $-2 = -6B$, so $B = \frac{1}{3}$. Setting $x = 5$ gives $4 = 6A$, so $A = \frac{2}{3}$. Now

$$\begin{aligned}\int_0^4 \frac{x-1}{x^2-4x-5} dx &= \int_0^4 \left(\frac{2/3}{x-5} + \frac{1/3}{x+1} \right) dx = \left[\frac{2}{3} \ln|x-5| + \frac{1}{3} \ln|x+1| \right]_0^4 \\ &= \frac{2}{3} \ln 1 + \frac{1}{3} \ln 5 - \frac{2}{3} \ln 5 - \frac{1}{3} \ln 1 = -\frac{1}{3} \ln 5\end{aligned}$$

9. $\int \frac{x-1}{x^2-4x+5} dx = \int \frac{(x-2)+1}{(x-2)^2+1} dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1} \right) du \quad [u = x-2, du = dx]$
 $= \frac{1}{2} \ln(u^2+1) + \tan^{-1} u + C = \frac{1}{2} \ln(x^2-4x+5) + \tan^{-1}(x-2) + C$

10. $\int \frac{x}{x^4+x^2+1} dx = \int \frac{\frac{1}{2} du}{u^2+u+1} \quad [u = x^2, du = 2x dx] = \frac{1}{2} \int \frac{du}{(u+\frac{1}{2})^2+\frac{3}{4}}$
 $= \frac{1}{2} \int \frac{\frac{\sqrt{3}}{2} dv}{\frac{3}{4}(v^2+1)} \quad [u+\frac{1}{2} = \frac{\sqrt{3}}{2}v, du = \frac{\sqrt{3}}{2} dv] = \frac{\sqrt{3}}{4} \cdot \frac{4}{3} \int \frac{dv}{v^2+1}$
 $= \frac{1}{\sqrt{3}} \tan^{-1} v + C = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(x^2 + \frac{1}{2} \right) \right) + C$

11. $\int \sin^3 \theta \cos^5 \theta d\theta = \int \cos^5 \theta \sin^2 \theta \sin \theta d\theta = -\int \cos^5 \theta (1-\cos^2 \theta)(-\sin \theta) d\theta$
 $= -\int u^5(1-u^2) du \quad \left[\begin{array}{l} u = \cos \theta, \\ du = -\sin \theta d\theta \end{array} \right]$
 $= \int (u^7 - u^5) du = \frac{1}{8} u^8 - \frac{1}{6} u^6 + C = \frac{1}{8} \cos^8 \theta - \frac{1}{6} \cos^6 \theta + C$

Another solution:

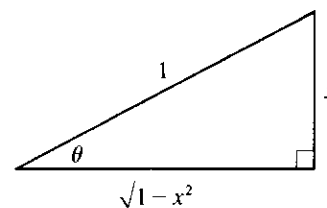
$$\begin{aligned}\int \sin^3 \theta \cos^5 \theta d\theta &= \int \sin^3 \theta (\cos^2 \theta)^2 \cos \theta d\theta = \int \sin^3 \theta (1-\sin^2 \theta)^2 \cos \theta d\theta \\ &= \int u^3(1-u^2)^2 du \quad \left[\begin{array}{l} u = \sin \theta, \\ du = \cos \theta d\theta \end{array} \right] = \int u^3(1-2u^2+u^4) du \\ &= \int (u^3 - 2u^5 + u^7) du = \frac{1}{4} u^4 - \frac{2}{6} u^6 + \frac{1}{8} u^8 + C = \frac{1}{4} \sin^4 \theta - \frac{1}{3} \sin^6 \theta + \frac{1}{8} \sin^8 \theta + C\end{aligned}$$

12. Let $u = \cos x$. Then $du = -\sin x dx \Rightarrow$

$$\int \sin x \cos(\cos x) dx = -\int \cos u du = -\sin u + C = -\sin(\cos x) + C.$$

13. Let $x = \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx = \cos \theta d\theta$ and

$$\begin{aligned}(1-x^2)^{1/2} &= \cos \theta, \text{ so} \\ \int \frac{dx}{(1-x^2)^{3/2}} &= \int \frac{\cos \theta d\theta}{(\cos \theta)^3} = \int \sec^2 \theta d\theta \\ &= \tan \theta + C = \frac{x}{\sqrt{1-x^2}} + C\end{aligned}$$



14. Let $u = \ln x$. Then $du = dx/x \Rightarrow$

$$\begin{aligned}\int \frac{\sqrt{1+\ln x}}{x \ln x} dx &= \int \frac{\sqrt{1+u}}{u} du = \int \frac{v}{v^2-1} 2v dv \quad [\text{put } v = \sqrt{1+u}, u = v^2-1, du = 2v dv] \\ &= 2 \int \left(1 + \frac{1}{v^2-1} \right) dv = 2v + \ln \left| \frac{v-1}{v+1} \right| + C = 2\sqrt{1+\ln x} + \ln \left(\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1} \right) + C\end{aligned}$$

15. Let $u = 1-x^2 \Rightarrow du = -2x dx$. Then

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int_1^{3/4} \frac{1}{\sqrt{u}} du = \frac{1}{2} \int_{3/4}^1 u^{-1/2} du = \frac{1}{2} \left[2u^{1/2} \right]_{3/4}^1 = [\sqrt{u}]_{3/4}^1 = 1 - \frac{\sqrt{3}}{2}$$

$$\begin{aligned}
 16. \int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}} dx &= \int_0^{\pi/4} \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \quad [x = \sin \theta, dx = \cos \theta d\theta] \\
 &= \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta) d\theta = \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/4} = \frac{1}{2} \left[\left(\frac{\pi}{4} - \frac{1}{2} \right) - (0 - 0) \right] = \frac{\pi}{8} - \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 17. \int x \sin^2 x dx &= \left[\begin{array}{l} u = x, \quad dv = \sin^2 x dx, \\ du = dx \quad v = \int \sin^2 x dx = \int \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2}x - \frac{1}{2} \sin x \cos x \end{array} \right] \\
 &= \frac{1}{2}x^2 - \frac{1}{2}x \sin x \cos x - \int \left(\frac{1}{2}x - \frac{1}{2} \sin x \cos x \right) dx \\
 &= \frac{1}{2}x^2 - \frac{1}{2}x \sin x \cos x - \frac{1}{4}x^2 + \frac{1}{4} \sin^2 x + C = \frac{1}{4}x^2 - \frac{1}{2}x \sin x \cos x + \frac{1}{4} \sin^2 x + C
 \end{aligned}$$

Note: $\int \sin x \cos x dx = \int s ds = \frac{1}{2}s^2 + C$ [where $s = \sin x$, $ds = \cos x dx$].

A slightly different method is to write $\int x \sin^2 x dx = \int x \cdot \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8} \cos 2x + C$.

18. Let $u = e^{2t}$, $du = 2e^{2t} dt$. Then

$$\int \frac{e^{2t}}{1 + e^{4t}} dt = \int \frac{\frac{1}{2}(2e^{2t}) dt}{1 + (e^{2t})^2} = \int \frac{\frac{1}{2} du}{1 + u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(e^{2t}) + C.$$

19. Let $u = e^x$. Then $\int e^{x+e^x} dx = \int e^{e^x} e^x dx = \int e^u du = e^u + C = e^{e^x} + C$.

20. Let $u = \sqrt[3]{x}$. Then $x = u^3 \Rightarrow \int e^{\sqrt[3]{x}} dx = \int e^u \cdot 3u^2 du$. Now use parts: let $w = u^2$, $dv = e^u du \Rightarrow$

$$dw = 2u du, v = e^u \Rightarrow 3 \int e^u u^2 du = 3(u^2 e^u - 2 \int u e^u du). \text{ Now use parts again with } W = u,$$

$$dV = e^u du \text{ to get } \int e^u 3u^2 du = e^u(3u^2 - 6u + 6) + C = 3e^{\sqrt[3]{x}}(x^{2/3} - 2\sqrt[3]{x} + 2) + C.$$

21. Integrate by parts three times, first with $u = t^3$, $dv = e^{-2t} dt$:

$$\begin{aligned}
 \int t^3 e^{-2t} dt &= -\frac{1}{2}t^3 e^{-2t} + \frac{1}{2} \int 3t^2 e^{-2t} dt = -\frac{1}{2}t^3 e^{-2t} - \frac{3}{4}t^2 e^{-2t} + \frac{1}{2} \int 3te^{-2t} dt \\
 &= -e^{-2t} \left[\frac{1}{2}t^3 + \frac{3}{4}t^2 \right] - \frac{3}{4}te^{-2t} + \frac{3}{4} \int e^{-2t} dt = -e^{-2t} \left[\frac{1}{2}t^3 + \frac{3}{4}t^2 + \frac{3}{4}t + \frac{3}{8} \right] + C \\
 &= -\frac{1}{8}e^{-2t}(4t^3 + 6t^2 + 6t + 3) + C
 \end{aligned}$$

22. Integrate by parts: $u = \sin^{-1} x$, $dv = x dx \Rightarrow du = (1/\sqrt{1-x^2}) dx$, $v = \frac{1}{2}x^2$, so

$$\begin{aligned}
 \int x \sin^{-1} x dx &= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2} \int \frac{\sin^2 \theta \cos \theta d\theta}{\cos \theta} \quad \left[\begin{array}{l} \text{where } x = \sin \theta \\ \text{for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{array} \right] \\
 &= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4} \int (1 - \cos 2\theta) d\theta = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}(\theta - \sin \theta \cos \theta) + C \\
 &= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4} \left[\sin^{-1} x - x\sqrt{1-x^2} \right] + C = \frac{1}{4} \left[(2x^2 - 1) \sin^{-1} x + x\sqrt{1-x^2} \right] + C
 \end{aligned}$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1) du \Rightarrow$

$$\begin{aligned}
 \int_0^1 (1 + \sqrt{x})^8 dx &= \int_1^2 u^8 \cdot 2(u-1) du = 2 \int_1^2 (u^9 - u^8) du = \left[\frac{1}{5}u^{10} - 2 \cdot \frac{1}{9}u^9 \right]_1^2 \\
 &= \frac{1024}{5} - \frac{1024}{9} - \frac{1}{5} + \frac{2}{9} = \frac{4097}{45}
 \end{aligned}$$

24. Let $u = \ln(x^2 - 1)$, $dv = dx \Leftrightarrow du = \frac{2x}{x^2 - 1}$, $v = x$. Then

$$\begin{aligned}\int \ln(x^2 - 1) dx &= x \ln(x^2 - 1) - \int \frac{2x^2}{x^2 - 1} dx = x \ln(x^2 - 1) - \int \left[2 + \frac{2}{(x-1)(x+1)} \right] dx \\ &= x \ln(x^2 - 1) - \int \left[2 + \frac{1}{x-1} - \frac{1}{x+1} \right] dx \\ &= x \ln(x^2 - 1) - 2x - \ln|x-1| + \ln|x+1| + C\end{aligned}$$

25. $\frac{3x^2 - 2}{x^2 - 2x - 8} = 3 + \frac{6x + 22}{(x-4)(x+2)} = 3 + \frac{A}{x-4} + \frac{B}{x+2} \Rightarrow 6x + 22 = A(x+2) + B(x-4)$. Setting $x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2 - 2}{x^2 - 2x - 8} dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2} \right) dx = 3x + \frac{23}{3} \ln|x-4| - \frac{5}{3} \ln|x+2| + C.$$

26. $\int \frac{3x^2 - 2}{x^3 - 2x - 8} dx = \int \frac{du}{u} \left[\begin{array}{l} u = x^3 - 2x - 8, \\ du = (3x^2 - 2) dx \end{array} \right] = \ln|u| + C = \ln|x^3 - 2x - 8| + C$

27. Let $u = \ln(\sin x)$. Then $du = \cot x dx \Rightarrow \int \cot x \ln(\sin x) dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}[\ln(\sin x)]^2 + C$.

28. $\int \sin \sqrt{at} dt = \int \sin u \cdot \frac{2}{a} u du \quad [u = \sqrt{at}, u^2 = at, 2u du = a dt] = \frac{2}{a} \int u \sin u du$
 $= \frac{2}{a} [-u \cos u + \sin u] + C \quad [\text{integration by parts}] = -\frac{2}{a} \sqrt{at} \cos \sqrt{at} + \frac{2}{a} \sin \sqrt{at} + C$
 $= -2\sqrt{\frac{t}{a}} \cos \sqrt{at} + \frac{2}{a} \sin \sqrt{at} + C$

29. $\int_0^5 \frac{3w-1}{w+2} dw = \int_0^5 \left(3 - \frac{7}{w+2} \right) dw = [3w - 7 \ln|w+2|]_0^5$
 $= 15 - 7 \ln 7 + 7 \ln 2 = 15 + 7(\ln 2 - \ln 7) = 15 + 7 \ln \frac{2}{7}$

30. $x^2 - 4x < 0$ on $[0, 4]$, so

$$\begin{aligned}\int_{-2}^2 |x^2 - 4x| dx &= \int_{-2}^0 (x^2 - 4x) dx + \int_0^2 (4x - x^2) dx = \left[\frac{1}{3}x^3 - 2x^2 \right]_{-2}^0 + \left[2x^2 - \frac{1}{3}x^3 \right]_0^2 \\ &= 0 - \left(-\frac{8}{3} - 8 \right) + \left(8 - \frac{8}{3} \right) - 0 = 16\end{aligned}$$

31. As in Example 5,

$$\begin{aligned}\int \sqrt{\frac{1+x}{1-x}} dx &= \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} dx = \int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x dx}{\sqrt{1-x^2}} \\ &= \sin^{-1} x - \sqrt{1-x^2} + C\end{aligned}$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

32. $\int \frac{\sqrt{2x-1}}{2x+3} dx = \int \frac{u \cdot u du}{u^2 + 4} \left[\begin{array}{l} u = \sqrt{2x-1}, 2x+3 = u^2 + 4, \\ u^2 = 2x-1, u du = dx \end{array} \right] = \int \left(1 - \frac{4}{u^2 + 4} \right) du$
 $= u - 4 \cdot \frac{1}{2} \tan^{-1} \left(\frac{1}{2}u \right) + C = \sqrt{2x-1} - 2 \tan^{-1} \left(\frac{1}{2}\sqrt{2x-1} \right) + C$

33. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x + 1)^2$. Let

$x + 1 = 2 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx = 2 \cos \theta d\theta$ and

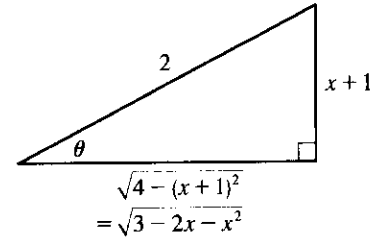
$$\int \sqrt{3 - 2x - x^2} dx = \int \sqrt{4 - (x + 1)^2} dx = \int \sqrt{4 - 4 \sin^2 \theta} 2 \cos \theta d\theta$$

$$= 4 \int \cos^2 \theta d\theta = 2 \int (1 + \cos 2\theta) d\theta$$

$$= 2\theta + \sin 2\theta + C = 2\theta + 2 \sin \theta \cos \theta + C$$

$$= 2 \sin^{-1} \left(\frac{x+1}{2} \right) + 2 \cdot \frac{x+1}{2} \cdot \frac{\sqrt{3-2x-x^2}}{2} + C$$

$$= 2 \sin^{-1} \left(\frac{x+1}{2} \right) + \frac{x+1}{2} \sqrt{3-2x-x^2} + C$$



34.
$$\int_{\pi/4}^{\pi/2} \frac{1 + 4 \cot x}{4 - \cot x} dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1 + 4 \cos x / \sin x)}{(4 - \cos x / \sin x)} \cdot \frac{\sin x}{\sin x} \right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4 \cos x}{4 \sin x - \cos x} dx$$

$$= \int_{3/\sqrt{2}}^4 \frac{1}{u} du \quad \left[\begin{array}{l} u = 4 \sin x - \cos x, \\ du = (4 \cos x + \sin x) dx \end{array} \right] = \left[\ln |u| \right]_{3/\sqrt{2}}^4$$

$$= \ln 4 - \ln \frac{3}{\sqrt{2}} = \ln \frac{4}{3/\sqrt{2}} = \ln \left(\frac{4}{3} \sqrt{2} \right)$$

35. Because $f(x) = x^8 \sin x$ is the product of an even function and an odd function, it is odd. Therefore,

$$\int_{-1}^1 x^8 \sin x dx = 0 \quad [\text{by (5.5.6)(b)}].$$

36. $\sin 4x \cos 3x = \frac{1}{2}(\sin x + \sin 7x)$ by Formula 8.2.2(a), so

$$\int \sin 4x \cos 3x dx = \frac{1}{2} \int (\sin x + \sin 7x) dx = \frac{1}{2} \left[-\cos x - \frac{1}{7} \cos 7x \right] + C = -\frac{1}{2} \cos x - \frac{1}{14} \cos 7x + C.$$

37.
$$\int_0^{\pi/4} \cos^2 \theta \tan^2 \theta d\theta = \int_0^{\pi/4} \sin^2 \theta d\theta = \int_0^{\pi/4} \frac{1}{2} (1 - \cos 2\theta) d\theta = \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{8} - \frac{1}{4} \right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$$

38.
$$\int_0^{\pi/4} \tan^5 \theta \sec^3 \theta d\theta = \int_0^{\pi/4} (\tan^2 \theta)^2 \sec^2 \theta \cdot \sec \theta \tan \theta d\theta = \int_1^{\sqrt{2}} (u^2 - 1)^2 u^2 du \quad \left[\begin{array}{l} u = \sec \theta \\ du = \sec \theta \tan \theta d\theta \end{array} \right]$$

$$= \int_1^{\sqrt{2}} (u^6 - 2u^4 + u^2) du = \left[\frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 \right]_1^{\sqrt{2}}$$

$$= \left(\frac{8}{7} \sqrt{2} - \frac{8}{5} \sqrt{2} + \frac{2}{3} \sqrt{2} \right) - \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) = \frac{22}{105} \sqrt{2} - \frac{8}{105} = \frac{2}{105} (11 \sqrt{2} - 4)$$

39. Let $u = 1 - x^2$. Then $du = -2x dx \Rightarrow$

$$\int \frac{x dx}{1 - x^2 + \sqrt{1 - x^2}} = -\frac{1}{2} \int \frac{du}{u + \sqrt{u}} = -\int \frac{v dv}{v^2 + v} \quad [v = \sqrt{u}, u = v^2, du = 2v dv]$$

$$= -\int \frac{dv}{v+1} = -\ln |v+1| + C = -\ln (\sqrt{1-x^2} + 1) + C$$

40. $4y^2 - 4y - 3 = (2y - 1)^2 - 2^2$, so let $u = 2y - 1 \Rightarrow du = 2 dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} = \int \frac{dy}{\sqrt{(2y - 1)^2 - 2^2}} = \frac{1}{2} \int \frac{du}{\sqrt{u^2 - 2^2}}$$

$$= \frac{1}{2} \ln \left| u + \sqrt{u^2 - 2^2} \right| \quad [\text{by Formula 20 in the table in this section}]$$

$$= \frac{1}{2} \ln \left| 2y - 1 + \sqrt{4y^2 - 4y - 3} \right| + C$$

41. Let $u = \theta$, $dv = \tan^2 \theta d\theta = (\sec^2 \theta - 1) d\theta \Rightarrow du = d\theta$ and $v = \tan \theta - \theta$. So

$$\begin{aligned}\int \theta \tan^2 \theta d\theta &= \theta(\tan \theta - \theta) - \int (\tan \theta - \theta) d\theta = \theta \tan \theta - \theta^2 - \ln |\sec \theta| + \frac{1}{2} \theta^2 + C \\ &= \theta \tan \theta - \frac{1}{2} \theta^2 - \ln |\sec \theta| + C\end{aligned}$$

42. Integrate by parts with $u = \tan^{-1} x$, $dv = x^2 dx \Rightarrow du = dx/(1+x^2)$, $v = \frac{1}{3}x^3$:

$$\begin{aligned}\int x^2 \tan^{-1} x dx &= \frac{1}{3}x^3 \tan^{-1} x - \int \frac{x^3}{3} \frac{dx}{1+x^2} = \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3} \int \left[x - \frac{x}{x^2+1} \right] dx \\ &= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6} \ln(x^2+1) + C\end{aligned}$$

43. Let $u = 1 + e^x$, so that $du = e^x dx$. Then

$$\int e^x \sqrt{1+e^x} dx = \int u^{1/2} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1+e^x)^{3/2} + C.$$

Or: Let $u = \sqrt{1+e^x}$, so that $u^2 = 1+e^x$ and $2u du = e^x dx$. Then

$$\int e^x \sqrt{1+e^x} dx = \int u \cdot 2u du = \int 2u^2 du = \frac{2}{3} u^3 + C = \frac{2}{3} (1+e^x)^{3/2} + C.$$

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1+e^x$, $2u du = e^x dx = (u^2-1) dx$, and $dx = \frac{2u}{u^2-1} du$, so

$$\begin{aligned}\int \sqrt{1+e^x} dx &= \int u \cdot \frac{2u}{u^2-1} du = \int \frac{2u^2}{u^2-1} du = \int \left(2 + \frac{2}{u^2-1} \right) du = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1} \right) du \\ &= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln(\sqrt{1+e^x}-1) - \ln(\sqrt{1+e^x}+1) + C\end{aligned}$$

45. Let $t = x^3$. Then $dt = 3x^2 dx \Rightarrow I = \int x^5 e^{-x^3} dx = \frac{1}{3} \int t e^{-t} dt$. Now integrate by parts with $u = t$,

$$dv = e^{-t} dt: I = -\frac{1}{3} t e^{-t} + \frac{1}{3} \int e^{-t} dt = -\frac{1}{3} t e^{-t} - \frac{1}{3} e^{-t} + C = -\frac{1}{3} e^{-x^3} (x^3 + 1) + C.$$

46. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\begin{aligned}\int \frac{1+e^x}{1-e^x} dx &= \int \frac{(1+u) du}{(1-u)u} = - \int \frac{(u+1) du}{(u-1)u} = - \int \left(\frac{2}{u-1} - \frac{1}{u} \right) du \\ &= \ln|u| - 2 \ln|u-1| + C = \ln e^x - 2 \ln|e^x-1| + C = x - 2 \ln|e^x-1| + C\end{aligned}$$

47. $\int \frac{x+a}{x^2+a^2} dx = \frac{1}{2} \int \frac{2x dx}{x^2+a^2} + a \int \frac{dx}{x^2+a^2} = \frac{1}{2} \ln(x^2+a^2) + a \cdot \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$
 $= \ln \sqrt{x^2+a^2} + \tan^{-1}(x/a) + C$

48. Let $u = x^2$. Then $du = 2x dx \Rightarrow$

$$\int \frac{x dx}{x^4 - a^4} = \int \frac{\frac{1}{2} du}{u^2 - (a^2)^2} = \frac{1}{4a^2} \ln \left| \frac{u-a^2}{u+a^2} \right| + C = \frac{1}{4a^2} \ln \left| \frac{x^2-a^2}{x^2+a^2} \right| + C.$$

49. Let $u = \sqrt{4x+1} \Rightarrow u^2 = 4x+1 \Rightarrow 2u du = 4 dx \Rightarrow dx = \frac{1}{2} u du$. So

$$\begin{aligned}\int \frac{1}{x \sqrt{4x+1}} dx &= \int \frac{\frac{1}{2} u du}{\frac{1}{4}(u^2-1)u} = 2 \int \frac{du}{u^2-1} = 2 \left(\frac{1}{2} \right) \ln \left| \frac{u-1}{u+1} \right| + C \quad [\text{by Formula 19}] \\ &= \ln \left| \frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1} \right| + C\end{aligned}$$

50. As in Exercise 49, let $u = \sqrt{4x+1}$. Then $\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u \, du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8 \int \frac{du}{(u^2-1)^2}$. Now

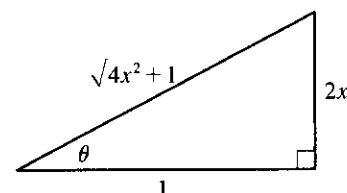
$$\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \Rightarrow$$

$$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2. \quad u=1 \Rightarrow D = \frac{1}{4}, u=-1 \Rightarrow B = \frac{1}{4}. \text{ Equating coefficients of } u^3 \text{ gives } A+C=0, \text{ and equating coefficients of } 1 \text{ gives } 1 = A+B-C+D \Rightarrow 1 = A + \frac{1}{4} - C + \frac{1}{4} \Rightarrow \frac{1}{2} = A - C. \text{ So } A = \frac{1}{4} \text{ and } C = -\frac{1}{4}. \text{ Therefore,}$$

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4x+1}} &= 8 \int \left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2} \right] du \\ &= \int \left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2} \right] du \\ &= 2 \ln|u+1| - \frac{2}{u+1} - 2 \ln|u-1| - \frac{2}{u-1} + C \\ &= 2 \ln(\sqrt{4x+1}+1) - \frac{2}{\sqrt{4x+1}+1} - 2 \ln|\sqrt{4x+1}-1| - \frac{2}{\sqrt{4x+1}-1} + C\end{aligned}$$

51. Let $2x = \tan \theta \Rightarrow x = \frac{1}{2} \tan \theta, dx = \frac{1}{2} \sec^2 \theta d\theta, \sqrt{4x^2 + 1} = \sec \theta$, so

$$\begin{aligned}\int \frac{dx}{x\sqrt{4x^2+1}} &= \int \frac{\frac{1}{2}\sec^2\theta d\theta}{\frac{1}{2}\tan\theta\sec\theta} = \int \frac{\sec\theta}{\tan\theta} d\theta = \int \csc\theta d\theta \\ &= -\ln|\csc\theta + \cot\theta| + C \quad [\text{or } \ln|\csc\theta - \cot\theta| + C] \\ &= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \quad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]\end{aligned}$$



52. Let $u = x^2$. Then $du = 2x \, dx \Rightarrow$

$$\begin{aligned}\int \frac{dx}{x(x^4+1)} &= \int \frac{x \, dx}{x^2(x^4+1)} = \frac{1}{2} \int \frac{du}{u(u^2+1)} = \frac{1}{2} \int \left[\frac{1}{u} - \frac{u}{u^2+1} \right] du = \frac{1}{2} \ln|u| - \frac{1}{4} \ln(u^2+1) + C \\ &= \frac{1}{2} \ln(x^2) - \frac{1}{4} \ln(x^4+1) + C = \frac{1}{4} [\ln(x^4) - \ln(x^4+1)] + C = \frac{1}{4} \ln\left(\frac{x^4}{x^4+1}\right) + C\end{aligned}$$

Or: Write $I = \int \frac{x^3 dx}{x^4(x^4 + 1)}$ and let $u = x^4$.

53. $\int x^2 \sinh(mx) dx = \frac{1}{m} x^2 \cosh(mx) - \frac{2}{m} \int x \cosh(mx) dx$ $\left[\begin{array}{ll} u = x^2, & dv = \sinh(mx) dx, \\ du = 2x dx & v = \frac{1}{m} \cosh(mx) \end{array} \right]$

$= \frac{1}{m} x^2 \cosh(mx) - \frac{2}{m} \left(\frac{1}{m} x \sinh(mx) - \frac{1}{m} \int \sinh(mx) dx \right)$ $\left[\begin{array}{ll} U = x, & dV = \cosh(mx) dx, \\ dU = dx & V = \frac{1}{m} \sinh(mx) \end{array} \right]$

$= \frac{1}{m} x^2 \cosh(mx) - \frac{2}{m^2} x \sinh(mx) + \frac{2}{m^3} \cosh(mx) + C$

54. $\int (x + \sin x)^2 dx = \int (x^2 + 2x \sin x + \sin^2 x) dx = \frac{1}{3}x^3 + 2(\sin x - x \cos x) + \frac{1}{2}(x - \sin x \cos x) + C$
 $= \frac{1}{3}x^3 + \frac{1}{2}x + 2 \sin x - \frac{1}{2} \sin x \cos x - 2x \cos x + C$

55. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1 \Rightarrow$

$$\begin{aligned}\int \frac{dx}{x+4+4\sqrt{x+1}} &= \int \frac{2u du}{u^2+3+4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3} \right] du \\ &= 3 \ln|u+3| - \ln|u+1| + C = 3 \ln(\sqrt{x+1}+3) - \ln(\sqrt{x+1}+1) + C\end{aligned}$$

56. Let $t = \sqrt{x^2-1}$. Then $dt = (x/\sqrt{x^2-1}) dx$, $x^2-1 = t^2$, $x = \sqrt{t^2+1}$, so

$$I = \int \frac{x \ln x}{\sqrt{x^2-1}} dx = \int \ln \sqrt{t^2+1} dt = \frac{1}{2} \int \ln(t^2+1) dt. \text{ Now use parts with } u = \ln(t^2+1), dv = dt:$$

$$\begin{aligned}I &= \frac{1}{2} t \ln(t^2+1) - \int \frac{t^2}{t^2+1} dt = \frac{1}{2} t \ln(t^2+1) - \int \left[1 - \frac{1}{t^2+1} \right] dt \\ &= \frac{1}{2} t \ln(t^2+1) - t + \tan^{-1} t + C = \sqrt{x^2-1} \ln x - \sqrt{x^2-1} + \tan^{-1} \sqrt{x^2-1} + C\end{aligned}$$

Another method: First integrate by parts with $u = \ln x$, $dv = (x/\sqrt{x^2-1}) dx$ and then use substitution ($x = \sec \theta$ or $u = \sqrt{x^2-1}$).

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \Rightarrow$

$$\begin{aligned}\int x \sqrt[3]{x+c} dx &= \int (u^3 - c)u \cdot 3u^2 du = 3 \int (u^6 - cu^3) du = \frac{3}{7} u^7 - \frac{3}{4} cu^4 + C \\ &= \frac{3}{7} (x+c)^{7/3} - \frac{3}{4} c(x+c)^{4/3} + C\end{aligned}$$

58. Integrate by parts with $u = \ln(1+x)$, $dv = x^2 dx \Rightarrow du = dx/(1+x)$, $v = \frac{1}{3}x^3$:

$$\begin{aligned}\int x^2 \ln(1+x) dx &= \frac{1}{3} x^3 \ln(1+x) - \int \frac{x^3 dx}{3(1+x)} = \frac{1}{3} x^3 \ln(1+x) - \frac{1}{3} \int \left(x^2 - x + 1 - \frac{1}{x+1} \right) dx \\ &= \frac{1}{3} x^3 \ln(1+x) - \frac{1}{9} x^3 + \frac{1}{6} x^2 - \frac{1}{3} x + \frac{1}{3} \ln(1+x) + C\end{aligned}$$

59. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\begin{aligned}\int \frac{dx}{e^{3x} - e^x} &= \int \frac{du/u}{u^3 - u} = \int \frac{du}{(u-1)u^2(u+1)} = \int \left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1} \right] du \\ &= \frac{1}{u} + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C = e^{-x} + \frac{1}{2} \ln \left| \frac{e^x-1}{e^x+1} \right| + C\end{aligned}$$

60. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 du \Rightarrow$

$$\int \frac{dx}{x + \sqrt[3]{x}} = \int \frac{3u^2 du}{u^3 + u} = \frac{3}{2} \int \frac{2u du}{u^2 + 1} = \frac{3}{2} \ln(u^2 + 1) + C = \frac{3}{2} \ln(x^{2/3} + 1) + C.$$

61. Let $u = x^5$. Then $du = 5x^4 dx \Rightarrow$

$$\int \frac{x^4 dx}{x^{10} + 16} = \int \frac{\frac{1}{5} du}{u^2 + 16} = \frac{1}{5} \cdot \frac{1}{4} \tan^{-1} \left(\frac{1}{4} u \right) + C = \frac{1}{20} \tan^{-1} \left(\frac{1}{4} x^5 \right) + C.$$

62. Let $u = x+1$. Then $du = dx \Rightarrow$

$$\begin{aligned}\int \frac{x^3}{(x+1)^{10}} dx &= \int \frac{(u-1)^3}{u^{10}} du = \int (u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}) du \\ &= -\frac{1}{6} u^{-6} + \frac{3}{7} u^{-7} - \frac{3}{8} u^{-8} + \frac{1}{9} u^{-9} + C \\ &= (x+1)^{-9} \left[-\frac{1}{6} (x+1)^3 + \frac{3}{7} (x+1)^2 - \frac{3}{8} (x+1) + \frac{1}{9} \right] + C\end{aligned}$$

63. Let $y = \sqrt{x}$ so that $dy = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2\sqrt{x} dy = 2y dy$. Then

$$\begin{aligned}\int \sqrt{x} e^{\sqrt{x}} dx &= \int y e^y (2y dy) = \int 2y^2 e^y dy \quad \left[\begin{array}{l} u = 2y^2, \quad dv = e^y dy, \\ du = 4y dy \quad v = e^y \end{array} \right] \\ &= 2y^2 e^y - \int 4y e^y dy \quad \left[\begin{array}{l} U = 4y, \quad dV = e^y dy, \\ dU = 4 dy \quad V = e^y \end{array} \right] \\ &= 2y^2 e^y - (4y e^y - \int 4e^y dy) = 2y^2 e^y - 4y e^y + 4e^y + C \\ &= 2(y^2 - 2y + 2)e^y + C = 2(x - 2\sqrt{x} + 2)e^{\sqrt{x}} + C\end{aligned}$$

64. Let $u = \tan x$. Then

$$\begin{aligned}\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x) dx}{\sin x \cos x} &= \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x} \sec^2 x dx = \int_1^{\sqrt{3}} \frac{\ln u}{u} du \\ &= \left[\frac{1}{2} (\ln u)^2 \right]_1^{\sqrt{3}} = \frac{1}{2} (\ln \sqrt{3})^2 = \frac{1}{8} (\ln 3)^2\end{aligned}$$

65.
$$\int \frac{dx}{\sqrt{x+1} + \sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} \right) dx = \int (\sqrt{x+1} - \sqrt{x}) dx$$

$$= \frac{2}{3} \left[(x+1)^{3/2} - x^{3/2} \right] + C$$

66.
$$\int \frac{u^3 + 1}{u^3 - u^2} du = \int \left[1 + \frac{u^2 + 1}{(u-1)u^2} \right] du = u + \int \left[\frac{2}{u-1} - \frac{1}{u} - \frac{1}{u^2} \right] du = u + 2 \ln |u-1| - \ln |u| + \frac{1}{u} + C.$$

Thus,

$$\begin{aligned}\int_2^3 \frac{u^3 + 1}{u^3 - u^2} du &= \left[u + 2 \ln(u-1) - \ln u + \frac{1}{u} \right]_2^3 = \left(3 + 2 \ln 2 - \ln 3 + \frac{1}{3} \right) - \left(2 + 2 \ln 1 - \ln 2 + \frac{1}{2} \right) \\ &= 1 + 3 \ln 2 - \ln 3 - \frac{1}{6} = \frac{5}{6} + \ln \frac{8}{3}\end{aligned}$$

67. Let $u = \sqrt{t}$. Then $du = dt/(2\sqrt{t}) \Rightarrow$

$$\begin{aligned}\int_1^3 \frac{\arctan \sqrt{t}}{\sqrt{t}} dt &= \int_1^{\sqrt{3}} \tan^{-1} u (2 du) = 2 \left[u \tan^{-1} u - \frac{1}{2} \ln(1+u^2) \right]_1^{\sqrt{3}} \quad [\text{Example 5 in Section 8.1}] \\ &= 2 \left[\left(\sqrt{3} \tan^{-1} \sqrt{3} - \frac{1}{2} \ln 4 \right) - \left(\tan^{-1} 1 - \frac{1}{2} \ln 2 \right) \right] \\ &= 2 \left[\left(\sqrt{3} \cdot \frac{\pi}{3} - \ln 2 \right) - \left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right) \right] = \frac{2}{3} \sqrt{3} \pi - \frac{1}{2} \pi - \ln 2\end{aligned}$$

68. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\begin{aligned}\int \frac{dx}{1+2e^x - e^{-x}} &= \int \frac{du/u}{1+2u-1/u} = \int \frac{du}{2u^2+u-1} = \int \left[\frac{2/3}{2u-1} - \frac{1/3}{u+1} \right] du \\ &= \frac{1}{3} \ln |2u-1| - \frac{1}{3} \ln |u+1| + C = \frac{1}{3} \ln |(2e^x-1)/(e^x+1)| + C\end{aligned}$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\begin{aligned}\int \frac{e^{2x}}{1+e^x} dx &= \int \frac{u^2}{1+u} \frac{du}{u} = \int \frac{u}{1+u} du = \int \left(1 - \frac{1}{1+u} \right) du \\ &= u - \ln |1+u| + C = e^x - \ln(1+e^x) + C\end{aligned}$$

70. Use parts with $u = \ln(x+1)$, $dv = dx/x^2$:

$$\begin{aligned}\int \frac{\ln(x+1)}{x^2} dx &= -\frac{1}{x} \ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x} \ln(x+1) + \int \left[\frac{1}{x} - \frac{1}{x+1} \right] dx \\ &= -\frac{1}{x} \ln(x+1) + \ln |x| - \ln(x+1) + C = -\left(1 + \frac{1}{x}\right) \ln(x+1) + \ln |x| + C\end{aligned}$$

$$71. \frac{x}{x^4 + 4x^2 + 3} = \frac{x}{(x^2 + 3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{x^2 + 1} \Rightarrow$$

$$\begin{aligned} x &= (Ax + B)(x^2 + 1) + (Cx + D)(x^2 + 3) = (Ax^3 + Bx^2 + Ax + B) + (Cx^3 + Dx^2 + 3Cx + 3D) \\ &= (A + C)x^3 + (B + D)x^2 + (A + 3C)x + (B + 3D) \Rightarrow \end{aligned}$$

$$A + C = 0, B + D = 0, A + 3C = 1, B + 3D = 0 \Rightarrow A = -\frac{1}{2}, C = \frac{1}{2}, B = 0, D = 0. \text{ Thus,}$$

$$\begin{aligned} \int \frac{x}{x^4 + 4x^2 + 3} dx &= \int \left(\frac{-\frac{1}{2}x}{x^2 + 3} + \frac{\frac{1}{2}x}{x^2 + 1} \right) dx \\ &= -\frac{1}{4} \ln(x^2 + 3) + \frac{1}{4} \ln(x^2 + 1) + C \quad \text{or} \quad \frac{1}{4} \ln \left(\frac{x^2 + 1}{x^2 + 3} \right) + C \end{aligned}$$

$$72. \text{ Let } u = \sqrt[6]{t}. \text{ Then } t = u^6, dt = 6u^5 du \Rightarrow$$

$$\begin{aligned} \int \frac{\sqrt{t} dt}{1 + \sqrt[3]{t}} &= \int \frac{u^3 \cdot 6u^5 du}{1 + u^2} = 6 \int \frac{u^8}{u^2 + 1} du = 6 \int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2 + 1} \right) du \\ &= 6 \left(\frac{1}{7} u^7 - \frac{1}{5} u^5 + \frac{1}{3} u^3 - u + \tan^{-1} u \right) + C \\ &= 6 \left(\frac{1}{7} t^{7/6} - \frac{1}{5} t^{5/6} + \frac{1}{3} t^{1/2} - t^{1/6} + \tan^{-1} t^{1/6} \right) + C \end{aligned}$$

$$73. \frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \Rightarrow$$

$$1 = A(x^2 + 4) + (Bx + C)(x - 2) = (A + B)x^2 + (C - 2B)x + (4A - 2C). \text{ So } 0 = A + B = C - 2B,$$

$$1 = 4A - 2C. \text{ Setting } x = 2 \text{ gives } A = \frac{1}{8} \Rightarrow B = -\frac{1}{8} \text{ and } C = -\frac{1}{4}. \text{ So}$$

$$\begin{aligned} \int \frac{1}{(x-2)(x^2+4)} dx &= \int \left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4} \right) dx = \frac{1}{8} \int \frac{dx}{x-2} - \frac{1}{16} \int \frac{2x dx}{x^2+4} - \frac{1}{4} \int \frac{dx}{x^2+4} \\ &= \frac{1}{8} \ln|x-2| - \frac{1}{16} \ln(x^2+4) - \frac{1}{8} \tan^{-1}(x/2) + C \end{aligned}$$

$$74. \text{ Let } u = e^x. \text{ Then } x = \ln u, dx = du/u \Rightarrow$$

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x dx}{e^{2x} - 1} = \int \frac{u}{u^2 - 1} \frac{du}{u} = \int \frac{du}{u^2 - 1} = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C = \frac{1}{2} \ln \left(\frac{|e^x - 1|}{e^x + 1} \right) + C.$$

$$\begin{aligned} 75. \int \sin x \sin 2x \sin 3x dx &= \int \sin x \cdot \frac{1}{2} [\cos(2x - 3x) - \cos(2x + 3x)] dx = \frac{1}{2} \int (\sin x \cos x - \sin x \cos 5x) dx \\ &= \frac{1}{4} \int \sin 2x dx - \frac{1}{2} \int \frac{1}{2} [\sin(x + 5x) + \sin(x - 5x)] dx \\ &= -\frac{1}{8} \cos 2x - \frac{1}{4} \int (\sin 6x - \sin 4x) dx = -\frac{1}{8} \cos 2x + \frac{1}{24} \cos 6x - \frac{1}{16} \cos 4x + C \end{aligned}$$

$$\begin{aligned} 76. \int (x^2 - bx) \sin 2x dx &= -\frac{1}{2} (x^2 - bx) \cos 2x + \frac{1}{2} \int (2x - b) \cos 2x dx \\ &\quad [u = x^2 - bx, dv = \sin 2x dx, du = (2x - b) dx, v = -\frac{1}{2} \cos 2x] \\ &= -\frac{1}{2} (x^2 - bx) \cos 2x + \frac{1}{2} \left[\frac{1}{2} (2x - b) \sin 2x - \int \sin 2x dx \right] \\ &\quad [U = 2x - b, dV = \cos 2x dx, dU = 2 dx, V = \frac{1}{2} \sin 2x] \\ &= -\frac{1}{2} (x^2 - bx) \cos 2x + \frac{1}{4} (2x - b) \sin 2x + \frac{1}{4} \cos 2x + C \end{aligned}$$

77. Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2} dx \Rightarrow \sqrt{x} dx = \frac{2}{3} du$. Then

$$\int \frac{\sqrt{x}}{1+x^3} dx = \int \frac{\frac{2}{3}}{1+u^2} du = \frac{2}{3} \tan^{-1} u + C = \frac{2}{3} \tan^{-1}(x^{3/2}) + C.$$

$$\begin{aligned} 78. \int \frac{\sec x \cos 2x}{\sin x + \sec x} dx &= \int \frac{\sec x \cos 2x}{\sin x + \sec x} \cdot \frac{2 \cos x}{2 \cos x} dx = \int \frac{2 \cos 2x}{2 \sin x \cos x + 2} dx \\ &= \int \frac{2 \cos 2x}{\sin 2x + 2} dx = \int \frac{1}{u} du \quad \left[\begin{array}{l} u = \sin 2x + 2, \\ du = 2 \cos 2x dx \end{array} \right] \\ &= \ln |u| + C = \ln |\sin 2x + 2| + C = \ln(\sin 2x + 2) + C \end{aligned}$$

79. Let $u = x$, $dv = \sin^2 x \cos x dx \Rightarrow du = dx$, $v = \frac{1}{3} \sin^3 x$. Then

$$\begin{aligned} \int x \sin^2 x \cos x dx &= \frac{1}{3} x \sin^3 x - \int \frac{1}{3} \sin^3 x dx = \frac{1}{3} x \sin^3 x - \frac{1}{3} \int (1 - \cos^2 x) \sin x dx \\ &= \frac{1}{3} x \sin^3 x + \frac{1}{3} \int (1 - y^2) dy \quad \left[\begin{array}{l} y = \cos x, \\ dy = -\sin x dx \end{array} \right] \\ &= \frac{1}{3} x \sin^3 x + \frac{1}{3} y - \frac{1}{9} y^3 + C = \frac{1}{3} x \sin^3 x + \frac{1}{3} \cos x - \frac{1}{9} \cos^3 x + C \end{aligned}$$

$$\begin{aligned} 80. \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx &= \int \frac{\sin x \cos x}{(\sin^2 x)^2 + (\cos^2 x)^2} dx = \int \frac{\sin x \cos x}{(\sin^2 x)^2 + (1 - \sin^2 x)^2} dx \\ &= \int \frac{1}{u^2 + (1-u)^2} \left(\frac{1}{2} du \right) \quad \left[\begin{array}{l} u = \sin^2 x, \\ du = 2 \sin x \cos x dx \end{array} \right] \\ &= \int \frac{1}{4u^2 - 4u + 2} du = \int \frac{1}{(4u^2 - 4u + 1) + 1} du \\ &= \int \frac{1}{(2u-1)^2 + 1} du = \frac{1}{2} \int \frac{1}{y^2 + 1} dy \quad \left[\begin{array}{l} y = 2u - 1, \\ dy = 2 du \end{array} \right] \\ &= \frac{1}{2} \tan^{-1} y + C = \frac{1}{2} \tan^{-1}(2u - 1) + C = \frac{1}{2} \tan^{-1}(2 \sin^2 x - 1) + C \end{aligned}$$

Another solution:

$$\begin{aligned} \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx &= \int \frac{(\sin x \cos x)/\cos^4 x}{(\sin^4 x + \cos^4 x)/\cos^4 x} dx = \int \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx \\ &= \int \frac{1}{u^2 + 1} \left(\frac{1}{2} du \right) \quad \left[\begin{array}{l} u = \tan^2 x, \\ du = 2 \tan x \sec^2 x dx \end{array} \right] \\ &= \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(\tan^2 x) + C \end{aligned}$$

81. The function $y = 2xe^{x^2}$ does have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\begin{aligned} \int (2x^2 + 1)e^{x^2} dx &= \int 2x^2 e^{x^2} dx + \int e^{x^2} dx = \int x(2xe^{x^2}) dx + \int e^{x^2} dx \\ &= xe^{x^2} - \int e^{x^2} dx + \int e^{x^2} dx \quad \left[\begin{array}{l} u = x, \quad dv = 2xe^{x^2} dx, \\ du = dx \quad v = e^{x^2} \end{array} \right] = xe^{x^2} + C \end{aligned}$$

8.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. We could make the substitution $u = \sqrt{2}x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a = \sqrt{7}$.
Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a = \sqrt{\frac{7}{2}}$.

$$\begin{aligned}\int \frac{\sqrt{7-2x^2}}{x^2} dx &= \sqrt{2} \int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2} dx \stackrel{33}{=} \sqrt{2} \left[-\frac{1}{x} \sqrt{\frac{7}{2}-x^2} - \sin^{-1} \frac{x}{\sqrt{\frac{7}{2}}} \right] + C \\ &= -\frac{1}{x} \sqrt{7-2x^2} - \sqrt{2} \sin^{-1} \left(\sqrt{\frac{2}{7}} x \right) + C\end{aligned}$$

$$\begin{aligned}2. \int \frac{3x}{\sqrt{3-2x}} dx &= 3 \int \frac{x}{\sqrt{3+(-2)x}} dx \stackrel{55}{=} 3 \left[\frac{2}{3(-2)^2} (-2x-2 \cdot 3) \sqrt{3+(-2)x} \right] + C \\ &= \frac{1}{2} (-2x-6) \sqrt{3-2x} + C = -(x+3) \sqrt{3-2x} + C\end{aligned}$$

3. Let $u = \pi x \Rightarrow du = \pi dx$, so

$$\begin{aligned}\int \sec^3(\pi x) dx &= \frac{1}{\pi} \int \sec^3 u du \stackrel{71}{=} \frac{1}{\pi} \left(\frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| \right) + C \\ &= \frac{1}{2\pi} \sec \pi x \tan \pi x + \frac{1}{2\pi} \ln |\sec \pi x + \tan \pi x| + C\end{aligned}$$

$$4. \int e^{2\theta} \sin 3\theta d\theta \stackrel{98}{=} \frac{e^{2\theta}}{2^2 + 3^2} (2 \sin 3\theta - 3 \cos 3\theta) + C = \frac{2}{13} e^{2\theta} \sin 3\theta - \frac{3}{13} e^{2\theta} \cos 3\theta + C$$

$$5. \int_0^1 2x \cos^{-1} x dx \stackrel{91}{=} 2 \left[\frac{2x^2-1}{4} \cos^{-1} x - \frac{x \sqrt{1-x^2}}{4} \right]_0^1 = 2 \left[\left(\frac{1}{4} \cdot 0 - 0 \right) - \left(-\frac{1}{4} \cdot \frac{\pi}{2} - 0 \right) \right] = 2 \left(\frac{\pi}{8} \right) = \frac{\pi}{4}$$

$$\begin{aligned}6. \int_2^3 \frac{1}{x^2 \sqrt{4x^2-7}} dx &= \int_4^6 \frac{1}{\left(\frac{1}{2}u\right)^2 \sqrt{u^2-7}} \left(\frac{1}{2} du\right) \quad [u = 2x, du = 2 dx] \\ &= 2 \int_4^6 \frac{du}{u^2 \sqrt{u^2-7}} \stackrel{45}{=} 2 \left[\frac{\sqrt{u^2-7}}{7u} \right]_4^6 \\ &= 2 \left(\frac{\sqrt{29}}{42} - \frac{3}{28} \right) = \frac{\sqrt{29}}{21} - \frac{3}{14}\end{aligned}$$

7. By Formula 99 with $a = -3$ and $b = 4$,

$$\int e^{-3x} \cos 4x dx = \frac{e^{-3x}}{(-3)^2 + 4^2} (-3 \cos 4x + 4 \sin 4x) + C = \frac{e^{-3x}}{25} (-3 \cos 4x + 4 \sin 4x) + C.$$

8. Let $u = x/2$, so $dx = 2 du$, and we use Formula 72:

$$\begin{aligned}\int \csc^3(x/2) dx &= 2 \int \csc^3 u du = -\csc u \cot u + \ln |\csc u - \cot u| + C \\ &= -\csc(x/2) \cot(x/2) + \ln |\csc(x/2) - \cot(x/2)| + C\end{aligned}$$

9. Let $u = 2x$ and $a = 3$. Then $du = 2 dx$ and

$$\begin{aligned}\int \frac{dx}{x^2 \sqrt{4x^2+9}} &= \int \frac{\frac{1}{2} du}{\frac{u^2}{4} \sqrt{u^2+a^2}} = 2 \int \frac{du}{u^2 \sqrt{a^2+u^2}} \stackrel{28}{=} -2 \frac{\sqrt{a^2+u^2}}{a^2 u} + C \\ &= -2 \frac{\sqrt{4x^2+9}}{9 \cdot 2x} + C = -\frac{\sqrt{4x^2+9}}{9x} + C\end{aligned}$$

10. Let $u = \sqrt{2}y$ and $a = \sqrt{3}$. Then $du = \sqrt{2}dy$ and

$$\begin{aligned}\int \frac{\sqrt{2y^2-3}}{y^2} dy &= \int \frac{\sqrt{u^2-a^2}}{\frac{1}{2}u^2} \frac{du}{\sqrt{2}} = \sqrt{2} \int \frac{\sqrt{u^2-a^2}}{u^2} du \\ &\stackrel{42}{=} \sqrt{2} \left(-\frac{\sqrt{u^2-a^2}}{u} + \ln|u + \sqrt{u^2-a^2}| \right) + C \\ &= \sqrt{2} \left(-\frac{\sqrt{2y^2-3}}{\sqrt{2}y} + \ln|\sqrt{2}y + \sqrt{2y^2-3}| \right) + C \\ &= -\frac{\sqrt{2y^2-3}}{y} + \sqrt{2} \ln|\sqrt{2}y + \sqrt{2y^2-3}| + C\end{aligned}$$

11. $\int_{-1}^0 t^2 e^{-t} dt \stackrel{97}{=} \left[\frac{1}{-1} t^2 e^{-t} \right]_{-1}^0 - \frac{2}{-1} \int_{-1}^0 t e^{-t} dt = e + 2 \int_{-1}^0 t e^{-t} dt \stackrel{96}{=} e + 2 \left[\frac{1}{(-1)^2} (-t-1) e^{-t} \right]_{-1}^0$
 $= e + 2[-e^0 + 0] = e - 2$

12. Let $u = 3x$. Then $du = 3dx$, so

$$\begin{aligned}\int x^2 \cos 3x dx &= \frac{1}{27} \int u^2 \cos u du \stackrel{85}{=} \frac{1}{27} (u^2 \sin u - 2 \int u \sin u du) \\ &\stackrel{82}{=} \frac{1}{3} x^2 \sin 3x - \frac{2}{27} (\sin 3x - 3x \cos 3x) + C \\ &= \frac{1}{27} [(9x^2 - 2) \sin 3x + 6x \cos 3x] + C\end{aligned}$$

$$\text{Thus, } \int_0^\pi x^2 \cos 3x dx = \frac{1}{27} [(9x^2 - 2) \sin 3x + 6x \cos 3x]_0^\pi = \frac{1}{27} [(0 + 6\pi(-1)) - (0 + 0)] = -\frac{6\pi}{27} = -\frac{2\pi}{9}.$$

13. $\int \frac{\tan^3(1/z)}{z^2} dz \quad \left[\begin{array}{l} u = 1/z, \\ du = -dz/z^2 \end{array} \right] = -\int \tan^3 u du \stackrel{69}{=} -\frac{1}{2} \tan^2 u - \ln|\cos u| + C$
 $= -\frac{1}{2} \tan^2\left(\frac{1}{z}\right) - \ln\left|\cos\left(\frac{1}{z}\right)\right| + C$

14. Let $u = \sqrt{x}$. Then $u^2 = x$ and $2u du = dx$, so

$$\begin{aligned}\int \sin^{-1} \sqrt{x} dx &= 2 \int u \sin^{-1} u du \stackrel{90}{=} \frac{2u^2-1}{2} \sin^{-1} u + \frac{u\sqrt{1-u^2}}{2} + C \\ &= \frac{2x-1}{2} \sin^{-1} \sqrt{x} + \frac{\sqrt{x(1-x)}}{2} + C\end{aligned}$$

15. Let $u = e^x$. Then $du = e^x dx$, so $\int e^x \operatorname{sech}(e^x) dx = \int \operatorname{sech} u du \stackrel{107}{=} \tan^{-1}|\sinh u| + C = \tan^{-1}[\sinh(e^x)] + C$

16. Let $u = x^2$, so that $du = 2x dx$. Then

$$\begin{aligned}\int x \sin(x^2) \cos(3x^2) dx &= \frac{1}{2} \int \sin u \cos 3u du \stackrel{81}{=} -\frac{1}{2} \frac{\cos(1-3)u}{2(1-3)} - \frac{1}{2} \frac{\cos(1+3)u}{2(1+3)} + C \\ &= \frac{1}{8} \cos 2u - \frac{1}{16} \cos 4u + C = \frac{1}{8} \cos(2x^2) - \frac{1}{16} \cos(4x^2) + C\end{aligned}$$

17. Let $z = 6 + 4y - 4y^2 = 6 - (4y^2 - 4y + 1) + 1 = 7 - (2y - 1)^2$, $u = 2y - 1$, and $a = \sqrt{7}$. Then $z = a^2 - u^2$, $du = 2 dy$, and

$$\begin{aligned} \int y \sqrt{6 + 4y - 4y^2} dy &= \int y \sqrt{z} dy = \int \frac{1}{2}(u + 1) \sqrt{a^2 - u^2} \frac{1}{2} du \\ &= \frac{1}{4} \int u \sqrt{a^2 - u^2} du + \frac{1}{4} \int \sqrt{a^2 - u^2} du \\ &= \frac{1}{4} \int \sqrt{a^2 - u^2} du - \frac{1}{8} \int (-2u) \sqrt{a^2 - u^2} du \\ &\stackrel{30}{=} \frac{u}{8} \sqrt{a^2 - u^2} + \frac{a^2}{8} \sin^{-1} \left(\frac{u}{a} \right) - \frac{1}{8} \int \sqrt{w} dw \quad \left[\begin{array}{l} w = a^2 - u^2, \\ dw = -2u du \end{array} \right] \\ &= \frac{2y - 1}{8} \sqrt{6 + 4y - 4y^2} + \frac{7}{8} \sin^{-1} \frac{2y - 1}{\sqrt{7}} - \frac{1}{8} \cdot \frac{2}{3} w^{3/2} + C \\ &= \frac{2y - 1}{8} \sqrt{6 + 4y - 4y^2} + \frac{7}{8} \sin^{-1} \frac{2y - 1}{\sqrt{7}} - \frac{1}{12} (6 + 4y - 4y^2)^{3/2} + C. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \sqrt{6 + 4y - 4y^2} \left[\frac{1}{8}(2y - 1) - \frac{1}{12}(6 + 4y - 4y^2) \right] + \frac{7}{8} \sin^{-1} \frac{2y - 1}{\sqrt{7}} + C \\ = \left(\frac{1}{3}y^2 - \frac{1}{12}y - \frac{5}{8} \right) \sqrt{6 + 4y - 4y^2} + \frac{7}{8} \sin^{-1} \left(\frac{2y - 1}{\sqrt{7}} \right) + C \\ = \frac{1}{24} (8y^2 - 2y - 15) \sqrt{6 + 4y - 4y^2} + \frac{7}{8} \sin^{-1} \left(\frac{2y - 1}{\sqrt{7}} \right) + C \end{aligned}$$

18. Let $u = x^2$. Then $du = 2x dx$, so by Formula 48,

$$\begin{aligned} \int \frac{x^5 dx}{x^2 + \sqrt{2}} &= \frac{1}{2} \int \frac{u^2}{u + \sqrt{2}} du = \frac{1}{2} \cdot \frac{1}{2} \left[(u + \sqrt{2})^2 - 4\sqrt{2}(u + \sqrt{2}) + 4 \ln|u + \sqrt{2}| \right] + C \\ &= \frac{1}{4} \left[(x^2 + \sqrt{2})^2 - 4\sqrt{2}(x^2 + \sqrt{2}) + 4 \ln(x^2 + \sqrt{2}) \right] + C \\ &= \frac{1}{4} x^4 - \frac{1}{\sqrt{2}} x^2 + \ln(x^2 + \sqrt{2}) + K \end{aligned}$$

Or: Let $u = x^2 + \sqrt{2}$.

19. Let $u = \sin x$. Then $du = \cos x dx$, so

$$\begin{aligned} \int \sin^2 x \cos x \ln(\sin x) dx &= \int u^2 \ln u du \stackrel{101}{=} \frac{u^{2+1}}{(2+1)^2} [(2+1) \ln u - 1] + C = \frac{1}{9} u^3 (3 \ln u - 1) + C \\ &= \frac{1}{9} \sin^3 x [3 \ln(\sin x) - 1] + C \end{aligned}$$

20. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\begin{aligned} \int \frac{dx}{e^x(1 + 2e^x)} &= \int \frac{du/u}{u(1 + 2u)} = \int \frac{du}{u^2(1 + 2u)} \stackrel{50}{=} -\frac{1}{u} + 2 \ln \left| \frac{1 + 2u}{u} \right| + C \\ &= -e^{-x} + 2 \ln(e^{-x} + 2) + C \end{aligned}$$

21. Let $u = e^x$ and $a = \sqrt{3}$. Then $du = e^x dx$ and

$$\int \frac{e^x}{3 - e^{2x}} dx = \int \frac{du}{a^2 - u^2} \stackrel{19}{=} \frac{1}{2a} \ln \left| \frac{u + a}{u - a} \right| + C = \frac{1}{2\sqrt{3}} \ln \left| \frac{e^x + \sqrt{3}}{e^x - \sqrt{3}} \right| + C.$$

22. Let $u = x^2$ and $a = 2$. Then $du = 2x dx$ and

$$\begin{aligned}\int_0^2 x^3 \sqrt{4x^2 - x^4} dx &= \frac{1}{2} \int_0^2 x^2 \sqrt{2 \cdot 2 \cdot x^2 - (x^2)^2} \cdot 2x dx = \frac{1}{2} \int_0^4 u \sqrt{2au - u^2} du \\ &\stackrel{114}{=} \left[\frac{2u^2 - au - 3a^2}{12} \sqrt{2au - u^2} + \frac{a^3}{4} \cos^{-1} \left(\frac{a-u}{a} \right) \right]_0^4 \\ &= \left[\frac{2u^2 - 2u - 12}{12} \sqrt{4u - u^2} + \frac{8}{4} \cos^{-1} \left(\frac{2-u}{2} \right) \right]_0^4 \\ &= \left[\frac{u^2 - u - 6}{6} \sqrt{4u - u^2} + 2 \cos^{-1} \left(\frac{2-u}{2} \right) \right]_0^4 \\ &= [0 + 2 \cos^{-1}(-1)] - (0 + 2 \cos^{-1} 1) = 2 \cdot \pi - 2 \cdot 0 = 2\pi\end{aligned}$$

23. $\int \sec^5 x dx \stackrel{77}{=} \frac{1}{4} \tan x \sec^3 x + \frac{3}{4} \int \sec^3 x dx \stackrel{77}{=} \frac{1}{4} \tan x \sec^3 x + \frac{3}{4} \left(\frac{1}{2} \tan x \sec x + \frac{1}{2} \int \sec x dx \right)$
 $\stackrel{14}{=} \frac{1}{4} \tan x \sec^3 x + \frac{3}{8} \tan x \sec x + \frac{3}{8} \ln |\sec x + \tan x| + C$

24. Let $u = 2x$. Then $du = 2 dx$, so

$$\begin{aligned}\int \sin^6 2x dx &= \frac{1}{2} \int \sin^6 u du \stackrel{73}{=} \frac{1}{2} \left(-\frac{1}{6} \sin^5 u \cos u + \frac{5}{6} \int \sin^4 u du \right) \\ &\stackrel{73}{=} -\frac{1}{12} \sin^5 u \cos u + \frac{5}{12} \left(-\frac{1}{4} \sin^3 u \cos u + \frac{3}{4} \int \sin^2 u du \right) \\ &\stackrel{63}{=} -\frac{1}{12} \sin^5 u \cos u - \frac{5}{48} \sin^3 u \cos u + \frac{5}{16} \left(\frac{1}{2} u - \frac{1}{4} \sin 2u \right) + C \\ &= -\frac{1}{12} \sin^5 2x \cos 2x - \frac{5}{48} \sin^3 2x \cos 2x - \frac{5}{64} \sin 4x + \frac{5}{16} x + C\end{aligned}$$

25. Let $u = \ln x$ and $a = 2$. Then $du = \frac{dx}{x}$ and

$$\begin{aligned}\int \frac{\sqrt{4 + (\ln x)^2}}{x} dx &= \int \sqrt{a^2 + u^2} du \stackrel{21}{=} \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C \\ &= \frac{1}{2} (\ln x) \sqrt{4 + (\ln x)^2} + 2 \ln [\ln x + \sqrt{4 + (\ln x)^2}] + C\end{aligned}$$

26. $\int x^4 e^{-x} dx \stackrel{97}{=} -x^4 e^{-x} + 4 \int x^3 e^{-x} dx \stackrel{97}{=} -x^4 e^{-x} + 4(-x^3 e^{-x} + 3 \int x^2 e^{-x} dx)$
 $\stackrel{97}{=} -(x^4 + 4x^3) e^{-x} + 12(-x^2 e^{-x} + 2 \int x e^{-x} dx)$
 $\stackrel{96}{=} -(x^4 + 4x^3 + 12x^2) e^{-x} + 24[(-x - 1) e^{-x}] + C$
 $= -(x^4 + 4x^3 + 12x^2 + 24x + 24) e^{-x} + C$

$$\begin{aligned}\text{So } \int_0^1 x^4 e^{-x} dx &= [-(x^4 + 4x^3 + 12x^2 + 24x + 24) e^{-x}]_0^1 \\ &= -(1 + 4 + 12 + 24 + 24) e^{-1} + 24 e^0 = 24 - 65 e^{-1}.\end{aligned}$$

27. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x} - 1} dx = \int \frac{\sqrt{u^2 - 1}}{u} du \stackrel{41}{=} \sqrt{u^2 - 1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x} - 1} - \cos^{-1}(e^{-x}) + C.$$

28. Let $u = \alpha t - 3$ and assume that $\alpha \neq 0$. Then $du = \alpha dt$ and

$$\begin{aligned}\int e^t \sin(\alpha t - 3) dt &= \frac{1}{\alpha} \int e^{(u+3)/\alpha} \sin u du = \frac{1}{\alpha} e^{3/\alpha} \int e^{(1/\alpha)u} \sin u du \\ &\stackrel{98}{=} \frac{1}{\alpha} e^{3/\alpha} \frac{e^{(1/\alpha)u}}{(1/\alpha)^2 + 1^2} \left(\frac{1}{\alpha} \sin u - \cos u \right) + C \\ &= \frac{1}{\alpha} e^{3/\alpha} e^{(1/\alpha)u} \frac{\alpha^2}{1 + \alpha^2} \left(\frac{1}{\alpha} \sin u - \cos u \right) + C \\ &= \frac{1}{1 + \alpha^2} e^{(u+3)/\alpha} (\sin u - \alpha \cos u) + C \\ &= \frac{1}{1 + \alpha^2} e^t [\sin(\alpha t - 3) - \alpha \cos(\alpha t - 3)] + C\end{aligned}$$

29. $\int \frac{x^4 dx}{\sqrt{x^{10} - 2}} = \int \frac{x^4 dx}{\sqrt{(x^5)^2 - 2}} = \frac{1}{5} \int \frac{du}{\sqrt{u^2 - 2}} \quad [u = x^5, du = 5x^4 dx]$

$$\stackrel{43}{=} \frac{1}{5} \ln|u + \sqrt{u^2 - 2}| + C = \frac{1}{5} \ln|x^5 + \sqrt{x^{10} - 2}| + C$$

30. Let $u = \tan \theta$ and $a = 3$. Then $du = \sec^2 \theta d\theta$ and

$$\begin{aligned}\int \frac{\sec^2 \theta \tan^2 \theta}{\sqrt{9 - \tan^2 \theta}} d\theta &= \int \frac{u^2}{\sqrt{a^2 - u^2}} du \stackrel{34}{=} -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{u}{a} \right) + C \\ &= -\frac{1}{2} \tan \theta \sqrt{9 - \tan^2 \theta} + \frac{9}{2} \sin^{-1} \left(\frac{\tan \theta}{3} \right) + C\end{aligned}$$

31. Using cylindrical shells, we get

$$\begin{aligned}V &= 2\pi \int_0^2 x \cdot x \sqrt{4 - x^2} dx = 2\pi \int_0^2 x^2 \sqrt{4 - x^2} dx \stackrel{31}{=} 2\pi \left[\frac{x}{8} (2x^2 - 4) \sqrt{4 - x^2} + \frac{16}{8} \sin^{-1} \frac{x}{2} \right]_0^2 \\ &= 2\pi [(0 + 2 \sin^{-1} 1) - (0 + 2 \sin^{-1} 0)] = 2\pi \left(2 \cdot \frac{\pi}{2} \right) = 2\pi^2\end{aligned}$$

32. Using disks, we get

$$\begin{aligned}\text{Volume} &= \int_0^{\pi/4} \pi \tan^4 x dx \stackrel{75}{=} \pi \left(\left[\frac{1}{3} \tan^3 x \right]_0^{\pi/4} - \int_0^{\pi/4} \tan^2 x dx \right) \stackrel{65}{=} \pi \left[\frac{1}{3} \tan^3 x - \tan x + x \right]_0^{\pi/4} \\ &= \pi \left(\frac{1}{3} - 1 + \frac{\pi}{4} \right) = \pi \left(\frac{\pi}{4} - \frac{2}{3} \right)\end{aligned}$$

33. (a) $\frac{d}{du} \left[\frac{1}{b^3} \left(a + bu - \frac{a^2}{a + bu} - 2a \ln |a + bu| \right) + C \right] = \frac{1}{b^3} \left[b + \frac{ba^2}{(a + bu)^2} - \frac{2ab}{(a + bu)} \right]$

$$= \frac{1}{b^3} \left[\frac{b(a + bu)^2 + ba^2 - (a + bu)2ab}{(a + bu)^2} \right] = \frac{1}{b^3} \left[\frac{b^3 u^2}{(a + bu)^2} \right] = \frac{u^2}{(a + bu)^2}$$

(b) Let $t = a + bu \Rightarrow dt = b du$. Note that $u = \frac{t - a}{b}$ and $du = \frac{1}{b} dt$.

$$\begin{aligned}\int \frac{u^2 du}{(a + bu)^2} &= \frac{1}{b^3} \int \frac{(t - a)^2}{t^2} dt = \frac{1}{b^3} \int \frac{t^2 - 2at + a^2}{t^2} dt \\ &= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2} \right) dt = \frac{1}{b^3} \left(t - 2a \ln |t| - \frac{a^2}{t} \right) + C \\ &= \frac{1}{b^3} \left(a + bu - \frac{a^2}{a + bu} - 2a \ln |a + bu| \right) + C\end{aligned}$$

$$\begin{aligned}
 34. \text{ (a) } \frac{d}{du} \left[\frac{u}{8} (2u^2 - a^2) \sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C \right] \\
 &= \frac{u}{8} (2u^2 - a^2) \frac{-u}{\sqrt{a^2 - u^2}} + \sqrt{a^2 - u^2} \left[\frac{u}{8} (4u) + (2u^2 - a^2) \frac{1}{8} \right] + \frac{a^4}{8} \frac{1/a}{\sqrt{1 - u^2/a^2}} \\
 &= -\frac{u^2(2u^2 - a^2)}{8\sqrt{a^2 - u^2}} + \sqrt{a^2 - u^2} \left[\frac{u^2}{2} + \frac{2u^2 - a^2}{8} \right] + \frac{a^4}{8\sqrt{a^2 - u^2}} \\
 &= \frac{1}{2} (a^2 - u^2)^{-1/2} \left[-\frac{u^2}{4} (2u^2 - a^2) + u^2 (a^2 - u^2) + \frac{1}{4} (a^2 - u^2) (2u^2 - a^2) + \frac{a^4}{4} \right] \\
 &= \frac{1}{2} (a^2 - u^2)^{-1/2} [2u^2 a^2 - 2u^4] = \frac{u^2(a^2 - u^2)}{\sqrt{a^2 - u^2}} = u^2 \sqrt{a^2 - u^2}
 \end{aligned}$$

(b) Let $u = a \sin \theta \Rightarrow du = a \cos \theta d\theta$. Then

$$\begin{aligned}
 \int u^2 \sqrt{a^2 - u^2} du &= \int a^2 \sin^2 \theta a \sqrt{1 - \sin^2 \theta} a \cos \theta d\theta = a^4 \int \sin^2 \theta \cos^2 \theta d\theta \\
 &= a^4 \int \frac{1}{2} (1 + \cos 2\theta) \frac{1}{2} (1 - \cos 2\theta) d\theta = \frac{1}{4} a^4 \int (1 - \cos^2 2\theta) d\theta \\
 &= \frac{1}{4} a^4 \int [1 - \frac{1}{2} (1 + \cos 4\theta)] d\theta = \frac{1}{4} a^4 (\frac{1}{2} \theta - \frac{1}{8} \sin 4\theta) + C \\
 &= \frac{1}{4} a^4 (\frac{1}{2} \theta - \frac{1}{8} 2 \sin 2\theta \cos 2\theta) + C = \frac{1}{4} a^4 [\frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta (1 - 2 \sin^2 \theta)] + C \\
 &= \frac{a^4}{8} \left[\sin^{-1} \frac{u}{a} - \frac{u}{a} \frac{\sqrt{a^2 - u^2}}{a} \left(1 - \frac{2u^2}{a^2} \right) \right] + C \\
 &= \frac{a^4}{8} \left[\sin^{-1} \frac{u}{a} - \frac{u}{a} \frac{\sqrt{a^2 - u^2}}{a} \frac{a^2 - 2u^2}{a^2} \right] + C \\
 &= \frac{u}{8} (2u^2 - a^2) \sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C
 \end{aligned}$$

35. Maple, Mathematica and Derive all give $\int x^2 \sqrt{5 - x^2} dx = -\frac{1}{4} x (5 - x^2)^{3/2} + \frac{5}{8} x \sqrt{5 - x^2} + \frac{25}{8} \sin^{-1} \left(\frac{1}{\sqrt{5}} x \right)$.

Using Formula 31, we get $\int x^2 \sqrt{5 - x^2} dx = \frac{1}{8} x (2x^2 - 5) \sqrt{5 - x^2} + \frac{1}{8} (5^2) \sin^{-1} \left(\frac{1}{\sqrt{5}} x \right) + C$. But

$-\frac{1}{4} x (5 - x^2)^{3/2} + \frac{5}{8} x \sqrt{5 - x^2} = \frac{1}{8} x \sqrt{5 - x^2} [5 - 2(5 - x^2)] = \frac{1}{8} x (2x^2 - 5) \sqrt{5 - x^2}$, and the $\sin^{-1}$ terms are the same in each expression, so the answers are equivalent.

36. Maple and Mathematica both give $\int x^2 (1 + x^3)^4 dx = \frac{1}{15} x^{15} + \frac{1}{3} x^{12} + \frac{2}{3} x^9 + \frac{2}{3} x^6 + \frac{1}{3} x^3$, while Derive gives

$\int x^2 (1 + x^3)^4 dx = \frac{1}{15} (x^3 + 1)^5$. Using the substitution $u = 1 + x^3 \Rightarrow du = 3x^2 dx$, we get

$\int x^2 (1 + x^3)^4 dx = \int u^4 (\frac{1}{3} du) = \frac{1}{15} u^5 + C = \frac{1}{15} (1 + x^3)^5 + C$. We can use the Binomial Theorem or a CAS to expand this expression, and we get $\frac{1}{15} (1 + x^3)^5 + C = \frac{1}{15} + \frac{1}{3} x^3 + \frac{2}{3} x^6 + \frac{2}{3} x^9 + \frac{1}{3} x^{12} + \frac{1}{15} x^{15} + C$.

37. Maple and Derive both give $\int \sin^3 x \cos^2 x dx = -\frac{1}{5} \sin^2 x \cos^3 x - \frac{2}{15} \cos^3 x$ (although Derive factors the expression), and Mathematica gives $\int \sin^3 x \cos^2 x dx = -\frac{1}{8} \cos x - \frac{1}{48} \cos 3x + \frac{1}{80} \cos 5x$. We can use a CAS to show that both of these expressions are equal to $-\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x$. Using Formula 86, we write

$$\begin{aligned}
 \int \sin^3 x \cos^2 x dx &= -\frac{1}{5} \sin^2 x \cos^3 x + \frac{2}{5} \int \sin x \cos^2 x dx = -\frac{1}{5} \sin^2 x \cos^3 x + \frac{2}{5} \left(-\frac{1}{3} \cos^3 x \right) + C \\
 &= -\frac{1}{5} \sin^2 x \cos^3 x - \frac{2}{15} \cos^3 x + C
 \end{aligned}$$

38. Maple gives $\int \tan^2 x \sec^4 x \, dx = \frac{1}{5} \frac{\sin^3 x}{\cos^5 x} + \frac{2}{15} \frac{\sin^3 x}{\cos^3 x}$,

Mathematica gives $\int \tan^2 x \sec^4 x \, dx = -\frac{1}{120} \sec^5 x (-20 \sin x + 5 \sin 3x + \sin 5x)$, and

Derive gives $\int \tan^2 x \sec^4 x \, dx = -\frac{2}{15} \tan x - \frac{\sin x}{15 \cos^3 x} + \frac{\sin x}{5 \cos^5 x}$. All of these expressions can be “simplified”

to $-\frac{1}{15} \frac{\sin x (\cos^2 x - 2 \cos^4 x - 3)}{\cos^5 x}$ using Maple. Using the identity $1 + \tan^2 x = \sec^2 x$, we write

$$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx = \int (\tan^2 x + \tan^4 x) \sec^2 x \, dx.$$

Now we substitute $u = \tan x \Rightarrow du = \sec^2 x \, dx$, and the integral becomes

$$\int (u^2 + u^4) \, du = \frac{1}{3} u^3 + \frac{1}{5} u^5 + C = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C. \text{ If we write}$$

$\sin^5 x = \sin^3 x (1 - \cos^2 x)$ and substitute into the numerator of the $\tan^5 x$ term, this becomes

$$\frac{1}{3} \frac{\sin^3 x}{\cos^3 x} + \frac{1}{5} \frac{\sin^3 x (1 - \cos^2 x)}{\cos^5 x} + C = \frac{1}{5} \frac{\sin^3 x}{\cos^5 x} + \left(\frac{1}{3} - \frac{1}{5} \right) \frac{\sin^3 x}{\cos^3 x} + C = \frac{1}{5} \frac{\sin^3 x}{\cos^5 x} + \frac{2}{15} \frac{\sin^3 x}{\cos^3 x} + C,$$

which is the same as Maple’s expression.

39. Maple gives $\int x \sqrt{1+2x} \, dx = \frac{1}{10} (1+2x)^{5/2} - \frac{1}{6} (1+2x)^{3/2}$, Mathematica gives $\sqrt{1+2x} (\frac{2}{5} x^2 + \frac{1}{15} x - \frac{1}{15})$, and Derive gives $\frac{1}{15} (1+2x)^{3/2} (3x-1)$. The first two expressions can be simplified to Derive’s result. If we use Formula 54, we get

$$\begin{aligned} \int x \sqrt{1+2x} \, dx &= \frac{2}{15(2)^2} (3 \cdot 2x - 2 \cdot 1) (1+2x)^{3/2} + C = \frac{1}{30} (6x-2) (1+2x)^{3/2} + C \\ &= \frac{1}{15} (3x-1) (1+2x)^{3/2} \end{aligned}$$

40. Maple and Derive both give $\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \cos x \sin x + \frac{3}{8} x$, while Mathematica gives $\frac{1}{32} (12x - 8 \sin 2x + \sin 4x)$, which can be expanded and simplified to give the other expression. Now

$$\begin{aligned} \int \sin^4 x \, dx &\stackrel{73}{=} -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx \stackrel{63}{=} -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left(\frac{1}{2} x - \frac{1}{4} \sin 2x \right) + C \\ &= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C \text{ since } \sin 2x = 2 \sin x \cos x \end{aligned}$$

41. Maple gives $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \frac{1}{2} \ln(1 + \tan^2 x)$, Mathematica gives $\int \tan^5 x \, dx = \frac{1}{4} [-1 - 2 \cos(2x)] \sec^4 x - \ln(\cos x)$, and Derive gives $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica’s and Derive’s expressions suggest that the integral is undefined where $\cos x < 0$, which is not the case. Using Formula 75, $\int \tan^5 x \, dx = \frac{1}{5-1} \tan^{5-1} x - \int \tan^{5-2} x \, dx = \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx$. Using Formula 69, $\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x + \ln |\cos x| + C$, so $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C$.
42. Maple gives $\int x^5 \sqrt{x^2+1} \, dx = \frac{1}{35} x^4 \sqrt{1+x^2} - \frac{4}{105} x^2 \sqrt{1+x^2} + \frac{8}{105} \sqrt{1+x^2} + \frac{1}{7} x^6 \sqrt{1+x^2}$. When we use the factor command on this expression, it becomes $\frac{1}{105} (1+x^2)^{3/2} (15x^4 - 12x^2 + 8)$. Mathematica gives $\sqrt{1+x^2} (\frac{8}{105} - \frac{4}{105} x^2 + \frac{1}{35} x^4 + \frac{1}{7} x^6)$, which again factors to give the above expression, and Derive gives the factored form immediately. If we substitute $u = \sqrt{x^2+1} \Rightarrow x^4 = (u^2-1)^2, x \, dx = u \, du$, then the integral

becomes

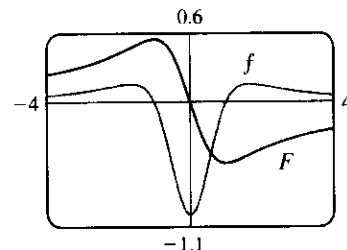
$$\begin{aligned} \int (u^2 - 1)^2 u(u \, du) &= \int (u^4 - 2u^2 + 1)u^2 \, du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\ &= (x^2 + 1)^{3/2} \left[\frac{1}{7}(x^2 + 1)^2 - \frac{2}{5}(x^2 + 1) + \frac{1}{3} \right] + C \\ &= \frac{1}{105}(x^2 + 1)^{3/2} \left[15(x^2 + 1)^2 - 42(x^2 + 1) + 35 \right] + C \\ &= \frac{1}{105}(x^2 + 1)^{3/2} (15x^4 - 12x^2 + 8) + C \end{aligned}$$

- 43.** Derive gives $I = \int 2^x \sqrt{4^x - 1} dx = \frac{2^{x-1} \sqrt{2^{2x} - 1}}{\ln 2} - \frac{\ln(\sqrt{2^{2x} - 1} + 2^x)}{2 \ln 2}$ immediately. Neither Maple nor Mathematica is able to evaluate I in its given form. However, if we instead write I as $\int 2^x \sqrt{(2^x)^2 - 1} dx$, both systems give the same answer as Derive (after minor simplification). Our trick works because the CAS now recognizes 2^x as a promising substitution.
- 44.** None of Maple, Mathematica and Derive is able to evaluate $\int (1 + \ln x) \sqrt{1 + (x \ln x)^2} dx$. However, if we let $u = x \ln x$, then $du = (1 + \ln x) dx$ and the integral is simply $\int \sqrt{1 + u^2} du$, which any CAS can evaluate. The antiderivative is $\frac{1}{2} \ln \left(x \ln x + \sqrt{1 + (x \ln x)^2} \right) + \frac{1}{2} x \ln x \sqrt{1 + (x \ln x)^2} + C$.

- 45.** Maple gives the antiderivative

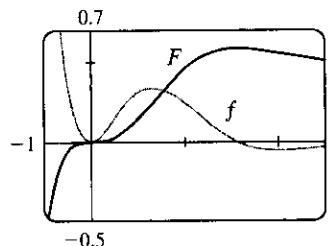
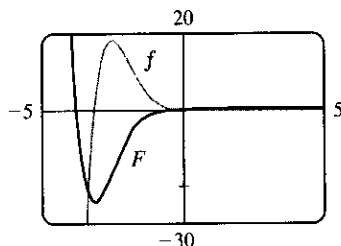
$$F(x) = \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx = -\frac{1}{2} \ln(x^2 + x + 1) + \frac{1}{2} \ln(x^2 - x + 1).$$

We can see that at 0, this antiderivative is 0. From the graphs, it appears that F has a maximum at $x = -1$ and a minimum at $x = 1$ [since $F'(x) = f(x)$ changes sign at these x -values], and that F has inflection points at $x \approx -1.7$, $x = 0$, and $x \approx 1.7$ [since $f(x)$ has extrema at these x -values].



- 46.** Maple gives the antiderivative which, after we use the `simplify` command, becomes

$\int x e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\cos x + x \cos x + x \sin x)$. At $x = 0$, this antiderivative has the value $-\frac{1}{2}$, so we use $F(x) = -\frac{1}{2} e^{-x} (\cos x + x \cos x + x \sin x) + \frac{1}{2}$ to make $F(0) = 0$.

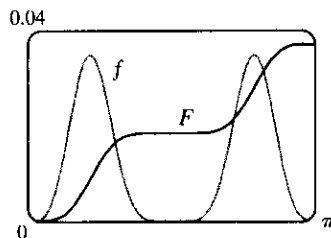


From the graphs, it appears that F has a minimum at $x \approx -3.1$ and a maximum at $x \approx 3.1$ [note that $f(x) = 0$ at $x = \pm\pi$], and that F has inflection points where f' changes sign, at $x \approx -2.5$, $x = 0$, $x \approx 1.3$ and $x \approx 4.1$.

47. Since $f(x) = \sin^4 x \cos^6 x$ is everywhere positive, we know that its antiderivative F is increasing. Maple gives

$$\int f(x) dx = -\frac{1}{10} \sin^3 x \cos^7 x - \frac{3}{80} \sin x \cos^7 x + \frac{1}{160} \cos^5 x \sin x + \frac{1}{128} \cos^3 x \sin x + \frac{3}{256} \cos x \sin x + \frac{3}{256} x$$

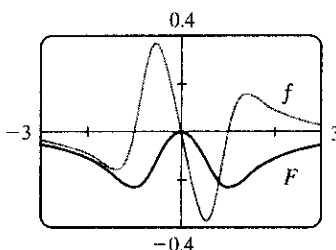
and this expression is 0 at $x = 0$.



F has a minimum at $x = 0$ and a maximum at $x = \pi$. F has inflection points where f' changes sign, that is, at $x \approx 0.7$, $x = \pi/2$, and $x \approx 2.5$.

48. From the graph of $f(x) = \frac{x^3 - x}{x^6 + 1}$, we can see that F has a maximum at $x = 0$, and minima at $x \approx \pm 1$. The

antiderivative given by Maple is $F(x) = -\frac{1}{3} \ln(x^2 + 1) + \frac{1}{6} \ln(x^4 - x^2 + 1)$, and $F(0) = 0$. Note that f is odd, and its antiderivative F is even.



F has inflection points where f' changes sign, that is, at $x \approx \pm 0.5$ and $x \approx \pm 1.4$.

DISCOVERY PROJECT Patterns in Integrals

1. (a) The CAS results are listed. Note that the absolute value symbols are missing, as is the familiar “+ C”.

$$(i) \int \frac{1}{(x+2)(x+3)} dx = \ln(x+2) - \ln(x+3)$$

$$(ii) \int \frac{1}{(x+1)(x+5)} dx = \frac{\ln(x+1)}{4} - \frac{\ln(x+5)}{4}$$

$$(iii) \int \frac{1}{(x+2)(x-5)} dx = \frac{\ln(x-5)}{7} - \frac{\ln(x+2)}{7}$$

$$(iv) \int \frac{1}{(x+2)^2} dx = -\frac{1}{x+2}$$

(b) If $a \neq b$, it appears that $\ln(x+a)$ is divided by $b-a$ and $\ln(x+b)$ is divided by $a-b$, so we guess that

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{\ln(x+a)}{b-a} + \frac{\ln(x+b)}{a-b} + C$$

If $a = b$, as in part (a)(iv), it appears that

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} + C$$

(c) The CAS verifies our guesses. Now

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} \Rightarrow 1 = A(x+b) + B(x+a)$$

Setting $x = -b$ gives $B = 1/(a-b)$ and setting $x = -a$ gives $A = 1/(b-a)$. So

$$\int \frac{1}{(x+a)(x+b)} dx = \int \left[\frac{1/(b-a)}{x+a} + \frac{1/(a-b)}{x+b} \right] dx = \frac{\ln|x+a|}{b-a} + \frac{\ln|x+b|}{a-b} + C$$

and our guess for $a \neq b$ is correct.

If $a = b$, then $\frac{1}{(x+a)(x+b)} = \frac{1}{(x+a)^2} = (x+a)^{-2}$. Letting $u = x+a \Rightarrow du = dx$, we have

$$\int (x+a)^{-2} dx = \int u^{-2} du = -\frac{1}{u} + C = -\frac{1}{x+a} + C, \text{ and our guess for } a = b \text{ is also correct.}$$

2. (a) (i) $\int \sin x \cos 2x dx = \frac{\cos x}{2} - \frac{\cos 3x}{6}$
 (ii) $\int \sin 3x \cos 7x dx = \frac{\cos 4x}{8} - \frac{\cos 10x}{20}$
 (iii) $\int \sin 8x \cos 3x dx = -\frac{\cos 11x}{22} - \frac{\cos 5x}{10}$

(b) Looking at the sums and differences of a and b in part (a), we guess that

$$\int \sin ax \cos bx dx = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

Note that $\cos((a-b)x) = \cos((b-a)x)$.

(c) The CAS verifies our guess. To integrate directly, we can use Formula 2(a) from Section 8.2.

$$\begin{aligned} \int \sin ax \cos bx dx &= \int \left\{ \frac{1}{2} [\sin(ax-bx) + \sin(ax+bx)] \right\} dx = \frac{1}{2} \int [\sin((a-b)x) + \sin((a+b)x)] dx \\ &= \frac{1}{2} \left[-\frac{\cos((a-b)x)}{b-a} - \frac{\cos((a+b)x)}{a+b} \right] + C = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C \end{aligned}$$

Our formula is valid for $a \neq b$.

3. (a) (i) $\int \ln x dx = x \ln x - x$
 (ii) $\int x \ln x dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$
 (iii) $\int x^2 \ln x dx = \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3$
 (iv) $\int x^3 \ln x dx = \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4$
 (v) $\int x^7 \ln x dx = \frac{1}{8} x^8 \ln x - \frac{1}{64} x^8$

(b) We guess that $\int x^n \ln x dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1}$.

(c) Let $u = \ln x$, $dv = x^n dx \Rightarrow du = \frac{dx}{x}$, $v = \frac{1}{n+1} x^{n+1}$.

$$\begin{aligned} \text{Then } \int x^n \ln x dx &= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \int x^n dx \\ &= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \cdot \frac{1}{n+1} x^{n+1}, \end{aligned}$$

which verifies our guess. We must have $n+1 \neq 0 \Leftrightarrow n \neq -1$.

4. (a) (i) $\int x e^x dx = e^x(x - 1)$
 (ii) $\int x^2 e^x dx = e^x(x^2 - 2x + 2)$
 (iii) $\int x^3 e^x dx = e^x(x^3 - 3x^2 + 6x - 6)$
 (iv) $\int x^4 e^x dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$
 (v) $\int x^5 e^x dx = e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)$

(b) Notice from part (a) that we can write

$$\int x^4 e^x dx = e^x(x^4 - 4x^3 + 4 \cdot 3x^2 - 4 \cdot 3 \cdot 2x + 4 \cdot 3 \cdot 2 \cdot 1)$$

and

$$\int x^5 e^x dx = e^x(x^5 - 5x^4 + 5 \cdot 4x^3 - 5 \cdot 4 \cdot 3x^2 + 5 \cdot 4 \cdot 3 \cdot 2x - 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$$

So we guess that

$$\begin{aligned} \int x^6 e^x dx &= e^x(x^6 - 6x^5 + 6 \cdot 5x^4 - 6 \cdot 5 \cdot 4x^3 + 6 \cdot 5 \cdot 4 \cdot 3x^2 - 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2x + 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \\ &= e^x(x^6 - 6x^5 + 30x^4 - 120x^3 + 360x^2 - 720x + 720) \end{aligned}$$

The CAS verifies our guess.

(c) From the results in part (a), as well as our prediction in part (b), we speculate that

$$\begin{aligned} \int x^n e^x dx &= e^x[x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \cdots \pm n!x \mp n!] \\ &= e^x \sum_{i=0}^n (-1)^{n-i} \frac{n!}{i!} x^i \end{aligned}$$

(We have reversed the order of the polynomial's terms.)

(d) Let S_n be the statement that $\int x^n e^x dx = e^x \sum_{i=0}^n (-1)^{n-i} \frac{n!}{i!} x^i$.

S_1 is true by part (a)(i). Suppose S_k is true for some k , and consider S_{k+1} . Integrating by parts with $u = x^{k+1}$, $dv = e^x dx \Rightarrow du = (k+1)x^k dx$, $v = e^x$, we get

$$\begin{aligned} \int x^{k+1} e^x dx &= x^{k+1} e^x - (k+1) \int x^k e^x dx \\ &= x^{k+1} e^x - (k+1) \left[e^x \sum_{i=0}^k (-1)^{k-i} \frac{k!}{i!} x^i \right] \\ &= e^x \left[x^{k+1} - (k+1) \sum_{i=0}^k (-1)^{k-i} \frac{k!}{i!} x^i \right] \\ &= e^x \left[x^{k+1} + \sum_{i=0}^k (-1)^{k-i+1} \frac{(k+1)k!}{i!} x^i \right] \\ &= e^x \sum_{i=0}^{k+1} (-1)^{(k+1)-i} \frac{(k+1)!}{i!} x^i \end{aligned}$$

This verifies S_n for $n = k + 1$. Thus, by mathematical induction, S_n is true for all n , where n is a positive integer.

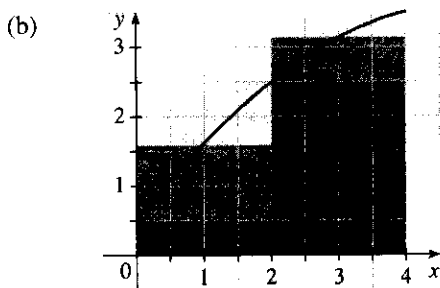
8.7 Approximate Integration

1. (a) $\Delta x = (b - a)/n = (4 - 0)/2 = 2$

$$L_2 = \sum_{i=1}^2 f(x_{i-1}) \Delta x = f(x_0) \cdot 2 + f(x_1) \cdot 2 = 2[f(0) + f(2)] = 2(0.5 + 2.5) = 6$$

$$R_2 = \sum_{i=1}^2 f(x_i) \Delta x = f(x_1) \cdot 2 + f(x_2) \cdot 2 = 2[f(2) + f(4)] = 2(2.5 + 3.5) = 12$$

$$M_2 = \sum_{i=1}^2 f(\bar{x}_i) \Delta x = f(\bar{x}_1) \cdot 2 + f(\bar{x}_2) \cdot 2 = 2[f(1) + f(3)] \approx 2(1.6 + 3.2) = 9.6$$



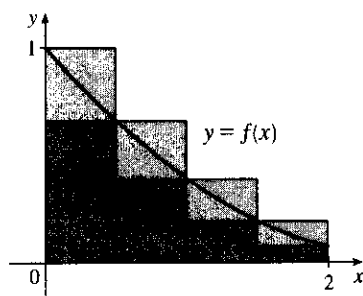
L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I . See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a, b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x) dx$.

(c) $T_2 = (\frac{1}{2} \Delta x)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9$.

This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n , we will have $L_n < T_n < I < M_n < R_n$.

2.



The diagram shows that $L_4 > T_4 > \int_0^2 f(x) dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x) dx$. In fact, for any function that is concave upward, it can be shown that

$$L_n > T_n > \int_0^2 f(x) dx > M_n > R_n.$$

(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that

$$L_n = 0.9540, T_n = 0.8675, M_n = 0.8632, \text{ and } R_n = 0.7811.$$

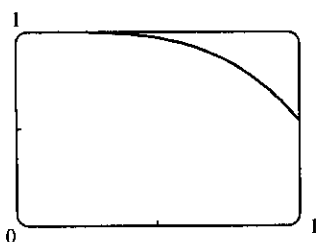
(b) Since $M_n < \int_0^2 f(x) dx < T_n$, we have

$$0.8632 < \int_0^2 f(x) dx < 0.8675.$$

3. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

(a) $T_4 = \frac{1}{4} \cdot \frac{1}{2} [f(0) + 2f(\frac{1}{4}) + 2f(\frac{3}{4}) + f(1)] \approx 0.895759$

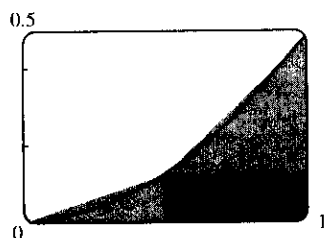
(b) $M_4 = \frac{1}{4} [f(\frac{1}{8}) + f(\frac{3}{8}) + f(\frac{5}{8}) + f(\frac{7}{8})] \approx 0.908907$



The graph shows that f is concave down on $[0, 1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that

$$0.895759 < \int_0^1 \cos(x^2) dx < 0.908907.$$

4.



(a) Since f is increasing on $[0, 1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I . Since f is concave upward on $[0, 1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n , we will have $L_n < M_n < I < T_n < R_n$.

$$(c) L_5 = \sum_{i=1}^5 f(x_{i-1}) \Delta x = \frac{1}{5} [f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$$

$$R_5 = \sum_{i=1}^5 f(x_i) \Delta x = \frac{1}{5} [f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$$

$$M_5 = \sum_{i=1}^5 f(\bar{x}_i) \Delta x = \frac{1}{5} [f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$$

$$T_5 = \left(\frac{1}{2} \Delta x\right) [f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 0.1666$$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

$$5. f(x) = x^2 \sin x, \Delta x = \frac{b-a}{n} = \frac{\pi-0}{8} = \frac{\pi}{8}$$

$$(a) M_8 = \frac{\pi}{8} \left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + \cdots + f\left(\frac{15\pi}{16}\right) \right] \approx 5.932957$$

$$(b) S_8 = \frac{\pi}{8 \cdot 3} \left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{4\pi}{8}\right) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{6\pi}{8}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi) \right] \\ \approx 5.869247$$

$$\text{Actual: } \int_0^\pi x^2 \sin x \, dx \stackrel{84}{=} [-x^2 \cos x]_0^\pi + 2 \int_0^\pi x \cos x \, dx \stackrel{83}{=} [-\pi^2(-1) - 0] + 2[\cos x + x \sin x]_0^\pi \\ = \pi^2 + 2[(-1 + 0) - (1 + 0)] = \pi^2 - 4 \approx 5.869604$$

$$\text{Errors: } E_M = \text{actual} - M_8 = \int_0^\pi x^2 \sin x \, dx - M_8 \approx -0.063353$$

$$E_S = \text{actual} - S_8 = \int_0^\pi x^2 \sin x \, dx - S_8 \approx 0.000357$$

$$6. f(x) = e^{-\sqrt{x}}, \Delta x = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

$$(a) M_6 = \frac{1}{6} \left[f\left(\frac{1}{12}\right) + f\left(\frac{3}{12}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{9}{12}\right) + f\left(\frac{11}{12}\right) \right] \approx 0.525100$$

$$(b) S_6 = \frac{1}{6 \cdot 3} \left[f(0) + 4f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 4f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + 4f\left(\frac{5}{6}\right) + f(1) \right] \approx 0.533979$$

$$\text{Actual: } \int_0^1 e^{-\sqrt{x}} \, dx = \int_0^1 e^u 2u \, du \quad [u = -\sqrt{x}, u^2 = x, 2u \, du = dx] \\ \stackrel{96}{=} 2[(u-1)e^u]_0^{-1} = 2[-2e^{-1} - (-1e^0)] = 2 - 4e^{-1} \approx 0.528482$$

$$\text{Errors: } E_M = \text{actual} - M_6 = \int_0^1 e^{-\sqrt{x}} \, dx - M_6 \approx 0.003382$$

$$E_S = \text{actual} - S_6 = \int_0^1 e^{-\sqrt{x}} \, dx - S_6 \approx -0.005497$$

$$7. f(x) = \sqrt[4]{1+x^2}, \Delta x = \frac{2-0}{8} = \frac{1}{4}$$

$$(a) T_8 = \frac{1}{4 \cdot 2} \left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + \cdots + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2) \right] \approx 2.413790$$

$$(b) M_8 = \frac{1}{4} \left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right) \right] \approx 2.411453$$

$$(c) S_8 = \frac{1}{4 \cdot 3} \left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + 2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2) \right] \approx 2.412232$$

$$8. f(x) = \sin(x^2), \Delta x = \frac{\frac{1}{2} - 0}{4} = \frac{1}{8}$$

$$(a) T_4 = \frac{1}{8 \cdot 2} [f(0) + 2f(\frac{1}{8}) + 2f(\frac{2}{8}) + 2f(\frac{3}{8}) + f(\frac{1}{2})] \approx 0.042743$$

$$(b) M_4 = \frac{1}{8} [f(\frac{1}{16}) + f(\frac{3}{16}) + f(\frac{5}{16}) + f(\frac{7}{16})] \approx 0.040850$$

$$(c) S_4 = \frac{1}{8 \cdot 3} [f(0) + 4f(\frac{1}{8}) + 2f(\frac{2}{8}) + 4f(\frac{3}{8}) + f(\frac{1}{2})] \approx 0.041478$$

$$9. f(x) = \frac{\ln x}{1+x}, \Delta x = \frac{2-1}{10} = \frac{1}{10}$$

$$(a) T_{10} = \frac{1}{10 \cdot 2} [f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 2f(1.9) + f(2)] \approx 0.146879$$

$$(b) M_{10} = \frac{1}{10} [f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.147391$$

$$(c) S_{10} = \frac{1}{10 \cdot 3} [f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)] \approx 0.147219$$

$$10. f(t) = \frac{1}{1+t^2+t^4}, \Delta t = \frac{3-0}{6} = \frac{1}{2}$$

$$(a) T_6 = \frac{1}{2 \cdot 2} [f(0) + 2f(\frac{1}{2}) + 2f(1) + 2f(\frac{3}{2}) + 2f(2) + 2f(\frac{5}{2}) + f(3)] \approx 0.895122$$

$$(b) M_6 = \frac{1}{2} [f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4})] \approx 0.895478$$

$$(c) S_6 = \frac{1}{2 \cdot 3} [f(0) + 4f(\frac{1}{2}) + 2f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + f(3)] \approx 0.898014$$

$$11. f(t) = \sin(e^{t/2}), \Delta t = \frac{\frac{1}{2} - 0}{8} = \frac{1}{16}$$

$$(a) T_8 = \frac{1}{16 \cdot 2} [f(0) + 2f(\frac{1}{16}) + 2f(\frac{2}{16}) + \cdots + 2f(\frac{7}{16}) + f(\frac{1}{2})] \approx 0.451948$$

$$(b) M_8 = \frac{1}{16} [f(\frac{1}{32}) + f(\frac{3}{32}) + f(\frac{5}{32}) + \cdots + f(\frac{13}{32}) + f(\frac{15}{32})] \approx 0.451991$$

$$(c) S_8 = \frac{1}{16 \cdot 3} [f(0) + 4f(\frac{1}{16}) + 2f(\frac{2}{16}) + \cdots + 4f(\frac{7}{16}) + f(\frac{1}{2})] \approx 0.451976$$

$$12. f(x) = \sqrt{1+\sqrt{x}}, \Delta x = \frac{4-0}{8} = \frac{1}{2}$$

$$(a) T_8 = \frac{1}{2 \cdot 2} [f(0) + 2f(\frac{1}{2}) + 2f(1) + \cdots + 2f(3) + 2f(\frac{7}{2}) + f(4)] \approx 6.042985$$

$$(b) M_8 = \frac{1}{2} [f(\frac{1}{4}) + f(\frac{3}{4}) + \cdots + f(\frac{13}{4}) + f(\frac{15}{4})] \approx 6.084778$$

$$(c) S_8 = \frac{1}{2 \cdot 3} [f(0) + 4f(\frac{1}{2}) + 2f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + f(4)] \approx 6.061678$$

$$13. f(x) = e^{1/x}, \Delta x = \frac{2-1}{4} = \frac{1}{4}$$

$$(a) T_4 = \frac{1}{4 \cdot 2} [f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)] \approx 2.031893$$

$$(b) M_4 = \frac{1}{4} [f(1.125) + f(1.375) + f(1.625) + f(1.875)] \approx 2.014207$$

$$(c) S_4 = \frac{1}{4 \cdot 3} [f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] \approx 2.020651$$

$$14. f(x) = \sqrt{x} \sin x, \Delta x = \frac{4-0}{8} = \frac{1}{2}$$

$$(a) T_8 = \frac{1}{2 \cdot 2} \{f(0) + 2[f(\frac{1}{2}) + f(1) + f(\frac{3}{2}) + f(2) + f(\frac{5}{2}) + f(3) + f(\frac{7}{2})] + f(4)\} \approx 1.732865$$

$$(b) M_8 = \frac{1}{2} [f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{5}{4}) + f(\frac{7}{4}) + \cdots + f(\frac{13}{4}) + f(\frac{15}{4})] \approx 1.787427$$

$$(c) S_8 = \frac{1}{2 \cdot 3} [f(0) + 4f(\frac{1}{2}) + 2f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + f(4)] \approx 1.772142$$

$$15. f(x) = \frac{\cos x}{x}, \Delta x = \frac{5-1}{8} = \frac{1}{2}$$

$$(a) T_8 = \frac{1}{2} \cdot \frac{1}{2} [f(1) + 2f(\frac{3}{2}) + 2f(2) + \cdots + 2f(4) + 2f(\frac{9}{2}) + f(5)] \approx -0.495333$$

$$(b) M_8 = \frac{1}{2} [f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4}) + f(\frac{13}{4}) + f(\frac{15}{4}) + f(\frac{17}{4}) + f(\frac{19}{4})] \approx -0.543321$$

$$(c) S_8 = \frac{1}{2} \cdot \frac{1}{2} [f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + 2f(4) + 4f(\frac{9}{2}) + f(5)] \\ \approx -0.526123$$

$$16. f(x) = \ln(x^3 + 2), \Delta x = \frac{6-4}{10} = \frac{1}{5}$$

$$(a) T_{10} = \frac{1}{5} \cdot \frac{1}{2} [f(4) + 2f(4.2) + 2f(4.4) + \cdots + 2f(5.6) + 2f(5.8) + f(6)] \approx 9.649753$$

$$(b) M_{10} = \frac{1}{5} [f(4.1) + f(4.3) + \cdots + f(5.7) + f(5.9)] \approx 9.650912$$

$$(c) S_{10} = \frac{1}{5} \cdot \frac{1}{3} [f(4) + 4f(4.2) + 2f(4.4) + 4f(4.6) + 2f(4.8) + 4f(5) \\ + 2f(5.2) + 4f(5.4) + 2f(5.6) + 4f(5.8) + f(6)] \\ \approx 9.650526$$

$$17. f(y) = \frac{1}{1+y^5}, \Delta y = \frac{3-0}{6} = \frac{1}{2}$$

$$(a) T_6 = \frac{1}{2} \cdot \frac{1}{2} [f(0) + 2f(\frac{1}{2}) + 2f(\frac{2}{2}) + 2f(\frac{3}{2}) + 2f(\frac{4}{2}) + 2f(\frac{5}{2}) + f(3)] \approx 1.064275$$

$$(b) M_6 = \frac{1}{2} [f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4})] \approx 1.067416$$

$$(c) S_6 = \frac{1}{2} \cdot \frac{1}{3} [f(0) + 4f(\frac{1}{2}) + 2f(\frac{2}{2}) + 4f(\frac{3}{2}) + 2f(\frac{4}{2}) + 4f(\frac{5}{2}) + f(3)] \approx 1.074915$$

$$18. f(x) = \frac{e^x}{x}, \Delta x = \frac{4-2}{10} = \frac{1}{5}$$

$$(a) T_{10} = \frac{1}{5} \cdot \frac{1}{2} \{f(2) + 2[f(2.2) + f(2.4) + f(2.6) + \cdots + f(3.8)] + f(4)\} \approx 14.704592$$

$$(b) M_{10} = \frac{1}{5} [f(2.1) + f(2.3) + f(2.5) + f(2.7) + \cdots + f(3.7) + f(3.9)] \approx 14.662669$$

$$(c) S_{10} = \frac{1}{5} \cdot \frac{1}{3} [f(2) + 4f(2.2) + 2f(2.4) + 4f(2.6) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 14.676696$$

$$19. f(x) = e^{-x^2}, \Delta x = \frac{2-0}{10} = \frac{1}{5}$$

$$(a) T_{10} = \frac{1}{5} \cdot \frac{1}{2} \{f(0) + 2[f(0.2) + f(0.4) + \cdots + f(1.8)] + f(2)\} \approx 0.881839$$

$$M_{10} = \frac{1}{5} [f(0.1) + f(0.3) + f(0.5) + \cdots + f(1.7) + f(1.9)] \approx 0.882202$$

$$(b) f(x) = e^{-x^2}, f'(x) = -2xe^{-x^2}, f''(x) = (4x^2 - 2)e^{-x^2}, f'''(x) = 4x(3 - 2x^2)e^{-x^2}.$$

$$f'''(x) = 0 \Leftrightarrow x = 0 \text{ or } x = \pm\sqrt{\frac{3}{2}}. \text{ So to find the maximum value of } |f''(x)| \text{ on } [0, 2], \text{ we need only}$$

$$\text{consider its values at } x = 0, x = 2, \text{ and } x = \sqrt{\frac{3}{2}}. |f''(0)| = 2, |f''(2)| \approx 0.2564 \text{ and } \left|f''\left(\sqrt{\frac{3}{2}}\right)\right| \approx 0.8925.$$

$$\text{Thus, taking } K = 2, a = 0, b = 2, \text{ and } n = 10 \text{ in Theorem 3, we get } |E_T| \leq 2 \cdot 2^3 / (12 \cdot 10^2) = \frac{1}{75} = 0.01\bar{3}, \\ \text{and } |E_M| \leq |E_T|/2 \leq 0.00\bar{6}.$$

$$(c) \text{ Take } K = 2 \text{ [as in part (b)] in Theorem 3. } |E_T| \leq \frac{K(b-a)^3}{12n^2} \leq 10^{-5} \Leftrightarrow \frac{2(2-0)^3}{12n^2} \leq 10^{-5} \Leftrightarrow$$

$$\frac{3}{4}n^2 \geq 10^5 \Leftrightarrow n \geq 365.1 \dots \Leftrightarrow n \geq 366. \text{ Take } n = 366 \text{ for } T_n. \text{ For } E_M, \text{ again take } K = 2 \text{ in}$$

$$\text{Theorem 3 to get } |E_M| \leq 10^{-5} \Leftrightarrow \frac{3}{2}n^2 \geq 10^5 \Leftrightarrow n \geq 258.2 \Rightarrow n \geq 259. \text{ Take } n = 259 \text{ for } M_n.$$

$$20. (a) T_8 = \frac{1}{8} \cdot \frac{1}{2} \{f(0) + 2[f(\frac{1}{8}) + f(\frac{2}{8}) + \cdots + f(\frac{7}{8})] + f(1)\} \approx 0.902333$$

$$M_8 = \frac{1}{8} [f(\frac{1}{16}) + f(\frac{3}{16}) + f(\frac{5}{16}) + \cdots + f(\frac{15}{16})] = 0.905620$$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x \sin(x^2)$, $f''(x) = -2 \sin(x^2) - 4x^2 \cos(x^2)$. For $0 \leq x \leq 1$, $\sin$ and $\cos$ are positive, so $|f''(x)| = 2 \sin(x^2) + 4x^2 \cos(x^2) \leq 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \leq 1$ and $\cos(x^2) \leq 1$ for all x , and $x^2 \leq 1$ for $0 \leq x \leq 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \leq 6 \cdot 1^3 / (12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \leq \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \leq 4$ for $0 \leq x \leq 1$.]

(c) Using $K = 6$ as in part (b), we have $|E_T| \leq 6 \cdot 1^3 / (12n^2) = 1 / (2n^2) \leq 10^{-5} \Rightarrow 2n^2 \geq 10^5 \Rightarrow n \geq \sqrt{\frac{1}{2} \cdot 10^5}$ or $n \geq 224$. To guarantee that $|E_M| \leq 0.00001$, we need $6 \cdot 1^3 / (24n^2) \leq 10^{-5} \Rightarrow 4n^2 \geq 10^5 \Rightarrow n \geq \sqrt{\frac{1}{4} \cdot 10^5}$ or $n \geq 159$.

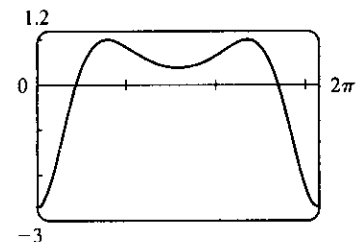
21. (a) $T_{10} = \frac{1}{10 \cdot 2} \{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.71971349$
 $S_{10} = \frac{1}{10 \cdot 3} [f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + \cdots + 4f(0.9) + f(1)] \approx 1.71828278$
 Since $I = \int_0^1 e^x dx = [e^x]_0^1 = e - 1 \approx 1.71828183$, $E_T = I - T_{10} \approx -0.00143166$ and $E_S = I - S_{10} \approx -0.00000095$.

(b) $f(x) = e^x \Rightarrow f''(x) = e^x \leq e$ for $0 \leq x \leq 1$. Taking $K = e$, $a = 0$, $b = 1$, and $n = 10$ in Theorem 3, we get $|E_T| \leq e(1)^3 / (12 \cdot 10^2) \approx 0.002265 > 0.00143166$ [actual $|E_T|$ from (a)]. $f^{(4)}(x) = e^x < e$ for $0 \leq x \leq 1$. Using Theorem 4, we have $|E_S| \leq e(1)^5 / (180 \cdot 10^4) \approx 0.0000015 > 0.00000095$ [actual $|E_S|$ from (a)]. We see that the actual errors are about two-thirds the size of the error estimates.

(c) From part (b), we take $K = e$ to get $|E_T| \leq \frac{K(b-a)^3}{12n^2} \leq 0.00001 \Rightarrow n^2 \geq \frac{e(1^3)}{12(0.00001)} \Rightarrow n \geq 150.5$. Take $n = 151$ for T_n . Now $|E_M| \leq \frac{K(b-a)^3}{24n^2} \leq 0.00001 \Rightarrow n \geq 106.4$. Take $n = 107$ for M_n . Finally, $|E_S| \leq \frac{K(b-a)^5}{180n^4} \leq 0.00001 \Rightarrow n^4 \geq \frac{e(1^5)}{180(0.00001)} \Rightarrow n \geq 6.23$. Take $n = 8$ for S_n (since n has to be even for Simpson's Rule).

22. From Example 7(b), we take $K = 76e$ to get $|E_S| \leq 76e(1)^5 / (180n^4) \leq 0.00001 \Rightarrow n^4 \geq 76e / [180(0.00001)] \Rightarrow n \geq 18.4$. Take $n = 20$ (since n must be even).

23. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x} (\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use student [middlesum].)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \leq \frac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$. With

$$K = 2.8, \text{ we get } |E_M| \leq \frac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916.$$

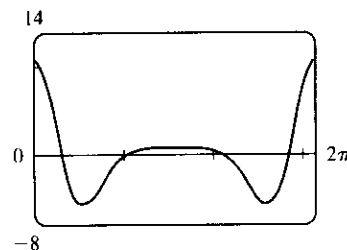
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9} , much less than the estimate in part (c).

- (f) We use the CAS to differentiate twice more, and then graph

$$f^{(4)}(x) = e^{\cos x} (\sin^4 x - 6 \sin^2 x \cos x + 3 - 7 \sin^2 x + \cos x).$$

From the graph, we see that the maximum value of $|f^{(4)}(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



- (g) A CAS gives
- $S_{10} \approx 7.953789422$
- . (In Maple, use
- `student [simpson]`
- .)

- (h) Using Theorem 4 with
- $K = 4e$
- , we get
- $|E_S| \leq \frac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$
- . With
- $K = 10.9$
- , we get

$$|E_S| \leq \frac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814.$$

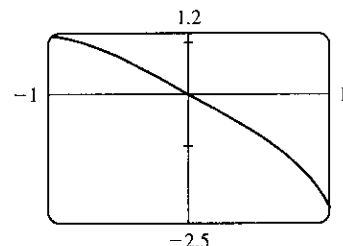
- (i) The actual error is about
- $7.954926521 - 7.953789422 \approx 0.00114$
- . This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

- (j) To ensure that
- $|E_S| \leq 0.0001$
- , we use Theorem 4:
- $|E_S| \leq \frac{4e(2\pi)^5}{180 \cdot n^4} \leq 0.0001 \Rightarrow \frac{4e(2\pi)^5}{180 \cdot 0.0001} \leq n^4 \Rightarrow n^4 \geq 5,915,362 \Leftrightarrow n \geq 49.3$
- . So we must take
- $n \geq 50$
- to ensure that
- $|I - S_n| \leq 0.0001$
- . (
- $K = 10.9$
- leads to the same value of
- n
- .)

24. (a) Using the CAS, we differentiate
- $f(x) = \sqrt{4 - x^3}$
- twice,

$$\text{and find that } f''(x) = -\frac{9x^4}{4(4 - x^3)^{3/2}} - \frac{3x}{(4 - x^3)^{1/2}}.$$

From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.



- (b) A CAS gives
- $M_{10} \approx 3.995804152$
- . (In Maple, use
- `student [middlesum]`
- .)

- (c) Using Theorem 3 for the Midpoint Rule, with
- $K = 2.2$
- , we get
- $|E_M| \leq \frac{2.2[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$
- .

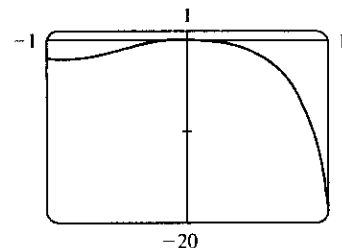
- (d) A CAS gives
- $I \approx 3.995487677$
- .

- (e) The actual error is about
- -0.0003165
- , much less than the estimate in part (c).

- (f) We use the CAS to differentiate twice more, and then graph

$$f^{(4)}(x) = \frac{9}{16} \frac{x^2(x^6 - 224x^3 - 1280)}{(4 - x^3)^{7/2}}.$$

From the graph, we see that $|f^{(4)}(x)| < 18.1$ on $[-1, 1]$.



- (g) A CAS gives
- $S_{10} \approx 3.995449790$
- . (In Maple, use
- `student [simpson]`
- .)

- (h) Using Theorem 4 with
- $K = 18.1$
- , we get
- $|E_S| \leq \frac{18.1[1 - (-1)]^5}{180 \cdot 10^4} \approx 0.000322$
- .

- (i) The actual error is about
- $3.995487677 - 3.995449790 \approx 0.0000379$
- . This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $|E_S| \leq 0.0001$, we use Theorem 4: $|E_S| \leq \frac{18.1(2)^5}{180 \cdot n^4} \leq 0.0001 \Rightarrow \frac{18.1(2)^5}{180 \cdot 0.0001} \leq n^4 \Rightarrow n^4 \geq 32,178 \Rightarrow n \geq 13.4$. So we must take $n \geq 14$ to ensure that $|I - S_n| \leq 0.0001$.

25. $I = \int_0^1 x^3 dx = \left[\frac{1}{4}x^4\right]_0^1 = 0.25$. $f(x) = x^3$.

$n = 4$: $L_4 = \frac{1}{4} \left[0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{2}{4}\right)^3 + \left(\frac{3}{4}\right)^3 \right] = 0.140625$

$R_4 = \frac{1}{4} \left[\left(\frac{1}{4}\right)^3 + \left(\frac{2}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3 \right] = 0.390625$

$T_4 = \frac{1}{4 \cdot 2} \left[0^3 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 1^3 \right] = 0.265625$,

$M_4 = \frac{1}{4} \left[\left(\frac{1}{8}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{7}{8}\right)^3 \right] = 0.2421875$,

$E_L = I - L_4 = \frac{1}{4} - 0.140625 = 0.109375$, $E_R = \frac{1}{4} - 0.390625 = -0.140625$,

$E_T = \frac{1}{4} - 0.265625 = -0.015625$, $E_M = \frac{1}{4} - 0.2421875 = 0.0078125$

$n = 8$: $L_8 = \frac{1}{8} \left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right) \right] \approx 0.191406$

$R_8 = \frac{1}{8} \left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right) + f(1) \right] \approx 0.316406$

$T_8 = \frac{1}{8 \cdot 2} \{ f(0) + 2[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)] + f(1) \} \approx 0.253906$

$M_8 = \frac{1}{8} \left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + \cdots + f\left(\frac{13}{16}\right) + f\left(\frac{15}{16}\right) \right] = 0.248047$

$E_L \approx \frac{1}{4} - 0.191406 \approx 0.058594$, $E_R \approx \frac{1}{4} - 0.316406 \approx -0.066406$,

$E_T \approx \frac{1}{4} - 0.253906 \approx -0.003906$, $E_M \approx \frac{1}{4} - 0.248047 \approx 0.001953$.

$n = 16$: $L_{16} = \frac{1}{16} \left[f(0) + f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right) \right] \approx 0.219727$

$R_{16} = \frac{1}{16} \left[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right) + f(1) \right] \approx 0.282227$

$T_{16} = \frac{1}{16 \cdot 2} \{ f(0) + 2[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right)] + f(1) \} \approx 0.250977$

$M_{16} = \frac{1}{16} \left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + \cdots + f\left(\frac{31}{32}\right) \right] \approx 0.249512$

$E_L \approx \frac{1}{4} - 0.219727 \approx 0.030273$, $E_R \approx \frac{1}{4} - 0.282227 \approx -0.032227$,

$E_T \approx \frac{1}{4} - 0.250977 \approx -0.000977$, $E_M \approx \frac{1}{4} - 0.249512 \approx 0.000488$.

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

n	E_L	E_R	E_T	E_M
4	0.109375	-0.140625	-0.015625	0.007813
8	0.058594	-0.066406	-0.003906	0.001953
16	0.030273	-0.032227	-0.000977	0.000488

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M .
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

$$26. \int_0^2 e^x dx = [e^x]_0^2 = e^2 - 1 \approx 6.389056. \quad f(x) = e^x$$

$$n = 4: \quad \Delta x = (2 - 0)/4 = \frac{1}{2}$$

$$L_4 = \frac{1}{2} [e^0 + e^{1/2} + e^1 + e^{3/2}] \approx 4.924346$$

$$R_4 = \frac{1}{2} [e^{1/2} + e^1 + e^{3/2} + e^2] \approx 8.118874$$

$$T_4 = \frac{1}{2 \cdot 2} [e^0 + 2e^{1/2} + 2e^1 + 2e^{3/2} + e^2] \approx 6.521610$$

$$M_4 = \frac{1}{2} [e^{1/4} + e^{3/4} + e^{5/4} + e^{7/4}] \approx 6.322986.$$

$$E_L \approx 6.389056 - 4.924346 \approx 1.464710, \quad E_R \approx 6.389056 - 8.118874 = -1.729818,$$

$$E_T \approx 6.389056 - 6.521610 \approx -0.132554, \quad E_M \approx 6.389056 - 6.322986 = 0.0660706.$$

$$n = 8: \quad \Delta x = (2 - 0)/8 = \frac{1}{4}$$

$$L_8 = \frac{1}{4} [e^0 + e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4}] \approx 5.623666$$

$$R_8 = \frac{1}{4} [e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4} + e^2] \approx 7.220930$$

$$T_8 = \frac{1}{4 \cdot 2} [e^0 + 2e^{1/4} + 2e^{1/2} + 2e^{3/4} + 2e^1 + 2e^{5/4} + 2e^{3/2} + 2e^{7/4} + e^2] \approx 6.422298$$

$$M_8 = \frac{1}{4} [e^{1/8} + e^{3/8} + e^{5/8} + e^{7/8} + e^{9/8} + e^{11/8} + e^{13/8} + e^{15/8}] \approx 6.372448$$

$$E_L \approx 6.389056 - 5.623666 \approx 0.765390, \quad E_R \approx 6.389056 - 7.220930 \approx -0.831874,$$

$$E_T \approx 6.389056 - 6.422298 \approx -0.033242, \quad E_M \approx 6.389056 - 6.372448 \approx 0.016608.$$

$$n = 16: \quad \Delta x = (2 - 0)/16 = \frac{1}{8}$$

$$L_{16} = \frac{1}{8} [f(0) + f(\frac{1}{8}) + f(\frac{2}{8}) + \cdots + f(\frac{14}{8}) + f(\frac{15}{8})] \approx 5.998057$$

$$R_{16} = \frac{1}{8} [f(\frac{1}{8}) + f(\frac{2}{8}) + f(\frac{3}{8}) + \cdots + f(\frac{15}{8}) + f(2)] \approx 6.796689$$

$$T_{16} = \frac{1}{8 \cdot 2} [f(0) + 2[f(\frac{1}{8}) + f(\frac{2}{8}) + f(\frac{3}{8}) + \cdots + f(\frac{15}{8})] + f(2)] \approx 6.397373$$

$$M_{16} = \frac{1}{8} [f(\frac{1}{16}) + f(\frac{3}{16}) + f(\frac{5}{16}) + \cdots + f(\frac{29}{16}) + f(\frac{31}{16})] \approx 6.384899$$

$$E_L \approx 6.389056 - 5.998057 \approx 0.390999, \quad E_R \approx 6.389056 - 6.796689 \approx -0.407633,$$

$$E_T \approx 6.389056 - 6.397373 \approx -0.008317, \quad E_M \approx 6.389056 - 6.384899 \approx 0.004158.$$

n	E_L	E_R	E_T	E_M
4	1.464710	-1.729818	-0.132554	0.066071
8	0.765390	-0.831874	-0.033242	0.016608
16	0.390999	-0.407633	-0.008317	0.004158

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M .
2. As n is doubled, E_L and E_R are decreased by a factor of about 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

$$27. \int_1^4 \sqrt{x} \, dx = \left[\frac{2}{3} x^{3/2} \right]_1^4 = \frac{2}{3} (8 - 1) = \frac{14}{3} \approx 4.666667$$

$$n = 6: \quad \Delta x = (4 - 1)/6 = \frac{1}{2}$$

$$T_6 = \frac{1}{2 \cdot 2} [\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4}] \approx 4.661488$$

$$M_6 = \frac{1}{2} [\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75}] \approx 4.669245$$

$$S_6 = \frac{1}{2 \cdot 3} [\sqrt{1} + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + 2\sqrt{3} + 4\sqrt{3.5} + \sqrt{4}] \approx 4.666563$$

$$E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178, \quad E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578,$$

$$E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104.$$

$$n = 12: \quad \Delta x = (4 - 1)/12 = \frac{1}{4}$$

$$T_{12} = \frac{1}{4 \cdot 2} (f(1) + 2[f(1.25) + f(1.5) + \cdots + f(3.5) + f(3.75)] + f(4)) \approx 4.665367$$

$$M_{12} = \frac{1}{4} [f(1.125) + f(1.375) + f(1.625) + \cdots + f(3.875)] \approx 4.667316$$

$$S_{12} = \frac{1}{4 \cdot 3} [f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + \cdots + 4f(3.75) + f(4)] \approx 4.666659$$

$$E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300, \quad E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649,$$

$$E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007.$$

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of T_n , M_n , and S_n with the rounded value of the actual integral.

n	T_n	M_n	S_n
6	4.661488	4.669245	4.666563
12	4.665367	4.667316	4.666659

n	E_T	E_M	E_S
6	0.005178	-0.002578	0.000104
12	0.001300	-0.000649	0.000007

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

$$28. I = \int_{-1}^2 x e^x \, dx = [x e^x - e^x]_{-1}^2 = e^2 + 2/e \approx 8.124815. \quad f(x) = x e^x.$$

$$n = 6: \quad \Delta x = [2 - (-1)]/6 = \frac{1}{2}$$

$$T_6 = \frac{1}{2 \cdot 2} \{f(-1) + 2[f(-0.5) + f(0) + \cdots + f(1.5)] + f(2)\} \approx 8.583514$$

$$M_6 = \frac{1}{2} [f(-0.75) + f(-0.25) + \cdots + f(1.75)] \approx 7.896632$$

$$S_6 = \frac{1}{2 \cdot 3} [f(-1) + 4f(-0.5) + 2f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] \approx 8.136885$$

$$E_T \approx I - 8.583514 \approx -0.458699, \quad E_M \approx I - 7.896632 \approx 0.228183,$$

$$E_S \approx I - 8.136885 \approx -0.012070.$$

$$n = 12: \quad \Delta x = [2 - (-1)]/12 = \frac{1}{4}$$

$$T_{12} = \frac{1}{4 \cdot 2} \{f(-1) + 2[f(-0.75) + f(-0.5) + \cdots + f(1.75)] + f(2)\} \approx 8.240073$$

$$M_{12} = \frac{1}{4} [f(-\frac{7}{8}) + f(-\frac{5}{8}) + \cdots + f(\frac{13}{8}) + f(\frac{15}{8})] \approx 8.067259$$

$$S_{12} = \frac{1}{4 \cdot 3} [f(-1) + 4f(-0.75) + 2f(-0.5) + \cdots + 2f(1.5) + 4f(1.75) + f(2)] \approx 8.125593$$

$$E_T \approx I - 8.240073 \approx -0.115258, E_M \approx I - 8.067259 \approx 0.057556,$$

$$E_S \approx I - 8.125593 \approx -0.000778$$

n	T_n	M_n	S_n
6	8.583514	7.896632	8.136885
12	8.240073	8.067259	8.125593

n	E_T	E_M	E_S
6	-0.458699	0.228183	-0.012070
12	-0.115258	0.057556	-0.000778

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

29. $\Delta x = (4 - 0) / 4 = 1$

(a) $T_4 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] \approx \frac{1}{2}[0 + 2(3) + 2(5) + 2(3) + 1] = 11.5$

(b) $M_4 = 1 \cdot [f(0.5) + f(1.5) + f(2.5) + f(3.5)] \approx 1 + 4.5 + 4.5 + 2 = 12$

(c) $S_4 = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx \frac{1}{3}[0 + 4(3) + 2(5) + 4(3) + 1] = 11.\bar{6}$

30. If x = distance from left end of pool and $w = w(x)$ = width at x , then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives Area = $\int_0^{16} w \, dx \approx \frac{2}{3}[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0] \approx 84 \text{ m}^2$.

31. (a) We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n = 8/2 = 4$ and $\Delta x = (3.2 - 0)/4 = 0.8$.

$$\begin{aligned} \int_0^{3.2} f(x) \, dx &\approx M_4 = 0.8[f(0.4) + f(1.2) + f(2.0) + f(2.8)] \\ &= 0.8[6.5 + 6.4 + 7.6 + 8.8] \\ &= 0.8(29.3) = 23.44 \end{aligned}$$

(b) $-4 \leq f''(x) \leq 1 \Rightarrow |f''(x)| \leq 4$, so use $K = 4$, $a = 0$, $b = 3.2$, and $n = 4$ in Theorem 3. So

$$|E_M| \leq \frac{4(3.2 - 0)^3}{24(4)^2} = \frac{128}{375} = 0.341\bar{3}.$$

32. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned} \text{distance} &= \int_0^5 v(t) \, dt \approx S_{10} = \frac{1}{2 \cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \frac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\ &\quad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\ &= \frac{1}{6}(268.41) = 44.735 \text{ m} \end{aligned}$$

33. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t) \, dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = (6 - 0)/6 = 1$ to estimate this integral:

$$\begin{aligned} \int_0^6 a(t) \, dt &\approx S_6 = \frac{1}{3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)] \\ &\approx \frac{1}{3}[0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \frac{1}{3}(113.2) = 37.7\bar{3} \text{ ft/s} \end{aligned}$$

34. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t) dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 r(t) dt &\approx S_6 = \frac{1}{3}[r(0) + 4r(1) + 2r(2) + 4r(3) + 2r(4) + 4r(5) + r(6)] \\ &\approx \frac{1}{3}[4 + 4(3) + 2(2.4) + 4(1.9) + 2(1.4) + 4(1.1) + 1] \\ &= \frac{1}{3}(36.6) = 12.2 \text{ liters}\end{aligned}$$

The function values were obtained from a high-resolution graph.

35. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t) dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = (6 - 0)/12 = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned}\int_0^6 P(t) dt &\approx S_{12} = \frac{1}{3}[P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + 4P(2.5) \\ &\quad + 2P(3) + 4P(3.5) + 2P(4) + 4P(4.5) + 2P(5) + 4P(5.5) + P(6)] \\ &= \frac{1}{6}[1814 + 4(1735) + 2(1686) + 4(1646) + 2(1637) + 4(1609) + 2(1604) \\ &\quad + 4(1611) + 2(1621) + 4(1666) + 2(1745) + 4(1886) + 2052] \\ &= \frac{1}{6}(61,064) = 10,177.\bar{3} \text{ megawatt-hours.}\end{aligned}$$

36. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t) dt \times 3600$ [since $D(t)$ is measured in megabits per second and t is in hours]. We use Simpson's Rule with $n = 8$ and $\Delta t = (8 - 0)/8 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^8 D(t) dt &\approx S_8 = \frac{1}{3}[D(0) + 4D(1) + 2D(2) + 4D(3) + 2D(4) + 4D(5) + 2D(6) + 4D(7) + D(8)] \\ &\approx \frac{1}{3}[0.35 + 4(0.32) + 2(0.41) + 4(0.50) + 2(0.51) + 4(0.56) + 2(0.56) + 4(0.83) + 0.88] \\ &= \frac{1}{3}(13.03) = 4.34\bar{3}\end{aligned}$$

Now multiply by 3600 to obtain 15,636 megabits.

37. Let $y = f(x)$ denote the curve. Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x) dx = 2\pi \int_2^{10} x f(x) dx = 2\pi I$. Now use Simpson's Rule to approximate I :

$$\begin{aligned}I &\approx S_8 = \frac{10-2}{3(8)} [2f(2) + 4 \cdot 3f(3) + 2 \cdot 4f(4) + 4 \cdot 5f(5) + 2 \cdot 6f(6) \\ &\quad + 4 \cdot 7f(7) + 2 \cdot 8f(8) + 4 \cdot 9f(9) + 10f(10)] \\ &\approx \frac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)] \\ &= \frac{1}{3}(395.2)\end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

38. $\text{Work} = \int_0^{18} f(x) dx \approx S_6 = \frac{18-0}{6 \cdot 3} [f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18)]$
 $= 1 \cdot [9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4] = 148 \text{ joules}$

39. $\text{Volume} = \pi \int_0^2 (\sqrt[3]{1+x^3})^2 dx = \pi \int_0^2 (1+x^3)^{2/3} dx$. $V \approx \pi \cdot S_{10}$ where $f(x) = (1+x^3)^{2/3}$ and $\Delta x = (2 - 0)/10 = \frac{1}{5}$. Therefore,

$$\begin{aligned}V &\approx \pi \cdot S_{10} = \pi \cdot \frac{1}{5 \cdot 3} [f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1) \\ &\quad + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \approx 12.325078\end{aligned}$$

40. Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8 \text{ m/s}^2$, $k^2 = \sin^2(\frac{1}{2}\theta_0)$, and $f(x) = 1/\sqrt{1 - k^2 \sin^2 x}$, we get

$$\begin{aligned} T &= 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \approx 4 \sqrt{\frac{L}{g}} S_{10} \\ &= 4 \sqrt{\frac{1}{9.8}} \left(\frac{\pi/2}{10 \cdot 3} \right) [f(0) + 4f(\frac{\pi}{20}) + 2f(\frac{2\pi}{20}) + \cdots + 4f(\frac{9\pi}{20}) + f(\frac{\pi}{2})] \approx 2.07665 \end{aligned}$$

41. $I(\theta) = \frac{N^2 \sin^2 k}{k^2}$, where $k = \frac{\pi N d \sin \theta}{\lambda}$, $N = 10,000$, $d = 10^{-4}$, and $\lambda = 632.8 \times 10^{-9}$. So
- $$I(\theta) = \frac{(10^4)^2 \sin^2 k}{k^2}, \text{ where } k = \frac{\pi (10^4) (10^{-4}) \sin \theta}{632.8 \times 10^{-9}}. \text{ Now } n = 10 \text{ and } \Delta\theta = \frac{10^{-6} - (-10^{-6})}{10} = 2 \times 10^{-7},$$
- so $M_{10} = 2 \times 10^{-7} [I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

42. $f(x) = \cos(\pi x)$, $\Delta x = \frac{20-0}{10} = 2 \Rightarrow$

$$\begin{aligned} T_{10} &= \frac{2}{2} \{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\} \\ &= 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi] \\ &= 1 + 2(1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + 1 = 20 \end{aligned}$$

The actual value is $\int_0^{20} \cos(\pi x) dx = \frac{1}{\pi} [\sin \pi x]_0^{20} = \frac{1}{\pi} (\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

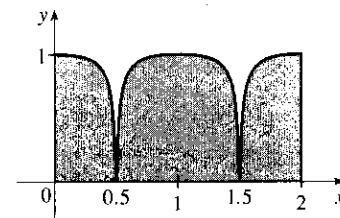
43. Consider the function f whose graph is shown. The area $\int_0^2 f(x) dx$ is close to 2. The Trapezoidal Rule gives

$$T_2 = \frac{2-0}{2-2} [f(0) + 2f(1) + f(2)] = \frac{1}{2} [1 + 2 \cdot 1 + 1] = 2.$$

The Midpoint Rule gives

$$M_2 = \frac{2-0}{2} [f(0.5) + f(1.5)] = 1[0 + 0] = 0,$$

so the Trapezoidal Rule is more accurate.

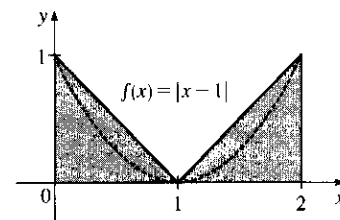


44. Consider the function $f(x) = |x - 1|$, $0 \leq x \leq 2$. The area

$\int_0^2 f(x) dx$ is exactly 1. So is the right endpoint approximation:

$R_2 = f(1) \Delta x + f(2) \Delta x = 0 \cdot 1 + 1 \cdot 1 = 1$. But Simpson's Rule approximates f with the parabola $y = (x - 1)^2$, shown dashed, and

$$S_2 = \frac{\Delta x}{3} [f(0) + 4f(1) + f(2)] = \frac{1}{3} [1 + 4 \cdot 0 + 1] = \frac{2}{3}.$$



45. Since the Trapezoidal and Midpoint approximations on the interval $[a, b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \leq x \leq b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x) dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x) dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a, b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x) dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x) dx$. Thus, $T_n < \int_a^b f(x) dx < M_n$.

46. Let f be a polynomial of degree ≤ 3 ; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\begin{aligned}\int_{-h}^h f(x) dx &\approx \frac{1}{3}h[f(-h) + 4f(0) + f(h)] \\ &= \frac{1}{3}h[(-Ah^3 + Bh^2 - Ch + D) + 4D + (Ah^3 + Bh^2 + Ch + D)] \\ &= \frac{1}{3}h[2Bh^2 + 6D] = \frac{2}{3}Bh^3 + 2Dh\end{aligned}$$

The exact value of the integral is

$$\begin{aligned}\int_{-h}^h (Ax^3 + Bx^2 + Cx + D) dx &= 2 \int_0^h (Bx^2 + D) dx \quad [\text{by Theorem 5.5.6(a) and (b)}] \\ &= 2\left[\frac{1}{3}Bx^3 + Dx\right]_0^h = \frac{2}{3}Bh^3 + 2Dh\end{aligned}$$

Thus, Simpson's Rule is exact.

47. $T_n = \frac{1}{2} \Delta x [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ and

$M_n = \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_{n-1}) + f(\bar{x}_n)]$, where $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$$\begin{aligned}T_{2n} &= \frac{1}{2} \left(\frac{1}{2} \Delta x \right) [f(x_0) + 2f(\bar{x}_1) + 2f(x_1) + 2f(\bar{x}_2) + 2f(x_2) + \cdots \\ &\quad + 2f(\bar{x}_{n-1}) + 2f(x_{n-1}) + 2f(\bar{x}_n) + f(x_n)]\end{aligned}$$

$$\begin{aligned}\text{so} \quad \frac{1}{2}(T_n + M_n) &= \frac{1}{2}T_n + \frac{1}{2}M_n \\ &= \frac{1}{4} \Delta x [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ &\quad + \frac{1}{4} \Delta x [2f(\bar{x}_1) + 2f(\bar{x}_2) + \cdots + 2f(\bar{x}_{n-1}) + 2f(\bar{x}_n)] \\ &= T_{2n}\end{aligned}$$

48. $T_n = \frac{\Delta x}{2} \left[f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$ and $M_n = \Delta x \sum_{i=1}^n f\left(x_i - \frac{\Delta x}{2}\right)$, so

$$\frac{1}{3}T_n + \frac{2}{3}M_n = \frac{1}{3}(T_n + 2M_n) = \frac{\Delta x}{3 \cdot 2} \left[f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4 \sum_{i=1}^n f\left(x_i - \frac{\Delta x}{2}\right) \right]$$

where $\Delta x = \frac{b-a}{n}$. Let $\delta x = \frac{b-a}{2n}$. Then $\Delta x = 2\delta x$, so

$$\begin{aligned}\frac{1}{3}T_n + \frac{2}{3}M_n &= \frac{\delta x}{3} \left[f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4 \sum_{i=1}^n f(x_i - \delta x) \right] \\ &= \frac{1}{3}\delta x [f(x_0) + 4f(x_1 - \delta x) + 2f(x_1) + 4f(x_2 - \delta x) \\ &\quad + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(x_n - \delta x) + f(x_n)]\end{aligned}$$

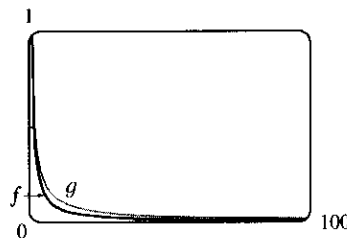
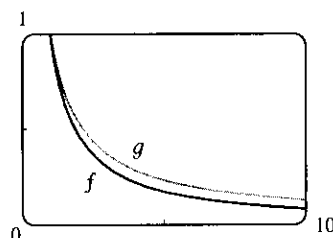
Since $x_0, x_1 - \delta x, x_1, x_2 - \delta x, x_2, \dots, x_{n-1}, x_n - \delta x, x_n$ are the subinterval endpoints for S_{2n} , and since

$\delta x = \frac{b-a}{2n}$ is the width of the subintervals for S_{2n} , the last expression for $\frac{1}{3}T_n + \frac{2}{3}M_n$ is the usual expression for S_{2n} . Therefore, $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

8.8 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4} dx$ has an infinite interval of integration, it is an improper integral of Type I.
- (b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x dx$ is a Type II improper integral.
- (c) Since $y = \frac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\int_0^2 \frac{x}{x^2 - 5x + 6} dx$ is a Type II improper integral.
- (d) Since $\int_{-\infty}^0 \frac{1}{x^2 + 5} dx$ has an infinite interval of integration, it is an improper integral of Type I.
2. (a) Since $y = 1/(2x - 1)$ is defined and continuous on $[1, 2]$, the integral is proper.
- (b) Since $y = \frac{1}{2x - 1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\int_0^1 \frac{1}{2x - 1} dx$ is a Type II improper integral.
- (c) Since $\int_{-\infty}^\infty \frac{\sin x}{1 + x^2} dx$ has an infinite interval of integration, it is an improper integral of Type I.
- (d) Since $y = \ln(x - 1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x - 1) dx$ is a Type II improper integral.
3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is
 $A(t) = \int_1^t x^{-3} dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \leq x \leq 10$ is
 $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \leq x \leq 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for
 $1 \leq x \leq 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \geq 1$ is
 $\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} \left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)

(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$\begin{aligned} F(t) &= \int_1^t f(x) dx = \int_1^t x^{-1.1} dx = \left[-\frac{1}{0.1}x^{-0.1}\right]_1^t \\ &= -10(t^{-0.1} - 1) = 10(1 - t^{-0.1}) \end{aligned}$$

and the area under the graph of g is

$$\begin{aligned} G(t) &= \int_1^t g(x) dx = \int_1^t x^{-0.9} dx = \left[\frac{1}{0.1}x^{0.1}\right]_1^t \\ &= 10(t^{0.1} - 1) \end{aligned}$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow \infty} 10(1 - t^{-0.1}) = 10$.The total area under the graph of g does not exist, since $\lim_{t \rightarrow \infty} G(t) = \lim_{t \rightarrow \infty} 10(t^{0.1} - 1) = \infty$.

$$5. I = \int_1^{\infty} \frac{1}{(3x+1)^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{(3x+1)^2} dx. \text{ Now}$$

$$\int \frac{1}{(3x+1)^2} dx = \frac{1}{3} \int \frac{1}{u^2} du \quad [u = 3x+1, du = 3 dx]$$

$$= -\frac{1}{3u} + C = -\frac{1}{3(3x+1)} + C,$$

$$\text{so } I = \lim_{t \rightarrow \infty} \left[-\frac{1}{3(3x+1)} \right]_1^t = \lim_{t \rightarrow \infty} \left[-\frac{1}{3(3t+1)} + \frac{1}{12} \right] = 0 + \frac{1}{12} = \frac{1}{12}. \quad \text{Convergent}$$

$$6. \int_{-\infty}^0 \frac{1}{2x-5} dx = \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{2x-5} dx = \lim_{t \rightarrow -\infty} \left[\frac{1}{2} \ln |2x-5| \right]_t^0 = \lim_{t \rightarrow -\infty} \left[\frac{1}{2} \ln 5 - \frac{1}{2} \ln |2t-5| \right] = -\infty.$$

Divergent

$$7. \int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}} dw = \lim_{t \rightarrow -\infty} \int_t^{-1} \frac{1}{\sqrt{2-w}} dw = \lim_{t \rightarrow -\infty} \left[-2\sqrt{2-w} \right]_t^{-1} \quad [u = 2-w, du = -dw]$$

$$= \lim_{t \rightarrow -\infty} \left[-2\sqrt{3} + 2\sqrt{2-t} \right] = \infty. \quad \text{Divergent}$$

$$8. \int_0^{\infty} \frac{x}{(x^2+2)^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x}{(x^2+2)^2} dx = \lim_{t \rightarrow \infty} \frac{1}{2} \left[\frac{-1}{x^2+2} \right]_0^t = \frac{1}{2} \lim_{t \rightarrow \infty} \left(\frac{-1}{t^2+2} + \frac{1}{2} \right)$$

$$= \frac{1}{2} \left(0 + \frac{1}{2} \right) = \frac{1}{4}. \quad \text{Convergent}$$

$$9. \int_4^{\infty} e^{-y/2} dy = \lim_{t \rightarrow \infty} \int_4^t e^{-y/2} dy = \lim_{t \rightarrow \infty} \left[-2e^{-y/2} \right]_4^t = \lim_{t \rightarrow \infty} (-2e^{-t/2} + 2e^{-2}) = 0 + 2e^{-2} = 2e^{-2}.$$

Convergent

$$10. \int_{-\infty}^{-1} e^{-2t} dt = \lim_{x \rightarrow -\infty} \int_x^{-1} e^{-2t} dt = \lim_{x \rightarrow -\infty} \left[-\frac{1}{2} e^{-2t} \right]_x^{-1} = \lim_{x \rightarrow -\infty} \left[-\frac{1}{2} e^2 + \frac{1}{2} e^{-2x} \right] = \infty. \quad \text{Divergent}$$

$$11. \int_{-\infty}^{\infty} \frac{x dx}{1+x^2} = \int_{-\infty}^0 \frac{x dx}{1+x^2} + \int_0^{\infty} \frac{x dx}{1+x^2} \text{ and}$$

$$\int_{-\infty}^0 \frac{x dx}{1+x^2} = \lim_{t \rightarrow -\infty} \left[\frac{1}{2} \ln(1+x^2) \right]_t^0 = \lim_{t \rightarrow -\infty} \left[0 - \frac{1}{2} \ln(1+t^2) \right] = -\infty. \quad \text{Divergent}$$

$$12. I = \int_{-\infty}^{\infty} (2-v^4) dv = I_1 + I_2 = \int_{-\infty}^0 (2-v^4) dv + \int_0^{\infty} (2-v^4) dv, \text{ but}$$

$$I_1 = \lim_{t \rightarrow -\infty} \left[2v - \frac{1}{5} v^5 \right]_t^0 = \lim_{t \rightarrow -\infty} \left(-2t + \frac{1}{5} t^5 \right) = -\infty. \text{ Since } I_1 \text{ is divergent, } I \text{ is divergent, and there is no need to evaluate } I_2. \quad \text{Divergent}$$

$$13. \int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^0 x e^{-x^2} dx + \int_0^{\infty} x e^{-x^2} dx.$$

$$\int_{-\infty}^0 x e^{-x^2} dx = \lim_{t \rightarrow -\infty} \left(-\frac{1}{2} \right) \left[e^{-x^2} \right]_t^0 = \lim_{t \rightarrow -\infty} \left(-\frac{1}{2} \right) (1 - e^{-t^2}) = -\frac{1}{2} \cdot 1 = -\frac{1}{2}, \text{ and}$$

$$\int_0^{\infty} x e^{-x^2} dx = \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \right) \left[e^{-x^2} \right]_0^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \right) (e^{-t^2} - 1) = -\frac{1}{2} \cdot (-1) = \frac{1}{2}.$$

$$\text{Therefore, } \int_{-\infty}^{\infty} x e^{-x^2} dx = -\frac{1}{2} + \frac{1}{2} = 0. \quad \text{Convergent}$$

$$14. \int_{-\infty}^{\infty} x^2 e^{-x^3} dx = \int_{-\infty}^0 x^2 e^{-x^3} dx + \int_0^{\infty} x^2 e^{-x^3} dx, \text{ and}$$

$$\int_{-\infty}^0 x^2 e^{-x^3} dx = \lim_{t \rightarrow -\infty} \left[-\frac{1}{3} e^{-x^3} \right]_t^0 = -\frac{1}{3} + \frac{1}{3} \left(\lim_{t \rightarrow -\infty} e^{-t^3} \right) = \infty. \quad \text{Divergent}$$

$$15. \int_{2\pi}^{\infty} \sin \theta d\theta = \lim_{t \rightarrow \infty} \int_{2\pi}^t \sin \theta d\theta = \lim_{t \rightarrow \infty} [-\cos \theta]_{2\pi}^t = \lim_{t \rightarrow \infty} (-\cos t + 1). \text{ This limit does not exist, so the integral is divergent. Divergent}$$

$$16. \int_0^\infty \cos^2 \alpha \, d\alpha = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{2} (1 + \cos 2\alpha) \, d\alpha = \lim_{t \rightarrow \infty} \left[\frac{1}{2} \alpha + \frac{1}{4} \sin 2\alpha \right]_0^t = \lim_{t \rightarrow \infty} \left[\frac{1}{2} t + \frac{1}{4} \sin 2t \right] = \infty \text{ since } \left| \frac{1}{4} \sin 2t \right| \leq \frac{1}{4} \text{ for all } t, \text{ but } \frac{1}{2} t \rightarrow \infty \text{ as } t \rightarrow \infty. \text{ Divergent}$$

$$17. \int_1^\infty \frac{x+1}{x^2+2x} \, dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x} \, dx = \frac{1}{2} \lim_{t \rightarrow \infty} [\ln(x^2+2x)]_1^t = \frac{1}{2} \lim_{t \rightarrow \infty} [\ln(t^2+2t) - \ln 3] = \infty. \text{ Divergent}$$

$$18. \int_0^\infty \frac{dz}{z^2+3z+2} = \lim_{t \rightarrow \infty} \int_0^t \left[\frac{1}{z+1} - \frac{1}{z+2} \right] dz = \lim_{t \rightarrow \infty} \left[\ln \left(\frac{z+1}{z+2} \right) \right]_0^t = \lim_{t \rightarrow \infty} \left[\ln \left(\frac{t+1}{t+2} \right) - \ln \left(\frac{1}{2} \right) \right] = \ln 1 + \ln 2 = \ln 2. \text{ Convergent}$$

$$19. \int_0^\infty s e^{-5s} \, ds = \lim_{t \rightarrow \infty} \int_0^t s e^{-5s} \, ds = \lim_{t \rightarrow \infty} \left[-\frac{1}{5} s e^{-5s} - \frac{1}{25} e^{-5s} \right]_0^t \quad \left[\begin{array}{l} \text{by integration by} \\ \text{parts with } u = s \end{array} \right] \\ = \lim_{t \rightarrow \infty} \left(-\frac{1}{5} t e^{-5t} - \frac{1}{25} e^{-5t} + \frac{1}{25} \right) = 0 - 0 + \frac{1}{25} \quad [\text{by l'Hospital's Rule}] \\ = \frac{1}{25}. \text{ Convergent}$$

$$20. \int_{-\infty}^6 r e^{r/3} \, dr = \lim_{t \rightarrow -\infty} \int_t^6 r e^{r/3} \, dr = \lim_{t \rightarrow -\infty} \left[3r e^{r/3} - 9e^{r/3} \right]_t^6 \quad \left[\begin{array}{l} \text{by integration by} \\ \text{parts with } u = r \end{array} \right] \\ = \lim_{t \rightarrow -\infty} (18e^2 - 9e^2 - 3te^{t/3} + 9e^{t/3}) = 9e^2 - 0 + 0 \quad [\text{by l'Hospital's Rule}] \\ = 9e^2. \text{ Convergent}$$

$$21. \int_1^\infty \frac{\ln x}{x} \, dx = \lim_{t \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^t \quad (\text{by substitution with } u = \ln x, du = dx/x) = \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} = \infty. \text{ Divergent}$$

$$22. \int_{-\infty}^\infty e^{-|x|} \, dx = \int_{-\infty}^0 e^x \, dx + \int_0^\infty e^{-x} \, dx, \int_{-\infty}^0 e^x \, dx = \lim_{t \rightarrow -\infty} [e^x]_t^0 = \lim_{t \rightarrow -\infty} (1 - e^t) = 1, \text{ and} \\ \int_0^\infty e^{-x} \, dx = \lim_{t \rightarrow \infty} [-e^{-x}]_0^t = \lim_{t \rightarrow \infty} (1 - e^{-t}) = 1. \text{ Therefore, } \int_{-\infty}^\infty e^{-|x|} \, dx = 1 + 1 = 2. \text{ Convergent}$$

$$23. \int_{-\infty}^\infty \frac{x^2}{9+x^6} \, dx = \int_{-\infty}^0 \frac{x^2}{9+x^6} \, dx + \int_0^\infty \frac{x^2}{9+x^6} \, dx = 2 \int_0^\infty \frac{x^2}{9+x^6} \, dx \quad [\text{since the integrand is even}].$$

$$\text{Now } \int \frac{x^2 \, dx}{9+x^6} \quad \left[\begin{array}{l} u = x^3 \\ du = 3x^2 \, dx \end{array} \right] = \int \frac{\frac{1}{3} du}{9+u^2} \quad \left[\begin{array}{l} u = 3v \\ du = 3 \, dv \end{array} \right] = \int \frac{\frac{1}{3}(3 \, dv)}{9+9v^2} = \frac{1}{9} \int \frac{dv}{1+v^2} \\ = \frac{1}{9} \tan^{-1} v + C = \frac{1}{9} \tan^{-1} \left(\frac{u}{3} \right) + C = \frac{1}{9} \tan^{-1} \left(\frac{x^3}{3} \right) + C,$$

$$\text{so } 2 \int_0^\infty \frac{x^2}{9+x^6} \, dx = 2 \lim_{t \rightarrow \infty} \int_0^t \frac{x^2}{9+x^6} \, dx = 2 \lim_{t \rightarrow \infty} \left[\frac{1}{9} \tan^{-1} \left(\frac{x^3}{3} \right) \right]_0^t \\ = 2 \lim_{t \rightarrow \infty} \frac{1}{9} \tan^{-1} \left(\frac{t^3}{3} \right) = \frac{2}{9} \cdot \frac{\pi}{2} = \frac{\pi}{9}. \text{ Convergent}$$

$$24. \text{ Integrate by parts with } u = \ln x, dv = dx/x^3 \Rightarrow du = dx/x, v = -1/(2x^2).$$

$$\int_1^\infty \frac{\ln x}{x^3} \, dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x^3} \, dx = \lim_{t \rightarrow \infty} \left(\left[-\frac{1}{2x^2} \ln x \right]_1^t + \frac{1}{2} \int_1^t \frac{1}{x^3} \, dx \right) \\ = \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4} \right) = \frac{1}{4}$$

$$\text{since } \lim_{t \rightarrow \infty} \frac{\ln t}{t^2} = \lim_{t \rightarrow \infty} \frac{1/t}{2t} = \lim_{t \rightarrow \infty} \frac{1}{2t^2} = 0. \text{ Convergent}$$

25. Integrate by parts with $u = \ln x$, $dv = dx/x^2 \Rightarrow du = dx/x$, $v = -1/x$.

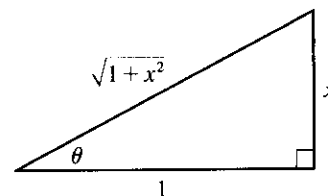
$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^t = \lim_{t \rightarrow \infty} \left(-\frac{\ln t}{t} - \frac{1}{t} + 0 + 1 \right) \\ &= -0 - 0 + 0 + 1 = 1\end{aligned}$$

since $\lim_{t \rightarrow \infty} \frac{\ln t}{t} \stackrel{H}{=} \lim_{t \rightarrow \infty} \frac{1/t}{1} = 0$. Convergent

26. $\int_0^\infty \frac{x \arctan x}{(1+x^2)^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x \arctan x}{(1+x^2)^2} dx$. Let $u = \arctan x$, $dv = \frac{x dx}{(1+x^2)^2}$. Then $du = \frac{dx}{1+x^2}$,

$$v = \frac{1}{2} \int \frac{2x dx}{(1+x^2)^2} = \frac{-1/2}{1+x^2}, \text{ and}$$

$$\begin{aligned}\int \frac{x \arctan x}{(1+x^2)^2} dx &= -\frac{1}{2} \frac{\arctan x}{1+x^2} + \frac{1}{2} \int \frac{dx}{(1+x^2)^2} \quad \left[\begin{array}{l} x = \tan \theta, \\ dx = \sec^2 \theta d\theta \end{array} \right] \\ &= -\frac{1}{2} \frac{\arctan x}{1+x^2} + \frac{1}{2} \int \frac{\sec^2 \theta d\theta}{(\sec^2 \theta)^2} \\ &= -\frac{1}{2} \frac{\arctan x}{1+x^2} + \frac{1}{2} \int \cos^2 \theta d\theta \\ &= -\frac{1}{2} \frac{\arctan x}{1+x^2} + \frac{\theta}{4} + \frac{\sin \theta \cos \theta}{4} + C \\ &= -\frac{1}{2} \frac{\arctan x}{1+x^2} + \frac{1}{4} \arctan x + \frac{1}{4} \frac{x}{1+x^2} + C\end{aligned}$$



It follows that

$$\begin{aligned}\int_0^\infty \frac{x \arctan x}{(1+x^2)^2} dx &= \lim_{t \rightarrow \infty} \left[-\frac{1}{2} \frac{\arctan x}{1+x^2} + \frac{1}{4} \arctan x + \frac{1}{4} \frac{x}{1+x^2} \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \frac{\arctan t}{1+t^2} + \frac{1}{4} \arctan t + \frac{1}{4} \frac{t}{1+t^2} \right) = 0 + \frac{1}{4} \cdot \frac{\pi}{2} + 0 = \frac{\pi}{8}.\end{aligned}$$

Convergent.

27. There is an infinite discontinuity at the left endpoint of $[0, 3]$.

$$\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{t \rightarrow 0^+} \int_t^3 \frac{dx}{\sqrt{x}} = \lim_{t \rightarrow 0^+} [2\sqrt{x}]_t^3 = \lim_{t \rightarrow 0^+} (2\sqrt{3} - 2\sqrt{t}) = 2\sqrt{3}. \text{ Convergent}$$

28. There is an infinite discontinuity at the left endpoint of $[0, 3]$.

$$\int_0^3 \frac{dx}{x\sqrt{x}} = \lim_{t \rightarrow 0^+} \int_t^3 \frac{dx}{x^{3/2}} = \lim_{t \rightarrow 0^+} \left[\frac{-2}{\sqrt{x}} \right]_t^3 = \frac{-2}{\sqrt{3}} + \lim_{t \rightarrow 0^+} \frac{2}{\sqrt{t}} = \infty. \text{ Divergent}$$

29. There is an infinite discontinuity at the right endpoint of $[-1, 0]$.

$$\int_{-1}^0 \frac{dx}{x^2} = \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{dx}{x^2} = \lim_{t \rightarrow 0^-} \left[\frac{-1}{x} \right]_{-1}^t = \lim_{t \rightarrow 0^-} \left[-\frac{1}{t} + \frac{1}{-1} \right] = \infty. \text{ Divergent}$$

30. $\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t \rightarrow 9^-} \int_1^t \frac{dx}{\sqrt[3]{x-9}} = \lim_{t \rightarrow 9^-} \left[\frac{3}{2}(x-9)^{2/3} \right]_1^t = \lim_{t \rightarrow 9^-} \left[\frac{3}{2}(t-9)^{2/3} - \frac{3}{2}(4) \right] = 0 - 6 = -6.$

Convergent

31. $\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, but $\int_{-2}^0 \frac{dx}{x^4} = \lim_{t \rightarrow 0^-} \left[-\frac{x^{-3}}{3} \right]_{-2}^t = \lim_{t \rightarrow 0^-} \left[-\frac{1}{3t^3} - \frac{1}{24} \right] = \infty. \text{ Divergent}$

32. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t \rightarrow 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t \rightarrow 1^-} [\sin^{-1} x]_0^t = \lim_{t \rightarrow 1^-} \sin^{-1} t = \frac{\pi}{2}. \text{ Convergent}$

33. There is an infinite discontinuity at $x = 1$. $\int_0^{33} (x-1)^{-1/5} dx = \int_0^1 (x-1)^{-1/5} dx + \int_1^{33} (x-1)^{-1/5} dx$. Here $\int_0^1 (x-1)^{-1/5} dx = \lim_{t \rightarrow 1^-} \int_0^t (x-1)^{-1/5} dx = \lim_{t \rightarrow 1^-} \left[\frac{5}{4} (x-1)^{4/5} \right]_0^t = \lim_{t \rightarrow 1^-} \left[\frac{5}{4} (t-1)^{4/5} - \frac{5}{4} \right] = -\frac{5}{4}$ and $\int_1^{33} (x-1)^{-1/5} dx = \lim_{t \rightarrow 1^+} \int_t^{33} (x-1)^{-1/5} dx = \lim_{t \rightarrow 1^+} \left[\frac{5}{4} (x-1)^{4/5} \right]_t^{33} = \lim_{t \rightarrow 1^+} \left[\frac{5}{4} \cdot 16 - \frac{5}{4} (t-1)^{4/5} \right] = 20$. Thus, $\int_0^{33} (x-1)^{-1/5} dx = -\frac{5}{4} + 20 = \frac{75}{4}$. Convergent

34. $f(y) = 1/(4y-1)$ has an infinite discontinuity at $y = \frac{1}{4}$.

$$\begin{aligned} \int_{1/4}^1 \frac{1}{4y-1} dy &= \lim_{t \rightarrow (1/4)^+} \int_t^1 \frac{1}{4y-1} dy = \lim_{t \rightarrow (1/4)^+} \left[\frac{1}{4} \ln |4y-1| \right]_t^1 \\ &= \lim_{t \rightarrow (1/4)^+} \left[\frac{1}{4} \ln 3 - \frac{1}{4} \ln(4t-1) \right] = \infty \end{aligned}$$

so $\int_{1/4}^1 \frac{1}{4y-1} dy$ diverges, and hence, $\int_0^1 \frac{1}{4y-1} dy$ diverges. Divergent

35. $\int_0^\pi \sec x dx = \int_0^{\pi/2} \sec x dx + \int_{\pi/2}^\pi \sec x dx$. $\int_0^{\pi/2} \sec x dx = \lim_{t \rightarrow \pi/2^-} \int_0^t \sec x dx$
 $= \lim_{t \rightarrow \pi/2^-} \left[\ln |\sec x + \tan x| \right]_0^t = \lim_{t \rightarrow \pi/2^-} \ln |\sec t + \tan t| = \infty$. Divergent

36. $\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \frac{dx}{(x+3)(x-2)} = \int_0^2 \frac{dx}{(x-2)(x+3)} + \int_2^4 \frac{dx}{(x-2)(x+3)}$, and
 $\int_0^2 \frac{dx}{(x-2)(x+3)} = \lim_{t \rightarrow 2^-} \int_0^t \left[\frac{1/5}{x-2} - \frac{1/5}{x+3} \right] dx$ [partial fractions] $= \lim_{t \rightarrow 2^-} \left[\frac{1}{5} \ln \left| \frac{x-2}{x+3} \right| \right]_0^t$
 $= \lim_{t \rightarrow 2^-} \frac{1}{5} \left[\ln \left| \frac{t-2}{t+3} \right| - \ln \frac{2}{3} \right] = -\infty$. Divergent

37. There is an infinite discontinuity at $x = 0$. $\int_{-1}^1 \frac{e^x}{e^x-1} dx = \int_{-1}^0 \frac{e^x}{e^x-1} dx + \int_0^1 \frac{e^x}{e^x-1} dx$.
 $\int_{-1}^0 \frac{e^x}{e^x-1} dx = \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{e^x}{e^x-1} dx = \lim_{t \rightarrow 0^-} \left[\ln |e^x-1| \right]_{-1}^t = \lim_{t \rightarrow 0^-} \left[\ln |e^t-1| - \ln |e^{-1}-1| \right] = -\infty$,
 so $\int_{-1}^1 \frac{e^x}{e^x-1} dx$ is divergent. The integral $\int_0^1 \frac{e^x}{e^x-1} dx$ also diverges since
 $\int_0^1 \frac{e^x}{e^x-1} dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{e^x}{e^x-1} dx = \lim_{t \rightarrow 0^+} \left[\ln |e^x-1| \right]_t^1 = \lim_{t \rightarrow 0^+} \left[\ln |e-1| - \ln |e^t-1| \right] = \infty$.
 Divergent

38. $\int_0^2 \frac{x-3}{2x-3} dx = \int_0^{3/2} \frac{x-3}{2x-3} dx + \int_{3/2}^2 \frac{x-3}{2x-3} dx$ and
 $\int \frac{x-3}{2x-3} dx = \frac{1}{2} \int \frac{2x-6}{2x-3} dx = \frac{1}{2} \int \left[1 - \frac{3}{2x-3} \right] dx = \frac{1}{2} x - \frac{3}{4} \ln |2x-3| + C$, so
 $\int_0^{3/2} \frac{x-3}{2x-3} dx = \lim_{t \rightarrow 3/2^-} \left[\frac{1}{2} t - \frac{3}{4} \ln |2t-3| \right]_0^t = \infty$. Divergent

$$\begin{aligned}
 39. I &= \int_0^2 z^2 \ln z \, dz = \lim_{t \rightarrow 0^+} \int_t^2 z^2 \ln z \, dz \stackrel{101}{=} \lim_{t \rightarrow 0^+} \left[\frac{z^3}{3^2} (3 \ln z - 1) \right]_t^2 \\
 &= \lim_{t \rightarrow 0^+} \left[\frac{8}{9} (3 \ln 2 - 1) - \frac{1}{9} t^3 (3 \ln t - 1) \right] = \frac{8}{3} \ln 2 - \frac{8}{9} - \frac{1}{9} \lim_{t \rightarrow 0^+} [t^3 (3 \ln t - 1)] = \frac{8}{3} \ln 2 - \frac{8}{9} - \frac{1}{9} L.
 \end{aligned}$$

$$\text{Now } L = \lim_{t \rightarrow 0^+} [t^3 (3 \ln t - 1)] = \lim_{t \rightarrow 0^+} \frac{3 \ln t - 1}{t^{-3}} \stackrel{H}{=} \lim_{t \rightarrow 0^+} \frac{3/t}{-3/t^4} = \lim_{t \rightarrow 0^+} (-t^3) = 0. \text{ Thus, } L = 0 \text{ and}$$

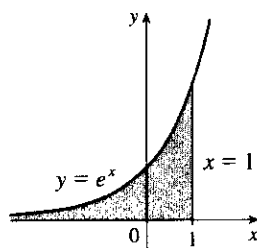
$$I = \frac{8}{3} \ln 2 - \frac{8}{9}. \quad \text{Convergent}$$

$$40. \text{ Integrate by parts with } u = \ln x, dv = dx/\sqrt{x} \Rightarrow du = dx/x, v = 2\sqrt{x}.$$

$$\begin{aligned}
 \int_0^1 \frac{\ln x}{\sqrt{x}} \, dx &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}} \, dx = \lim_{t \rightarrow 0^+} \left([2\sqrt{x} \ln x]_t^1 - 2 \int_t^1 \frac{dx}{\sqrt{x}} \right) = \lim_{t \rightarrow 0^+} \left(-2\sqrt{t} \ln t - 4[\sqrt{x}]_t^1 \right) \\
 &= \lim_{t \rightarrow 0^+} (-2\sqrt{t} \ln t - 4 + 4\sqrt{t}) = -4
 \end{aligned}$$

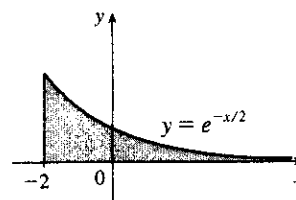
$$\text{since } \lim_{t \rightarrow 0^+} \sqrt{t} \ln t = \lim_{t \rightarrow 0^+} \frac{\ln t}{t^{-1/2}} \stackrel{H}{=} \lim_{t \rightarrow 0^+} \frac{1/t}{-t^{-3/2}/2} = \lim_{t \rightarrow 0^+} (-2\sqrt{t}) = 0. \quad \text{Convergent}$$

41.



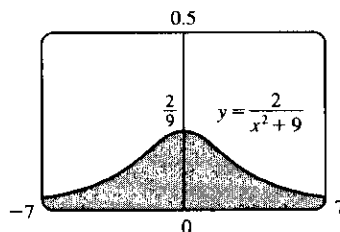
$$\begin{aligned}
 \text{Area} &= \int_{-\infty}^1 e^x \, dx = \lim_{t \rightarrow -\infty} [e^x]_t^1 \\
 &= e - \lim_{t \rightarrow -\infty} e^t = e
 \end{aligned}$$

42.



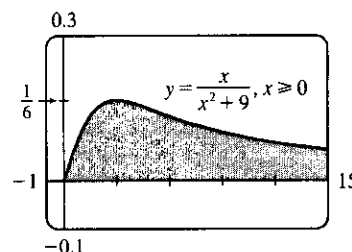
$$\begin{aligned}
 \text{Area} &= \int_{-2}^{\infty} e^{-x/2} \, dx = -2 \lim_{t \rightarrow \infty} [e^{-x/2}]_{-2}^t \\
 &= -2 \lim_{t \rightarrow \infty} e^{-t/2} + 2e = 2e
 \end{aligned}$$

43.



$$\begin{aligned}
 \text{Area} &= \int_{-\infty}^{\infty} \frac{2}{x^2 + 9} \, dx = 2 \cdot 2 \int_0^{\infty} \frac{1}{x^2 + 9} \, dx \\
 &= 4 \lim_{t \rightarrow \infty} \int_0^t \frac{1}{x^2 + 9} \, dx = 4 \lim_{t \rightarrow \infty} \left[\frac{1}{3} \tan^{-1} \frac{x}{3} \right]_0^t \\
 &= \frac{4}{3} \lim_{t \rightarrow \infty} \left[\tan^{-1} \frac{t}{3} - 0 \right] = \frac{4}{3} \cdot \frac{\pi}{2} = \frac{2\pi}{3}
 \end{aligned}$$

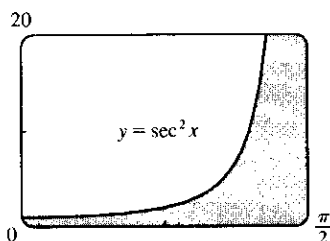
44.



$$\begin{aligned}
 \text{Area} &= \int_0^{\infty} \frac{x}{x^2 + 9} \, dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x}{x^2 + 9} \, dx \\
 &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} \ln(x^2 + 9) \right]_0^t \\
 &= \frac{1}{2} \lim_{t \rightarrow \infty} [\ln(t^2 + 9) - \ln 9] = \infty
 \end{aligned}$$

Infinite area

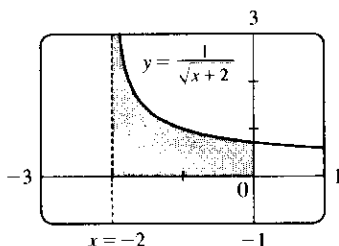
45.



$$\begin{aligned} \text{Area} &= \int_0^{\pi/2} \sec^2 x \, dx = \lim_{t \rightarrow (\pi/2)^-} \int_0^t \sec^2 x \, dx \\ &= \lim_{t \rightarrow (\pi/2)^-} [\tan x]_0^t = \lim_{t \rightarrow (\pi/2)^-} (\tan t - 0) \\ &= \infty \end{aligned}$$

Infinite area

46.



$$\begin{aligned} \text{Area} &= \int_{-2}^0 \frac{1}{\sqrt{x+2}} \, dx = \lim_{t \rightarrow -2^+} \int_t^0 \frac{1}{\sqrt{x+2}} \, dx \\ &= \lim_{t \rightarrow -2^+} [2\sqrt{x+2}]_t^0 = \lim_{t \rightarrow -2^+} (2\sqrt{2} - 2\sqrt{t+2}) \\ &= 2\sqrt{2} - 0 = 2\sqrt{2} \end{aligned}$$

47. (a)

t	$\int_1^t g(x) \, dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

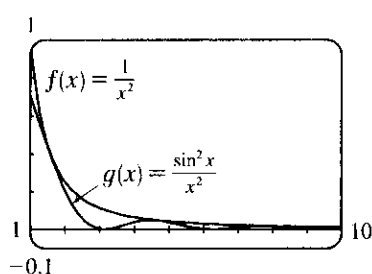
$$g(x) = \frac{\sin^2 x}{x^2}.$$

It appears that the integral is convergent.

$$(b) -1 \leq \sin x \leq 1 \Rightarrow 0 \leq \sin^2 x \leq 1 \Rightarrow 0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}. \text{ Since } \int_1^\infty \frac{1}{x^2} \, dx \text{ is convergent}$$

(Equation 2 with $p = 2 > 1$), $\int_1^\infty \frac{\sin^2 x}{x^2} \, dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x) \, dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x) \, dx$ must be finite; that is, the integral is convergent.

48. (a)

t	$\int_2^t g(x) \, dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10,000	208.124560

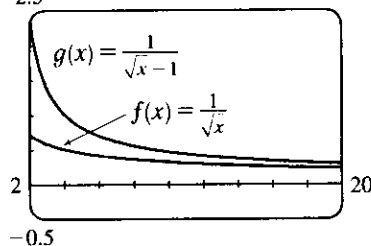
$$g(x) = \frac{1}{\sqrt{x} - 1}.$$

It appears that the integral is divergent.

(b) For $x \geq 2$, $\sqrt{x} > \sqrt{x} - 1 \Rightarrow \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x} - 1}$. Since $\int_2^\infty \frac{1}{\sqrt{x}} dx$ is divergent (Equation 2 with

$p = \frac{1}{2} \leq 1$), $\int_2^\infty \frac{1}{\sqrt{x} - 1} dx$ is divergent by the Comparison Theorem.

(c) 2.5



Since $\int_2^\infty f(x) dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2, t]$, $\int_2^\infty g(x) dx$ must be infinite; that is, the integral is divergent.

49. For $x \geq 1$, $\frac{\cos^2 x}{1+x^2} \leq \frac{1}{1+x^2} < \frac{1}{x^2}$. $\int_1^\infty \frac{1}{x^2} dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\int_1^\infty \frac{\cos^2 x}{1+x^2} dx$ is convergent by the Comparison Theorem.

50. For $x \geq 1$, $\frac{2+e^{-x}}{x} > \frac{2}{x}$ [since $e^{-x} > 0$] $> \frac{1}{x}$. $\int_1^\infty \frac{1}{x} dx$ is divergent by Equation 2 with $p = 1 \leq 1$, so

$\int_1^\infty \frac{2+e^{-x}}{x} dx$ is divergent by the Comparison Theorem.

51. For $x \geq 1$, $x + e^{2x} > e^{2x} > 0 \Rightarrow \frac{1}{x + e^{2x}} \leq \frac{1}{e^{2x}} = e^{-2x}$ on $[1, \infty)$.

$\int_1^\infty e^{-2x} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2} e^{-2x} \right]_1^t = \lim_{t \rightarrow \infty} \left[-\frac{1}{2} e^{-2t} + \frac{1}{2} e^{-2} \right] = \frac{1}{2} e^{-2}$. Therefore, $\int_1^\infty e^{-2x} dx$ is convergent,

and by the Comparison Theorem, $\int_1^\infty \frac{dx}{x + e^{2x}}$ is also convergent.

52. For $x \geq 1$, $0 < \frac{x}{\sqrt{1+x^6}} < \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$. $\int_1^\infty \frac{1}{x^2} dx$ is convergent by Equation 2 with $p = 2 > 1$, so

$\int_1^\infty \frac{x}{\sqrt{1+x^6}} dx$ is convergent by the Comparison Theorem.

53. $\frac{1}{x \sin x} \geq \frac{1}{x}$ on $(0, \frac{\pi}{2}]$ since $0 \leq \sin x \leq 1$. $\int_0^{\pi/2} \frac{dx}{x} = \lim_{t \rightarrow 0^+} \int_t^{\pi/2} \frac{dx}{x} = \lim_{t \rightarrow 0^+} [\ln x]_t^{\pi/2} = \lim_{t \rightarrow 0^+} (\ln \frac{\pi}{2} - \ln t) = -\infty$.

But $\ln t \rightarrow -\infty$ as $t \rightarrow 0^+$, so $\int_0^{\pi/2} \frac{dx}{x}$ is divergent, and by the Comparison Theorem, $\int_0^{\pi/2} \frac{dx}{x \sin x}$ is also divergent.

54. For $0 \leq x \leq 1$, $e^{-x} \leq 1 \Rightarrow \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$.

$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^+} [2\sqrt{x}]_t^1 = \lim_{t \rightarrow 0^+} (2 - 2\sqrt{t}) = 2$ is convergent. Therefore, $\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx$

is convergent by the Comparison Theorem.

$$\begin{aligned}
 55. \int_0^\infty \frac{dx}{\sqrt{x}(1+x)} &= \int_0^1 \frac{dx}{\sqrt{x}(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}(1+x)} = \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{\sqrt{x}(1+x)} + \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{\sqrt{x}(1+x)}. \text{ Now} \\
 \int \frac{dx}{\sqrt{x}(1+x)} &= \int \frac{2u du}{u(1+u^2)} \quad [u = \sqrt{x}, x = u^2, dx = 2u du] \\
 &= 2 \int \frac{du}{1+u^2} = 2 \tan^{-1} u + C = 2 \tan^{-1} \sqrt{x} + C,
 \end{aligned}$$

$$\begin{aligned}
 \text{so } \int_0^\infty \frac{dx}{\sqrt{x}(1+x)} &= \lim_{t \rightarrow 0^+} [2 \tan^{-1} \sqrt{x}]_t^1 + \lim_{t \rightarrow \infty} [2 \tan^{-1} \sqrt{x}]_1^t \\
 &= \lim_{t \rightarrow 0^+} [2(\frac{\pi}{4}) - 2 \tan^{-1} \sqrt{t}] + \lim_{t \rightarrow \infty} [2 \tan^{-1} \sqrt{t} - 2(\frac{\pi}{4})] = \frac{\pi}{2} - 0 + 2(\frac{\pi}{2}) - \frac{\pi}{2} = \pi.
 \end{aligned}$$

$$\begin{aligned}
 56. \int_2^\infty \frac{dx}{x\sqrt{x^2-4}} &= \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t \rightarrow 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t \rightarrow \infty} \int_3^t \frac{dx}{x\sqrt{x^2-4}}. \text{ Now} \\
 \int \frac{dx}{x\sqrt{x^2-4}} &= \int \frac{2 \sec \theta \tan \theta d\theta}{2 \sec \theta \tan \theta} \quad [x = 2 \sec \theta, \text{ where } 0 \leq \theta < \pi/2 \text{ or } \pi \leq \theta < 3\pi/2] \\
 &= \frac{1}{2} \theta + C = \frac{1}{2} \sec^{-1}(\frac{1}{2}x) + C, \text{ so} \\
 \int_2^\infty \frac{dx}{x\sqrt{x^2-4}} &= \lim_{t \rightarrow 2^+} [\frac{1}{2} \sec^{-1}(\frac{1}{2}x)]_t^3 + \lim_{t \rightarrow \infty} [\frac{1}{2} \sec^{-1}(\frac{1}{2}x)]_3^t \\
 &= \frac{1}{2} \sec^{-1}(\frac{3}{2}) - 0 + \frac{1}{2}(\frac{\pi}{2}) - \frac{1}{2} \sec^{-1}(\frac{3}{2}) = \frac{\pi}{4}
 \end{aligned}$$

$$57. \text{ If } p = 1, \text{ then } \int_0^1 \frac{dx}{x^p} = \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{x} = \lim_{t \rightarrow 0^+} [\ln x]_t^1 = \infty. \text{ Divergent.}$$

$$\text{If } p \neq 1, \text{ then } \int_0^1 \frac{dx}{x^p} = \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{x^p} \quad (\text{note that the integral is not improper if } p < 0)$$

$$= \lim_{t \rightarrow 0^+} \left[\frac{x^{-p+1}}{-p+1} \right]_t^1 = \lim_{t \rightarrow 0^+} \frac{1}{1-p} \left[1 - \frac{1}{t^{p-1}} \right]$$

$$\text{If } p > 1, \text{ then } p-1 > 0, \text{ so } \frac{1}{t^{p-1}} \rightarrow \infty \text{ as } t \rightarrow 0^+, \text{ and the integral diverges.}$$

$$\text{If } p < 1, \text{ then } p-1 < 0, \text{ so } \frac{1}{t^{p-1}} \rightarrow 0 \text{ as } t \rightarrow 0^+ \text{ and } \int_0^1 \frac{dx}{x^p} = \frac{1}{1-p} \left[\lim_{t \rightarrow 0^+} (1 - t^{1-p}) \right] = \frac{1}{1-p}.$$

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\frac{1}{1-p}$.

$$58. \text{ Let } u = \ln x. \text{ Then } du = dx/x \Rightarrow \int_e^\infty \frac{dx}{x(\ln x)^p} = \int_1^\infty \frac{du}{u^p}. \text{ By Example 4, this converges to } \frac{1}{p-1} \text{ if } p > 1 \text{ and diverges otherwise.}$$

$$59. \text{ First suppose } p = -1. \text{ Then}$$

$$\int_0^1 x^p \ln x \, dx = \int_0^1 \frac{\ln x}{x} \, dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{\ln x}{x} \, dx = \lim_{t \rightarrow 0^+} \left[\frac{1}{2} (\ln x)^2 \right]_t^1 = -\frac{1}{2} \lim_{t \rightarrow 0^+} (\ln t)^2 = -\infty,$$

so the integral diverges. Now suppose $p \neq -1$. Then integration by parts gives

$$\int x^p \ln x \, dx = \frac{x^{p+1}}{p+1} \ln x - \int \frac{x^p}{p+1} \, dx = \frac{x^{p+1}}{p+1} \ln x - \frac{x^{p+1}}{(p+1)^2} + C. \text{ If } p < -1, \text{ then } p+1 < 0, \text{ so}$$

$$\int_0^1 x^p \ln x \, dx = \lim_{t \rightarrow 0^+} \left[\frac{x^{p+1}}{p+1} \ln x - \frac{x^{p+1}}{(p+1)^2} \right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1} \right) \lim_{t \rightarrow 0^+} \left[t^{p+1} \left(\ln t - \frac{1}{p+1} \right) \right] = \infty.$$

If $p > -1$, then $p + 1 > 0$ and

$$\begin{aligned}\int_0^1 x^p \ln x \, dx &= \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1} \right) \lim_{t \rightarrow 0^+} \frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \stackrel{H}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1} \right) \lim_{t \rightarrow 0^+} \frac{1/t}{-(p+1)t^{-(p+2)}} \\ &= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2} \lim_{t \rightarrow 0^+} t^{p+1} = \frac{-1}{(p+1)^2}\end{aligned}$$

Thus, the integral converges to $-\frac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

$$\begin{aligned}60. \text{ (a) } n = 0: \quad \int_0^\infty x^n e^{-x} \, dx &= \lim_{t \rightarrow \infty} \int_0^t e^{-x} \, dx = \lim_{t \rightarrow \infty} [-e^{-x}]_0^t \\ &= \lim_{t \rightarrow \infty} [-e^{-t} + 1] = 0 + 1 = 1\end{aligned}$$

$$\begin{aligned}n = 1: \quad \int_0^\infty x^n e^{-x} \, dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-x} \, dx. \text{ To evaluate } \int x e^{-x} \, dx, \text{ we'll use integration by parts} \\ \text{with } u = x, dv = e^{-x} \, dx \Rightarrow du = dx, v = -e^{-x}.\end{aligned}$$

$$\text{So } \int x e^{-x} \, dx = -x e^{-x} - \int -e^{-x} \, dx = -x e^{-x} - e^{-x} + C = (-x - 1)e^{-x} + C \text{ and}$$

$$\begin{aligned}\lim_{t \rightarrow \infty} \int_0^t x e^{-x} \, dx &= \lim_{t \rightarrow \infty} [(-x - 1)e^{-x}]_0^t \\ &= \lim_{t \rightarrow \infty} [(-t - 1)e^{-t} + 1] = \lim_{t \rightarrow \infty} [-te^{-t} - e^{-t} + 1] \\ &= 0 - 0 + 1 \quad [\text{use l'Hospital's Rule}] = 1\end{aligned}$$

$$\begin{aligned}n = 2: \quad \int_0^\infty x^n e^{-x} \, dx &= \lim_{t \rightarrow \infty} \int_0^t x^2 e^{-x} \, dx. \text{ To evaluate } \int x^2 e^{-x} \, dx, \text{ we could use integration by parts} \\ \text{again or Formula 97. Thus,}\end{aligned}$$

$$\begin{aligned}\lim_{t \rightarrow \infty} \int_0^t x^2 e^{-x} \, dx &= \lim_{t \rightarrow \infty} [-x^2 e^{-x}]_0^t + 2 \lim_{t \rightarrow \infty} \int_0^t x e^{-x} \, dx \\ &= 0 + 0 + 2(1) \quad [\text{use l'Hospital's Rule and the result for } n = 1] = 2\end{aligned}$$

$$\begin{aligned}n = 3: \quad \int_0^\infty x^n e^{-x} \, dx &= \lim_{t \rightarrow \infty} \int_0^t x^3 e^{-x} \, dx \stackrel{97}{=} \lim_{t \rightarrow \infty} [-x^3 e^{-x}]_0^t + 3 \lim_{t \rightarrow \infty} \int_0^t x^2 e^{-x} \, dx \\ &= 0 + 0 + 3(2) \quad [\text{use l'Hospital's Rule and the result for } n = 2] = 6\end{aligned}$$

(b) For $n = 1, 2$, and 3 , we have $\int_0^\infty x^n e^{-x} \, dx = 1, 2$, and 6 . The values for the integral are equal to the factorials for n , so we guess $\int_0^\infty x^n e^{-x} \, dx = n!$.

(c) Suppose that $\int_0^\infty x^k e^{-x} \, dx = k!$ for some positive integer k . Then $\int_0^\infty x^{k+1} e^{-x} \, dx = \lim_{t \rightarrow \infty} \int_0^t x^{k+1} e^{-x} \, dx$.

To evaluate $\int x^{k+1} e^{-x} \, dx$, we use parts with $u = x^{k+1}$, $dv = e^{-x} \, dx \Rightarrow du = (k+1)x^k \, dx$, $v = -e^{-x}$.

So $\int x^{k+1} e^{-x} \, dx = -x^{k+1} e^{-x} - \int -(k+1)x^k e^{-x} \, dx = -x^{k+1} e^{-x} + (k+1) \int x^k e^{-x} \, dx$ and

$$\begin{aligned}\lim_{t \rightarrow \infty} \int_0^t x^{k+1} e^{-x} \, dx &= \lim_{t \rightarrow \infty} [-x^{k+1} e^{-x}]_0^t + (k+1) \lim_{t \rightarrow \infty} \int_0^t x^k e^{-x} \, dx \\ &= \lim_{t \rightarrow \infty} [-t^{k+1} e^{-t} + 0] + (k+1)k! = 0 + 0 + (k+1)k! = (k+1)!,\end{aligned}$$

so the formula holds for $k + 1$. By induction, the formula holds for all positive integers. (Since $0! = 1$, the formula holds for $n = 0$, too.)

61. (a) $I = \int_{-\infty}^{\infty} x \, dx = \int_{-\infty}^0 x \, dx + \int_0^{\infty} x \, dx$, and

$$\int_0^{\infty} x \, dx = \lim_{t \rightarrow \infty} \int_0^t x \, dx = \lim_{t \rightarrow \infty} \left[\frac{1}{2} x^2 \right]_0^t = \lim_{t \rightarrow \infty} \left[\frac{1}{2} t^2 - 0 \right] = \infty, \text{ so } I \text{ is divergent.}$$

(b) $\int_{-t}^t x \, dx = \left[\frac{1}{2} x^2 \right]_{-t}^t = \frac{1}{2} t^2 - \frac{1}{2} t^2 = 0$, so $\lim_{t \rightarrow \infty} \int_{-t}^t x \, dx = 0$. Therefore, $\int_{-\infty}^{\infty} x \, dx \neq \lim_{t \rightarrow \infty} \int_{-t}^t x \, dx$.

62. Let $k = \frac{M}{2RT}$ so that $\bar{v} = \frac{4}{\sqrt{\pi}} k^{3/2} \int_0^{\infty} v^3 e^{-kv^2} \, dv$. Let I denote the integral and use parts to integrate I .

$$\text{Let } \alpha = v^2, d\beta = v e^{-kv^2} \, dv \Rightarrow d\alpha = 2v \, dv, \beta = -\frac{1}{2k} e^{-kv^2}.$$

$$\begin{aligned} I &= \lim_{t \rightarrow \infty} \left[-\frac{1}{2k} v^2 e^{-kv^2} \right]_0^t + \frac{1}{k} \int_0^{\infty} v e^{-kv^2} \, dv = -\frac{1}{2k} \lim_{t \rightarrow \infty} (t^2 e^{-kt^2}) + \frac{1}{k} \lim_{t \rightarrow \infty} \left[-\frac{1}{2k} e^{-kv^2} \right]_0^t \\ &\stackrel{H}{=} -\frac{1}{2k} \cdot 0 - \frac{1}{2k^2} (0 - 1) = \frac{1}{2k^2} \end{aligned}$$

$$\text{Thus, } \bar{v} = \frac{4}{\sqrt{\pi}} k^{3/2} \cdot \frac{1}{2k^2} = \frac{2}{(k\pi)^{1/2}} = \frac{2}{[\pi M / (2RT)]^{1/2}} = \frac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\frac{8RT}{\pi M}}.$$

63. Volume $= \int_1^{\infty} \pi \left(\frac{1}{x} \right)^2 \, dx = \pi \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2} = \pi \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_1^t = \pi \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t} \right) = \pi < \infty$.

64. Work $= \int_R^{\infty} \frac{GMm}{r^2} \, dr = \lim_{t \rightarrow \infty} \int_R^t \frac{GMm}{r^2} \, dr = \lim_{t \rightarrow \infty} GMm \left[-\frac{1}{r} \right]_R^t = GMm \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + \frac{1}{R} \right) = \frac{GMm}{R}$,

where M = mass of Earth $= 5.98 \times 10^{24}$ kg, m = mass of satellite $= 10^3$ kg,

R = radius of Earth $= 6.37 \times 10^6$ m, and G = gravitational constant $= 6.67 \times 10^{-11}$ N·m²/kg.

$$\text{Therefore, Work} = \frac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24} \cdot 10^3}{6.37 \times 10^6} \approx 6.26 \times 10^{10} \text{ J.}$$

65. Work $= \int_R^{\infty} F \, dr = \lim_{t \rightarrow \infty} \int_R^t \frac{GmM}{r^2} \, dr = \lim_{t \rightarrow \infty} GmM \left(\frac{1}{R} - \frac{1}{t} \right) = \frac{GmM}{R}$. The initial kinetic energy

$$\text{provides the work, so } \frac{1}{2} m v_0^2 = \frac{GmM}{R} \Rightarrow v_0 = \sqrt{\frac{2GM}{R}}.$$

66. $y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}} x(r) \, dr$ and $x(r) = \frac{1}{2}(R - r)^2 \Rightarrow$

$$\begin{aligned} y(s) &= \lim_{t \rightarrow s^+} \int_t^R \frac{r(R - r)^2}{\sqrt{r^2 - s^2}} \, dr = \lim_{t \rightarrow s^+} \int_t^R \frac{r^3 - 2Rr^2 + R^2r}{\sqrt{r^2 - s^2}} \, dr \\ &= \lim_{t \rightarrow s^+} \left[\int_t^R \frac{r^3 \, dr}{\sqrt{r^2 - s^2}} - 2R \int_t^R \frac{r^2 \, dr}{\sqrt{r^2 - s^2}} + R^2 \int_t^R \frac{r \, dr}{\sqrt{r^2 - s^2}} \right] \\ &= \lim_{t \rightarrow s^+} (I_1 - 2RI_2 + R^2 I_3) = L \end{aligned}$$

For I_1 : Let $u = \sqrt{r^2 - s^2} \Rightarrow u^2 = r^2 - s^2, r^2 = u^2 + s^2, 2r \, dr = 2u \, du$, so, omitting limits and constant of integration,

$$\begin{aligned} I_1 &= \int \frac{(u^2 + s^2)u}{u} \, du = \int (u^2 + s^2) \, du = \frac{1}{3} u^3 + s^2 u = \frac{1}{3} u(u^2 + 3s^2) \\ &= \frac{1}{3} \sqrt{r^2 - s^2} (r^2 - s^2 + 3s^2) = \frac{1}{3} \sqrt{r^2 - s^2} (r^2 + 2s^2) \end{aligned}$$

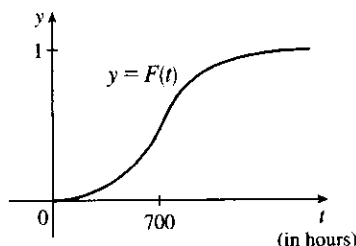
For I_2 : Using Formula 44, $I_2 = \frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2}\ln|r + \sqrt{r^2 - s^2}|$.

For I_3 : Let $u = r^2 - s^2 \Rightarrow du = 2r dr$. Then $I_3 = \frac{1}{2} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \cdot 2\sqrt{u} = \sqrt{r^2 - s^2}$.

Thus,

$$\begin{aligned} L &= \lim_{t \rightarrow s^+} \left[\frac{1}{3}\sqrt{r^2 - s^2}(r^2 + 2s^2) - 2R \left(\frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2}\ln|r + \sqrt{r^2 - s^2}| \right) + R^2\sqrt{r^2 - s^2} \right]_t^R \\ &= \lim_{t \rightarrow s^+} \left[\frac{1}{3}\sqrt{R^2 - s^2}(R^2 + 2s^2) - 2R \left(\frac{R}{2}\sqrt{R^2 - s^2} + \frac{s^2}{2}\ln|R + \sqrt{R^2 - s^2}| \right) + R^2\sqrt{R^2 - s^2} \right] \\ &\quad - \lim_{t \rightarrow s^+} \left[\frac{1}{3}\sqrt{t^2 - s^2}(t^2 + 2s^2) - 2R \left(\frac{t}{2}\sqrt{t^2 - s^2} + \frac{s^2}{2}\ln|t + \sqrt{t^2 - s^2}| \right) + R^2\sqrt{t^2 - s^2} \right] \\ &= \left[\frac{1}{3}\sqrt{R^2 - s^2}(R^2 + 2s^2) - Rs^2\ln|R + \sqrt{R^2 - s^2}| \right] - [-Rs^2\ln|s|] \\ &= \frac{1}{3}\sqrt{R^2 - s^2}(R^2 + 2s^2) - Rs^2\ln\left(\frac{R + \sqrt{R^2 - s^2}}{s}\right) \end{aligned}$$

67. (a) We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t) dt = \lim_{x \rightarrow \infty} F(x) = 1$, since all of the bulbs will eventually burn out.

$$\begin{aligned} 68. I &= \int_0^\infty te^{kt} dt = \lim_{s \rightarrow \infty} \left[\frac{1}{k^2}(kt - 1)e^{kt} \right]_0^s \quad [\text{Formula 96, or parts}] \\ &= \lim_{s \rightarrow \infty} \left[\left(\frac{1}{k}se^{ks} - \frac{1}{k^2}e^{ks} \right) - \left(-\frac{1}{k^2} \right) \right]. \end{aligned}$$

Since $k < 0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M = -kI = -k(1/k^2) = -1/k = -1/(-0.000121) \approx 8264.5$ years.

$$\begin{aligned} 69. I &= \int_a^\infty \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \int_a^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} [\tan^{-1} x]_a^t = \lim_{t \rightarrow \infty} (\tan^{-1} t - \tan^{-1} a) = \frac{\pi}{2} - \tan^{-1} a. \\ I < 0.001 &\Rightarrow \frac{\pi}{2} - \tan^{-1} a < 0.001 \Rightarrow \tan^{-1} a > \frac{\pi}{2} - 0.001 \Rightarrow a > \tan\left(\frac{\pi}{2} - 0.001\right) \approx 1000. \end{aligned}$$

$$70. f(x) = e^{-x^2} \text{ and } \Delta x = \frac{4-0}{8} = \frac{1}{2}.$$

$$\begin{aligned} \int_0^4 f(x) dx &\approx S_8 = \frac{1}{2} \cdot \frac{1}{8} [f(0) + 4f(0.5) + 2f(1) + \cdots + 2f(3) + 4f(3.5) + f(4)] \\ &\approx \frac{1}{8} (5.31717808) \approx 0.8862 \end{aligned}$$

$$\text{Now } x > 4 \Rightarrow -x \cdot x < -x \cdot 4 \Rightarrow e^{-x^2} < e^{-4x} \Rightarrow \int_4^\infty e^{-x^2} dx < \int_4^\infty e^{-4x} dx.$$

$$\int_4^\infty e^{-4x} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{4}e^{-4x} \right]_4^t = -\frac{1}{4}(0 - e^{-16}) = 1/(4e^{16}) \approx 0.0000000281 < 0.0000001, \text{ as desired.}$$

71. (a) $F(s) = \int_0^\infty f(t)e^{-st} dt = \int_0^\infty e^{-st} dt = \lim_{n \rightarrow \infty} \left[-\frac{e^{-st}}{s} \right]_0^n = \lim_{n \rightarrow \infty} \left(\frac{e^{-sn}}{-s} + \frac{1}{s} \right)$. This converges to $\frac{1}{s}$ only if $s > 0$. Therefore $F(s) = \frac{1}{s}$ with domain $\{s \mid s > 0\}$.

$$(b) F(s) = \int_0^\infty f(t)e^{-st} dt = \int_0^\infty e^t e^{-st} dt = \lim_{n \rightarrow \infty} \int_0^n e^{t(1-s)} dt = \lim_{n \rightarrow \infty} \left[\frac{1}{1-s} e^{t(1-s)} \right]_0^n \\ = \lim_{n \rightarrow \infty} \left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s} \right)$$

This converges only if $1-s < 0 \Rightarrow s > 1$, in which case $F(s) = \frac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

$$(c) F(s) = \int_0^\infty f(t)e^{-st} dt = \lim_{n \rightarrow \infty} \int_0^n te^{-st} dt. \text{ Use integration by parts: let } u = t, dv = e^{-st} dt \Rightarrow \\ du = dt, v = -\frac{e^{-st}}{s}. \text{ Then } F(s) = \lim_{n \rightarrow \infty} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^n = \lim_{n \rightarrow \infty} \left(\frac{-n}{se^{sn}} - \frac{1}{s^2 e^{sn}} + 0 + \frac{1}{s^2} \right) = \frac{1}{s^2} \\ \text{only if } s > 0. \text{ Therefore, } F(s) = \frac{1}{s^2} \text{ and the domain of } F \text{ is } \{s \mid s > 0\}.$$

72. $0 \leq f(t) \leq Me^{at} \Rightarrow 0 \leq f(t)e^{-st} \leq Me^{at}e^{-st}$ for $t \geq 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st} dt = \lim_{n \rightarrow \infty} M \int_0^n e^{t(a-s)} dt = M \cdot \lim_{n \rightarrow \infty} \left[\frac{1}{a-s} e^{t(a-s)} \right]_0^n = M \cdot \lim_{n \rightarrow \infty} \frac{1}{a-s} [e^{n(a-s)} - 1]$$

This is convergent only when $a-s < 0 \Rightarrow s > a$. Therefore, by the Comparison Theorem,

$F(s) = \int_0^\infty f(t)e^{-st} dt$ is also convergent for $s > a$.

73. $G(s) = \int_0^\infty f'(t)e^{-st} dt$. Integrate by parts with $u = e^{-st}$, $dv = f'(t) dt \Rightarrow du = -se^{-st}$, $v = f(t)$:

$$G(s) = \lim_{n \rightarrow \infty} [f(t)e^{-st}]_0^n + s \int_0^\infty f(t)e^{-st} dt = \lim_{n \rightarrow \infty} f(n)e^{-sn} - f(0) + sF(s)$$

But $0 \leq f(t) \leq Me^{at} \Rightarrow 0 \leq f(t)e^{-st} \leq Me^{at}e^{-st}$ and $\lim_{t \rightarrow \infty} Me^{t(a-s)} = 0$ for $s > a$. So by the Squeeze

Theorem, $\lim_{t \rightarrow \infty} f(t)e^{-st} = 0$ for $s > a \Rightarrow G(s) = 0 - f(0) + sF(s) = sF(s) - f(0)$ for $s > a$.

74. Assume without loss of generality that $a < b$. Then

$$\int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx + \lim_{u \rightarrow \infty} \int_a^u f(x) dx \\ = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx + \lim_{u \rightarrow \infty} \left[\int_a^b f(x) dx + \int_b^u f(x) dx \right] \\ = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx + \int_a^b f(x) dx + \lim_{u \rightarrow \infty} \int_b^u f(x) dx \\ = \lim_{t \rightarrow -\infty} \left[\int_t^a f(x) dx + \int_a^b f(x) dx \right] + \int_b^\infty f(x) dx \\ = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx + \int_b^\infty f(x) dx \\ = \int_{-\infty}^b f(x) dx + \int_b^\infty f(x) dx$$

75. We use integration by parts: let $u = x$, $dv = xe^{-x^2} dx \Rightarrow du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\begin{aligned}\int_0^\infty x^2 e^{-x^2} dx &= \lim_{t \rightarrow \infty} \left[-\frac{1}{2} x e^{-x^2} \right]_0^t + \frac{1}{2} \int_0^\infty e^{-x^2} dx \\ &= \lim_{t \rightarrow \infty} \left[-t / (2e^{t^2}) \right] + \frac{1}{2} \int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-x^2} dx\end{aligned}$$

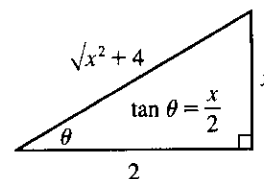
(The limit is 0 by l'Hospital's Rule.)

76. $\int_0^\infty e^{-x^2} dx$ is the area under the curve $y = e^{-x^2}$ for $0 \leq x < \infty$ and $0 < y \leq 1$. Solving $y = e^{-x^2}$ for x , we get $y = e^{-x^2} \Rightarrow \ln y = -x^2 \Rightarrow -\ln y = x^2 \Rightarrow x = \pm \sqrt{-\ln y}$. Since x is positive, choose $x = \sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y} dy$. Therefore, each integral represents the same area, so the integrals are equal.

77. For the first part of the integral, let $x = 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta d\theta$.

$$\int \frac{1}{\sqrt{x^2 + 4}} dx = \int \frac{2 \sec^2 \theta}{2 \sec \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|. \text{ From the}$$

figure, $\tan \theta = \frac{x}{2}$, and $\sec \theta = \frac{\sqrt{x^2 + 4}}{2}$. So



$$\begin{aligned}I &= \int_0^\infty \left(\frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) dx = \lim_{t \rightarrow \infty} \left[\ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| - C \ln |x + 2| \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left[\ln \frac{\sqrt{t^2 + 4} + t}{2} - C \ln(t + 2) - (\ln 1 - C \ln 2) \right] \\ &= \lim_{t \rightarrow \infty} \left[\ln \left(\frac{\sqrt{t^2 + 4} + t}{2(t + 2)^C} \right) + \ln 2^C \right] \\ &= \ln \left(\lim_{t \rightarrow \infty} \frac{t + \sqrt{t^2 + 4}}{(t + 2)^C} \right) + \ln 2^{C-1}\end{aligned}$$

$$\text{Now } L = \lim_{t \rightarrow \infty} \frac{t + \sqrt{t^2 + 4}}{(t + 2)^C} \stackrel{H}{=} \lim_{t \rightarrow \infty} \frac{1 + t/\sqrt{t^2 + 4}}{C(t + 2)^{C-1}} = \frac{2}{C \lim_{t \rightarrow \infty} (t + 2)^{C-1}}.$$

If $C < 1$, $L = \infty$ and I diverges. If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$. If $C > 1$, $L = 0$ and I diverges to $-\infty$.

$$\begin{aligned}78. I &= \int_0^\infty \left(\frac{x}{x^2 + 1} - \frac{C}{3x + 1} \right) dx = \lim_{t \rightarrow \infty} \left[\frac{1}{2} \ln(x^2 + 1) - \frac{1}{3} C \ln(3x + 1) \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left[\ln(t^2 + 1)^{1/2} - \ln(3t + 1)^{C/3} \right] \\ &= \lim_{t \rightarrow \infty} \left(\ln \frac{(t^2 + 1)^{1/2}}{(3t + 1)^{C/3}} \right) = \ln \left(\lim_{t \rightarrow \infty} \frac{\sqrt{t^2 + 1}}{(3t + 1)^{C/3}} \right)\end{aligned}$$

For $C \leq 0$, the integral diverges. For $C > 0$, we have

$$L = \lim_{t \rightarrow \infty} \frac{\sqrt{t^2 + 1}}{(3t + 1)^{C/3}} \stackrel{H}{=} \lim_{t \rightarrow \infty} \frac{t/\sqrt{t^2 + 1}}{C(3t + 1)^{(C/3)-1}} = \frac{1}{C} \lim_{t \rightarrow \infty} \frac{1}{(3t + 1)^{(C/3)-1}}.$$

For $C/3 < 1 \Leftrightarrow C < 3$, $L = \infty$ and I diverges. For $C = 3$, $L = \frac{1}{3}$ and $I = \ln \frac{1}{3}$. For $C > 3$, $L = 0$ and I diverges to $-\infty$.

8 Review

CONCEPT CHECK

1. See Formula 8.1.1 or 8.1.2. We try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x) dx$ can be readily integrated to give v .
2. See the Strategy for Evaluating $\int \sin^m x \cos^n x dx$ on page 520.
3. If $\sqrt{a^2 - x^2}$ occurs, try $x = a \sin \theta$; if $\sqrt{a^2 + x^2}$ occurs, try $x = a \tan \theta$, and if $\sqrt{x^2 - a^2}$ occurs, try $x = a \sec \theta$. See the Table of Trigonometric Substitutions on page 526.
4. See Equation 2 and Expressions 7, 9, and 11 in Section 8.4.
5. See the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule, as well as their associated error bounds, all in Section 8.7. We would expect the best estimate to be given by Simpson's Rule.
6. See Definitions 1(a), (b), and (c) in Section 8.8.
7. See Definitions 3(b), (a), and (c) in Section 8.8.
8. See the Comparison Theorem after Example 8 in Section 8.8.

TRUE-FALSE QUIZ

1. False. Since the numerator has a higher degree than the denominator,

$$\frac{x(x^2 + 4)}{x^2 - 4} = x + \frac{8x}{x^2 - 4} = x + \frac{A}{x + 2} + \frac{B}{x - 2}.$$

2. True. In fact, $A = -1$, $B = C = 1$.

3. False. It can be put in the form $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 4}$.

4. False. The form is $\frac{A}{x} + \frac{Bx + C}{x^2 + 4}$.

5. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2 - 1} dx = \int_0^1 \frac{x}{x^2 - 1} dx + \int_1^4 \frac{x}{x^2 - 1} dx \text{ and}$$

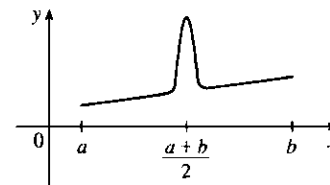
$$\int_0^1 \frac{x}{x^2 - 1} dx = \lim_{t \rightarrow 1^-} \int_0^t \frac{x}{x^2 - 1} dx = \lim_{t \rightarrow 1^-} \left[\frac{1}{2} \ln|x^2 - 1| \right]_0^t = \lim_{t \rightarrow 1^-} \frac{1}{2} \ln|t^2 - 1| = \infty$$

So the integral diverges.

6. True by Theorem 8.8.2 with $p = \sqrt{2} > 1$.

7. False. See Exercise 61 in Section 8.8.

8. False. For example, with $n = 1$ the Trapezoidal Rule is much more accurate than the Midpoint Rule for the function in the diagram.



9. (a) True. See the end of Section 8.5.

- (b) False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \frac{\sin x}{x}$.

10. True. If f is continuous on $[0, \infty)$, then $\int_0^1 f(x) dx$ is finite. Since $\int_1^\infty f(x) dx$ is finite, so is $\int_0^\infty f(x) dx = \int_0^1 f(x) dx + \int_1^\infty f(x) dx$.
11. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1, \infty)$ with $\lim_{x \rightarrow \infty} f(x) = 0$, but $\int_1^\infty f(x) dx$ is divergent.
12. True.
$$\begin{aligned}\int_a^\infty [f(x) + g(x)] dx &= \lim_{t \rightarrow \infty} \int_a^t [f(x) + g(x)] dx = \lim_{t \rightarrow \infty} \left(\int_a^t f(x) dx + \int_a^t g(x) dx \right) \\ &= \lim_{t \rightarrow \infty} \int_a^t f(x) dx + \lim_{t \rightarrow \infty} \int_a^t g(x) dx \quad \left[\begin{array}{l} \text{since both limits} \\ \text{in the sum exist} \end{array} \right] \\ &= \int_a^\infty f(x) dx + \int_a^\infty g(x) dx\end{aligned}$$
Since the two integrals are finite, so is their sum.
13. False. Take $f(x) = 1$ for all x and $g(x) = -1$ for all x . Then $\int_a^\infty f(x) dx = \infty$ [divergent] and $\int_a^\infty g(x) dx = -\infty$ [divergent], but $\int_a^\infty [f(x) + g(x)] dx = 0$ [convergent].
14. False. $\int_0^\infty f(x) dx$ could converge or diverge. For example, if $g(x) = 1$, then $\int_0^\infty f(x) dx$ diverges if $f(x) = 1$ and converges if $f(x) = 0$.

EXERCISES

1.
$$\begin{aligned}\int_0^5 \frac{x}{x+10} dx &= \int_0^5 \left(1 - \frac{10}{x+10} \right) dx = \left[x - 10 \ln(x+10) \right]_0^5 \\ &= 5 - 10 \ln 15 + 10 \ln 10 = 5 + 10 \ln \frac{10}{15} = 5 + 10 \ln \frac{2}{3}\end{aligned}$$
2.
$$\begin{aligned}\int_0^5 ye^{-0.6y} dy \quad \left[\begin{array}{l} u = y, \quad dv = e^{-0.6y} dy \\ du = dy \quad v = -\frac{5}{3}e^{-0.6y} \end{array} \right] &= \left[-\frac{5}{3}ye^{-0.6y} \right]_0^5 - \int_0^5 \left(-\frac{5}{3}e^{-0.6y} \right) dy \\ &= -\frac{25}{3}e^{-3} - \frac{25}{9} \left[e^{-0.6y} \right]_0^5 = -\frac{25}{3}e^{-3} - \frac{25}{9}(e^{-3} - 1) \\ &= -\frac{25}{3}e^{-3} - \frac{25}{9}e^{-3} + \frac{25}{9} = \frac{25}{9} - \frac{100}{9}e^{-3}\end{aligned}$$
3.
$$\int_0^{\pi/2} \frac{\cos \theta}{1 + \sin \theta} d\theta = \left[\ln(1 + \sin \theta) \right]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2$$
4.
$$\int_1^4 \frac{dt}{(2t+1)^3} \quad \left[\begin{array}{l} u = 2t+1, \\ du = 2 dt \end{array} \right] = \int_3^9 \frac{\frac{1}{2} du}{u^3} = \frac{-1}{4} \left[\frac{1}{u^2} \right]_3^9 = -\frac{1}{4} \left(\frac{1}{81} - \frac{1}{9} \right) = -\frac{1}{4} \left(-\frac{8}{81} \right) = \frac{2}{81}$$
5. Let $u = \sec x$. Then $du = \sec x \tan x dx$, so
- $$\begin{aligned}\int \tan^7 x \sec^3 x dx &= \int \tan^6 x \sec^2 x \sec x \tan x dx = \int (u^2 - 1)^3 u^2 du = \int (u^8 - 3u^6 + 3u^4 - u^2) du \\ &= \frac{1}{9}u^9 - \frac{3}{7}u^7 + \frac{3}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{9}\sec^9 x - \frac{3}{7}\sec^7 x + \frac{3}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C\end{aligned}$$
6.
$$\frac{1}{y^2 - 4y - 12} = \frac{1}{(y-6)(y+2)} = \frac{A}{y-6} + \frac{B}{y+2} \Rightarrow 1 = A(y+2) + B(y-6). \text{ Letting } y = -2 \Rightarrow B = -\frac{1}{8} \text{ and letting } y = 6 \Rightarrow A = \frac{1}{8}. \text{ So}$$

$$\int \frac{1}{y^2 - 4y - 12} dy = \int \left(\frac{1/8}{y-6} + \frac{-1/8}{y+2} \right) dy = \frac{1}{8} \ln |y-6| - \frac{1}{8} \ln |y+2| + C$$

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7. Let $u = \ln t$, $du = dt/t$. Then $\int \frac{\sin(\ln t)}{t} dt = \int \sin u du = -\cos u + C = -\cos(\ln t) + C$.

8. Let $x = \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{1+x^2}} &= \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta \sec \theta} = \int \frac{\sec \theta d\theta}{\tan^2 \theta} = \int \frac{\cos \theta d\theta}{\sin^2 \theta} = \int \frac{du}{u^2} \quad [\text{put } u = \sin \theta] \\ &= -\frac{1}{u} + C = -\frac{1}{\sin \theta} + C = -\frac{\sqrt{1+x^2}}{x} + C \end{aligned}$$

9. $\int_1^4 x^{3/2} \ln x dx \quad \left[\begin{array}{l} u = \ln x, \quad dv = x^{3/2} dx, \\ du = dx/x \quad v = \frac{2}{5} x^{5/2} \end{array} \right] = \frac{2}{5} \left[x^{5/2} \ln x \right]_1^4 - \frac{2}{5} \int_1^4 x^{3/2} dx$

$$\begin{aligned} &= \frac{2}{5} (32 \ln 4 - \ln 1) - \frac{2}{5} \left[\frac{2}{5} x^{5/2} \right]_1^4 \\ &= \frac{2}{5} (64 \ln 2) - \frac{4}{25} (32 - 1) \\ &= \frac{128}{5} \ln 2 - \frac{124}{25} \quad (\text{or } \frac{64}{5} \ln 4 - \frac{124}{25}) \end{aligned}$$

10. Let $u = \arctan x$, $du = dx/(1+x^2)$. Then

$$\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2} dx = \int_0^{\pi/4} \sqrt{u} du = \frac{2}{3} \left[u^{3/2} \right]_0^{\pi/4} = \frac{2}{3} \left[\frac{\pi^{3/2}}{4^{3/2}} - 0 \right] = \frac{2}{3} \cdot \frac{1}{8} \pi^{3/2} = \frac{1}{12} \pi^{3/2}.$$

11. Let $x = \sec \theta$. Then

$$\begin{aligned} \int_1^2 \frac{\sqrt{x^2-1}}{x} dx &= \int_0^{\pi/3} \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta d\theta = \int_0^{\pi/3} \tan^2 \theta d\theta = \int_0^{\pi/3} (\sec^2 \theta - 1) d\theta \\ &= [\tan \theta - \theta]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \end{aligned}$$

12. $\int_{-1}^1 \frac{\sin x}{1+x^2} dx = 0$ by Theorem 5.5.6(b), since $f(x) = \frac{\sin x}{1+x^2}$ is an odd function.

13. $\int \frac{dx}{x^3+x} = \int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx = \ln|x| - \frac{1}{2} \ln(x^2+1) + C$

14. $\int \frac{x^2+2}{x+2} dx = \int \left(x-2 + \frac{6}{x+2} \right) dx = \frac{1}{2}x^2 - 2x + 6 \ln|x+2| + C$

15. $\int \sin^2 \theta \cos^5 \theta d\theta = \int \sin^2 \theta (\cos^2 \theta)^2 \cos \theta d\theta = \int \sin^2 \theta (1 - \sin^2 \theta)^2 \cos \theta d\theta$

$$\begin{aligned} &= \int u^2 (1-u^2)^2 du \quad [u = \sin \theta, du = \cos \theta d\theta] = \int u^2 (1-2u^2+u^4) du \\ &= \int (u^2 - 2u^4 + u^6) du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C = \frac{1}{3}\sin^3 \theta - \frac{2}{5}\sin^5 \theta + \frac{1}{7}\sin^7 \theta + C \end{aligned}$$

16. $\int \frac{\sec^6 \theta}{\tan^2 \theta} d\theta = \int \frac{(\tan^2 \theta + 1)^2 \sec^2 \theta}{\tan^2 \theta} d\theta \quad \left[\begin{array}{l} u = \tan \theta, \\ du = \sec^2 \theta d\theta \end{array} \right]$

$$\begin{aligned} &= \int \frac{(u^2+1)^2}{u^2} du = \int \frac{u^4+2u^2+1}{u^2} du = \int \left(u^2 + 2 + \frac{1}{u^2} \right) du \\ &= \frac{u^3}{3} + 2u - \frac{1}{u} + C = \frac{1}{3} \tan^3 \theta + 2 \tan \theta - \cot \theta + C \end{aligned}$$

17. Integrate by parts with $u = x$, $dv = \sec x \tan x \, dx \Rightarrow du = dx$, $v = \sec x$:

$$\int x \sec x \tan x \, dx = x \sec x - \int \sec x \, dx \stackrel{14}{=} x \sec x - \ln|\sec x + \tan x| + C.$$

18. $\frac{x^2 + 8x - 3}{x^3 + 3x^2} = \frac{x^2 + 8x - 3}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} \Rightarrow x^2 + 8x - 3 = Ax(x+3) + B(x+3) + Cx^2.$

Taking $x = 0$, we get $-3 = 3B$, so $B = -1$. Taking $x = -3$, we get $-18 = 9C$, so $C = -2$. Taking

$x = 1$, we get $6 = 4A + 4B + C = 4A - 4 - 2$, so $4A = 12$ and $A = 3$. Now

$$\int \frac{x^2 + 8x - 3}{x^3 + 3x^2} \, dx = \int \left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3} \right) \, dx = 3 \ln|x| + \frac{1}{x} - 2 \ln|x+3| + C$$

19. $\int \frac{x+1}{9x^2+6x+5} \, dx = \int \frac{x+1}{(9x^2+6x+1)+4} \, dx = \int \frac{x+1}{(3x+1)^2+4} \, dx \quad \begin{cases} u = 3x+1, \\ du = 3 \, dx \end{cases}$

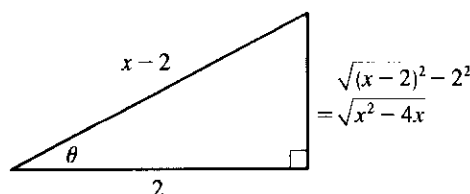
$$= \int \frac{\left[\frac{1}{3}(u-1)\right] + 1}{u^2+4} \left(\frac{1}{3} du\right) = \frac{1}{3} \cdot \frac{1}{3} \int \frac{(u-1)+3}{u^2+4} \, du$$

$$= \frac{1}{9} \int \frac{u}{u^2+4} \, du + \frac{1}{9} \int \frac{2}{u^2+2^2} \, du = \frac{1}{9} \cdot \frac{1}{2} \ln(u^2+4) + \frac{2}{9} \cdot \frac{1}{2} \tan^{-1}\left(\frac{1}{2}u\right) + C$$

$$= \frac{1}{18} \ln(9x^2+6x+5) + \frac{1}{9} \tan^{-1}\left[\frac{1}{2}(3x+1)\right] + C$$

20. $\int \frac{dt}{\sin^2 t + \cos 2t} = \int \frac{dt}{\sin^2 t + (\cos^2 t - \sin^2 t)} = \int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt = \tan t + C$

21.



$$\int \frac{dx}{\sqrt{x^2 - 4x}} = \int \frac{dx}{\sqrt{(x^2 - 4x + 4) - 4}} = \int \frac{dx}{\sqrt{(x-2)^2 - 2^2}}$$

$$= \int \frac{2 \sec \theta \tan \theta \, d\theta}{2 \tan \theta} \left[\begin{array}{l} x-2 = 2 \sec \theta, \\ dx = 2 \sec \theta \tan \theta \, d\theta \end{array} \right] = \int \sec \theta \, d\theta = \ln|\sec \theta + \tan \theta| + C_1$$

$$= \ln \left| \frac{x-2}{2} + \frac{\sqrt{x^2 - 4x}}{2} \right| + C_1 = \ln \left| x-2 + \sqrt{x^2 - 4x} \right| + C, \text{ where } C = C_1 - \ln 2$$

22. Let $u = x + 1$. Then

$$\int \frac{x^3}{(x+1)^{10}} \, dx = \int \frac{(u-1)^3}{u^{10}} \, dx = \int \frac{u^3 - 3u^2 + 3u - 1}{u^{10}} \, du$$

$$= \int (u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}) \, du = -\frac{1}{6}u^{-6} + \frac{3}{7}u^{-7} - \frac{3}{8}u^{-8} + \frac{1}{9}u^{-9} + C$$

$$= -\frac{1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{3}{8(x+1)^8} + \frac{1}{9(x+1)^9} + C$$

23. Let $u = \cot 4x$. Then $du = -4 \csc^2 4x \, dx \Rightarrow$

$$\begin{aligned} \int \csc^4 4x \, dx &= \int (\cot^2 4x + 1) \csc^2 4x \, dx = \int (u^2 + 1) \left(-\frac{1}{4} du\right) \\ &= -\frac{1}{4} \left(\frac{1}{3} u^3 + u\right) + C = -\frac{1}{12} (\cot^3 4x + 3 \cot 4x) + C \end{aligned}$$

24. Let $u = \cos x$, $dv = e^x \, dx \Rightarrow du = -\sin x \, dx$, $v = e^x$: (*) $I = \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$.

To integrate $\int e^x \sin x \, dx$, let $U = \sin x$, $dV = e^x \, dx \Rightarrow dU = \cos x \, dx$, $V = e^x$. Then

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - I. \text{ By substitution in (*), } I = e^x \cos x + e^x \sin x - I \Rightarrow \\ 2I &= e^x (\cos x + \sin x) \Rightarrow I = \frac{1}{2} e^x (\cos x + \sin x) + C. \end{aligned}$$

25. $\frac{3x^3 - x^2 + 6x - 4}{(x^2 + 1)(x^2 + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 2} \Rightarrow$

$$\begin{aligned} 3x^3 - x^2 + 6x - 4 &= (Ax + B)(x^2 + 2) + (Cx + D)(x^2 + 1). \text{ Equating the coefficients gives } A + C = 3, \\ B + D &= -1, 2A + C = 6, \text{ and } 2B + D = -4 \Rightarrow A = 3, C = 0, B = -3, \text{ and } D = 2. \text{ Now} \end{aligned}$$

$$\begin{aligned} \int \frac{3x^3 - x^2 + 6x - 4}{(x^2 + 1)(x^2 + 2)} \, dx &= 3 \int \frac{x - 1}{x^2 + 1} \, dx + 2 \int \frac{dx}{x^2 + 2} \\ &= \frac{3}{2} \ln(x^2 + 1) - 3 \tan^{-1} x + \sqrt{2} \tan^{-1} \left(\frac{1}{\sqrt{2}} x\right) + C \end{aligned}$$

26. Let $u = e^x$. Then $x = \ln u$, $dx = \frac{du}{u}$, so

$$\begin{aligned} \int \frac{dx}{1 + e^x} &= \int \frac{du/u}{1 + u} = \int \left[\frac{1}{u} - \frac{1}{u + 1} \right] du = \ln u - \ln(u + 1) + C = \ln e^x - \ln(1 + e^x) + C \\ &= x - \ln(1 + e^x) + C \end{aligned}$$

27. $\int_0^{\pi/2} \cos^3 x \sin 2x \, dx = \int_0^{\pi/2} \cos^3 x (2 \sin x \cos x) \, dx = \int_0^{\pi/2} 2 \cos^4 x \sin x \, dx = \left[-\frac{2}{5} \cos^5 x\right]_0^{\pi/2} = \frac{2}{5}$

28. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 \, du \Rightarrow$

$$\begin{aligned} \int \frac{\sqrt[3]{x} + 1}{\sqrt[3]{x} - 1} \, dx &= \int \frac{u + 1}{u - 1} 3u^2 \, du = 3 \int \left(u^2 + 2u + 2 + \frac{2}{u - 1}\right) du \\ &= u^3 + 3u^2 + 6u + 6 \ln |u - 1| + C = x + 3x^{2/3} + 6\sqrt[3]{x} + 6 \ln |\sqrt[3]{x} - 1| + C \end{aligned}$$

29. The product of an odd function and an even function is an odd function, so $f(x) = x^5 \sec x$ is an odd function. By Theorem 5.5.6(b), $\int_{-1}^1 x^5 \sec x \, dx = 0$.

30. Let $u = e^{-x}$, $du = -e^{-x} \, dx$. Then

$$\int \frac{dx}{e^x \sqrt{1 - e^{-2x}}} = \int \frac{e^{-x} \, dx}{\sqrt{1 - (e^{-x})^2}} = \int \frac{-du}{\sqrt{1 - u^2}} = -\sin^{-1} u + C = -\sin^{-1}(e^{-x}) + C.$$

31. Let $u = \sqrt{e^x - 1}$. Then $u^2 = e^x - 1$ and $2u du = e^x dx$. Also, $e^x + 8 = u^2 + 9$. Thus,

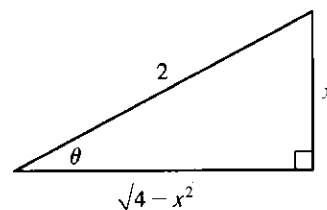
$$\begin{aligned} \int_0^{\ln 10} \frac{e^x \sqrt{e^x - 1}}{e^x + 8} dx &= \int_0^3 \frac{u \cdot 2u du}{u^2 + 9} = 2 \int_0^3 \frac{u^2}{u^2 + 9} du = 2 \int_0^3 \left(1 - \frac{9}{u^2 + 9}\right) du \\ &= 2 \left[u - \frac{9}{3} \tan^{-1} \left(\frac{u}{3} \right) \right]_0^3 = 2[(3 - 3 \tan^{-1} 1) - 0] = 2 \left(3 - 3 \cdot \frac{\pi}{4} \right) = 6 - \frac{3\pi}{2} \end{aligned}$$

32. $\int_0^{\pi/4} \frac{x \sin x}{\cos^3 x} dx = \int_0^{\pi/4} x \tan x \sec^2 x dx \quad \left[\begin{array}{l} u = x, \quad dv = \tan x \sec^2 x dx, \\ du = dx \quad v = \frac{1}{2} \tan^2 x \end{array} \right]$

$$\begin{aligned} &= \left[\frac{x}{2} \tan^2 x \right]_0^{\pi/4} - \frac{1}{2} \int_0^{\pi/4} \tan^2 x dx = \frac{\pi}{8} \cdot 1^2 - 0 - \frac{1}{2} \int_0^{\pi/4} (\sec^2 x - 1) dx \\ &= \frac{\pi}{8} - \frac{1}{2} \left[\tan x - x \right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2} \left(1 - \frac{\pi}{4} \right) = \frac{\pi}{4} - \frac{1}{2} \end{aligned}$$

33. Let $x = 2 \sin \theta \Rightarrow (4 - x^2)^{3/2} = (2 \cos \theta)^3$, $dx = 2 \cos \theta d\theta$, so

$$\begin{aligned} \int \frac{x^2}{(4 - x^2)^{3/2}} dx &= \int \frac{4 \sin^2 \theta}{8 \cos^3 \theta} 2 \cos \theta d\theta \\ &= \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta \\ &= \tan \theta - \theta + C = \frac{x}{\sqrt{4 - x^2}} - \sin^{-1} \left(\frac{x}{2} \right) + C \end{aligned}$$



34. Integrate by parts twice, first with $u = (\arcsin x)^2$, $dv = dx$:

$$I = \int (\arcsin x)^2 dx = x(\arcsin x)^2 - \int 2x \arcsin x \left(\frac{dx}{\sqrt{1 - x^2}} \right)$$

Now let $U = \arcsin x$, $dV = \frac{x}{\sqrt{1 - x^2}} dx \Rightarrow dU = \frac{1}{\sqrt{1 - x^2}} dx$, $V = -\sqrt{1 - x^2}$. So

$$I = x(\arcsin x)^2 - 2 \left[\arcsin x \left(-\sqrt{1 - x^2} \right) + \int dx \right] = x(\arcsin x)^2 + 2\sqrt{1 - x^2} \arcsin x - 2x + C$$

35. $\int \frac{1}{\sqrt{x + x^{3/2}}} dx = \int \frac{dx}{\sqrt{x(1 + \sqrt{x})}} = \int \frac{dx}{\sqrt{x} \sqrt{1 + \sqrt{x}}} \quad \left[\begin{array}{l} u = 1 + \sqrt{x}, \\ du = \frac{dx}{2\sqrt{x}} \end{array} \right] = \int \frac{2 du}{\sqrt{u}} = \int 2u^{-1/2} du$

$$= 4\sqrt{u} + C = 4\sqrt{1 + \sqrt{x}} + C$$

36. $\int \frac{1 - \tan \theta}{1 + \tan \theta} d\theta = \int \frac{\frac{\cos \theta}{\cos \theta} - \frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\cos \theta} + \frac{\sin \theta}{\cos \theta}} d\theta = \int \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} d\theta = \ln |\cos \theta + \sin \theta| + C$

37. $\int (\cos x + \sin x)^2 \cos 2x dx = \int (\cos^2 x + 2 \sin x \cos x + \sin^2 x) \cos 2x dx$

$$= \int (1 + \sin 2x) \cos 2x dx = \int \cos 2x dx + \frac{1}{2} \int \sin 4x dx = \frac{1}{2} \sin 2x - \frac{1}{8} \cos 4x + C$$

Or: $\int (\cos x + \sin x)^2 \cos 2x dx = \int (\cos x + \sin x)^2 (\cos^2 x - \sin^2 x) dx$

$$= \int (\cos x + \sin x)^3 (\cos x - \sin x) dx = \frac{1}{4} (\cos x + \sin x)^4 + C_1$$

38. Let $u = (\tan^{-1} x)^2$, $dv = x dx \Rightarrow du = 2(\tan^{-1} x)/(1+x^2) dx$, $v = \frac{1}{2}x^2$. Then

$$I = \int x(\tan^{-1} x)^2 dx = \frac{1}{2}x^2(\tan^{-1} x)^2 - \int \frac{x^2 \tan^{-1} x}{1+x^2} dx$$

Now let $w = \tan^{-1} x$, $dw = 1/(1+x^2) dx$, and $x^2 = \tan^2 w$. So

$$\begin{aligned} I &= \frac{1}{2}x^2(\tan^{-1} x)^2 - \int w \tan^2 w dw = \frac{1}{2}x^2(\tan^{-1} x)^2 - \int w \sec^2 w dw + \int w dw \\ &= \frac{1}{2}x^2(\tan^{-1} x)^2 - (x \tan^{-1} x - \ln \sqrt{x^2+1}) + \frac{1}{2}(\tan^{-1} x)^2 \quad [\text{parts with } u = w, dv = \sec^2 w dw] \\ &= \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x \tan^{-1} x + \ln \sqrt{x^2+1} + C \\ &\text{or } \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x \tan^{-1} x + \frac{1}{2} \ln(x^2+1) + C \end{aligned}$$

39. We'll integrate $I = \int \frac{xe^{2x}}{(1+2x)^2} dx$ by parts with $u = xe^{2x}$ and $dv = \frac{dx}{(1+2x)^2}$. Then

$$du = (x \cdot 2e^{2x} + e^{2x} \cdot 1) dx \text{ and } v = -\frac{1}{2} \cdot \frac{1}{1+2x}, \text{ so}$$

$$\begin{aligned} I &= -\frac{1}{2} \cdot \frac{xe^{2x}}{1+2x} - \int \left[-\frac{1}{2} \cdot \frac{e^{2x}(2x+1)}{1+2x} \right] dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C \\ &= e^{2x} \left(\frac{1}{4} - \frac{x}{4x+2} \right) + C. \end{aligned}$$

$$\text{Thus, } \int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2} dx = \left[e^{2x} \left(\frac{1}{4} - \frac{x}{4x+2} \right) \right]_0^{1/2} = e \left(\frac{1}{4} - \frac{1}{8} \right) - 1 \left(\frac{1}{4} - 0 \right) = \frac{1}{8}e - \frac{1}{4}.$$

40.
$$\begin{aligned} \int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan \theta}}{\sin 2\theta} d\theta &= \int_{\pi/4}^{\pi/3} \frac{\sqrt{\frac{\sin \theta}{\cos \theta}}}{2 \sin \theta \cos \theta} d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2} (\sin \theta)^{-1/2} (\cos \theta)^{-3/2} d\theta \\ &= \int_{\pi/4}^{\pi/3} \frac{1}{2} \left(\frac{\sin \theta}{\cos \theta} \right)^{-1/2} (\cos \theta)^{-2} d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2} (\tan \theta)^{-1/2} \sec^2 \theta d\theta \\ &= \left[\sqrt{\tan \theta} \right]_{\pi/4}^{\pi/3} = \sqrt{\sqrt{3}} - \sqrt{1} = \sqrt[4]{3} - 1 \end{aligned}$$

41.
$$\begin{aligned} \int_1^\infty \frac{1}{(2x+1)^3} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{(2x+1)^3} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{2} (2x+1)^{-3} 2 dx \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{4(2x+1)^2} \right]_1^t = -\frac{1}{4} \lim_{t \rightarrow \infty} \left[\frac{1}{(2t+1)^2} - \frac{1}{9} \right] = -\frac{1}{4} \left(0 - \frac{1}{9} \right) = \frac{1}{36} \end{aligned}$$

42.
$$\frac{t^2+1}{t^2-1} = \frac{(t^2-1)+2}{t^2-1} = 1 + \frac{2}{(t+1)(t-1)}.$$
 Now $\frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} \Rightarrow$
 $2 = A(t-1) + B(t+1).$ Letting $t = 1 \Rightarrow B = 1$ and letting $t = -1 \Rightarrow A = -1$. So

$$\begin{aligned} \int_0^1 \frac{t^2+1}{t^2-1} dt &= \lim_{b \rightarrow 1^-} \int_0^b \left(1 + \frac{-1}{t+1} + \frac{1}{t-1} \right) dt = \lim_{b \rightarrow 1^-} \left[t - \ln|t+1| + \ln|t-1| \right]_0^b \\ &= \lim_{b \rightarrow 1^-} \left[t + \ln \left| \frac{t-1}{t+1} \right| \right]_0^b = \lim_{b \rightarrow 1^-} \left[\left(b + \ln \left| \frac{b-1}{b+1} \right| \right) - (0+0) \right] = -\infty. \quad \text{Divergent} \end{aligned}$$

43. $\int \frac{dx}{x \ln x} \left[\begin{array}{l} u = \ln x, \\ du = \frac{dx}{x} \end{array} \right] = \int \frac{du}{u} = \ln |u| + C = \ln |\ln x| + C$, so

$$\int_2^\infty \frac{dx}{x \ln x} = \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x \ln x} = \lim_{t \rightarrow \infty} [\ln |\ln x|]_2^t = \lim_{t \rightarrow \infty} [\ln(\ln t) - \ln(\ln 2)] = \infty$$
, so the integral is divergent.

44. Let $u = \sqrt{y-2}$. Then $y = u^2 + 2$ and $dy = 2u \, du$, so

$$\int \frac{y \, dy}{\sqrt{y-2}} = \int \frac{(u^2 + 2)2u \, du}{u} = 2 \int (u^2 + 2) \, du = 2 \left[\frac{1}{3} u^3 + 2u \right] + C$$

Thus, $\int_2^6 \frac{y \, dy}{\sqrt{y-2}} = \lim_{t \rightarrow 2^+} \int_t^6 \frac{y \, dy}{\sqrt{y-2}} = \lim_{t \rightarrow 2^+} \left[\frac{2}{3} (y-2)^{3/2} + 4 \sqrt{y-2} \right]_t^6$

$$= \lim_{t \rightarrow 2^+} \left[\frac{16}{3} + 8 - \frac{2}{3} (t-2)^{3/2} - 4 \sqrt{t-2} \right] = \frac{40}{3}.$$

45. $\int_0^4 \frac{\ln x}{\sqrt{x}} \, dx = \lim_{t \rightarrow 0^+} \int_t^4 \frac{\ln x}{\sqrt{x}} \, dx \stackrel{*}{=} \lim_{t \rightarrow 0^+} [2\sqrt{x} \ln x - 4\sqrt{x}]_t^4$

$$= \lim_{t \rightarrow 0^+} [(2 \cdot 2 \ln 4 - 4 \cdot 2) - (2\sqrt{t} \ln t - 4\sqrt{t})] \stackrel{**}{=} (4 \ln 4 - 8) - (0 - 0) = 4 \ln 4 - 8$$

(*) Let $u = \ln x$, $dv = \frac{1}{\sqrt{x}} \, dx \Rightarrow du = \frac{1}{x} \, dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}} \, dx = 2\sqrt{x} \ln x - 2 \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} \ln x - 4\sqrt{x} + C$$

(**) $\lim_{t \rightarrow 0^+} (2\sqrt{t} \ln t) = \lim_{t \rightarrow 0^+} \frac{2 \ln t}{t^{-1/2}} \stackrel{H}{=} \lim_{t \rightarrow 0^+} \frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t \rightarrow 0^+} (-4\sqrt{t}) = 0$

46. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3} \frac{1}{2-3x} \, dx = \lim_{t \rightarrow (2/3)^-} \int_0^t \frac{1}{2-3x} \, dx = \lim_{t \rightarrow (2/3)^-} \left[-\frac{1}{3} \ln |2-3x| \right]_0^t$$

$$= -\frac{1}{3} \lim_{t \rightarrow (2/3)^-} [\ln |2-3t| - \ln 2] = \infty$$

Since $\int_0^{2/3} \frac{1}{2-3x} \, dx$ diverges, so does $\int_0^1 \frac{1}{2-3x} \, dx$.

47. $\int_0^3 \frac{dx}{x^2 - x - 2} = \int_0^3 \frac{dx}{(x+1)(x-2)} = \int_0^2 \frac{dx}{(x+1)(x-2)} + \int_2^3 \frac{dx}{(x+1)(x-2)}$, and

$$\int_2^3 \frac{dx}{x^2 - x - 2} = \lim_{t \rightarrow 2^+} \int_t^3 \left[\frac{-1/3}{x+1} + \frac{1/3}{x-2} \right] dx = \lim_{t \rightarrow 2^+} \left[\frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| \right]_t^3$$

$$= \lim_{t \rightarrow 2^+} \left[\frac{1}{3} \ln \frac{1}{4} - \frac{1}{3} \ln \left| \frac{t-2}{t+1} \right| \right] = \infty$$

so $\int_0^3 \frac{dx}{x^2 - x - 2}$ diverges.

18. $\int_{-1}^1 \frac{x+1}{\sqrt[3]{x^4}} dx = \int_{-1}^1 (x^{-1/3} + x^{-4/3}) dx = \int_{-1}^0 (x^{-1/3} + x^{-4/3}) dx + \int_0^1 (x^{-1/3} + x^{-4/3}) dx$. But

$$\begin{aligned} \int_0^1 (x^{-1/3} + x^{-4/3}) dx &= \lim_{t \rightarrow 0^+} \int_t^1 (x^{-1/3} + x^{-4/3}) dx = \lim_{t \rightarrow 0^+} \left[\frac{3}{2} x^{2/3} - 3x^{-1/3} \right]_t^1 \\ &= \lim_{t \rightarrow 0^+} \left[\frac{3}{2} - 3 - \frac{3}{2} t^{2/3} + 3t^{-1/3} \right] = \infty. \quad \text{Divergent} \end{aligned}$$

19. Let $u = 2x + 1$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{4x^2 + 4x + 5} &= \int_{-\infty}^{\infty} \frac{\frac{1}{2} du}{u^2 + 4} = \frac{1}{2} \int_{-\infty}^0 \frac{du}{u^2 + 4} + \frac{1}{2} \int_0^{\infty} \frac{du}{u^2 + 4} \\ &= \frac{1}{2} \lim_{t \rightarrow -\infty} \left[\frac{1}{2} \tan^{-1} \left(\frac{1}{2} u \right) \right]_t^0 + \frac{1}{2} \lim_{t \rightarrow \infty} \left[\frac{1}{2} \tan^{-1} \left(\frac{1}{2} u \right) \right]_0^t \\ &= \frac{1}{4} \left[0 - \left(-\frac{\pi}{2} \right) \right] + \frac{1}{4} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{4} \end{aligned}$$

50. $\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\tan^{-1} x}{x^2} dx$. Integrate by parts:

$$\begin{aligned} \int \frac{\tan^{-1} x}{x^2} dx &= \frac{-\tan^{-1} x}{x} + \int \frac{1}{x} \frac{dx}{1+x^2} = \frac{-\tan^{-1} x}{x} + \int \left[\frac{1}{x} - \frac{x}{x^2+1} \right] dx \\ &= \frac{-\tan^{-1} x}{x} + \ln|x| - \frac{1}{2} \ln(x^2+1) + C = \frac{-\tan^{-1} x}{x} + \frac{1}{2} \ln \frac{x^2}{x^2+1} + C \end{aligned}$$

Thus,

$$\begin{aligned} \int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx &= \lim_{t \rightarrow \infty} \left[-\frac{\tan^{-1} x}{x} + \frac{1}{2} \ln \frac{x^2}{x^2+1} \right]_1^t = \lim_{t \rightarrow \infty} \left[-\frac{\tan^{-1} t}{t} + \frac{1}{2} \ln \frac{t^2}{t^2+1} + \frac{\pi}{4} - \frac{1}{2} \ln \frac{1}{2} \right] \\ &= 0 + \frac{1}{2} \ln 1 + \frac{\pi}{4} + \frac{1}{2} \ln 2 = \frac{\pi}{4} + \frac{1}{2} \ln 2 \end{aligned}$$

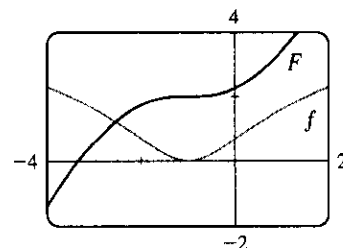
51. We first make the substitution $t = x + 1$, so $\ln(x^2 + 2x + 2) = \ln[(x+1)^2 + 1] = \ln(t^2 + 1)$. Then we use parts with $u = \ln(t^2 + 1)$, $dv = dt$:

$$\begin{aligned} \int \ln(t^2 + 1) dt &= t \ln(t^2 + 1) - \int \frac{t(2t) dt}{t^2 + 1} = t \ln(t^2 + 1) - 2 \int \frac{t^2 dt}{t^2 + 1} \\ &= t \ln(t^2 + 1) - 2 \int \left(1 - \frac{1}{t^2 + 1} \right) dt = t \ln(t^2 + 1) - 2t + 2 \arctan t + C \\ &= (x+1) \ln(x^2 + 2x + 2) - 2x + 2 \arctan(x+1) + K, \quad \text{where } K = C - 2 \end{aligned}$$

[Alternatively, we could have integrated by parts immediately with

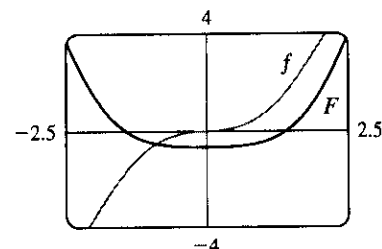
$u = \ln(x^2 + 2x + 2)$.] Notice from the graph that $f = 0$ where F has a

horizontal tangent. Also, F is always increasing, and $f \geq 0$.



52. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x dx = \frac{1}{2} du$, so

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}} dx &= \int \frac{(u-1)}{\sqrt{u}} \left(\frac{1}{2} du\right) = \frac{1}{2} \int (u^{1/2} - u^{-1/2}) du \\&= \frac{1}{2} \left(\frac{2}{3} u^{3/2} - 2u^{1/2}\right) + C \\&= \frac{1}{3} (x^2+1)^{3/2} - (x^2+1)^{1/2} + C \\&= \frac{1}{3} (x^2+1)^{1/2} [(x^2+1) - 3] + C \\&= \frac{1}{3} \sqrt{x^2+1} (x^2-2) + C.\end{aligned}$$

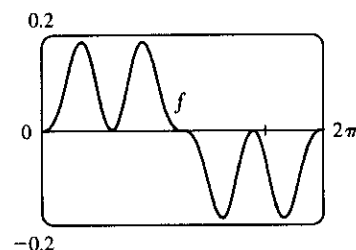


53. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \sin^3 x dx$ is equal to 0.

To evaluate the integral, we write the integral as

$$I = \int_0^{2\pi} \cos^2 x (1 - \cos^2 x) \sin x dx \text{ and let } u = \cos x \Rightarrow$$

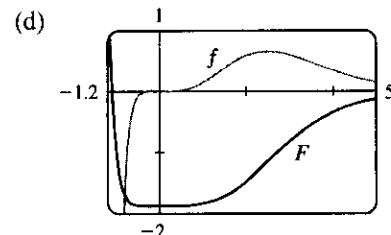
$$du = -\sin x dx. \text{ Thus, } I = \int_1^{-1} u^2 (1 - u^2) (-du) = 0.$$



54. (a) To evaluate $\int x^5 e^{-2x} dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce the degree of the x -factor by 1.

- (b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(c) $\int x^5 e^{-2x} dx$
 $= -\frac{1}{8} e^{-2x} (4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15) + C$



55. $u = e^x \Rightarrow du = e^x dx$, so

$$\int e^x \sqrt{1 - e^{2x}} dx = \int \sqrt{1 - u^2} du \stackrel{30}{=} \frac{1}{2} u \sqrt{1 - u^2} + \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} [e^x \sqrt{1 - e^{2x}} + \sin^{-1}(e^x)] + C$$

56. $\int \csc^5 t dt \stackrel{78}{=} -\frac{1}{4} \cot t \csc^3 t + \frac{3}{4} \int \csc^3 t dt$
 $\stackrel{72}{=} -\frac{1}{4} \cot t \csc^3 t + \frac{3}{4} \left[-\frac{1}{2} \csc t \cot t + \frac{1}{2} \ln |\csc t - \cot t| \right] + C$
 $= -\frac{1}{4} \cot t \csc^3 t - \frac{3}{8} \csc t \cot t + \frac{3}{8} \ln |\csc t - \cot t| + C$

57. $\int \sqrt{x^2 + x + 1} dx = \int \sqrt{x^2 + x + \frac{1}{4} + \frac{3}{4}} dx = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx$
 $= \int \sqrt{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du \quad [u = x + \frac{1}{2}, du = dx]$
 $\stackrel{21}{=} \frac{1}{2} u \sqrt{u^2 + \frac{3}{4}} + \frac{3}{8} \ln \left(u + \sqrt{u^2 + \frac{3}{4}}\right) + C$
 $= \frac{2x+1}{4} \sqrt{x^2 + x + 1} + \frac{3}{8} \ln \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + C$

58. Let $u = \sin x$. Then $du = \cos x \, dx$, so

$$\int \frac{\cot x \, dx}{\sqrt{1+2\sin x}} = \int \frac{du}{u\sqrt{1+2u}} \stackrel{57 \text{ with } a=1, b=2}{=} \ln \left| \frac{\sqrt{1+2u}-1}{\sqrt{1+2u}+1} \right| + C = \ln \left| \frac{\sqrt{1+2\sin x}-1}{\sqrt{1+2\sin x}+1} \right| + C$$

$$\begin{aligned} 59. (a) \frac{d}{du} \left[-\frac{1}{u} \sqrt{a^2-u^2} - \sin^{-1} \left(\frac{u}{a} \right) + C \right] &= \frac{1}{u^2} \sqrt{a^2-u^2} + \frac{1}{\sqrt{a^2-u^2}} - \frac{1}{\sqrt{1-u^2/a^2}} \cdot \frac{1}{a} \\ &= (a^2-u^2)^{-1/2} \left[\frac{1}{u^2} (a^2-u^2) + 1-1 \right] = \frac{\sqrt{a^2-u^2}}{u^2} \end{aligned}$$

(b) Let $u = a \sin \theta \Rightarrow du = a \cos \theta \, d\theta$, $a^2 - u^2 = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta$.

$$\begin{aligned} \int \frac{\sqrt{a^2-u^2}}{u^2} \, du &= \int \frac{a^2 \cos^2 \theta}{a^2 \sin^2 \theta} \, d\theta = \int \frac{1 - \sin^2 \theta}{\sin^2 \theta} \, d\theta = \int (\csc^2 \theta - 1) \, d\theta = -\cot \theta - \theta + C \\ &= -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1} \left(\frac{u}{a} \right) + C \end{aligned}$$

60. Work backward, and use integration by parts with $U = u^{-(n-1)}$ and $dV = (a+bu)^{-1/2} \, du \Rightarrow$

$$dU = \frac{-(n-1) \, du}{u^n} \text{ and } V = \frac{2}{b} \sqrt{a+bu}, \text{ to get}$$

$$\begin{aligned} \int \frac{du}{u^{n-1} \sqrt{a+bu}} &= \int U \, dV = UV - \int V \, dU = \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b} \int \frac{\sqrt{a+bu}}{u^n} \, du \\ &= \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b} \int \frac{a+bu}{u^n \sqrt{a+bu}} \, du \\ &= \frac{2\sqrt{a+bu}}{bu^{n-1}} + 2(n-1) \int \frac{du}{u^{n-1} \sqrt{a+bu}} + \frac{2a(n-1)}{b} \int \frac{du}{u^n \sqrt{a+bu}} \end{aligned}$$

$$\begin{aligned} \text{Rearranging the equation gives } \frac{2a(n-1)}{b} \int \frac{du}{u^n \sqrt{a+bu}} &= -\frac{2\sqrt{a+bu}}{bu^{n-1}} - (2n-3) \int \frac{du}{u^{n-1} \sqrt{a+bu}} \Rightarrow \\ \int \frac{du}{u^n \sqrt{a+bu}} &= \frac{-\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{du}{u^{n-1} \sqrt{a+bu}} \end{aligned}$$

61. For $n \geq 0$, $\int_0^\infty x^n \, dx = \lim_{t \rightarrow \infty} [x^{n+1}/(n+1)]_0^t = \infty$. For $n < 0$, $\int_0^\infty x^n \, dx = \int_0^1 x^n \, dx + \int_1^\infty x^n \, dx$. Both integrals are improper. By (8.8.2), the second integral diverges if $-1 \leq n < 0$. By Exercise 8.8.57, the first integral diverges if $n \leq -1$. Thus, $\int_0^\infty x^n \, dx$ is divergent for all values of n .

$$\begin{aligned} 62. I &= \int_0^\infty e^{ax} \cos x \, dx = \lim_{t \rightarrow \infty} \int_0^t e^{ax} \cos x \, dx \stackrel{99 \text{ with } b=1}{=} \lim_{t \rightarrow \infty} \left[\frac{e^{ax}}{a^2+1} (a \cos x + \sin x) \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left[\frac{e^{at}}{a^2+1} (a \cos t + \sin t) - \frac{1}{a^2+1} (a) \right] = \frac{1}{a^2+1} \lim_{t \rightarrow \infty} [e^{at} (a \cos t + \sin t) - a]. \end{aligned}$$

For $a \geq 0$, the limit does not exist due to oscillation. For $a < 0$, $\lim_{t \rightarrow \infty} [e^{at} (a \cos t + \sin t)] = 0$ by the Squeeze

Theorem, because $|e^{at} (a \cos t + \sin t)| \leq e^{at} (|a| + 1)$, so $I = \frac{1}{a^2+1} (-a) = -\frac{a}{a^2+1}$.

$$63. f(x) = \sqrt{1+x^4}, \Delta x = \frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10}.$$

$$(a) T_{10} = \frac{1}{10 \cdot 2} \{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.090608$$

$$(b) M_{10} = \frac{1}{10} [f(\frac{1}{20}) + f(\frac{3}{20}) + f(\frac{5}{20}) + \cdots + f(\frac{19}{20})] \approx 1.088840$$

$$(c) S_{10} = \frac{1}{10 \cdot 3} [f(0) + 4f(0.1) + 2f(0.2) + \cdots + 4f(0.9) + f(1)] \approx 1.089429$$

f is concave upward, so the Trapezoidal Rule gives us an overestimate, the Midpoint Rule gives an underestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

$$64. f(x) = \sqrt{\sin x}, \Delta x = \frac{\frac{\pi}{2} - 0}{10} = \frac{\pi}{20}.$$

$$(a) T_{10} = \frac{\pi}{20 \cdot 2} \{f(0) + 2[f(\frac{\pi}{20}) + f(\frac{2\pi}{20}) + \cdots + f(\frac{9\pi}{20})] + f(\frac{\pi}{2})\} \approx 1.185197$$

$$(b) M_{10} = \frac{\pi}{20} [f(\frac{\pi}{40}) + f(\frac{3\pi}{40}) + f(\frac{5\pi}{40}) + \cdots + f(\frac{17\pi}{40}) + f(\frac{19\pi}{40})] \approx 1.201932$$

$$(c) S_{10} = \frac{\pi}{20 \cdot 3} [f(0) + 4f(\frac{\pi}{20}) + 2f(\frac{2\pi}{20}) + \cdots + 4f(\frac{9\pi}{20}) + f(\frac{\pi}{2})] \approx 1.193089$$

f is concave downward, so the Trapezoidal Rule gives us an underestimate, the Midpoint Rule gives an overestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

$$65. f(x) = (1+x^4)^{1/2}, f'(x) = \frac{1}{2}(1+x^4)^{-1/2}(4x^3) = 2x^3(1+x^4)^{-1/2}, f''(x) = (2x^6 + 6x^2)(1+x^4)^{-3/2}.$$

A graph of f'' on $[0, 1]$ shows that it has its maximum at $x = 1$, so $|f''(x)| \leq f''(1) = \sqrt{8}$ on $[0, 1]$. By taking

$K = \sqrt{8}$, we find that the error in Exercise 63(a) is bounded by $\frac{K(b-a)^3}{12n^2} = \frac{\sqrt{8}}{1200} \approx 0.0024$, and in (b) by about $\frac{1}{2}(0.0024) = 0.0012$.

Note: Another way to estimate K is to let $x = 1$ in the factor $2x^6 + 6x^2$ (maximizing the numerator) and let $x = 0$ in the factor $(1+x^4)^{-3/2}$ (minimizing the denominator). Doing so gives us $K = 8$ and errors of $0.00\bar{6}$ and $0.00\bar{3}$.

Using $K = 8$ for the Trapezoidal Rule, we have $|E_T| \leq \frac{K(b-a)^3}{12n^2} \leq 0.00001 \Leftrightarrow \frac{8(1-0)^3}{12n^2} \leq \frac{1}{100,000}$

$$\Leftrightarrow n^2 \geq \frac{800,000}{12} \Leftrightarrow n \gtrsim 258.2, \text{ so we should take } n = 259.$$

For the Midpoint Rule, $|E_M| \leq \frac{K(b-a)^3}{24n^2} \leq 0.00001 \Leftrightarrow n^2 \geq \frac{800,000}{24} \Leftrightarrow n \gtrsim 182.6$, so we should take $n = 183$.

$$66. \int_1^4 \frac{e^x}{x} dx \approx S_6 = \frac{(4-1)/6}{3} [f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 17.739438$$

$$67. \Delta t = (\frac{10}{60} - 0)/10 = \frac{1}{60}.$$

$$\text{Distance traveled} = \int_0^{10} v dt \approx S_{10}$$

$$\begin{aligned} &= \frac{1}{60 \cdot 3} [40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56] \\ &= \frac{1}{180} (1544) = 8.5\bar{7} \text{ mi} \end{aligned}$$

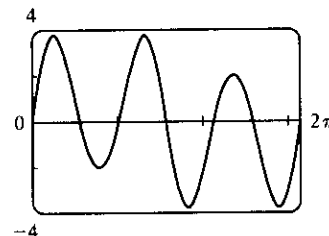
58. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$:

$$\begin{aligned} \text{Increase in bee population} &= \int_0^{24} r(t) dt \approx S_6 \\ &= \frac{4}{3}[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)] \\ &= \frac{4}{3}[0 + 4(300) + 2(3000) + 4(11,000) + 2(4000) + 4(400) + 0] \\ &= \frac{4}{3}(60,800) \approx 81,067 \text{ bees} \end{aligned}$$

59. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$\begin{aligned} f^{(4)}(x) &= \sin(\sin x)[\cos^4 x + 7\cos^2 x - 3] \\ &\quad + \cos(\sin x)[6\cos^2 x \sin x + \sin x] \end{aligned}$$

From the graph, we see that $|f^{(4)}(x)| < 3.8$ for $x \in [0, \pi]$.



(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^\pi f(x) dx \approx \frac{\pi}{10 \cdot 3} [f(0) + 4f(\frac{\pi}{10}) + 2f(\frac{2\pi}{10}) + \cdots + 4f(\frac{9\pi}{10}) + f(\pi)] \approx 1.786721$$

From part (a), we know that $|f^{(4)}(x)| < 3.8$ on $[0, \pi]$, so we use Theorem 8.7.4 with $K = 3.8$, and estimate the error as $|E_S| \leq \frac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \leq \frac{3.8\pi^5}{180n^4} \leq 0.00001$, so

$$\begin{aligned} n^4 &\geq \frac{3.8\pi^5}{180(0.00001)} \approx 646,041.6 \Rightarrow n \geq 28.35. \text{ Since } n \text{ must be even for Simpson's Rule, we must have} \\ n &\geq 30 \text{ to ensure the desired accuracy.} \end{aligned}$$

70. With an x -axis in the normal position, at $x = 7$ we have $C = 2\pi r = 45 \Rightarrow r(7) = \frac{2\pi}{45}$. Using Simpson's Rule with $n = 4$ and $\Delta x = 7$, we have

$$V = \int_0^{28} \pi[r(x)]^2 dx \approx S_4 = \frac{\pi}{3} \left[0 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 2\pi\left(\frac{53}{2\pi}\right)^2 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 0 \right] = \frac{\pi}{3} \left(\frac{21,818}{4\pi} \right) \approx 4051 \text{ cm}^3$$

71. $\frac{x^3}{x^5 + 2} \leq \frac{x^3}{x^5} = \frac{1}{x^2}$ for x in $[1, \infty)$. $\int_1^\infty \frac{1}{x^2} dx$ is convergent by (8.8.2) with $p = 2 > 1$. Therefore,

$$\int_1^\infty \frac{x^3}{x^5 + 2} dx \text{ is convergent by the Comparison Theorem.}$$

72. The line $y = 3$ intersects the hyperbola $y^2 - x^2 = 1$ at two points on its upper branch, namely $(-\sqrt{2}, 3)$ and $(\sqrt{2}, 3)$. The desired area is

$$\begin{aligned} A &= \int_{-\sqrt{2}}^{\sqrt{2}} (3 - \sqrt{x^2 + 1}) dx = 2 \int_0^{\sqrt{2}} (3 - \sqrt{x^2 + 1}) dx \\ &\stackrel{21}{=} 2 \left[3x - \frac{1}{2} x \sqrt{x^2 + 1} - \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) \right]_0^{\sqrt{2}} \\ &= [6x - x \sqrt{x^2 + 1} - \ln(x + \sqrt{x^2 + 1})]_0^{2\sqrt{2}} = 12\sqrt{2} - 2\sqrt{2} \cdot 3 - \ln(2\sqrt{2} + 3) \\ &= 6\sqrt{2} - \ln(3 + 2\sqrt{2}) \end{aligned}$$

Another method: $A = 2 \int_1^3 \sqrt{y^2 - 1} dy$ and use Formula 39.

73. For x in $[0, \frac{\pi}{2}]$, $0 \leq \cos^2 x \leq \cos x$. For x in $[\frac{\pi}{2}, \pi]$, $\cos x \leq 0 \leq \cos^2 x$. Thus,

$$\begin{aligned}\text{area} &= \int_0^{\pi/2} (\cos x - \cos^2 x) dx + \int_{\pi/2}^{\pi} (\cos^2 x - \cos x) dx \\ &= [\sin x - \frac{1}{2}x - \frac{1}{4}\sin 2x]_0^{\pi/2} + [\frac{1}{2}x + \frac{1}{4}\sin 2x - \sin x]_{\pi/2}^{\pi} \\ &= [(1 - \frac{\pi}{4}) - 0] + [\frac{\pi}{2} - (\frac{\pi}{4} - 1)] = 2\end{aligned}$$

74. The curves $y = \frac{1}{2 \pm \sqrt{x}}$ are defined for $x \geq 0$. For $x > 0$, $\frac{1}{2 - \sqrt{x}} > \frac{1}{2 + \sqrt{x}}$. Thus, the required area is

$$\begin{aligned}\int_0^1 \left(\frac{1}{2 - \sqrt{x}} - \frac{1}{2 + \sqrt{x}} \right) dx &= \int_0^1 \left(\frac{1}{2 - u} - \frac{1}{2 + u} \right) 2u du \quad [\text{put } u = \sqrt{x}] \\ &= 2 \int_0^1 \left(-\frac{u}{u-2} - \frac{u}{u+2} \right) du = 2 \int_0^1 \left(-1 - \frac{2}{u-2} - 1 + \frac{2}{u+2} \right) du \\ &= 2 \left[2 \ln \left| \frac{u+2}{u-2} \right| - 2u \right]_0^1 = 4 \ln 3 - 4\end{aligned}$$

75. Using the formula for disks, the volume is

$$\begin{aligned}V &= \int_0^{\pi/2} \pi [f(x)]^2 dx = \pi \int_0^{\pi/2} (\cos^2 x)^2 dx = \pi \int_0^{\pi/2} [\frac{1}{2}(1 + \cos 2x)]^2 dx \\ &= \frac{\pi}{4} \int_0^{\pi/2} (1 + \cos^2 2x + 2 \cos 2x) dx = \frac{\pi}{4} \int_0^{\pi/2} [1 + \frac{1}{2}(1 + \cos 4x) + 2 \cos 2x] dx \\ &= \frac{\pi}{4} [\frac{3}{2}x + \frac{1}{2}(\frac{1}{4}\sin 4x) + 2(\frac{1}{2}\sin 2x)]_0^{\pi/2} = \frac{\pi}{4} [(\frac{3\pi}{4} + \frac{1}{8} \cdot 0 + 0) - 0] = \frac{3\pi^2}{16}\end{aligned}$$

76. Using the formula for cylindrical shells, the volume is

$$\begin{aligned}V &= \int_0^{\pi/2} 2\pi x f(x) dx = 2\pi \int_0^{\pi/2} x \cos^2 x dx \\ &= 2\pi \int_0^{\pi/2} x [\frac{1}{2}(1 + \cos 2x)] dx = 2(\frac{1}{2}) \pi \int_0^{\pi/2} (x + x \cos 2x) dx \\ &= \pi \left([\frac{1}{2}x^2]_0^{\pi/2} + [x(\frac{1}{2}\sin 2x)]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{2}\sin 2x dx \right) \quad [\text{parts with } u = x, dv = \cos 2x dx] \\ &= \pi \left[\frac{1}{2}(\frac{\pi}{2})^2 + 0 - \frac{1}{2}[-\frac{1}{2}\cos 2x]_0^{\pi/2} \right] = \frac{\pi^3}{8} + \frac{\pi}{4}(-1 - 1) = \frac{1}{8}(\pi^3 - 4\pi)\end{aligned}$$

77. By the Fundamental Theorem of Calculus,

$$\int_0^{\infty} f'(x) dx = \lim_{t \rightarrow \infty} \int_0^t f'(x) dx = \lim_{t \rightarrow \infty} [f(t) - f(0)] = \lim_{t \rightarrow \infty} f(t) - f(0) = 0 - f(0) = -f(0).$$

$$\begin{aligned}78. \text{ (a) } (\tan^{-1} x)_{\text{ave}} &= \lim_{t \rightarrow \infty} \frac{1}{t-0} \int_0^t \tan^{-1} x dx \stackrel{89}{=} \lim_{t \rightarrow \infty} \left\{ \frac{1}{t} [x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)]_0^t \right\} \\ &= \lim_{t \rightarrow \infty} \left[\frac{1}{t} (t \tan^{-1} t - \frac{1}{2} \ln(1+t^2)) \right] = \lim_{t \rightarrow \infty} \left[\tan^{-1} t - \frac{\ln(1+t^2)}{2t} \right] \\ &\stackrel{H}{=} \frac{\pi}{2} - \lim_{t \rightarrow \infty} \frac{2t/(1+t^2)}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

2 □ CHAPTER 8 TECHNIQUES OF INTEGRATION

(b) $f(x) \geq 0$ and $\int_a^\infty f(x) dx$ is divergent $\Rightarrow \lim_{t \rightarrow \infty} \int_a^t f(x) dx = \infty$.

$$f_{\text{ave}} = \lim_{t \rightarrow \infty} \frac{\int_a^t f(x) dx}{t - a} \stackrel{H}{=} \lim_{t \rightarrow \infty} \frac{f(t)}{1} \quad [\text{by FTC1}] = \lim_{x \rightarrow \infty} f(x), \text{ if this limit exists.}$$

(c) Suppose $\int_a^\infty f(x) dx$ converges; that is, $\lim_{t \rightarrow \infty} \int_a^t f(x) dx = L < \infty$. Then

$$f_{\text{ave}} = \lim_{t \rightarrow \infty} \left[\frac{1}{t - a} \int_a^t f(x) dx \right] = \lim_{t \rightarrow \infty} \frac{1}{t - a} \cdot \lim_{t \rightarrow \infty} \int_a^t f(x) dx = 0 \cdot L = 0$$

$$(d) (\sin x)_{\text{ave}} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sin x dx = \lim_{t \rightarrow \infty} \left(\frac{1}{t} [-\cos x]_0^t \right) = \lim_{t \rightarrow \infty} \left(-\frac{\cos t}{t} + \frac{1}{t} \right) = \lim_{t \rightarrow \infty} \frac{1 - \cos t}{t} = 0$$

9. Let $u = 1/x \Rightarrow x = 1/u \Rightarrow dx = -(1/u^2) du$.

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2} \left(-\frac{du}{u^2} \right) = \int_\infty^0 \frac{-\ln u}{u^2+1} (-du) = \int_\infty^0 \frac{\ln u}{1+u^2} du = -\int_0^\infty \frac{\ln u}{1+u^2} du$$

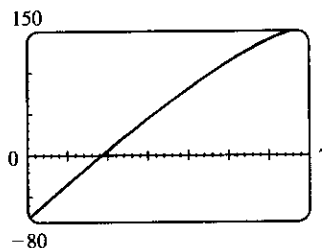
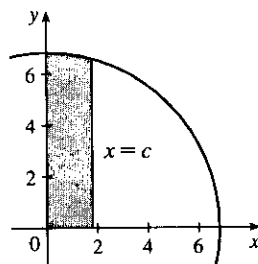
$$\text{Therefore, } \int_0^\infty \frac{\ln x}{1+x^2} dx = -\int_0^\infty \frac{\ln x}{1+x^2} dx = 0.$$

10. If the distance between P and the point charge is d , then the potential V at P is

$$V = W = \int_\infty^d F dr = \int_\infty^d \frac{q}{4\pi\epsilon_0 r^2} dr = \lim_{t \rightarrow \infty} \frac{q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_t^d = \frac{q}{4\pi\epsilon_0} \lim_{t \rightarrow \infty} \left(-\frac{1}{d} + \frac{1}{t} \right) = -\frac{q}{4\pi\epsilon_0 d}$$

□ PROBLEMS PLUS

1.



By symmetry, the problem can be reduced to finding the line $x = c$ such that the shaded area is one-third of the area of the quarter-circle. The equation of the circle is $y = \sqrt{49 - x^2}$, so we require that $\int_0^c \sqrt{49 - x^2} dx = \frac{1}{3} \cdot \frac{1}{4} \pi (7)^2$
 $\Leftrightarrow \left[\frac{1}{2} x \sqrt{49 - x^2} + \frac{49}{2} \sin^{-1}(x/7) \right]_0^c = \frac{49}{12} \pi$ [by Formula 30] $\Leftrightarrow \frac{1}{2} c \sqrt{49 - c^2} + \frac{49}{2} \sin^{-1}(c/7) = \frac{49}{12} \pi$.
 This equation would be difficult to solve exactly, so we plot the left-hand side as a function of c , and find that the equation holds for $c \approx 1.85$. So the cuts should be made at distances of about 1.85 inches from the center of the pizza.

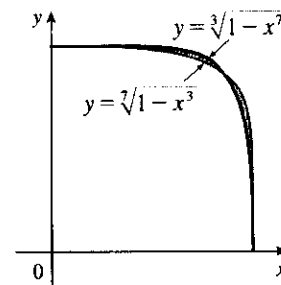
$$\begin{aligned} 2. \int \frac{1}{x^7 - x} dx &= \int \frac{dx}{x(x^6 - 1)} = \int \frac{x^5}{x^6(x^6 - 1)} dx = \frac{1}{6} \int \frac{1}{u(u - 1)} du \quad [u = x^6, du = 6x^5 dx] \\ &= \frac{1}{6} \int \left(\frac{1}{u - 1} - \frac{1}{u} \right) du = \frac{1}{6} (\ln |u - 1| - \ln |u|) + C = \frac{1}{6} \ln \left| \frac{u - 1}{u} \right| + C \\ &= \frac{1}{6} \ln \left| \frac{x^6 - 1}{x^6} \right| + C \end{aligned}$$

Alternate method: $\int \frac{1}{x^7 - x} dx = \int \frac{x^{-7}}{1 - x^{-6}} dx \quad [u = 1 - x^{-6}, du = 6x^{-7} dx]$

$$= \frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \ln |u| + C = \frac{1}{6} \ln |1 - x^{-6}| + C$$

Other methods: Substitute $u = x^3$ or $x^3 = \sec \theta$.

3. The given integral represents the difference of the shaded areas, which appears to be 0. It can be calculated by integrating with respect to either x or y , so we find x in terms of y for each curve: $y = \sqrt[3]{1 - x^7} \Rightarrow x = \sqrt[7]{1 - y^3}$ and $y = \sqrt[7]{1 - x^3} \Rightarrow x = \sqrt[3]{1 - y^7}$, so
- $$\int_0^1 \left(\sqrt[3]{1 - y^7} - \sqrt[7]{1 - y^3} \right) dy = \int_0^1 \left(\sqrt[7]{1 - x^3} - \sqrt[3]{1 - x^7} \right) dx.$$
- But this equation is of the form $z = -z$. So $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3} \right) dx = 0$.

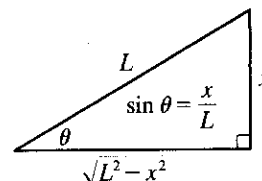


4. (a) The tangent to the curve $y = f(x)$ at $x = x_0$ has the equation $y - f(x_0) = f'(x_0)(x - x_0)$. The y -intercept of this tangent line is $f(x_0) - f'(x_0)x_0$. Thus, L is the distance from the point $(0, f(x_0) - f'(x_0)x_0)$ to the point $(x_0, f(x_0))$; that is, $L^2 = x_0^2 + [f'(x_0)]^2 x_0^2$, so $[f'(x_0)]^2 = \frac{L^2 - x_0^2}{x_0^2}$ and $f'(x_0) = -\frac{\sqrt{L^2 - x_0^2}}{x_0}$ for $0 < x_0 < L$.

$$(b) \frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x} \Rightarrow y = \int \left(-\frac{\sqrt{L^2 - x^2}}{x} \right) dx.$$

Let $x = L \sin \theta$. Then $dx = L \cos \theta d\theta$ and

$$\begin{aligned} y &= \int \frac{-L \cos \theta L \cos \theta d\theta}{L \sin \theta} = L \int \frac{\sin^2 \theta - 1}{\sin \theta} d\theta \\ &= L \int (\sin \theta - \csc \theta) d\theta \\ &= -L \cos \theta - L \ln |\csc \theta - \cot \theta| + C \\ &= -\sqrt{L^2 - x^2} - L \ln \left(\frac{L}{x} - \frac{\sqrt{L^2 - x^2}}{x} \right) + C. \end{aligned}$$



When $x = L$, $y = 0$, and $0 = -0 - L \ln(1 - 0) + C$, so $C = 0$. Therefore,

$$y = -\sqrt{L^2 - x^2} - L \ln \left(\frac{L - \sqrt{L^2 - x^2}}{x} \right)$$

5. Recall that $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$. So

$$\begin{aligned} f(x) &= \int_0^\pi \cos t \cos(x - t) dt = \frac{1}{2} \int_0^\pi [\cos(t + x - t) + \cos(t - x + t)] dt \\ &= \frac{1}{2} \int_0^\pi [\cos x + \cos(2t - x)] dt = \frac{1}{2} \left[t \cos x + \frac{1}{2} \sin(2t - x) \right]_0^\pi \\ &= \frac{\pi}{2} \cos x + \frac{1}{4} \sin(2\pi - x) - \frac{1}{4} \sin(-x) = \frac{\pi}{2} \cos x + \frac{1}{4} \sin(-x) - \frac{1}{4} \sin(-x) \\ &= \frac{\pi}{2} \cos x \end{aligned}$$

The minimum of $\cos x$ on this domain is -1 , so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

6. n is a positive integer, so

$$\int (\ln x)^n dx = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1} (dx/x) \quad [\text{by parts}] = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

Thus,

$$\begin{aligned} \int_0^1 (\ln x)^n dx &= \lim_{t \rightarrow 0^+} \int_t^1 (\ln x)^n dx = \lim_{t \rightarrow 0^+} [x(\ln x)^n]_t^1 - n \lim_{t \rightarrow 0^+} \int_t^1 (\ln x)^{n-1} dx \\ &= - \lim_{t \rightarrow 0^+} \frac{(\ln t)^n}{1/t} - n \int_0^1 (\ln x)^{n-1} dx = -n \int_0^1 (\ln x)^{n-1} dx \end{aligned}$$

by repeated application of l'Hospital's Rule. We want to prove that $\int_0^1 (\ln x)^n dx = (-1)^n n!$ for every positive integer n . For $n = 1$, we have

$$\int_0^1 (\ln x)^1 dx = (-1) \int_0^1 (\ln x)^0 dx = - \int_0^1 dx = -1 \quad \left[\text{or } \int_0^1 \ln x dx = \lim_{t \rightarrow 0^+} [x \ln x - x]_t^1 = -1 \right]$$

Assuming that the formula holds for n , we find that

$$\int_0^1 (\ln x)^{n+1} dx = -(n+1) \int_0^1 (\ln x)^n dx = -(n+1)(-1)^n n! = (-1)^{n+1} (n+1)!$$

This is the formula for $n+1$. Thus, the formula holds for all positive integers n by induction.

7. In accordance with the hint, we let $I_k = \int_0^1 (1 - x^2)^k dx$, and we find an expression for I_{k+1} in terms of I_k . We integrate I_{k+1} by parts with $u = (1 - x^2)^{k+1} \Rightarrow du = (k+1)(1 - x^2)^k(-2x) dx$, $dv = dx \Rightarrow v = x$, and then split the remaining integral into identifiable quantities:

$$\begin{aligned} I_{k+1} &= x(1 - x^2)^{k+1} \Big|_0^1 + 2(k+1) \int_0^1 x^2(1 - x^2)^k dx = (2k+2) \int_0^1 (1 - x^2)^k [1 - (1 - x^2)] dx \\ &= (2k+2)(I_k - I_{k+1}) \end{aligned}$$

So $I_{k+1}[1 + (2k + 2)] = (2k + 2)I_k \Rightarrow I_{k+1} = \frac{2k+2}{2k+3}I_k$. Now to complete the proof, we use induction:

$I_0 = 1 = \frac{2^0(0!)^2}{1!}$, so the formula holds for $n = 0$. Now suppose it holds for $n = k$. Then

$$\begin{aligned} I_{k+1} &= \frac{2k+2}{2k+3}I_k = \frac{2k+2}{2k+3} \left[\frac{2^{2k}(k!)^2}{(2k+1)!} \right] = \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{2(k+1)}{2k+2} \cdot \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} \\ &= \frac{[2(k+1)]^2 2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} = \frac{2^{2(k+1)}[(k+1)!]^2}{[2(k+1)+1]!} \end{aligned}$$

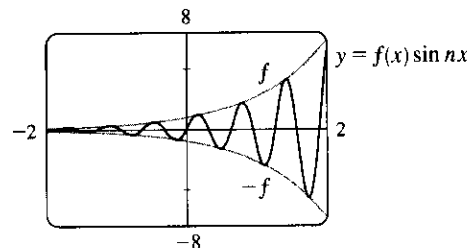
So by induction, the formula holds for all integers $n \geq 0$.

8. (a) Since $-1 \leq \sin \leq 1$, we have

$-f(x) \leq f(x) \sin nx \leq f(x)$, and the graph of

$y = f(x) \sin nx$ oscillates between $f(x)$ and $-f(x)$.

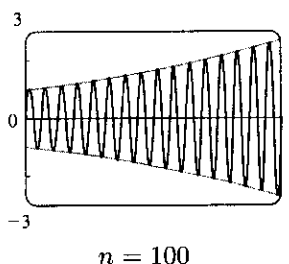
(The diagram shows the case $f(x) = e^x$ and $n = 10$.) As $n \rightarrow \infty$, the graph oscillates more and more frequently; see the graphs in part (b).



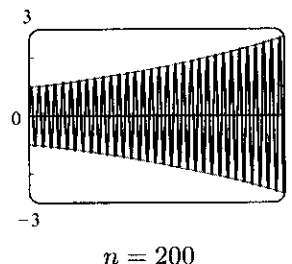
- (b) From the graphs of the integrand, it seems

that $\lim_{n \rightarrow \infty} \int_0^1 f(x) \sin nx \, dx = 0$, since as n

increases, the integrand oscillates more and more rapidly, and thus (since f' is continuous) it makes sense that the areas above the x -axis and below it during each oscillation approach equality.



$n = 100$



$n = 200$

- (c) We integrate by parts with $u = f(x) \Rightarrow du = f'(x) \, dx$, $dv = \sin nx \, dx \Rightarrow v = -\frac{\cos nx}{n}$:

$$\begin{aligned} \int_0^1 f(x) \sin nx \, dx &= \left[-\frac{f(x) \cos nx}{n} \right]_0^1 + \int_0^1 \frac{\cos nx}{n} f'(x) \, dx \\ &= \frac{1}{n} \left(\int_0^1 \cos nx f'(x) \, dx - [f(x) \cos nx]_0^1 \right) \\ &= \frac{1}{n} \left[\int_0^1 \cos nx f'(x) \, dx + f(0) - f(1) \cos n \right] \end{aligned}$$

Taking absolute values of the first and last terms in this equality, and using the facts that $|\alpha \pm \beta| \leq |\alpha| + |\beta|$,

$\int_0^1 f(x) \, dx \leq \int_0^1 |f(x)| \, dx$, $|f(0)| = f(0)$ [f is positive], $|f'(x)| \leq M$ for $0 \leq x \leq 1$, and $|\cos nx| \leq 1$,

$$\left| \int_0^1 f(x) \sin nx \, dx \right| \leq \frac{1}{n} \left[\int_0^1 M \, dx + |f(0)| + |f(1)| \right] = \frac{1}{n} [M + |f(0)| + |f(1)|]$$

which approaches 0 as $n \rightarrow \infty$. The result follows by the Squeeze Theorem.

9. $0 < a < b$. Now

$$\int_0^1 [bx + a(1-x)]^t dx = \int_a^b \frac{u^t}{(b-a)} du \quad [\text{put } u = bx + a(1-x)] = \left[\frac{u^{t+1}}{(t+1)(b-a)} \right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}.$$

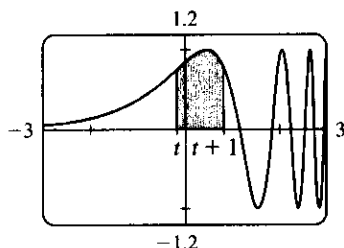
Now let $y = \lim_{t \rightarrow 0} \left[\frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)} \right]^{1/t}$. Then $\ln y = \lim_{t \rightarrow 0} \left[\frac{1}{t} \ln \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)} \right]$. This limit is of the form $0/0$,

so we can apply l'Hospital's Rule to get

$$\ln y = \lim_{t \rightarrow 0} \left[\frac{b^{t+1} \ln b - a^{t+1} \ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1} \right] = \frac{b \ln b - a \ln a}{b-a} - 1 = \frac{b \ln b}{b-a} - \frac{a \ln a}{b-a} - \ln e = \ln \frac{b^{b/(b-a)}}{ea^{a/(b-a)}}.$$

Therefore, $y = e^{-1} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}.$

10.



From the graph, it appears that the area under the graph of

$f(x) = \sin(e^x)$ on the interval $[t, t+1]$ is greatest when

$t \approx -0.2$. To find the exact value, we write the integral as

$$I = \int_t^{t+1} f(x) dx = \int_0^{t+1} f(x) dx - \int_0^t f(x) dx, \text{ and use FTC1}$$

to find $dI/dt = f(t+1) - f(t) = \sin(e^{t+1}) - \sin(e^t) = 0$

when $\sin(e^{t+1}) = \sin(e^t)$.

Now we have $\sin x = \sin y$ whenever $x - y = 2k\pi$ and also whenever x and y are the same distance from

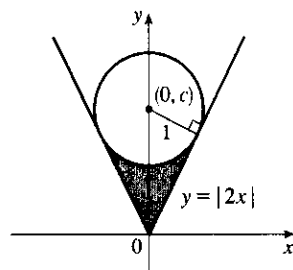
$(k + \frac{1}{2})\pi$, k any integer, since $\sin x$ is symmetric about the line $x = (k + \frac{1}{2})\pi$. The first possibility is the more

obvious one, but if we calculate $e^{t+1} - e^t = 2k\pi$, we get $t = \ln(2k\pi/(e-1))$, which is about 1.3 for $k = 1$ (the least possible value of k). From the graph, this looks unlikely to give the maximum we are looking for. So instead

we set $e^{t+1} - (k + \frac{1}{2})\pi = (k + \frac{1}{2})\pi - e^t \Leftrightarrow e^{t+1} + e^t = (2k+1)\pi \Leftrightarrow e^t(e+1) = (2k+1)\pi \Leftrightarrow$

$t = \ln((2k+1)\pi/(e+1))$. Now $k = 0 \Rightarrow t = \ln(\pi/(e+1)) \approx -0.16853$, which does give the maximum value, as we have seen from the graph of f .

11.



An equation of the circle with center $(0, c)$ and radius 1 is

$x^2 + (y - c)^2 = 1^2$, so an equation of the lower semicircle is

$y = c - \sqrt{1 - x^2}$. At the points of tangency, the slopes of the line

and semicircle must be equal. For $x \geq 0$, we must have

$$y' = 2 \Rightarrow \frac{x}{\sqrt{1-x^2}} = 2 \Rightarrow x = 2\sqrt{1-x^2} \Rightarrow$$

$$x^2 = 4(1-x^2) \Rightarrow 5x^2 = 4 \Rightarrow x^2 = \frac{4}{5} \Rightarrow x = \frac{2}{5}\sqrt{5}$$

and so $y = 2(\frac{2}{5}\sqrt{5}) = \frac{4}{5}\sqrt{5}$. The slope of the perpendicular line segment is $-\frac{1}{2}$, so an equation of the line

segment is $y - \frac{4}{5}\sqrt{5} = -\frac{1}{2}(x - \frac{2}{5}\sqrt{5}) \Leftrightarrow y = -\frac{1}{2}x + \frac{1}{5}\sqrt{5} + \frac{4}{5}\sqrt{5} \Leftrightarrow y = -\frac{1}{2}x + \sqrt{5}$, so $c = \sqrt{5}$ and

an equation of the lower semicircle is $y = \sqrt{5} - \sqrt{1 - x^2}$. Thus, the shaded area is

$$\begin{aligned} 2 \int_0^{(2/5)\sqrt{5}} \left[\left(\sqrt{5} - \sqrt{1 - x^2} \right) - 2x \right] dx &\stackrel{30}{=} 2 \left[\sqrt{5}x - \frac{x}{2}\sqrt{1 - x^2} - \frac{1}{2}\sin^{-1}x - x^2 \right]_0^{(2/5)\sqrt{5}} \\ &= 2 \left[2 - \frac{\sqrt{5}}{5} \cdot \frac{1}{\sqrt{5}} - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{4}{5} \right] - 2(0) \\ &= 2 \left[1 - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \right] = 2 - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \end{aligned}$$

12. (a) $M \frac{dv}{dt} - ub = -Mg \Rightarrow (M_0 - bt) \frac{dv}{dt} = ub - (M_0 - bt)g \Rightarrow \frac{dv}{dt} = \frac{ub}{M_0 - bt} - g \Rightarrow$
 $v(t) = -u \ln(M_0 - bt) - gt + C$. Now $0 = v(0) = -u \ln M_0 + C$, so $C = u \ln M_0$. Thus
 $v(t) = u \ln M_0 - u \ln(M_0 - bt) - gt = u \ln \frac{M_0}{M_0 - bt} - gt$.

(b) Burnout velocity $= v\left(\frac{M_2}{b}\right) = u \ln \frac{M_0}{M_0 - M_2} - g \frac{M_2}{b} = u \ln \frac{M_0}{M_1} - g \frac{M_2}{b}$.

Note: The reason for the term "burnout velocity" is that M_2 kilograms of fuel is used in M_2/b seconds, so $v(M_2/b)$ is the rocket's velocity when the fuel is used up.

(c) Height at burnout time $= y\left(\frac{M_2}{b}\right)$. Now $\frac{dy}{dt} = v(t) = u \ln M_0 - gt - u \ln(M_0 - bt)$, so

$y(t) = (u \ln M_0)t - \frac{gt^2}{2} - \frac{u}{b}(M_0 - bt) \ln(M_0 - bt) + ut + C$. Since $0 = y(0) = \frac{u}{b}M_0 \ln M_0 + C$, we get
 $C = -\frac{u}{b}M_0 \ln M_0$ and $y(t) = u(1 + \ln M_0)t - \frac{gt^2}{2} + \frac{u}{b}(M_0 - bt) \ln(M_0 - bt) - \frac{u}{b}M_0 \ln M_0$. Therefore,
the height at burnout is

$$\begin{aligned} y\left(\frac{M_2}{b}\right) &= u(1 + \ln M_0) \frac{M_2}{b} - \frac{g}{2} \left(\frac{M_2}{b}\right)^2 + \frac{u}{b}M_1 \ln M_1 - \frac{u}{b}M_0 \ln M_0 \\ &= \frac{u}{b}M_2 - \frac{u}{b}M_1 \ln M_0 + \frac{u}{b}M_1 \ln M_1 - \frac{g}{2} \left(\frac{M_2}{b}\right)^2 = \frac{u}{b}M_2 + \frac{u}{b}M_1 \ln \frac{M_1}{M_0} - \frac{g}{2} \left(\frac{M_2}{b}\right)^2 \end{aligned}$$

[In the calculation of $y(M_2/b)$, repeated use was made of the relation $M_0 = M_1 + M_2$. In particular,
 $t = M_2/b \Rightarrow M_0 - bt = M_1$.

(d) The formula for $y(t)$ in part (c) holds while there is still fuel. Once the fuel is used up, gravity is the only force acting on the rocket. $-M_1g = M_1 \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = -g \Rightarrow v(t) = -gt + c_1$, where $c_1 = v\left(\frac{M_2}{b}\right) + \frac{gM_2}{b}$

$$\Rightarrow v(t) = v\left(\frac{M_2}{b}\right) - g\left(t - \frac{M_2}{b}\right) \Rightarrow$$

$$y(t) = v\left(\frac{M_2}{b}\right)\left(t - \frac{M_2}{b}\right) - \frac{g}{2}\left(t - \frac{M_2}{b}\right)^2 + c_2, \text{ where } c_2 = y\left(\frac{M_2}{b}\right), \text{ so}$$

$$y(t) = y\left(\frac{M_2}{b}\right) + v\left(\frac{M_2}{b}\right)\left(t - \frac{M_2}{b}\right) - \frac{g}{2}\left(t - \frac{M_2}{b}\right)^2, \quad t \geq \frac{M_2}{b}.$$

To summarize: For $0 \leq t \leq \frac{M_2}{b}$, $y(t) = u(1 + \ln M_0)t - \frac{gt^2}{2} + \frac{u}{b}(M_0 - bt) \ln(M_0 - bt) - \frac{u}{b}M_0 \ln M_0$

[from part (c)], and for $t \geq \frac{M_2}{b}$, $y(t) = y\left(\frac{M_2}{b}\right) + v\left(\frac{M_2}{b}\right)\left(t - \frac{M_2}{b}\right) - \frac{g}{2}\left(t - \frac{M_2}{b}\right)^2$ [from above].

$y\left(\frac{M_2}{b}\right)$ and $v\left(\frac{M_2}{b}\right)$ are given in parts (c) and (b), respectively.

3. We integrate by parts with $u = \frac{1}{\ln(1+x+t)}$, $dv = \sin t \, dt$, so $du = \frac{-1}{(1+x+t)[\ln(1+x+t)]^2}$ and $v = -\cos t$. The integral becomes

$$\begin{aligned} I &= \int_0^\infty \frac{\sin t \, dt}{\ln(1+x+t)} = \lim_{b \rightarrow \infty} \left(\left[\frac{-\cos t}{\ln(1+x+t)} \right]_0^b - \int_0^b \frac{\cos t \, dt}{(1+x+t)[\ln(1+x+t)]^2} \right) \\ &= \lim_{b \rightarrow \infty} \frac{-\cos b}{\ln(1+x+b)} + \frac{1}{\ln(1+x)} + \int_0^\infty \frac{-\cos t \, dt}{(1+x+t)[\ln(1+x+t)]^2} = \frac{1}{\ln(1+x)} + J \end{aligned}$$

where $J = \int_0^\infty \frac{-\cos t \, dt}{(1+x+t)[\ln(1+x+t)]^2}$. Now $-1 \leq -\cos t \leq 1$ for all t ; in fact, the inequality is strict

except at isolated points. So $-\int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2} < J < \int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2} \Leftrightarrow$

$$-\frac{1}{\ln(1+x)} < J < \frac{1}{\ln(1+x)} \Leftrightarrow 0 < I < \frac{2}{\ln(1+x)}.$$

14. (a) $T_n(x) = \cos(n \arccos x)$. The domain of $\arccos$ is $[-1, 1]$, and the domain of $\cos$ is $\mathbb{R}$, so the domain of $T_n(x)$ is $[-1, 1]$. As for the range, $T_0(x) = \cos 0 = 1$, so the range of $T_0(x)$ is $\{1\}$. But since the range of $n \arccos x$ is at least $[0, \pi]$ for $n > 0$, and since $\cos y$ takes on all values in $[-1, 1]$ for $y \in [0, \pi]$, the range of $T_n(x)$ is $[-1, 1]$ for $n > 0$.

- (b) Using the usual trigonometric identities, $T_2(x) = \cos(2 \arccos x) = 2[\cos(\arccos x)]^2 - 1 = 2x^2 - 1$, and

$$\begin{aligned} T_3(x) &= \cos(3 \arccos x) = \cos(\arccos x + 2 \arccos x) \\ &= \cos(\arccos x) \cos(2 \arccos x) - \sin(\arccos x) \sin(2 \arccos x) \\ &= x(2x^2 - 1) - \sin(\arccos x)[2 \sin(\arccos x) \cos(\arccos x)] \\ &= 2x^3 - x - 2[\sin^2(\arccos x)]x = 2x^3 - x - 2x[1 - \cos^2(\arccos x)] \\ &= 2x^3 - x - 2x(1 - x^2) = 4x^3 - 3x \end{aligned}$$

- (c) Let $y = \arccos x$. Then

$$\begin{aligned} T_{n+1}(x) &= \cos[(n+1)y] = \cos(y + ny) = \cos y \cos ny - \sin y \sin ny \\ &= 2 \cos y \cos ny - (\cos y \cos ny + \sin y \sin ny) = 2xT_n(x) - \cos(ny - y) \\ &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

- (d) Here we use induction. $T_0(x) = 1$, a polynomial of degree 0. Now assume that $T_k(x)$ is a polynomial of degree k . Then $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$. By assumption, the leading term of T_k is $a_k x^k$, say, so the leading term of T_{k+1} is $2xa_k x^k = 2a_k x^{k+1}$, and so T_{k+1} has degree $k+1$.

- (e) $T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$,

$$T_5(x) = 2xT_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x,$$

$$T_6(x) = 2xT_5(x) - T_4(x) = 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) = 32x^6 - 48x^4 + 18x^2 - 1,$$

$$\begin{aligned} T_7(x) &= 2xT_6(x) - T_5(x) = 2x(32x^6 - 48x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x) \\ &= 64x^7 - 112x^5 + 56x^3 - 7x \end{aligned}$$

(f) The zeros of $T_n(x) = \cos(n \arccos x)$ occur where $n \arccos x = k\pi + \frac{\pi}{2}$ for some integer k , since then

$\cos(n \arccos x) = \cos(k\pi + \frac{\pi}{2}) = 0$. Note that there will be restrictions on k , since $0 \leq \arccos x \leq \pi$. We

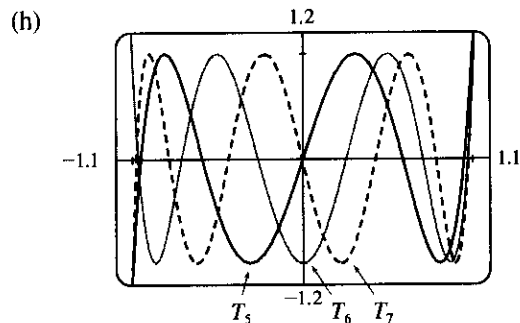
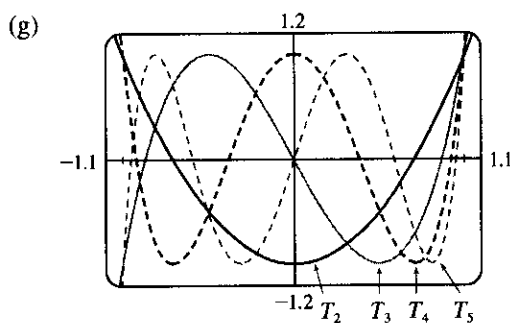
continue: $n \arccos x = k\pi + \frac{\pi}{2} \Leftrightarrow \arccos x = \frac{k\pi + \frac{\pi}{2}}{n}$. This only has solutions for $0 \leq \frac{k\pi + \frac{\pi}{2}}{n} \leq \pi$

$\Leftrightarrow 0 < k\pi + \frac{\pi}{2} < n\pi \Leftrightarrow 0 \leq k < n$. [This makes sense, because then $T_n(x)$ has n zeros, and it is a polynomial of degree n .] So, taking cosines of both sides of the last equation, we find that the zeros of $T_n(x)$

occur at $x = \cos \frac{k\pi + \frac{\pi}{2}}{n}$, k an integer with $0 \leq k < n$. To find the values of x at which $T_n(x)$ has local

extrema, we set $0 = T'_n(x) = -\sin(n \arccos x) \frac{-n}{\sqrt{1-x^2}} = \frac{n \sin(n \arccos x)}{\sqrt{1-x^2}} \Leftrightarrow \sin(n \arccos x) = 0$

$\Leftrightarrow n \arccos x = k\pi, k \text{ some integer} \Leftrightarrow \arccos x = k\pi/n$. This has solutions for $0 \leq k \leq n$, but we disallow the cases $k = 0$ and $k = n$, since these give $x = 1$ and $x = -1$ respectively. So the local extrema of $T_n(x)$ occur at $x = \cos(k\pi/n)$, k an integer with $0 < k < n$. [Again, this seems reasonable, since a polynomial of degree n has at most $(n - 1)$ extrema.] By the First Derivative Test, the cases where k is even give maxima of $T_n(x)$, since then $n \arccos [\cos(k\pi/n)] = k\pi$ is an even multiple of π , so $\sin(n \arccos x)$ goes from negative to positive at $x = \cos(k\pi/n)$. Similarly, the cases where k is odd represent minima of $T_n(x)$.



(i) From the graphs, it seems that the zeros of T_n and T_{n+1} alternate; that is, between two adjacent zeros of T_n , there is a zero of T_{n+1} , and vice versa. The same is true of the x -coordinates of the extrema of T_n and T_{n+1} : between the x -coordinates of any two adjacent extrema of one, there is the x -coordinate of an extremum of the other.

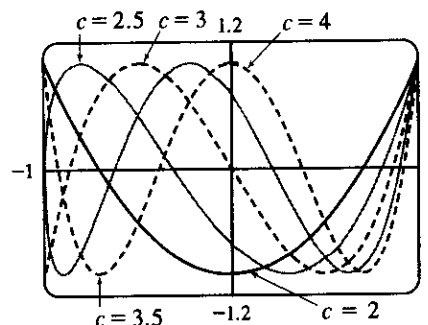
(j) When n is odd, the function $T_n(x)$ is odd, since all of its terms have odd degree, and so $\int_{-1}^1 T_n(x) dx = 0$.

When n is even, $T_n(x)$ is even, and it appears that the integral is negative, but decreases in absolute value as n gets larger.

- (k) $\int_{-1}^1 T_n(x) dx = \int_{-1}^1 \cos(n \arccos x) dx$. We substitute $u = \arccos x \Rightarrow x = \cos u \Rightarrow dx = -\sin u du$, $x = -1 \Rightarrow u = \pi$, and $x = 1 \Rightarrow u = 0$. So the integral becomes

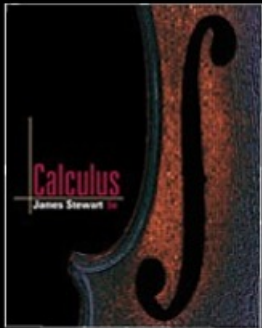
$$\begin{aligned} & \int_0^\pi \cos(nu) \sin u du \\ &= \int_0^\pi \frac{1}{2} [\sin(u - nu) + \sin(u + nu)] du \\ &= \frac{1}{2} \left[\frac{\cos[(1-n)u]}{n-1} - \frac{\cos[(1+n)u]}{n+1} \right]_0^\pi \\ &= \begin{cases} \frac{1}{2} \left[\left(\frac{-1}{n-1} - \frac{-1}{n+1} \right) - \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right] & \text{if } n \text{ is even} \\ \frac{1}{2} \left[\left(\frac{1}{n-1} - \frac{1}{n+1} \right) - \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right] & \text{if } n \text{ is odd} \end{cases} = \begin{cases} -\frac{2}{n^2-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

- (l) From the graph, we see that as c increases through an integer, the graph of f gains a local extremum, which starts at $x = -1$ and moves rightward, compressing the graph of f as c continues to increase.

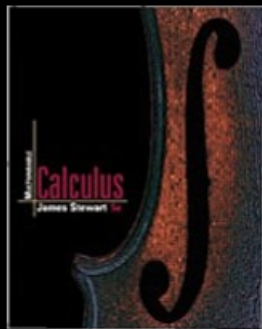


Chapter 9

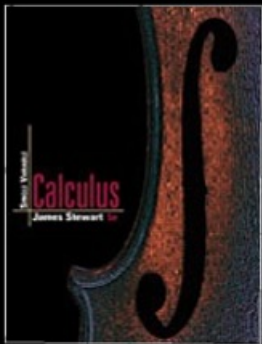
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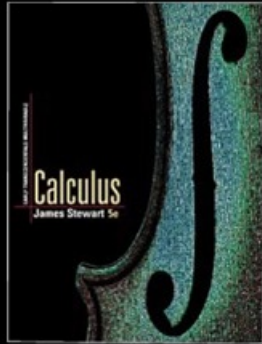
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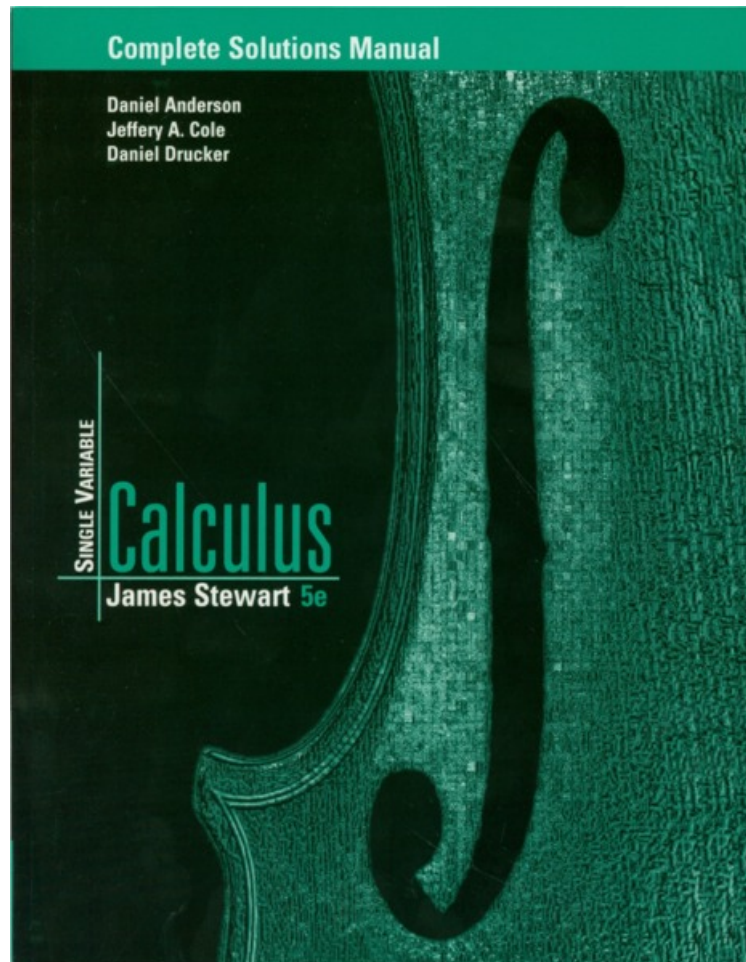
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9 □ FURTHER APPLICATIONS OF INTEGRATION

9.1 Arc Length

1. $y = 2 - 3x \Rightarrow L = \int_{-2}^1 \sqrt{1 + (dy/dx)^2} dx = \int_{-2}^1 \sqrt{1 + (-3)^2} dx = \sqrt{10} [1 - (-2)] = 3\sqrt{10}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

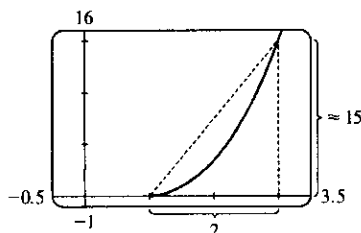
$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

2. Using the arc length formula with $y = \sqrt{4 - x^2} \Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}}$, we get

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}} dx = \int_0^2 \frac{2 dx}{\sqrt{4 - x^2}} = 2 \lim_{t \rightarrow 2^-} \int_0^t \frac{dx}{\sqrt{2^2 - x^2}} \\ &= 2 \lim_{t \rightarrow 2^-} [\sin^{-1}(x/2)]_0^t = 2 \lim_{t \rightarrow 2^-} [\sin^{-1}(t/2) - \sin^{-1} 0] = 2\left(\frac{\pi}{2} - 0\right) = \pi \end{aligned}$$

The curve is a quarter of a circle with radius 2, so the length of the arc is $\frac{1}{4}(2\pi \cdot 2) = \pi$, as above.

3.



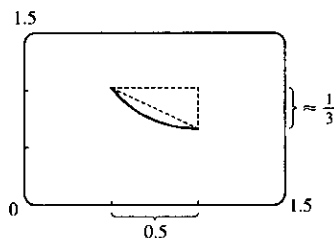
From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1, 0)$, $(3, 0)$, and $(3, f(3)) \approx (3, 15)$, where $y = f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$. This length is about $\sqrt{15^2 + 2^2} \approx 15$, so we might estimate the length to

$$\text{be } 15.5. \quad y = \frac{2}{3}(x^2 - 1)^{3/2} \Rightarrow y' = (x^2 - 1)^{1/2}(2x) \Rightarrow$$

$1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2$, so, using the fact that $2x^2 - 1 > 0$ for $1 \leq x \leq 3$,

$$\begin{aligned} L &= \int_1^3 \sqrt{(2x^2 - 1)^2} dx = \int_1^3 |2x^2 - 1| dx = \int_1^3 (2x^2 - 1) dx = \left[\frac{2}{3}x^3 - x\right]_1^3 \\ &= (18 - 3) - \left(\frac{2}{3} - 1\right) = \frac{46}{3} = 15.\bar{3} \end{aligned}$$

4.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(0.5, f(0.5) \approx 1)$, $(1, f(0.5) \approx 1)$ and $(1, \frac{2}{3})$, where $y = f(x) = x^3/6 + 1/(2x)$.

This length is about $\sqrt{(\frac{1}{2})^2 + (\frac{1}{3})^2} \approx 0.6$, so we might estimate the length to be 0.65.

$$y = \frac{x^3}{6} + \frac{1}{2x} \Rightarrow y' = \frac{x^2}{2} - \frac{x^{-2}}{2} \Rightarrow$$

$$1 + (y')^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{x^{-4}}{4} = \frac{x^4}{4} + \frac{1}{2} + \frac{x^{-4}}{4} = \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2$$

so, using the fact that the parenthetical expression is positive,

$$\begin{aligned} L &= \int_{1/2}^1 \sqrt{\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2} dx = \int_{1/2}^1 \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_{1/2}^1 \\ &= \left(\frac{1}{6} - \frac{1}{2}\right) - \left(\frac{1}{48} - 1\right) = \frac{31}{48} = 0.6458\bar{3} \end{aligned}$$

5. $y = 1 + 6x^{3/2} \Rightarrow dy/dx = 9x^{1/2} \Rightarrow 1 + (dy/dx)^2 = 1 + 81x$. So

$$L = \int_0^1 \sqrt{1 + 81x} dx = \int_1^{82} u^{1/2} \left(\frac{1}{81} du\right) \quad [\text{where } u = 1 + 81x \text{ and } du = 81 dx]$$

$$= \frac{1}{81} \cdot \frac{2}{3} \left[u^{3/2} \right]_1^{82} = \frac{2}{243} (82\sqrt{82} - 1)$$

6. $y^2 = 4(x + 4)^3, y > 0 \Rightarrow y = 2(x + 4)^{3/2} \Rightarrow dy/dx = 3(x + 4)^{1/2} \Rightarrow$
 $1 + (dy/dx)^2 = 1 + 9(x + 4) = 9x + 37$. So

$$L = \int_0^2 \sqrt{9x + 37} dx \quad \left[\begin{array}{l} u = 9x + 37, \\ du = 9 dx \end{array} \right] = \int_{37}^{55} u^{1/2} \left(\frac{1}{9} du\right)$$

$$= \frac{1}{9} \cdot \frac{2}{3} \left[u^{3/2} \right]_{37}^{55} = \frac{2}{27} (55\sqrt{55} - 37\sqrt{37})$$

7. $y = \frac{x^5}{6} + \frac{1}{10x^3} \Rightarrow \frac{dy}{dx} = \frac{5}{6}x^4 - \frac{3}{10}x^{-4} \Rightarrow$

$$1 + (dy/dx)^2 = 1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100}x^{-8} = \frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100}x^{-8} = \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2. \text{ So}$$

$$L = \int_1^2 \sqrt{\left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2} dx = \int_1^2 \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right) dx = \left[\frac{1}{6}x^5 - \frac{1}{10}x^{-3}\right]_1^2$$

$$= \left(\frac{32}{6} - \frac{1}{80}\right) - \left(\frac{1}{6} - \frac{1}{10}\right) = \frac{31}{6} + \frac{7}{80} = \frac{1261}{240}$$

8. $y = \frac{x^2}{2} - \frac{\ln x}{4} \Rightarrow \frac{dy}{dx} = x - \frac{1}{4x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = x^2 + \frac{1}{2} + \frac{1}{16x^2}$. So

$$L = \int_2^4 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left(8 + \frac{2\ln 2}{4}\right) - \left(2 + \frac{\ln 2}{4}\right) = 6 + \frac{\ln 2}{4}.$$

9. $x = \frac{1}{3}\sqrt{y}(y - 3) = \frac{1}{3}y^{3/2} - y^{1/2} \Rightarrow dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow$

$$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2. \text{ So}$$

$$L = \int_1^9 \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right) dy = \frac{1}{2} \left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \frac{1}{2} \left[\left(\frac{2}{3} \cdot 27 + 2 \cdot 3\right) - \left(\frac{2}{3} \cdot 1 + 2 \cdot 1\right)\right]$$

$$= \frac{1}{2} \left(24 - \frac{8}{3}\right) = \frac{1}{2} \left(\frac{64}{3}\right) = \frac{32}{3}$$

10. $y = \ln(\cos x) \Rightarrow dy/dx = -\tan x \Rightarrow 1 + (dy/dx)^2 = 1 + \tan^2 x = \sec^2 x$. So

$$L = \int_0^{\pi/3} \sqrt{\sec^2 x} dx = \int_0^{\pi/3} \sec x dx = [\ln|\sec x + \tan x|]_0^{\pi/3} = \ln(2 + \sqrt{3}) - \ln(1 + 0) = \ln(2 + \sqrt{3}).$$

11. $y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$, so

$$L = \int_0^{\pi/4} \sqrt{\sec^2 x} dx = \int_0^{\pi/4} |\sec x| dx = \int_0^{\pi/4} \sec x dx = [\ln(\sec x + \tan x)]_0^{\pi/4}$$

$$= \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1)$$

12. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{x}\right)^2} = \frac{\sqrt{1 + x^2}}{x}$. So $L = \int_1^{\sqrt{3}} \frac{\sqrt{1 + x^2}}{x} dx$. Now

let $v = \sqrt{1 + x^2}$, so $v^2 = 1 + x^2$ and $v dv = x dx$. Thus

$$L = \int_{\sqrt{2}}^2 \frac{v}{v^2 - 1} v dv = \int_{\sqrt{2}}^2 \left(1 + \frac{1/2}{v - 1} - \frac{1/2}{v + 1}\right) dv = \left[v + \frac{1}{2} \ln|v - 1| - \frac{1}{2} \ln|v + 1|\right]_{\sqrt{2}}^2$$

$$= \left[v - \frac{1}{2} \ln \left|\frac{v + 1}{v - 1}\right|\right]_{\sqrt{2}}^2 = 2 - \frac{1}{2} \ln 3 - \sqrt{2} + \frac{1}{2} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right) = 2 - \sqrt{2} + \ln(\sqrt{2} + 1) - \frac{1}{2} \ln 3$$

Or: Use Formula 23 in the table of integrals.

$$13. y = \cosh x \Rightarrow y' = \sinh x \Rightarrow 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x.$$

$$\text{So } L = \int_0^1 \cosh x \, dx = [\sinh x]_0^1 = \sinh 1 = \frac{1}{2}(e - 1/e).$$

$$14. y^2 = 4x, x = \frac{1}{4}y^2 \Rightarrow \frac{dx}{dy} = \frac{1}{2}y \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y^2. \text{ So}$$

$$L = \int_0^2 \sqrt{1 + \frac{1}{4}y^2} \, dy = \int_0^1 \sqrt{1 + u^2} \cdot 2 \, du \quad [u = \frac{1}{2}y, dy = 2 \, du]$$

$$\stackrel{21}{=} [u\sqrt{1+u^2} + \ln|u + \sqrt{1+u^2}|]_0^1 = \sqrt{2} + \ln(1 + \sqrt{2})$$

$$15. y = e^x \Rightarrow y' = e^x \Rightarrow 1 + (y')^2 = 1 + e^{2x}. \text{ So}$$

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + e^{2x}} \, dx = \int_1^e \sqrt{1 + u^2} \frac{du}{u} \quad [u = e^x, \text{ so } x = \ln u, dx = du/u] \\ &= \int_1^e \frac{\sqrt{1+u^2}}{u^2} u \, du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2-1} v \, dv \quad [v = \sqrt{1+u^2}, \text{ so } v^2 = 1+u^2, v \, dv = u \, du] \\ &= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1/2}{v-1} - \frac{1/2}{v+1}\right) dv = \left[v + \frac{1}{2} \ln \frac{v-1}{v+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}} \\ &= \sqrt{1+e^2} + \frac{1}{2} \ln \frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \sqrt{2} - \frac{1}{2} \ln \frac{\sqrt{2}-1}{\sqrt{2}+1} \\ &= \sqrt{1+e^2} - \sqrt{2} + \ln(\sqrt{1+e^2}-1) - 1 - \ln(\sqrt{2}-1) \end{aligned}$$

Or: Use Formula 23 for $\int (\sqrt{1+u^2}/u) \, du$, or substitute $u = \tan \theta$.

$$16. y = \ln\left(\frac{e^x+1}{e^x-1}\right) = \ln(e^x+1) - \ln(e^x-1) \Rightarrow y' = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1} = \frac{-2e^x}{e^{2x}-1} \Rightarrow$$

$$1 + (y')^2 = 1 + \frac{4e^{2x}}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2} \Rightarrow \sqrt{1+(y')^2} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{\cosh x}{\sinh x}.$$

$$\text{So } L = \int_a^b \frac{\cosh x}{\sinh x} \, dx = [\ln \sinh x]_a^b = \ln \sinh b - \ln \sinh a = \ln\left(\frac{\sinh b}{\sinh a}\right) = \ln\left(\frac{e^b - e^{-b}}{e^a - e^{-a}}\right).$$

$$17. y = \cos x \Rightarrow dy/dx = -\sin x \Rightarrow 1 + (dy/dx)^2 = 1 + \sin^2 x. \text{ So } L = \int_0^{2\pi} \sqrt{1 + \sin^2 x} \, dx.$$

$$18. y = 2^x \Rightarrow dy/dx = (2^x) \ln 2 \Rightarrow L = \int_0^3 \sqrt{1 + (\ln 2)^2 2^{2x}} \, dx$$

$$19. x = y + y^3 \Rightarrow dx/dy = 1 + 3y^2 \Rightarrow 1 + (dx/dy)^2 = 1 + (1 + 3y^2)^2 = 9y^4 + 6y^2 + 2.$$

$$\text{So } L = \int_1^4 \sqrt{9y^4 + 6y^2 + 2} \, dy.$$

$$20. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, y = \pm b\sqrt{1 - x^2/a^2} = \pm \frac{b}{a}\sqrt{a^2 - x^2} \text{ [assume } a > 0].$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2 - x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{b^2 x^2}{a^2(a^2 - x^2)}.$$

$$\text{So } L = 2 \int_{-a}^a \left[1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}\right]^{1/2} dx = \frac{4}{a} \int_0^a \left[\frac{(b^2 - a^2)x^2 + a^4}{a^2 - x^2}\right]^{1/2} dx.$$

21. $y = xe^{-x} \Rightarrow dy/dx = e^{-x} - xe^{-x} = e^{-x}(1-x) \Rightarrow 1 + (dy/dx)^2 = 1 + e^{-2x}(1-x)^2$. Let $f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + e^{-2x}(1-x)^2}$. Then $L = \int_0^5 f(x) dx$. Since $n = 10$, $\Delta x = \frac{5-0}{10} = \frac{1}{2}$. Now
- $$L \approx S_{10} = \frac{1}{3} [f(0) + 4f(\frac{1}{2}) + 2f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + 2f(4) + 4f(\frac{9}{2}) + f(5)] \approx 5.115840$$

The value of the integral produced by a calculator is 5.113568 (to six decimal places).

22. $x = y + \sqrt{y} \Rightarrow dx/dy = 1 + \frac{1}{2\sqrt{y}} \Rightarrow 1 + (dx/dy)^2 = 1 + \left(1 + \frac{1}{2\sqrt{y}}\right)^2 = 2 + \frac{1}{\sqrt{y}} + \frac{1}{4y}$.

Let $g(y) = \sqrt{1 + (dx/dy)^2}$. Then $L = \int_1^2 g(y) dy$. Since $n = 10$, $\Delta y = \frac{2-1}{10} = \frac{1}{10}$. Now

$$L \approx S_{10} = \frac{1}{3} [g(1) + 4g(1.1) + 2g(1.2) + 4g(1.3) + 2g(1.4) + 4g(1.5) + 2g(1.6) + 4g(1.7) + 2g(1.8) + 4g(1.9) + g(2)] \approx 1.732215,$$

which is the same value of the integral produced by a calculator to six decimal places.

23. $y = \sec x \Rightarrow dy/dx = \sec x \tan x \Rightarrow L = \int_0^{\pi/3} f(x) dx$, where $f(x) = \sqrt{1 + \sec^2 x \tan^2 x}$.

Since $n = 10$, $\Delta x = \frac{\pi/3 - 0}{10} = \frac{\pi}{30}$. Now

$$L \approx S_{10} = \frac{\pi/30}{3} \left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + 4f\left(\frac{3\pi}{30}\right) + 2f\left(\frac{4\pi}{30}\right) + 4f\left(\frac{5\pi}{30}\right) + 2f\left(\frac{6\pi}{30}\right) + 4f\left(\frac{7\pi}{30}\right) + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right) \right] \approx 1.569619.$$

The value of the integral produced by a calculator is 1.569259 (to six decimal places).

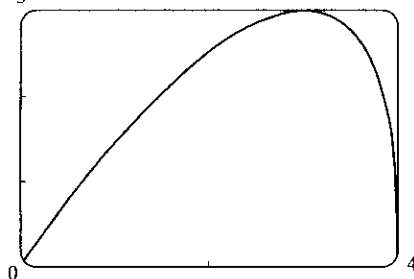
24. $y = x \ln x \Rightarrow dy/dx = 1 + \ln x$. Let $f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + (1 + \ln x)^2}$.

Then $L = \int_1^3 f(x) dx$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Now

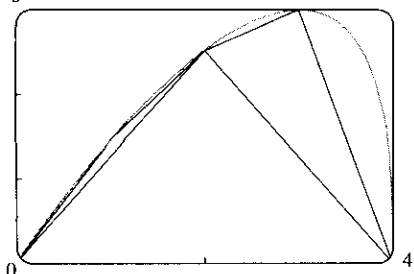
$$L \approx S_{10} = \frac{1}{3} [f(1) + 4f(1.2) + 2f(1.4) + 4f(1.6) + 2f(1.8) + 4f(2) + 2f(2.2) + 4f(2.4) + 2f(2.6) + 4f(2.8) + f(3)] \approx 3.869618.$$

The value of the integral produced by a calculator is 3.869617 (to six decimal places).

25. (a) 3



- (b) 3



Let $f(x) = y = x\sqrt{4-x}$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(4, f(4)) = (4, 0)$, and its length is 4. The polygon with two sides joins the points $(0, 0)$, $(2, f(2)) = (2, 2\sqrt{2})$ and $(4, 0)$.

Its length is

$$\sqrt{(2-0)^2 + \left(2\sqrt[3]{2} - 0\right)^2} + \sqrt{(4-2)^2 + \left(0 - 2\sqrt[3]{2}\right)^2} = 2\sqrt{4 + 2^{8/3}} \approx 6.43$$

Similarly, the inscribed polygon with four sides joins the points $(0, 0)$, $(1, \sqrt[3]{3})$, $(2, 2\sqrt[3]{2})$, $(3, 3)$, and $(4, 0)$, so its length is

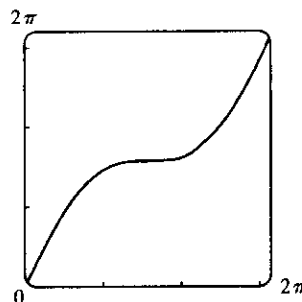
$$\sqrt{1 + \left(\sqrt[3]{3}\right)^2} + \sqrt{1 + \left(2\sqrt[3]{2} - \sqrt[3]{3}\right)^2} + \sqrt{1 + \left(3 - 2\sqrt[3]{2}\right)^2} + \sqrt{1 + 9} \approx 7.50$$

(c) Using the arc length formula with $\frac{dy}{dx} = x \left[\frac{1}{3}(4-x)^{-2/3}(-1) \right] + \sqrt[3]{4-x} = \frac{12-4x}{3(4-x)^{2/3}}$, the length of the

curve is
$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^4 \sqrt{1 + \left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2} dx.$$

(d) According to a CAS, the length of the curve is $L \approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

26. (a) Let $f(x) = y = x + \sin x$ with $0 \leq x \leq 2\pi$.

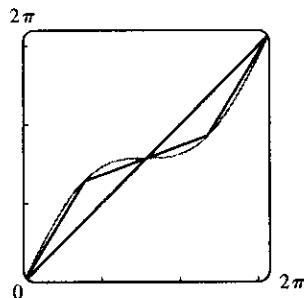


(b) The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(2\pi, f(2\pi)) = (2\pi, 2\pi)$, and its length is $\sqrt{(2\pi - 0)^2 + (2\pi - 0)^2} = 2\sqrt{2}\pi \approx 8.9$.

The polygon with two sides joins the points $(0, 0)$, $(\pi, f(\pi)) = (\pi, \pi)$, and $(2\pi, 2\pi)$. Its length is

$$\sqrt{(\pi - 0)^2 + (\pi - 0)^2} + \sqrt{(2\pi - \pi)^2 + (2\pi - \pi)^2} = \sqrt{2}\pi + \sqrt{2}\pi = 2\sqrt{2}\pi \approx 8.9$$

Note from the diagram that the two approximations are the same because the sides of the 2-sided polygon are in fact on the same line, since $f(\pi) = \pi = \frac{1}{2}f(2\pi)$.



The four-sided polygon joins the points $(0, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2} + 1)$, (π, π) , $(\frac{3\pi}{2}, \frac{3\pi}{2} - 1)$, and $(2\pi, 2\pi)$, so its length is

$$\sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2} + 1\right)^2} + \sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2} - 1\right)^2} + \sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2} - 1\right)^2} + \sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2} + 1\right)^2} \approx 9.4$$

(c) Using the arc length formula with $dy/dx = 1 + \cos x$, the length of the curve is

$$L = \int_0^{2\pi} \sqrt{1 + (1 + \cos x)^2} dx = \int_0^{2\pi} \sqrt{2 + 2 \cos x + \cos^2 x} dx$$

(d) The CAS approximates the integral as 9.5076. The actual length is larger than the approximations in part (b).

27. $x = \ln(1 - y^2) \Rightarrow \frac{dx}{dy} = \frac{-2y}{1 - y^2} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{4y^2}{(1 - y^2)^2} = \frac{(1 + y^2)^2}{(1 - y^2)^2}$. So

$$L = \int_0^{1/2} \sqrt{\frac{(1 + y^2)^2}{(1 - y^2)^2}} dy = \int_0^{1/2} \frac{1 + y^2}{1 - y^2} dy = \ln 3 - \frac{1}{2} \quad [\text{from a CAS}] \approx 0.599$$

28. $y = x^{4/3} \Rightarrow dy/dx = \frac{4}{3}x^{1/3} \Rightarrow 1 + (dy/dx)^2 = 1 + \frac{16}{9}x^{2/3} \Rightarrow$

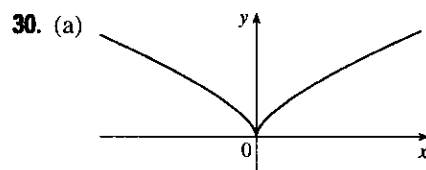
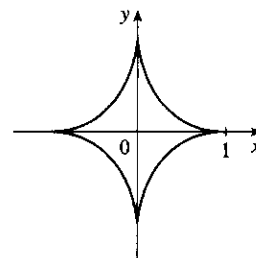
$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \frac{16}{9}x^{2/3}} dx = \int_0^{4/3} \sqrt{1 + u^2} \frac{81}{64} u^2 du \quad \left[\begin{array}{l} u = \frac{4}{3}x^{1/3}, du = \frac{4}{9}x^{-2/3} dx, \\ dx = \frac{9}{4}x^{2/3} du = \frac{9}{4} \cdot \frac{9}{16}u^2 du = \frac{81}{64}u^2 du \end{array} \right] \\ &\stackrel{22}{=} \frac{81}{64} \left[\frac{1}{8}u(1 + 2u^2) \sqrt{1 + u^2} - \frac{1}{8} \ln(u + \sqrt{1 + u^2}) \right]_0^{4/3} \\ &= \frac{81}{64} \left[\frac{1}{6} \left(1 + \frac{32}{9}\right) \sqrt{\frac{25}{9}} - \frac{1}{8} \ln\left(\frac{4}{3} + \sqrt{\frac{25}{9}}\right) \right] = \frac{81}{64} \left(\frac{1}{6} \cdot \frac{41}{9} \cdot \frac{5}{3} - \frac{1}{8} \ln 3 \right) \\ &= \frac{205}{128} - \frac{81}{512} \ln 3 \approx 1.4277586 \end{aligned}$$

29. $y^{2/3} = 1 - x^{2/3} \Rightarrow y = (1 - x^{2/3})^{3/2} \Rightarrow$

$$\frac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2} \left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1 - x^{2/3})^{1/2} \Rightarrow$$

$$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}(1 - x^{2/3}) = x^{-2/3} - 1. \text{ Thus}$$

$$L = 4 \int_0^1 \sqrt{1 + (x^{-2/3} - 1)} dx = 4 \int_0^1 x^{-1/3} dx = 4 \lim_{t \rightarrow 0^+} \left[\frac{3}{2}x^{2/3} \right]_t^1 = 6.$$



(b) $y = x^{2/3} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{2}{3}x^{-1/3}\right)^2 = 1 + \frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}} dx$

[an improper integral]. $x = y^{3/2} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{3}{2}y^{1/2}\right)^2 = 1 + \frac{9}{4}y$. So $L = \int_0^1 \sqrt{1 + \frac{9}{4}y} dy$.

The second integral equals $\frac{4}{9} \cdot \frac{2}{3} \left[\left(1 + \frac{9}{4}y\right)^{3/2} \right]_0^1 = \frac{8}{27} \left(\frac{13\sqrt{13}}{8} - 1 \right) = \frac{13\sqrt{13} - 8}{27}$. The first integral can be evaluated as follows:

$$\begin{aligned} \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}} dx &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}} dx = \lim_{t \rightarrow 0^+} \int_{9t^{2/3}}^9 \frac{\sqrt{u + 4}}{18} du \quad \left[\begin{array}{l} u = 9x^{2/3}, \\ du = 6x^{-1/3} dx \end{array} \right] \\ &= \int_0^9 \frac{\sqrt{u + 4}}{18} du = \frac{1}{18} \cdot \left[\frac{2}{3}(u + 4)^{3/2} \right]_0^9 = \frac{1}{27} (13^{3/2} - 4^{3/2}) = \frac{13\sqrt{13} - 8}{27} \end{aligned}$$

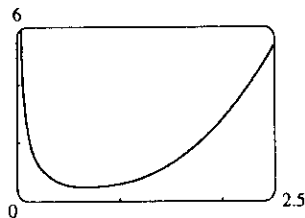
(c) L = length of the arc of this curve from $(-1, 1)$ to $(8, 4)$

$$\begin{aligned} &= \int_0^1 \sqrt{1 + \frac{9}{4}y} dy + \int_0^4 \sqrt{1 + \frac{9}{4}y} dy = \frac{13\sqrt{13}-8}{27} + \frac{8}{27} \left[\left(1 + \frac{9}{4}y\right)^{3/2} \right]_0^4 \quad [\text{from part (b)}] \\ &= \frac{13\sqrt{13}-8}{27} + \frac{8}{27} (10\sqrt{10} - 1) = \frac{13\sqrt{13}+80\sqrt{10}-16}{27} \end{aligned}$$

31. $y = 2x^{3/2} \Rightarrow y' = 3x^{1/2} \Rightarrow 1 + (y')^2 = 1 + 9x$. The arc length function with starting point $P_0(1, 2)$ is

$$s(x) = \int_1^x \sqrt{1 + 9t} dt = \left[\frac{2}{27} (1 + 9t)^{3/2} \right]_1^x = \frac{2}{27} \left[(1 + 9x)^{3/2} - 10\sqrt{10} \right]$$

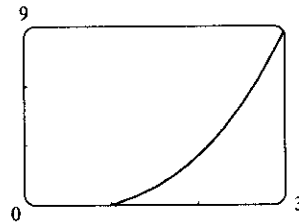
32. (a)



(b) $1 + \left(\frac{dy}{dx}\right)^2 = x^4 + \frac{1}{2} + \frac{1}{16x^4}$,

$$\begin{aligned} s(x) &= \int_1^x \left[t^2 + 1/(4t^2) \right] dt \\ &= \left[\frac{1}{3}t^3 - 1/(4t) \right]_1^x \\ &= \frac{1}{3}x^3 - 1/(4x) - \left(\frac{1}{3} - \frac{1}{4} \right) \\ &= \frac{1}{3}x^3 - 1/(4x) - \frac{1}{12} \quad \text{for } x \geq 1 \end{aligned}$$

(c)



33. The prey hits the ground when $y = 0 \Leftrightarrow 180 - \frac{1}{45}x^2 = 0 \Leftrightarrow x^2 = 45 \cdot 180 \Rightarrow x = \sqrt{8100} = 90$, since x must be positive. $y' = -\frac{2}{45}x \Rightarrow 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$\begin{aligned} L &= \int_0^{90} \sqrt{1 + \frac{4}{45^2}x^2} dx = \int_0^4 \sqrt{1 + u^2} \left(\frac{45}{2} du \right) \quad [u = \frac{2}{45}x, du = \frac{2}{45}dx] \\ &\stackrel{21}{=} \frac{45}{2} \left[\frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u + \sqrt{1+u^2}) \right]_0^4 \\ &= \frac{45}{2} \left[2\sqrt{17} + \frac{1}{2}\ln(4 + \sqrt{17}) \right] = 45\sqrt{17} + \frac{45}{4}\ln(4 + \sqrt{17}) \approx 209.1 \text{ m} \end{aligned}$$

34. $y = 150 - \frac{1}{40}(x - 50)^2 \Rightarrow y' = -\frac{1}{20}(x - 50) \Rightarrow 1 + (y')^2 = 1 + \frac{1}{20^2}(x - 50)^2$, so the distance traveled by the kite is

$$\begin{aligned} L &= \int_0^{80} \sqrt{1 + \frac{1}{20^2}(x - 50)^2} dx = \int_{-5/2}^{3/2} \sqrt{1 + u^2} (20 du) \quad [u = \frac{1}{20}(x - 50), du = \frac{1}{20}dx] \\ &\stackrel{21}{=} 20 \left[\frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u + \sqrt{1+u^2}) \right]_{-5/2}^{3/2} \\ &= 10 \left[\frac{3}{2}\sqrt{\frac{13}{4}} + \ln\left(\frac{3}{2} + \sqrt{\frac{13}{4}}\right) + \frac{5}{2}\sqrt{\frac{29}{4}} - \ln\left(-\frac{5}{2} + \sqrt{\frac{29}{4}}\right) \right] \\ &= \frac{15}{2}\sqrt{13} + \frac{25}{2}\sqrt{29} + 10\ln\left(\frac{3+\sqrt{13}}{-5+\sqrt{29}}\right) \approx 122.8 \text{ ft} \end{aligned}$$

35. The sine wave has amplitude 1 and period 14, since it goes through two periods in a distance of 28 in., so its equation is $y = 1 \sin\left(\frac{2\pi}{14}x\right) = \sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x = 28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{\pi}{7} \cos\left(\frac{\pi}{7}x\right)$: $L = \int_0^{28} \sqrt{1 + \left[\frac{\pi}{7} \cos\left(\frac{\pi}{7}x\right)\right]^2} dx = 2 \int_0^{14} \sqrt{1 + \left[\frac{\pi}{7} \cos\left(\frac{\pi}{7}x\right)\right]^2} dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 29.36$ inches.

36. (a) $y = c + a \cosh\left(\frac{x}{a}\right) \Rightarrow y' = \sinh\left(\frac{x}{a}\right) \Rightarrow 1 + (y')^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$. So

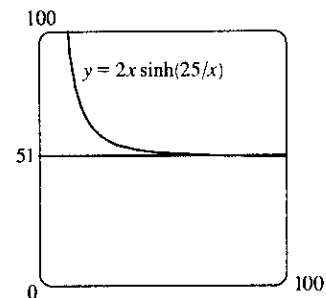
$$L = \int_{-b}^b \sqrt{\cosh^2\left(\frac{x}{a}\right)} dx = 2 \int_0^b \cosh\left(\frac{x}{a}\right) dx = 2 \left[a \sinh\left(\frac{x}{a}\right) \right]_0^b = 2a \sinh\left(\frac{b}{a}\right)$$

(b) At $x = 0$, $y = c + a$, so $c + a = 20$. The poles are 50 ft apart, so

$$b = 25, \text{ and } L = 51 \Rightarrow 51 = 2a \sinh(b/a) \text{ [from part (a)].}$$

From the figure, we see that $y = 51$ intersects $y = 2x \sinh(25/x)$ at $x \approx 72.3843$ for $x > 0$. So $a \approx 72.3843$ and the wire should be attached at a distance of

$$y = c + a \cosh(25/a) = 20 - a + a \cosh(25/a) \approx 24.36 \text{ ft above the ground.}$$



37. $y = \int_1^x \sqrt{t^3 - 1} dt \Rightarrow \frac{dy}{dx} = \sqrt{x^3 - 1}$ [by FTC1] $\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + (\sqrt{x^3 - 1})^2 = x^3 \Rightarrow$

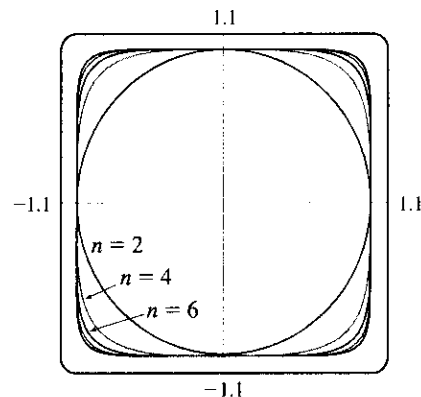
$$L = \int_1^4 \sqrt{x^3} dx = \int_1^4 x^{3/2} dx = \frac{2}{5} \left[x^{5/2} \right]_1^4 = \frac{2}{5} (32 - 1) = \frac{62}{5} = 12.4$$

38. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4.

$$x^{2k} + y^{2k} = 1 \Rightarrow y^{2k} = 1 - x^{2k} \Rightarrow y = (1 - x^{2k})^{1/(2k)}$$

(in the first quadrant), so we use the arc length formula with

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2k} (1 - x^{2k})^{1/(2k)-1} (-2kx^{2k-1}) \\ &= -x^{2k-1} (1 - x^{2k})^{1/(2k)-1} \end{aligned}$$



The total length is therefore

$$L_{2k} = 4 \int_0^1 \sqrt{1 + \left[-x^{2k-1} (1 - x^{2k})^{1/(2k)-1} \right]^2} dx = 4 \int_0^1 \sqrt{1 + x^{2(2k-1)} (1 - x^{2k})^{1/k-2}} dx$$

Now from the graph, we see that as k increases, the “corners” of these fat circles get closer to the points $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, and the “edges” of the fat circles approach the lines joining these four points. It seems plausible that as $k \rightarrow \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \rightarrow \infty$ of the equation of the fat circle in the first quadrant: $\lim_{k \rightarrow \infty} (1 - x^{2k})^{1/(2k)} = 1$ for $0 \leq x < 1$. So we guess that $\lim_{k \rightarrow \infty} L_{2k} = 4 \cdot 2 = 8$.

DISCOVERY PROJECT Arc Length Contest

For advice on how to run the contest and a list of student entries, see the article “Arc Length Contest” by Larry Riddle in *The College Mathematics Journal*, Volume 29, No. 4, September 1998, pages 314–320.

9.2 Area of a Surface of Revolution

$$1. y = \ln x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + (1/x)^2} dx \Rightarrow S = \int_1^3 2\pi(\ln x) \sqrt{1 + (1/x)^2} dx \quad [\text{by (7)}]$$

$$2. y = \sin^2 x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + (2 \sin x \cos x)^2} dx \Rightarrow \\ S = \int_0^{\pi/2} 2\pi \sin^2 x \sqrt{1 + (2 \sin x \cos x)^2} dx \quad [\text{by (7)}]$$

$$3. y = \sec x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + (\sec x \tan x)^2} dx \Rightarrow \\ S = \int_0^{\pi/4} 2\pi x \sqrt{1 + (\sec x \tan x)^2} dx \quad [\text{by (8)}]$$

$$4. y = e^x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + e^{2x}} dx \Rightarrow S = \int_0^{\ln 2} 2\pi x \sqrt{1 + e^{2x}} dx \quad [\text{by (8)}] \quad \text{or} \\ \int_1^2 2\pi(\ln y) \sqrt{1 + (1/y)^2} dy \quad [\text{by (6)}]$$

$$5. y = x^3 \Rightarrow y' = 3x^2. \text{ So}$$

$$S = \int_0^2 2\pi y \sqrt{1 + (y')^2} dx = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4} dx \quad [u = 1 + 9x^4, du = 36x^3 dx] \\ = \frac{2\pi}{36} \int_1^{145} \sqrt{u} du = \frac{\pi}{18} \left[\frac{2}{3} u^{3/2} \right]_1^{145} = \frac{\pi}{27} (145 \sqrt{145} - 1)$$

6. The curve $9x = y^2 + 18$ is symmetric about the x -axis, so we only use its top half, given by

$$y = 3\sqrt{x-2}. \quad dy/dx = \frac{3}{2\sqrt{x-2}}, \text{ so } 1 + (dy/dx)^2 = 1 + \frac{9}{4(x-2)}. \text{ Thus,}$$

$$S = \int_2^6 2\pi \cdot 3\sqrt{x-2} \sqrt{1 + \frac{9}{4(x-2)}} dx = 6\pi \int_2^6 \sqrt{x-2 + \frac{9}{4}} dx = 6\pi \int_2^6 \left(x + \frac{1}{4}\right)^{1/2} dx \\ = 6\pi \cdot \frac{2}{3} \left[\left(x + \frac{1}{4}\right)^{3/2} \right]_2^6 = 4\pi \left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2} \right] = 4\pi \left(\frac{125}{8} - \frac{27}{8} \right) = 4\pi \cdot \frac{98}{8} = 49\pi$$

$$7. y = \sqrt{x} \Rightarrow 1 + (dy/dx)^2 = 1 + [1/(2\sqrt{x})]^2 = 1 + 1/(4x). \text{ So}$$

$$S = \int_4^9 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_4^9 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} dx = 2\pi \int_4^9 \sqrt{x + \frac{1}{4}} dx \\ = 2\pi \left[\frac{2}{3} \left(x + \frac{1}{4}\right)^{3/2} \right]_4^9 = \frac{4\pi}{3} \left[\frac{1}{8} (4x + 1)^{3/2} \right]_4^9 = \frac{\pi}{6} (37\sqrt{37} - 17\sqrt{17})$$

$$8. y = \cos 2x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + (-2 \sin 2x)^2} dx \Rightarrow$$

$$S = \int_0^{\pi/6} 2\pi \cos 2x \sqrt{1 + 4 \sin^2 2x} dx = 2\pi \int_0^{\sqrt{3}} \sqrt{1 + u^2} \left(\frac{1}{4} du\right) \quad [u = 2 \sin 2x, du = 4 \cos 2x dx] \\ \stackrel{21}{=} \frac{\pi}{2} \left[\frac{1}{2} u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}) \right]_0^{\sqrt{3}} = \frac{\pi}{2} \left[\frac{\sqrt{3}}{2} \cdot 2 + \frac{1}{2} \ln(\sqrt{3} + 2) \right] = \frac{\pi\sqrt{3}}{2} + \frac{\pi}{4} \ln(2 + \sqrt{3})$$

9. $y = \cosh x \Rightarrow 1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$S = 2\pi \int_0^1 \cosh x \cosh x \, dx = 2\pi \int_0^1 \frac{1}{2}(1 + \cosh 2x) \, dx = \pi \left[x + \frac{1}{2} \sinh 2x \right]_0^1 \\ = \pi \left(1 + \frac{1}{2} \sinh 2 \right) \quad \text{or} \quad \pi \left[1 + \frac{1}{4}(e^2 - e^{-2}) \right]$$

10. $y = \frac{x^3}{6} + \frac{1}{2x} \Rightarrow \frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} \Rightarrow$

$$\sqrt{1 + (dy/dx)^2} = \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} = \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2} \Rightarrow$$

$$S = \int_{1/2}^1 2\pi \left(\frac{x^3}{6} + \frac{1}{2x} \right) \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx = 2\pi \int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3} \right) dx \\ = 2\pi \int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{x^{-3}}{4} \right) dx = 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{x^{-2}}{8} \right]_{1/2}^1 \\ = 2\pi \left[\left(\frac{1}{72} + \frac{1}{6} - \frac{1}{8} \right) - \left(\frac{1}{64 \cdot 72} + \frac{1}{24} - \frac{1}{2} \right) \right] = 2\pi \left(\frac{263}{512} \right) = \frac{263}{256}\pi$$

11. $x = \frac{1}{3}(y^2 + 2)^{3/2} \Rightarrow dx/dy = \frac{1}{2}(y^2 + 2)^{1/2}(2y) = y\sqrt{y^2 + 2} \Rightarrow$

$$1 + (dx/dy)^2 = 1 + y^2(y^2 + 2) = (y^2 + 1)^2. \text{ So}$$

$$S = 2\pi \int_1^2 y(y^2 + 1) \, dy = 2\pi \left[\frac{1}{4}y^4 + \frac{1}{2}y^2 \right]_1^2 = 2\pi \left(4 + 2 - \frac{1}{4} - \frac{1}{2} \right) = \frac{21\pi}{2}$$

12. $x = 1 + 2y^2 \Rightarrow 1 + (dx/dy)^2 = 1 + (4y)^2 = 1 + 16y^2$. So

$$S = 2\pi \int_1^2 y \sqrt{1 + 16y^2} \, dy = \frac{\pi}{16} \int_1^2 (16y^2 + 1)^{1/2} 32y \, dy = \frac{\pi}{16} \left[\frac{2}{3}(16y^2 + 1)^{3/2} \right]_1^2 \\ = \frac{\pi}{24} (65\sqrt{65} - 17\sqrt{17})$$

13. $y = \sqrt[3]{x} \Rightarrow x = y^3 \Rightarrow 1 + (dx/dy)^2 = 1 + 9y^4$. So

$$S = 2\pi \int_1^2 x \sqrt{1 + (dx/dy)^2} \, dy = 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4} \, dy = \frac{2\pi}{36} \int_1^2 \sqrt{1 + 9y^4} 36y^3 \, dy \\ = \frac{\pi}{18} \left[\frac{2}{3}(1 + 9y^4)^{3/2} \right]_1^2 = \frac{\pi}{27} (145\sqrt{145} - 10\sqrt{10})$$

14. $y = 1 - x^2 \Rightarrow 1 + (dy/dx)^2 = 1 + 4x^2 \Rightarrow$

$$S = 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx = \frac{\pi}{4} \int_0^1 8x \sqrt{4x^2 + 1} \, dx = \frac{\pi}{4} \left[\frac{2}{3}(4x^2 + 1)^{3/2} \right]_0^1 = \frac{\pi}{6} (5\sqrt{5} - 1)$$

15. $x = \sqrt{a^2 - y^2} \Rightarrow dx/dy = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = -y/\sqrt{a^2 - y^2} \Rightarrow$

$$1 + (dx/dy)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \Rightarrow$$

$$S = \int_0^{a/2} 2\pi \sqrt{a^2 - y^2} \frac{a}{\sqrt{a^2 - y^2}} \, dy = 2\pi \int_0^{a/2} a \, dy = 2\pi a[y]_0^{a/2} = 2\pi a \left(\frac{a}{2} - 0 \right) = \pi a^2. \text{ Note that this is}$$

$\frac{1}{4}$ the surface area of a sphere of radius a , and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

16. $x = a \cosh(y/a) \Rightarrow 1 + (dx/dy)^2 = 1 + \sinh^2(y/a) = \cosh^2(y/a)$. So

$$\begin{aligned} S &= 2\pi \int_{-a}^a a \cosh\left(\frac{y}{a}\right) \cosh\left(\frac{y}{a}\right) dy = 4\pi a \int_0^a \cosh^2\left(\frac{y}{a}\right) dy = 2\pi a \int_0^a \left[1 + \cosh\left(\frac{2y}{a}\right)\right] dy \\ &= 2\pi a \left[y + \frac{a}{2} \sinh\left(\frac{2y}{a}\right) \right]_0^a = 2\pi a \left[a + \frac{a}{2} \sinh 2 \right] = 2\pi a^2 \left[1 + \frac{1}{2} \sinh 2 \right] \quad \text{or} \quad \frac{\pi a^2 (e^2 + 4 - e^{-2})}{2} \end{aligned}$$

17. $y = \ln x \Rightarrow dy/dx = 1/x \Rightarrow 1 + (dy/dx)^2 = 1 + 1/x^2 \Rightarrow S = \int_1^3 2\pi \ln x \sqrt{1 + 1/x^2} dx$.

Let $f(x) = \ln x \sqrt{1 + 1/x^2}$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Then

$$S \approx S_{10} = 2\pi \cdot \frac{1/5}{3} [f(1) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3)] \approx 9.023754.$$

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

18. $y = x + \sqrt{x} \Rightarrow dy/dx = 1 + \frac{1}{2}x^{-1/2} \Rightarrow 1 + (dy/dx)^2 = 2 + x^{-1/2} + \frac{1}{4}x^{-1} \Rightarrow$

$$S = \int_1^2 2\pi (x + \sqrt{x}) \sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}} dx. \text{ Let } f(x) = (x + \sqrt{x}) \sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}}.$$

Since $n = 10$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$. Then

$$S \approx S_{10} = 2\pi \cdot \frac{1/10}{3} [f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \approx 29.506566.$$

The value of the integral produced by a calculator is 29.506568 (to six decimal places).

19. $y = \sec x \Rightarrow dy/dx = \sec x \tan x \Rightarrow 1 + (dy/dx)^2 = 1 + \sec^2 x \tan^2 x \Rightarrow$

$$S = \int_0^{\pi/3} 2\pi \sec x \sqrt{1 + \sec^2 x \tan^2 x} dx. \text{ Let } f(x) = \sec x \sqrt{1 + \sec^2 x \tan^2 x}.$$

Since $n = 10$, $\Delta x = \frac{\pi/3 - 0}{10} = \frac{\pi}{30}$. Then

$$S \approx S_{10} = 2\pi \cdot \frac{\pi/30}{3} \left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + \cdots + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right) \right] \approx 13.527296.$$

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

20. $y = (1 + e^x)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(1 + e^x)^{-1/2} \cdot e^x = \frac{e^x}{2(1 + e^x)^{1/2}} \Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2x}}{4(1 + e^x)} = \frac{4 + 4e^x + e^{2x}}{4(1 + e^x)} = \frac{(e^x + 2)^2}{4(1 + e^x)} \Rightarrow$$

$$S = \int_0^1 2\pi \sqrt{1 + e^x} \cdot \frac{e^x + 2}{2\sqrt{1 + e^x}} dx = \pi \int_0^1 (e^x + 2) dx = \pi [e^x + 2x]_0^1 = \pi[(e + 2) - (1 + 0)] = \pi(e + 1).$$

Let $f(x) = \frac{1}{2}(e^x + 2)$. Since $n = 10$, $\Delta x = \frac{1-0}{10} = \frac{1}{10}$. Then

$$S \approx S_{10} = 2\pi \cdot \frac{1/10}{3} [f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)] \approx 11.681330.$$

The value of the integral produced by a calculator is 11.681327 (to six decimal places).

$$21. y = 1/x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + (-1/x^2)^2} dx = \sqrt{1 + 1/x^4} dx \Rightarrow$$

$$\begin{aligned} S &= \int_1^2 2\pi \cdot \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx = 2\pi \int_1^2 \frac{\sqrt{x^4 + 1}}{x^3} dx = 2\pi \int_1^4 \frac{\sqrt{u^2 + 1}}{u^2} \left(\frac{1}{2} du\right) \quad [u = x^2, du = 2x dx] \\ &= \pi \int_1^4 \frac{\sqrt{1 + u^2}}{u^2} du \stackrel{24}{=} \pi \left[-\frac{\sqrt{1 + u^2}}{u} + \ln(u + \sqrt{1 + u^2}) \right]_1^4 \\ &= \pi \left[-\frac{\sqrt{17}}{4} + \ln(4 + \sqrt{17}) + \frac{\sqrt{2}}{1} - \ln(1 + \sqrt{2}) \right] = \pi \left[\sqrt{2} - \frac{\sqrt{17}}{4} + \ln\left(\frac{4 + \sqrt{17}}{1 + \sqrt{2}}\right) \right] \end{aligned}$$

$$22. y = \sqrt{x^2 + 1} \Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}} \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{x^2}{x^2 + 1}} dx \Rightarrow$$

$$\begin{aligned} S &= \int_0^3 2\pi \sqrt{x^2 + 1} \sqrt{1 + \frac{x^2}{x^2 + 1}} dx = 2\pi \int_0^3 \sqrt{2x^2 + 1} dx = 2\sqrt{2}\pi \int_0^3 \sqrt{x^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dx \\ &\stackrel{21}{=} 2\sqrt{2}\pi \left[\frac{1}{2}x \sqrt{x^2 + \frac{1}{2}} + \frac{1}{4} \ln\left(x + \sqrt{x^2 + \frac{1}{2}}\right) \right]_0^3 \\ &= 2\sqrt{2}\pi \left[\frac{3}{2}\sqrt{9 + \frac{1}{2}} + \frac{1}{4} \ln\left(3 + \sqrt{9 + \frac{1}{2}}\right) - \frac{1}{4} \ln \frac{1}{\sqrt{2}} \right] = 2\sqrt{2}\pi \left[\frac{3}{2}\sqrt{\frac{19}{2}} + \frac{1}{4} \ln\left(3 + \sqrt{\frac{19}{2}}\right) + \frac{1}{4} \ln \sqrt{2} \right] \\ &= 2\sqrt{2}\pi \left[\frac{3}{2}\frac{\sqrt{19}}{\sqrt{2}} + \frac{1}{4} \ln(3\sqrt{2} + \sqrt{19}) \right] = 3\sqrt{19}\pi + \frac{\pi}{\sqrt{2}} \ln(3\sqrt{2} + \sqrt{19}) \end{aligned}$$

$$23. y = x^3 \text{ and } 0 \leq y \leq 1 \Rightarrow y' = 3x^2 \text{ and } 0 \leq x \leq 1.$$

$$\begin{aligned} S &= \int_0^1 2\pi x \sqrt{1 + (3x^2)^2} dx = 2\pi \int_0^3 \sqrt{1 + u^2} \frac{1}{6} du \quad [u = 3x^2, du = 6x dx] \\ &= \frac{\pi}{3} \int_0^3 \sqrt{1 + u^2} du \stackrel{21}{=} \quad [\text{or use CAS}] \quad \frac{\pi}{3} \left[\frac{1}{2}u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}) \right]_0^3 \\ &= \frac{\pi}{3} \left[\frac{3}{2}\sqrt{10} + \frac{1}{2} \ln(3 + \sqrt{10}) \right] = \frac{\pi}{6} \left[3\sqrt{10} + \ln(3 + \sqrt{10}) \right] \end{aligned}$$

$$24. y = \ln(x + 1), 0 \leq x \leq 1. ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{1}{x + 1}\right)^2} dx, \text{ so}$$

$$\begin{aligned} S &= \int_0^1 2\pi x \sqrt{1 + \frac{1}{(x + 1)^2}} dx = \int_1^2 2\pi(u - 1) \sqrt{1 + \frac{1}{u^2}} du \quad [u = x + 1, du = dx] \\ &= 2\pi \int_1^2 u \frac{\sqrt{1 + u^2}}{u} du - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u} du = 2\pi \int_1^2 \sqrt{1 + u^2} du - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u} du \\ &\stackrel{21, 23}{=} \quad [\text{or use CAS}] \quad 2\pi \left[\frac{1}{2}u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}) \right]_1^2 - 2\pi \left[\sqrt{1 + u^2} - \ln\left(\frac{1 + \sqrt{1 + u^2}}{u}\right) \right]_1^2 \\ &= 2\pi \left[\sqrt{5} + \frac{1}{2} \ln(2 + \sqrt{5}) - \frac{1}{2}\sqrt{2} - \frac{1}{2} \ln(1 + \sqrt{2}) \right] - 2\pi \left[\sqrt{5} - \ln\left(\frac{1 + \sqrt{5}}{2}\right) - \sqrt{2} + \ln(1 + \sqrt{2}) \right] \\ &= 2\pi \left[\frac{1}{2} \ln(2 + \sqrt{5}) + \ln\left(\frac{1 + \sqrt{5}}{2}\right) + \frac{\sqrt{2}}{2} - \frac{3}{2} \ln(1 + \sqrt{2}) \right] \end{aligned}$$

$$25. S = 2\pi \int_1^\infty y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx = 2\pi \int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3} dx. \text{ Rather than trying to}$$

evaluate this integral, note that $\sqrt{x^4 + 1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$$S = 2\pi \int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3} dx > 2\pi \int_1^\infty \frac{x^2}{x^3} dx = 2\pi \int_1^\infty \frac{1}{x} dx$$

But we know that this integral diverges, so the area S is infinite.

$$26. S = \int_0^\infty 2\pi y \sqrt{1 + (dy/dx)^2} dx = 2\pi \int_0^\infty e^{-x} \sqrt{1 + (-e^{-x})^2} dx \quad [y = e^{-x}, y' = -e^{-x}].$$

Evaluate $I = \int e^{-x} \sqrt{1 + (-e^{-x})^2} dx$ by using the substitution $u = -e^{-x}$, $du = e^{-x} dx$.

$$\begin{aligned} I &= \int \sqrt{1 + u^2} du \stackrel{21}{=} \frac{1}{2}u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}) + C \\ &= \frac{1}{2}(-e^{-x})\sqrt{1 + e^{-2x}} + \frac{1}{2} \ln(-e^{-x} + \sqrt{1 + e^{-2x}}) + C \end{aligned}$$

Returning to the surface area integral, we have

$$\begin{aligned} S &= 2\pi \lim_{t \rightarrow \infty} \int_0^t e^{-x} \sqrt{1 + (-e^{-x})^2} dx \\ &= 2\pi \lim_{t \rightarrow \infty} \left[\frac{1}{2}(-e^{-x})\sqrt{1 + e^{-2x}} + \frac{1}{2} \ln(-e^{-x} + \sqrt{1 + e^{-2x}}) \right]_0^t \\ &= 2\pi \lim_{t \rightarrow \infty} \left\{ \left[\frac{1}{2}(-e^{-t})\sqrt{1 + e^{-2t}} + \frac{1}{2} \ln(-e^{-t} + \sqrt{1 + e^{-2t}}) \right] - \left[\frac{1}{2}(-1)\sqrt{1 + 1} + \frac{1}{2} \ln(-1 + \sqrt{1 + 1}) \right] \right\} \\ &= 2\pi \left\{ \left[\frac{1}{2}(0)\sqrt{1} + \frac{1}{2} \ln(0 + \sqrt{1}) \right] - \left[-\frac{1}{2}\sqrt{2} + \frac{1}{2} \ln(-1 + \sqrt{2}) \right] \right\} \\ &= 2\pi \left\{ [0] + \frac{1}{2}[\sqrt{2} - \ln(\sqrt{2} - 1)] \right\} = \pi[\sqrt{2} - \ln(\sqrt{2} - 1)] \end{aligned}$$

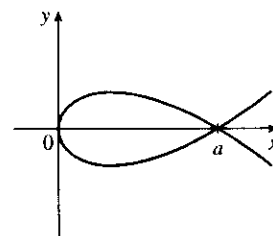
27. Since $a > 0$, the curve $3ay^2 = x(a - x)^2$ only has points with

$$x \geq 0. (3ay^2 \geq 0 \Rightarrow x(a - x)^2 \geq 0 \Rightarrow x \geq 0.) \text{ The}$$

curve is symmetric about the x -axis (since the equation is

unchanged when y is replaced by $-y$). $y = 0$ when $x = 0$ or a ,

so the curve's loop extends from $x = 0$ to $x = a$.



$$\frac{d}{dx}(3ay^2) = \frac{d}{dx}[x(a - x)^2] \Rightarrow 6ay \frac{dy}{dx} = x \cdot 2(a - x)(-1) + (a - x)^2 \Rightarrow$$

$$\frac{dy}{dx} = \frac{(a - x)[-2x + a - x]}{6ay} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(a - x)^2(a - 3x)^2}{36a^2y^2} = \frac{(a - x)^2(a - 3x)^2}{36a^2} \cdot \frac{3a}{x(a - x)^2}$$

$$\left[\begin{array}{l} \text{the last fraction} \\ \text{is } 1/y^2 \end{array} \right] = \frac{(a - 3x)^2}{12ax} \Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{12ax}{12ax} + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{a^2 + 6ax + 9x^2}{12ax} = \frac{(a + 3x)^2}{12ax} \text{ for } x \neq 0.$$

$$\begin{aligned} \text{(a) } S &= \int_{x=0}^a 2\pi y \, ds = 2\pi \int_0^a \frac{\sqrt{x}(a-x)}{\sqrt{3a}} \cdot \frac{a+3x}{\sqrt{12ax}} \, dx = 2\pi \int_0^a \frac{(a-x)(a+3x)}{6a} \, dx \\ &= \frac{\pi}{3a} \int_0^a (a^2 + 2ax - 3x^2) \, dx = \frac{\pi}{3a} [a^2x + ax^2 - x^3]_0^a = \frac{\pi}{3a} (a^3 + a^3 - a^3) = \frac{\pi}{3a} \cdot a^3 = \frac{\pi a^2}{3}. \end{aligned}$$

Note that we have rotated the top half of the loop about the x -axis. This generates the full surface.

(b) We must rotate the full loop about the y -axis, so we get double the area obtained by rotating the top half of the loop:

$$\begin{aligned} S &= 2 \cdot 2\pi \int_{x=0}^a x \, ds = 4\pi \int_0^a x \frac{a+3x}{\sqrt{12ax}} \, dx = \frac{4\pi}{2\sqrt{3a}} \int_0^a x^{1/2}(a+3x) \, dx \\ &= \frac{2\pi}{\sqrt{3a}} \int_0^a (ax^{1/2} + 3x^{3/2}) \, dx = \frac{2\pi}{\sqrt{3a}} \left[\frac{2}{3}ax^{3/2} + \frac{6}{5}x^{5/2} \right]_0^a = \frac{2\pi\sqrt{3}}{3\sqrt{a}} \left(\frac{2}{3}a^{5/2} + \frac{6}{5}a^{5/2} \right) \\ &= \frac{2\pi\sqrt{3}}{3} \left(\frac{2}{3} + \frac{6}{5} \right) a^2 = \frac{2\pi\sqrt{3}}{3} \left(\frac{28}{15} \right) a^2 = \frac{56\pi\sqrt{3}a^2}{45} \end{aligned}$$

28. In general, if the parabola $y = ax^2$, $-c \leq x \leq c$, is rotated about the y -axis, the surface area it generates is

$$\begin{aligned} 2\pi \int_0^c x \sqrt{1 + (2ax)^2} \, dx &= 2\pi \int_0^{2ac} \frac{u}{2a} \sqrt{1 + u^2} \frac{1}{2a} \, du \quad \left[\begin{array}{l} u = 2ax, \\ du = 2a \, dx \end{array} \right] \\ &= \frac{\pi}{4a^2} \int_0^{2ac} (1 + u^2)^{1/2} 2u \, du = \frac{\pi}{4a^2} \left[\frac{2}{3} (1 + u^2)^{3/2} \right]_0^{2ac} \\ &= \frac{\pi}{6a^2} \left[(1 + 4a^2c^2)^{3/2} - 1 \right] \end{aligned}$$

Here $2c = 10$ ft and $ac^2 = 2$ ft, so $c = 5$ and $a = \frac{2}{25}$. Thus, the surface area is

$$\begin{aligned} S &= \frac{\pi}{6} \frac{625}{4} \left[\left(1 + 4 \cdot \frac{4}{625} \cdot 25 \right)^{3/2} - 1 \right] = \frac{625\pi}{24} \left[\left(1 + \frac{16}{25} \right)^{3/2} - 1 \right] = \frac{625\pi}{24} \left(\frac{41\sqrt{41}}{125} - 1 \right) \\ &= \frac{5\pi}{24} \left(41\sqrt{41} - 125 \right) \approx 90.01 \text{ ft}^2 \end{aligned}$$

$$29. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y(dy/dx)}{b^2} = -\frac{x}{a^2} \Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y} \Rightarrow$$

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{b^4x^2}{a^4y^2} = \frac{b^4x^2 + a^4y^2}{a^4y^2} = \frac{b^4x^2 + a^4b^2(1 - x^2/a^2)}{a^4b^2(1 - x^2/a^2)} = \frac{a^4b^2 + b^4x^2 - a^2b^2x^2}{a^4b^2 - a^2b^2x^2} \\ &= \frac{a^4 + b^2x^2 - a^2x^2}{a^4 - a^2x^2} = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)} \end{aligned}$$

The ellipsoid's surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x -axis. Thus,

$$\begin{aligned}
 S &= 2 \int_0^a 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 4\pi \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a \sqrt{a^2 - x^2}} dx \\
 &= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - (a^2 - b^2)x^2} dx = \frac{4\pi b}{a^2} \int_0^{a\sqrt{a^2 - b^2}} \sqrt{a^4 - u^2} \frac{du}{\sqrt{a^2 - b^2}} \quad [u = \sqrt{a^2 - b^2} x] \\
 &\stackrel{30}{=} \frac{4\pi b}{a^2 \sqrt{a^2 - b^2}} \left[\frac{u}{2} \sqrt{a^4 - u^2} + \frac{a^4}{2} \sin^{-1} \frac{u}{a^2} \right]_0^{a\sqrt{a^2 - b^2}} \\
 &= \frac{4\pi b}{a^2 \sqrt{a^2 - b^2}} \left[\frac{a \sqrt{a^2 - b^2}}{2} \sqrt{a^4 - a^2(a^2 - b^2)} + \frac{a^4}{2} \sin^{-1} \frac{\sqrt{a^2 - b^2}}{a} \right] = 2\pi \left[b^2 + \frac{a^2 b \sin^{-1} \frac{\sqrt{a^2 - b^2}}{a}}{\sqrt{a^2 - b^2}} \right]
 \end{aligned}$$

30. The upper half of the torus is generated by rotating the curve $(x - R)^2 + y^2 = r^2$, $y > 0$, about the y -axis.

$$y \frac{dy}{dx} = -(x - R) \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x - R)^2}{y^2} = \frac{y^2 + (x - R)^2}{y^2} = \frac{r^2}{r^2 - (x - R)^2}. \text{ Thus,}$$

$$\begin{aligned}
 S &= 2 \int_{R-r}^{R+r} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 4\pi \int_{R-r}^{R+r} \frac{rx}{\sqrt{r^2 - (x - R)^2}} dx \\
 &= 4\pi r \int_{-r}^r \frac{u + R}{\sqrt{r^2 - u^2}} du \quad [u = x - R] \\
 &= 4\pi r \int_{-r}^r \frac{u du}{\sqrt{r^2 - u^2}} + 4\pi Rr \int_{-r}^r \frac{du}{\sqrt{r^2 - u^2}} \\
 &= 4\pi r \cdot 0 + 8\pi Rr \int_0^r \frac{du}{\sqrt{r^2 - u^2}} \quad [\text{since the first integrand is odd and the second is even}] \\
 &= 8\pi Rr [\sin^{-1}(u/r)]_0^r = 8\pi Rr \left(\frac{\pi}{2}\right) = 4\pi^2 Rr
 \end{aligned}$$

31. The analogue of $f(x_i^*)$ in the derivation of (4) is now $c - f(x_i^*)$, so

$$S = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi [c - f(x_i^*)] \sqrt{1 + [f'(x_i^*)]^2} \Delta x = \int_a^b 2\pi [c - f(x)] \sqrt{1 + [f'(x)]^2} dx.$$

32. $y = x^{1/2} \Rightarrow y' = \frac{1}{2}x^{-1/2} \Rightarrow 1 + (y')^2 = 1 + 1/4x$, so by Exercise 31,

$$S = \int_0^4 2\pi(4 - \sqrt{x}) \sqrt{1 + 1/(4x)} dx. \text{ Using a CAS, we get}$$

$$S = 2\pi \ln(\sqrt{17} + 4) + \frac{\pi}{6} (31\sqrt{17} + 1) \approx 80.6095.$$

33. For the upper semicircle, $f(x) = \sqrt{r^2 - x^2}$, $f'(x) = -x/\sqrt{r^2 - x^2}$. The surface area generated is

$$\begin{aligned} S_1 &= \int_{-r}^r 2\pi \left(r - \sqrt{r^2 - x^2} \right) \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = 4\pi \int_0^r \left(r - \sqrt{r^2 - x^2} \right) \frac{r}{\sqrt{r^2 - x^2}} dx \\ &= 4\pi \int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} - r \right) dx \end{aligned}$$

For the lower semicircle, $f(x) = -\sqrt{r^2 - x^2}$ and $f'(x) = \frac{x}{\sqrt{r^2 - x^2}}$, so $S_2 = 4\pi \int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} + r \right) dx$.

Thus, the total area is $S = S_1 + S_2 = 8\pi \int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} \right) dx = 8\pi \left[r^2 \sin^{-1} \left(\frac{x}{r} \right) \right]_0^r = 8\pi r^2 \left(\frac{\pi}{2} \right) = 4\pi^2 r^2$.

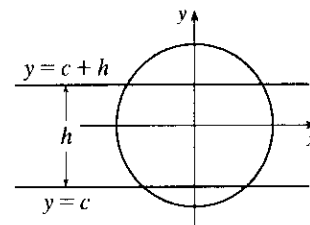
34. Take the sphere $x^2 + y^2 + z^2 = \frac{1}{4}d^2$ and let the intersecting

planes be $y = c$ and $y = c + h$, where $-\frac{1}{2}d \leq c \leq \frac{1}{2}d - h$.

The sphere intersects the xy -plane in the circle

$x^2 + y^2 = \frac{1}{4}d^2$. From this equation, we get $x \frac{dx}{dy} + y = 0$,

so $\frac{dx}{dy} = -\frac{y}{x}$. The desired surface area is



$$\begin{aligned} S &= 2\pi \int x ds = 2\pi \int_c^{c+h} x \sqrt{1 + (dx/dy)^2} dy = 2\pi \int_c^{c+h} x \sqrt{1 + y^2/x^2} dy = 2\pi \int_c^{c+h} \sqrt{x^2 + y^2} dy \\ &= 2\pi \int_c^{c+h} \frac{1}{2}d dy = \pi d \int_c^{c+h} dy = \pi dh \end{aligned}$$

35. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$, the

area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i = f(x_i) \approx f(x_i^*)$ and $y_{i-1} = f(x_{i-1}) \approx f(x_{i-1}^*)$ must be replaced by $y_i = |f(x_i)| \approx |f(x_i^*)|$ and $y_{i-1} = |f(x_{i-1})| \approx |f(x_{i-1}^*)|$. Thus,

$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi |f(x_i^*)| \sqrt{1 + [f'(x_i^*)]^2} \Delta x$. Continuing with the rest of the derivation as before, we

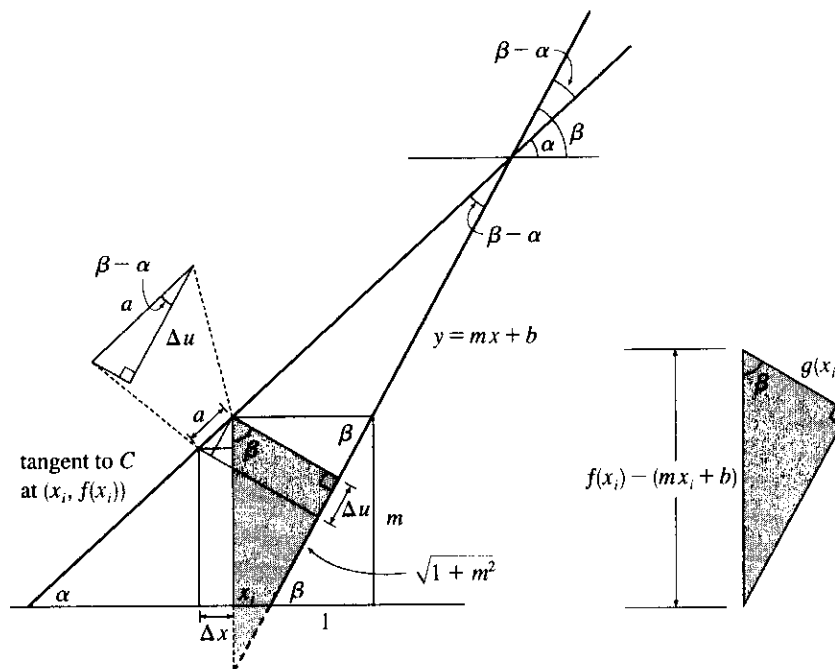
obtain $S = \int_a^b 2\pi |f(x)| \sqrt{1 + [f'(x)]^2} dx$.

36. Since $g(x) = f(x) + c$, we have $g'(x) = f'(x)$. Thus,

$$\begin{aligned} S_g &= \int_a^b 2\pi g(x) \sqrt{1 + [g'(x)]^2} dx = \int_a^b 2\pi [f(x) + c] \sqrt{1 + [f'(x)]^2} dx \\ &= \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx + 2\pi c \int_a^b \sqrt{1 + [f'(x)]^2} dx = S_f + 2\pi cL \end{aligned}$$

DISCOVERY PROJECT Rotating on a Slant

1.



In the figure, the segment a lying above the interval $[x_i - \Delta x, x_i]$ along the tangent to C has length

$\Delta x \sec \alpha = \Delta x \sqrt{1 + \tan^2 \alpha} = \sqrt{1 + [f'(x_i)]^2} \Delta x$. The segment from $(x_i, f(x_i))$ drawn perpendicular to the line $y = mx + b$ has length

$$g(x_i) = [f(x_i) - mx_i - b] \cos \beta = \frac{f(x_i) - mx_i - b}{\sec \beta} = \frac{f(x_i) - mx_i - b}{\sqrt{1 + \tan^2 \beta}} = \frac{f(x_i) - mx_i - b}{\sqrt{1 + m^2}}$$

Also, $\cos(\beta - \alpha) = \frac{\Delta u}{\Delta x \sec \alpha} \Rightarrow$

$$\begin{aligned} \Delta u &= \Delta x \sec \alpha \cos(\beta - \alpha) = \Delta x \frac{\cos \beta \cos \alpha + \sin \beta \sin \alpha}{\cos \alpha} = \Delta x (\cos \beta + \sin \beta \tan \alpha) \\ &= \Delta x \left[\frac{1}{\sqrt{1 + m^2}} + \frac{m}{\sqrt{1 + m^2}} f'(x_i) \right] = \frac{1 + m f'(x_i)}{\sqrt{1 + m^2}} \Delta x \end{aligned}$$

Thus,

$$\begin{aligned} \text{Area}(\mathcal{R}) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n g(x_i) \Delta u = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{f(x_i) - mx_i - b}{\sqrt{1 + m^2}} \cdot \frac{1 + m f'(x_i)}{\sqrt{1 + m^2}} \Delta x \\ &= \frac{1}{1 + m^2} \int_p^q [f(x) - mx - b] [1 + m f'(x)] dx \end{aligned}$$

2. From Problem 1 with $m = 1$, $f(x) = x + \sin x$, $mx + b = x - 2$, $p = 0$, and $q = 2\pi$,

$$\begin{aligned}\text{Area} &= \frac{1}{1+1^2} \int_0^{2\pi} [x + \sin x - (x - 2)] [1 + 1(1 + \cos x)] dx = \frac{1}{2} \int_0^{2\pi} (\sin x + 2)(2 + \cos x) dx \\ &= \frac{1}{2} \int_0^{2\pi} (2 \sin x + \sin x \cos x + 4 + 2 \cos x) dx = \frac{1}{2} [-2 \cos x + \frac{1}{2} \sin^2 x + 4x + 2 \sin x]_0^{2\pi} \\ &= \frac{1}{2} [(-2 + 0 + 8\pi + 0) - (-2 + 0 + 0 + 0)] = \frac{1}{2}(8\pi) = 4\pi\end{aligned}$$

$$\begin{aligned}3. V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi [g(x_i)]^2 \Delta u = \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi \left[\frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}} \right]^2 \frac{1+mf'(x_i)}{\sqrt{1+m^2}} \Delta x \\ &= \frac{\pi}{(1+m^2)^{3/2}} \int_p^q [f(x) - mx - b]^2 [1+mf'(x)] dx\end{aligned}$$

$$\begin{aligned}4. V &= \frac{\pi}{(1+1^2)^{3/2}} \int_0^{2\pi} (x + \sin x - x + 2)^2 (1 + 1 + \cos x) dx \\ &= \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} (\sin x + 2)^2 (\cos x + 2) dx \\ &= \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} (\sin^2 x + 4 \sin x + 4)(\cos x + 2) dx \\ &= \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} (\sin^2 x \cos x + 4 \sin x \cos x + 4 \cos x + 2 \sin^2 x + 8 \sin x + 8) dx \\ &= \frac{\pi}{2\sqrt{2}} \left[\frac{1}{3} \sin^3 x + 2 \sin^2 x + 4 \sin x + x - \frac{1}{2} \sin 2x - 8 \cos x + 8x \right]_0^{2\pi} \quad [\text{since } 2 \sin^2 x = 1 - \cos 2x] \\ &= \frac{\pi}{2\sqrt{2}} [(2\pi - 8 + 16\pi) - (-8)] = \frac{9\sqrt{2}}{2} \pi^2\end{aligned}$$

$$5. S = \int_p^q 2\pi g(x) \sqrt{1+[f'(x)]^2} dx = \frac{2\pi}{\sqrt{1+m^2}} \int_p^q [f(x) - mx - b] \sqrt{1+[f'(x)]^2} dx$$

6. From Problem 5 with $f(x) = \sqrt{x}$, $p = 0$, $q = 4$, $m = \frac{1}{2}$, and $b = 0$,

$$\begin{aligned}S &= \frac{2\pi}{\sqrt{1+(\frac{1}{2})^2}} \int_0^4 (\sqrt{x} - \frac{1}{2}x) \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2} dx \\ &= \frac{\pi}{\sqrt{5}} \left[\frac{\ln(\sqrt{17}+4)}{32} + \frac{37\sqrt{17}}{24} - \frac{1}{3} \right] \quad [\text{from CAS}] \\ &\approx 8.554\end{aligned}$$

9.3 Applications to Physics and Engineering

1. The weight density of water is $\delta = 62.5 \text{ lb/ft}^3$.

$$(a) P = \delta d \approx (62.5 \text{ lb/ft}^3)(3 \text{ ft}) = 187.5 \text{ lb/ft}^2$$

$$(b) F = PA \approx (187.5 \text{ lb/ft}^2)(5 \text{ ft})(2 \text{ ft}) = 1875 \text{ lb. } (A \text{ is the area of the bottom of the tank.})$$

(c) As in Example 1, the area of the i th strip is $2(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

$$F = \int_0^3 \delta x \cdot 2 dx \approx (62.5)(2) \int_0^3 x dx = 125 \left[\frac{1}{2} x^2 \right]_0^3 = 125 \left(\frac{9}{2} \right) = 562.5 \text{ lb}$$

2. (a) $P = \rho g d = 1030(9.8)(2.5) = 25,235 \approx 2.52 \times 10^4 \text{ Pa} = 25.2 \text{ kPa}$

$$(b) F = PA \approx (2.52 \times 10^4 \text{ N/m}^2)(50 \text{ m}^2) = 1.26 \times 10^6 \text{ N}$$

$$(c) F = \int_0^{2.5} \rho g x \cdot 5 dx = (1030)(9.8)(5) \int_0^{2.5} x dx \approx 2.52 \times 10^4 \left[\frac{1}{2} x^2 \right]_0^{2.5} \approx 1.58 \times 10^5 \text{ N}$$

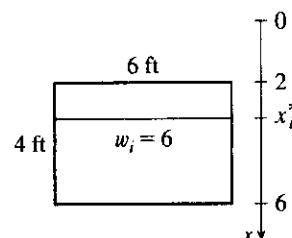
In Exercises 3–9, n is the number of subintervals of length Δx and x_i^* is a sample point in the i th subinterval $[x_{i-1}, x_i]$.

3. Set up a vertical x -axis as shown, with $x = 0$ at the water's surface and x increasing in the downward direction. Then the area of the i th rectangular strip is $6 \Delta x$ and the pressure on the strip is δx_i^* (where $\delta \approx 62.5 \text{ lb/ft}^3$).

Thus, the hydrostatic force on the strip is $\delta x_i^* \cdot 6 \Delta x$ and the total

hydrostatic force $\approx \sum_{i=1}^n \delta x_i^* \cdot 6 \Delta x$. The total force

$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \delta x_i^* \cdot 6 \Delta x = \int_2^6 \delta x \cdot 6 dx = 6\delta \int_2^6 x dx \\ &= 6\delta \left[\frac{1}{2} x^2 \right]_2^6 = 6\delta(18 - 2) = 96\delta \approx 6000 \text{ lb} \end{aligned}$$



4. Set up a vertical x -axis as shown. Then the area of the i th rectangular strip

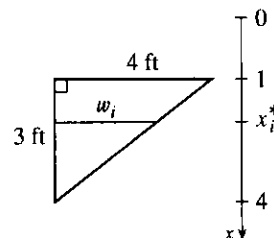
is $\frac{4}{3}(4 - x_i^*) \Delta x$. [By similar triangles, $\frac{w_i}{4 - x_i^*} = \frac{4}{3}$, so

$w_i = \frac{4}{3}(4 - x_i^*)$.] The pressure on the strip is δx_i^* , so the hydrostatic force

on the strip is $\delta x_i^* \cdot \frac{4}{3}(4 - x_i^*) \Delta x$ and the total force on the

plate $\approx \sum_{i=1}^n \delta x_i^* \cdot \frac{4}{3}(4 - x_i^*) \Delta x$. The total force

$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \delta x_i^* \cdot \frac{4}{3}(4 - x_i^*) \Delta x = \int_1^4 \delta x \cdot \frac{4}{3}(4 - x) dx = \frac{4}{3}\delta \int_1^4 (4 - x) dx \\ &= \frac{4}{3}\delta \left[2x^2 - \frac{1}{3}x^3 \right]_1^4 = \frac{4}{3}\delta \left[\left(32 - \frac{64}{3} \right) - \left(2 - \frac{1}{3} \right) \right] = \frac{4}{3}\delta(9) = 12\delta \approx 750 \text{ lb} \end{aligned}$$



5. Since an equation for the shape is $x^2 + y^2 = 10^2$ ($x \geq 0$), we have

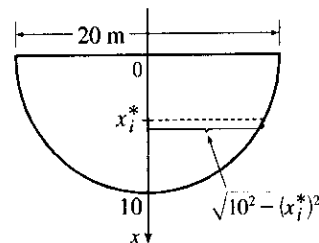
$$y = \sqrt{100 - x^2}. \text{ Thus, the area of the } i\text{th strip is } 2\sqrt{100 - (x_i^*)^2} \Delta x$$

and the pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the

strip is $\rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2} \Delta x$ and the total force on the

$$\text{plate} \approx \sum_{i=1}^n \rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2} \Delta x. \text{ The total force}$$

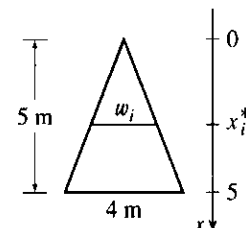
$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2} \Delta x = \int_0^{10} 2\rho g x \sqrt{100 - x^2} dx \\ &= -\rho g \int_0^{10} (100 - x^2)^{1/2} (-2x) dx = -\rho g \left[\frac{2}{3} (100 - x^2)^{3/2} \right]_0^{10} = -\frac{2}{3} \rho g (0 - 1000) \\ &= \frac{2000}{3} \rho g \approx \frac{2000}{3} \cdot 1000 \cdot 9.8 \approx 6.5 \times 10^6 \text{ N} \quad [\rho \approx 1000 \text{ kg/m}^3 \text{ and } g \approx 9.8 \text{ m/s}^2.] \end{aligned}$$



6. By similar triangles, $w_i/4 = x_i^*/5$, so $w_i = \frac{4}{5}x_i^*$ and the area of the i th strip is $\frac{4}{5}x_i^* \Delta x$. The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot \frac{4}{5}x_i^* \Delta x$ and the total force on the

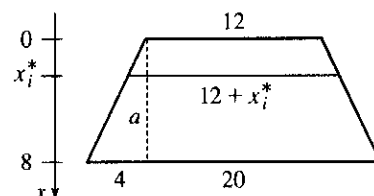
$$\text{plate} \approx \sum_{i=1}^n \rho g x_i^* \cdot \frac{4}{5}x_i^* \Delta x. \text{ The total force}$$

$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho g x_i^* \cdot \frac{4}{5}x_i^* \Delta x = \int_0^5 \rho g x \cdot \frac{4}{5}x dx = \frac{4}{5} \rho g \left[\frac{1}{3} x^3 \right]_0^5 = \frac{4}{5} \rho g \cdot \frac{125}{3} = \frac{100}{3} \rho g \\ &\approx \frac{100}{3} \cdot 1000 \cdot 9.8 \approx 3.3 \times 10^5 \text{ N}. \end{aligned}$$



7. Using similar triangles, $\frac{4 \text{ ft wide}}{8 \text{ ft high}} = \frac{a \text{ ft wide}}{x_i^* \text{ ft high}}$, so $a = \frac{1}{2}x_i^*$ and the width of the i th rectangular strip is $12 + 2a = 12 + x_i^*$. The area of the strip is $(12 + x_i^*) \Delta x$. The pressure on the strip is δx_i^* .

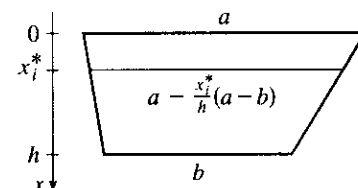
$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \delta x_i^* (12 + x_i^*) \Delta x = \int_0^8 \delta x \cdot (12 + x) dx \\ &= \delta \int_0^8 (12x + x^2) dx = \delta \left[6x^2 + \frac{x^3}{3} \right]_0^8 = \delta (384 + \frac{512}{3}) \\ &= (62.5) \frac{1664}{3} \approx 3.47 \times 10^4 \text{ lb} \end{aligned}$$



8. In the figure, deleting a $b \times h$ rectangle leaves a triangle with base $a - b$ and height h . By similar triangles, $\frac{(a - b) \text{ ft wide}}{h \text{ ft high}} = \frac{d \text{ ft wide}}{(h - x_i^*) \text{ ft high}}$, so the width of the triangle is

$$d = \frac{h - x_i^*}{h} (a - b) = \left(1 - \frac{x_i^*}{h} \right) (a - b) = a - b - \frac{x_i^*}{h} (a - b)$$

and the width of the trapezoid is $b + d = a - \frac{x_i^*}{h} (a - b)$. The area of the i th rectangular strip is



$\left[a - \frac{x_i^*}{h} (a - b) \right] \Delta x$ and the pressure on it is $\rho g x_i^*$.

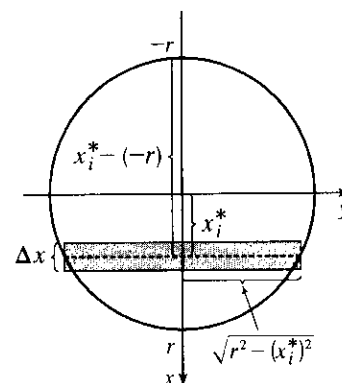
$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho g x_i^* \left[a - \frac{x_i^*}{h} (a - b) \right] \Delta x = \int_0^h \rho g x \left[a - \frac{x}{h} (a - b) \right] dx \\ &= \rho g a \int_0^h x dx + \frac{\rho g (b - a)}{h} \int_0^h x^2 dx = \rho g a \frac{h^2}{2} + \rho g \frac{b - a}{h} \frac{h^3}{3} \\ &= \rho g h^2 \left(\frac{a}{2} + \frac{b - a}{3} \right) = \rho g h^2 \frac{a + 2b}{6} \approx \frac{500}{3} g h^2 (a + 2b) \text{ N} \end{aligned}$$

9. From the figure, the area of the i th rectangular strip is $2\sqrt{r^2 - (x_i^*)^2} \Delta x$ and the pressure on it is $\rho g(x_i^* + r)$.

$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho g(x_i^* + r) 2\sqrt{r^2 - (x_i^*)^2} \Delta x \\ &= \int_{-r}^r \rho g(x + r) \cdot 2\sqrt{r^2 - x^2} dx \\ &= \rho g \int_{-r}^r \sqrt{r^2 - x^2} 2x dx + 2\rho g r \int_{-r}^r \sqrt{r^2 - x^2} dx \end{aligned}$$

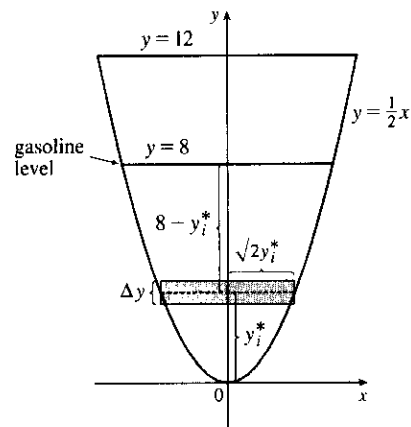
The first integral is 0 because the integrand is an odd function. The second integral can be interpreted as the area of a semicircular disk with radius r , or we could make the trigonometric substitution $x = r \sin \theta$. Continuing:

$$F = \rho g \cdot 0 + 2\rho g r \cdot \frac{1}{2} \pi r^2 = \rho g \pi r^3 = 1000 g \pi r^3 \text{ N (SI units assumed)}.$$



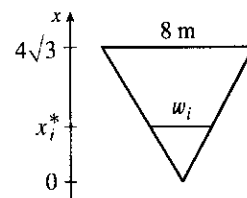
10. The area of the i th rectangular strip is $2\sqrt{2y_i^*} \Delta y$ and the pressure on it is $\delta d_i = \delta(8 - y_i^*)$.

$$\begin{aligned} F &= \int_0^8 \delta(8 - y) 2\sqrt{2y} dy = 42 \cdot 2 \cdot \sqrt{2} \int_0^8 (8 - y)y^{1/2} dy \\ &= 84\sqrt{2} \int_0^8 (8y^{1/2} - y^{3/2}) dy = 84\sqrt{2} \left[8 \cdot \frac{2}{3} y^{3/2} - \frac{2}{5} y^{5/2} \right]_0^8 \\ &= 84\sqrt{2} \left[8 \cdot \frac{2}{3} \cdot 16\sqrt{2} - \frac{2}{5} \cdot 128\sqrt{2} \right] \\ &= 84\sqrt{2} \cdot 256\sqrt{2} \left(\frac{1}{3} - \frac{1}{5} \right) = 43,008 \cdot \frac{2}{15} = 5734.4 \text{ lb} \end{aligned}$$

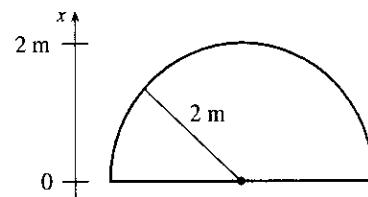


11. By similar triangles, $\frac{8}{4\sqrt{3}} = \frac{w_i}{x_i^*} \Rightarrow w_i = \frac{2x_i^*}{\sqrt{3}}$. The area of the i th rectangular strip is $\frac{2x_i^*}{\sqrt{3}} \Delta x$ and the pressure on it is $\rho g(4\sqrt{3} - x_i^*)$.

$$\begin{aligned} F &= \int_0^{4\sqrt{3}} \rho g(4\sqrt{3} - x) \frac{2x}{\sqrt{3}} dx = 8\rho g \int_0^{4\sqrt{3}} x dx - \frac{2\rho g}{\sqrt{3}} \int_0^{4\sqrt{3}} x^2 dx \\ &= 4\rho g [x^2]_0^{4\sqrt{3}} - \frac{2\rho g}{3\sqrt{3}} [x^3]_0^{4\sqrt{3}} = 192\rho g - \frac{2\rho g}{3\sqrt{3}} 64 \cdot 3\sqrt{3} \\ &= 192\rho g - 128\rho g = 64\rho g \approx 64(840)(9.8) \approx 5.27 \times 10^5 \text{ N} \end{aligned}$$



$$\begin{aligned}
 12. \quad F &= \int_0^2 \rho g(10-x)2\sqrt{4-x^2} dx \\
 &= 20\rho g \int_0^2 \sqrt{4-x^2} dx - \rho g \int_0^2 \sqrt{4-x^2} 2x dx \\
 &= 20\rho g \frac{1}{4}\pi(2^2) - \rho g \int_0^4 u^{1/2} du \quad [u = 4-x^2, du = -2x dx] \\
 &= 20\pi\rho g - \frac{2}{3}\rho g \left[u^{3/2} \right]_0^4 = 20\pi\rho g - \frac{16}{3}\rho g = \rho g \left(20\pi - \frac{16}{3} \right) \\
 &= (1000)(9.8) \left(20\pi - \frac{16}{3} \right) \approx 5.63 \times 10^5 \text{ N}
 \end{aligned}$$



13. (a) The top of the cube has depth $d = 1 \text{ m} - 20 \text{ cm} = 80 \text{ cm} = 0.8 \text{ m}$.

$$F = \rho g d A \approx (1000)(9.8)(0.8)(0.2)^2 = 313.6 \approx 314 \text{ N}$$

- (b) The area of a strip is $0.2 \Delta x$ and the pressure on it is $\rho g x_i^*$.

$$\begin{aligned}
 F &= \int_{0.8}^1 \rho g x(0.2) dx = 0.2\rho g \left[\frac{1}{2}x^2 \right]_{0.8}^1 = (0.2\rho g)(0.18) = 0.036\rho g = 0.036(1000)(9.8) \\
 &= 352.8 \approx 353 \text{ N}
 \end{aligned}$$

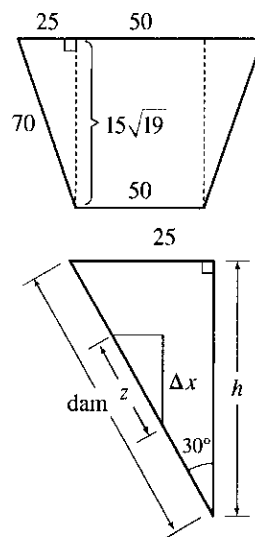
14. The height of the dam is $h = \sqrt{70^2 - 25^2} \cos 30^\circ = 15\sqrt{19} \left(\frac{\sqrt{3}}{2} \right)$.

From the solution for Exercise 8, the width of the trapezoid is

$$100 - \frac{x}{h}(100 - 50) = 100 - \frac{50x}{h}. \text{ From the small triangle in the}$$

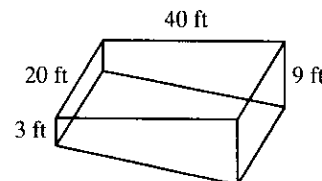
$$\text{second figure, } \cos 30^\circ = \frac{\Delta x}{z} \Rightarrow z = \Delta x \sec 30^\circ = 2\Delta x/\sqrt{3}.$$

$$\begin{aligned}
 F &= \int_0^h \delta x \left(100 - \frac{50x}{h} \right) \frac{2}{\sqrt{3}} dx = \frac{200\delta}{\sqrt{3}} \int_0^h x dx - \frac{100\delta}{h\sqrt{3}} \int_0^h x^2 dx \\
 &= \frac{200\delta}{\sqrt{3}} \frac{h^2}{2} - \frac{100\delta}{h\sqrt{3}} \frac{h^3}{3} = \frac{200\delta h^2}{3\sqrt{3}} = \frac{200(62.5)}{3\sqrt{3}} \cdot \frac{12,825}{4} \\
 &\approx 7.71 \times 10^6 \text{ lb}
 \end{aligned}$$



15. (a) The area of a strip is $20 \Delta x$ and the pressure on it is δx_i .

$$\begin{aligned}
 F &= \int_0^3 \delta x 20 dx = 20\delta \left[\frac{1}{2}x^2 \right]_0^3 = 20\delta \cdot \frac{9}{2} = 90\delta \\
 &= 90(62.5) = 5625 \text{ lb} \approx 5.63 \times 10^3 \text{ lb}
 \end{aligned}$$



(b) $F = \int_0^9 \delta x 20 dx = 20\delta \left[\frac{1}{2}x^2 \right]_0^9 = 20\delta \cdot \frac{81}{2} = 810\delta = 810(62.5) = 50,625 \text{ lb} \approx 5.06 \times 10^4 \text{ lb}.$

- (c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \leq 9$, we can use similar triangles to find the length a : $\frac{a}{40} = \frac{9-x}{6} \Rightarrow a = 40 \cdot \frac{9-x}{6}$.

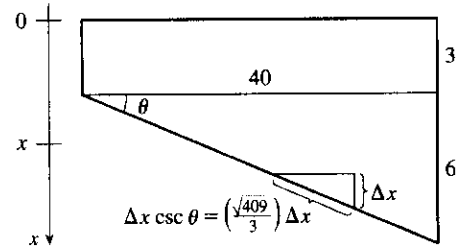
$$\begin{aligned} F &= \int_0^3 \delta x 40 dx + \int_3^9 \delta x (40) \frac{9-x}{6} dx = 40\delta \left[\frac{1}{2} x^2 \right]_0^3 + \frac{20}{3} \delta \int_3^9 (9-x) dx \\ &= 180\delta + \frac{20}{3} \delta \left[\frac{9}{2} x^2 - \frac{1}{3} x^3 \right]_3^9 = 180\delta + \frac{20}{3} \delta \left[\left(\frac{729}{2} - 243 \right) - \left(\frac{81}{2} - 9 \right) \right] \\ &= 180\delta + 600\delta = 780\delta = 780(62.5) = 48,750 \text{ lb} \approx 4.88 \times 10^4 \text{ lb} \end{aligned}$$

- (d) For any right triangle with hypotenuse on the bottom,

$$\csc \theta = \frac{\Delta x}{\text{hypotenuse}} \Rightarrow$$

$$\text{hypotenuse} = \Delta x \csc \theta = \Delta x \frac{\sqrt{40^2 + 6^2}}{6} = \frac{\sqrt{409}}{3} \Delta x.$$

$$\begin{aligned} F &= \int_3^9 \delta x 20 \frac{\sqrt{409}}{3} dx = \frac{1}{3} (20\sqrt{409}) \delta \left[\frac{1}{2} x^2 \right]_3^9 \\ &= \frac{1}{3} \cdot 10 \sqrt{409} \delta (81 - 9) \\ &\approx 303,356 \text{ lb} \approx 3.03 \times 10^5 \text{ lb} \end{aligned}$$



16. Partition the interval $[a, b]$ by points x_i as usual and choose $x_i^* \in [x_{i-1}, x_i]$ for each i . The i th horizontal strip of the immersed plate is approximated by a rectangle of height Δx_i and width $w(x_i^*)$, so its area is $A_i \approx w(x_i^*) \Delta x_i$. For small Δx_i , the pressure P_i on the i th strip is almost constant and $P_i \approx \rho g x_i^*$ by Equation 1. The hydrostatic force F_i acting on the i th strip is $F_i = P_i A_i \approx \rho g x_i^* w(x_i^*) \Delta x_i$. Adding these forces and taking the limit as $n \rightarrow \infty$, we obtain the hydrostatic force on the immersed plate:

$$F = \lim_{n \rightarrow \infty} \sum_{i=1}^n F_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho g x_i^* w(x_i^*) \Delta x_i = \int_a^b \rho g x w(x) dx$$

17. $F = \int_2^5 \rho g x \cdot w(x) dx$, where $w(x)$ is the width of the plate at depth x . Since $n = 6$, $\Delta x = \frac{5-2}{6} = \frac{1}{2}$, and $F \approx S_6 = \rho g \cdot \frac{1}{2} [2 \cdot w(2) + 4 \cdot 2.5 \cdot w(2.5) + 2 \cdot 3 \cdot w(3) + 4 \cdot 3.5 \cdot w(3.5) + 2 \cdot 4 \cdot w(4) + 4 \cdot 4.5 \cdot w(4.5) + 5 \cdot w(5)]$
- $$\begin{aligned} &= \frac{1}{6} \rho g (2 \cdot 0 + 10 \cdot 0.8 + 6 \cdot 1.7 + 14 \cdot 2.4 + 8 \cdot 2.9 + 18 \cdot 3.3 + 5 \cdot 3.6) \\ &= \frac{1}{6} (1000) (9.8) (152.4) \approx 2.5 \times 10^5 \text{ N} \end{aligned}$$

18. (a) From Equation 8, $\bar{x} = \frac{1}{A} \int_a^b x w(x) dx \Rightarrow A \bar{x} = \int_a^b x w(x) dx \Rightarrow \rho g A \bar{x} = \rho g \int_a^b x w(x) dx \Rightarrow (\rho g \bar{x}) A = \int_a^b \rho g x w(x) dx = F$ by Exercise 16.

- (b) The centroid of a circle is its center. In this case, the center is at a depth of r meters, so $\bar{x} = r$. Thus,

$$F = (\rho g \bar{x}) A = (\rho g r) (\pi r^2) = \rho g \pi r^3.$$

19. The moment M of the system about the origin is $M = \sum_{i=1}^2 m_i x_i = m_1 x_1 + m_2 x_2 = 40 \cdot 2 + 30 \cdot 5 = 230$.

The mass m of the system is $m = \sum_{i=1}^2 m_i = m_1 + m_2 = 40 + 30 = 70$. The center of mass of the system is

$$M/m = \frac{230}{70} = \frac{23}{7}.$$

20. $M = m_1x_1 + m_2x_2 + m_3x_3 = 25(-2) + 20(3) + 10(7) = 80$;

$$\bar{x} = M/(m_1 + m_2 + m_3) = \frac{80}{55} = \frac{16}{11}.$$

21. $m = \sum_{i=1}^3 m_i = 6 + 5 + 10 = 21$. $M_x = \sum_{i=1}^3 m_i y_i = 6(5) + 5(-2) + 10(-1) = 10$;

$$M_y = \sum_{i=1}^3 m_i x_i = 6(1) + 5(3) + 10(-2) = 1. \quad \bar{x} = \frac{M_y}{m} = \frac{1}{21} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{10}{21}, \text{ so the center of mass of the system is } \left(\frac{1}{21}, \frac{10}{21}\right).$$

22. $M_x = \sum_{i=1}^4 m_i y_i = 6(-2) + 5(4) + 1(-7) + 4(-1) = -3$, $M_y = \sum_{i=1}^4 m_i x_i = 6(1) + 5(3) + 1(-3) + 4(6) = 42$,

$$\text{and } m = \sum_{i=1}^4 m_i = 16, \text{ so } \bar{x} = \frac{M_y}{m} = \frac{42}{16} = \frac{21}{8} \text{ and } \bar{y} = \frac{M_x}{m} = -\frac{3}{16}; \text{ the center of mass is } (\bar{x}, \bar{y}) = \left(\frac{21}{8}, -\frac{3}{16}\right).$$

23. Since the region in the figure is symmetric about the y -axis, we know

that $\bar{x} = 0$. The region is “bottom-heavy,” so we know that $\bar{y} < 2$,

and we might guess that $\bar{y} = 1.5$.

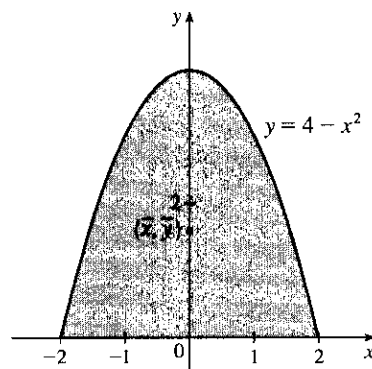
$$A = \int_{-2}^2 (4 - x^2) dx = 2 \int_0^2 (4 - x^2) dx = 2 \left[4x - \frac{1}{3}x^3 \right]_0^2 \\ = 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3}$$

$$\bar{x} = \frac{1}{A} \int_{-2}^2 x(4 - x^2) dx = 0 \text{ since } f(x) = x(4 - x^2) \text{ is an odd}$$

function (or since the region is symmetric about the y -axis).

$$\bar{y} = \frac{1}{A} \int_{-2}^2 \frac{1}{2} (4 - x^2)^2 dx = \frac{3}{32} \cdot \frac{1}{2} \cdot 2 \int_0^2 (16 - 8x^2 + x^4) dx = \frac{3}{32} \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2 \\ = \frac{3}{32} \left(32 - \frac{64}{3} + \frac{32}{5} \right) = 3 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = 3 \left(\frac{8}{15} \right) = \frac{8}{5}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$.



24. The region in the figure is “left-heavy” and “bottom-heavy,” so we know

$\bar{x} < 1$ and $\bar{y} < 1.5$, and we might guess that $\bar{x} = 0.7$ and $\bar{y} = 1.2$.

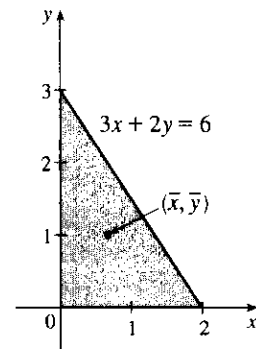
$$3x + 2y = 6 \Leftrightarrow 2y = 6 - 3x \Leftrightarrow y = 3 - \frac{3}{2}x.$$

$$A = \int_0^2 \left(3 - \frac{3}{2}x\right) dx = \left[3x - \frac{3}{4}x^2\right]_0^2 = 6 - 3 = 3.$$

$$\bar{x} = \frac{1}{A} \int_0^2 x \left(3 - \frac{3}{2}x\right) dx = \frac{1}{3} \int_0^2 \left(3x - \frac{3}{2}x^2\right) dx \\ = \frac{1}{3} \left[\frac{3}{2}x^2 - \frac{1}{2}x^3 \right]_0^2 = \frac{1}{3} (6 - 4) = \frac{2}{3};$$

$$\bar{y} = \frac{1}{A} \int_0^2 \frac{1}{2} \left(3 - \frac{3}{2}x\right)^2 dx = \frac{1}{3} \cdot \frac{1}{2} \int_0^2 \left(9 - 9x + \frac{9}{4}x^2\right) dx = \frac{1}{6} \left[9x - \frac{9}{2}x^2 + \frac{3}{4}x^3 \right]_0^2 = \frac{1}{6} (18 - 18 + 6) = 1.$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{2}{3}, 1\right)$.



25. The region in the figure is “right-heavy” and “bottom-heavy,” so we know

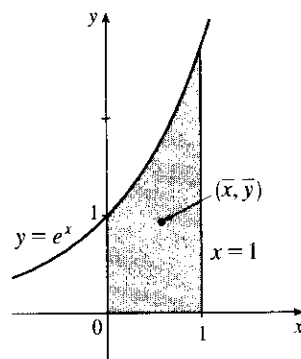
$\bar{x} > 0.5$ and $\bar{y} < 1$, and we might guess that $\bar{x} = 0.6$ and $\bar{y} = 0.9$.

$$A = \int_0^1 e^x dx = [e^x]_0^1 = e - 1,$$

$$\begin{aligned}\bar{x} &= \frac{1}{A} \int_0^1 x e^x dx = \frac{1}{e-1} [x e^x - e^x]_0^1 \quad [\text{by parts}] \\ &= \frac{1}{e-1} [0 - (-1)] = \frac{1}{e-1},\end{aligned}$$

$$\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2} (e^x)^2 dx = \frac{1}{e-1} \cdot \frac{1}{4} [e^{2x}]_0^1 = \frac{1}{4(e-1)} (e^2 - 1) = \frac{e+1}{4}.$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right) \approx (0.58, 0.93)$.



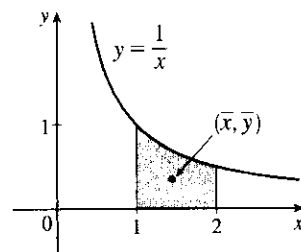
26. The region in the figure is “left-heavy” and “bottom-heavy,” so we know

$\bar{x} < 1.5$ and $\bar{y} < 0.5$, and we might guess that $\bar{x} = 1.4$ and $\bar{y} = 0.4$.

$$A = \int_1^2 \frac{1}{x} dx = [\ln x]_1^2 = \ln 2, \quad \bar{x} = \frac{1}{A} \int_1^2 x \cdot \frac{1}{x} dx = \frac{1}{A} [x]_1^2 = \frac{1}{A} = \frac{1}{\ln 2},$$

$$\begin{aligned}\bar{y} &= \frac{1}{A} \int_1^2 \frac{1}{2} \left(\frac{1}{x}\right)^2 dx = \frac{1}{2A} \int_1^2 x^{-2} dx = \frac{1}{2A} \left[-\frac{1}{x}\right]_1^2 \\ &= \frac{1}{2 \ln 2} \left(-\frac{1}{2} + 1\right) = \frac{1}{4 \ln 2}.\end{aligned}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{1}{\ln 2}, \frac{1}{4 \ln 2}\right) \approx (1.44, 0.36)$.

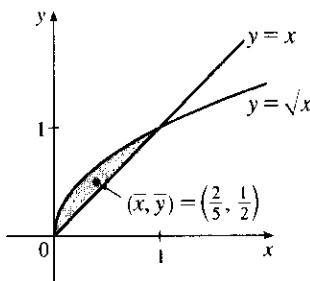


27. $A = \int_0^1 (\sqrt{x} - x) dx = \left[\frac{2}{3} x^{3/2} - \frac{1}{2} x^2\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$

$$\begin{aligned}\bar{x} &= \frac{1}{A} \int_0^1 x(\sqrt{x} - x) dx = 6 \int_0^1 (x^{3/2} - x^2) dx \\ &= 6 \left[\frac{2}{5} x^{5/2} - \frac{1}{3} x^3\right]_0^1 = 6 \left(\frac{2}{5} - \frac{1}{3}\right) = 6 \left(\frac{1}{15}\right) = \frac{2}{5};\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{1}{A} \int_0^1 \frac{1}{2} [(\sqrt{x})^2 - x^2] dx = 6 \cdot \frac{1}{2} \int_0^1 (x - x^2) dx \\ &= 3 \left[\frac{1}{2} x^2 - \frac{1}{3} x^3\right]_0^1 = 3 \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{2}.\end{aligned}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{2}{5}, \frac{1}{2}\right)$.



28. $A = \int_{-1}^2 (x + 2 - x^2) dx = \left[\frac{1}{2} x^2 + 2x - \frac{1}{3} x^3\right]_{-1}^2$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}.$$

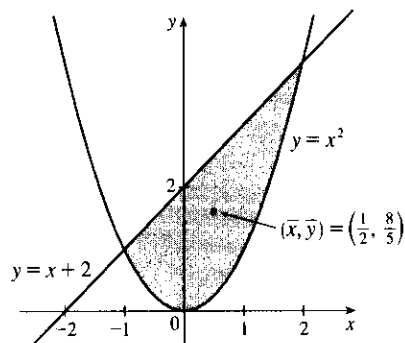
$$\bar{x} = \frac{1}{A} \int_{-1}^2 x(x + 2 - x^2) dx = \frac{2}{9} \int_{-1}^2 (x^2 + 2x - x^3) dx$$

$$\begin{aligned}&= \frac{2}{9} \left[\frac{1}{3} x^3 + x^2 - \frac{1}{4} x^4\right]_{-1}^2 \\ &= \frac{2}{9} \left[\left(\frac{8}{3} + 4 - 4\right) - \left(-\frac{1}{3} + 1 - \frac{1}{4}\right)\right] = \frac{2}{9} \cdot \frac{9}{4} = \frac{1}{2};\end{aligned}$$

$$\bar{y} = \frac{1}{A} \int_{-1}^2 \frac{1}{2} [(x+2)^2 - (x^2)^2] dx = \frac{2}{9} \cdot \frac{1}{2} \int_{-1}^2 (x^2 + 4x + 4 - x^4) dx = \frac{1}{9} \left[\frac{1}{3} x^3 + 2x^2 + 4x - \frac{1}{5} x^5\right]_{-1}^2$$

$$= \frac{1}{9} \left[\left(\frac{8}{3} + 8 + 8 - \frac{32}{5}\right) - \left(-\frac{1}{3} + 2 - 4 + \frac{1}{5}\right)\right] = \frac{1}{9} \left(18 + \frac{9}{3} - \frac{33}{5}\right) = \frac{1}{9} \cdot \frac{72}{5} = \frac{8}{5}.$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{8}{5}\right)$.

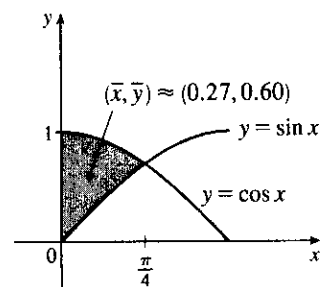


$$29. A = \int_0^{\pi/4} (\cos x - \sin x) dx = [\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1,$$

$$\begin{aligned}\bar{x} &= A^{-1} \int_0^{\pi/4} x(\cos x - \sin x) dx \\ &= A^{-1} [x(\sin x + \cos x) + \cos x - \sin x]_0^{\pi/4} \quad [\text{integration by parts}] \\ &= A^{-1} \left(\frac{\pi}{4} \sqrt{2} - 1 \right) = \frac{\frac{1}{4} \pi \sqrt{2} - 1}{\sqrt{2} - 1}\end{aligned}$$

$$\bar{y} = A^{-1} \int_0^{\pi/4} \frac{1}{2} (\cos^2 x - \sin^2 x) dx = \frac{1}{2A} \int_0^{\pi/4} \cos 2x dx = \frac{1}{4A} [\sin 2x]_0^{\pi/4} = \frac{1}{4A} = \frac{1}{4(\sqrt{2}-1)}$$

$$\text{Thus, the centroid is } (\bar{x}, \bar{y}) = \left(\frac{\pi \sqrt{2} - 4}{4(\sqrt{2} - 1)}, \frac{1}{4(\sqrt{2} - 1)} \right) \approx (0.27, 0.60).$$

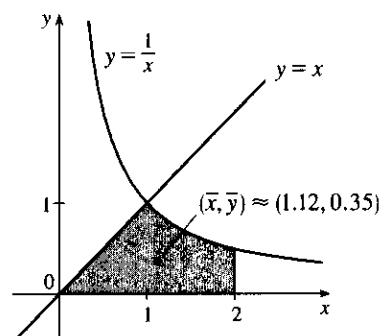


$$30. A = \int_0^1 x dx + \int_1^2 \frac{1}{x} dx = \left[\frac{1}{2} x^2 \right]_0^1 + [\ln x]_1^2 = \frac{1}{2} + \ln 2,$$

$$\begin{aligned}\bar{x} &= \frac{1}{A} \left[\int_0^1 x^2 dx + \int_1^2 1 dx \right] = \frac{1}{A} \left(\left[\frac{1}{3} x^3 \right]_0^1 + [x]_1^2 \right) \\ &= \frac{1}{A} \left(\frac{1}{3} + 1 \right) = \frac{2}{1 + 2 \ln 2} \cdot \frac{4}{3} = \frac{8}{3(1 + 2 \ln 2)},\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{1}{A} \left[\int_0^1 \frac{1}{2} x^2 dx + \int_1^2 \frac{1}{2x^2} dx \right] = \frac{1}{2A} \left(\left[\frac{1}{3} x^3 \right]_0^1 + \left[-\frac{1}{x} \right]_1^2 \right) \\ &= \frac{1}{2A} \left(\frac{1}{3} + \frac{1}{2} \right) = \frac{5}{12A} = \frac{5}{6 + 12 \ln 2}.\end{aligned}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{8}{3(1 + 2 \ln 2)}, \frac{5}{6(1 + 2 \ln 2)} \right) \approx (1.12, 0.35)$. The principle used in this problem is stated after Example 3: the moment of the union of two nonoverlapping regions is the sum of the moments of the individual regions.



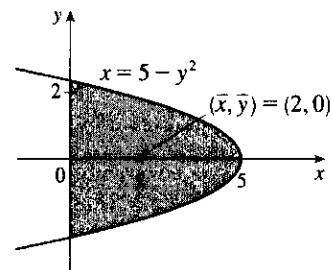
31. From the figure we see that $\bar{y} = 0$. Now

$$\begin{aligned}A &= \int_0^5 2\sqrt{5-x} dx = 2 \left[-\frac{2}{3} (5-x)^{3/2} \right]_0^5 \\ &= 2 \left(0 + \frac{2}{3} \cdot 5^{3/2} \right) = \frac{20}{3} \sqrt{5}\end{aligned}$$

so

$$\begin{aligned}\bar{x} &= \frac{1}{A} \int_0^5 x [\sqrt{5-x} - (-\sqrt{5-x})] dx = \frac{1}{A} \int_0^5 2x \sqrt{5-x} dx \\ &= \frac{1}{A} \int_{\sqrt{5}}^0 2(5-u^2)u(-2u) du \quad [u = \sqrt{5-x}, x = 5-u^2, u^2 = 5-x, dx = -2u du] \\ &= \frac{4}{A} \int_0^{\sqrt{5}} u^2 (5-u^2) du = \frac{4}{A} \left[\frac{5}{3} u^3 - \frac{1}{5} u^5 \right]_0^{\sqrt{5}} = \frac{3}{5\sqrt{5}} \left(\frac{25}{3} \sqrt{5} - 5 \sqrt{5} \right) = 5 - 3 = 2\end{aligned}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = (2, 0)$.



32. By symmetry, $M_y = 0$ and $\bar{x} = 0$; $A = \frac{1}{2}\pi \cdot 1^2 + 4$, so $m = \rho A = 5(\frac{\pi}{2} + 4) = \frac{5}{2}(\pi + 8)$;

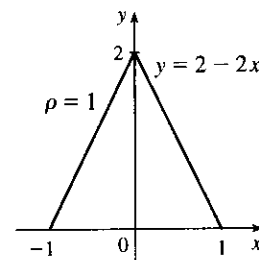
$$M_x = \rho \cdot 2 \int_0^1 \frac{1}{2}[(\sqrt{1-x^2})^2 - (-2)^2] dx = 5 \int_0^1 (-x^2 - 3) dx = -5 \left[\frac{1}{3}x^3 + 3x \right]_0^1 = -5 \cdot \frac{10}{3} = -\frac{50}{3};$$

$$\bar{y} = \frac{1}{m} M_x = \frac{2}{5(\pi + 8)} \cdot \frac{-50}{3} = -\frac{20}{3(\pi + 8)}. \text{ Thus, the centroid is } (\bar{x}, \bar{y}) = \left(0, \frac{-20}{3(\pi + 8)} \right).$$

33. By symmetry, $M_y = 0$ and $\bar{x} = 0$. $A = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$.

$$\begin{aligned} M_x &= \rho \int_{-1}^1 \frac{1}{2}(2-2x)^2 dx = 2\rho \int_0^1 \frac{1}{2}(2-2x)^2 dx \\ &= (2 \cdot 1 \cdot \frac{1}{2} \cdot 2^2) \int_0^1 (1-x)^2 dx \\ &= 4 \int_1^0 u^2 (-du) \quad [u = 1-x, du = -dx] \\ &= -4 \left[\frac{1}{3}u^3 \right]_1^0 = -4 \left(-\frac{1}{3} \right) = \frac{4}{3} \end{aligned}$$

$$\bar{y} = \frac{1}{m} M_x = \frac{1}{\rho A} M_x = \frac{1}{1 \cdot 2} \cdot \frac{4}{3} = \frac{2}{3}. \text{ Thus, the centroid is } (\bar{x}, \bar{y}) = \left(0, \frac{2}{3} \right).$$



34. By symmetry about the line $y = x$, we expect that $\bar{x} = \bar{y}$. $A = \frac{1}{4}\pi r^2$, so $m = \rho A = 2A = \frac{1}{2}\pi r^2$.

$$M_x = \rho \int_0^r \frac{1}{2}(\sqrt{r^2 - x^2})^2 dx = 2 \cdot \frac{1}{2} \int_0^r (r^2 - x^2) dx = \left[r^2 x - \frac{1}{3}x^3 \right]_0^r = \frac{2}{3}r^3.$$

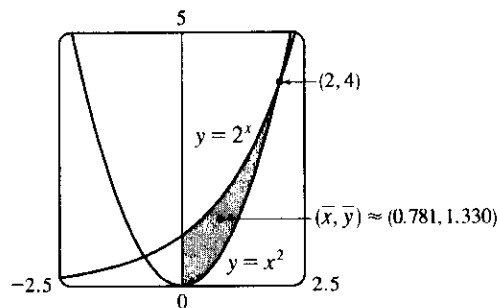
$$M_y = \rho \int_0^r x \sqrt{r^2 - x^2} dx = \int_0^r (r^2 - x^2)^{1/2} 2x dx = \int_0^{r^2} u^{1/2} du \quad [u = r^2 - x^2] = \left[\frac{2}{3}u^{3/2} \right]_0^{r^2} = \frac{2}{3}r^3.$$

$$\bar{x} = \frac{1}{m} M_y = \frac{2}{\pi r^2} \left(\frac{2}{3}r^3 \right) = \frac{4}{3\pi}r; \quad \bar{y} = \frac{1}{m} M_x = \frac{2}{\pi r^2} \left(\frac{2}{3}r^3 \right) = \frac{4}{3\pi}r. \text{ Thus, the centroid is } (\bar{x}, \bar{y}) = \left(\frac{4}{3\pi}r, \frac{4}{3\pi}r \right).$$

$$\begin{aligned} 35. A &= \int_0^2 (2^x - x^2) dx = \left[\frac{2^x}{\ln 2} - \frac{x^3}{3} \right]_0^2 \\ &= \left(\frac{4}{\ln 2} - \frac{8}{3} \right) - \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{8}{3} \approx 1.661418. \end{aligned}$$

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^2 x(2^x - x^2) dx = \frac{1}{A} \int_0^2 (x2^x - x^3) dx \\ &= \frac{1}{A} \left[\frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} - \frac{x^4}{4} \right]_0^2 \quad [\text{use parts}] \\ &= \frac{1}{A} \left[\frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} - 4 + \frac{1}{(\ln 2)^2} \right] \\ &= \frac{1}{A} \left[\frac{8}{\ln 2} - \frac{3}{(\ln 2)^2} - 4 \right] \approx \frac{1}{A} (1.297453) \approx 0.781 \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int_0^2 \frac{1}{2}[(2^x)^2 - (x^2)^2] dx = \frac{1}{A} \int_0^2 \frac{1}{2}(2^{2x} - x^4) dx = \frac{1}{A} \cdot \frac{1}{2} \left[\frac{2^{2x}}{2 \ln 2} - \frac{x^5}{5} \right]_0^2 \\ &= \frac{1}{A} \cdot \frac{1}{2} \left(\frac{16}{2 \ln 2} - \frac{32}{5} - \frac{1}{2 \ln 2} \right) = \frac{1}{A} \left(\frac{15}{4 \ln 2} - \frac{16}{5} \right) \approx \frac{1}{A} (2.210106) \approx 1.330 \end{aligned}$$



Since the position of a centroid is independent of density when the density is constant, we will assume for convenience that $\rho = 1$ in Exercises 36 and 37.

36. The curves $y = x + \ln x$ and $y = x^3 - x$ intersect at

$$(a, c) \approx (0.447141, -0.357742) \text{ and}$$

$$(b, d) \approx (1.507397, 1.917782).$$

$$A = \int_a^b (x + \ln x - x^3 + x) dx = \int_a^b (2x + \ln x - x^3) dx$$

$$\stackrel{(100)}{=} \left[x^2 + x \ln x - x - \frac{1}{4}x^4 \right]_a^b \approx 0.709781$$

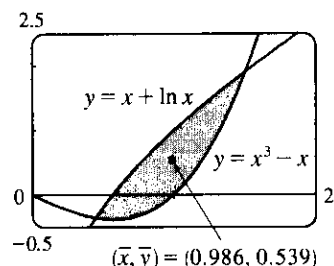
$$\bar{x} = \frac{1}{A} \int_a^b x (2x + \ln x - x^3) dx = \frac{1}{A} \int_a^b (2x^2 + x \ln x - x^4) dx$$

$$\stackrel{(101)}{=} \frac{1}{A} \left[\frac{2}{3}x^3 + \frac{1}{4}x^2 (2 \ln x - 1) - \frac{1}{5}x^5 \right]_a^b \approx \frac{1}{A} (0.699489) \approx 0.985501$$

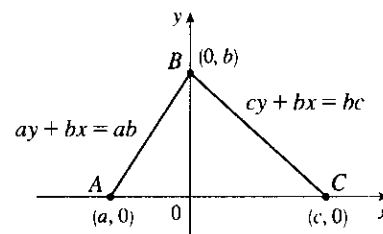
$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} [(x + \ln x)^2 - (x^3 - x)^2] dx = \frac{1}{2A} \int_a^b [2x \ln x + (\ln x)^2 - x^6 + 2x^4] dx$$

$$\stackrel{(101 \text{ and } \text{parts})}{=} \frac{1}{2A} \left[x^2 \ln x - \frac{1}{2}x^2 + x (\ln x)^2 - 2x \ln x + 2x - \frac{1}{7}x^7 + \frac{2}{5}x^5 \right]_a^b \approx \frac{1}{2A} (0.765092) \approx 0.538964$$

Thus, the centroid is $(\bar{x}, \bar{y}) \approx (0.986, 0.539)$.



37. Choose x - and y -axes so that the base (one side of the triangle) lies along the x -axis with the other vertex along the positive y -axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $(\frac{1}{2}(a+c), 0)$ of side AC , so the point of intersection of the medians is $(\frac{2}{3} \cdot \frac{1}{2}(a+c), \frac{1}{3}b) = (\frac{1}{3}(a+c), \frac{1}{3}b)$.



This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A = \frac{1}{2}(c-a)b$.

$$\begin{aligned} \bar{x} &= \frac{1}{A} \left[\int_a^0 x \cdot \frac{b}{a} (a-x) dx + \int_0^c x \cdot \frac{b}{c} (c-x) dx \right] = \frac{1}{A} \left[\frac{b}{a} \int_a^0 (ax - x^2) dx + \frac{b}{c} \int_0^c (cx - x^2) dx \right] \\ &= \frac{b}{Aa} \left[\frac{1}{2}ax^2 - \frac{1}{3}x^3 \right]_a^0 + \frac{b}{Ac} \left[\frac{1}{2}cx^2 - \frac{1}{3}x^3 \right]_0^c = \frac{b}{Aa} \left[-\frac{1}{2}a^3 + \frac{1}{3}a^3 \right] + \frac{b}{Ac} \left[\frac{1}{2}c^3 - \frac{1}{3}c^3 \right] \\ &= \frac{2}{a(c-a)} \cdot \frac{-a^3}{6} + \frac{2}{c(c-a)} \cdot \frac{c^3}{6} = \frac{1}{3(c-a)} (c^2 - a^2) = \frac{a+c}{3} \end{aligned}$$

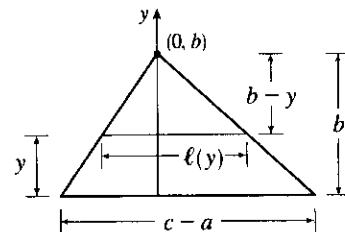
and

$$\begin{aligned} \bar{y} &= \frac{1}{A} \left[\int_a^0 \frac{1}{2} \left(\frac{b}{a} (a-x) \right)^2 dx + \int_0^c \frac{1}{2} \left(\frac{b}{c} (c-x) \right)^2 dx \right] \\ &= \frac{1}{A} \left[\frac{b^2}{2a^2} \int_a^0 (a^2 - 2ax + x^2) dx + \frac{b^2}{2c^2} \int_0^c (c^2 - 2cx + x^2) dx \right] \\ &= \frac{1}{A} \left[\frac{b^2}{2a^2} \left[a^2x - ax^2 + \frac{1}{3}x^3 \right]_a^0 + \frac{b^2}{2c^2} \left[c^2x - cx^2 + \frac{1}{3}x^3 \right]_0^c \right] \\ &= \frac{1}{A} \left[\frac{b^2}{2a^2} (-a^3 + a^3 - \frac{1}{3}a^3) + \frac{b^2}{2c^2} (c^3 - c^3 + \frac{1}{3}c^3) \right] = \frac{1}{A} \left[\frac{b^2}{6} (-a + c) \right] = \frac{2}{(c-a)b} \cdot \frac{(c-a)b^2}{6} = \frac{b}{3} \end{aligned}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{a+c}{3}, \frac{b}{3}\right)$, as claimed.

Remarks: Actually the computation of $\bar{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\bar{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y) \Delta y$, its mass is $\rho \ell(y) \Delta y$, and its moment about the x -axis is $\Delta M_x = \rho y \ell(y) \Delta y$. Thus,



$$M_x = \int \rho y \ell(y) dy \quad \text{and} \quad \bar{y} = \frac{\int \rho y \ell(y) dy}{\rho A} = \frac{1}{A} \int y \ell(y) dy$$

In this problem, $\ell(y) = \frac{c-a}{b}(b-y)$ by similar triangles, so

$$\bar{y} = \frac{1}{A} \int_0^b \frac{c-a}{b} y(b-y) dy = \frac{2}{b^2} \int_0^b (by - y^2) dy = \frac{2}{b^2} \left[\frac{1}{2} by^2 - \frac{1}{3} y^3 \right]_0^b = \frac{2}{b^2} \cdot \frac{b^3}{6} = \frac{b}{3}$$

Notice that only one integral is needed when this method is used.

38. Divide the lamina into three rectangles with masses 2, 2 and 6, with centroids $(-\frac{3}{2}, 1)$, $(0, \frac{1}{2})$ and $(2, \frac{3}{2})$, respectively. The total mass of the lamina is 10. So, using Formulas 5, 6, and 7, we have

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \sum_{i=1}^3 m_i x_i = \frac{1}{10} [2(-\frac{3}{2}) + 2(0) + 6(2)] = \frac{1}{10}(9), \text{ and}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \sum_{i=1}^3 m_i y_i = \frac{1}{10} [2(1) + 2(\frac{1}{2}) + 6(\frac{3}{2})] = \frac{1}{10}(12).$$

Thus, the centroid is $(\bar{x}, \bar{y}) = (\frac{9}{10}, \frac{6}{5})$.

39. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 37, the triangles have centroids $(-1, \frac{2}{3})$ and $(1, \frac{2}{3})$. The centroid of the rectangle (its center) is $(0, -\frac{1}{2})$. So, using Formulas 5 and 7, we have

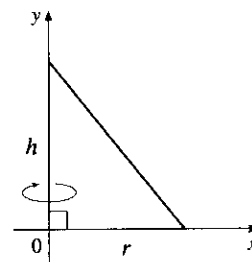
$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \sum_{i=1}^3 m_i y_i = \frac{1}{8} [2(\frac{2}{3}) + 2(\frac{2}{3}) + 4(-\frac{1}{2})] = \frac{1}{8}(\frac{2}{3}) = \frac{1}{12}, \text{ and } \bar{x} = 0, \text{ since the lamina is symmetric about the line } x = 0. \text{ Thus, the centroid is } (\bar{x}, \bar{y}) = (0, \frac{1}{12}).$$

40. A sphere can be generated by rotating a semicircle about its diameter. By Example 4, the center of mass travels a distance $2\pi\bar{y} = 2\pi(\frac{4r}{3\pi}) = \frac{8\pi}{3}$, so by the Theorem of Pappus, the volume of the sphere is

$$V = Ad = \frac{\pi r^2}{2} \cdot \frac{8r}{3} = \frac{4}{3}\pi r^3.$$

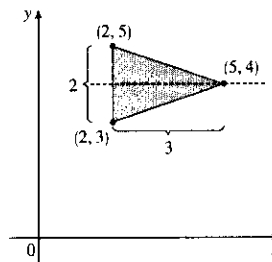
41. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 37, $\bar{x} = \frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is

$$V = Ad = \left(\frac{1}{2} \cdot \text{base} \cdot \text{height}\right) \cdot (2\pi\bar{x}) = \frac{1}{2}rh \cdot 2\pi\left(\frac{1}{3}r\right) = \frac{1}{3}\pi r^2 h.$$



42. From the symmetry in the figure, $\bar{y} = 4$. So the distance traveled by the centroid when rotating the triangle about the x -axis is $d = 2\pi \cdot 4 = 8\pi$.

The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3$. By the Theorem of Pappus, the volume of the resulting solid is $Ad = 3(8\pi) = 24\pi$.



43. Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$, as illustrated in Figure 13. Choose points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose x_i^* to be the midpoint of the i th subinterval; that is, $x_i^* = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the i th approximating rectangle R_i is its

center $C_i = (\bar{x}_i, \frac{1}{2}[f(\bar{x}_i) + g(\bar{x}_i)])$. Its area is $[f(\bar{x}_i) - g(\bar{x}_i)] \Delta x$, so its mass is

$\rho[f(\bar{x}_i) - g(\bar{x}_i)] \Delta x$. Thus, $M_y(R_i) = \rho[f(\bar{x}_i) - g(\bar{x}_i)] \Delta x \cdot \bar{x}_i = \rho\bar{x}_i [f(\bar{x}_i) - g(\bar{x}_i)] \Delta x$ and

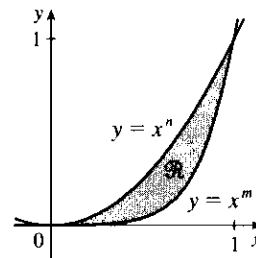
$M_x(R_i) = \rho[f(\bar{x}_i) - g(\bar{x}_i)] \Delta x \cdot \frac{1}{2}[f(\bar{x}_i) + g(\bar{x}_i)] = \rho \cdot \frac{1}{2}[f(\bar{x}_i)^2 - g(\bar{x}_i)^2] \Delta x$. Summing over i and taking

the limit as $n \rightarrow \infty$, we get $M_y = \lim_{n \rightarrow \infty} \sum_i \rho\bar{x}_i [f(\bar{x}_i) - g(\bar{x}_i)] \Delta x = \rho \int_a^b x[f(x) - g(x)] dx$ and

$M_x = \lim_{n \rightarrow \infty} \sum_i \rho \cdot \frac{1}{2}[f(\bar{x}_i)^2 - g(\bar{x}_i)^2] \Delta x = \rho \int_a^b \frac{1}{2}[f(x)^2 - g(x)^2] dx$. Thus,

$$\bar{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} = \frac{1}{A} \int_a^b x[f(x) - g(x)] dx \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{M_x}{\rho A} = \frac{1}{A} \int_a^b \frac{1}{2}[f(x)^2 - g(x)^2] dx$$

44. (a) Let $0 \leq x \leq 1$. If $n < m$, then $x^n > x^m$; that is, raising x to a larger power produces a smaller number.



(b) Using Formulas 9 and the fact that the area of $\mathcal{R}$ is

$$A = \int_0^1 (x^n - x^m) dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}, \text{ we get}$$

$$\begin{aligned} \bar{x} &= \frac{(n+1)(m+1)}{m-n} \int_0^1 x[x^n - x^m] dx = \frac{(n+1)(m+1)}{m-n} \int_0^1 (x^{n+1} - x^{m+1}) dx \\ &= \frac{(n+1)(m+1)}{m-n} \left[\frac{1}{n+2} - \frac{1}{m+2} \right] = \frac{(n+1)(m+1)}{(n+2)(m+2)} \end{aligned}$$

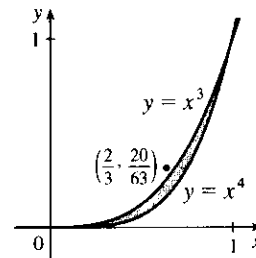
and

$$\begin{aligned}\bar{y} &= \frac{(n+1)(m+1)}{m-n} \int_0^1 \frac{1}{2} [(x^n)^2 - (x^m)^2] dx = \frac{(n+1)(m+1)}{2(m-n)} \int_0^1 (x^{2n} - x^{2m}) dx \\ &= \frac{(n+1)(m+1)}{2(m-n)} \left[\frac{1}{2n+1} - \frac{1}{2m+1} \right] = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}\end{aligned}$$

(c) If we take $n = 3$ and $m = 4$, then

$$(\bar{x}, \bar{y}) = \left(\frac{4 \cdot 5}{5 \cdot 6}, \frac{4 \cdot 5}{7 \cdot 9} \right) = \left(\frac{2}{3}, \frac{20}{63} \right)$$

which lies outside $\mathcal{R}$ since $(\frac{2}{3})^3 = \frac{8}{27} < \frac{20}{63}$. This is the simplest of many possibilities.



9.4 Applications to Economics and Biology

1. By the Net Change Theorem, $C'(2000) - C(0) = \int_0^{2000} C'(x) dx \Rightarrow$

$$\begin{aligned}C(2000) &= 20,000 + \int_0^{2000} (5 - 0.008x + 0.000009x^2) dx = 20,000 + [5x - 0.004x^2 + 0.000003x^3]_0^{2000} \\ &= 20,000 + 10,000 - 0.004(4,000,000) + 0.000003(8,000,000,000) = 30,000 - 16,000 + 24,000 \\ &= \$38,000\end{aligned}$$

2. By the Net Change Theorem, $R(5000) - R(1000) = \int_{1000}^{5000} R'(x) dx \Rightarrow$

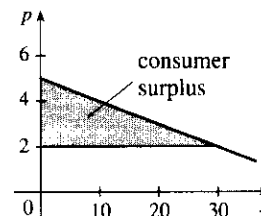
$$\begin{aligned}R(5000) &= 12,400 + \int_{1000}^{5000} (12 - 0.0004x) dx = 12,400 + [12x - 0.0002x^2]_{1000}^{5000} \\ &= 12,400 + (60,000 - 5,000) - (12,000 - 200) = \$55,600\end{aligned}$$

3. If the production level is raised from 1200 units to 1600 units, then the increase in cost is

$$\begin{aligned}C(1600) - C(1200) &= \int_{1200}^{1600} C'(x) dx = \int_{1200}^{1600} (74 + 1.1x - 0.002x^2 + 0.00004x^3) dx \\ &= [74x + 0.55x^2 - \frac{0.002}{3}x^3 + 0.00001x^4]_{1200}^{1600} \\ &= 64,331,733.33 - 20,464,800 = \$43,866,933.33\end{aligned}$$

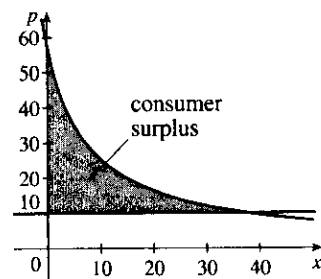
4. Consumer surplus $= \int_0^{30} [p(x) - p(30)] dx$

$$\begin{aligned}&= \int_0^{30} \left[5 - \frac{1}{10}x - \left(5 - \frac{30}{10} \right) \right] dx \\ &= \left[3x - \frac{1}{20}x^2 \right]_0^{30} = 90 - 45 = \$45\end{aligned}$$



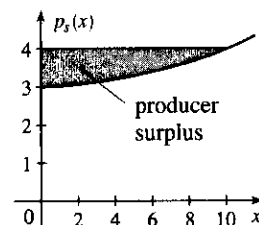
$$5. p(x) = 10 \Rightarrow \frac{450}{x+8} = 10 \Rightarrow x+8 = 45 \Rightarrow x = 37.$$

$$\begin{aligned}\text{Consumer surplus} &= \int_0^{37} [p(x) - 10] dx = \int_0^{37} \left(\frac{450}{x+8} - 10 \right) dx \\ &= [450 \ln(x+8) - 10x]_0^{37} \\ &= (450 \ln 45 - 370) - 450 \ln 8 \\ &= 450 \ln\left(\frac{45}{8}\right) - 370 \approx \$407.25\end{aligned}$$



$$6. p_S(x) = 3 + 0.01x^2. P = p_S(10) = 3 + 1 = 4.$$

$$\begin{aligned}\text{Producer surplus} &= \int_0^{10} [P - p_S(x)] dx \\ &= \int_0^{10} [4 - 3 - 0.01x^2] dx = \left[x - \frac{0.01}{3}x^3 \right]_0^{10} \\ &\approx 10 - 3.33 = \$6.67\end{aligned}$$



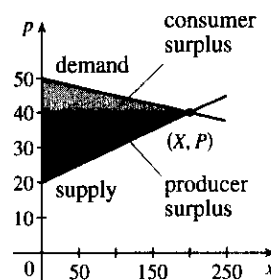
$$7. P = p_S(x) \Rightarrow 400 = 200 + 0.2x^{3/2} \Rightarrow 200 = 0.2x^{3/2} \Rightarrow 1000 = x^{3/2} \Rightarrow x = 1000^{2/3} = 100.$$

$$\begin{aligned}\text{Producer surplus} &= \int_0^{100} [P - p_S(x)] dx = \int_0^{100} \left[400 - \left(200 + 0.2x^{3/2} \right) \right] dx = \int_0^{100} \left(200 - \frac{1}{5}x^{3/2} \right) dx \\ &= \left[200x - \frac{2}{25}x^{5/2} \right]_0^{100} = 20,000 - 8,000 = \$12,000\end{aligned}$$

$$8. p = 50 - \frac{1}{20}x \text{ and } p = 20 + \frac{1}{10}x \text{ intersect at } p = 40 \text{ and } x = 200.$$

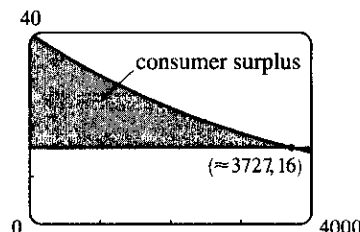
$$\begin{aligned}\text{Consumer surplus} &= \int_0^{200} \left(50 - \frac{1}{20}x - 40 \right) dx \\ &= \left[10x - \frac{1}{40}x^2 \right]_0^{200} = \$1000\end{aligned}$$

$$\begin{aligned}\text{Producer surplus} &= \int_0^{200} \left(40 - 20 - \frac{1}{10}x \right) dx \\ &= \left[20x - \frac{1}{20}x^2 \right]_0^{200} = \$2000\end{aligned}$$



$$9. p(x) = \frac{800,000e^{-x/5000}}{x + 20,000} = 16 \Rightarrow x = x_1 \approx 3727.04.$$

$$\text{Consumer surplus} = \int_0^{x_1} [p(x) - 16] dx \approx \$37,753$$



$$10. \text{The demand function is linear with slope } \frac{-0.5}{35} = -\frac{1}{70} \text{ and } p(400) = 7.5, \text{ so an equation is}$$

$$p - 7.5 = -\frac{1}{70}(x - 400) \text{ or } p = -\frac{1}{70}x + \frac{185}{14}. \text{ A selling price of \$6 implies that } 6 = -\frac{1}{70}x + \frac{185}{14} \Rightarrow$$

$$\frac{1}{70}x = \frac{185}{14} - \frac{84}{14} = \frac{101}{14} \Rightarrow x = 505.$$

$$\text{Consumer surplus} = \int_0^{505} \left(-\frac{1}{70}x + \frac{185}{14} - 6 \right) dx = \left[-\frac{1}{140}x^2 + \frac{101}{14}x \right]_0^{505} \approx \$1821.61$$

$$11. f(8) - f(4) = \int_4^8 f'(t) dt = \int_4^8 \sqrt{t} dt = \left[\frac{2}{3} t^{3/2} \right]_4^8 = \frac{2}{3} (16\sqrt{2} - 8) \approx \$9.75 \text{ million}$$

$$12. n(9) - n(5) = \int_5^9 (2200 + 10e^{0.8t}) dt = \left[2200t + \frac{10e^{0.8t}}{0.8} \right]_5^9 = [2200t]_5^9 + \frac{25}{2} [e^{0.8t}]_5^9 \\ = 2200(9 - 5) + 12.5(e^{7.2} - e^4) \approx 24,860$$

$$13. F = \frac{\pi PR^4}{8\eta l} = \frac{\pi(4000)(0.008)^4}{8(0.027)(2)} \approx 1.19 \times 10^{-4} \text{ cm}^3/\text{s}$$

$$14. \text{ If the flux remains constant, then } \frac{\pi P_0 R_0^4}{8\eta l} = \frac{\pi PR^4}{8\eta l} \Rightarrow P_0 R_0^4 = PR^4 \Rightarrow \frac{P}{P_0} = \left(\frac{R_0}{R} \right)^4.$$

$$R = \frac{3}{4} R_0 \Rightarrow \frac{P}{P_0} = \left(\frac{R_0}{\frac{3}{4} R_0} \right)^4 \Rightarrow P = P_0 \left(\frac{4}{3} \right)^4 \approx 3.1605 P_0 > 3P_0; \text{ that is, the blood pressure is more than tripled.}$$

$$15. \int_0^{12} c(t) dt = \int_0^{12} \frac{1}{4} t(12 - t) dt = \int_0^{12} \left(3t - \frac{1}{4} t^2 \right) dt = \left[\frac{3}{2} t^2 - \frac{1}{12} t^3 \right]_0^{12} = (216 - 144) = 72 \text{ mg} \cdot \text{s/L}.$$

$$\text{Thus, the cardiac output is } F = \frac{A}{\int_0^{12} c(t) dt} = \frac{8 \text{ mg}}{72 \text{ mg} \cdot \text{s/L}} = \frac{1}{9} \text{ L/s} = \frac{60}{9} \text{ L/min}.$$

16. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = 2$.

$$\int_0^{20} c(t) dt \approx \frac{2}{3} [1(0) + 4(2.4) + 2(5.1) + 4(7.8) + 2(7.6) \\ + 4(5.4) + 2(3.9) + 4(2.3) + 2(1.6) + 4(0.7) + 1(0)] \\ = \frac{2}{3} (110.8) \approx 73.87 \text{ mg} \cdot \text{s/L}$$

$$\text{Therefore, } F \approx \frac{A}{73.87} = \frac{8}{73.87} \approx 0.1083 \text{ L/s or } 6.498 \text{ L/min}.$$

9.5 Probability

1. (a) $\int_{30,000}^{40,000} f(x) dx$ is the probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles.

(b) $\int_{25,000}^{\infty} f(x) dx$ is the probability that a randomly chosen tire will have a lifetime of at least 25,000 miles.

2. (a) The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t) dt$.

(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t) dt$.

3. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x , and (2) $\int_{-\infty}^{\infty} f(x) dx = 1$. For $0 \leq x \leq 4$, we have $f(x) = \frac{3}{64} x \sqrt{16 - x^2} \geq 0$, so $f(x) \geq 0$ for all x .

$$\text{Also, } \int_{-\infty}^{\infty} f(x) dx = \int_0^4 \frac{3}{64} x \sqrt{16 - x^2} dx = -\frac{3}{128} \int_0^4 (16 - x^2)^{1/2} (-2x) dx = -\frac{3}{128} \left[\frac{2}{3} (16 - x^2)^{3/2} \right]_0^4 \\ = -\frac{1}{64} \left[(16 - x^2)^{3/2} \right]_0^4 = -\frac{1}{64} (0 - 64) = 1.$$

Therefore, f is a probability density function.

$$\begin{aligned}
 (b) \ P(X < 2) &= \int_{-\infty}^2 f(x) \, dx = \int_0^2 \frac{3}{64} x \sqrt{16 - x^2} \, dx = -\frac{3}{128} \int_0^2 (16 - x^2)^{1/2} (-2x) \, dx \\
 &= -\frac{3}{128} \left[\frac{2}{3} (16 - x^2)^{3/2} \right]_0^2 = -\frac{1}{64} \left[(16 - x^2)^{3/2} \right]_0^2 = -\frac{1}{64} (12^{3/2} - 16^{3/2}) \\
 &= \frac{1}{64} (64 - 12\sqrt{12}) = \frac{1}{64} (64 - 24\sqrt{3}) = 1 - \frac{3}{8}\sqrt{3} \approx 0.350481
 \end{aligned}$$

4. (a) For $0 \leq x \leq 1$, we have $f(x) = kx^2(1 - x)$, which is nonnegative if and only if $k \geq 0$. Also,
 $\int_{-\infty}^{\infty} f(x) \, dx = \int_0^1 kx^2(1 - x) \, dx = k \int_0^1 (x^2 - x^3) \, dx = k \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = k/12$. Now $k/12 = 1 \Leftrightarrow k = 12$. Therefore, f is a probability density function if and only if $k = 12$.

(b) Let $k = 12$.

$$\begin{aligned}
 P(X \geq \tfrac{1}{2}) &= \int_{1/2}^{\infty} f(x) \, dx = \int_{1/2}^1 12x^2(1 - x) \, dx = \int_{1/2}^1 (12x^2 - 12x^3) \, dx = [4x^3 - 3x^4]_{1/2}^1 \\
 &= (4 - 3) - (\tfrac{1}{2} - \tfrac{3}{16}) = 1 - \tfrac{5}{16} = \tfrac{11}{16}
 \end{aligned}$$

(c) The mean

$$\begin{aligned}
 \mu &= \int_{-\infty}^{\infty} xf(x) \, dx = \int_0^1 x \cdot 12x^2(1 - x) \, dx = 12 \int_0^1 (x^3 - x^4) \, dx = 12 \left[\tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 \right]_0^1 \\
 &= 12 \left(\tfrac{1}{4} - \tfrac{1}{5} \right) = \tfrac{12}{20} = \tfrac{3}{5}
 \end{aligned}$$

5. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x , and (2) $\int_{-\infty}^{\infty} f(x) \, dx = 1$. Since $f(x) = 0$ or $f(x) = 0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty} f(x) \, dx = \int_0^{10} 0.1 \, dx = \left[\tfrac{1}{10}x \right]_0^{10} = 1$. Thus, $f(x)$ is a probability density function for the spinner's values.

(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x = 5$.

$$\mu = \int_{-\infty}^{\infty} xf(x) \, dx = \int_0^{10} x(0.1) \, dx = \left[\tfrac{1}{20}x^2 \right]_0^{10} = \tfrac{100}{20} = 5, \text{ as expected.}$$

6. (a) As in the preceding exercise, (1) $f(x) \geq 0$ and

(2) $\int_{-\infty}^{\infty} f(x) \, dx = \int_0^{10} f(x) \, dx = \tfrac{1}{2}(10)(0.2)$ [area of a triangle] $= 1$. So $f(x)$ is a probability density function.

(b) (i) $P(X < 3) = \int_0^3 f(x) \, dx = \tfrac{1}{2}(3)(0.1) = \tfrac{3}{20} = 0.15$

(ii) We first compute $P(X > 8)$ and then subtract that value and our answer in (i) from 1 (the total probability).

$$P(X > 8) = \int_8^{10} f(x) \, dx = \tfrac{1}{2}(2)(0.1) = \tfrac{2}{20} = 0.10. \text{ So } P(3 \leq X \leq 8) = 1 - 0.15 - 0.10 = 0.75.$$

(c) We find equations of the lines from $(0, 0)$ to $(6, 0.2)$ and from $(6, 0.2)$ to $(10, 0)$, and find that

$$f(x) = \begin{cases} \frac{1}{30}x & \text{if } 0 \leq x < 6 \\ -\frac{1}{20}x + \frac{1}{2} & \text{if } 6 \leq x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 \mu &= \int_{-\infty}^{\infty} xf(x) \, dx = \int_0^6 x \left(\tfrac{1}{30}x \right) \, dx + \int_6^{10} x \left(-\tfrac{1}{20}x + \tfrac{1}{2} \right) \, dx = \left[\tfrac{1}{90}x^3 \right]_0^6 + \left[-\tfrac{1}{60}x^3 + \tfrac{1}{4}x^2 \right]_6^{10} \\
 &= \tfrac{216}{90} + \left(-\tfrac{1000}{60} + \tfrac{100}{4} \right) - \left(-\tfrac{216}{60} + \tfrac{36}{4} \right) = \tfrac{16}{3} = 5.\bar{3}
 \end{aligned}$$

7. We need to find m so that $\int_m^{\infty} f(t) \, dt = \tfrac{1}{2} \Rightarrow \lim_{x \rightarrow \infty} \int_m^x \tfrac{1}{5} e^{-t/5} \, dt = \tfrac{1}{2} \Rightarrow \lim_{x \rightarrow \infty} \left[\tfrac{1}{5}(-5)e^{-t/5} \right]_m^x = \tfrac{1}{2} \Rightarrow$
 $(-1)(0 - e^{-m/5}) = \tfrac{1}{2} \Rightarrow e^{-m/5} = \tfrac{1}{2} \Rightarrow -m/5 = \ln \tfrac{1}{2} \Rightarrow m = -5 \ln \tfrac{1}{2} = 5 \ln 2 \approx 3.47 \text{ min.}$

$$8. (a) \mu = 1000 \Rightarrow f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

$$(i) P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000} dt = \left[-e^{-t/1000}\right]_0^{200} = -e^{-1/5} + 1 \approx 0.181$$

$$(ii) P(X > 800) = \int_{800}^{\infty} \frac{1}{1000}e^{-t/1000} dt = \lim_{x \rightarrow \infty} \left[-e^{-t/1000}\right]_{800}^x = 0 + e^{-4/5} \approx 0.449$$

$$(b) \text{ We need to find } m \text{ so that } \int_m^{\infty} f(t) dt = \frac{1}{2} \Rightarrow \lim_{x \rightarrow \infty} \int_m^x \frac{1}{1000}e^{-t/1000} dt = \frac{1}{2} \Rightarrow$$

$$\lim_{x \rightarrow \infty} \left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \Rightarrow 0 + e^{-m/1000} = \frac{1}{2} \Rightarrow -m/1000 = \ln \frac{1}{2} \Rightarrow$$

$$m = -1000 \ln \frac{1}{2} = 1000 \ln 2 \approx 693.1 \text{ h.}$$

9. We use an exponential density function with $\mu = 2.5$ min.

$$(a) P(X > 4) = \int_4^{\infty} f(t) dt = \lim_{x \rightarrow \infty} \int_4^x \frac{1}{2.5}e^{-t/2.5} dt = \lim_{x \rightarrow \infty} \left[-e^{-t/2.5}\right]_4^x = 0 + e^{-4/2.5} \approx 0.202$$

$$(b) P(0 \leq X \leq 2) = \int_0^2 f(t) dt = \left[-e^{-t/2.5}\right]_0^2 = -e^{-2/2.5} + 1 \approx 0.551$$

(c) We need to find a value a so that $P(X \geq a) = 0.02$, or, equivalently, $P(0 \leq X \leq a) = 0.98 \Leftrightarrow$

$$\int_0^a f(t) dt = 0.98 \Leftrightarrow \left[-e^{-t/2.5}\right]_0^a = 0.98 \Leftrightarrow -e^{-a/2.5} + 1 = 0.98 \Leftrightarrow e^{-a/2.5} = 0.02 \Leftrightarrow$$

$$-a/2.5 = \ln 0.02 \Leftrightarrow a = -2.5 \ln \frac{1}{50} = 2.5 \ln 50 \approx 9.78 \text{ min} \approx 10 \text{ min. The ad should say that if you aren't served within 10 minutes, you get a free hamburger.}$$

$$10. (a) \text{ With } \mu = 69 \text{ and } \sigma = 2.8, \text{ we have } P(65 \leq X \leq 73) = \int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}} \exp\left(-\frac{(x-69)^2}{2 \cdot 2.8^2}\right) dx \approx 0.847 \text{ (using a calculator or computer to estimate the integral).}$$

$$(b) P(X > 6 \text{ feet}) = P(X > 72 \text{ inches}) = 1 - P(0 \leq X \leq 72) \approx 1 - 0.858 = 0.142, \text{ so } 14.2\% \text{ of the adult male population is more than 6 feet tall.}$$

$$11. P(X \geq 10) = \int_{10}^{\infty} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2 \cdot 4.2^2}\right) dx. \text{ To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, } P(X \geq 10) \approx \int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2 \cdot 4.2^2}\right) dx \approx 0.443 \text{ (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week.}$$

Note: We can't evaluate $1 - P(0 \leq X \leq 10)$ for this problem since a significant amount of area lies to the left of $X = 0$.

$$12. (a) P(0 \leq X \leq 480) = \int_0^{480} \frac{1}{12\sqrt{2\pi}} \exp\left(-\frac{(x-500)^2}{2 \cdot 12^2}\right) dx \approx 0.0478 \text{ (using a calculator or computer to estimate the integral), so there is about a 4.78\% chance that a particular box contains less than 480 g of cereal.}$$

(b) We need to find μ so that $P(0 \leq X < 500) = 0.05$. Using our calculator or computer to find $P(0 \leq X \leq 500)$ for various values of μ , we find that if $\mu = 519.73$, $P = 0.05007$; and if $\mu = 519.74$, $P = 0.04998$. So a good target weight is at least 519.74 g.

$$13. P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \text{ Substituting } t = \frac{x-\mu}{\sigma} \text{ and } dt = \frac{1}{\sigma} dx \text{ gives us}$$

$$\int_{-2}^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2} (\sigma dt) = \frac{1}{\sqrt{2\pi}} \int_{-2}^2 e^{-t^2/2} dt \approx 0.9545$$

14. Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0 \end{cases}$ where $c = 1/\mu$. By using parts, tables, or a CAS, we find that

$$(1): \int x e^{bx} dx = (e^{bx}/b^2)(bx - 1)$$

$$(2): \int x^2 e^{bx} dx = (e^{bx}/b^3)(b^2 x^2 - 2bx + 2)$$

Now

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^0 (x - \mu)^2 f(x) dx + \int_0^{\infty} (x - \mu)^2 f(x) dx \\ &= 0 + \lim_{t \rightarrow \infty} c \int_0^t (x - \mu)^2 e^{-cx} dx = c \cdot \lim_{t \rightarrow \infty} \int_0^t (x^2 e^{-cx} - 2x\mu e^{-cx} + \mu^2 e^{-cx}) dx \end{aligned}$$

Next we use (2) and (1) with $b = -c$ to get

$$\sigma^2 = c \lim_{t \rightarrow \infty} \left[-\frac{e^{-cx}}{c^3} (c^2 x^2 + 2cx + 2) - 2\mu \frac{e^{-cx}}{c^2} (-cx - 1) + \mu^2 \frac{e^{-cx}}{-c} \right]_0^t$$

Using l'Hospital's Rule several times, along with the fact that $\mu = 1/c$, we get

$$\sigma^2 = c \left[0 - \left(-\frac{2}{c^3} + \frac{2}{c} \cdot \frac{1}{c^2} + \frac{1}{c^2} \cdot \frac{1}{-c} \right) \right] = c \left(\frac{1}{c^3} \right) = \frac{1}{c^2} \Rightarrow \sigma = \frac{1}{c} = \mu$$

15. (a) First $p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0$ for $r \geq 0$. Next,

$$\int_{-\infty}^{\infty} p(r) dr = \int_0^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0} dr = \frac{4}{a_0^3} \lim_{t \rightarrow \infty} \int_0^t r^2 e^{-2r/a_0} dr$$

By using parts, tables, or a CAS [or as in Exercise 14], we find that

$$\int x^2 e^{bx} dx = (e^{bx}/b^3)(b^2 x^2 - 2bx + 2). (*)$$

Next, we use (*) (with $b = -2/a_0$) and l'Hospital's Rule to get $\frac{4}{a_0^3} \left[\frac{a_0^3}{-8} (-2) \right] = 1$. This satisfies the second condition for a function to be a probability density function.

- (b) Using l'Hospital's Rule, $\frac{4}{a_0^3} \lim_{r \rightarrow \infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r \rightarrow \infty} \frac{2r}{(2/a_0)e^{2r/a_0}} = \frac{2}{a_0^2} \lim_{r \rightarrow \infty} \frac{2}{(2/a_0)e^{2r/a_0}} = 0$.

To find the maximum of p , we differentiate:

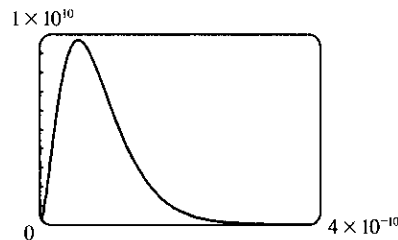
$$p'(r) = \frac{4}{a_0^3} \left[r^2 e^{-2r/a_0} \left(-\frac{2}{a_0} \right) + e^{-2r/a_0} (2r) \right] = \frac{4}{a_0^3} e^{-2r/a_0} (2r) \left(-\frac{r}{a_0} + 1 \right)$$

$p'(r) = 0 \Leftrightarrow r = 0$ or $1 = \frac{r}{a_0} \Leftrightarrow r = a_0$ [$a_0 \approx 5.59 \times 10^{-11}$ m]. $p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

- (c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p(a_0) = \frac{4}{a_0^3} a_0^2 e^{-2a_0/a_0} = \frac{4}{a_0} e^{-2} \approx 9,684,098,979$$

With a maximum of nearly 10 billion and a total area under the curve of 1, we know that the "hump" in the graph must be extremely narrow.



(d) $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds \Rightarrow P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds$. Using (*) from part (a) (with $b = -2/a_0$),

$$P(4a_0) = \frac{4}{a_0^3} \left[\frac{e^{-2s/a_0}}{-8/a_0^3} \left(\frac{4}{a_0^2} s^2 + \frac{4}{a_0} s + 2 \right) \right]_0^{4a_0} = \frac{4}{a_0^3} \left(\frac{a_0^3}{-8} \right) [e^{-8}(64 + 16 + 2) - 1(2)]$$

$$= -\frac{1}{2} (82e^{-8} - 2) = 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \int_{-\infty}^{\infty} r p(r) dr = \frac{4}{a_0^3} \lim_{t \rightarrow \infty} \int_0^t r^3 e^{-2r/a_0} dr$. Integrating by parts three times or using a CAS, we find that

$$\int x^3 e^{bx} dx = \frac{e^{bx}}{b^4} (b^3 x^3 - 3b^2 x^2 + 6bx - 6). \text{ So with } b = -\frac{2}{a_0}, \text{ we use l'Hospital's Rule, and get}$$

$$\mu = \frac{4}{a_0^3} \left[-\frac{a_0^4}{16} (-6) \right] = \frac{3}{2} a_0.$$

9 Review

CONCEPT CHECK

- (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 9.1.

(b) See Equation 9.1.2.

(c) See Equation 9.1.4.
- (a) $S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$

(b) If $x = g(y)$, $c \leq y \leq d$, then $S = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2} dy$.

(c) $S = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2} dx$ or $S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy$
- Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth x . Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x) dx$, where a and b are the lower and upper limits for x at points of the wall and δ is the weight density of the fluid.
- (a) The center of mass is the point at which the plate balances horizontally.

(b) See Equations 9.3.8.
- If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ , then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.
- See Figure 3 in Section 9.4, and the discussion which precedes it.
- (a) See the definition in the first paragraph of the subsection *Cardiac Output* in Section 9.4.

(b) See the discussion in the second paragraph of the subsection *Cardiac Output* in Section 9.4.
- A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x) dx$ measures the probability that X lies between a and b . Such a function f has nonnegative values and satisfies the relation $\int_D f(x) dx = 1$, where D is the domain of the corresponding random variable X . If $D = \mathbb{R}$, or if we define $f(x) = 0$ for real numbers $x \notin D$, then $\int_{-\infty}^{\infty} f(x) dx = 1$. (Of course, to work with f in this way, we must assume that the integrals of f exist.)

9. (a) $\int_0^{100} f(x) dx$ represents the probability that the weight of a randomly chosen female college student is less than 100 pounds.
- (b) $\mu = \int_{-\infty}^{\infty} xf(x) dx = \int_0^{\infty} xf(x) dx$
- (c) The median of f is the number m such that $\int_m^{\infty} f(x) dx = \frac{1}{2}$.
10. See the discussion near Equation 3 in Section 9.5.

EXERCISES

1. $y = \frac{1}{6}(x^2 + 4)^{3/2} \Rightarrow dy/dx = \frac{1}{4}(x^2 + 4)^{1/2}(2x) \Rightarrow$
 $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left[\frac{1}{2}x(x^2 + 4)^{1/2}\right]^2 = 1 + \frac{1}{4}x^2(x^2 + 4) = \frac{1}{4}x^4 + x^2 + 1 = \left(\frac{1}{2}x^2 + 1\right)^2.$
 Thus, $L = \int_0^3 \sqrt{\left(\frac{1}{2}x^2 + 1\right)^2} dx = \int_0^3 \left(\frac{1}{2}x^2 + 1\right) dx = \left[\frac{1}{6}x^3 + x\right]_0^3 = \frac{15}{2}.$
2. $y = 2 \ln(\sin \frac{1}{2}x) \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{\sin(\frac{1}{2}x)} \cdot \cos(\frac{1}{2}x) \cdot \frac{1}{2} = \cot(\frac{1}{2}x) \Rightarrow$
 $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cot^2(\frac{1}{2}x) = \csc^2(\frac{1}{2}x).$ Thus,
 $L = \int_{\pi/3}^{\pi} \sqrt{\csc^2(\frac{1}{2}x)} dx = \int_{\pi/3}^{\pi} |\csc(\frac{1}{2}x)| dx = \int_{\pi/3}^{\pi} \csc(\frac{1}{2}x) dx = \int_{\pi/6}^{\pi/2} \csc u (2 du) \quad \left[\begin{array}{l} u = \frac{1}{2}x, \\ du = \frac{1}{2} dx \end{array} \right]$
 $= 2[\ln |\csc u - \cot u|]_{\pi/6}^{\pi/2} = 2[\ln |\csc \frac{\pi}{2} - \cot \frac{\pi}{2}| - \ln |\csc \frac{\pi}{6} - \cot \frac{\pi}{6}|]$
 $= 2[\ln |1 - 0| - \ln |2 - \sqrt{3}|] = -2 \ln(2 - \sqrt{3}) \approx 2.63$
3. (a) $y = \frac{x^4}{16} + \frac{1}{2x^2} = \frac{1}{16}x^4 + \frac{1}{2}x^{-2} \Rightarrow \frac{dy}{dx} = \frac{1}{4}x^3 - x^{-3} \Rightarrow$
 $1 + (dy/dx)^2 = 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \frac{1}{16}x^6 - \frac{1}{2} + x^{-6} = \frac{1}{16}x^6 + \frac{1}{2} + x^{-6} = \left(\frac{1}{4}x^3 + x^{-3}\right)^2.$
 Thus, $L = \int_1^2 \left(\frac{1}{4}x^3 + x^{-3}\right) dx = \left[\frac{1}{16}x^4 - \frac{1}{2}x^{-2}\right]_1^2 = \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right) = \frac{21}{16}.$
- (b) $S = \int_1^2 2\pi x \left(\frac{1}{4}x^3 + x^{-3}\right) dx = 2\pi \int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right) dx = 2\pi \left[\frac{1}{20}x^5 - \frac{1}{x}\right]_1^2$
 $= 2\pi \left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi \left(\frac{8}{5} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi \left(\frac{41}{20}\right) = \frac{41}{10}\pi$
4. (a) $y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2 \Rightarrow$
 $S = \int_0^1 2\pi x \sqrt{1 + 4x^2} dx = \int_1^5 \frac{\pi}{4} \sqrt{u} du \quad [u = 1 + 4x^2] = \frac{\pi}{6} \left[u^{3/2}\right]_1^5 = \frac{\pi}{6} (5^{3/2} - 1)$
- (b) $y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2.$ So
 $S = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} dx = 2\pi \int_0^2 \frac{1}{4} u^2 \sqrt{1 + u^2} \frac{1}{2} du \quad [u = 2x] = \frac{\pi}{4} \int_0^2 u^2 \sqrt{1 + u^2} du$
 $= \frac{\pi}{4} \left[\frac{1}{8} u(1 + 2u^2) \sqrt{1 + u^2} - \frac{1}{8} \ln |u + \sqrt{1 + u^2}| \right]_0^2 \quad [u = \tan \theta \text{ or use Formula 22}]$
 $= \frac{\pi}{4} \left[\frac{1}{4} (9) \sqrt{5} - \frac{1}{8} \ln(2 + \sqrt{5}) - 0 \right] = \frac{\pi}{32} [18\sqrt{5} - \ln(2 + \sqrt{5})]$

$$5. y = e^{-x^2} \Rightarrow dy/dx = -2xe^{-x^2} \Rightarrow 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2}.$$

Let $f(x) = \sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$L = \int_0^3 f(x) dx \approx S_6 = \frac{(3-0)/6}{3} [f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)] \\ \approx 3.292287$$

$$6. S = \int_0^3 2\pi y ds = \int_0^3 2\pi e^{-x^2} \sqrt{1 + 4x^2e^{-2x^2}} dx. \text{ Let } g(x) = 2\pi e^{-x^2} \sqrt{1 + 4x^2e^{-2x^2}}. \text{ Then}$$

$$S = \int_0^3 g(x) dx \approx S_6 = \frac{(3-0)/6}{3} [g(0) + 4g(0.5) + 2g(1) + 4g(1.5) + 2g(2) + 4g(2.5) + g(3)] \\ \approx 6.648327$$

$$7. y = \int_1^x \sqrt{\sqrt{t}-1} dt \Rightarrow dy/dx = \sqrt{\sqrt{x}-1} \Rightarrow 1 + (dy/dx)^2 = 1 + (\sqrt{x}-1) = \sqrt{x}.$$

$$\text{Thus, } L = \int_1^{16} \sqrt{\sqrt{x}} dx = \int_1^{16} x^{1/4} dx = \frac{4}{5} \left[x^{5/4} \right]_1^{16} = \frac{4}{5} (32 - 1) = \frac{124}{5}.$$

$$8. S = \int_1^{16} 2\pi x ds = 2\pi \int_1^{16} x \cdot x^{1/4} dx = 2\pi \int_1^{16} x^{5/4} dx = 2\pi \cdot \frac{4}{9} \left[x^{9/4} \right]_1^{16} = \frac{8\pi}{9} (512 - 1) = \frac{4088\pi}{9}$$

$$9. \text{ As in Example 1 of Section 9.3, } \frac{a}{2-x} = \frac{1}{2} \Rightarrow 2a = 2-x \text{ and}$$

$$w = 2(1.5 + a) = 3 + 2a = 3 + 2 - x = 5 - x. \text{ Thus,}$$

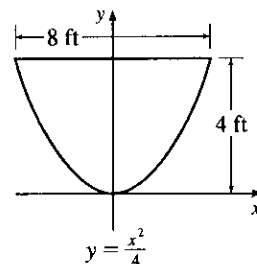
$$F = \int_0^2 \rho g x(5-x) dx = \rho g \left[\frac{5}{2}x^2 - \frac{1}{3}x^3 \right]_0^2 = \rho g \left(10 - \frac{8}{3} \right) = \frac{22}{3} \delta [\rho g = \delta] \approx \frac{22}{3} \cdot 62.5 \approx 458 \text{ lb.}$$

$$10. F = \int_0^4 \delta(4-y)2(2\sqrt{y}) dy = 4\delta \int_0^4 (4y^{1/2} - y^{3/2}) dy$$

$$= 4\delta \left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2} \right]_0^4 = 4\delta \left(\frac{64}{3} - \frac{64}{5} \right)$$

$$= 256\delta \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{512}{15} \delta$$

$$\approx 2133.3 \text{ lb } [\delta \approx 62.5 \text{ lb/ft}^3]$$



$$11. A = \int_{-2}^1 [(4-x^2) - (x+2)] dx = \int_{-2}^1 (2-x-x^2) dx = \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-2}^1 \\ = \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - 2 + \frac{8}{3} \right) = \frac{9}{2} \Rightarrow$$

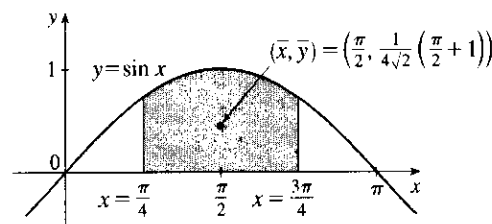
$$\bar{x} = A^{-1} \int_{-2}^1 x(2-x-x^2) dx = \frac{2}{9} \int_{-2}^1 (2x - x^2 - x^3) dx = \frac{2}{9} \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_{-2}^1 \\ = \frac{2}{9} \left[\left(1 - \frac{1}{3} - \frac{1}{4} \right) - \left(4 + \frac{8}{3} - 4 \right) \right] = -\frac{1}{2}$$

$$\text{and } \bar{y} = A^{-1} \int_{-2}^1 \frac{1}{2} [(4-x^2)^2 - (x+2)^2] dx = \frac{1}{9} \int_{-2}^1 (x^4 - 9x^2 - 4x + 12) dx \\ = \frac{1}{9} \left[\frac{1}{5}x^5 - 3x^3 - 2x^2 + 12x \right]_{-2}^1 = \frac{1}{9} \left[\left(\frac{1}{5} - 3 - 2 + 12 \right) - \left(-\frac{32}{5} + 24 - 8 - 24 \right) \right] = \frac{12}{5}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(-\frac{1}{2}, \frac{12}{5} \right)$.

12. From the symmetry of the region, $\bar{x} = \frac{\pi}{2}$. $A = \int_{\pi/4}^{3\pi/4} \sin x \, dx = [-\cos x]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$$\begin{aligned}\bar{y} &= \frac{1}{A} \int_{\pi/4}^{3\pi/4} \frac{1}{2} \sin^2 x \, dx = \frac{1}{A} \int_{\pi/4}^{3\pi/4} \frac{1}{4} (1 - \cos 2x) \, dx \\ &= \frac{1}{4\sqrt{2}} \left[x - \frac{1}{2} \sin 2x \right]_{\pi/4}^{3\pi/4} \\ &= \frac{1}{4\sqrt{2}} \left[\frac{3\pi}{4} - \frac{1}{2}(-1) - \frac{\pi}{4} + \frac{1}{2} \cdot 1 \right] \\ &= \frac{1}{4\sqrt{2}} \left(\frac{\pi}{2} + 1 \right)\end{aligned}$$



Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{1}{4\sqrt{2}} \left(\frac{\pi}{2} + 1 \right) \right) \approx (1.57, 0.45)$.

13. An equation of the line passing through $(0, 0)$ and $(3, 2)$ is $y = \frac{2}{3}x$. $A = \frac{1}{2} \cdot 3 \cdot 2 = 3$. Therefore, using

$$\text{Equations 9.3.8, } \bar{x} = \frac{1}{3} \int_0^3 x \left(\frac{2}{3}x \right) dx = \frac{2}{27} [x^3]_0^3 = 2, \text{ and } \bar{y} = \frac{1}{3} \int_0^3 \frac{1}{2} \left(\frac{2}{3}x \right)^2 dx = \frac{2}{81} [x^3]_0^3 = \frac{2}{3}.$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(2, \frac{2}{3} \right)$.

14. Suppose first that the large rectangle were complete, so that its mass would be $6 \cdot 3 = 18$. Its centroid would be $\left(1, \frac{3}{2} \right)$. The mass removed from this object to create the one being studied is 3. The centroid of the cut-out piece is $\left(\frac{3}{2}, \frac{3}{2} \right)$. Therefore, for the actual lamina, whose mass is 15, $\bar{x} = \frac{18}{15} (1) - \frac{3}{15} \left(\frac{3}{2} \right) = \frac{9}{10}$, and $\bar{y} = \frac{3}{2}$, since the lamina is symmetric about the line $y = \frac{3}{2}$. Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{9}{10}, \frac{3}{2} \right)$.

15. The centroid of this circle, $(1, 0)$, travels a distance $2\pi(1)$ when the lamina is rotated about the y -axis. The area of the circle is $\pi(1)^2$. So by the Theorem of Pappus, $V = A(2\pi\bar{x}) = \pi(1)^2 2\pi(1) = 2\pi^2$.

16. The semicircular region has an area of $\frac{1}{2}\pi r^2$, and sweeps out a sphere of radius r when rotated about the x -axis.

$$\bar{x} = 0 \text{ because of symmetry about the line } x = 0. \text{ And by the Theorem of Pappus, } V = A(2\pi\bar{y}) \Rightarrow$$

$$\frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2(2\pi\bar{y}) \Rightarrow \bar{y} = \frac{4}{3\pi}r. \text{ Thus, the centroid is } (\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}r \right).$$

17. $x = 100 \Rightarrow P = 2000 - 0.1(100) - 0.01(100)^2 = 1890$

$$\begin{aligned}\text{Consumer surplus} &= \int_0^{100} [p(x) - P] \, dx = \int_0^{100} (2000 - 0.1x - 0.01x^2 - 1890) \, dx \\ &= [110x - 0.05x^2 - \frac{0.01}{3}x^3]_0^{100} = 11,000 - 500 - \frac{10,000}{3} \approx \$7166.67\end{aligned}$$

18. $\int_0^{24} c(t) \, dt \approx S_{12} = \frac{24-0}{12 \cdot 3} [1(0) + 4(1.9) + 2(3.3) + 4(5.1) + 2(7.6) + 4(7.1) + 2(5.8) + 4(4.7) + 2(3.3) + 4(2.1) + 2(1.1) + 4(0.5) + 1(0)]$
 $= \frac{2}{3}(127.8) = 85.2 \text{ mg} \cdot \text{s/L}$

Therefore, $F \approx A/85.2 \approx 6/85.2 \approx 0.0704 \text{ L/s}$ or 4.225 L/min .

$$19. f(x) = \begin{cases} \frac{\pi}{20} \sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \leq x \leq 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

(a) $f(x) \geq 0$ for all real numbers x and

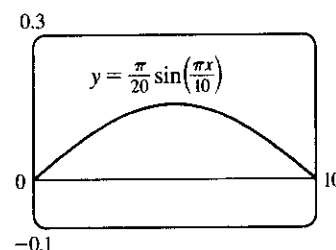
$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_0^{10} \frac{\pi}{20} \sin\left(\frac{\pi}{10}x\right) dx = \frac{\pi}{20} \cdot \frac{10}{\pi} \left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^{10} \\ &= \frac{1}{2}(-\cos \pi + \cos 0) = \frac{1}{2}(1 + 1) = 1 \end{aligned}$$

Therefore, f is a probability density function.

$$\begin{aligned} (b) P(X < 4) &= \int_{-\infty}^4 f(x) dx = \int_0^4 \frac{\pi}{20} \sin\left(\frac{\pi}{10}x\right) dx = \frac{1}{2} \left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^4 = \frac{1}{2}(-\cos \frac{2\pi}{5} + \cos 0) \\ &\approx \frac{1}{2}(-0.309017 + 1) \approx 0.3455 \end{aligned}$$

$$\begin{aligned} (c) \mu &= \int_{-\infty}^{\infty} x f(x) dx = \int_0^{10} \frac{\pi}{20} x \sin\left(\frac{\pi}{10}x\right) dx \\ &= \int_0^{\pi} \frac{\pi}{20} \cdot \frac{10}{\pi} u (\sin u) \left(\frac{10}{\pi}\right) du \quad [u = \frac{\pi}{10}x, du = \frac{\pi}{10} dx] \\ &= \frac{5}{\pi} \int_0^{\pi} u \sin u du \stackrel{82}{=} \frac{5}{\pi} [\sin u - u \cos u]_0^{\pi} = \frac{5}{\pi} [0 - \pi(-1)] = 5 \end{aligned}$$

This answer is expected because the graph of f is symmetric about the line $x = 5$.



20. $P(250 \leq X \leq 280) = \int_{250}^{280} \frac{1}{15\sqrt{2\pi}} \exp\left(\frac{-(x-268)^2}{2 \cdot 15^2}\right) dx \approx 0.673$. Thus, the percentage of pregnancies that last between 250 and 280 days is about 67.3%.

21. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \geq 0 \end{cases}$

$$P(0 \leq X \leq 3) = \int_0^3 \frac{1}{8}e^{-t/8} dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$$

$$(b) P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8} dt = \lim_{x \rightarrow \infty} \left[-e^{-t/8}\right]_{10}^x = \lim_{x \rightarrow \infty} (-e^{-x/8} + e^{-10/8}) = 0 + e^{-5/4} \approx 0.2865$$

$$\begin{aligned} (c) \text{ We need to find } m \text{ such that } P(X \geq m) &= \frac{1}{2} \Rightarrow \int_m^{\infty} \frac{1}{8}e^{-t/8} dt = \frac{1}{2} \Rightarrow \lim_{x \rightarrow \infty} \left[-e^{-t/8}\right]_m^x = \frac{1}{2} \Rightarrow \\ \lim_{x \rightarrow \infty} (-e^{-x/8} + e^{-m/8}) &= \frac{1}{2} \Rightarrow e^{-m/8} = \frac{1}{2} \Rightarrow -m/8 = \ln \frac{1}{2} \Rightarrow \\ m &= -8 \ln \frac{1}{2} = 8 \ln 2 \approx 5.55 \text{ minutes.} \end{aligned}$$

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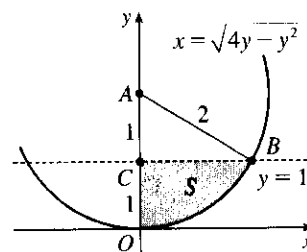
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□ PROBLEMS PLUS

1. $x^2 + y^2 \leq 4y \Leftrightarrow x^2 + (y - 2)^2 \leq 4$, so S is part of a circle, as shown in the diagram. The area of S is

$$\begin{aligned} \int_0^1 \sqrt{4y - y^2} dy &\stackrel{113}{=} \left[\frac{y-2}{2} \sqrt{4y - y^2} + 2 \cos^{-1} \left(\frac{2-y}{2} \right) \right]_0^1 \quad [a = 2] \\ &= -\frac{1}{2} \sqrt{3} + 2 \cos^{-1} \left(\frac{1}{2} \right) - 2 \cos^{-1} 1 \\ &= -\frac{\sqrt{3}}{2} + 2 \left(\frac{\pi}{3} \right) - 2(0) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \end{aligned}$$



Another method (without calculus): Note that $\theta = \angle CAB = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB) - (\text{area of } \triangle ABC) = \frac{1}{2} (2^2) \frac{\pi}{3} - \frac{1}{2} (1) \sqrt{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

2. $y = \pm \sqrt{x^3 - x^4} \Rightarrow$ The loop of the curve is symmetric about $y = 0$, and therefore $\bar{y} = 0$. At each point x where $0 \leq x \leq 1$, the lamina has a vertical length of $\sqrt{x^3 - x^4} - (-\sqrt{x^3 - x^4}) = 2\sqrt{x^3 - x^4}$. Therefore,

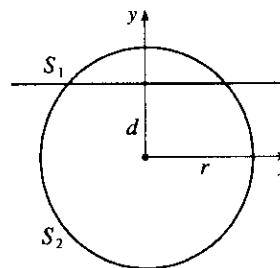
$$\bar{x} = \frac{\int_0^1 x \cdot 2\sqrt{x^3 - x^4} dx}{\int_0^1 2\sqrt{x^3 - x^4} dx} = \frac{\int_0^1 x \sqrt{x^3 - x^4} dx}{\int_0^1 \sqrt{x^3 - x^4} dx}. \text{ We evaluate the integrals separately:}$$

$$\begin{aligned} \int_0^1 x \sqrt{x^3 - x^4} dx &= \int_0^1 x^{5/2} \sqrt{1 - x} dx \\ &= \int_0^{\pi/2} 2 \sin^6 \theta \cos \theta \sqrt{1 - \sin^2 \theta} d\theta \quad \left[\begin{array}{l} \sin \theta = \sqrt{x}, \cos \theta d\theta = dx/(2\sqrt{x}), \\ 2 \sin \theta \cos \theta d\theta = dx \end{array} \right] \\ &= \int_0^{\pi/2} 2 \sin^6 \theta \cos^2 \theta d\theta = \int_0^{\pi/2} 2 \left[\frac{1}{2} (1 - \cos 2\theta) \right]^3 \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \int_0^{\pi/2} \frac{1}{8} (1 - 2 \cos 2\theta + 2 \cos^3 2\theta - \cos^4 2\theta) d\theta \\ &= \int_0^{\pi/2} \frac{1}{8} [1 - 2 \cos 2\theta + 2 \cos 2\theta (1 - \sin^2 2\theta) - \frac{1}{4} (1 + \cos 4\theta)^2] d\theta \\ &= \frac{1}{8} \left[\theta - \frac{1}{3} \sin^3 2\theta \right]_0^{\pi/2} - \frac{1}{32} \int_0^{\pi/2} (1 + 2 \cos 4\theta + \cos^2 4\theta) d\theta \\ &= \frac{\pi}{16} - \frac{1}{32} \left[\theta + \frac{1}{2} \sin 4\theta \right]_0^{\pi/2} - \frac{1}{64} \int_0^{\pi/2} (1 + \cos 8\theta) d\theta \\ &= \frac{3\pi}{64} - \frac{1}{64} \left[\theta + \frac{1}{8} \sin 8\theta \right]_0^{\pi/2} = \frac{5\pi}{128} \end{aligned}$$

$$\begin{aligned} \int_0^1 \sqrt{x^3 - x^4} dx &= \int_0^1 x^{3/2} \sqrt{1 - x} dx = \int_0^{\pi/2} 2 \sin^4 \theta \cos \theta \sqrt{1 - \sin^2 \theta} d\theta \quad [\sin \theta = \sqrt{x}] \\ &= \int_0^{\pi/2} 2 \sin^4 \theta \cos^2 \theta d\theta = \int_0^{\pi/2} 2 \cdot \frac{1}{4} (1 - \cos 2\theta)^2 \cdot \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \int_0^{\pi/2} \frac{1}{4} (1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta) d\theta \\ &= \int_0^{\pi/2} \frac{1}{4} [1 - \cos 2\theta - \frac{1}{2} (1 + \cos 4\theta) + \cos 2\theta (1 - \sin^2 2\theta)] d\theta \\ &= \frac{1}{4} \left[\frac{\theta}{2} - \frac{1}{8} \sin 4\theta - \frac{1}{6} \sin^3 2\theta \right]_0^{\pi/2} = \frac{\pi}{16} \end{aligned}$$

Therefore, $\bar{x} = \frac{5\pi/128}{\pi/16} = \frac{5}{8}$, and $(\bar{x}, \bar{y}) = (\frac{5}{8}, 0)$.

3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2 , are generated by rotation about the y -axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation $x = \sqrt{r^2 - y^2}$ for $d \leq y \leq r$. Thus, $dx/dy = -y/\sqrt{r^2 - y^2}$ and



$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}} dy = \sqrt{\frac{r^2}{r^2 - y^2}} dy = \frac{r dy}{\sqrt{r^2 - y^2}}$$

From Formula 9.2.8 we have

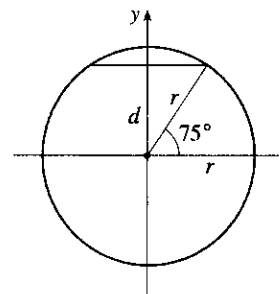
$$S_1 = \int_d^r 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_d^r 2\pi \sqrt{r^2 - y^2} \frac{r dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r dy = 2\pi r(r - d)$$

Similarly, we can compute $S_2 = \int_{-r}^d 2\pi x \sqrt{1 + (dx/dy)^2} dy = \int_{-r}^d 2\pi r dy = 2\pi r(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

- (b) $r = 3960$ mi and $d = r(\sin 75^\circ) \approx 3825$ mi,

so the surface area of the Arctic Ocean is about

$$2\pi r(r - d) \approx 2\pi(3960)(135) \approx 3.36 \times 10^6 \text{ mi}^2.$$

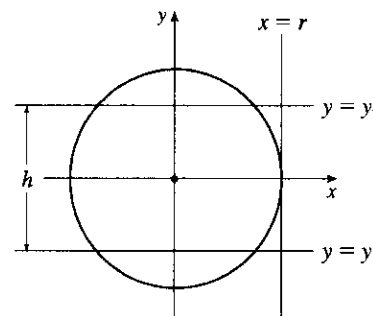


- (c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the

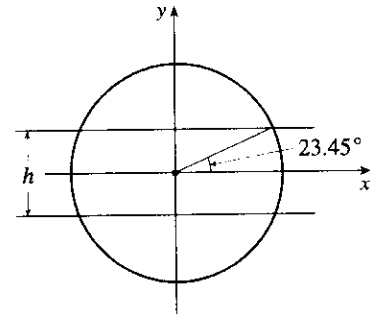
$$\text{surface area on the sphere to be } S = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_{y_1}^{y_2} 2\pi r dy = 2\pi r(y_2 - y_1) = 2\pi r h.$$

This equals the lateral area of a cylinder of radius r and height h , since such a cylinder is obtained by rotating the line $x = r$ about the y -axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is

$$\begin{aligned} A &= \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2} dy \\ &= 2\pi r y \Big|_{y=y_1}^{y_2} = 2\pi r(y_2 - y_1) = 2\pi r h \end{aligned}$$



- (d) $h = 2r \sin 23.45^\circ \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi rh \approx 2\pi(3960)(3152) \approx 7.84 \times 10^7$ mi².



4. (a) Since the right triangles OAT and OTB are similar, we have

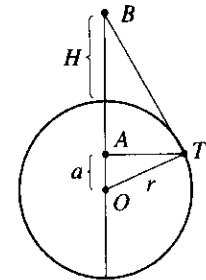
$$\frac{r+H}{r} = \frac{r}{a} \Rightarrow a = \frac{r^2}{r+H}. \text{ The surface area visible from } B \text{ is}$$

$$S = \int_a^r 2\pi x \sqrt{1 + (dx/dy)^2} dy. \text{ From } x^2 + y^2 = r^2, \text{ we get}$$

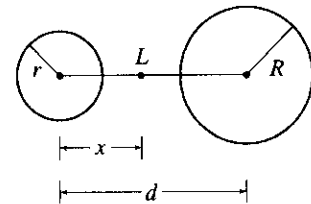
$$\frac{d}{dy}(x^2 + y^2) = \frac{d}{dy}(r^2) \Rightarrow 2x \frac{dx}{dy} + 2y = 0 \Rightarrow$$

$$\frac{dx}{dy} = -\frac{y}{x} \text{ and } 1 + \left(\frac{dx}{dy}\right)^2 = \frac{x^2 + y^2}{x^2} = \frac{r^2}{x^2}. \text{ Thus,}$$

$$S = \int_a^r 2\pi x \cdot \frac{r}{x} dy = 2\pi r(r-a) = 2\pi r\left(r - \frac{r^2}{r+H}\right) = 2\pi r^2\left(1 - \frac{r}{r+H}\right) = 2\pi r^2 \cdot \frac{H}{r+H} = \frac{2\pi r^2 H}{r+H}.$$



- (b) Assume $R \geq r$. If a light is placed at point L , at a distance x from the center of the sphere of radius r , then from part (a) we find that the total illuminated area A on the two spheres is [with $r+H = x$ and $r+H = d-x$].



$$A(x) = \frac{2\pi r^2(x-r)}{x} + \frac{2\pi R^2(d-x-R)}{d-x} \quad [r \leq x \leq d-R]. \quad \frac{A(x)}{2\pi} = r^2\left(1 - \frac{r}{x}\right) + R^2\left(1 - \frac{R}{d-x}\right), \text{ so}$$

$$A'(x) = 0 \Leftrightarrow 0 = r^2 \cdot \frac{r}{x^2} + R^2 \cdot \frac{-R}{(d-x)^2} \Leftrightarrow \frac{r^3}{x^2} = \frac{R^3}{(d-x)^2} \Leftrightarrow \frac{(d-x)^2}{x^2} = \frac{R^3}{r^3} \Leftrightarrow$$

$$\left(\frac{d}{x} - 1\right)^2 = \left(\frac{R}{r}\right)^3 \Rightarrow \frac{d}{x} - 1 = \left(\frac{R}{r}\right)^{3/2} \Leftrightarrow \frac{d}{x} = 1 + \left(\frac{R}{r}\right)^{3/2} \Leftrightarrow x = x^* = \frac{d}{1 + (R/r)^{3/2}}.$$

Now $A'(x) = 2\pi\left(\frac{r^3}{x^2} - \frac{R^3}{(d-x)^2}\right) \Rightarrow A''(x) = 2\pi\left(-\frac{2r^3}{x^3} - \frac{2R^3}{(d-x)^3}\right)$ and $A''(x^*) < 0$, so we have a local maximum at $x = x^*$.

However, x^* may not be an allowable value of x —we must show that x^* is between r and $d-R$.

$$(1) \quad x^* \geq r \Leftrightarrow \frac{d}{1 + (R/r)^{3/2}} \geq r \Leftrightarrow d \geq r + R\sqrt{R/r}$$

$$(2) \ x^* \leq d - R \Leftrightarrow \frac{d}{1 + (R/r)^{3/2}} \leq d - R \Leftrightarrow d \leq d - R + d \left(\frac{R}{r} \right)^{3/2} - R \left(\frac{R}{r} \right)^{3/2} \Leftrightarrow$$

$$R + R \left(\frac{R}{r} \right)^{3/2} \leq d \left(\frac{R}{r} \right)^{3/2} \Leftrightarrow d \geq \frac{R}{(R/r)^{3/2}} + R = R + r\sqrt{r/R}, \text{ but}$$

$$R + r\sqrt{r/R} \leq R + r, \text{ and since } d > r + R \text{ [given], we conclude that } x^* \leq d - R.$$

Thus, from (1) and (2), x^* is not an allowable value of x if $d < r + R\sqrt{R/r}$.

So A may have a maximum at $x = r, x^*$, or $d - R$.

$$A(r) = \frac{2\pi R^2(d - r - R)}{d - r} \quad \text{and} \quad A(d - R) = \frac{2\pi r^2(d - r - R)}{d - R}$$

$$A(r) > A(d - R) \Leftrightarrow \frac{R^2}{d - r} > \frac{r^2}{d - R} \Leftrightarrow R^2(d - R) > r^2(d - r) \Leftrightarrow R^2d - R^3 > r^2d - r^3 \Leftrightarrow$$

$$R^2d - r^2d > R^3 - r^3 \Leftrightarrow d(R - r)(R + r) > (R - r)(R^2 + Rr + r^2) \Leftrightarrow$$

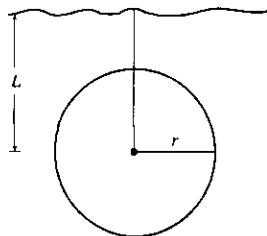
$$d > (R^2 + Rr + r^2)/(R + r) \Leftrightarrow d > [(R + r)^2 - Rr]/(R + r) \Leftrightarrow d > R + r - Rr/(R + r). \text{ Now}$$

$R + r - Rr/(R + r) < R + r$, and we know that $d > R + r$, so we conclude that $A(r) > A(d - R)$.

In conclusion, A has an absolute maximum at $x = x^*$ provided $d \geq r + R\sqrt{R/r}$; otherwise, A has its maximum at $x = r$.

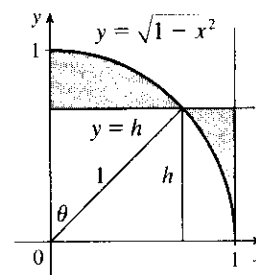
5. (a) Choose a vertical x -axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z , consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i . The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g \Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum_{i=1}^n \rho(x_i^*)g \Delta x$. The estimate becomes exact if we take the limit as $n \rightarrow \infty$; weight (or force) per unit area at depth z is $W = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(x_i^*)g \Delta x$. In other words, $P(z) = \int_0^z \rho(x)g dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)g dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)g$.

(b)



$$\begin{aligned} F &= \int_{-r}^r P(L+x) \cdot 2\sqrt{r^2 - x^2} dx \\ &= \int_{-r}^r \left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g dz \right) \cdot 2\sqrt{r^2 - x^2} dx \\ &= P_0 \int_{-r}^r 2\sqrt{r^2 - x^2} dx + \rho_0 g H \int_{-r}^r \left(e^{(L+x)/H} - 1 \right) \cdot 2\sqrt{r^2 - x^2} dx \\ &= (P_0 - \rho_0 g H) \int_{-r}^r 2\sqrt{r^2 - x^2} dx + \rho_0 g H \int_{-r}^r e^{(L+x)/H} \cdot 2\sqrt{r^2 - x^2} dx \\ &= (P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \int_{-r}^r e^{x/H} \cdot 2\sqrt{r^2 - x^2} dx \end{aligned}$$

6. The problem can be reduced to finding the line which minimizes the shaded area in the diagram. The equation of the circle in the first quadrant is $y = \sqrt{1-x^2}$, so if the equation of the line is $y = h$, then the circle and the line intersect where $h = \sqrt{1-x^2} \Rightarrow x = \sqrt{1-h^2}$. So the shaded area is



$$\begin{aligned}
 A &= \int_0^{\sqrt{1-h^2}} (\sqrt{1-x^2} - h) dx + \int_{\sqrt{1-h^2}}^1 (h - \sqrt{1-x^2}) dx \\
 &\stackrel{*}{=} [-hx]_0^{\sqrt{1-h^2}} + [hx]_{\sqrt{1-h^2}}^1 + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2} dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2} dx \\
 &= -h\sqrt{1-h^2} + h - h\sqrt{1-h^2} + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2} dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2} dx \\
 &= h(1 - 2\sqrt{1-h^2}) + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2} dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2} dx
 \end{aligned}$$

Note that at (*), we reversed the limits of integration and changed the sign in the last integral. We are interested in the minimum of

$A(h) = h(1 - 2\sqrt{1-h^2}) + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2} dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2} dx$, so we find dA/dh using FTC1 and the Chain Rule:

$$\begin{aligned}
 \frac{dA}{dh} &= h \left(-2 \frac{-h}{\sqrt{1-h^2}} \right) + (1 - 2\sqrt{1-h^2}) + 2 \left[\sqrt{1 - (\sqrt{1-h^2})^2} \right] \frac{d}{dh} (\sqrt{1-h^2}) \\
 &= \frac{1}{\sqrt{1-h^2}} [2h^2 + \sqrt{1-h^2} - 2(1-h^2)] + 2h \frac{-h}{\sqrt{1-h^2}} \\
 &= \frac{1}{\sqrt{1-h^2}} [\sqrt{1-h^2} - 2(1-h^2)]
 \end{aligned}$$

This is 0 when $\sqrt{1-h^2} - 2(1-h^2) = 0 \Leftrightarrow u - 2u^2 = 0$ (where $u = \sqrt{1-h^2}$) $\Leftrightarrow u = 0$ or $\frac{1}{2} \Leftrightarrow$

$h = 1$ or $\frac{\sqrt{3}}{2}$. By the First Derivative Test, $h = \frac{\sqrt{3}}{2}$ represents a minimum for $A(h)$, since $A'(h) = 1 - \frac{2}{\sqrt{1-h^2}}$

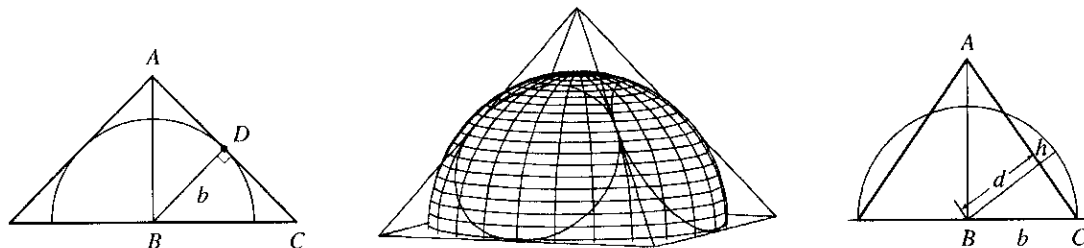
goes from negative to positive at $h = \frac{\sqrt{3}}{2}$.

Another method: Use FTC2 to evaluate all of the integrals before differentiating.

Note: Another strategy is to use the angle θ as the variable (see diagram above) and show that

$A = \theta + \cos \theta - \frac{\pi}{4} - \frac{1}{2} \sin 2\theta$, which is minimized when $\theta = \frac{\pi}{6}$.

7. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}b$.



We first observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 6.2.49 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that

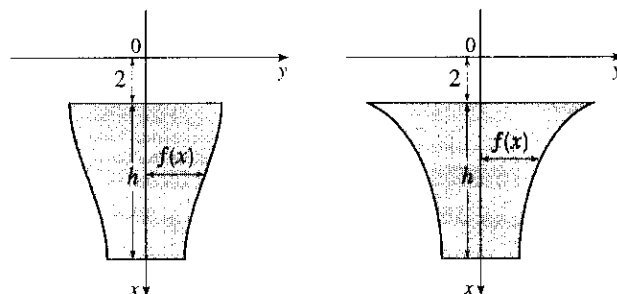
$$\frac{d}{b} = \frac{|AB|}{|AC|} = \frac{\sqrt{2}b}{\sqrt{b^2 + (\sqrt{2}b)^2}} \Rightarrow d = \frac{\sqrt{2}b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}b$$

So $h = b - d = b - \frac{\sqrt{6}}{3}b = \frac{3-\sqrt{6}}{3}b$. So, using the formula $V = \pi h^2(r - h/3)$ from Exercise 6.2.49 with $r = b$, we find that the volume of each of the caps is

$$\pi \left(\frac{3-\sqrt{6}}{3}b \right)^2 \left(b - \frac{3-\sqrt{6}}{3}b \right) = \frac{15-6\sqrt{6}}{9} \cdot \frac{6+\sqrt{6}}{9} \pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6} \right) \pi b^3. \text{ So, using our first observation, the}$$

$$\text{shared volume is } V = \frac{1}{2} \left(\frac{4}{3} \pi b^3 \right) - 4 \left(\frac{2}{3} - \frac{7}{27}\sqrt{6} \right) \pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2 \right) \pi b^3.$$

8. Orient the positive x -axis as in the figure.



Suppose that the plate has height h and is symmetric about the x -axis. At depth x below the water ($2 \leq x \leq 2 + h$), let the width of the plate be $2f(x)$. Now each of the n horizontal strips has height h/n and the i th strip ($1 \leq i \leq n$) goes from $x = 2 + \left(\frac{i-1}{n}\right)h$ to $x = 2 + \left(\frac{i}{n}\right)h$. The hydrostatic force on the i th strip is

$F(i) = \int_{2+(i-1)/n}^{2+i/n} 62.5x[2f(x)]dx$. If we now let $x[2f(x)] = k$ (a constant) so that $f(x) = k/(2x)$, then

$$\begin{aligned} F(i) &= \int_{2+(i-1)/n}^{2+i/n} 62.5k dx = 62.5k [x]_{2+(i-1)/n}^{2+i/n} \\ &= 62.5k \left[\left(2 + \frac{i}{n}h\right) - \left(2 + \frac{i-1}{n}h\right) \right] = 62.5k \left(\frac{h}{n}\right) \end{aligned}$$

So the hydrostatic force on the i th strip is independent of i , that is, the force on each strip is the same. So the plate can be shaped as shown in the figure. (In fact, the required condition is satisfied whenever the plate has width C/x at depth x , for some constant C . Many shapes are possible.)

9. We can assume that the cut is made along a vertical line $x = b > 0$,

that the disk's boundary is the circle $x^2 + y^2 = 1$, and that the center

of mass of the smaller piece (to the right of $x = b$) is $(\frac{1}{2}, 0)$. We

wish to find b to two decimal places. We have

$$\frac{1}{2} = \bar{x} = \frac{\int_b^1 x \cdot 2\sqrt{1-x^2} dx}{\int_b^1 2\sqrt{1-x^2} dx}. \text{ Evaluating the numerator gives us}$$

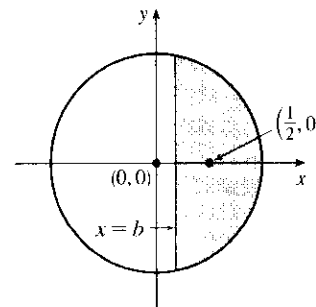
$$-\int_b^1 (1-x^2)^{1/2}(-2x) dx = -\frac{2}{3} \left[(1-x^2)^{3/2} \right]_b^1 = -\frac{2}{3} \left[0 - (1-b^2)^{3/2} \right] = \frac{2}{3}(1-b^2)^{3/2}. \text{ Using}$$

Formula 30 in the table of integrals, we find that the denominator is

$$[x\sqrt{1-x^2} + \sin^{-1}x]_b^1 = \left(0 + \frac{\pi}{2}\right) - (b\sqrt{1-b^2} + \sin^{-1}b). \text{ Thus, we have}$$

$$\frac{1}{2} = \bar{x} = \frac{\frac{2}{3}(1-b^2)^{3/2}}{\frac{\pi}{2} - b\sqrt{1-b^2} - \sin^{-1}b}, \text{ or, equivalently, } \frac{2}{3}(1-b^2)^{3/2} = \frac{\pi}{4} - \frac{1}{2}b\sqrt{1-b^2} - \frac{1}{2}\sin^{-1}b. \text{ Solving this}$$

equation numerically with a calculator or CAS, we obtain $b \approx 0.138173$, or $b = 0.14$ m to two decimal places.

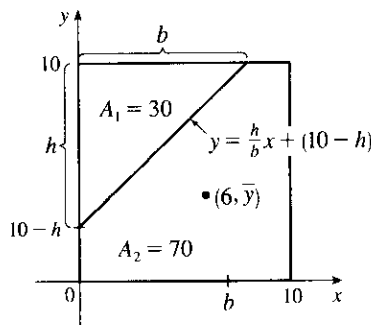


10. $A_1 = 30 \Rightarrow \frac{1}{2}bh = 30 \Rightarrow bh = 60$.

$$\bar{x} = 6 \Rightarrow \frac{1}{A_2} \int_0^{10} xf(x) dx = 6 \Rightarrow \int_0^b x \left(\frac{h}{b}x + 10 - h \right) dx + \int_b^{10} x(10) dx = 6(70) \Rightarrow$$

$$\int_0^b \left(\frac{h}{b}x^2 + 10x - hx \right) dx + 10 \cdot \frac{1}{2}[x^2]_b^{10} = 420 \Rightarrow \left[\frac{h}{3b}x^3 + 5x^2 - \frac{h}{2}x^2 \right]_0^b + 5(100 - b^2) = 420 \Rightarrow$$

$$\frac{1}{3}hb^2 + 5b^2 - \frac{1}{2}hb^2 + 500 - 5b^2 = 420 \Rightarrow 80 = \frac{1}{6}hb^2 \Rightarrow 480 = (hb)b \Rightarrow 480 = 60b \Rightarrow b = 8.$$



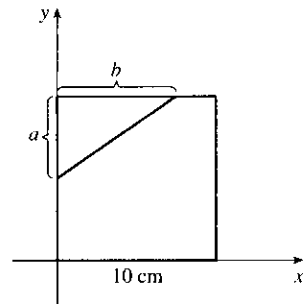
So $h = \frac{60}{8} = \frac{15}{2}$ and an equation of the line is $y = \frac{15/2}{8}x + \left(10 - \frac{15}{2}\right) = \frac{15}{16}x + \frac{5}{2}$. Now

$$\begin{aligned}\bar{y} &= \frac{1}{A_2} \int_0^{10} \frac{1}{2} [f(x)]^2 dx = \frac{1}{70 \cdot 2} \left[\int_0^8 \left(\frac{15}{16}x + \frac{5}{2} \right)^2 dx + \int_8^{10} (10)^2 dx \right] \\ &= \frac{1}{140} \left[\int_0^8 \left(\frac{225}{256}x^2 + \frac{75}{16}x + \frac{25}{4} \right) dx + 100(10 - 8) \right] = \frac{1}{140} \left(\left[\frac{225}{768}x^3 + \frac{75}{32}x^2 + \frac{25}{4}x \right]_0^8 + 200 \right) \\ &= \frac{1}{140} (150 + 150 + 50 + 200) = \frac{550}{140} = \frac{55}{14}\end{aligned}$$

Another solution:

Assume that the right triangle cut from the square has legs a cm and b cm long as shown. The triangle has area 30 cm^2 , so $\frac{1}{2}ab = 30$ and $ab = 60$. We place the square in the first quadrant of the xy -plane as shown, and we let T , R , and S denote the triangle, the remaining portion of the square, and the full square, respectively. By symmetry, the centroid of S is $(5, 5)$. By Exercise 9.3.37, the centroid of T

is $\left(\frac{b}{3}, 10 - \frac{a}{3}\right)$.



We are given that the centroid of R is $(6, c)$, where c is to be determined. We take the density of the square to be 1, so that areas can be used as masses. Then T has mass $m_T = 30$, S has mass $m_S = 100$, and R has mass $m_R = m_S - m_T = 70$. As in Exercises 38 and 39 of Section 9.3, we view S as consisting of a mass m_T at the centroid $(\bar{x}_T, \bar{y}_T)$ of T and a mass R at the centroid $(\bar{x}_R, \bar{y}_R)$ of R . Then $\bar{x}_S = \frac{m_T \bar{x}_T + m_R \bar{x}_R}{m_T + m_R}$ and

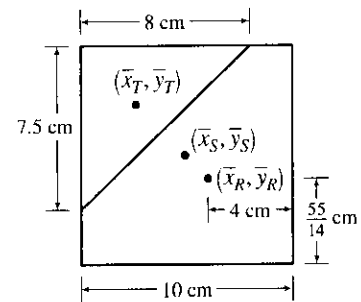
$$\bar{y}_S = \frac{m_T \bar{y}_T + m_R \bar{y}_R}{m_T + m_R}; \text{ that is, } 5 = \frac{30(b/3) + 70(6)}{100} \text{ and } 5 = \frac{30(10 - a/3) + 70c}{100}.$$

Solving the first equation for b , we get $b = 8$ cm. Since

$ab = 60 \text{ cm}^2$, it follows that $a = \frac{60}{8} = 7.5$ cm. Now the second

equation says that $70c = 200 + 10a$, so $7c = 20 + a = \frac{55}{2}$ and

$c = \frac{55}{14} = 3.9285714$ cm. The solution is depicted in the figure.



11. If $h = L$, then

$$P = \frac{\text{area under } y = L \sin \theta}{\text{area of rectangle}} = \frac{\int_0^\pi L \sin \theta \, d\theta}{\pi L} = \frac{[-\cos \theta]_0^\pi}{\pi} = \frac{-(-1) + 1}{\pi} = \frac{2}{\pi}$$

If $h = L/2$, then

$$P = \frac{\text{area under } y = \frac{1}{2}L \sin \theta}{\text{area of rectangle}} = \frac{\int_0^\pi \frac{1}{2}L \sin \theta \, d\theta}{\pi L} = \frac{[-\cos \theta]_0^\pi}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$$

12. (a) The total set of possibilities can be identified with the rectangular

region $\mathcal{R} = \{(\theta, y) \mid 0 \leq y < L, 0 \leq \theta < \pi\}$. Even when $h > L$,

the needle intersects at least one line if and only if $y \leq h \sin \theta$. Let

$\mathcal{R}_1 = \{(\theta, y) \mid 0 \leq y \leq h \sin \theta, 0 \leq \theta < \pi\}$. When $h \leq L$, $\mathcal{R}_1$ is

contained in $\mathcal{R}$, but that is no longer true when $h > L$. Thus, the probability that the needle intersects a line becomes

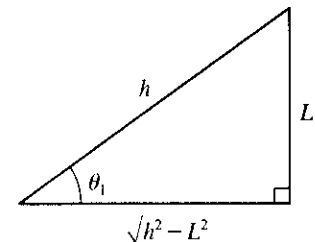
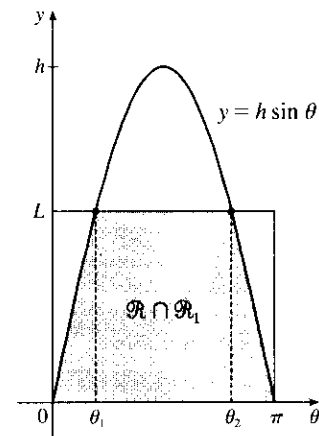
$$P = \frac{\text{area}(\mathcal{R} \cap \mathcal{R}_1)}{\text{area}(\mathcal{R})} = \frac{\text{area}(\mathcal{R} \cap \mathcal{R}_1)}{\pi L}$$

When $h > L$, the curve $y = h \sin \theta$ intersects the line $y = L$

twice — at $(\sin^{-1}(L/h), L)$ and at $(\pi - \sin^{-1}(L/h), L)$. Set

$\theta_1 = \sin^{-1}(L/h)$ and $\theta_2 = \pi - \theta_1$. Then

$$\begin{aligned} \text{area}(\mathcal{R} \cap \mathcal{R}_1) &= \int_0^{\theta_1} h \sin \theta \, d\theta + \int_{\theta_1}^{\theta_2} L \, d\theta + \int_{\theta_2}^\pi h \sin \theta \, d\theta \\ &= 2 \int_0^{\theta_1} h \sin \theta \, d\theta + L(\theta_2 - \theta_1) = 2h[-\cos \theta]_0^{\theta_1} + L(\pi - 2\theta_1) \\ &= 2h(1 - \cos \theta_1) + L(\pi - 2\theta_1) \\ &= 2h\left(1 - \frac{\sqrt{h^2 - L^2}}{h}\right) + L\left[\pi - 2\sin^{-1}\left(\frac{L}{h}\right)\right] \\ &= 2h - 2\sqrt{h^2 - L^2} + \pi L - 2L\sin^{-1}\left(\frac{L}{h}\right) \end{aligned}$$



We are told that $L = 4$ and $h = 7$, so $\text{area}(\mathcal{R} \cap \mathcal{R}_1) = 14 - 2\sqrt{33} + 4\pi - 8\sin^{-1}\left(\frac{4}{7}\right) \approx 10.21128$ and

$P = \frac{1}{4\pi} \text{area}(\mathcal{R} \cap \mathcal{R}_1) \approx 0.812588$. (By comparison, $P = \frac{2}{\pi} \approx 0.636620$ when $h = L$, as shown in the solution to Problem 11.)

- (b) The needle intersects at least two lines when $y + L \leq h \sin \theta$;

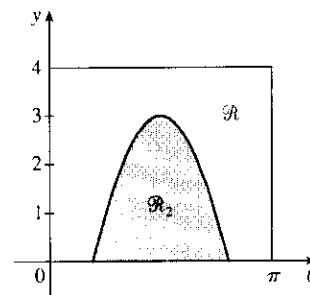
that is, when $y \leq h \sin \theta - L$. Set

$\mathcal{R}_2 = \{(\theta, y) \mid 0 \leq y \leq h \sin \theta - L, 0 \leq \theta < \pi\}$. Then the

probability that the needle intersects at least two lines is

$$P_2 = \frac{\text{area}(\mathcal{R} \cap \mathcal{R}_2)}{\text{area}(\mathcal{R})} = \frac{\text{area}(\mathcal{R} \cap \mathcal{R}_2)}{\pi L}. \text{ When } L = 4 \text{ and } h = 7, \mathcal{R}_2$$

is contained in $\mathcal{R}$ (see the figure). Thus,



$$\begin{aligned} P_2 &= \frac{1}{4\pi} \text{area}(\mathcal{R}_2) = \frac{1}{4\pi} \int_{\sin^{-1}(4/7)}^{\pi - \sin^{-1}(4/7)} (7 \sin \theta - 4) d\theta = \frac{1}{4\pi} \cdot 2 \int_{\sin^{-1}(4/7)}^{\pi/2} (7 \sin \theta - 4) d\theta \\ &= \frac{1}{2\pi} [-7 \cos \theta - 4\theta]_{\sin^{-1}(4/7)}^{\pi/2} = \frac{1}{2\pi} \left[0 - 2\pi + 7 \frac{\sqrt{33}}{7} + 4 \sin^{-1}(4/7) \right] \\ &= \frac{\sqrt{33} + 4 \sin^{-1}(4/7) - 2\pi}{2\pi} \approx 0.301497 \end{aligned}$$

- (c) The needle intersects at least three lines when $y + 2L \leq h \sin \theta$; that is, when $y \leq h \sin \theta - 2L$. Set

$\mathcal{R}_3 = \{(\theta, y) \mid 0 \leq y \leq h \sin \theta - 2L, 0 \leq \theta < \pi\}$. Then the probability that the needle intersects at least three

lines is $P_3 = \frac{\text{area}(\mathcal{R} \cap \mathcal{R}_3)}{\text{area}(\mathcal{R})} = \frac{\text{area}(\mathcal{R} \cap \mathcal{R}_3)}{\pi L}$. (At this point, the generalization to P_n , n any positive integer,

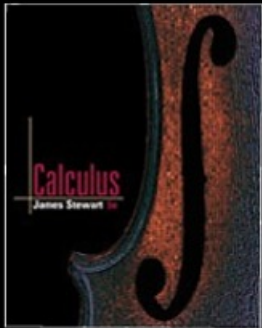
should be clear.) Under the given assumption,

$$\begin{aligned} P_3 &= \frac{1}{\pi L} \text{area}(\mathcal{R}_3) = \frac{1}{\pi L} \int_{\sin^{-1}(2L/h)}^{\pi - \sin^{-1}(2L/h)} (h \sin \theta - 2L) d\theta \\ &= \frac{2}{\pi L} \int_{\sin^{-1}(2L/h)}^{\pi/2} (h \sin \theta - 2L) d\theta \\ &= \frac{2}{\pi L} [-h \cos \theta - 2L\theta]_{\sin^{-1}(2L/h)}^{\pi/2} \\ &= \frac{2}{\pi L} \left[-\pi L + \sqrt{h^2 - 4L^2} + 2L \sin^{-1}(2L/h) \right] \end{aligned}$$

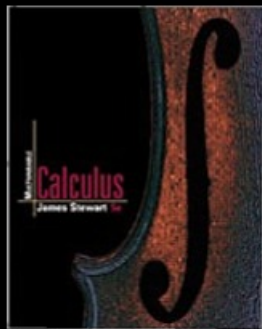
Note that the probability that a needle touches exactly one line is $P_1 - P_2$, the probability that it touches exactly two lines is $P_2 - P_3$, and so on.

Chapter 10

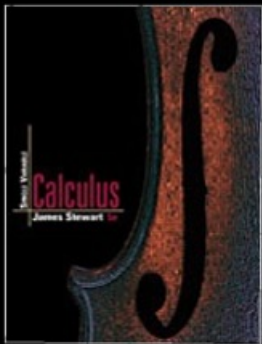
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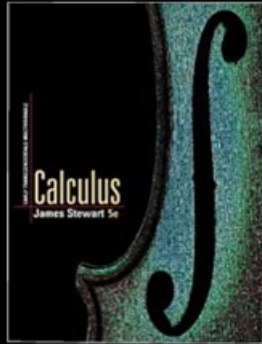
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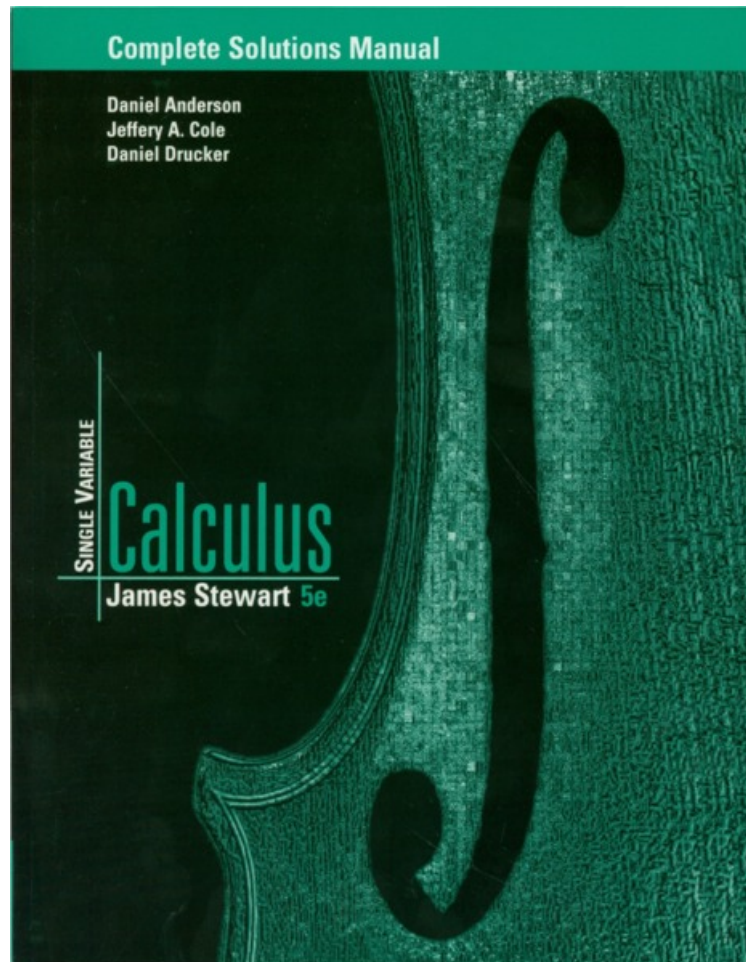
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10 □ DIFFERENTIAL EQUATIONS

10.1 Modeling with Differential Equations

1. $y = x - x^{-1} \Rightarrow y' = 1 + x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\text{LHS} = xy' + y = x(1 + x^{-2}) + (x - x^{-1}) = x + x^{-1} + x - x^{-1} = 2x = \text{RHS}$$

2. $y = \sin x \cos x - \cos x \Rightarrow y' = \sin x(-\sin x) + \cos x(\cos x) - (-\sin x) = \cos^2 x - \sin^2 x + \sin x$.

$$\begin{aligned}\text{LHS} &= y' + (\tan x)y = \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x \cos x - \cos x) \\ &= \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x = \text{RHS},\end{aligned}$$

so y is a solution of the differential equation. Also, $y(0) = \sin 0 \cos 0 - \cos 0 = 0 \cdot 1 - 1 = -1$, so the initial condition is satisfied.

3. (a) $y = \sin kt \Rightarrow y' = k \cos kt \Rightarrow y'' = -k^2 \sin kt$. $y'' + 9y = 0 \Rightarrow -k^2 \sin kt + 9 \sin kt = 0$ for all $t \Leftrightarrow (9 - k^2) \sin kt = 0$ for all $t \Leftrightarrow 9 - k^2 = 0 \Leftrightarrow k = \pm 3$

(b) $y = A \sin kt + B \cos kt \Rightarrow y' = Ak \cos kt - Bk \sin kt \Rightarrow y'' = -Ak^2 \sin kt - Bk^2 \cos kt$.
Thus, $y'' + 9y = 0 \Rightarrow -Ak^2 \sin kt - Bk^2 \cos kt + 9(A \sin kt + B \cos kt) = 0 \Rightarrow$
 $(9 - k^2)A \sin kt + (9 - k^2)B \cos kt = 0$. The last equation is true for all values of A and B if $k = \pm 3$.

4. $y = e^{rt} \Rightarrow y' = re^{rt} \Rightarrow y'' = r^2 e^{rt}$. $y'' + y' - 6y = 0 \Rightarrow r^2 e^{rt} + re^{rt} - 6e^{rt} = 0 \Rightarrow$
 $(r^2 + r - 6)e^{rt} = 0 \Rightarrow (r + 3)(r - 2) = 0 \Rightarrow r = -3 \text{ or } 2$

5. (a) $y = e^t \Rightarrow y' = e^t \Rightarrow y'' = e^t$. $\text{LHS} = y'' + 2y' + y = e^t + 2e^t + e^t = 4e^t \neq 0$, so $y = e^t$ is not a solution of the differential equation.

(b) $y = e^{-t} \Rightarrow y' = -e^{-t} \Rightarrow y'' = e^{-t}$. $\text{LHS} = y'' + 2y' + y = e^{-t} - 2e^{-t} + e^{-t} = 0 = \text{RHS}$, so $y = e^{-t}$ is a solution.

(c) $y = te^{-t} \Rightarrow y' = t(-e^{-t}) + e^{-t}(1) = e^{-t}(1 - t) \Rightarrow y'' = e^{-t}(t - 2)$.

$$\begin{aligned}\text{LHS} &= y'' + 2y' + y = e^{-t}(t - 2) + 2e^{-t}(1 - t) + te^{-t} \\ &= e^{-t}[(t - 2) + 2(1 - t) + t] = e^{-t}(0) = 0 = \text{RHS},\end{aligned}$$

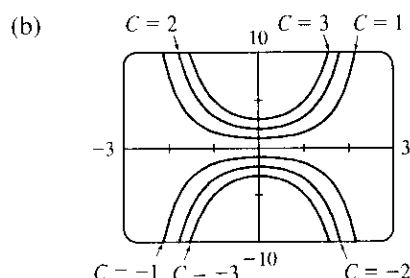
so $y = te^{-t}$ is a solution.

(d) $y = t^2 e^{-t} \Rightarrow y' = te^{-t}(2 - t) \Rightarrow y'' = e^{-t}(t^2 - 4t + 2)$.

$$\begin{aligned}\text{LHS} &= y'' + 2y' + y = e^{-t}(t^2 - 4t + 2) + 2te^{-t}(2 - t) + t^2 e^{-t} \\ &= e^{-t}[(t^2 - 4t + 2) + 2t(2 - t) + t^2] = e^{-t}(2) \neq 0,\end{aligned}$$

so $y = t^2 e^{-t}$ is not a solution.

6. (a) $y = Ce^{x^2/2} \Rightarrow y' = Ce^{x^2/2}(2x/2) = xCe^{x^2/2} = xy.$



(c) $y(0) = 5 \Rightarrow Ce^0 = 5 \Rightarrow C = 5$, so the solution is

$$y = 5e^{x^2/2}.$$

(d) $y(1) = 2 \Rightarrow Ce^{1/2} = 2 \Rightarrow C = 2e^{-1/2}$, so the solution is

$$y = 2e^{-1/2}e^{x^2/2} = 2e^{(x^2-1)/2}.$$

7. (a) Since the derivative $y' = -y^2$ is always negative (or 0 if $y = 0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y = \frac{1}{x+C} \Rightarrow y' = -\frac{1}{(x+C)^2}$. LHS = $y' = -\frac{1}{(x+C)^2} = -\left(\frac{1}{x+C}\right)^2 = -y^2 = \text{RHS}$

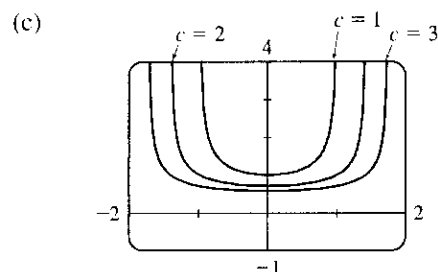
(c) $y = 0$ is a solution of $y' = -y^2$ that is not a member of the family in part (b).

(d) If $y(x) = \frac{1}{x+C}$, then $y(0) = \frac{1}{0+C} = \frac{1}{C}$. Since $y(0) = 0.5$, $\frac{1}{C} = \frac{1}{2} \Rightarrow C = 2$, so $y = \frac{1}{x+2}$.

8. (a) If x is close to 0, then xy^3 is close to 0, and hence, y' is close to 0. Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y .)

(b) $y = (c - x^2)^{-1/2} \Rightarrow y' = x(c - x^2)^{-3/2}.$

$$\text{RHS} = xy^3 = x[(c - x^2)^{-1/2}]^3 = x(c - x^2)^{-3/2} = y' = \text{LHS}$$



When x is close to 0, y' is also close to 0.

As x gets larger, so does $|y'|$.

(d) $y(0) = (c - 0)^{-1/2} = 1/\sqrt{c}$ and $y(0) = 2 \Rightarrow \sqrt{c} = \frac{1}{2} \Rightarrow c = \frac{1}{4}$, so $y = (\frac{1}{4} - x^2)^{-1/2}.$

9. (a) $\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$. Now $\frac{dP}{dt} > 0 \Rightarrow 1 - \frac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow \frac{P}{4200} < 1 \Rightarrow P < 4200 \Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\frac{dP}{dt} < 0 \Rightarrow P > 4200$

(c) $\frac{dP}{dt} = 0 \Rightarrow P = 4200$ or $P = 0$

10. (a) $y = k \Rightarrow y' = 0$, so $\frac{dy}{dt} = y^4 - 6y^3 + 5y^2 \Leftrightarrow 0 = k^4 - 6k^3 + 5k^2 \Leftrightarrow k^2(k^2 - 6k + 5) = 0 \Leftrightarrow k^2(k - 1)(k - 5) = 0 \Leftrightarrow k = 0, 1, \text{ or } 5$

(b) y is increasing $\Leftrightarrow \frac{dy}{dt} > 0 \Leftrightarrow y^2(y - 1)(y - 5) > 0 \Leftrightarrow y \in (-\infty, 0) \cup (0, 1) \cup (5, \infty)$

(c) y is decreasing $\Leftrightarrow \frac{dy}{dt} < 0 \Leftrightarrow y \in (1, 5)$

11. (a) This function is increasing *and* also decreasing. But $dy/dt = e^t(y - 1)^2 \geq 0$ for all t , implying that the graph of the solution of the differential equation cannot be decreasing on any interval.

(b) When $y = 1$, $dy/dt = 0$, but the graph does not have a horizontal tangent line.

12. The graph for this exercise is shown in the figure at the right.

A. $y' = 1 + xy > 1$ for points in the first quadrant, but we can see that $y' < 0$ for some points in the first quadrant. So equation A is incorrect.

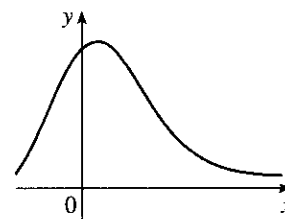
B. $y' = -2xy = 0$ when $x = 0$, but we can see that $y' > 0$ for $x = 0$. So equation B is incorrect.

C. $y' = 1 - 2xy$ seems reasonable since:

(1) When $x = 0$, y' could be 1.

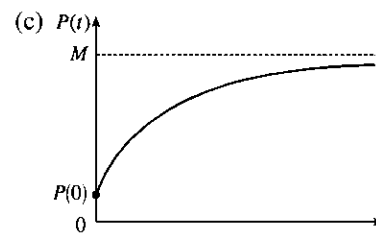
(2) When $x < 0$, y' could be greater than 1.

(3) Solving $y' = 1 - 2xy$ for y gives us $y = \frac{1 - y'}{2x}$. If y' takes on small negative values, then as $x \rightarrow \infty$, $y \rightarrow 0^+$, as shown in the figure. Thus, the correct equation is C.



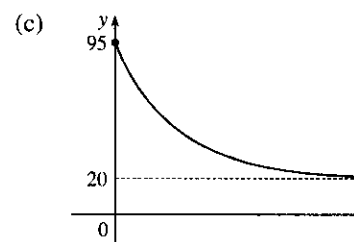
13. (a) P increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) $\frac{dP}{dt} = k(M - P)$ is always positive, so the level of performance P is increasing. As P gets close to M , dP/dt gets close to 0; that is, the performance levels off, as explained in part (a).



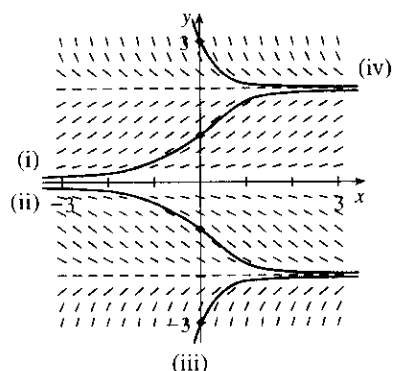
14. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

(b) $\frac{dy}{dt} = k(y - R)$, where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The initial condition is $y(0) = 95^\circ\text{C}$. The answer and the model support each other because as y approaches R , dy/dt approaches 0, so the model seems appropriate.



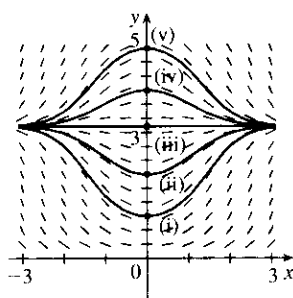
10.2 Direction Fields and Euler's Method

1. (a)



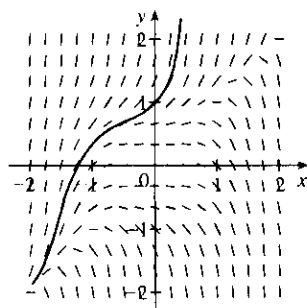
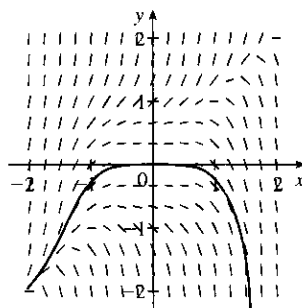
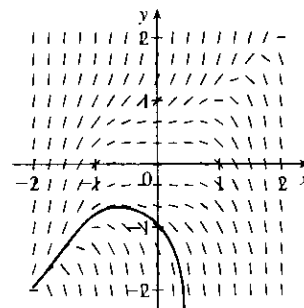
(b) It appears that the constant functions $y = 0$, $y = -2$, and $y = 2$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y' = y(1 - \frac{1}{4}y^2)$.

2. (a)

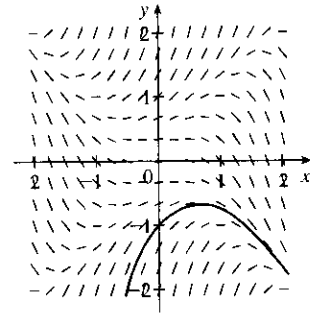
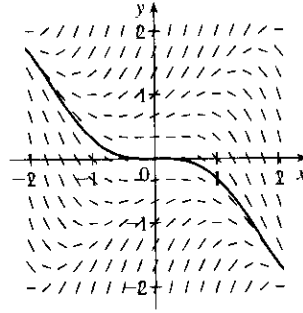
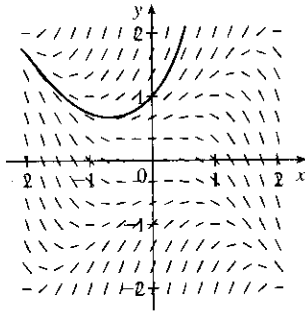


(b) From the figure, it appears that $y = \pi$ is an equilibrium solution. From the equation $y' = x \sin y$, we see that $y = n\pi$ (n an integer) describes all the equilibrium solutions.

3. $y' = y - 1$. The slopes at each point are independent of x , so the slopes are the same along each line parallel to the x -axis. Thus, IV is the direction field for this equation. Note that for $y = 1$, $y' = 0$.
4. $y' = y - x = 0$ on the line $y = x$, when $x = 0$ the slope is y , and when $y = 0$ the slope is $-x$. Direction field II satisfies these conditions. [Looking at the slope at the point $(0, 2)$, II looks more like it has a slope of 2 than does direction field I.]
5. $y' = y^2 - x^2 = 0 \Rightarrow y = \pm x$. There are horizontal tangents on these lines only in graph III, so this equation corresponds to direction field III.
6. $y' = y^3 - x^3 = 0$ on the line $y = x$, when $x = 0$ the slope is y^3 , and when $y = 0$ the slope is $-x^3$. The graph is similar to the graph for Exercise 4, but the segments must get steeper very rapidly as they move away from the origin, because x and y are raised to the third power. This is the case in direction field I.

7. (a) $y(0) = 1$ (b) $y(0) = 0$ (c) $y(0) = -1$ 

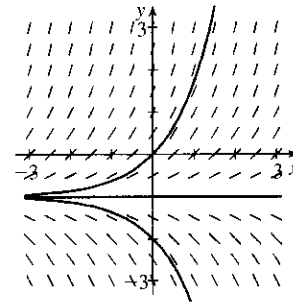
(c) $y(0) = -1$



9.

x	y	$y' = 1 + y$
0	0	1
0	1	2
0	2	3
0	-3	-2
0	-2	-1

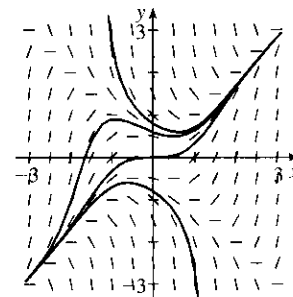
Note that for $y = -1$, $y' = 0$. The three solution curves sketched go through $(0, 0)$, $(0, -1)$, and $(0, -2)$.



10.

x	y	$y' = x^2 - y^2$
± 1	± 3	-8
± 3	± 1	8
± 1	± 0.5	0.75
± 0.5	± 1	-0.75

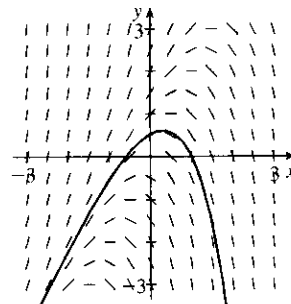
Note that $y' = 0$ for $y = \pm x$. If $|x| < |y|$, then $y' < 0$; that is, the slopes are negative for all points in quadrants I and II above both of the lines $y = x$ and $y = -x$, and all points in quadrants III and IV below both of the lines $y = -x$ and $y = x$. A similar statement holds for positive slopes.



11.

x	y	$y' = y - 2x$
-2	-2	2
-2	2	6
2	2	-2
2	-2	-6

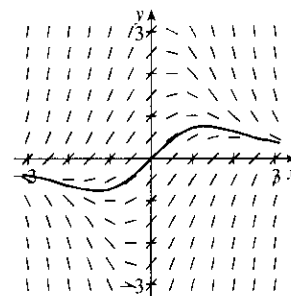
Note that $y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1, 0)$.



12.

x	y	$y' = 1 - xy$
± 1	± 1	0
± 2	± 2	-3
± 2	∓ 2	5

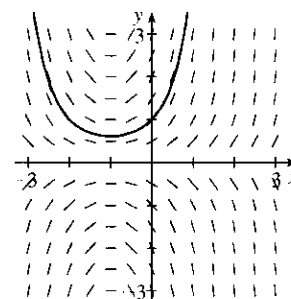
Note that $y' = 0$ for any point on the hyperbola $xy = 1$ (or $y = 1/x$). The slopes are negative at points “inside” the branches and positive at points everywhere else. The solution curve in the graph passes through $(0, 0)$.



13.

x	y	$y' = y + xy$
0	± 2	± 2
1	± 2	± 4
-3	± 2	∓ 4

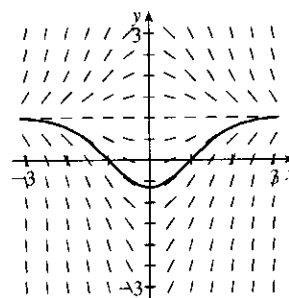
Note that $y' = y(x + 1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and $x + 1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0, 1)$.



14.

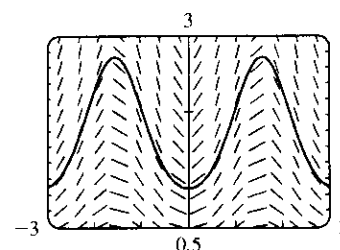
x	y	$y' = x - xy$
± 2	0	± 2
± 2	3	∓ 4
± 2	-1	± 4

Note that $y' = x(1 - y) = 0$ for any point on $x = 0$ or on $y = 1$. The slopes are positive when the factors x and $1 - y$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(1, 0)$.



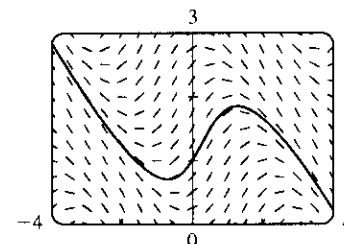
15. In Maple, we can use either `directionfield` (in Maple's share library) or `plots[fieldplot]` to plot the direction field. To plot the solution, we can either use the initial-value option in `directionfield`, or actually solve the equation. In Mathematica, we use `PlotVectorField` for the direction field, and the `Plot[Evaluate[...]]` construction to plot the solution, which

is $y = e^{(1 - \cos 2x)/2}$. In Derive, use `Direction_Field` (in utility file `ODE_APPR`) to plot the direction field. Then use `DSOLVE1(-y*SIN(2*x), 1, x, y, 0, 1)` (in utility file `ODE1`) to solve the equation. Simplify each result.

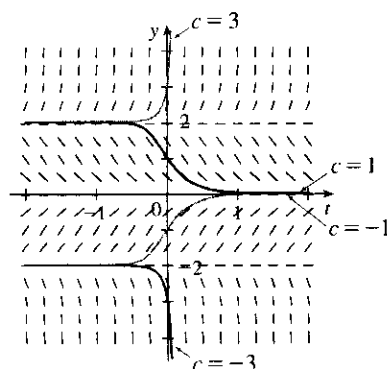


16. See Exercise 15 for specific CAS directions. The exact solution is

$$y = -x - 2 \arctan \frac{2 + x - \frac{2}{1 + \tan(1/2)}}{x - \frac{2}{1 + \tan(1/2)}}$$

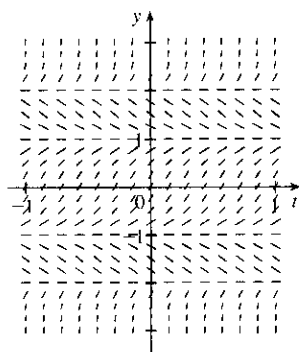


17.



$L = \lim_{t \rightarrow \infty} y(t)$ exists for $-2 \leq c \leq 2$; $L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$. For other values of c , L does not exist.

18.



Note that when $f(y) = 0$ on the graph in the text, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1, \pm 2$. We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t \rightarrow \infty} y = \infty$ and $\lim_{t \rightarrow -\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t \rightarrow \infty} y = 1$ and $\lim_{t \rightarrow -\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t \rightarrow \infty} y = 1$ and $\lim_{t \rightarrow -\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t \rightarrow \infty} y = -2$ and $\lim_{t \rightarrow -\infty} y = -1$.
- If $y < -2$, then $\lim_{t \rightarrow \infty} y = -2$ and $\lim_{t \rightarrow -\infty} y = -\infty$.

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \Rightarrow y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$,
so $y_1 = y(0.4) = 1.4$.

(ii) $h = 0.2 \Rightarrow x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2 .

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2,$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44,$$

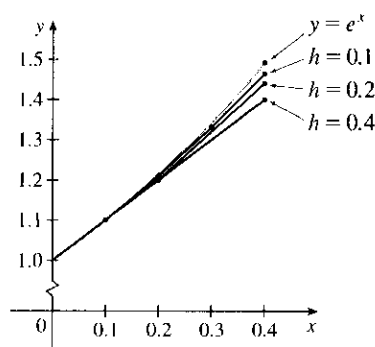
(iii) $h = 0.1 \Rightarrow x_4 = 0.4$, so we need to find y_4 . $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21,$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331,$$

$$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641.$$

(b)



We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

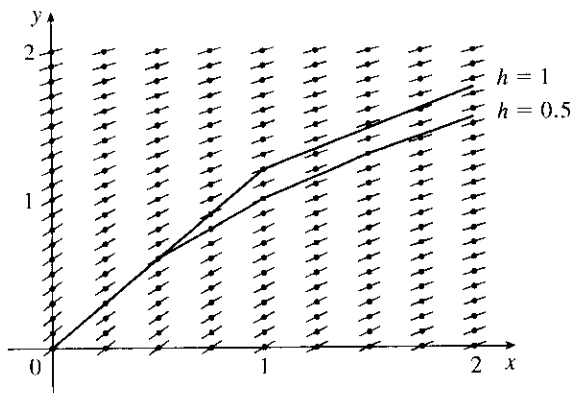
(c) (i) For $h = 0.4$: (exact value) - (approximate value) = $e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) - (approximate value) = $e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) - (approximate value) = $e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

20.



As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h = 0.5$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = y - 2x$.

Note that $x_1 = x_0 + h = 1 + 0.5 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$.

$$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.5F(1, 0) = 0.5[0 - 2(1)] = -1.$$

$$y_2 = y_1 + hF(x_1, y_1) = -1 + 0.5F(1.5, -1) = -1 + 0.5[-1 - 2(1.5)] = -3.$$

$$y_3 = y_2 + hF(x_2, y_2) = -3 + 0.5F(2, -3) = -3 + 0.5[-3 - 2(2)] = -6.5.$$

$$y_4 = y_3 + hF(x_3, y_3) = -6.5 + 0.5F(2.5, -6.5) = -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25.$$

22. $h = 0.2$, $x_0 = 0$, $y_0 = 0$, and $F(x, y) = 1 - xy$.

Note that $x_1 = x_0 + h = 0 + 0.2 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, and $x_4 = 0.8$.

$$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2F(0, 0) = 0.2[1 - (0)(0)] = 0.2.$$

$$y_2 = y_1 + hF(x_1, y_1) = 0.2 + 0.2F(0.2, 0.2) = 0.2 + 0.2[1 - (0.2)(0.2)] = 0.392.$$

$$y_3 = y_2 + hF(x_2, y_2) = 0.392 + 0.2F(0.4, 0.392) = 0.392 + 0.2[1 - (0.4)(0.392)] = 0.56064.$$

$$y_4 = y_3 + hF(x_3, y_3) = 0.56064 + 0.2[1 - (0.6)(0.56064)] = 0.6933632.$$

$$y_5 = y_4 + hF(x_4, y_4) = 0.6933632 + 0.2[1 - (0.8)(0.6933632)] = 0.782425088.$$

Thus, $y(1) \approx 0.7824$.

23. $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = y + xy$.

Note that $x_1 = x_0 + h = 0 + 0.1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, and $x_4 = 0.4$.

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1F(0, 1) = 1 + 0.1[1 + (0)(1)] = 1.1.$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1F(0.1, 1.1) = 1.1 + 0.1[1.1 + (0.1)(1.1)] = 1.221.$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.221 + 0.1F(0.2, 1.221) = 1.221 + 0.1[1.221 + (0.2)(1.221)] = 1.36752.$$

$$\begin{aligned} y_4 = y_3 + hF(x_3, y_3) &= 1.36752 + 0.1F(0.3, 1.36752) = 1.36752 + 0.1[1.36752 + (0.3)(1.36752)] \\ &= 1.5452976. \end{aligned}$$

$$\begin{aligned} y_5 = y_4 + hF(x_4, y_4) &= 1.5452976 + 0.1F(0.4, 1.5452976) \\ &= 1.5452976 + 0.1[1.5452976 + (0.4)(1.5452976)] = 1.761639264. \end{aligned}$$

Thus, $y(0.5) \approx 1.7616$.

24. (a) $h = 0.2$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = x - xy$.

We need to find y_2 , because $x_1 = 1.2$ and $x_2 = 1.4$.

$$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2F(1, 0) = 0.2[1 - (1)(0)] = 0.2.$$

$$y_2 = y_1 + hF(x_1, y_1) = 0.2 + 0.2F(1.2, 0.2) = 0.2 + 0.2[1.2 - (1.2)(0.2)] = 0.392 \approx y(1.4).$$

(b) Now $h = 0.1$, so we need to find y_4 .

$$y_1 = 0 + 0.1[1 - (1)(0)] = 0.1,$$

$$y_2 = 0.1 + 0.1[1.1 - (1.1)(0.1)] = 0.199,$$

$$y_3 = 0.199 + 0.1[1.2 - (1.2)(0.199)] = 0.29512, \text{ and}$$

$$y_4 = 0.29512 + 0.1[1.3 - (1.3)(0.29512)] = 0.3867544 \approx y(1.4).$$

25. (a) $dy/dx + 3x^2y = 6x^2 \Rightarrow y' = 6x^2 - 3x^2y$. Store this expression in Y_1 and use the following simple program to evaluate $y(1)$ for each part, using $H = h = 1$ and $N = 1$ for part (i), $H = 0.1$ and $N = 10$ for part (ii), and so forth.

$h \rightarrow H: 0 \rightarrow X: 3 \rightarrow Y:$

For(1, 1, N): $Y + H \times Y_1 \rightarrow Y: X + H \rightarrow X:$

End(loop):

Display Y. [To see all iterations, include this statement in the loop.]

(i) $H = 1, N = 1 \Rightarrow y(1) = 3$

(ii) $H = 0.1, N = 10 \Rightarrow y(1) \approx 2.3928$

(iii) $H = 0.01, N = 100 \Rightarrow y(1) \approx 2.3701$

(iv) $H = 0.001, N = 1000 \Rightarrow y(1) \approx 2.3681$

(b) $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$

$$\text{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2(2 + e^{-x^3}) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \text{RHS}$$

$$y(0) = 2 + e^{-0} = 2 + 1 = 3$$

(c) The exact value of $y(1)$ is $2 + e^{-1^3} = 2 + e^{-1}$.

(i) For $h = 1$: (exact value) - (approximate value) = $2 + e^{-1} - 3 \approx -0.6321$

(ii) For $h = 0.1$: (exact value) - (approximate value) = $2 + e^{-1} - 2.3928 \approx -0.0249$

(iii) For $h = 0.01$: (exact value) - (approximate value) = $2 + e^{-1} - 2.3701 \approx -0.0022$

(iv) For $h = 0.001$: (exact value) - (approximate value) = $2 + e^{-1} - 2.3681 \approx -0.0002$

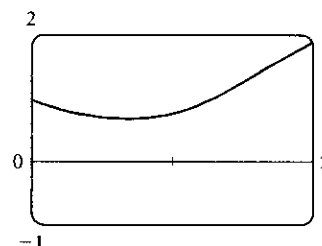
In (ii)–(iv), it seems that when the step size is divided by 10, the error estimate is also divided by 10 (approximately).

26. (a) We use the program from the solution to

Exercise 25 with $Y_1 = x^3 - y^3$, $H = 0.01$, and

$N = \frac{2-0}{0.01} = 200$. With $(x_0, y_0) = (0, 1)$, we get
 $y(2) \approx 1.9000$.

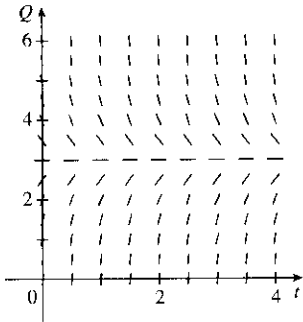
(b)



Notice from the graph that $y(2) \approx 1.9$, which serves as a check on our calculation in part (a).

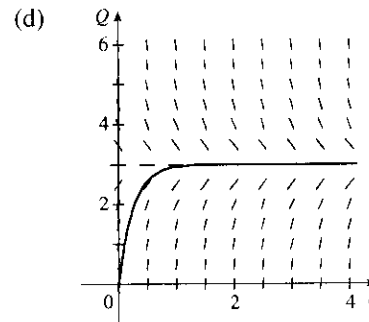
27. (a) $R \frac{dQ}{dt} + \frac{1}{C}Q = E(t)$ becomes

$$5Q' + \frac{1}{0.05}Q = 60 \text{ or } Q' + 4Q = 12.$$



(b) From the graph, it appears that the limiting value of the charge Q is about 3.

(c) If $Q' = 0$, then $4Q = 12 \Rightarrow Q = 3$ is an equilibrium solution.



(e) $Q' + 4Q = 12 \Rightarrow Q' = 12 - 4Q$. Now $Q(0) = 0$, so $t_0 = 0$ and $Q_0 = 0$.

$$Q_1 = Q_0 + hF(t_0, Q_0) = 0 + 0.1(12 - 4 \cdot 0) = 1.2$$

$$Q_2 = Q_1 + hF(t_1, Q_1) = 1.2 + 0.1(12 - 4 \cdot 1.2) = 1.92$$

$$Q_3 = Q_2 + hF(t_2, Q_2) = 1.92 + 0.1(12 - 4 \cdot 1.92) = 2.352$$

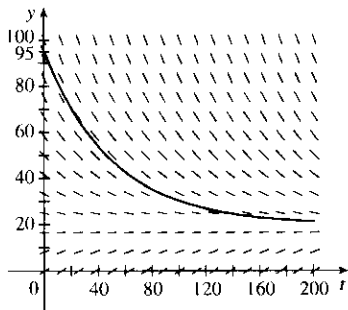
$$Q_4 = Q_3 + hF(t_3, Q_3) = 2.352 + 0.1(12 - 4 \cdot 2.352) = 2.6112$$

$$Q_5 = Q_4 + hF(t_4, Q_4) = 2.6112 + 0.1(12 - 4 \cdot 2.6112) = 2.76672$$

Thus, $Q_5 = Q(0.5) \approx 2.77$ C.

28. (a) From Exercise 10.1.14, we have $dy/dt = k(y - R)$. We are given that $R = 20^\circ\text{C}$ and $dy/dt = -1^\circ\text{C}/\text{min}$ when $y = 70^\circ\text{C}$. Thus, $-1 = k(70 - 20) \Rightarrow k = -\frac{1}{50}$ and the differential equation becomes $dy/dt = -\frac{1}{50}(y - 20)$.

(b)



The limiting value of the temperature is 20°C ; that is, the temperature of the room.

(c) From part (a), $dy/dt = -\frac{1}{50}(y - 20)$. With $t_0 = 0$, $y_0 = 95$, and $h = 2$ min, we get

$$y_1 = y_0 + hF(t_0, y_0) = 95 + 2\left[-\frac{1}{50}(95 - 20)\right] = 92$$

$$y_2 = y_1 + hF(t_1, y_1) = 92 + 2\left[-\frac{1}{50}(92 - 20)\right] = 89.12$$

$$y_3 = y_2 + hF(t_2, y_2) = 89.12 + 2\left[-\frac{1}{50}(89.12 - 20)\right] = 86.3552$$

$$y_4 = y_3 + hF(t_3, y_3) = 86.3552 + 2\left[-\frac{1}{50}(86.3552 - 20)\right] = 83.700992$$

$$y_5 = y_4 + hF(t_4, y_4) = 83.700992 + 2\left[-\frac{1}{50}(83.700992 - 20)\right] = 81.15295232$$

Thus, $y(10) \approx 81.15^\circ\text{C}$.

10.3 Separable Equations

1. $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \quad [y \neq 0] \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln |y| = \ln |x| + C \Rightarrow$
 $|y| = e^{\ln|x|+C} = e^{\ln|x|} e^C = e^C |x| \Rightarrow y = Kx$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)
2. $\frac{dy}{dx} = \frac{e^{2x}}{4y^3} \Rightarrow 4y^3 dy = e^{2x} dx \Rightarrow \int 4y^3 dy = \int e^{2x} dx \Rightarrow y^4 = \frac{1}{2}e^{2x} + C \Rightarrow$
 $y = \pm \sqrt[4]{\frac{1}{2}e^{2x} + C}$
3. $(x^2 + 1)y' = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2 + 1} \Rightarrow \frac{dy}{y} = \frac{x dx}{x^2 + 1} \quad [y \neq 0] \Rightarrow \int \frac{dy}{y} = \int \frac{x dx}{x^2 + 1} \Rightarrow$
 $\ln |y| = \frac{1}{2} \ln(x^2 + 1) + C \quad [u = x^2 + 1, du = 2x dx] = \ln(x^2 + 1)^{1/2} + \ln e^C = \ln(e^C \sqrt{x^2 + 1}) \Rightarrow$
 $|y| = e^C \sqrt{x^2 + 1} \Rightarrow y = K \sqrt{x^2 + 1}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)
4. $y' = y^2 \sin x \Rightarrow \frac{dy}{dx} = y^2 \sin x \Rightarrow \frac{dy}{y^2} = \sin x dx \quad [y \neq 0] \Rightarrow \int \frac{dy}{y^2} = \int \sin x dx \Rightarrow$
 $-\frac{1}{y} = -\cos x + C \Rightarrow \frac{1}{y} = \cos x - C \Rightarrow y = \frac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.
5. $(1 + \tan y)y' = x^2 + 1 \Rightarrow (1 + \tan y)\frac{dy}{dx} = x^2 + 1 \Rightarrow \left(1 + \frac{\sin y}{\cos y}\right) dy = (x^2 + 1) dx \Rightarrow$
 $\int \left(1 - \frac{\sin y}{\cos y}\right) dy = \int (x^2 + 1) dx \Rightarrow y - \ln |\cos y| = \frac{1}{3}x^3 + x + C$. Note: The left side is equivalent to $y + \ln |\sec y|$.
6. $\frac{du}{dr} = \frac{1 + \sqrt{r}}{1 + \sqrt{u}} \Rightarrow (1 + \sqrt{u}) du = (1 + \sqrt{r}) dr \Rightarrow \int (1 + u^{1/2}) du = \int (1 + r^{1/2}) dr \Rightarrow$
 $u + \frac{2}{3}u^{3/2} = r + \frac{2}{3}r^{3/2} + C$
7. $\frac{dy}{dt} = \frac{te^t}{y\sqrt{1+y^2}} \Rightarrow y\sqrt{1+y^2} dy = te^t dt \Rightarrow \int y\sqrt{1+y^2} dy = \int te^t dt \Rightarrow$
 $\frac{1}{3}(1+y^2)^{3/2} = te^t - e^t + C$ [where the first integral is evaluated by substitution and the second by parts] $\Rightarrow$
 $1+y^2 = [3(te^t - e^t + C)]^{2/3} \Rightarrow y = \pm \sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$
8. $y' = \frac{xy}{2 \ln y} \Rightarrow \frac{2 \ln y}{y} dy = x dx \Rightarrow \int \frac{2 \ln y}{y} dy = \int x dx \Rightarrow (\ln y)^2 = \frac{x^2}{2} + C \Rightarrow$
 $\ln y = \pm \sqrt{x^2/2 + C} \Rightarrow y = e^{\pm \sqrt{x^2/2 + C}}$

9. $\frac{du}{dt} = 2 + 2u + t + tu \Rightarrow \frac{du}{dt} = (1+u)(2+t) \Rightarrow \int \frac{du}{1+u} = \int (2+t)dt \quad [u \neq -1] \Rightarrow$
 $\ln|1+u| = \frac{1}{2}t^2 + 2t + C \Rightarrow |1+u| = e^{t^2/2+2t+C} = Ke^{t^2/2+2t}, \text{ where } K = e^C \Rightarrow$
 $1+u = \pm Ke^{t^2/2+2t} \Rightarrow u = -1 \pm Ke^{t^2/2+2t} \text{ where } K > 0. u = -1 \text{ is also a solution, so}$
 $u = -1 + Ae^{t^2/2+2t}, \text{ where } A \text{ is an arbitrary constant.}$
10. $\frac{dz}{dt} + e^t + z = 0 \Rightarrow \frac{dz}{dt} = -e^t e^z \Rightarrow \int e^{-z} dz = -\int e^t dt \Rightarrow -e^{-z} = -e^t + C \Rightarrow e^{-z} = e^t - C$
 $\Rightarrow \frac{1}{e^z} = e^t - C \Rightarrow e^z = \frac{1}{e^t - C} \Rightarrow z = \ln\left(\frac{1}{e^t - C}\right) \Rightarrow z = -\ln(e^t - C)$
11. $\frac{dy}{dx} = y^2 + 1, y(1) = 0. \int \frac{dy}{y^2 + 1} = \int dx \Rightarrow \tan^{-1} y = x + C. y = 0 \text{ when } x = 1, \text{ so}$
 $1 + C = \tan^{-1} 0 = 0 \Rightarrow C = -1. \text{ Thus, } \tan^{-1} y = x - 1 \text{ and } y = \tan(x - 1).$
12. $\frac{dy}{dx} = \frac{y \cos x}{1 + y^2}, y(0) = 1. (1 + y^2) dy = y \cos x dx \Rightarrow \frac{1 + y^2}{y} dy = \cos x dx \Rightarrow$
 $\int \left(\frac{1}{y} + y\right) dy = \int \cos x dx \Rightarrow \ln|y| + \frac{1}{2}y^2 = \sin x + C. y(0) = 1 \Rightarrow \ln 1 + \frac{1}{2} = \sin 0 + C \Rightarrow$
 $C = \frac{1}{2}, \text{ so } \ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}. \text{ We cannot solve explicitly for } y.$
13. $x \cos x = (2y + e^{3y}) y' \Rightarrow x \cos x dx = (2y + e^{3y}) dy \Rightarrow \int (2y + e^{3y}) dy = \int x \cos x dx \Rightarrow$
 $y^2 + \frac{1}{3}e^{3y} = x \sin x + \cos x + C \quad [\text{where the second integral is evaluated using integration by parts}]. \text{ Now}$
 $y(0) = 0 \Rightarrow 0 + \frac{1}{3} = 0 + 1 + C \Rightarrow C = -\frac{2}{3}. \text{ Thus, a solution is } y^2 + \frac{1}{3}e^{3y} = x \sin x + \cos x - \frac{2}{3}.$
 We cannot solve explicitly for y .
14. $\frac{dP}{dt} = \sqrt{Pt} \Rightarrow dP/\sqrt{P} = \sqrt{t} dt \Rightarrow \int P^{-1/2} dP = \int t^{1/2} dt \Rightarrow 2P^{1/2} = \frac{2}{3}t^{3/2} + C.$
 $P(1) = 2 \Rightarrow 2\sqrt{2} = \frac{2}{3} + C \Rightarrow C = 2\sqrt{2} - \frac{2}{3}, \text{ so } 2P^{1/2} = \frac{2}{3}t^{3/2} + 2\sqrt{2} - \frac{2}{3} \Rightarrow$
 $\sqrt{P} = \frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3} \Rightarrow P = \left(\frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2.$
15. $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}, u(0) = -5. \int 2u du = \int (2t + \sec^2 t) dt \Rightarrow u^2 = t^2 + \tan t + C, \text{ where}$
 $[u(0)]^2 = 0^2 + \tan 0 + C \Rightarrow C = (-5)^2 = 25. \text{ Therefore, } u^2 = t^2 + \tan t + 25, \text{ so } u = \pm\sqrt{t^2 + \tan t + 25}.$
 Since $u(0) = -5$, we must have $u = -\sqrt{t^2 + \tan t + 25}.$
16. $\frac{dy}{dt} = te^y, y(1) = 0. \int e^{-y} dy = \int t dt \Rightarrow -e^{-y} = \frac{1}{2}t^2 + C. \text{ Since } y(1) = 0, -e^0 = \frac{1}{2} \cdot 1^2 + C. \text{ Therefore,}$
 $C = -1 - \frac{1}{2} = -\frac{3}{2} \text{ and } -e^{-y} = \frac{1}{2}t^2 - \frac{3}{2}. \text{ So } e^{-y} = \frac{3}{2} - \frac{1}{2}t^2 = \frac{3-t^2}{2} \Rightarrow e^y = \frac{2}{3-t^2} \Rightarrow$
 $y = \ln 2 - \ln(3-t^2) \text{ for } |t| < \sqrt{3}.$

$$17. y' \tan x = a + y, 0 < x < \pi/2 \Rightarrow \frac{dy}{dx} = \frac{a+y}{\tan x} \Rightarrow \frac{dy}{a+y} = \cot x \, dx \quad [a+y \neq 0] \Rightarrow$$

$$\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x} \, dx \Rightarrow \ln|a+y| = \ln|\sin x| + C \Rightarrow$$

$|a+y| = e^{\ln|\sin x|+C} = e^{\ln|\sin x|} \cdot e^C = e^C |\sin x| \Rightarrow a+y = K \sin x$, where $K = \pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case $y = -a$ by allowing K to be zero.) $y(\pi/3) = a \Rightarrow$

$$a+a = K \sin\left(\frac{\pi}{3}\right) \Rightarrow 2a = K \frac{\sqrt{3}}{2} \Rightarrow K = \frac{4a}{\sqrt{3}}. \text{ Thus, } a+y = \frac{4a}{\sqrt{3}} \sin x \text{ and so } y = \frac{4a}{\sqrt{3}} \sin x - a.$$

$$18. xy' + y = y^2 \Rightarrow x \frac{dy}{dx} = y^2 - y \Rightarrow x \, dy = (y^2 - y) \, dx \Rightarrow \frac{dy}{y^2 - y} = \frac{dx}{x} \Rightarrow$$

$$\int \frac{dy}{y(y-1)} = \int \frac{dx}{x} \quad [y \neq 0, 1] \Rightarrow \int \left(\frac{1}{y-1} - \frac{1}{y} \right) dy = \int \frac{dx}{x} \Rightarrow \ln|y-1| - \ln|y| = \ln|x| + C$$

$$\Rightarrow \ln \left| \frac{y-1}{y} \right| = \ln(e^C |x|) \Rightarrow \left| \frac{y-1}{y} \right| = e^C |x| \Rightarrow \frac{y-1}{y} = Kx, \text{ where } K = \pm e^C \Rightarrow$$

$$1 - \frac{1}{y} = Kx \Rightarrow \frac{1}{y} = 1 - Kx \Rightarrow y = \frac{1}{1-Kx}. \text{ [The excluded cases, } y=0 \text{ and } y=1, \text{ are ruled out by}$$

$$\text{the initial condition } y(1) = -1.] \text{ Now } y(1) = -1 \Rightarrow -1 = \frac{1}{1-K} \Rightarrow 1-K = -1 \Rightarrow K = 2,$$

$$\text{so } y = \frac{1}{1-2x}.$$

$$19. \frac{dy}{dx} = 4x^3 y, y(0) = 7. \frac{dy}{y} = 4x^3 \, dx \text{ [if } y \neq 0] \Rightarrow \int \frac{dy}{y} = \int 4x^3 \, dx \Rightarrow \ln|y| = x^4 + C \Rightarrow$$

$$e^{\ln|y|} = e^{x^4+C} \Rightarrow |y| = e^{x^4} e^C \Rightarrow y = Ae^{x^4}; y(0) = 7 \Rightarrow A = 7 \Rightarrow y = 7e^{x^4}.$$

$$20. \frac{dy}{dx} = \frac{y^2}{x^3}, y(1) = 1. \int \frac{dy}{y^2} = \int \frac{dx}{x^3} \Rightarrow -\frac{1}{y} = -\frac{1}{2x^2} + C. y(1) = 1 \Rightarrow -1 = -\frac{1}{2} + C \Rightarrow$$

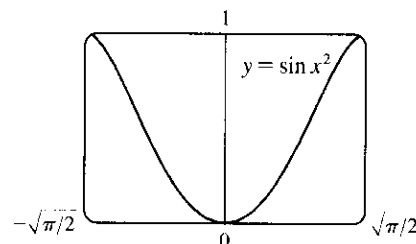
$$C = -\frac{1}{2}. \text{ So } \frac{1}{y} = \frac{1}{2x^2} + \frac{1}{2} = \frac{2+2x^2}{2 \cdot 2x^2} \Rightarrow y = \frac{2x^2}{x^2+1}.$$

$$21. (a) y' = 2x \sqrt{1-y^2} \Rightarrow \frac{dy}{dx} = 2x \sqrt{1-y^2} \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x \, dx \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int 2x \, dx \Rightarrow$$

$$\sin^{-1} y = x^2 + C \text{ for } -\frac{\pi}{2} \leq x^2 + C \leq \frac{\pi}{2}.$$

$$(b) y(0) = 0 \Rightarrow \sin^{-1} 0 = 0^2 + C \Rightarrow C = 0, \text{ so } \sin^{-1} y = x^2$$

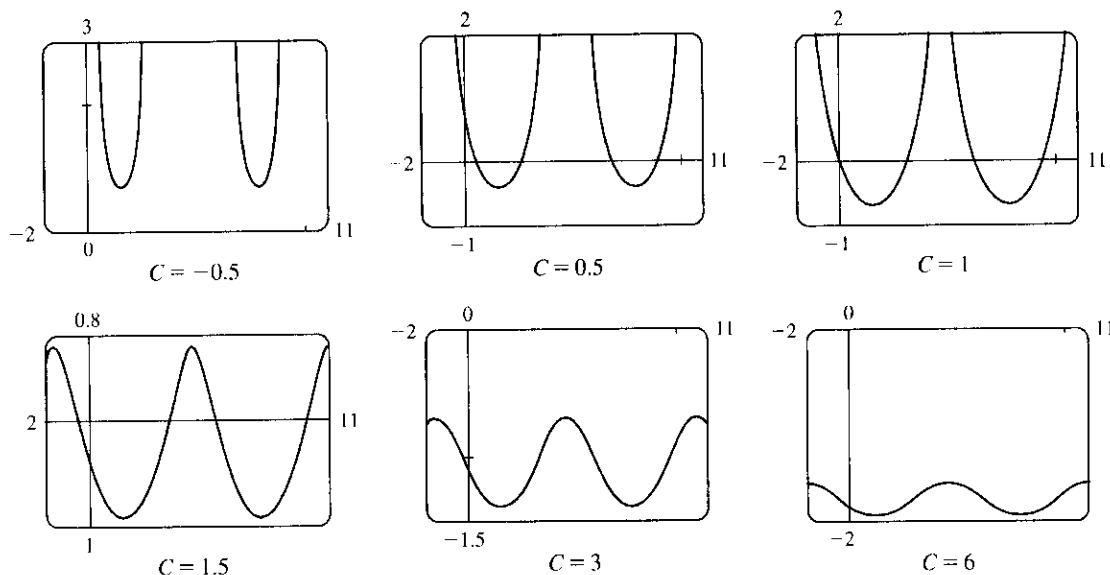
$$\text{and } y = \sin(x^2) \text{ for } -\sqrt{\pi/2} \leq x \leq \sqrt{\pi/2}.$$



(c) For $\sqrt{1-y^2}$ to be a real number, we must have $-1 \leq y \leq 1$; that is, $-1 \leq y(0) \leq 1$. Thus, the initial-value problem $y' = 2x \sqrt{1-y^2}$, $y(0) = 2$ does *not* have a solution.

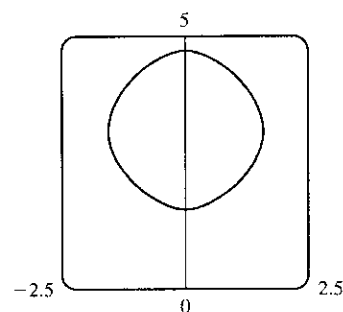
22. $e^{-y}y' + \cos x = 0 \Leftrightarrow \int e^{-y} dy = -\int \cos x dx \Leftrightarrow -e^{-y} = -\sin x + C_1 \Leftrightarrow y = -\ln(\sin x + C).$

The solution is periodic, with period 2π . Note that for $C > 1$, the domain of the solution is $\mathbb{R}$, but for $-1 < C \leq 1$ it is only defined on the intervals where $\sin x + C > 0$, and it is meaningless for $C \leq -1$, since then $\sin x + C \leq 0$, and the logarithm is undefined.



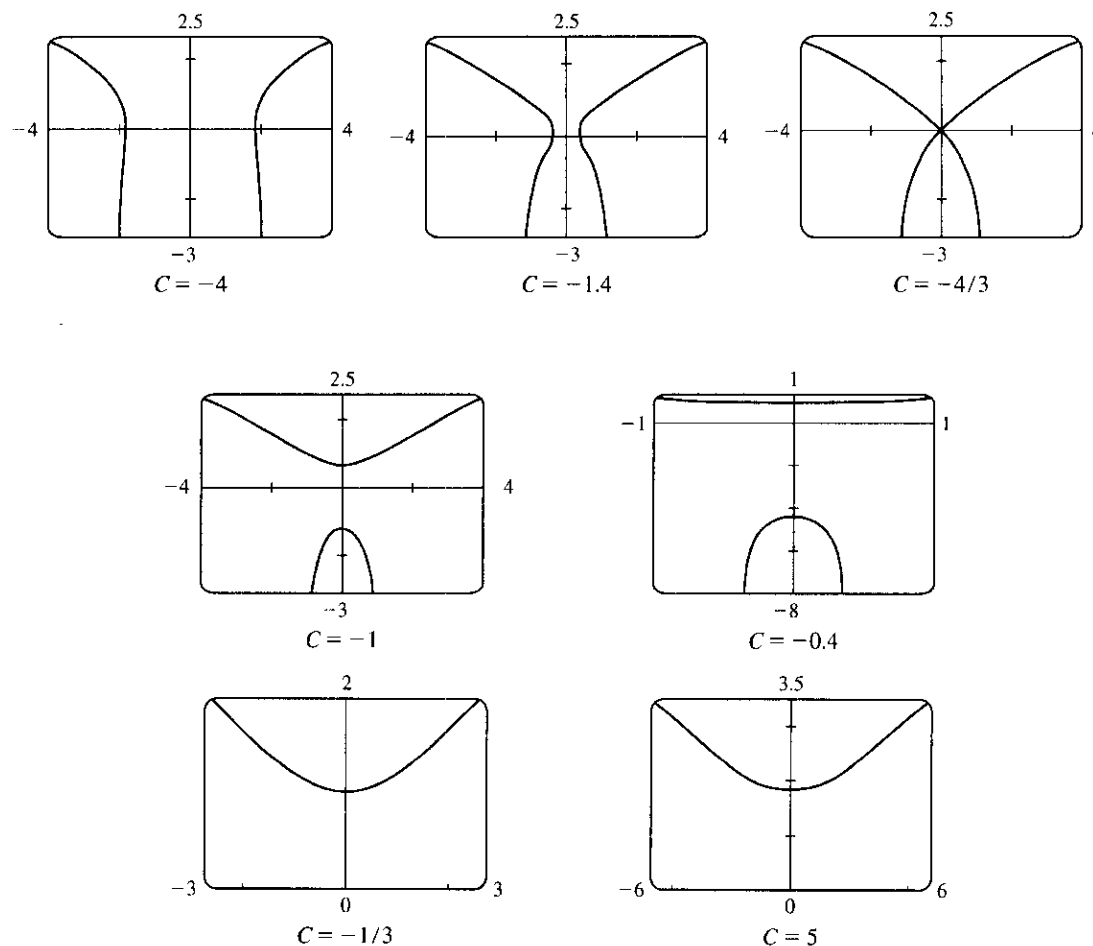
For $-1 < C < 1$, the solution curve consists of concave-up pieces separated by intervals on which the solution is not defined (where $\sin x + C \leq 0$). For $C = 1$, the solution curve consists of concave-up pieces separated by vertical asymptotes at the points where $\sin x + C = 0 \Leftrightarrow \sin x = -1$. For $C > 1$, the curve is continuous, and as C increases, the graph moves downward, and the amplitude of the oscillations decreases.

23. $\frac{dy}{dx} = \frac{\sin x}{\sin y}, y(0) = \frac{\pi}{2}$. So $\int \sin y dy = \int \sin x dx \Leftrightarrow -\cos y = -\cos x + C \Leftrightarrow \cos y = \cos x - C$. From the initial condition, we need $\cos \frac{\pi}{2} = \cos 0 - C \Rightarrow 0 = 1 - C \Rightarrow C = 1$, so the solution is $\cos y = \cos x - 1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1 \leq \cos x - 1 \Leftrightarrow 0 \leq \cos x$, as $\cos^{-1}$ is defined only on $[-1, 1]$. Instead we plot the graph using Maple's `plots[implicitplot]` or Mathematica's `Plot[Evaluate[...]]`.



24. $\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{ye^y} \Leftrightarrow \int ye^y dy = \int x\sqrt{x^2+1} dx$. We use parts on the LHS with $u = y, dv = e^y dy$, and on the RHS we use the substitution $z = x^2 + 1$, so $dz = 2x dx$. The equation becomes $ye^y - \int e^y dy = \frac{1}{2} \int \sqrt{z} dz \Leftrightarrow e^y(y-1) = \frac{1}{3}(x^2+1)^{3/2} + C$, so we see that the curves are symmetric about the y -axis. Every point (x, y) in the plane lies on one of the curves, namely the one for which $C = (y-1)e^y - \frac{1}{3}(x^2+1)^{3/2}$. For example,

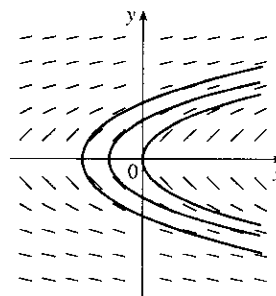
along the y -axis, $C = (y - 1)e^y - \frac{1}{3}$, so the origin lies on the curve with $C = -\frac{4}{3}$. We use Maple's `plots[implicitplot]` command or `Plot[Evaluate[...]]` in Mathematica to plot the solution curves for various values of C .



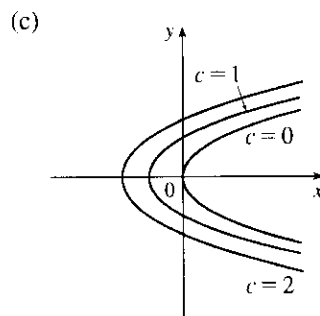
It seems that the transitional values of C are $-\frac{4}{3}$ and $-\frac{1}{3}$. For $C < -\frac{4}{3}$, the graph consists of left and right branches. At $C = -\frac{4}{3}$, the two branches become connected at the origin, and as C increases, the graph splits into top and bottom branches. At $C = -\frac{1}{3}$, the bottom half disappears. As C increases further, the graph moves upward, but doesn't change shape much.

25. (a)

x	y	$y' = 1/y$	x	y	$y' = 1/y$
0	0.5	2	0	-2	-0.5
0	-0.5	-2	0	4	0.25
0	1	1	0	3	$0.\bar{3}$
0	-1	-1	0	0.25	4
0	2	0.5	0	$0.\bar{3}$	3

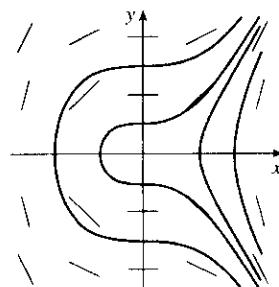


(b) $y' = 1/y \Rightarrow dy/dx = 1/y \Rightarrow$
 $y dy = dx \Rightarrow \int y dy = \int dx \Rightarrow$
 $\frac{1}{2}y^2 = x + c \Rightarrow y^2 = 2(x + c)$
 or $y = \pm\sqrt{2(x + c)}.$

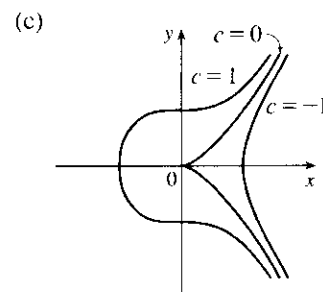


26. (a)

x	y	$y' = x^2/y$
1	1	1
-1	1	1
-1	-1	-1
1	-1	-1
1	2	0.5
2	1	4
2	2	2
1	0.5	2
0.5	1	0.25
2	0.5	8

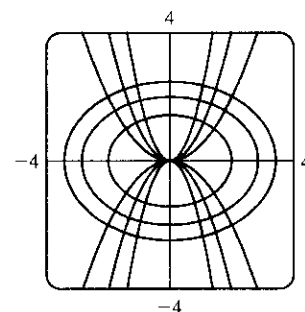


(b) $y' = x^2/y \Rightarrow y dy = x^2 dx,$
 so $\frac{1}{2}y^2 = \frac{1}{3}x^3 + c_1$, or
 $y = \pm(\frac{2}{3}x^3 + c)^{1/2}.$

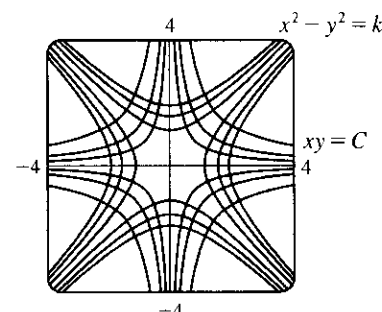


27. The curves $y = kx^2$ form a family of parabolas with axis the y -axis.

Differentiating gives $y' = 2kx$, but $k = y/x^2$, so $y' = 2y/x$. Thus, the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = 2y/x$, so the orthogonal trajectories must satisfy $y' = -x/(2y)$
 $\Leftrightarrow 2y dy = -x dx \Leftrightarrow y^2 = -x^2/2 + C_1 \Leftrightarrow x^2 + 2y^2 = C.$
 This is a family of ellipses.



28. The curves $x^2 - y^2 = k$ form a family of hyperbolas. Differentiating gives $2x - 2y(dy/dx) = 0$ or $y' = x/y$, the slope of the tangent line at (x, y) on one of the hyperbolas. Thus, the orthogonal trajectories must satisfy $y' = -y/x \Leftrightarrow dy/y = -dx/x \Leftrightarrow \ln|y| = -\ln|x| + C_1 \Leftrightarrow \ln|x| + \ln|y| = C_1 \Leftrightarrow \ln|xy| = C_1 \Leftrightarrow |xy| = e^{C_1} \Leftrightarrow xy = C.$ This is a family of hyperbolas.

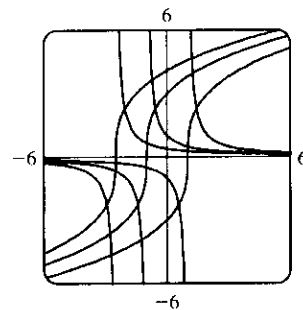


29. Differentiating $y = (x + k)^{-1}$ gives $y' = -\frac{1}{(x + k)^2}$, but $k = \frac{1}{y} - x$, so

$$y' = -\frac{1}{(1/y)^2} = -y^2. \text{ Thus, the orthogonal trajectories must satisfy}$$

$$y' = -\frac{1}{-y^2} = \frac{1}{y^2} \Leftrightarrow y^2 dy = dx \Leftrightarrow \frac{y^3}{3} = x + C \text{ or}$$

$$y = [3(x + C)]^{1/3}$$

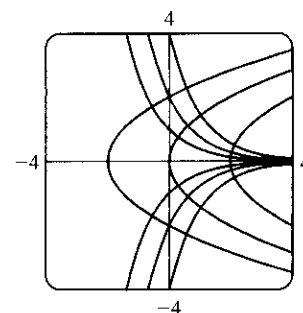


30. Differentiating $y = ke^{-x}$ gives $y' = -ke^{-x}$, but $k = ye^x$, so $y' = -y$.

Thus, the orthogonal trajectories must satisfy $y' = -1/(-y) = 1/y \Leftrightarrow$

$$y dy = dx \Leftrightarrow \frac{1}{2}y^2 = x + C \Leftrightarrow y = \pm [2(C + x)]^{1/2}. \text{ This is a}$$

family of parabolas with axis the x -axis.



31. From Exercise 10.2.27, $\frac{dQ}{dt} = 12 - 4Q \Leftrightarrow \int \frac{dQ}{12 - 4Q} = \int dt \Leftrightarrow -\frac{1}{4} \ln|12 - 4Q| = t + C \Leftrightarrow$

$$\ln|12 - 4Q| = -4t - 4C \Leftrightarrow |12 - 4Q| = e^{-4t-4C} \Leftrightarrow 12 - 4Q = Ke^{-4t} [K = \pm e^{-4C}] \Leftrightarrow$$

$$4Q = 12 - Ke^{-4t} \Leftrightarrow Q = 3 - Ae^{-4t} [A = K/4]. Q(0) = 0 \Leftrightarrow 0 = 3 - A \Leftrightarrow A = 3 \Leftrightarrow$$

$$Q(t) = 3 - 3e^{-4t}. \text{ As } t \rightarrow \infty, Q(t) \rightarrow 3 - 0 = 3 \text{ (the limiting value).}$$

32. From Exercise 10.2.28, $\frac{dy}{dt} = -\frac{1}{50}(y - 20) \Leftrightarrow \int \frac{dy}{y - 20} = \int (-\frac{1}{50}) dt \Leftrightarrow \ln|y - 20| = -\frac{1}{50}t + C \Leftrightarrow$

$$y - 20 = Ke^{-t/50} \Leftrightarrow y(t) = Ke^{-t/50} + 20. y(0) = 95 \Leftrightarrow 95 = K + 20 \Leftrightarrow K = 75 \Leftrightarrow$$

$$y(t) = 75e^{-t/50} + 20.$$

33. $\frac{dP}{dt} = k(M - P) \Leftrightarrow \int \frac{dP}{P - M} = \int (-k) dt \Leftrightarrow \ln|P - M| = -kt + C \Leftrightarrow |P - M| = e^{-kt+C}$

$$\Leftrightarrow P - M = Ae^{-kt} [A = \pm e^C] \Leftrightarrow P = M + Ae^{-kt}. \text{ If we assume that performance is at level 0 when}$$

$$t = 0, \text{ then } P(0) = 0 \Leftrightarrow 0 = M + A \Leftrightarrow A = -M \Leftrightarrow P(t) = M - Me^{-kt}.$$

$$\lim_{t \rightarrow \infty} P(t) = M - M \cdot 0 = M.$$

34. (a) $\frac{dx}{dt} = k(a - x)(b - x), a \neq b$. Using partial fractions, $\frac{1}{(a - x)(b - x)} = \frac{1/(b - a)}{a - x} - \frac{1/(b - a)}{b - x}$, so

$$\int \frac{dx}{(a - x)(b - x)} = \int k dt \Rightarrow \frac{1}{b - a}(-\ln|a - x| + \ln|b - x|) = kt + C \Rightarrow$$

$$\ln \left| \frac{b - x}{a - x} \right| = (b - a)(kt + C). \text{ The concentrations } [A] = a - x \text{ and } [B] = b - x \text{ cannot be negative, so}$$

$$\frac{b - x}{a - x} \geq 0 \text{ and } \left| \frac{b - x}{a - x} \right| = \frac{b - x}{a - x}. \text{ We now have } \ln \left(\frac{b - x}{a - x} \right) = (b - a)(kt + C). \text{ Since } x(0) = 0, \text{ we get}$$

$$\ln\left(\frac{b}{a}\right) = (b-a)C. \text{ Hence, } \ln\left(\frac{b-x}{a-x}\right) = (b-a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt} \Rightarrow$$

$$x = \frac{b[e^{(b-a)kt} - 1]}{be^{(b-a)kt}/a - 1} = \frac{ab[e^{(b-a)kt} - 1]}{be^{(b-a)kt} - a} \text{ moles/L.}$$

(b) If $b = a$, then $\frac{dx}{dt} = k(a-x)^2$, so $\int \frac{dx}{(a-x)^2} = \int k dt$ and $\frac{1}{a-x} = kt + C$. Since $x(0) = 0$, we get

$$C = \frac{1}{a}. \text{ Thus, } a-x = \frac{1}{kt + 1/a} \text{ and } x = a - \frac{a}{akt + 1} = \frac{a^2 kt}{akt + 1} \text{ moles/L.}$$

$$\text{Suppose } x = [C] = a/2 \text{ when } t = 20. \text{ Then } x(20) = a/2 \Rightarrow \frac{a}{2} = \frac{20a^2 k}{20ak + 1} \Rightarrow 40a^2 k = 20a^2 k + a$$

$$\Rightarrow 20a^2 k = a \Rightarrow k = \frac{1}{20a}, \text{ so } x = \frac{a^2 t / (20a)}{1 + at / (20a)} = \frac{at/20}{1 + t/20} = \frac{at}{t+20} \text{ moles/L.}$$

35. (a) If $a = b$, then $\frac{dx}{dt} = k(a-x)(b-x)^{1/2}$ becomes $\frac{dx}{dt} = k(a-x)^{3/2} \Rightarrow (a-x)^{-3/2} dx = k dt \Rightarrow$

$$\int (a-x)^{-3/2} dx = \int k dt \Rightarrow 2(a-x)^{-1/2} = kt + C \text{ [by substitution]} \Rightarrow \frac{2}{kt + C} = \sqrt{a-x} \Rightarrow$$

$$\left(\frac{2}{kt + C}\right)^2 = a-x \Rightarrow x(t) = a - \frac{4}{(kt + C)^2}. \text{ The initial concentration of HBr is 0, so } x(0) = 0 \Rightarrow$$

$$0 = a - \frac{4}{C^2} \Rightarrow \frac{4}{C^2} = a \Rightarrow C^2 = \frac{4}{a} \Rightarrow C = 2/\sqrt{a} \text{ (C is positive since}$$

$$kt + C = 2(a-x)^{-1/2} > 0). \text{ Thus, } x(t) = a - \frac{4}{(kt + 2/\sqrt{a})^2}.$$

(b) $\frac{dx}{dt} = k(a-x)(b-x)^{1/2} \Rightarrow \frac{dx}{(a-x)\sqrt{b-x}} = k dt \Rightarrow \int \frac{dx}{(a-x)\sqrt{b-x}} = \int k dt \text{ (*)}. \text{ From the}$

$$\text{hint, } u = \sqrt{b-x} \Rightarrow u^2 = b-x \Rightarrow 2u du = -dx, \text{ so } \int \frac{dx}{(a-x)\sqrt{b-x}} = \int \frac{-2u du}{[a - (b-u^2)]u} =$$

$$-2 \int \frac{du}{a-b+u^2} = -2 \int \frac{du}{(\sqrt{a-b})^2 + u^2} \stackrel{17}{=} -2 \left(\frac{1}{\sqrt{a-b}} \tan^{-1} \frac{u}{\sqrt{a-b}} \right). \text{ So (*) becomes}$$

$$\frac{-2}{\sqrt{a-b}} \tan^{-1} \frac{\sqrt{b-x}}{\sqrt{a-b}} = kt + C. \text{ Now } x(0) = 0 \Rightarrow C = \frac{-2}{\sqrt{a-b}} \tan^{-1} \frac{\sqrt{b}}{\sqrt{a-b}}$$

$$\text{and we have } \frac{-2}{\sqrt{a-b}} \tan^{-1} \frac{\sqrt{b-x}}{\sqrt{a-b}} = kt - \frac{2}{\sqrt{a-b}} \tan^{-1} \frac{\sqrt{b}}{\sqrt{a-b}} \Rightarrow$$

$$\frac{2}{\sqrt{a-b}} \left(\tan^{-1} \sqrt{\frac{b}{a-b}} - \tan^{-1} \sqrt{\frac{b-x}{a-b}} \right) = kt \Rightarrow$$

$$t(x) = \frac{2}{k\sqrt{a-b}} \left(\tan^{-1} \sqrt{\frac{b}{a-b}} - \tan^{-1} \sqrt{\frac{b-x}{a-b}} \right).$$

36. If $S = \frac{dT}{dr}$, then $\frac{dS}{dr} = \frac{d^2 T}{dr^2}$. The differential equation $\frac{d^2 T}{dr^2} + \frac{2}{r} \frac{dT}{dr} = 0$ can be written as $\frac{dS}{dr} + \frac{2}{r} S = 0$. Thus,

$$\frac{dS}{dr} = -\frac{2S}{r} \Rightarrow \frac{dS}{S} = -\frac{2}{r} dr \Rightarrow \int \frac{1}{S} dS = \int -\frac{2}{r} dr \Rightarrow \ln|S| = -2 \ln|r| + C. \text{ Assuming}$$

$$S = dT/dr > 0 \text{ and } r > 0, \text{ we have } S = e^{-2 \ln r + C} = e^{\ln r^{-2}} e^C = r^{-2} k \text{ [} k = e^C \text{]} \Rightarrow S = \frac{1}{r^2} k \Rightarrow$$

$$\frac{dT}{dr} = \frac{1}{r^2} k \Rightarrow dT = \frac{1}{r^2} k dr \Rightarrow \int dT = \int \frac{1}{r^2} k dr \Rightarrow T(r) = -\frac{k}{r} + A.$$

$$T(1) = 15 \Rightarrow 15 = -k + A \quad (1) \text{ and } T(2) = 25 \Rightarrow 25 = -\frac{1}{2}k + A \quad (2).$$

Now solve for k and A : $-2(2) + (1) \Rightarrow -35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.

$$\begin{aligned} 37. (a) \frac{dC}{dt} &= r - kC \Rightarrow \frac{dC}{dt} = -(kC - r) \Rightarrow \int \frac{dC}{kC - r} = \int -dt \Rightarrow (1/k) \ln|kC - r| = -t + M_1 \\ &\Rightarrow \ln|kC - r| = -kt + M_2 \Rightarrow |kC - r| = e^{-kt+M_2} \Rightarrow kC - r = M_3 e^{-kt} \Rightarrow \\ kC &= M_3 e^{-kt} + r \Rightarrow C(t) = M_4 e^{-kt} + r/k. C(0) = C_0 \Rightarrow C_0 = M_4 + r/k \Rightarrow \\ M_4 &= C_0 - r/k \Rightarrow C(t) = (C_0 - r/k)e^{-kt} + r/k. \end{aligned}$$

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t \rightarrow \infty} C(t) = r/k$.

As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

38. (a) Use 1 billion dollars as the x -unit and 1 day as the t -unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t , the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is

$\frac{10 - x(t)}{10} \cdot 0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so

$$\frac{dx}{dt} = \frac{10 - x}{10} \cdot 0.05 = 0.005(10 - x) \text{ billion dollars per day.}$$

$$(b) \frac{dx}{10 - x} = 0.005 dt \Rightarrow \frac{-dx}{10 - x} = -0.005 dt \Rightarrow \ln(10 - x) = -0.005t + c \Rightarrow$$

$$10 - x = Ce^{-0.005t}, \text{ where } C = e^c \Rightarrow x(t) = 10 - Ce^{-0.005t}. \text{ From } x(0) = 0, \text{ we get } C = 10, \text{ so}$$

$$x(t) = 10(1 - e^{-0.005t}).$$

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$$9 = 10(1 - e^{-0.005t}) \Rightarrow 0.9 = 1 - e^{-0.005t} \Rightarrow e^{-0.005t} = 0.1 \Rightarrow -0.005t = -\ln 10 \Rightarrow$$

$$t = 200 \ln 10 \approx 460.517 \text{ days} \approx 1.26 \text{ years.}$$

39. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and

$$\frac{dy}{dt} = -\left[\frac{y(t)}{1000} \frac{\text{kg}}{\text{L}}\right]\left(10 \frac{\text{L}}{\text{min}}\right) + \left(0.04 \frac{\text{kg}}{\text{L}}\right)\left(10 \frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000} \frac{\text{kg}}{\text{L}}\right)\left(15 \frac{\text{L}}{\text{min}}\right)$$

$$\frac{dy}{dt} = -\frac{y(t)}{100} + \frac{4}{100} - \frac{3y(t)}{200} = \frac{130 - 3y}{200}$$

$$y(0) = 15 \Rightarrow \ln 15 = C, \text{ so } \ln y = \ln 15 - \frac{t}{100}. \text{ It follows that } \ln\left(\frac{y}{15}\right) = -\frac{t}{100} \text{ and } \frac{y}{15} = e^{-t/100}, \text{ so}$$

$$y = 15e^{-t/100} \text{ kg.}$$

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

40. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\begin{aligned} \frac{dy}{dt} &= \left(0.05 \frac{\text{kg}}{\text{L}}\right)\left(5 \frac{\text{L}}{\text{min}}\right) + \left(0.04 \frac{\text{kg}}{\text{L}}\right)\left(10 \frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000} \frac{\text{kg}}{\text{L}}\right)\left(15 \frac{\text{L}}{\text{min}}\right) \\ &= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200} \frac{\text{kg}}{\text{min}} \end{aligned}$$

$$\text{so } \int \frac{dy}{130 - 3y} = \int \frac{dt}{200} \text{ and } -\frac{1}{3} \ln|130 - 3y| = \frac{1}{200}t + C; \text{ since } y(0) = 0, \text{ we have } -\frac{1}{3} \ln 130 = C,$$

so $-\frac{1}{3} \ln|130 - 3y| = \frac{1}{200}t - \frac{1}{3} \ln 130 \Rightarrow \ln|130 - 3y| = -\frac{3}{200}t + \ln 130 = \ln(130e^{-3t/200})$, and $|130 - 3y| = 130e^{-3t/200}$. Since y is continuous, $y(0) = 0$, and the right-hand side is never zero, we deduce that $130 - 3y$ is always positive. Thus, $130 - 3y = 130e^{-3t/200}$ and $y = \frac{130}{3}(1 - e^{-3t/200})$ kg.

(b) After one hour, $y = \frac{130}{3}(1 - e^{-3 \cdot 60/200}) = \frac{130}{3}(1 - e^{-0.9}) \approx 25.7$ kg.

Note: As $t \rightarrow \infty$, $y(t) \rightarrow \frac{130}{3} = 43\frac{1}{3}$ kg.

41. Assume that the raindrop begins at rest, so that $v(0) = 0$. $dm/dt = km$ and $(mv)' = gm \Rightarrow mv' + vm' = gm \Rightarrow mv' + v(km) = gm \Rightarrow v' + vk = g \Rightarrow dv/dt = g - kv \Rightarrow \int \frac{dv}{g - kv} = \int dt \Rightarrow -(1/k) \ln|g - kv| = t + C \Rightarrow \ln|g - kv| = -kt - kC \Rightarrow g - kv = Ae^{-kt}$. $v(0) = 0 \Rightarrow A = g$. So $kv = g - ge^{-kt} \Rightarrow v = (g/k)(1 - e^{-kt})$. Since $k > 0$, as $t \rightarrow \infty$, $e^{-kt} \rightarrow 0$ and therefore, $\lim_{t \rightarrow \infty} v(t) = g/k$.

42. (a) $m \frac{dv}{dt} = -kv \Rightarrow \frac{dv}{v} = -\frac{k}{m} dt \Rightarrow \ln|v| = -\frac{k}{m}t + C$. Since $v(0) = v_0$, $\ln|v_0| = C$. Therefore, $\ln\left|\frac{v}{v_0}\right| = -\frac{k}{m}t \Rightarrow \left|\frac{v}{v_0}\right| = e^{-kt/m} \Rightarrow v(t) = \pm v_0 e^{-kt/m}$. The sign is + when $t = 0$, and we assume v is continuous, so that the sign is + for all t . Thus, $v(t) = v_0 e^{-kt/m}$. $ds/dt = v_0 e^{-kt/m} \Rightarrow s(t) = -\frac{mv_0}{k} e^{-kt/m} + C'$. From $s(0) = s_0$, we get $s_0 = -\frac{mv_0}{k} + C'$, so $C' = s_0 + \frac{mv_0}{k}$ and $s(t) = s_0 + \frac{mv_0}{k}(1 - e^{-kt/m})$. The distance traveled from time 0 to time t is $s(t) - s_0$, so the total distance traveled is $\lim_{t \rightarrow \infty} [s(t) - s_0] = \frac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k > 0$ to conclude that $\lim_{t \rightarrow \infty} e^{-kt/m} = 0$.

(b) $m \frac{dv}{dt} = -kv^2 \Rightarrow \frac{dv}{v^2} = -\frac{k}{m} dt \Rightarrow \frac{-1}{v} = -\frac{kt}{m} + C \Rightarrow \frac{1}{v} = \frac{kt}{m} - C$. Since $v(0) = v_0$, $C = -\frac{1}{v_0}$

and $\frac{1}{v} = \frac{kt}{m} + \frac{1}{v_0}$. Therefore, $v(t) = \frac{1}{kt/m + 1/v_0} = \frac{mv_0}{kv_0 t + m}$. $\frac{ds}{dt} = \frac{mv_0}{kv_0 t + m} \Rightarrow$

$s(t) = \frac{m}{k} \int \frac{kv_0 dt}{kv_0 t + m} = \frac{m}{k} \ln|kv_0 t + m| + C'$. Since $s(0) = s_0$, we get $s_0 = \frac{m}{k} \ln m + C' \Rightarrow$

$C' = s_0 - \frac{m}{k} \ln m \Rightarrow s(t) = s_0 + \frac{m}{k} (\ln|kv_0 t + m| - \ln m) = s_0 + \frac{m}{k} \ln \left| \frac{kv_0 t + m}{m} \right|$. We can rewrite

the formulas for $v(t)$ and $s(t)$ as $v(t) = \frac{v_0}{1 + (kv_0/m)t}$ and $s(t) = s_0 + \frac{m}{k} \ln \left| 1 + \frac{kv_0}{m}t \right|$.

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which $v_0 > 0$. Then the term $-kv^2$ representing the resisting force causes the object to decelerate. The absolute value in the expression for $s(t)$ is unnecessary (since k , v_0 , and m are all positive), and $\lim_{t \rightarrow \infty} s(t) = \infty$. In other

words, the object travels infinitely far. However, $\lim_{t \rightarrow \infty} v(t) = 0$. When $v_0 < 0$, the term $-kv^2$ increases the

magnitude of the object's negative velocity. According to the formula for $s(t)$, the position of the object

approaches $-\infty$ as t approaches $m/k(-v_0)$: $\lim_{t \rightarrow -m/(kv_0)} s(t) = -\infty$. Again the object travels infinitely far,

but this time the feat is accomplished in a finite amount of time. Notice also that $\lim_{t \rightarrow -m/(kv_0)} v(t) = -\infty$ when $v_0 < 0$, showing that the speed of the object increases without limit.

43. (a) The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k , $dA/dt = k\sqrt{A}(M - A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for dA/dt from the differential equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{dA}{dt}\right) &= k\left[\sqrt{A}(-1)\frac{dA}{dt} + (M - A) \cdot \frac{1}{2}A^{-1/2}\frac{dA}{dt}\right] = \frac{1}{2}kA^{-1/2}\frac{dA}{dt}[-2A + (M - A)] \\ &= \frac{1}{2}kA^{-1/2}\left[k\sqrt{A}(M - A)\right][M - 3A] = \frac{1}{2}k^2(M - A)(M - 3A)\end{aligned}$$

This is 0 when $M - A = 0$ [this situation never actually occurs, since the graph of $A(t)$ is asymptotic to the line $y = M$, as in the logistic model] and when $M - 3A = 0 \Leftrightarrow A(t) = M/3$. This represents a maximum by the First Derivative Test, since $\frac{d}{dt}\left(\frac{dA}{dt}\right)$ goes from positive to negative when $A(t) = M/3$.

- (b) From the CAS, we get $A(t) = M\left(\frac{Ce^{\sqrt{M}kt} - 1}{Ce^{\sqrt{M}kt} + 1}\right)^2$. To get C in terms of the initial area A_0 and the maximum

area M , we substitute $t = 0$ and $A = A_0 = A(0)$: $A_0 = M\left(\frac{C - 1}{C + 1}\right)^2 \Leftrightarrow (C + 1)\sqrt{A_0} = (C - 1)\sqrt{M}$

$$\Leftrightarrow C\sqrt{A_0} + \sqrt{A_0} = C\sqrt{M} - \sqrt{M} \Leftrightarrow \sqrt{M} + \sqrt{A_0} = C\sqrt{M} - C\sqrt{A_0} \Leftrightarrow$$

$$\sqrt{M} + \sqrt{A_0} = C(\sqrt{M} - \sqrt{A_0}) \Leftrightarrow C = \frac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}. \text{ (Notice that if } A_0 = 0, \text{ then } C = 1.)$$

44. (a) According to the hint we use the Chain Rule: $m\frac{dv}{dt} = m\frac{dv}{dx} \cdot \frac{dx}{dt} = mv\frac{dv}{dx} = -\frac{mgR^2}{(x + R)^2} \Rightarrow$

$$\int v dv = \int \frac{-gR^2 dx}{(x + R)^2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{x + R} + C. \text{ When } x = 0, v = v_0, \text{ so } \frac{v_0^2}{2} = \frac{gR^2}{0 + R} + C \Rightarrow$$

$$C = \frac{1}{2}v_0^2 - gR \Rightarrow \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \frac{gR^2}{x + R} - gR. \text{ Now at the top of its flight, the rocket's velocity will be 0,}$$

$$\text{and its height will be } x = h. \text{ Solving for } v_0: -\frac{1}{2}v_0^2 = \frac{gR^2}{h + R} - gR \Rightarrow$$

$$\frac{v_0^2}{2} = g\left[-\frac{R^2}{R + h} + \frac{R(R + h)}{R + h}\right] = \frac{gRh}{R + h} \Rightarrow v_0 = \sqrt{\frac{2gRh}{R + h}}.$$

$$(b) v_e = \lim_{h \rightarrow \infty} v_0 = \lim_{h \rightarrow \infty} \sqrt{\frac{2gRh}{R + h}} = \lim_{h \rightarrow \infty} \sqrt{\frac{2gR}{(R/h) + 1}} = \sqrt{2gR}$$

$$(c) v_e = \sqrt{2 \cdot 32 \text{ ft/s}^2 \cdot 3960 \text{ mi} \cdot 5280 \text{ ft/mi}} \approx 36,581 \text{ ft/s} \approx 6.93 \text{ mi/s}$$

APPLIED PROJECT How Fast Does a Tank Drain?

1. (a) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$ [implicit differentiation] $\Rightarrow$

$$\frac{dh}{dt} = \frac{1}{\pi r^2} \frac{dV}{dt} = \frac{1}{\pi r^2} \left(-a\sqrt{2gh} \right) = \frac{1}{\pi 2^2} \left[-\pi \left(\frac{1}{12} \right)^2 \sqrt{2 \cdot 32} \sqrt{h} \right] = -\frac{1}{72} \sqrt{h}$$

(b) $\frac{dh}{dt} = -\frac{1}{72} \sqrt{h} \Rightarrow h^{-1/2} dh = -\frac{1}{72} dt \Rightarrow 2\sqrt{h} = -\frac{1}{72}t + C.$

$$h(0) = 6 \Rightarrow 2\sqrt{6} = 0 + C \Rightarrow C = 2\sqrt{6} \Rightarrow h(t) = \left(-\frac{1}{144}t + \sqrt{6} \right)^2.$$

(c) We want to find t when $h = 0$, so we set $h = 0 = \left(-\frac{1}{144}t + \sqrt{6} \right)^2 \Rightarrow t = 144\sqrt{6} \approx 5 \text{ min } 53 \text{ s}.$

2. (a) $\frac{dh}{dt} = k\sqrt{h} \Rightarrow h^{-1/2} dh = k dt$ [$h \neq 0$] $\Rightarrow 2\sqrt{h} = kt + C$

$$\Rightarrow h(t) = \frac{1}{4}(kt + C)^2. \text{ Since } h(0) = 10 \text{ cm, the relation}$$

$$2\sqrt{h(t)} = kt + C \text{ gives us } 2\sqrt{10} = C. \text{ Also, } h(68) = 3 \text{ cm,}$$

$$\text{so } 2\sqrt{3} = 68k + 2\sqrt{10} \text{ and } k = -\frac{\sqrt{10} - \sqrt{3}}{34}. \text{ Thus,}$$

$$h(t) = \frac{1}{4} \left(2\sqrt{10} - \frac{\sqrt{10} - \sqrt{3}}{34}t \right)^2 \approx 10 - 0.133t + 0.00044t^2.$$

Here is a table of values of $h(t)$ correct to one decimal place.

t (in s)	$h(t)$ (in cm)
10	8.7
20	7.5
30	6.4
40	5.4
50	4.5
60	3.6

(b) The answers to this part are to be obtained experimentally. See the article by Tom Farmer and Fred Gass,

Physical Demonstrations in the Calculus Classroom, College Mathematics Journal 1992, pp. 146–148.

3. $V(t) = \pi r^2 h(t) = 100\pi h(t) \Rightarrow \frac{dV}{dh} = 100\pi$ and $\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = 100\pi \frac{dh}{dt}.$

$$\text{Diameter} = 2.5 \text{ inches} \Rightarrow \text{radius} = 1.25 \text{ inches} = \frac{5}{4} \cdot \frac{1}{12} \text{ foot} = \frac{5}{48} \text{ foot. Thus,}$$

$$\frac{dV}{dt} = -a\sqrt{2gh} \Rightarrow 100\pi \frac{dh}{dt} = -\pi \left(\frac{5}{48} \right)^2 \sqrt{2 \cdot 32} \sqrt{h} = -\frac{25\pi}{288} \sqrt{h} \Rightarrow \frac{dh}{dt} = -\frac{\sqrt{h}}{1152} \Rightarrow$$

$$\int h^{-1/2} dh = \int -\frac{1}{1152} dt \Rightarrow 2\sqrt{h} = -\frac{1}{1152}t + C \Rightarrow \sqrt{h} = -\frac{1}{2304}t + k \Rightarrow h(t) = \left(-\frac{1}{2304}t + k \right)^2.$$

The water pressure after t seconds is $62.5h(t)$ lb/ft², so the condition that the pressure be at least 2160 lb/ft²

for 10 minutes (600 seconds) is the condition $62.5 \cdot h(600) \geq 2160$; that is, $\left(k - \frac{600}{2304} \right)^2 \geq \frac{2160}{62.5} \Rightarrow$

$$\left| k - \frac{25}{96} \right| \geq \sqrt{34.56} \Rightarrow k \geq \frac{25}{96} + \sqrt{34.56}. \text{ Now } h(0) = k^2, \text{ so the height of the tank should be at least}$$

$$\left(\frac{25}{96} + \sqrt{34.56} \right)^2 \approx 37.69 \text{ ft.}$$

4. (a) If the radius of the circular cross-section at height h is r , then the Pythagorean Theorem gives

$$r^2 = 2^2 - (2 - h)^2 \text{ since the radius of the tank is 2 m. So } A(h) = \pi r^2 = \pi [4 - (2 - h)^2] = \pi(4h - h^2).$$

$$\text{Thus, } A(h) \frac{dh}{dt} = -a\sqrt{2gh} \Rightarrow \pi(4h - h^2) \frac{dh}{dt} = -\pi(0.01)^2 \sqrt{2 \cdot 10} \sqrt{h} \Rightarrow$$

$$(4h - h^2) \frac{dh}{dt} = -0.0001\sqrt{20} \sqrt{h}.$$

(b) From part (a) we have $(4h^{1/2} - h^{3/2})dh = (-0.0001\sqrt{20})dt \Rightarrow$

$$\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} = (-0.0001\sqrt{20})t + C. \quad h(0) = 2 \Rightarrow \frac{8}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = C \Rightarrow$$

$C = (\frac{16}{3} - \frac{8}{5})\sqrt{2} = \frac{56}{15}\sqrt{2}$. To find out how long it will take to drain all the water we evaluate t when $h = 0$:

$$0 = (-0.0001\sqrt{20})t + C \Rightarrow$$

$$t = \frac{C}{0.0001\sqrt{20}} = \frac{56\sqrt{2}/15}{0.0001\sqrt{20}} = \frac{11,200\sqrt{10}}{3} \approx 11,806 \text{ s} \approx 3 \text{ h } 17 \text{ min.}$$

APPLIED PROJECT Which is Faster, Going Up or Coming Down?

$$1. \quad mv' = -pv - mg \Rightarrow m \frac{dv}{dt} = -(pv + mg) \Rightarrow \int \frac{dv}{pv + mg} = \int -\frac{1}{m} dt \Rightarrow$$

$$\frac{1}{p} \ln(pv + mg) = -\frac{1}{m}t + C \quad [pv + mg > 0]. \text{ At } t = 0, v = v_0, \text{ so } C = \frac{1}{p} \ln(pv_0 + mg). \text{ Thus,}$$

$$\frac{1}{p} \ln(pv + mg) = -\frac{1}{m}t + \frac{1}{p} \ln(pv_0 + mg) \Rightarrow \ln(pv + mg) = -\frac{p}{m}t + \ln(pv_0 + mg) \Rightarrow$$

$$pv + mg = e^{-pt/m}(pv_0 + mg) \Rightarrow pv = (pv_0 + mg)e^{-pt/m} - mg \Rightarrow$$

$$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}.$$

$$2. \quad y(t) = \int v(t) dt = \int \left[\left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}\right] dt = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m}\left(-\frac{m}{p}\right) - \frac{mg}{p}t + C$$

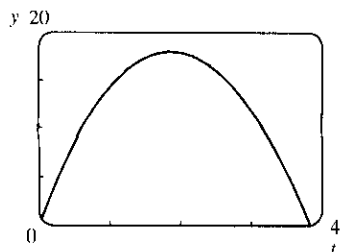
$$\text{At } t = 0, y = 0, \text{ so } C = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}. \text{ Thus,}$$

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p} - \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}e^{-pt/m} - \frac{mgt}{p} = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}(1 - e^{-pt/m}) - \frac{mgt}{p}$$

$$3. \quad v(t) = 0 \Rightarrow \frac{mg}{p} = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} \Rightarrow e^{pt/m} = \frac{pv_0}{mg} + 1 \Rightarrow \frac{pt}{m} = \ln\left(\frac{pv_0}{mg} + 1\right) \Rightarrow$$

$$t_1 = \frac{m}{p} \ln\left(\frac{mg + pv_0}{mg}\right). \text{ With } m = 1, v_0 = 20, p = \frac{1}{10}, \text{ and } g = 9.8, \text{ we have } t_1 = 10 \ln\left(\frac{11.8}{9.8}\right) \approx 1.86 \text{ s.}$$

4.



The figure shows the graph of $y = 1180(1 - e^{-0.1t}) - 98t$. The zeros are at $t = 0$ and $t_2 \approx 3.84$. Thus, $t_1 - 0 \approx 1.86$ and $t_2 - t_1 \approx 1.98$. So the time it takes to come down is about 0.12 s longer than the time it takes to go up; hence, going up is faster.

$$\begin{aligned} 5. y(2t_1) &= \left(v_0 + \frac{mg}{p}\right) \frac{m}{p} \left(1 - e^{-2pt_1/m}\right) - \frac{mg}{p} \cdot 2t_1 \\ &= \left(\frac{pv_0 + mg}{p}\right) \frac{m}{p} \left[1 - \left(e^{pt_1/m}\right)^{-2}\right] - \frac{mg}{p} \cdot 2 \frac{m}{p} \ln\left(\frac{pv_0 + mg}{mg}\right) \end{aligned}$$

Substituting $x = e^{pt_1/m} = \frac{pv_0}{mg} + 1 = \frac{pv_0 + mg}{mg}$ (from Problem 3), we get

$$y(2t_1) = \left(x \cdot \frac{mg}{p}\right) \frac{m}{p} (1 - x^{-2}) - \frac{m^2 g}{p^2} \cdot 2 \ln x = \frac{m^2 g}{p^2} \left(x - \frac{1}{x} - 2 \ln x\right). \text{ Now } p > 0, m > 0, t_1 > 0 \Rightarrow$$

$$x = e^{pt_1/m} > e^0 = 1. f(x) = x - \frac{1}{x} - 2 \ln x \Rightarrow f'(x) = 1 + \frac{1}{x^2} - \frac{2}{x} = \frac{x^2 - 2x + 1}{x^2} = \frac{(x-1)^2}{x^2} > 0 \text{ for}$$

$x > 1 \Rightarrow f(x)$ is increasing for $x > 1$. Since $f(1) = 0$, it follows that $f(x) > 0$ for every $x > 1$. Therefore,

$$y(2t_1) = \frac{m^2 g}{p^2} f(x) \text{ is positive, which means that the ball has not yet reached the ground at time } 2t_1. \text{ This tells us}$$

that the time spent going up is always less than the time spent coming down, so *ascent is faster*.

10.4 Exponential Growth and Decay

1. The relative growth rate is $\frac{1}{P} \frac{dP}{dt} = 0.7944$, so $\frac{dP}{dt} = 0.7944P$ and, by Theorem 2,

$$P(t) = P(0)e^{0.7944t} = 2e^{0.7944t}. \text{ Thus, } P(6) = 2e^{0.7944(6)} \approx 234.99 \text{ or about 235 members.}$$

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 60e^{kt}$. In 20 minutes ($\frac{1}{3}$ hour), there are 120 cells, so

$$P\left(\frac{1}{3}\right) = 60e^{k/3} = 120 \Rightarrow e^{k/3} = 2 \Rightarrow k/3 = \ln 2 \Rightarrow k = 3 \ln 2 = \ln(2^3) = \ln 8.$$

$$(b) P(t) = 60e^{(\ln 8)t} = 60 \cdot 8^t$$

$$(c) P(8) = 60 \cdot 8^8 = 60 \cdot 2^{24} = 1,006,632,960$$

$$(d) dP/dt = kP \Rightarrow P'(8) = kP(8) = (\ln 8)P(8) \approx 2.093 \text{ billion cells/h}$$

$$(e) P(t) = 20,000 \Rightarrow 60 \cdot 8^t = 20,000 \Rightarrow 8^t = 1000/3 \Rightarrow t \ln 8 = \ln(1000/3) \Rightarrow t = \frac{\ln(1000/3)}{\ln 8} \approx 2.79 \text{ h}$$

3. (a) By Theorem 2, $y(t) = y(0)e^{kt} = 500e^{kt}$. Now $y(3) = 500e^{k(3)} = 8000 \Rightarrow e^{3k} = \frac{8000}{500} \Rightarrow$

$$3k = \ln 16 \Rightarrow k = (\ln 16)/3. \text{ So } y(t) = 500e^{(\ln 16)t/3} = 500 \cdot 16^{t/3}$$

$$(b) y(4) = 500 \cdot 16^{4/3} \approx 20,159$$

$$(c) dy/dt = ky \Rightarrow y'(4) = ky(4) = \frac{1}{3} \ln 16 \left(500 \cdot 16^{4/3}\right) \text{ [from part (a)]} \approx 18,631 \text{ cells/h}$$

$$(d) y(t) = 500 \cdot 16^{t/3} = 30,000 \Rightarrow 16^{t/3} = 60 \Rightarrow \frac{1}{3}t \ln 16 = \ln 60 \Rightarrow t = 3(\ln 60)/(\ln 16) \approx 4.4 \text{ h}$$

4. (a) $y(t) = y(0)e^{kt} \Rightarrow y(2) = y(0)e^{2k} = 600, y(8) = y(0)e^{8k} = 75,000$. Dividing these equations, we get $e^{8k}/e^{2k} = 75,000/600 \Rightarrow e^{6k} = 125 \Rightarrow 6k = \ln 125 = \ln 5^3 = 3 \ln 5 \Rightarrow k = \frac{3}{6} \ln 5 = \frac{1}{2} \ln 5$.

$$\text{Thus, } y(0) = 600/e^{2k} = 600/e^{\ln 5} = \frac{600}{5} = 120.$$

$$(b) y(t) = y(0)e^{kt} = 120e^{(\ln 5)t/2} \text{ or } y = 120 \cdot 5^{t/2}$$

$$(c) y(5) = 120 \cdot 5^{5/2} = 120 \cdot 25 \sqrt{5} = 3000 \sqrt{5} \approx 6708 \text{ bacteria.}$$

$$(d) y(t) = 120 \cdot 5^{t/2} \Rightarrow y'(t) = 120 \cdot 5^{t/2} \cdot \ln 5 \cdot \frac{1}{2} = 60 \cdot \ln 5 \cdot 5^{t/2}.$$

$$y'(5) = 60 \cdot \ln 5 \cdot 5^{5/2} = 60 \cdot \ln 5 \cdot 25 \sqrt{5} \approx 5398 \text{ bacteria/hour.}$$

$$(e) y(t) = 200,000 \Leftrightarrow 120e^{(\ln 5)t/2} = 200,000 \Leftrightarrow e^{(\ln 5)t/2} = \frac{5000}{3} \Leftrightarrow (\ln 5)t/2 = \ln \frac{5000}{3} \Leftrightarrow t = (2 \ln \frac{5000}{3}) / \ln 5 \approx 9.2 \text{ h.}$$

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then

$$P(1800) = 980 = 790e^{k(1800-1750)} \Rightarrow \frac{980}{790} = e^{k(50)} \Rightarrow \ln \frac{980}{790} = 50k \Rightarrow$$

$$k = \frac{1}{50} \ln \frac{980}{790} \approx 0.0043104. \text{ So with this model, we have } P(1900) = 790e^{k(1900-1750)} \approx 1508 \text{ million, and}$$

$$P(1950) = 790e^{k(1950-1750)} \approx 1871 \text{ million. Both of these estimates are much too low.}$$

- (b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \Rightarrow$

$$P(1900) = 1650 = 1260e^{k(1900-1850)} \Rightarrow \ln \frac{1650}{1260} = k(50) \Rightarrow k = \frac{1}{50} \ln \frac{1650}{1260} \approx 0.005393. \text{ So with}$$

this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

- (c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \Rightarrow P(1950) = 2560 = 1650e^{k(1950-1900)} \Rightarrow \ln \frac{2560}{1650} = k(50) \Rightarrow k = \frac{1}{50} \ln \frac{2560}{1650} \approx 0.008785. \text{ With this model, we estimate}$

$P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t . Since the initial time is the year 1900, we substitute $t - 1900$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1900)e^{k(t-1900)} \Rightarrow$

$$P(1910) = 92 = 76e^{k(1910-1900)} \Rightarrow k = \frac{1}{10} \ln \frac{92}{76} \approx 0.0191. \text{ With this model, we estimate}$$

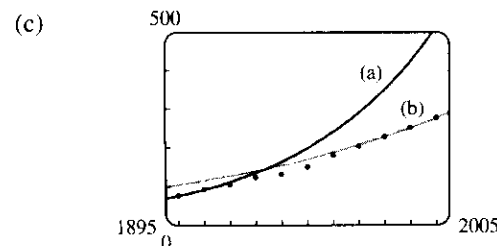
$P(2000) = 76e^{k(2000-1900)} \approx 514$ million. This estimate is much too high. The discrepancy is explained by the fact that, between the years 1900 and 1910, an enormous number of immigrants (compared to the total population) came to the United States. Since that time, immigration (as a proportion of total population) has been much lower. Also, the birth rate in the United States has declined since the turn of the century. So our calculation of the constant k was based partly on factors which no longer exist.

- (b) Substituting $t - 1980$ for t in Theorem 2, we find that the exponential model gives $P(t) = P(1980)e^{k(t-1980)}$

$$\Rightarrow P(1990) = 250 = 227e^{k(1990-1980)} \Rightarrow k = \frac{1}{10} \ln \frac{250}{227} \approx 0.00965. \text{ With this model, we estimate}$$

$$P(2000) = 227e^{k(2000-1980)} \approx 275.3 \text{ million. This is quite accurate. The further estimates are}$$

$$P(2010) = 227e^{30k} \approx 303 \text{ million and } P(2020) = 227e^{40k} \approx 334 \text{ million.}$$



The model in part (a) is quite inaccurate after 1910 (off by 5 million in 1920 and 12 million in 1930). The model in part (b) is more accurate (which is not surprising, since it is based on more recent information).

7. (a) If $y = [\text{N}_2\text{O}_5]$ then by Theorem 2, $\frac{dy}{dt} = -0.0005y \Rightarrow y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

(b) $y(t) = Ce^{-0.0005t} = 0.9C \Rightarrow e^{-0.0005t} = 0.9 \Rightarrow -0.0005t = \ln 0.9 \Rightarrow t = -2000 \ln 0.9 \approx 211 \text{ s}$

8. (a) The mass remaining after t days is

$$y(t) = y(0)e^{kt} = 800e^{kt}. \text{ Since the half-life is 5.0 days,}$$

$$y(5) = 800e^{5k} = 400 \Rightarrow e^{5k} = \frac{1}{2} \Rightarrow$$

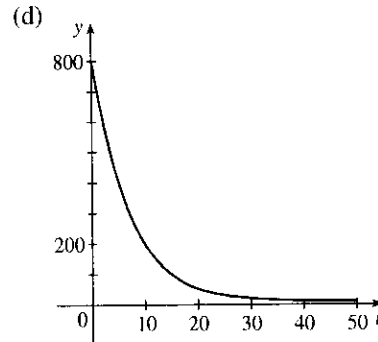
$$5k = \ln \frac{1}{2} \Rightarrow k = -(\ln 2)/5, \text{ so}$$

$$y(t) = 800e^{-(\ln 2)t/5} = 800 \cdot 2^{-t/5}.$$

(b) $y(30) = 800 \cdot 2^{-30/5} = 12.5 \text{ mg}$

(c) $800e^{-(\ln 2)t/5} = 1 \Leftrightarrow -(\ln 2) \frac{t}{5} = \ln \frac{1}{800} = -\ln 800$

$$\Leftrightarrow t = 5 \frac{\ln 800}{\ln 2} \approx 48 \text{ days}$$



9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(30) = 100e^{30k} = \frac{1}{2}(100)$

$$\Rightarrow e^{30k} = \frac{1}{2} \Rightarrow k = -(\ln 2)/30 \Rightarrow y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92 \text{ mg}$

(c) $100e^{-(\ln 2)t/30} = 1 \Rightarrow -(\ln 2)t/30 = \ln \frac{1}{100} \Rightarrow t = -30 \frac{\ln 0.01}{\ln 2} \approx 199.3 \text{ years}$

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$. $y(3) = Ae^{3k} = 0.58A \Rightarrow$

$$e^{3k} = 0.58 \Rightarrow 3k = \ln 0.58 \Rightarrow k = \frac{1}{3} \ln 0.58. \text{ Then } Ae^{(\ln 0.58)t/3} = \frac{1}{2}A \Leftrightarrow$$

$$\ln e^{(\ln 0.58)t/3} = \ln \frac{1}{2} \Leftrightarrow \frac{(\ln 0.58)t}{3} = \ln \frac{1}{2}, \text{ so the half-life is } t = -\frac{3 \ln 2}{\ln 0.58} \approx 3.82 \text{ days.}$$

(b) $Ae^{(\ln 0.58)t/3} = 0.10A \Leftrightarrow \frac{(\ln 0.58)t}{3} = \ln \frac{1}{10} \Leftrightarrow t = -\frac{3 \ln 10}{\ln 0.58} \approx 12.68 \text{ days}$

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:

$$y(5730) = \frac{1}{2}y(0) \Rightarrow y(0)e^{-k(5730)} = \frac{1}{2}y(0) \Rightarrow e^{-5730k} = \frac{1}{2} \Rightarrow$$

$$-5730k = \ln \frac{1}{2} \Rightarrow k = -\frac{\ln \frac{1}{2}}{5730} = \frac{\ln 2}{5730}. \text{ If 74\% of the } ^{14}\text{C} \text{ remains, then we know that } y(t) = 0.74y(0)$$

$$\Rightarrow 0.74 = e^{-t(\ln 2)/5730} \Rightarrow \ln 0.74 = -\frac{t \ln 2}{5730} \Rightarrow t = -\frac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500 \text{ years.}$$

12. From the information given, we know that $\frac{dy}{dx} = 2y \Rightarrow y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \Rightarrow C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

13. (a) Using Newton's Law of Cooling, $\frac{dT}{dt} = k(T - T_s)$, we have $\frac{dT}{dt} = k(T - 75)$.

Now let $y = T - 75$, so $y(0) = T(0) - 75 = 185 - 75 = 110$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 110$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 110e^{kt}$.

$$y(30) = 110e^{30k} = 150 - 75 \Rightarrow e^{30k} = \frac{75}{110} = \frac{15}{22} \Rightarrow k = \frac{1}{30} \ln \frac{15}{22},$$

$$\text{so } y(t) = 110e^{\frac{1}{30}t \ln(\frac{15}{22})} \text{ and } y(45) = 110e^{\frac{45}{30} \ln(\frac{15}{22})} \approx 62^\circ\text{F. Thus, } T(45) \approx 62 + 75 = 137^\circ\text{F.}$$

(b) $T(t) = 100 \Rightarrow y(t) = 25$. $y(t) = 110e^{\frac{1}{30}t \ln(\frac{15}{22})} = 25 \Rightarrow e^{\frac{1}{30}t \ln(\frac{15}{22})} = \frac{25}{110} \Rightarrow$

$$\frac{1}{30}t \ln \frac{15}{22} = \ln \frac{25}{110} \Rightarrow t = \frac{30 \ln \frac{25}{110}}{\ln \frac{15}{22}} \approx 116 \text{ min.}$$

14. (a) Let $T(t)$ = temperature after t minutes. Newton's Law of Cooling implies that $\frac{dT}{dt} = k(T - 5)$. Let

$$y(t) = T(t) - 5. \text{ Then } \frac{dy}{dt} = ky, \text{ so } y(t) = y(0)e^{kt} = 15e^{kt} \Rightarrow T(t) = 5 + 15e^{kt} \Rightarrow$$

$$T(1) = 5 + 15e^k = 12 \Rightarrow e^k = \frac{7}{15} \Rightarrow k = \ln \frac{7}{15}, \text{ so } T(t) = 5 + 15e^{\ln(7/15)t} \text{ and}$$

$$T(2) = 5 + 15e^{2\ln(7/15)} \approx 8.3^\circ\text{C}.$$

(b) $5 + 15e^{\ln(7/15)t} = 6$ when $e^{\ln(7/15)t} = \frac{1}{15} \Rightarrow \ln\left(\frac{7}{15}\right)t = \ln \frac{1}{15} \Rightarrow t = \frac{\ln \frac{1}{15}}{\ln \frac{7}{15}} \approx 3.6$ min.

15. $\frac{dT}{dt} = k(T - 20)$. Letting $y = T - 20$, we get $\frac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$.

$$y(0) = T(0) - 20 = 5 - 20 = -15, \text{ so } y(25) = y(0)e^{25k} = -15e^{25k}, \text{ and}$$

$$y(25) = T(25) - 20 = 10 - 20 = -10, \text{ so } -15e^{25k} = -10 \Rightarrow e^{25k} = \frac{2}{3}. \text{ Thus, } 25k = \ln\left(\frac{2}{3}\right) \text{ and}$$

$$k = \frac{1}{25} \ln\left(\frac{2}{3}\right), \text{ so } y(t) = y(0)e^{kt} = -15e^{(\ln(2/3)/25)t}. \text{ More simply, } e^{25k} = \frac{2}{3} \Rightarrow e^k = \left(\frac{2}{3}\right)^{1/25} \Rightarrow$$

$$e^{kt} = \left(\frac{2}{3}\right)^{t/25} \Rightarrow y(t) = -15 \cdot \left(\frac{2}{3}\right)^{t/25}.$$

(a) $T(50) = 20 + y(50) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{50/25} = 20 - 15 \cdot \left(\frac{2}{3}\right)^2 = 20 - \frac{20}{3} = 13.\bar{3}^\circ\text{C}$

(b) $15 = T(t) = 20 + y(t) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{t/25} \Rightarrow 15 \cdot \left(\frac{2}{3}\right)^{t/25} = 5 \Rightarrow \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \Rightarrow$
 $(t/25) \ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \Rightarrow t = 25 \ln\left(\frac{1}{3}\right) / \ln\left(\frac{2}{3}\right) \approx 67.74$ min.

16. $\frac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\frac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$,

so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\frac{dT}{dt} = -1^\circ\text{C/min}$. Equivalently, $\frac{dy}{dt} = -1$ when $y(t) = 50$. Thus,

$$-1 = \frac{dy}{dt} = ky(t) = 50k \text{ and } 50 = y(t) = 75e^{kt}. \text{ The first relation implies } k = -1/50, \text{ so the second relation}$$

says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \Rightarrow -t/50 = \ln\left(\frac{2}{3}\right) \Rightarrow t = -50 \ln\left(\frac{2}{3}\right) \approx 20.27$ min.

17. (a) Let $P(h)$ be the pressure at altitude h . Then $dP/dh = kP \Rightarrow P(h) = P(0)e^{kh} = 101.3e^{kh}$.

$$P(1000) = 101.3e^{1000k} = 87.14 \Rightarrow 1000k = \ln\left(\frac{87.14}{101.3}\right) \Rightarrow$$

$$k = \frac{1}{1000} \ln\left(\frac{87.14}{101.3}\right) \Rightarrow P(h) = 101.3 e^{\frac{1}{1000}h \ln\left(\frac{87.14}{101.3}\right)}, \text{ so } P(3000) = 101.3e^{3 \ln\left(\frac{87.14}{101.3}\right)} \approx 64.5 \text{ kPa}.$$

(b) $P(6187) = 101.3 e^{\frac{6187}{1000} \ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

18. (a) Using $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$ with $A_0 = 500$, $r = 0.14$, and $t = 2$,

we have:

(i) Annually: $n = 1$; $A = 500\left(1 + \frac{0.14}{1}\right)^{1 \cdot 2} = \649.80

(ii) Quarterly: $n = 4$; $A = 500\left(1 + \frac{0.14}{4}\right)^{4 \cdot 2} = \658.40

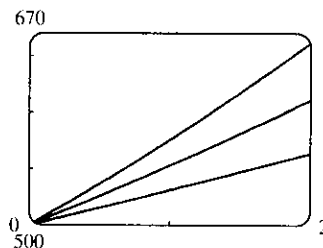
(iii) Monthly: $n = 12$; $A = 500\left(1 + \frac{0.14}{12}\right)^{12 \cdot 2} = \660.49

(iv) Daily: $n = 365$; $A = 500\left(1 + \frac{0.14}{365}\right)^{365 \cdot 2} = \661.53

(v) Hourly: $n = 365 \cdot 24$; $A = 500\left(1 + \frac{0.14}{365 \cdot 24}\right)^{365 \cdot 24 \cdot 2} = \661.56

(vi) Continuously: $A = 500e^{(0.14)2} = \$661.56$

(b)



$$A_{0.14}(2) = \$661.56,$$

$$A_{0.10}(2) = \$610.70, \text{ and}$$

$$A_{0.06}(2) = \$563.75.$$

19. (a) Using $A = A_0 \left(1 + \frac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000 \left(1 + \frac{0.05}{1}\right)^{1 \cdot 5} = \3828.84

(ii) Semiannually: $n = 2$; $A = 3000 \left(1 + \frac{0.05}{2}\right)^{2 \cdot 5} = \3840.25

(iii) Monthly: $n = 12$; $A = 3000 \left(1 + \frac{0.05}{12}\right)^{12 \cdot 5} = \3850.08

(iv) Weekly: $n = 52$; $A = 3000 \left(1 + \frac{0.05}{52}\right)^{52 \cdot 5} = \3851.61

(v) Daily: $n = 365$; $A = 3000 \left(1 + \frac{0.05}{365}\right)^{365 \cdot 5} = \3852.01

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

20. (a) $A_0 e^{0.06t} = 2A_0 \Leftrightarrow e^{0.06t} = 2 \Leftrightarrow 0.06t = \ln 2 \Leftrightarrow t = \frac{50}{3} \ln 2 \approx 11.55$, so the investment will double in about 11.55 years.

(b) The annual interest rate in $A = A_0(1+r)^t$ is r . From part (a), we have $A = A_0 e^{0.06t}$. These amounts must be equal, so $(1+r)^t = e^{0.06t} \Rightarrow 1+r = e^{0.06} \Rightarrow r = e^{0.06} - 1 \approx 0.0618 = 6.18\%$, which is the equivalent annual interest rate.

21. (a) $\frac{dP}{dt} = kP - m = k\left(P - \frac{m}{k}\right)$. Let $y = P - \frac{m}{k}$, so $\frac{dy}{dt} = \frac{dP}{dt}$ and the differential equation becomes $\frac{dy}{dt} = ky$. The solution is $y = y_0 e^{kt} \Rightarrow P - \frac{m}{k} = \left(P_0 - \frac{m}{k}\right)e^{kt} \Rightarrow P(t) = \frac{m}{k} + \left(P_0 - \frac{m}{k}\right)e^{kt}$.

(b) Since $k > 0$, there will be an exponential expansion $\Leftrightarrow P_0 - \frac{m}{k} > 0 \Leftrightarrow m < kP_0$.

(c) The population will be constant if $P_0 - \frac{m}{k} = 0 \Leftrightarrow m = kP_0$. It will decline if $P_0 - \frac{m}{k} < 0 \Leftrightarrow m > kP_0$.

(d) $P_0 = 8,000,000$, $k = \alpha - \beta = 0.016$, $m = 210,000 \Rightarrow m > kP_0 (= 128,000)$, so by part (c), the population was declining.

22. (a) $\frac{dy}{dt} = ky^{1+c} \Rightarrow y^{-1-c} dy = k dt \Rightarrow \frac{y^{-c}}{-c} = kt + C$. Since $y(0) = y_0$, we have $C = \frac{y_0^{-c}}{-c}$. Thus, $\frac{y^{-c}}{-c} = kt + \frac{y_0^{-c}}{-c}$, or $y^{-c} = y_0^{-c} - ckt$. So $y^c = \frac{1}{y_0^{-c} - ckt} = \frac{y_0^c}{1 - cy_0^c kt}$ and $y(t) = \frac{y_0}{(1 - cy_0^c kt)^{1/c}}$.

(b) $y(t) \rightarrow \infty$ as $1 - cy_0^c kt \rightarrow 0$, that is, as $t \rightarrow \frac{1}{cy_0^c k}$. Define $T = \frac{1}{cy_0^c k}$. Then $\lim_{t \rightarrow T^-} y(t) = \infty$.

(c) According to the data given, we have $c = 0.01$, $y(0) = 2$, and $y(3) = 16$, where the time t is given in months.

Thus, $y_0 = 2$ and $16 = y(3) = \frac{y_0}{(1 - cy_0^c k \cdot 3)^{1/c}}$. Since $T = \frac{1}{cy_0^c k}$, we will solve for $cy_0^c k$.

$$16 = \frac{2}{(1 - 3cy_0^c k)^{100}} \Rightarrow 1 - 3cy_0^c k = \left(\frac{1}{8}\right)^{0.01} = 8^{-0.01} \Rightarrow cy_0^c k = \frac{1}{3}(1 - 8^{-0.01}). \text{ Thus, doomsday}$$

$$\text{occurs when } t = T = \frac{1}{cy_0^c k} = \frac{3}{1 - 8^{-0.01}} \approx 145.77 \text{ months or 12.15 years.}$$

APPLIED PROJECT Calculus and Baseball

1. (a) $F = ma = m \frac{dv}{dt}$, so by the Substitution Rule we have

$$\int_{t_0}^{t_1} F(t) dt = \int_{t_0}^{t_1} m \left(\frac{dv}{dt} \right) dt = m \int_{v_0}^{v_1} dv = [mv]_{v_0}^{v_1} = mv_1 - mv_0 = p(t_1) - p(t_0)$$

- (b) (i) We have $v_1 = 110 \text{ mi/h} = \frac{110(5280)}{3600} \text{ ft/s} = 161.\bar{3} \text{ ft/s}$, $v_0 = -90 \text{ mi/h} = -132 \text{ ft/s}$, and the mass of the baseball is $m = \frac{w}{g} = \frac{5/16}{32} = \frac{5}{512}$. So the change in momentum is

$$p(t_1) - p(t_0) = mv_1 - mv_0 = \frac{5}{512} [161.\bar{3} - (-132)] \approx 2.86 \text{ slug-ft/s.}$$

- (ii) From part (a) and part (b)(i), we have $\int_0^{0.001} F(t) dt = p(0.001) - p(0) \approx 2.86$, so the average force over the interval $[0, 0.001]$ is $\frac{1}{0.001} \int_0^{0.001} F(t) dt \approx \frac{1}{0.001} (2.86) = 2860 \text{ lb.}$

2. (a) $W = \int_{s_0}^{s_1} F(s) ds$, where $F(s) = m \frac{dv}{dt} = m \frac{dv}{ds} \frac{ds}{dt} = mv \frac{dv}{ds}$ and so, by the Substitution Rule,

$$W = \int_{s_0}^{s_1} F(s) ds = \int_{s_0}^{s_1} mv \frac{dv}{ds} ds = \int_{v(s_0)}^{v(s_1)} mv dv = \left[\frac{1}{2} mv^2 \right]_{v_0}^{v_1} = \frac{1}{2} mv_1^2 - \frac{1}{2} mv_0^2$$

- (b) From part (b)(i), $90 \text{ mi/h} = 132 \text{ ft/s}$. Assume $v_0 = v(s_0) = 0$ and $v_1 = v(s_1) = 132 \text{ ft/s}$ (note that s_1 is the point of release of the baseball). $m = \frac{5}{512}$, so the work done is

$$W = \frac{1}{2} mv_1^2 - \frac{1}{2} mv_0^2 = \frac{1}{2} \cdot \frac{5}{512} \cdot (132)^2 \approx 85 \text{ ft-lb}$$

3. (a) Here we have a differential equation of the form $dv/dt = kv$, so by Theorem 10.4.2, the solution is

$v(t) = v(0)e^{kt}$. In this case $k = -\frac{1}{10}$ and $v(0) = 100 \text{ ft/s}$, so $v(t) = 100e^{-t/10}$. We are interested in the time t that the ball takes to travel 280 ft, so we find the distance function

$$\begin{aligned} s(t) &= \int_0^t v(x) dx = \int_0^t 100e^{-x/10} dx = 100 \left[-10e^{-x/10} \right]_0^t = -1000(e^{-t/10} - 1) \\ &= 1000(1 - e^{-t/10}) \end{aligned}$$

Now we set $s(t) = 280$ and solve for t : $280 = 1000(1 - e^{-t/10}) \Rightarrow 1 - e^{-t/10} = \frac{7}{25} \Rightarrow$

$$-\frac{1}{10}t = \ln\left(1 - \frac{7}{25}\right) \Rightarrow t \approx 3.285 \text{ seconds.}$$

- (b) Let x be the distance of the shortstop from home plate. We calculate the time for the ball to reach home plate as a function of x , then differentiate with respect to x to find the value of x which corresponds to the minimum time. The total time that it takes the ball to reach home is the sum of the times of the two throws, plus the relay time ($\frac{1}{2}$ s). The distance from the fielder to the shortstop is $280 - x$, so to find the time t_1 taken by the first

$$\text{throw, we solve the equation } s_1(t_1) = 280 - x \Leftrightarrow 1 - e^{-t_1/10} = \frac{280 - x}{1000} \Leftrightarrow t_1 = -10 \ln \frac{720 + x}{1000}.$$

We find the time t_2 taken by the second throw if the shortstop throws with velocity w , since we see that this velocity varies in the rest of the problem. We use $v = we^{-t/10}$ and isolate t_2 in the equation

$s(t_2) = 10w(1 - e^{-t_2/10}) = x \Leftrightarrow e^{-t_2/10} = 1 - \frac{x}{10w} \Leftrightarrow t_2 = -10 \ln \frac{10w - x}{10w}$, so the total time is

$t_w(x) = \frac{1}{2} - 10 \left[\ln \frac{720 + x}{1000} + \ln \frac{10w - x}{10w} \right]$. To find the minimum, we differentiate:

$\frac{dt_w}{dx} = -10 \left[\frac{1}{720 + x} - \frac{1}{10w - x} \right]$, which changes from negative to positive when $720 + x = 10w - x \Leftrightarrow$

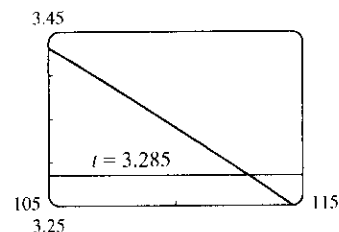
$x = 5w - 360$. By the First Derivative Test, t_w has a minimum at this distance from the shortstop to home plate. So if the shortstop throws at $w = 105$ ft/s from a point $x = 5(105) - 360 = 165$ ft from home plate, the minimum time is $t_{105}(165) = \frac{1}{2} - 10 \left(\ln \frac{720 + 165}{1000} + \ln \frac{1050 - 165}{1050} \right) \approx 3.431$ seconds. This is longer than the time taken in part (a), so in this case the manager should encourage a direct throw.

If $w = 115$ ft/s, then $x = 215$ ft from home, and the minimum time is

$t_{115}(215) = \frac{1}{2} - 10 \left(\ln \frac{720 + 215}{1000} + \ln \frac{1150 - 215}{1150} \right) \approx 3.242$ seconds. This is less than the time taken in part (a), so in this case, the manager should encourage a relayed throw.

(c) In general, the minimum time is

$$\begin{aligned} t_w(5w - 360) &= \frac{1}{2} - 10 \left[\ln \frac{360 + 5w}{1000} + \ln \frac{360 + 5w}{10w} \right] \\ &= \frac{1}{2} - 10 \ln \frac{(w + 72)^2}{400w} \end{aligned}$$

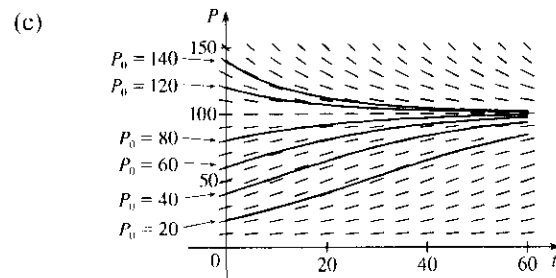


We want to find out when this is about 3.285 seconds, the same time as the direct throw. From the graph, we estimate that this is the case for $w \approx 112.8$ ft/s. So if the shortstop can throw the ball with this velocity, then a relayed throw takes the same time as a direct throw.

10.5 The Logistic Equation

1. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100)$. Comparing to Equation 1, $dP/dt = kP(1 - P/K)$, we see that the carrying capacity is $K = 100$ and the value of k is 0.05.

- (b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

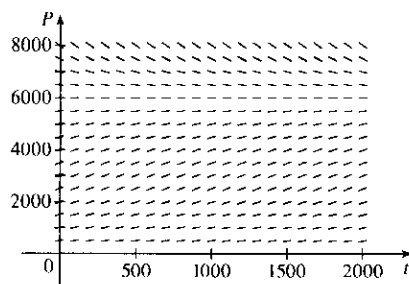


All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

- (d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \rightarrow \infty$.

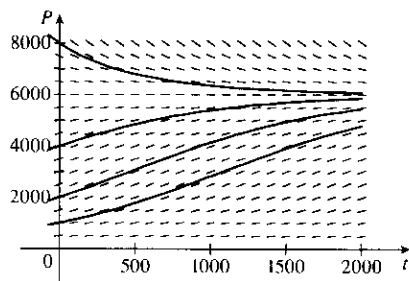
2. (a) $K = 6000$ and $k = 0.0015 \Rightarrow dP/dt = 0.0015P(1 - P/6000)$.

(b)



All of the solution curves approach 6000 as $t \rightarrow \infty$.

(c)



The curves with $P_0 = 1000$ and $P_0 = 2000$ appear to be concave upward at first and then concave downward. The curve with $P_0 = 4000$ appears to be concave downward everywhere. The curve with $P_0 = 8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

- (d) See the solution to Exercise 10.2.25 for a possible program to calculate $P(50)$. [In this case, we use $X = 0$, $H = 1$, $N = 50$, $Y_1 = 0.0015y(1 - y/6000)$, and $Y = 1000$.] We find that $P(50) \approx 1064$.

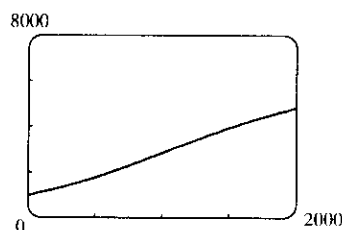
- (e) Using Equation 4 with $K = 6000$, $k = 0.0015$, and $P_0 = 1000$, we

$$\text{have } P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{6000}{1 + Ae^{-0.0015t}}, \text{ where}$$

$$A = \frac{K - P_0}{P_0} = \frac{6000 - 1000}{1000} = 5. \text{ Thus,}$$

$$P(50) = \frac{6000}{1 + 5e^{-0.0015(50)}} \approx 1064.1, \text{ which is extremely close to the estimate obtained in part (d).}$$

(f)



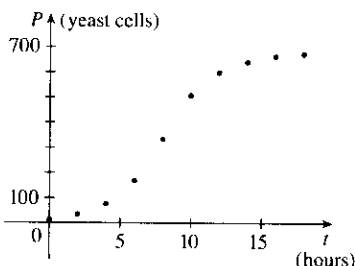
The curves are very similar.

3. (a) $\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right) \Rightarrow y(t) = \frac{K}{1 + Ae^{-kt}}$ with $A = \frac{K - y(0)}{y(0)}$. With $K = 8 \times 10^7$, $k = 0.71$, and

$$y(0) = 2 \times 10^7, \text{ we get the model } y(t) = \frac{8 \times 10^7}{1 + 3e^{-0.71t}}, \text{ so } y(1) = \frac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7 \text{ kg.}$$

$$\begin{aligned} \text{(b) } y(t) &= 4 \times 10^7 \Rightarrow \frac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \Rightarrow 2 = 1 + 3e^{-0.71t} \Rightarrow e^{-0.71t} = \frac{1}{3} \Rightarrow \\ -0.71t &= \ln \frac{1}{3} \Rightarrow t = \frac{\ln 3}{0.71} \approx 1.55 \text{ years} \end{aligned}$$

4. (a)



- (b) An estimate of the initial relative growth rate is

$$\frac{1}{P_0} \frac{dP}{dt} = \frac{1}{18} \cdot \frac{39 - 18}{2 - 0} = \frac{7}{12} = 0.58\bar{3}.$$

- (c) An exponential model is $P(t) = 18e^{7t/12}$. A

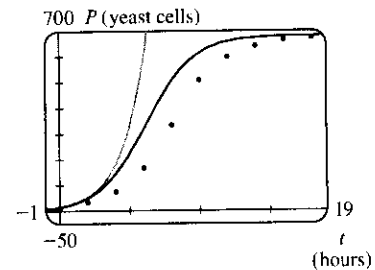
$$\text{logistic model is } P(t) = \frac{680}{1 + Ae^{-7t/12}}, \text{ where}$$

$$A = \frac{680 - 18}{18} = \frac{331}{9}.$$

From the graph, we estimate the carrying capacity K for the yeast population to be 680.

(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19,739	658
14	640	63,389	673
16	664	203,558	678
18	672	653,679	679



The exponential model is a poor fit for anything beyond the first two observed values. The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

$$(e) P(7) = \frac{680}{1 + \frac{331}{9}e^{-7(7/12)}} \approx 420 \text{ yeast cells}$$

5. (a) We will assume that the difference in the birth and death rates is 20 million/year. Let $t = 0$ correspond to the year 1990 and use a unit of 1 billion for all calculations. $k \approx \frac{1}{P} \frac{dP}{dt} = \frac{1}{5.3}(0.02) = \frac{1}{265}$, so

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) = \frac{1}{265}P \left(1 - \frac{P}{100}\right), \quad P \text{ in billions}$$

$$(b) A = \frac{K - P_0}{P_0} = \frac{100 - 5.3}{5.3} = \frac{947}{53} \approx 17.8679. \quad P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{100}{1 + \frac{947}{53}e^{-(1/265)t}}, \text{ so}$$

$$P(10) \approx 5.49 \text{ billion.}$$

- (c) $P(110) \approx 7.81$, and $P(510) \approx 27.72$. The predictions are 7.81 billion in the year 2100 and 27.72 billion in 2500.

- (d) If $K = 50$, then $P(t) = \frac{50}{1 + \frac{447}{53}e^{-(1/265)t}}$. So $P(10) \approx 5.48$, $P(110) \approx 7.61$, and $P(510) \approx 22.41$. The predictions become 5.48 billion in the year 2000, 7.61 billion in 2100, and 22.41 billion in the year 2500.

6. (a) If we assume that the carrying capacity for the world population is 100 billion, it would seem reasonable that the carrying capacity for the U.S. is 3–5 billion by using current populations and simple proportions. We will use $K = 4$ billion or 4000 million. With $t = 0$ corresponding to 1980, we have

$$P(t) = \frac{4000}{1 + \left(\frac{4000 - 250}{250}\right)e^{-kt}} = \frac{4000}{1 + 15e^{-kt}}$$

$$(b) P(10) = 275 \Rightarrow \frac{4000}{1 + 15e^{-10k}} = 275 \Rightarrow 1 + 15e^{-10k} = \frac{4000}{275} \Rightarrow e^{-10k} = \frac{\frac{4000}{275} - 1}{15} \Rightarrow$$

$$-10k = \ln \frac{149}{165} \Rightarrow k = -\frac{1}{10} \ln \frac{149}{165} \approx 0.01019992.$$

- (c) $2100 - 1990 = 110$ and $P(110) \approx 680$ million.
 $2200 - 1990 = 210$ and $P(210) \approx 1449$ million, or about 1.4 billion.

$$(d) P(t) = 300 \Rightarrow \frac{4000}{1 + 15e^{-kt}} = 300 \Rightarrow 1 + 15e^{-kt} = \frac{40}{3} \Rightarrow e^{-kt} = \frac{37}{3} \cdot \frac{1}{15} \Rightarrow -kt = \ln \frac{37}{45}$$

$$\Rightarrow t = 10 \frac{\ln \frac{37}{45}}{\ln \frac{149}{165}} \approx 19.19 \approx 19. \text{ So we predict that the U.S. population will exceed 300 million in the year } 1990 + 19 = 2009.$$

7. (a) Our assumption is that $\frac{dy}{dt} = ky(1-y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (1), $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$, we substitute $y = \frac{P}{K}$, $P = Ky$, and $\frac{dP}{dt} = K\frac{dy}{dt}$, to obtain $K\frac{dy}{dt} = k(Ky)(1-y) \Leftrightarrow \frac{dy}{dt} = ky(1-y)$, our equation in part (a). Now the solution to (1) is $P(t) = \frac{K}{1 + Ae^{-kt}}$, where $A = \frac{K - P_0}{P_0}$. We use the same substitution to obtain $Ky = \frac{K}{1 + \frac{K - Ky_0}{Ky_0}e^{-kt}}$

$$\Rightarrow y = \frac{y_0}{y_0 + (1 - y_0)e^{-kt}}.$$

Alternatively, we could use the same steps as outlined in "The Analytic Solution," following Example 2.

(c) Let t be the number of hours since 8 A.M. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so $\frac{1}{2} = y(4) = \frac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$, so $y = \frac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \frac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t , we get $2y + 23y\left(\frac{2}{23}\right)^{t/4} = 2 \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2 - 2y}{23y} \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23} \cdot \frac{1 - y}{y} \Rightarrow \left(\frac{2}{23}\right)^{t/4 - 1} = \frac{1 - y}{y}$. It follows that $\frac{t}{4} - 1 = \frac{\ln[(1 - y)/y]}{\ln \frac{2}{23}}$, so $t = 4\left[1 + \frac{\ln((1 - y)/y)}{\ln \frac{2}{23}}\right]$. When $y = 0.9$, $\frac{1 - y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \frac{\ln 9}{\ln \frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.

8. (a) $P(0) = P_0 = 400$, $P(1) = 1200$ and $K = 10,000$. From the solution to the logistic differential equation

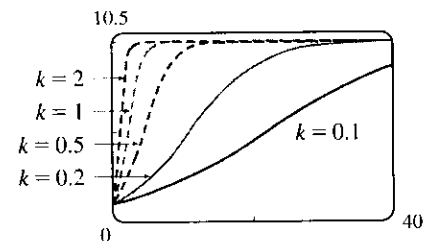
$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-kt}}, \text{ we get } P = \frac{400(10,000)}{400 + (9600)e^{-kt}} = \frac{10,000}{1 + 24e^{-kt}}. P(1) = 1200 \Rightarrow 1 + 24e^{-k} = \frac{100}{12} \Rightarrow e^k = \frac{288}{88} \Rightarrow k = \ln \frac{36}{11}. \text{ So } P = \frac{10,000}{1 + 24e^{-t \ln(36/11)}} = \frac{10,000}{1 + 24 \cdot (11/36)^t}.$$

(b) $5000 = \frac{10,000}{1 + 24(11/36)^t} \Rightarrow 24\left(\frac{11}{36}\right)^t = 1 \Rightarrow t \ln \frac{11}{36} = \ln \frac{1}{24} \Rightarrow t \approx 2.68$ years.

9. (a) $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \Rightarrow$

$$\begin{aligned} \frac{d^2P}{dt^2} &= k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right)\frac{dP}{dt}\right] = k\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right) \\ &= k\left[kP\left(1 - \frac{P}{K}\right)\right]\left(1 - \frac{2P}{K}\right) = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right) \end{aligned}$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \Leftrightarrow P = 0$, $P = K$, or $P = K/2$. Since $0 < P < K$, we see that $P'' = 0 \Leftrightarrow P = K/2$.



Now we instead keep P_0 constant (at $P_0 = 1$) and change k in the function $P = \frac{10}{1 + 9e^{-kt}}$. It seems that as k increases, the graph approaches the line $P = 10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x -scales, the graphs all look the same.)

-

(d)

Figure 1(d) shows a plot of pressure P versus distance x for various initial pressures P_0 . The y-axis (P) ranges from 0 to 1200, and the x-axis (x) ranges from 0 to 120. The curves represent the pressure profile for $P_0 = 1000, 600, 400, 300, 200$. The curves for higher P_0 values show a more pronounced oscillatory behavior, with multiple peaks and troughs, while the curve for $P_0 = 200$ is a smooth, monotonic decay.

$$\begin{aligned} \text{(e)} \quad \frac{dP}{dt} &= 0.08P \left(1 - \frac{P}{1000}\right) - 15 \Leftrightarrow -\frac{100,000}{8} \cdot \frac{dP}{dt} = (0.08P - 0.00008P^2 - 15) \cdot \left(-\frac{100,000}{8}\right) \Leftrightarrow \\ &-12,500 \frac{dP}{dt} = P^2 - 1000P + 187,500 \Leftrightarrow \frac{dP}{(P-250)(P-750)} = -\frac{1}{12,500} dt \Leftrightarrow \\ &\int \left(\frac{-1/500}{P-250} + \frac{1/500}{P-750} \right) dP = -\frac{1}{12,500} dt \Leftrightarrow \int \left(\frac{1}{P-250} - \frac{1}{P-750} \right) dP = \frac{1}{25} dt \Leftrightarrow \end{aligned}$$

$$\ln|P - 250| - \ln|P - 750| = \frac{1}{25}t + C \Leftrightarrow \ln\left|\frac{P - 250}{P - 750}\right| = \frac{1}{25}t + C \Leftrightarrow$$

$$\left|\frac{P - 250}{P - 750}\right| = e^{t/25 + C} = ke^{t/25} \Leftrightarrow \frac{P - 250}{P - 750} = ke^{t/25} \Leftrightarrow P - 250 = Pke^{t/25} - 750ke^{t/25} \Leftrightarrow$$

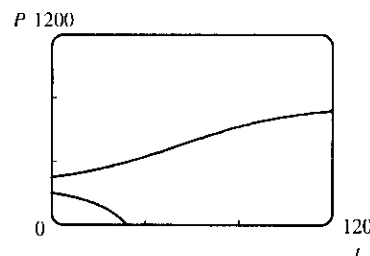
$$P - Pke^{t/25} = 250 - 750ke^{t/25} \Leftrightarrow P(t) = \frac{250 - 750ke^{t/25}}{1 - ke^{t/25}}. \text{ If } t = 0 \text{ and } P = 200, \text{ then}$$

$$200 = \frac{250 - 750k}{1 - k} \Leftrightarrow 200 - 200k = 250 - 750k \Leftrightarrow$$

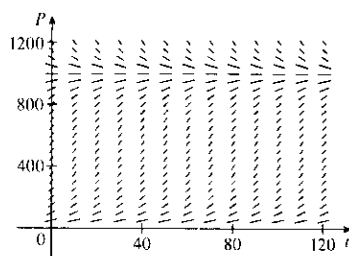
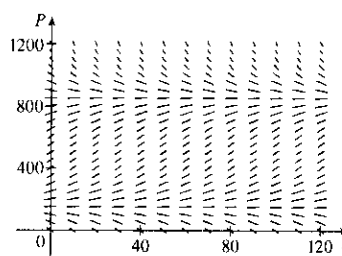
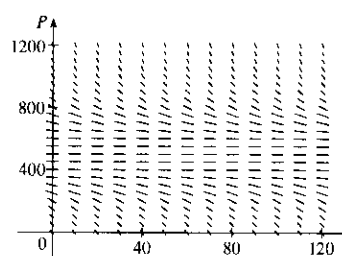
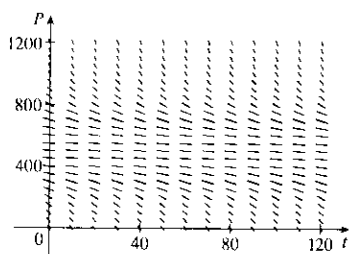
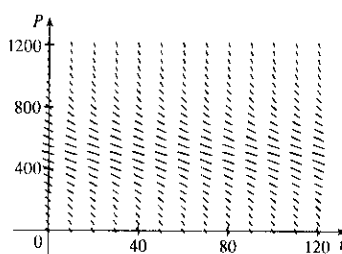
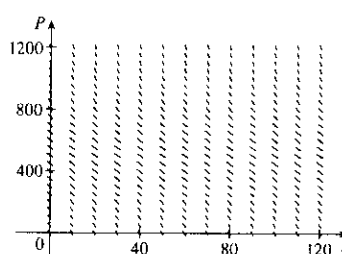
$$550k = 50 \Leftrightarrow k = \frac{1}{11}. \text{ Similarly, if } t = 0 \text{ and } P = 300, \text{ then}$$

$$k = -\frac{1}{9}. \text{ Simplifying } P \text{ with these two values of } k \text{ gives us}$$

$$P(t) = \frac{250(3e^{t/25} - 11)}{e^{t/25} - 11} \text{ and } P(t) = \frac{750(e^{t/25} + 3)}{e^{t/25} + 9}.$$



12. (a)

 $c = 0$  $c = 10$  $c = 20$  $c = 21$  $c = 25$  $c = 30$

(b) For $0 \leq c \leq 20$, there is at least one equilibrium solution. For $c > 20$, the population always dies out.

$$(c) \frac{dP}{dt} = 0.08P - 0.00008P^2 - c. \quad \frac{dP}{dt} = 0 \Leftrightarrow P = \frac{-0.08 \pm \sqrt{(0.08)^2 - 4(-0.00008)(-c)}}{2(-0.00008)}, \text{ which has at}$$

least one solution when the discriminant is nonnegative $\Rightarrow 0.0064 - 0.00032c \geq 0 \Leftrightarrow c \leq 20$. For

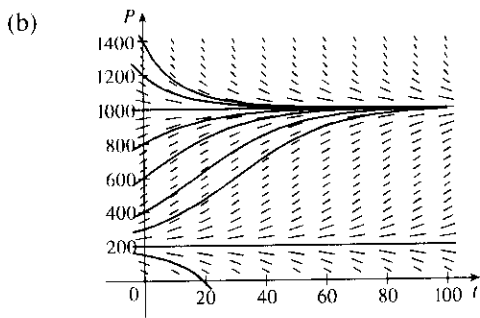
$0 \leq c \leq 20$, there is at least one value of P such that $dP/dt = 0$ and hence, at least one equilibrium solution.

For $c > 20$, $dP/dt < 0$ and the population always dies out.

(d) The weekly catch should be less than 20 fish per week.

$$13. (a) \frac{dP}{dt} = (kP)\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right). \text{ If } m < P < K, \text{ then } dP/dt = (+)(+)(+) = + \Rightarrow P \text{ is increasing.}$$

If $0 < P < m$, then $dP/dt = (+)(+)(-) = - \Rightarrow P$ is decreasing.



$$k = 0.08, K = 1000, \text{ and } m = 200 \Rightarrow$$

$$\frac{dP}{dt} = 0.08P \left(1 - \frac{P}{1000} \right) \left(1 - \frac{200}{P} \right)$$

For $0 < P_0 < 200$, the population dies out. For $P_0 = 200$, the population is steady. For $200 < P_0 < 1000$, the population increases and approaches 1000. For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t) = 200$ and $P(t) = 1000$.

$$(c) \frac{dP}{dt} = kP \left(1 - \frac{P}{K} \right) \left(1 - \frac{m}{P} \right) = kP \left(\frac{K-P}{K} \right) \left(\frac{P-m}{P} \right) = \frac{k}{K} (K-P)(P-m) \Leftrightarrow$$

$$\int \frac{dP}{(K-P)(P-m)} = \int \frac{k}{K} dt.$$

By partial fractions, $\frac{1}{(K-P)(P-m)} = \frac{A}{K-P} + \frac{B}{P-m}$, so $A(P-m) + B(K-P) = 1$.

$$\text{If } P = m, B = \frac{1}{K-m}; \text{ if } P = K, A = \frac{1}{K-m}, \text{ so } \frac{1}{K-m} \int \left(\frac{1}{K-P} + \frac{1}{P-m} \right) dP = \int \frac{k}{K} dt \Rightarrow$$

$$\frac{1}{K-m} (-\ln|K-P| + \ln|P-m|) = \frac{k}{K} t + M \Rightarrow \frac{1}{K-m} \ln \left| \frac{P-m}{K-P} \right| = \frac{k}{K} t + M \Rightarrow$$

$$\ln \left| \frac{P-m}{K-P} \right| = (K-m) \frac{k}{K} t + M_1 \Leftrightarrow \frac{P-m}{K-P} = D e^{(K-m)(k/K)t} \quad [D = \pm e^{M_1}].$$

Let $t = 0$: $\frac{P_0 - m}{K - P_0} = D$. So $\frac{P-m}{K-P} = \frac{P_0 - m}{K - P_0} e^{(K-m)(k/K)t}$. Solving for P , we get

$$P(t) = \frac{m(K - P_0) + K(P_0 - m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0 - m)e^{(K-m)(k/K)t}}.$$

(d) If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then

$$N(0) = P_0(K - m) > 0, \text{ and } P_0 - m < 0 \Leftrightarrow \lim_{t \rightarrow \infty} K(P_0 - m)e^{(K-m)(k/K)t} = -\infty \Rightarrow$$

$\lim_{t \rightarrow \infty} N(t) = -\infty$. Since N is continuous, there is a number t such that $N(t) = 0$ and thus $P(t) = 0$. So the species will become extinct.

$$14. (a) \frac{dP}{dt} = c \ln \left(\frac{K}{P} \right) P \Rightarrow \int \frac{dP}{P \ln(K/P)} = \int c dt. \text{ Let } u = \ln \left(\frac{K}{P} \right) = \ln K - \ln P \Rightarrow du = -\frac{dP}{P}$$

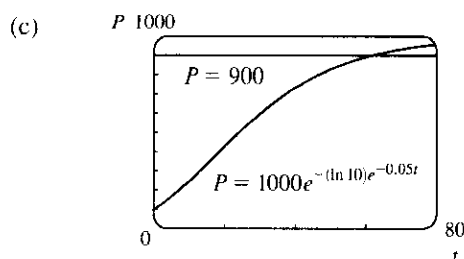
$$\Rightarrow \int -\frac{du}{u} = ct + D \Rightarrow \ln|u| = -ct - D \Rightarrow |u| = e^{-(ct+D)} \Rightarrow |\ln(K/P)| = e^{-(ct+D)} \Rightarrow$$

$$\ln(K/P) = \pm e^{-(ct+D)}. \text{ Letting } t = 0, \text{ we get } \ln(K/P_0) = \pm e^{-D}, \text{ so}$$

$$\ln(K/P) = \pm e^{-ct-D} = \pm e^{-ct} e^{-D} = \ln(K/P_0) e^{-ct} \Rightarrow K/P = e^{\ln(K/P_0) e^{-ct}} \Rightarrow$$

$$P(t) = K e^{-\ln(K/P_0) e^{-ct}}, c \neq 0.$$

$$(b) \lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} K e^{-\ln(K/P_0) e^{-ct}} = K e^{-\ln(K/P_0) \cdot 0} = K e^0 = K$$



The graphs look very similar. For the Gompertz function, $P(40) \approx 732$, nearly the same as the logistic function. The Gompertz function reaches $P = 900$ at $t \approx 61.7$ and its value at $t = 80$ is about 959, so it doesn't increase quite as fast as the logistic curve.

$$(d) \frac{dP}{dt} = c \ln\left(\frac{K}{P}\right) P = cP(\ln K - \ln P) \Rightarrow$$

$$\begin{aligned} \frac{d^2P}{dt^2} &= c \left[P \left(-\frac{1}{P} \frac{dP}{dt} \right) + (\ln K - \ln P) \frac{dP}{dt} \right] = c \frac{dP}{dt} \left[-1 + \ln\left(\frac{K}{P}\right) \right] \\ &= c [c \ln(K/P) P] [\ln(K/P) - 1] = c^2 P \ln(K/P) [\ln(K/P) - 1] \end{aligned}$$

Since $0 < P < K$, $P'' = 0 \Leftrightarrow \ln(K/P) = 1 \Leftrightarrow K/P = e \Leftrightarrow P = K/e$. $P'' > 0$ for

$0 < P < K/e$ and $P'' < 0$ for $K/e < P < K$, so P' is a maximum (and P grows fastest) when $P = K/e$.

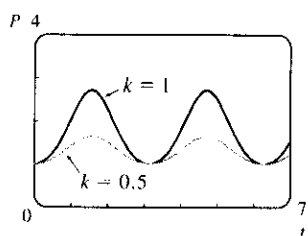
Note: If $P > K$, then $\ln(K/P) < 0$, so $P''(t) > 0$.

15. (a) $dP/dt = kP \cos(rt - \phi) \Rightarrow (dP)/P = k \cos(rt - \phi) dt \Rightarrow \int (dP)/P = k \int \cos(rt - \phi) dt \Rightarrow$
 $\ln P = (k/r) \sin(rt - \phi) + C$. (Since this is a growth model, $P > 0$ and we can write $\ln P$ instead of $\ln|P|$.)
 Since $P(0) = P_0$, we obtain $\ln P_0 = (k/r) \sin(-\phi) + C = -(k/r) \sin \phi + C \Rightarrow$
 $C = \ln P_0 + (k/r) \sin \phi$. Thus, $\ln P = (k/r) \sin(rt - \phi) + \ln P_0 + (k/r) \sin \phi$, which we can rewrite as
 $\ln(P/P_0) = (k/r) [\sin(rt - \phi) + \sin \phi]$ or, after exponentiation, $P(t) = P_0 e^{(k/r) [\sin(rt - \phi) + \sin \phi]}$.

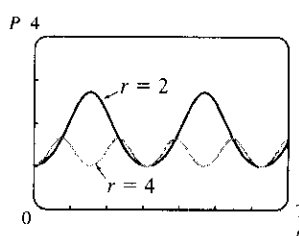
(b) As k increases, the amplitude increases, but the minimum value stays the same.

As r increases, the amplitude and the period decrease.

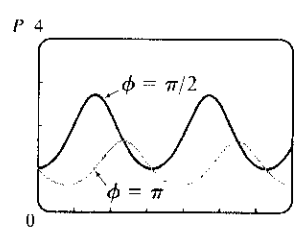
A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0 e^{(k/r)(1+\sin \phi)}$ and $P_0 e^{(k/r)(-1+\sin \phi)}$ (the extreme values are attained when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim_{t \rightarrow \infty} P(t)$ does not exist.

16. (a) $dP/dt = kP \cos^2(rt - \phi) \Rightarrow (dP)/P = k \cos^2(rt - \phi) dt \Rightarrow \int (dP)/P = k \int \cos^2(rt - \phi) dt \Rightarrow$
 $\ln P = k \int \frac{1 + \cos(2(rt - \phi))}{2} dt = \frac{k}{2} t + \frac{k}{4r} \sin(2(rt - \phi)) + C$. From $P(0) = P_0$, we get
 $\ln P_0 = \frac{k}{4r} \sin(-2\phi) + C = C - \frac{k}{4r} \sin 2\phi$, so $C = \ln P_0 + \frac{k}{4r} \sin 2\phi$ and

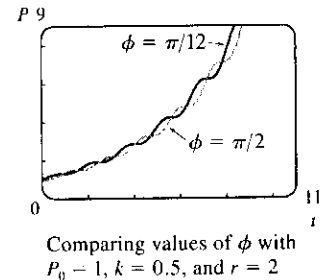
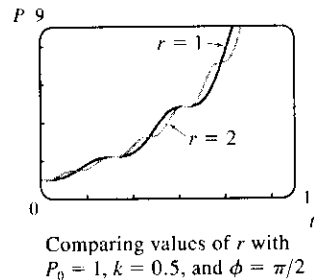
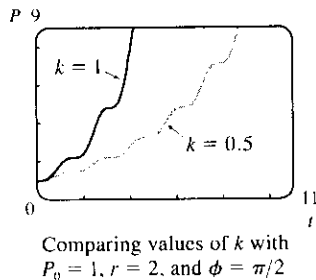
$\ln P = \frac{k}{2}t + \frac{k}{4r} \sin(2(rt - \phi)) + \ln P_0 + \frac{k}{4r} \sin 2\phi$. Simplifying, we get

$$\ln \frac{P}{P_0} = \frac{k}{2}t + \frac{k}{4r} [\sin(2(rt - \phi)) + \sin 2\phi] = f(t), \text{ or } P(t) = P_0 e^{f(t)}.$$

- (b) An increase in k stretches the graph of P vertically while maintaining $P(0) = P_0$.

An increase in r compresses the graph of P horizontally—similar to changing the period in Exercise 15.

As in Exercise 15, a change in ϕ only makes slight adjustments in the growth of P , as shown in the figure.



$f'(t) = k/2 + [k/(4r)][2r \cos(2(rt - \phi))] = (k/2)[1 + \cos(2(rt - \phi))] \geq 0$. Since $P(t) = P_0 e^{f(t)}$, we have $P'(t) = P_0 f'(t) e^{f(t)} \geq 0$, with equality only when $\cos(2(rt - \phi)) = -1$; that is, when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0, \infty)$. P can also be written as

$P(t) = P_0 e^{kt/2} e^{(k/4r)[\sin(2(rt - \phi)) + \sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1 + \sin 2\phi)}$ and $e^{(k/4r)(-1 + \sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim_{t \rightarrow \infty} P(t) = \infty$.

17. By Equation (4), $P(t) = \frac{K}{1 + Ae^{-kt}}$. By comparison, if $c = (\ln A)/k$ and $u = \frac{1}{2}k(t - c)$, then

$$1 + \tanh u = 1 + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{e^u + e^{-u}}{e^u + e^{-u}} + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2e^u}{e^u + e^{-u}} = \frac{2}{1 + e^{-2u}}$$

and $e^{-2u} = e^{-k(t-c)} = e^{kc} e^{-kt} = e^{\ln A} e^{-kt} = Ae^{-kt}$, so

$$\frac{1}{2}K[1 + \tanh(\frac{1}{2}k(t - c))] = \frac{K}{2}[1 + \tanh u] = \frac{K}{2} \cdot \frac{2}{1 + e^{-2u}} = \frac{K}{1 + e^{-2u}} = \frac{K}{1 + Ae^{-kt}} = P(t).$$

10.6 Linear Equations

- $y' + e^x y = x^2 y^2$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)y = Q(x)$.
- $y + \sin x = x^3 y'$ $\Rightarrow$ $x^3 y' - y = \sin x$ $\Rightarrow$ $y' + \left(-\frac{1}{x^3}\right)y = \frac{\sin x}{x^3}$. This equation is in the standard linear form (1), so it is linear.
- $xy' + \ln x - x^2 y = 0$ $\Rightarrow$ $xy' - x^2 y = -\ln x$ $\Rightarrow$ $y' + (-x)y = -\frac{\ln x}{x}$, which is in the standard linear form (1), so this equation is linear.
- $y' + \cos y = \tan x$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)y = Q(x)$. [$\cos y$ is not of the form $P(x)y$.]

5. Comparing the given equation, $y' + 2y = 2e^x$, with the general form, $y' + P(x)y = Q(x)$, we see that $P(x) = 2$ and the integrating factor is $I(x) = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$. Multiplying the differential equation by $I(x)$ gives $e^{2x}y' + 2e^{2x}y = 2e^{3x} \Rightarrow (e^{2x}y)' = 2e^{3x} \Rightarrow e^{2x}y = \int 2e^{3x}dx \Rightarrow e^{2x}y = \frac{2}{3}e^{3x} + C \Rightarrow y = \frac{2}{3}e^x + Ce^{-2x}$.
6. $y' = x + 5y \Rightarrow y' - 5y = x$. $I(x) = e^{\int P(x)dx} = e^{\int (-5)dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x}y' - 5e^{-5x}y = xe^{-5x} \Rightarrow (e^{-5x}y)' = xe^{-5x} \Rightarrow e^{-5x}y = \int xe^{-5x}dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$ [by parts] $\Rightarrow y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$.
7. $xy' - 2y = x^2$ [divide by x] $\Rightarrow y' + \left(-\frac{2}{x}\right)y = x$ (*).
 $I(x) = e^{\int P(x)dx} = e^{\int (-2/x)dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = e^{\ln(1/x^2)} = 1/x^2$. Multiplying the differential equation (*) by $I(x)$ gives $\frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{1}{x} \Rightarrow \left(\frac{1}{x^2}y\right)' = \frac{1}{x} \Rightarrow \frac{1}{x^2}y = \ln|x| + C \Rightarrow y = x^2(\ln|x| + C) = x^2\ln|x| + Cx^2$.
8. $x^2y' + 2xy = \cos^2 x \Rightarrow y' + \frac{2}{x}y = \frac{\cos^2 x}{x^2}$. $I(x) = e^{\int P(x)dx} = e^{\int 2/x dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$. Multiplying by $I(x)$ gives us our original equation back. You may have noticed this immediately, since $P(x)$ is the derivative of the coefficient of y' . We rewrite it as $(x^2y)' = \cos^2 x$. Thus,
 $x^2y = \int \cos^2 x dx = \int \frac{1}{2}(1 + \cos 2x) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \Rightarrow y = \frac{1}{2x} + \frac{1}{4x^2}\sin 2x + \frac{C}{x^2}$ or
 $y = \frac{1}{2x} + \frac{1}{2x^2}\sin x \cos x + \frac{C}{x^2}$.
9. Since $P(x)$ is the derivative of the coefficient of y' [$P(x) = 1$ and the coefficient is x], we can write the differential equation $xy' + y = \sqrt{x}$ in the easily integrable form $(xy)' = \sqrt{x} \Rightarrow xy = \frac{2}{3}x^{3/2} + C \Rightarrow y = \frac{2}{3}\sqrt{x} + C/x$.
10. $y' - y = 1/x$ [$x \neq 0$], so $I(x) = e^{\int (-1)dx} = e^{-x}$. Multiplying the differential equation by $I(x)$ gives $e^{-x}y' - e^{-x}y = e^{-x}/x \Rightarrow (e^{-x}y)' = e^{-x}/x \Rightarrow y = e^x \left[\int (e^{-x}/x) dx + C \right]$.
11. $I(x) = e^{\int 2x dx} = e^{x^2}$. Multiplying the differential equation $y' + 2xy = x^2$ by $I(x)$ gives $e^{x^2}y' + 2xe^{x^2}y = x^2e^{x^2} \Rightarrow (e^{x^2}y)' = x^2e^{x^2}$. Thus
 $y = e^{-x^2} \left[\int x^2e^{x^2} dx + C \right] = e^{-x^2} \left[\frac{1}{2}xe^{x^2} - \int \frac{1}{2}e^{x^2} dx + C \right] = \frac{1}{2}x + Ce^{-x^2} - e^{-x^2} \int \frac{1}{2}e^{x^2} dx$.
12. $I(x) = e^{\int -\tan x dx} = e^{\ln|\cos x|} = \cos x$ (since $-\frac{\pi}{2} < x < \frac{\pi}{2}$). Multiplying the differential equation by $I(x)$ gives $y' \cos x - y \tan x \cos x = x \cos x \sin 2x \Rightarrow (y \cos x)' = x \cos x \sin 2x$. So

$$y = \frac{1}{\cos x} \left[\int x \cos x \sin 2x dx + C \right] = \frac{1}{\cos x} \left[\int 2x \cos^2 x \sin x dx + C \right]$$

$$= \frac{1}{\cos x} \left[\frac{-2x \cos^3 x}{3} + \frac{2}{3} \left(\sin x - \frac{\sin^3 x}{3} \right) + C \right] = \frac{-2x \cos^2 x}{3} + \frac{C}{\cos x} + 2 \tan x \frac{3 - \sin^2 x}{9}$$
13. $(1+t) \frac{du}{dt} + u = 1+t$, $t > 0$ [divide by $1+t$] $\Rightarrow \frac{du}{dt} + \frac{1}{1+t}u = 1$ (*), which has the form $u' + P(t)u = Q(t)$. The integrating factor is $I(t) = e^{\int P(t)dt} = e^{\int [1/(1+t)]dt} = e^{\ln(1+t)} = 1+t$.

Multiplying (*) by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)u]' = 1+t$. Thus,

$$(1+t)u = \int (1+t) dt = t + \frac{1}{2}t^2 + C \Rightarrow u = \frac{t + \frac{1}{2}t^2 + C}{1+t} \text{ or } u = \frac{t^2 + 2t + 2C}{2(t+1)}.$$

14. $t \ln t \frac{dr}{dt} + r = te^t \Rightarrow \frac{dr}{dt} + \frac{1}{t \ln t} r = \frac{e^t}{\ln t}$. $I(t) = e^{\int dt/(t \ln t)} = e^{\ln(\ln t)} = \ln t$. Multiplying by $\ln t$ gives

$$\ln t \frac{dr}{dt} + \frac{1}{t} r = e^t \Rightarrow [(\ln t)r]' = e^t \Rightarrow (\ln t)r = e^t + C \Rightarrow r = \frac{e^t + C}{\ln t}.$$

15. $y' = x + y \Rightarrow y' + (-1)y = x$. $I(x) = e^{\int (-1) dx} = e^{-x}$. Multiplying by e^{-x} gives $e^{-x}y' - e^{-x}y = xe^{-x}$

$$\Rightarrow (e^{-x}y)' = xe^{-x} \Rightarrow e^{-x}y = \int xe^{-x} dx = -xe^{-x} - e^{-x} + C \quad [\text{integration by parts with } u = x,$$

$$dv = e^{-x} dx] \Rightarrow y = -x - 1 + Ce^x. \quad y(0) = 2 \Rightarrow -1 + C = 2 \Rightarrow C = 3, \text{ so } y = -x - 1 + 3e^x.$$

16. $t \frac{dy}{dt} + 2y = t^3, t > 0, y(1) = 0$. Divide by t to get $\frac{dy}{dt} + \frac{2}{t}y = t^2$, which is linear.

$$I(t) = e^{\int (2/t) dt} = e^{2 \ln t} = t^2. \text{ Multiplying by } t^2 \text{ gives } t^2 \frac{dy}{dt} + 2ty = t^4 \Rightarrow (t^2y)' = t^4 \Rightarrow$$

$$t^2y = \frac{1}{5}t^5 + C \Rightarrow y = \frac{t^3}{5} + \frac{C}{t^2}. \text{ Thus, } 0 = y(1) = \frac{1}{5} + C \Rightarrow C = -\frac{1}{5}, \text{ so } y = \frac{t^3}{5} - \frac{1}{5t^2}.$$

17. $\frac{dv}{dt} - 2tv = 3t^2e^{t^2}, v(0) = 5$. $I(t) = e^{\int (-2t) dt} = e^{-t^2}$. Multiply the differential equation by $I(t)$ to get

$$e^{-t^2} \frac{dv}{dt} - 2te^{-t^2}v = 3t^2 \Rightarrow (e^{-t^2}v)' = 3t^2 \Rightarrow e^{-t^2}v = \int 3t^2 dt = t^3 + C \Rightarrow v = t^3e^{t^2} + Ce^{t^2}.$$

$$5 = v(0) = 0 \cdot 1 + C \cdot 1 = C, \text{ so } v = t^3e^{t^2} + 5e^{t^2}.$$

18. $2xy' + y = 6x, x > 0 \Rightarrow y' + \frac{1}{2x}y = 3$. $I(x) = e^{\int 1/(2x) dx} = e^{(1/2) \ln x} = e^{\ln x^{1/2}} = \sqrt{x}$. Multiplying

$$\text{by } \sqrt{x} \text{ gives } \sqrt{x}y' + \frac{1}{2\sqrt{x}}y = 3\sqrt{x} \Rightarrow (\sqrt{x}y)' = 3\sqrt{x} \Rightarrow \sqrt{x}y = \int 3\sqrt{x} dx = 2x^{3/2} + C \Rightarrow$$

$$y = 2x + \frac{C}{\sqrt{x}}. \quad y(4) = 20 \Rightarrow 8 + \frac{C}{2} = 20 \Rightarrow C = 24, \text{ so } y = 2x + \frac{24}{\sqrt{x}}.$$

19. $xy' = y + x^2 \sin x \Rightarrow y' - \frac{1}{x}y = x \sin x$. $I(x) = e^{\int (-1/x) dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$.

$$\text{Multiplying by } \frac{1}{x} \text{ gives } \frac{1}{x}y' - \frac{1}{x^2}y = \sin x \Rightarrow \left(\frac{1}{x}y\right)' = \sin x \Rightarrow \frac{1}{x}y = -\cos x + C \Rightarrow$$

$$y = -x \cos x + Cx. \quad y(\pi) = 0 \Rightarrow -\pi \cdot (-1) + C\pi = 0 \Rightarrow C = -1, \text{ so } y = -x \cos x - x.$$

20. $x \frac{dy}{dx} - \frac{y}{x+1} = x \Rightarrow y' - \frac{y}{x(x+1)} = 1 (x > 0)$, so $I(x) = e^{-\int 1/[x(x+1)] dx} = e^{-(\ln|x| - \ln|x+1|)} = \frac{x+1}{x}$.

$$\text{Multiplying the differential equation by } I(x) \text{ gives } \frac{x+1}{x}y' - \frac{y}{x(x+1)} \frac{x+1}{x} = \frac{x+1}{x} \Rightarrow$$

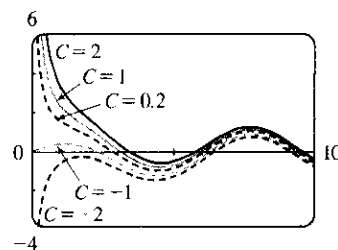
$$\left(\frac{x+1}{x}y\right)' = \frac{x+1}{x}. \text{ Then } y = \frac{x}{x+1} \left[\int \left(1 + \frac{1}{x}\right) dx + C \right] = \frac{x}{x+1} (x + \ln x + C). \text{ But}$$

$$0 = y(1) = \frac{1}{2} [1 + C] \text{ so } C = -1 \text{ and the solution to the initial-value problem is } y = \frac{x}{x+1} (x - 1 + \ln x).$$

21. $y' + \frac{1}{x}y = \cos x$ ($x \neq 0$), so $I(x) = e^{\int (1/x)dx} = e^{\ln|x|} = x$ (for $x > 0$). Multiplying the differential equation by $I(x)$ gives
- $$xy' + y = x \cos x \Rightarrow (xy)' = x \cos x. \text{ Thus,}$$

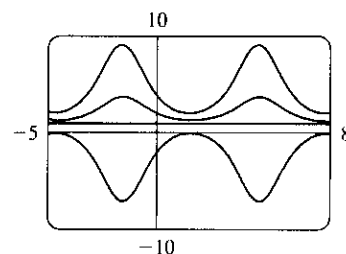
$$y = \frac{1}{x} \left[\int x \cos x dx + C \right] = \frac{1}{x} [x \sin x + \cos x + C]$$

$$= \sin x + \frac{\cos x}{x} + \frac{C}{x}$$



The solutions are asymptotic to the y -axis (except for $C = -1$). In fact, for $C > -1$, $y \rightarrow \infty$ as $x \rightarrow 0^+$, whereas for $C < -1$, $y \rightarrow -\infty$ as $x \rightarrow 0^+$. As x gets larger, the solutions approximate $y = \sin x$ more closely. The graphs for larger C lie above those for smaller C . The distance between the graphs lessens as x increases.

22. $I(x) = e^{\int \cos x dx} = e^{\sin x}$. Multiplying the differential equation by $I(x)$ gives $e^{\sin x}y' + \cos x \cdot e^{\sin x}y = \cos x \cdot e^{\sin x} \Rightarrow$
- $$(e^{\sin x}y)' = \cos x \cdot e^{\sin x} \Rightarrow$$
- $$y = e^{-\sin x} \left[\int \cos x \cdot e^{\sin x} dx + C \right] = 1 + Ce^{-\sin x}.$$
- The graphs for $C = -3, 0, 1$, and 3 are shown. As the values of C get further from zero the graph is stretched away from the line $y = 1$, which is the value for $C = 0$. The graphs are all periodic in x , with a period of 2π .



23. Setting $u = y^{1-n}$, $\frac{du}{dx} = (1-n)y^{-n} \frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{y^n}{1-n} \frac{du}{dx} = \frac{u^{n/(1-n)}}{1-n} \frac{du}{dx}$. Then the Bernoulli differential equation becomes $\frac{u^{n/(1-n)}}{1-n} \frac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$ or $\frac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$.

24. Here $y' + \frac{y}{x} = -y^2$, so $n = 2$, $P(x) = \frac{1}{x}$ and $Q(x) = -1$. Setting $u = y^{-1}$, u satisfies $u' - \frac{1}{x}u = 1$. Then

$$I(x) = e^{\int (-1/x)dx} = \frac{1}{x} \text{ (for } x > 0 \text{) and } u = x \left(\int \frac{1}{x} dx + C \right) = x(\ln|x| + C). \text{ Thus, } y = \frac{1}{x(C + \ln|x|)}.$$

25. $y' + \frac{2}{x}y = \frac{y^3}{x^2}$. Here $n = 3$, $P(x) = \frac{2}{x}$, $Q(x) = \frac{1}{x^2}$ and setting $u = y^{-2}$, u satisfies $u' - \frac{4u}{x} = -\frac{2}{x^2}$.

$$\text{Then } I(x) = e^{\int (-4/x)dx} = x^{-4} \text{ and } u = x^4 \left(\int -\frac{2}{x^6} dx + C \right) = x^4 \left(\frac{2}{5x^5} + C \right) = Cx^4 + \frac{2}{5x}.$$

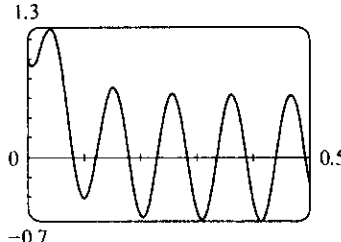
$$\text{Thus, } y = \pm \left(Cx^4 + \frac{2}{5x} \right)^{-1/2}.$$

26. Here $n = 3$, $P(x) = 1$, $Q(x) = x$ and setting $u = y^{-2}$, u satisfies $u' - 2u = -2x$. Then

$$I(x) = e^{\int (-2)dx} = e^{-2x} \text{ and } u = e^{2x} \left[\int -2xe^{-2x} dx + C \right] = e^{2x} \left(xe^{-2x} + \frac{1}{2}e^{-2x} + C \right) = x + \frac{1}{2} + Ce^{2x}.$$

$$\text{So } y^{-2} = x + \frac{1}{2} + Ce^{2x} \Rightarrow y = \pm \left[x + \frac{1}{2} + Ce^{2x} \right]^{-1/2}.$$

27. (a) $2\frac{dI}{dt} + 10I = 40$ or $\frac{dI}{dt} + 5I = 20$. Then the integrating factor is $e^{\int 5 dt} = e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\frac{dI}{dt} + 5Ie^{5t} = 20e^{5t} \Rightarrow (e^{5t}I)' = 20e^{5t} \Rightarrow I(t) = e^{-5t} [\int 20e^{5t} dt + C] = 4 + Ce^{-5t}$. But $0 = I(0) = 4 + C$, so $I(t) = 4 - 4e^{-5t}$.
- (b) $I(0.1) = 4 - 4e^{-0.5} \approx 1.57$ A

28. (a) $\frac{dI}{dt} + 20I = 40 \sin 60t$, so the integrating factor is e^{20t} . Multiplying the differential equation by the integrating factor gives $e^{20t}\frac{dI}{dt} + 20Ie^{20t} = 40e^{20t} \sin 60t \Rightarrow (e^{20t}I)' = 40e^{20t} \sin 60t \Rightarrow$
- $$I(t) = e^{-20t} \left[\int 40e^{20t} \sin 60t dt + C \right]$$
- $$= e^{-20t} \left[40e^{20t} \left(\frac{1}{4000} \right) (20 \sin 60t - 60 \cos 60t) \right] + Ce^{-20t}$$
- $$= \frac{\sin 60t - 3 \cos 60t}{5} + Ce^{-20t}$$
- But $1 = I(0) = -\frac{3}{5} + C$, so
- $$I(t) = \frac{\sin 60t - 3 \cos 60t + 8e^{-20t}}{5}.$$
- (b) $I(0.1) = \frac{\sin 6 - 3 \cos 6 + 8e^{-2}}{5} \approx -0.42$ A
- (c) 

29. $5\frac{dQ}{dt} + 20Q = 60$ with $Q(0) = 0$ C. Then the integrating factor is $e^{\int 4 dt} = e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\frac{dQ}{dt} + 4e^{4t}Q = 12e^{4t} \Rightarrow (e^{4t}Q)' = 12e^{4t} \Rightarrow$
- $$Q(t) = e^{-4t} [\int 12e^{4t} dt + C] = 3 + Ce^{-4t}.$$
- But $0 = Q(0) = 3 + C$ so $Q(t) = 3(1 - e^{-4t})$ is the charge at time t and $I = dQ/dt = 12e^{-4t}$ is the current at time t .

30. $2\frac{dQ}{dt} + 100Q = 10 \sin 60t$ or $\frac{dQ}{dt} + 50Q = 5 \sin 60t$. Then the integrating factor is $e^{\int 50 dt} = e^{50t}$, and multiplying the differential equation by the integrating factor gives $e^{50t}\frac{dQ}{dt} + 50e^{50t}Q = 5e^{50t} \sin 60t \Rightarrow$
- $$(e^{50t}Q)' = 5e^{50t} \sin 60t \Rightarrow$$

$$Q(t) = e^{-50t} \left[\int 5e^{50t} \sin 60t dt + C \right] = e^{-50t} \left[5e^{50t} \left(\frac{1}{6100} \right) (50 \sin 60t - 60 \cos 60t) \right] + Ce^{-50t}$$

$$= \frac{1}{122} (5 \sin 60t - 6 \cos 60t) + Ce^{-50t}$$

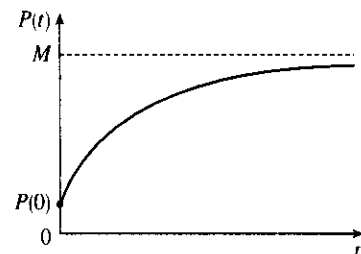
But $0 = Q(0) = -\frac{6}{122} + C$ so $C = \frac{3}{61}$ and $Q(t) = \frac{5 \sin 60t - 6 \cos 60t}{122} + \frac{3e^{-50t}}{61}$ is the charge at time t , while the current is $I(t) = \frac{dQ}{dt} = \frac{150 \cos 60t + 180 \sin 60t - 150e^{-50t}}{61}$.

31. $\frac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k dt} = e^{kt}$. Multiplying the differential

equation by $I(t)$ gives $e^{kt}\frac{dP}{dt} + kPe^{kt} = kMe^{kt} \Rightarrow$

$$(e^{kt}P)' = kMe^{kt} \Rightarrow$$

$P(t) = e^{-kt} (\int kMe^{kt} dt + C) = M + Ce^{-kt}$, $k > 0$. Furthermore, it is reasonable to assume that $0 \leq P(0) \leq M$, so $-M \leq C \leq 0$.



32. Since $P(0) = 0$, we have $P(t) = M_1(1 - e^{-kt})$. If $P_1(t)$ is Jim's learning curve, then $P_1(1) = 25$ and $P_1(2) = 45$. Hence, $25 = M_1(1 - e^{-k})$ and $45 = M_1(1 - e^{-2k})$, so $1 - 25/M_1 = e^{-k}$ or
- $$k = -\ln\left(1 - \frac{25}{M_1}\right) = \ln\left(\frac{M_1}{M_1 - 25}\right).$$
- But $45 = M_1(1 - e^{-2k})$ so $45 = M_1\left[1 - \left(\frac{M_1 - 25}{M_1}\right)^2\right]$ or
- $$45 = \frac{50M_1 - 625}{M_1}.$$
- Thus, $M_1 = 125$ is the maximum number of units per hour Jim is capable of processing. Similarly, if $P_2(t)$ is Mark's learning curve, then $P_2(1) = 35$ and $P_2(2) = 50$. So $k = \ln\left(\frac{M_2}{M_2 - 35}\right)$ and
- $$50 = M_2\left[1 - \left(\frac{M_2 - 35}{M_2}\right)^2\right]$$
- or $M_2 = 61.25$. Hence the maximum number of units per hour for Mark is approximately 61. Another approach would be to use the midpoints of the intervals so that $P_1(0.5) = 25$ and $P_1(1.5) = 45$. Doing so gives us $M_1 \approx 52.6$ and $M_2 \approx 51.8$.

33. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4 \frac{\text{kg}}{\text{L}}\right)\left(5 \frac{\text{L}}{\text{min}}\right) = 2 \frac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\frac{y(t)}{100 + 2t} \frac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\frac{y(t)}{100 + 2t} \frac{\text{kg}}{\text{L}}\right)\left(3 \frac{\text{L}}{\text{min}}\right) = \frac{3y}{100 + 2t} \frac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$. Rewriting this equation as $\frac{dy}{dt} + \left(\frac{3}{100 + 2t}\right)y = 2$, we see that it is linear. $I(t) = \exp\left(\int \frac{3 dt}{100 + 2t}\right) = \exp\left(\frac{3}{2} \ln(100 + 2t)\right) = (100 + 2t)^{3/2}$. Multiplying the differential equation by $I(t)$ gives $(100 + 2t)^{3/2} \frac{dy}{dt} + 3(100 + 2t)^{1/2}y = 2(100 + 2t)^{3/2} \Rightarrow$
- $$\left[(100 + 2t)^{3/2}y\right]' = 2(100 + 2t)^{3/2} \Rightarrow (100 + 2t)^{3/2}y = \frac{2}{5}(100 + 2t)^{5/2} + C \Rightarrow$$
- $$y = \frac{2}{5}(100 + 2t) + C(100 + 2t)^{-3/2}.$$
- Now $0 = y(0) = \frac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \frac{1}{1000}C \Rightarrow C = -40,000$, so $y = \left[\frac{2}{5}(100 + 2t) - 40,000(100 + 2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t) = \frac{y(t)}{100 + 2t} = \left[\frac{-40,000}{(100 + 2t)^{5/2}} + \frac{2}{5}\right] \frac{\text{kg}}{\text{L}}$. In particular,
- $$C(20) = \frac{-40,000}{140^{5/2}} + \frac{2}{5} \approx 0.2275 \frac{\text{kg}}{\text{L}} \text{ and } y(20) = \frac{2}{5}(140) - 40,000(140)^{-3/2} \approx 31.85 \text{ kg}.$$

34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0) = (0.05 \text{ g/L})(400 \text{ L}) = 20 \text{ g}$. The amount of liquid in the tank at time t is $(400 - 6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\frac{y(t)}{400 - 6t} \frac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\frac{y(t)}{400 - 6t} \frac{\text{g}}{\text{L}}\right]\left[10 \frac{\text{L}}{\text{s}}\right] = \frac{10y(t)}{400 - 6t} \frac{\text{g}}{\text{s}} = \frac{5y(t)}{200 - 3t} \frac{\text{g}}{\text{s}}$. Therefore,

$$\frac{dy}{dt} = -\frac{5y}{200-3t} \Rightarrow \int \frac{dy}{y} = \int \frac{-5 dt}{200-3t} \Rightarrow \ln y = \frac{5}{3} \ln(200-3t) + C \Rightarrow$$

$$y = \exp\left(\frac{5}{3} \ln(200-3t) + C\right) = e^C (200-3t)^{5/3}. \text{ Now } 20 = y(0) = e^C \cdot 200^{5/3} \Rightarrow e^C = \frac{20}{200^{5/3}}, \text{ so}$$

$$y(t) = 20 \frac{(200-3t)^{5/3}}{200^{5/3}} = 20(1-0.015t)^{5/3} \text{ g for } 0 \leq t \leq 66\frac{2}{3} \text{ s, at which time the tank is empty.}$$

35. (a) $\frac{dv}{dt} + \frac{c}{m}v = g$ and $I(t) = e^{\int (c/m) dt} = e^{(c/m)t}$, and multiplying the differential equation by $I(t)$ gives

$$e^{(c/m)t} \frac{dv}{dt} + \frac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \Rightarrow \left[e^{(c/m)t} v \right]' = ge^{(c/m)t}. \text{ Hence,}$$

$$v(t) = e^{-(c/m)t} \left[\int ge^{(c/m)t} dt + K \right] = mg/c + Ke^{-(c/m)t}. \text{ But the object is dropped from rest, so } v(0) = 0$$

$$\text{and } K = -mg/c. \text{ Thus, the velocity at time } t \text{ is } v(t) = (mg/c) \left[1 - e^{-(c/m)t} \right].$$

(b) $\lim_{t \rightarrow \infty} v(t) = mg/c$

(c) $s(t) = \int v(t) dt = (mg/c) \left[t + (m/c)e^{-(c/m)t} \right] + c_1$ where $c_1 = s(0) - m^2g/c^2$. $s(0)$ is the initial position,

$$\text{so } s(0) = 0 \text{ and } s(t) = (mg/c) \left[t + (m/c)e^{-(c/m)t} \right] - m^2g/c^2.$$

36. $v = (mg/c)(1 - e^{-ct/m}) \Rightarrow \frac{dv}{dm} = \frac{mg}{c} \left(0 - e^{-ct/m} \cdot \frac{ct}{m^2} \right) + \frac{g}{c} (1 - e^{-ct/m}) \cdot 1 =$

$$-\frac{gt}{m} e^{-ct/m} + \frac{g}{c} - \frac{g}{c} e^{-ct/m} = \frac{g}{c} \left(1 - e^{-ct/m} - \frac{ct}{m} e^{-ct/m} \right) \Rightarrow$$

$$\frac{c}{g} \frac{dv}{dm} = 1 - \left(1 + \frac{ct}{m} \right) e^{-ct/m} = 1 - \frac{1 + ct/m}{e^{ct/m}} = 1 - \frac{1+Q}{e^Q}, \text{ where } Q = \frac{ct}{m} \geq 0. \text{ Since } e^Q > 1+Q \text{ for all}$$

$Q > 0$, it follows that $dv/dm > 0$ for $t > 0$. In other words, for all $t > 0$, v increases as m increases.

10.7 Predator-Prey Systems

- (a) $dx/dt = -0.05x + 0.0001xy$. If $y = 0$, we have $dx/dt = -0.05x$, which indicates that in the absence of y , x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt = 0.1y - 0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.
- (b) $dy/dt = -0.015y + 0.00008xy$. If $x = 0$, we have $dy/dt = -0.015y$, which indicates that in the absence of x , y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt = 0.2x - 0.0002x^2 - 0.006xy = 0.2x(1 - 0.001x) - 0.006xy$), is restricted by a carrying capacity of 1000 [from the term $1 - 0.001x = 1 - x/1000$] and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

2. (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$, $dy/dt = 0.08y + 0.00004xy$.

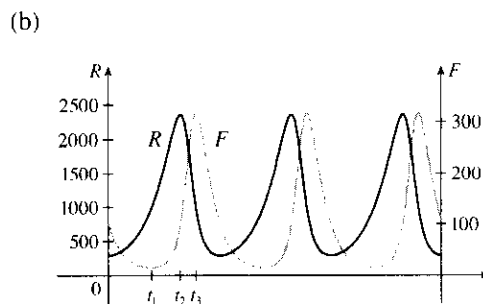
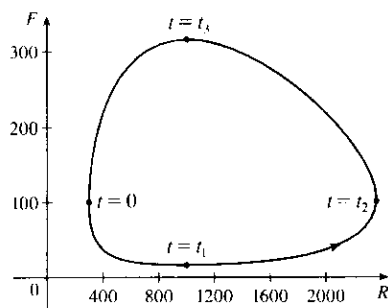
The xy terms represent encounters between the two species x and y . An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x(1 - x/750) - 0.0006xy$.

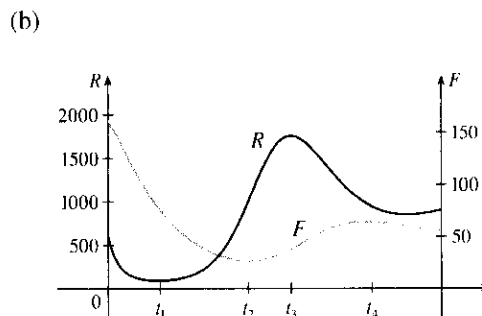
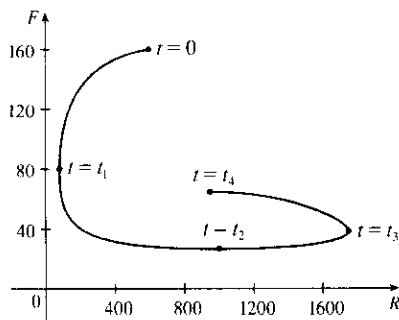
$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y(1 - y/2500) - 0.0002xy$.

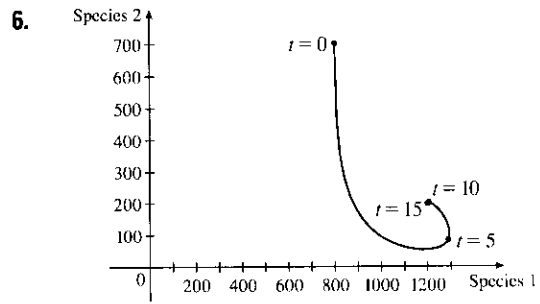
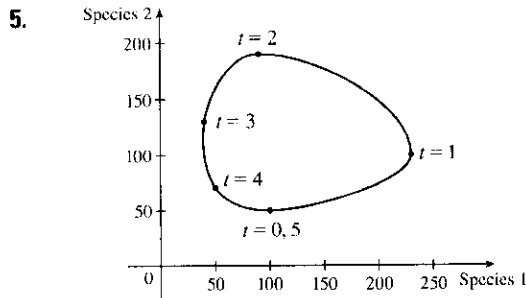
The system shows that x and y have carrying capacities of 750 and 2500. An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

3. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



4. (a) At $t = 0$, there are about 600 rabbits and 160 foxes. At $t = t_1$, the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80. At $t = t_2$, the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000. At $t = t_3$, the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750. The curve ends at $t = t_4$, where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950.





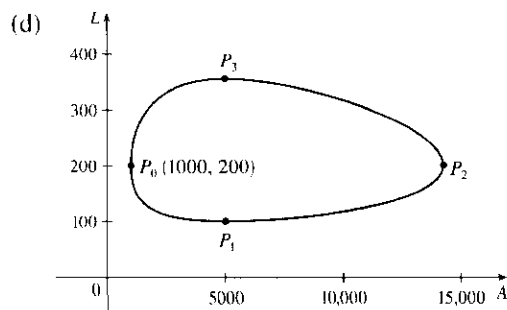
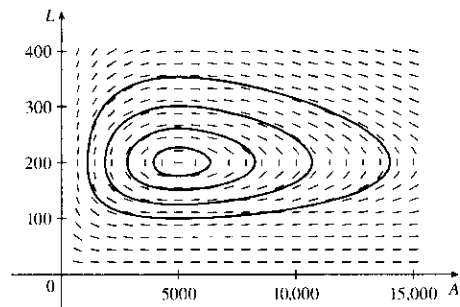
7. $\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW} \Leftrightarrow (0.08 - 0.001W)R dW = (-0.02 + 0.00002R)W dR \Leftrightarrow$
 $\frac{0.08 - 0.001W}{W} dW = \frac{-0.02 + 0.00002R}{R} dR \Leftrightarrow \int \left(\frac{0.08}{W} - 0.001 \right) dW = \int \left(-\frac{0.02}{R} + 0.00002 \right) dR$
 $\Leftrightarrow 0.08 \ln|W| - 0.001W = -0.02 \ln|R| + 0.00002R + K \Leftrightarrow$
 $0.08 \ln W + 0.02 \ln R = 0.001W + 0.00002R + K \Leftrightarrow \ln(W^{0.08} R^{0.02}) = 0.00002R + 0.001W + K \Leftrightarrow$
 $W^{0.08} R^{0.02} = e^{0.00002R + 0.001W + K} \Leftrightarrow R^{0.02} W^{0.08} = C e^{0.00002R} e^{0.001W} \Leftrightarrow \frac{R^{0.02} W^{0.08}}{e^{0.00002R} e^{0.001W}} = C.$
 In general, if $\frac{dy}{dx} = \frac{-ry + bxy}{kx - axy}$, then $C = \frac{x^r y^k}{e^{bx} e^{ay}}.$

8. (a) A and L are constant $\Rightarrow A' = 0$ and $L' = 0 \Rightarrow \begin{cases} 0 = 2A - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{cases} \Rightarrow \begin{cases} 0 = A(2 - 0.01L) \\ 0 = L(-0.5 + 0.0001A) \end{cases}$

So either $A = L = 0$ or $L = \frac{2}{0.01} = 200$ and $A = \frac{0.5}{0.0001} = 5000$. The trivial solution $A = L = 0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non-trivial solution, $L = 200$ and $A = 5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\frac{dL}{dA} = \frac{dL/dt}{dA/dt} = \frac{-0.5L + 0.0001AL}{2A - 0.01AL}$

(c) The solution curves (phase trajectories) are all closed curves that have the equilibrium point (5000, 200) inside them.

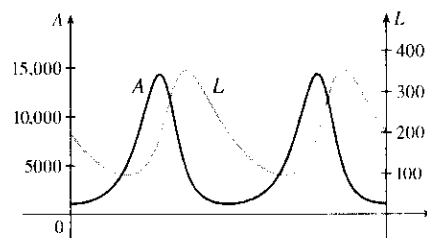


At $P_0(1000, 200)$, $dA/dt = 0$ and $dL/dt = -80 < 0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0 , there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000, 100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14,250, 200)$.

Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000, 355)$, and as we pass through P_2 , the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum

population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0 , where the cycle starts over again.

- (e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A .



9. (a) Letting $W = 0$ gives us $dR/dt = 0.08R(1 - 0.0002R)$. $dR/dt = 0 \Leftrightarrow R = 0$ or 5000 . Since $dR/dt > 0$ for $0 < R < 5000$, we would expect the rabbit population to *increase* to 5000 for these values of R . Since $dR/dt < 0$ for $R > 5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R . Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000.

- (b) R and W are constant $\Rightarrow R' = 0$ and $W' = 0 \Rightarrow$

$$\begin{cases} 0 = 0.08R(1 - 0.0002R) - 0.001RW \\ 0 = -0.02W + 0.00002RW \end{cases} \Rightarrow \begin{cases} 0 = R[0.08(1 - 0.0002R) - 0.001W] \\ 0 = W(-0.02 + 0.00002R) \end{cases}$$

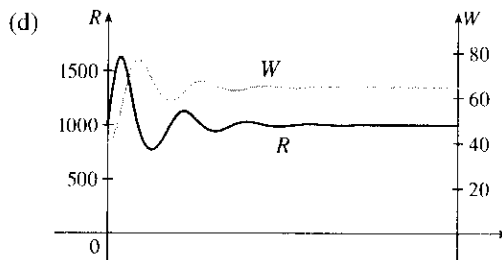
The second equation is true if $W = 0$ or $R = \frac{0.02}{0.00002} = 1000$. If $W = 0$ in the first equation, then either $R = 0$ or $R = \frac{1}{0.0002} = 5000$ [as in part (a)]. If $R = 1000$, then $0 = 1000[0.08(1 - 0.0002 \cdot 1000) - 0.001W] \Leftrightarrow 0 = 80(1 - 0.2) - W \Leftrightarrow W = 64$.

Case (i): $W = 0, R = 0$: both populations are zero

Case (ii): $W = 0, R = 5000$: see part (a)

Case (iii): $R = 1000, W = 64$: the predator/prey interaction balances and the populations are stable.

- (c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.



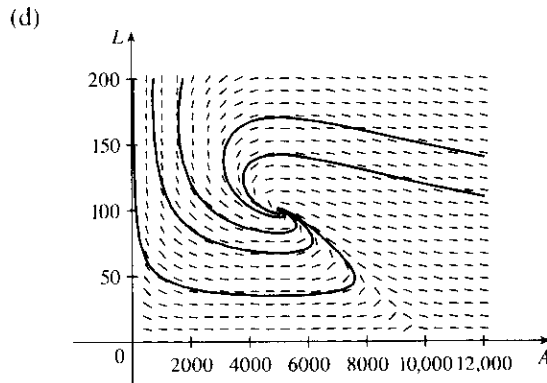
10. (a) If $L = 0$, $dA/dt = 2A(1 - 0.0001A)$, so $dA/dt = 0 \Leftrightarrow A = 0$ or $A = \frac{1}{0.0001} = 10,000$. Since $dA/dt > 0$ for $0 < A < 10,000$, we expect the aphid population to *increase* to 10,000 for these values of A . Since $dA/dt < 0$ for $A > 10,000$, we expect the aphid population to *decrease* to 10,000 for these values of A . Hence, in the absence of ladybugs we expect the aphid population to stabilize at 10,000.

- (b) A and L are constant $\Rightarrow A' = 0$ and $L' = 0 \Rightarrow$

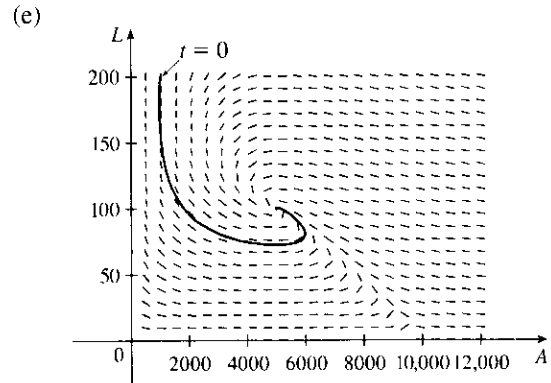
$$\begin{cases} 0 = 2A(1 - 0.0001A) - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{cases} \Rightarrow \begin{cases} 0 = A[2(1 - 0.0001A) - 0.01L] \\ 0 = L(-0.5 + 0.0001A) \end{cases}$$

The second equation is true if $L = 0$ or $A = \frac{0.5}{0.0001} = 5000$. If $L = 0$ in the first equation, then either $A = 0$ or $A = \frac{1}{0.0001} = 10,000$. If $A = 5000$, then $0 = 5000[2(1 - 0.0001 \cdot 5000) - 0.01L] \Leftrightarrow 0 = 10,000(1 - 0.5) - 50L \Leftrightarrow 50L = 5000 \Leftrightarrow L = 100$. The equilibrium solutions are:
 (i) $L = 0, A = 0$ (ii) $L = 0, A = 10,000$ (iii) $A = 5000, L = 100$

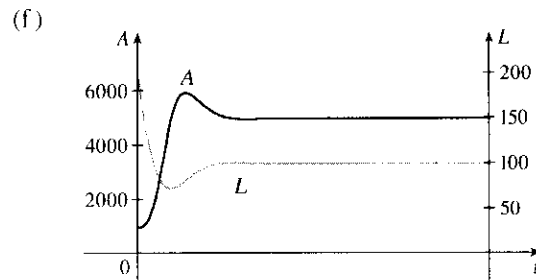
$$(c) \frac{dL}{dA} = \frac{dL/dt}{dA/dt} = \frac{-0.5L + 0.0001AL}{2A(1 - 0.0001A) - 0.01AL}$$



All of the phase trajectories spiral tightly around the equilibrium solution (5000, 100).



At $t = 0$, the ladybug population decreases rapidly and the aphid population decreases slightly before beginning to increase. As the aphid population continues to increase, the ladybug population reaches a minimum at about (5000, 75). The ladybug population starts to increase and quickly stabilizes at 100, while the aphid population stabilizes at 5000.



The graph of A peaks just after the graph of L has a minimum.

10 Review

CONCEPT CHECK

- (a) A differential equation is an equation that contains an unknown function and one or more of its derivatives.

(b) The order of a differential equation is the order of the highest derivative that occurs in the equation.

(c) An initial condition is a condition of the form $y(t_0) = y_0$.
- $y' = x^2 + y^2 \geq 0$ for all x and y . $y' = 0$ only at the origin, so there is a horizontal tangent at $(0, 0)$, but nowhere else. The graph of the solution is increasing on every interval.
- See the paragraph preceding Example 1 in Section 10.2.

4. See the paragraph after Figure 14 in Section 10.2.
5. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y , that is, $dy/dx = g(x)f(y)$. We can solve the equation by integrating both sides of the equation $dy/f(y) = g(x)dx$ and solving for y .
6. A first-order linear differential equation is a differential equation that can be put in the form $\frac{dy}{dx} + P(x)y = Q(x)$, where P and Q are continuous functions on a given interval. To solve such an equation, multiply it by the integrating factor $I(x) = e^{\int P(x)dx}$ to put it in the form $[I(x)y]' = I(x)Q(x)$ and then integrate both sides to get $I(x)y = \int I(x)Q(x)dx$, that is, $e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx$. Solving for y gives us $y = e^{-\int P(x)dx} \int e^{\int P(x)dx}Q(x)dx$.
7. (a) $\frac{dy}{dt} = ky$; the relative growth rate, $\frac{1}{y} \frac{dy}{dt}$, is constant.
 (b) The equation in part (a) is an appropriate model for population growth, assuming that there is enough room and nutrition to support the growth.
 (c) If $y(0) = y_0$, then the solution is $y(t) = y_0 e^{kt}$.
8. (a) $dP/dt = kP(1 - P/K)$, where K is the carrying capacity.
 (b) The equation in part (a) is an appropriate model for population growth, assuming that the population grows at a rate proportional to the size of the population in the beginning, but eventually levels off and approaches its carrying capacity because of limited resources.
9. (a) $dF/dt = kF - aFS$ and $dS/dt = -rS + bFS$.
 (b) In the absence of sharks, an ample food supply would support exponential growth of the fish population, that is, $dF/dt = kF$, where k is a positive constant. In the absence of fish, we assume that the shark population would decline at a rate proportional to itself, that is, $dS/dt = -rS$, where r is a positive constant.

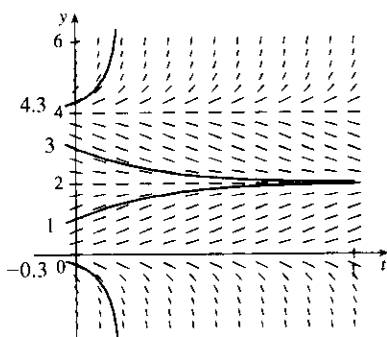
 TRUE-FALSE QUIZ

1. True. Since $y^4 \geq 0$, $y' = -1 - y^4 < 0$ and the solutions are decreasing functions.
2. True. $y = \frac{\ln x}{x} \Rightarrow y' = \frac{1 - \ln x}{x^2}$.

$$\text{LHS} = x^2 y' + xy = x^2 \cdot \frac{1 - \ln x}{x^2} + x \cdot \frac{\ln x}{x} = (1 - \ln x) + \ln x = 1 = \text{RHS, so } y = \frac{\ln x}{x} \text{ is a solution}$$
 of $x^2 y' + xy = 1$.
3. False. $x + y$ cannot be written in the form $g(x)f(y)$.
4. True. $y' = 3y - 2x + 6xy - 1 = 6xy - 2x + 3y - 1 = 2x(3y - 1) + 1(3y - 1) = (2x + 1)(3y - 1)$, so y' can be written in the form $g(x)f(y)$, and hence, is separable.
5. True. $e^x y' = y \Rightarrow y' = e^{-x} y \Rightarrow y' + (-e^{-x})y = 0$, which is of the form $y' + P(x)y = Q(x)$, so the equation is linear.
6. False. $y' + xy = e^y$ cannot be put in the form $y' + P(x)y = Q(x)$, so it is not linear.
7. True. By comparing $\frac{dy}{dt} = 2y\left(1 - \frac{y}{5}\right)$ with the logistic differential equation (10.5.1), we see that the carrying capacity is 5; that is, $\lim_{t \rightarrow \infty} y = 5$.

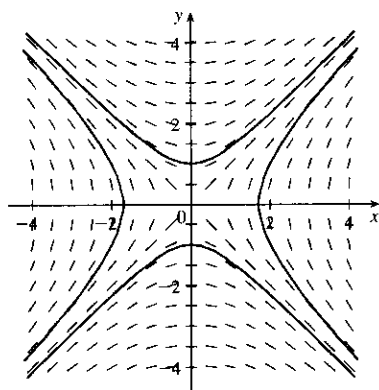
EXERCISES

1. (a)



- (b) $\lim_{t \rightarrow \infty} y(t)$ appears to be finite for $0 \leq c \leq 4$. In fact $\lim_{t \rightarrow \infty} y(t) = 4$ for $c = 4$, $\lim_{t \rightarrow \infty} y(t) = 2$ for $0 < c < 4$, and $\lim_{t \rightarrow \infty} y(t) = 0$ for $c = 0$. The equilibrium solutions are $y(t) = 0$, $y(t) = 2$, and $y(t) = 4$.

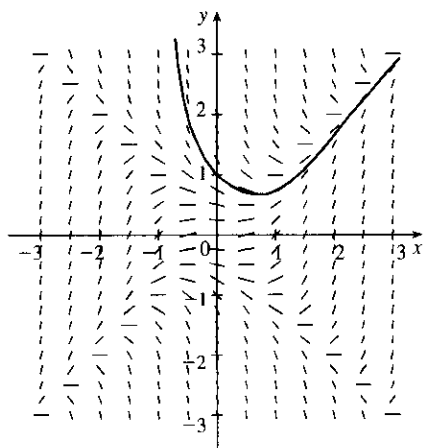
2. (a)



We sketch the direction field and four solution curves, as shown. Note that the slope $y' = x/y$ is not defined on the line $y = 0$.

- (b) $y' = x/y \Leftrightarrow y \, dy = x \, dx \Leftrightarrow y^2 = x^2 + C$. For $C = 0$, this is the pair of lines $y = \pm x$. For $C \neq 0$, it is the hyperbola $x^2 - y^2 = -C$.

3. (a)



- (b) $h = 0.1$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = x^2 - y^2$. So $y_n = y_{n-1} + 0.1(x_{n-1}^2 - y_{n-1}^2)$. Thus,
 $y_1 = 1 + 0.1(0^2 - 1^2) = 0.9$,
 $y_2 = 0.9 + 0.1(0.1^2 - 0.9^2) = 0.82$,
 $y_3 = 0.82 + 0.1(0.2^2 - 0.82^2) = 0.75676$. This is close to our graphical estimate of $y(0.3) \approx 0.8$.

- (c) The centers of the horizontal line segments of the direction field are located on the lines $y = x$ and $y = -x$. When a solution curve crosses one of these lines, it has a local maximum or minimum.

We estimate that when $x = 0.3$, $y = 0.8$, so $y(0.3) \approx 0.8$.

4. (a) $h = 0.2$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = 2xy^2$. We need y_2 .

$$y_1 = 1 + 0.2(2 \cdot 0 \cdot 1^2) = 1, \quad y_2 = 1 + 0.2(2 \cdot 0.2 \cdot 1^2) = 1.08 \approx y(0.4).$$

(b) $h = 0.1$ now, so $y_1 = 1 + 0.1(2 \cdot 0 \cdot 1^2) = 1$, $y_2 = 1 + 0.1(2 \cdot 0.1 \cdot 1^2) = 1.02$,

$$y_3 = 1.02 + 0.1(2 \cdot 0.2 \cdot 1.02^2) \approx 1.06162, \quad y_4 = 1.06162 + 0.1(2 \cdot 0.3 \cdot 1.06162^2) \approx 1.1292 \approx y(0.4).$$

(c) The equation is separable, so we write $\frac{dy}{y^2} = 2x dx \Rightarrow \int \frac{dy}{y^2} = \int 2x dx \Leftrightarrow -\frac{1}{y} = x^2 + C$, but

$y(0) = 1$, so $C = -1$ and $y(x) = \frac{1}{1-x^2} \Leftrightarrow y(0.4) = \frac{1}{1-0.16} \approx 1.1905$. From this we see that the approximation was greatly improved by increasing the number of steps, but the approximations were still far off.

5. $y' = xe^{-\sin x} - y \cos x \Rightarrow y' + (\cos x)y = xe^{-\sin x}$ (*). This is a linear equation and the integrating factor is $I(x) = e^{\int \cos x dx} = e^{\sin x}$. Multiplying (*) by $e^{\sin x}$ gives $e^{\sin x} y' + e^{\sin x} (\cos x)y = x \Rightarrow (e^{\sin x} y)' = x \Rightarrow e^{\sin x} y = \frac{1}{2}x^2 + C \Rightarrow y = (\frac{1}{2}x^2 + C)e^{-\sin x}$.

6. $\frac{dx}{dt} = 1 - t + x - tx = 1(1-t) + x(1-t) = (1+x)(1-t) \Rightarrow \frac{dx}{1+x} = (1-t) dt \Rightarrow \int \frac{dx}{1+x} = \int (1-t) dt \Rightarrow \ln|1+x| = t - \frac{1}{2}t^2 + C \Rightarrow |1+x| = e^{t-t^2/2+C} \Rightarrow 1+x = \pm e^{t-t^2/2} \cdot e^C \Rightarrow x = -1 + Ke^{t-t^2/2}$, where K is any nonzero constant.

7. $(3y^2 + 2y)y' = x \cos x \Rightarrow (3y^2 + 2y) dy = (x \cos x) dx \Rightarrow \int (3y^2 + 2y) dy = \int (x \cos x) dx \Rightarrow y^3 + y^2 = \cos x + x \sin x + C$. For the last step, use integration by parts or Formula 83 in the Table of Integrals.

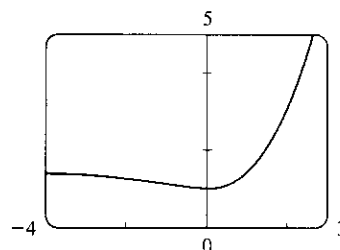
8. $x^2 y' - y = 2x^3 e^{-1/x} \Rightarrow y' - \frac{1}{x^2} y = 2xe^{-1/x}$ (*). This is a linear equation and the integrating factor is $I(x) = e^{\int (-1/x^2) dx} = e^{1/x}$. Multiplying (*) by $e^{1/x}$ gives $e^{1/x} y' - e^{1/x} \cdot \frac{1}{x^2} y = 2x \Rightarrow (e^{1/x} y)' = 2x \Rightarrow e^{1/x} y = x^2 + C \Rightarrow y = e^{-1/x}(x^2 + C)$.

9. $xyy' = \ln x \Rightarrow y dy = \frac{\ln x}{x} dx \Rightarrow \int y dy = \int \frac{\ln x}{x} dx$ (Make the substitution $u = \ln x$; then $du = dx/x$.) So $\int y dy = \int u du \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}u^2 + C \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C$. $y(1) = 2 \Rightarrow \frac{1}{2}2^2 = \frac{1}{2}(\ln 1)^2 + C = C \Leftrightarrow C = 2$. Therefore, $\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + 2$, or $y = \sqrt{(\ln x)^2 + 4}$. The negative square root is inadmissible, since $y(1) > 0$.

10. $1 + x = 2xyy' \Rightarrow y' = \frac{1+x}{2xy} \Leftrightarrow y dy = \frac{1+x}{2x} dx \Rightarrow \frac{y^2}{2} = \frac{\ln|x|}{2} + \frac{x}{2} + c_1$. But $x > 0$, so $y^2 = \ln x + x + c \Leftrightarrow y(x) = \pm \sqrt{c + x + \ln x}$. But $-2 = y(1)$ so choose the negative square root and $-2 = -\sqrt{c+1}$ so $c = 3$. Thus, the solution is $y(x) = -\sqrt{3 + x + \ln x}$.

11. Since the equation is linear, let $I(x) = e^{\int dx} = e^x$. Then multiplying by $I(x)$ gives $e^x y' + e^x y = \sqrt{x} \Rightarrow (e^x y)' = \sqrt{x} \Rightarrow y(x) = e^{-x} (\int \sqrt{x} dx + c) = e^{-x} (\frac{2}{3}x^{3/2} + c)$. But $3 = y(0) = c$, so the solution to the initial-value problem is $y(x) = e^{-x} (\frac{2}{3}x^{3/2} + 3)$.

12. $2yy' = xe^x \Rightarrow \int 2y dy = \int xe^x dx \Rightarrow y^2 = xe^x - \int e^x dx$ (by parts) $= (x-1)e^x + C$. We substitute the initial condition: $1^2 = (0-1)e^0 + C \Rightarrow C = 2$. So the solution is $y = \sqrt{(x-1)e^x + 2}$. The negative square root is inadmissible due to the initial condition.



13. The curves $kx^2 + y^2 = 1$ form a family of ellipses for $k > 0$, a family of hyperbolas for $k < 0$, and two parallel lines $y = \pm 1$ for $k = 0$. Solving $kx^2 + y^2 = 1$ for k gives $k = \frac{1-y^2}{x^2}$. Differentiating gives $2kx + 2yy' = 0 \Leftrightarrow y' = -\frac{kx}{y} = -(1-y^2)\frac{x}{yx^2} = \frac{y^2-1}{xy}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = -\frac{xy}{y^2-1} \Rightarrow \frac{y^2-1}{y} dy = -x dx \Rightarrow \frac{y^2}{2} - \ln|y| = \frac{-x^2}{2} + K \Rightarrow y^2 - 2\ln|y| + x^2 = C$. For $k = 0$, the orthogonal trajectories are given by $x = C_1$ for C_1 an arbitrary constant.
14. Differentiating both sides of $y = \frac{k}{1+x^2}$ gives $y' = -\frac{2kx}{(1+x^2)^2} = -2xy \frac{1+x^2}{(1+x^2)^2} = -\frac{2xy}{1+x^2}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = \frac{1+x^2}{2xy} \Rightarrow 2y dy = \left(\frac{1}{x} + x\right) dx \Rightarrow y^2 = \frac{x^2}{2} + \ln|x| + C$. For $k = 0$, the orthogonal trajectories are given by $x = C_2$ for C_2 an arbitrary constant.
15. (a) $y(t) = y(0)e^{kt} = 1000e^{kt} \Rightarrow y(2) = 1000e^{2k} = 9000 \Rightarrow e^{2k} = 9 \Rightarrow 2k = \ln 9 \Rightarrow k = \frac{1}{2} \ln 9 = \ln 3 \Rightarrow y(t) = 1000e^{(\ln 3)t} = 1000 \cdot 3^t$
 (b) $y(3) = 1000 \cdot 3^3 = 27,000$
 (c) $y'(t) = 1000 \cdot 3^t \cdot \ln 3$, so $y'(3) = 27,000 \ln 3 \approx 29,663$ bacteria per hour
 (d) $1000 \cdot 3^t = 2 \cdot 1000 \Rightarrow 3^t = 2 \Rightarrow t \ln 3 = \ln 2 \Rightarrow t = (\ln 2)/\ln 3 \approx 0.63$ h
16. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 18e^{kt}$. $y(25) = 18e^{25k} = \frac{1}{2} \cdot 18 \Rightarrow e^{25k} = \frac{1}{2} \Rightarrow 25k = -\ln 2 \Rightarrow k = -\frac{1}{25} \ln 2 \Rightarrow y(t) = 18e^{-(\ln 2)t/25} = 18 \cdot 2^{-t/25}$.
 (b) $18 \cdot 2^{-t/25} = 2 \Rightarrow 2^{-t/25} = \frac{1}{9} \Rightarrow -\frac{1}{25}t \ln 2 = -\ln 9 \Rightarrow t = 25 \frac{\ln 9}{\ln 2} \approx 79$ years
17. (a) $C'(t) = -kC(t) \Rightarrow C(t) = C(0)e^{-kt}$ by Theorem 10.4.2. But $C(0) = C_0$, so $C(t) = C_0e^{-kt}$.
 (b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \Rightarrow \ln \frac{1}{2} = -30k \Rightarrow k = -\frac{1}{30} \ln \frac{1}{2} = \frac{1}{30} \ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \Rightarrow \ln 0.1 = -t(\ln 2)/30 \Rightarrow t = -\frac{30}{\ln 2} \ln 0.1 \approx 100$ h.
18. (a) Let $t = 0$ correspond to 1990 so that $P(t) = 5.28e^{kt}$ is a starting point for the model. When $t = 10$, $P = 6.07$. So $6.07 = 5.28e^{10k} \Rightarrow 10k = \ln \frac{6.07}{5.28} \Rightarrow k = \frac{1}{10} \ln \frac{6.07}{5.28} \approx 0.01394$. For the year 2020, $t = 30$, and $P(30) = 5.28e^{30k} \approx 8.02$ billion.
- (b) $P = 10 \Rightarrow 5.28e^{kt} = 10 \Rightarrow \frac{10}{5.28} = e^{kt} \Rightarrow kt = \ln \frac{10}{5.28} \Rightarrow t = 10 \frac{\ln \frac{10}{5.28}}{\ln \frac{6.07}{5.28}} \approx 45.8$ years; that is, in $1990 + 45 = 2035$.
- (c) $P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{100}{1 + Ae^{-kt}}$, where $A = \frac{100 - 5.28}{5.28} \approx 17.94$. Using $k = \frac{1}{10} \ln \frac{6.07}{5.28}$ from part (a), a model is $P(t) \approx \frac{100}{1 + 17.94e^{-0.01394t}}$ and $P(30) \approx 7.81$ billion, slightly lower than our estimate of 8.02 billion in part (a).
- (d) $P = 10 \Rightarrow 1 + Ae^{-kt} = \frac{100}{10} \Rightarrow Ae^{-kt} = 9 \Rightarrow e^{-kt} = 9/A \Rightarrow -kt = \ln(9/A) \Rightarrow t = -\frac{1}{k} \ln \frac{9}{A} \approx 49.47$ years (that is, in 2039), which is later than the prediction of 2035 in part (b).

19. (a) $\frac{dL}{dt} \propto L_\infty - L \Rightarrow \frac{dL}{dt} = k(L_\infty - L) \Rightarrow \int \frac{dL}{L_\infty - L} = \int k dt \Rightarrow -\ln|L_\infty - L| = kt + C \Rightarrow \ln|L_\infty - L| = -kt - C \Rightarrow |L_\infty - L| = e^{-kt-C} \Rightarrow L_\infty - L = Ae^{-kt} \Rightarrow L = L_\infty - Ae^{-kt}$.
 At $t = 0$, $L = L(0) = L_\infty - A \Rightarrow A = L_\infty - L(0) \Rightarrow L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$.
 (b) $L_\infty = 53$ cm, $L(0) = 10$ cm, and $k = 0.2 \Rightarrow L(t) = 53 - (53 - 10)e^{-0.2t} = 53 - 43e^{-0.2t}$.

20. Denote the amount of salt in the tank (in kg) by y . $y(0) = 0$ since initially there is only water in the tank. The rate at which y increases is equal to the rate at which salt flows into the tank minus the rate at which it flows out. That

$$\text{rate is } \frac{dy}{dt} = 0.1 \frac{\text{kg}}{\text{L}} \times 10 \frac{\text{L}}{\text{min}} - \frac{y}{100} \frac{\text{kg}}{\text{L}} \times 10 \frac{\text{L}}{\text{min}} = 1 - \frac{y}{10} \frac{\text{kg}}{\text{min}} \Rightarrow \int \frac{dy}{10 - y} = \int \frac{1}{10} dt \Rightarrow$$

$$-\ln|10 - y| = \frac{1}{10}t + C \Rightarrow 10 - y = Ae^{-t/10}. \quad y(0) = 0 \Rightarrow 10 = A \Rightarrow y = 10(1 - e^{-t/10}).$$

$$\text{At } t = 6 \text{ minutes, } y = 10(1 - e^{-6/10}) \approx 4.512 \text{ kg.}$$

21. Let P be the population and I be the number of infected people. The rate of spread dI/dt is jointly proportional to

$$I \text{ and to } P - I, \text{ so for some constant } k, dI/dt = kI(P - I) \Rightarrow I = \frac{I_0 P}{I_0 + (P - I_0)e^{-kPt}} \quad (\text{from the}$$

discussion of logistic growth in Section 10.5).

Now, measuring t in days, we substitute $t = 7$, $P = 5000$, $I_0 = 160$ and $I(7) = 1200$ to find k :

$$1200 = \frac{160 \cdot 5000}{160 + (5000 - 160)e^{-5000 \cdot 7 \cdot k}} \Leftrightarrow k \approx 0.00006448. \text{ So, putting } I = 5000 \times 80\% = 4000, \text{ we solve}$$

$$\text{for } t: 4000 = \frac{160 \cdot 5000}{160 + (5000 - 160)e^{-0.00006448 \cdot 5000 \cdot t}} \Leftrightarrow 160 + 4840e^{-0.3224t} = 200 \Leftrightarrow$$

$$-0.3224t = \ln \frac{40}{4840} \Leftrightarrow t \approx 14.9. \text{ So it takes about 15 days for 80\% of the population to be infected.}$$

22. $\frac{1}{R} \frac{dR}{dt} = \frac{k}{S} \frac{dS}{dt} \Rightarrow \frac{d}{dt}(\ln R) = \frac{d}{dt}(k \ln S) \Rightarrow \ln R = k \ln S + C \Rightarrow$

$$R = e^{k \ln S + C} = e^C (e^{\ln S})^k \Rightarrow R = AS^k, \text{ where } A = e^C \text{ is a positive constant.}$$

23. $\frac{dh}{dt} = -\frac{R}{V} \left(\frac{h}{k+h} \right) \Rightarrow \int \frac{k+h}{h} dh = \int \left(-\frac{R}{V} \right) dt \Rightarrow \int \left(1 + \frac{k}{h} \right) dh = -\frac{R}{V} \int 1 dt \Rightarrow$

$$h + k \ln h = -\frac{R}{V}t + C. \text{ This equation gives a relationship between } h \text{ and } t, \text{ but it is not possible to isolate } h \text{ and express it in terms of } t.$$

24. $dx/dt = 0.4x - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$

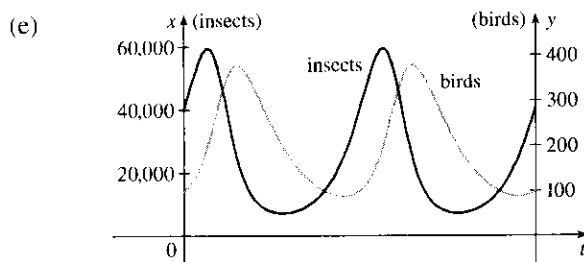
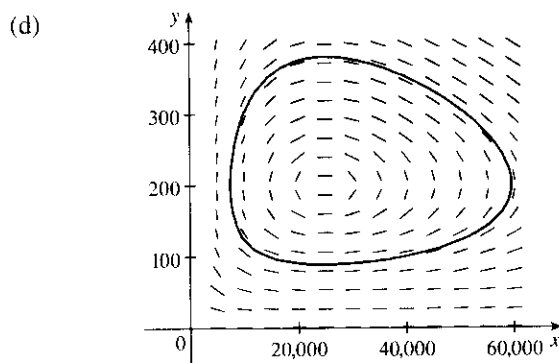
(a) The xy terms represent encounters between the birds and the insects. Since the y -population increases from these terms and the x -population decreases, we expect y to represent the birds and x the insects.

(b) x and y are constant $\Rightarrow x' = 0$ and $y' = 0 \Rightarrow$

$$\left\{ \begin{array}{l} 0 = 0.4x - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} 0 = 0.4x(1 - 0.005y) \\ 0 = -0.2y(1 - 0.00004x) \end{array} \right\} \Rightarrow y = 0 \text{ and } x = 0 \text{ (zero populations)}$$

or $y = \frac{1}{0.005} = 200$ and $x = \frac{1}{0.00004} = 25,000$. The non-trivial solution represents the population sizes needed so that there are no changes in either the number of birds or the number of insects.

$$(c) \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$$



At $(x, y) = (40,000, 100)$, $dx/dt = 8000 > 0$, so as t increases we are proceeding in a counterclockwise direction. The populations increase to approximately $(59,646, 200)$, at which point the insect population starts to decrease. The birds attain a maximum population of about 380 when the insect population is 25,000. The populations decrease to about $(7370, 200)$, at which point the insect population starts to increase. The birds attain a minimum population of about 88 when the insect population is 25,000, and then the cycle repeats.

Both graphs have the same period and the bird population peaks about a quarter-cycle after the insect population.

25. (a) $dx/dt = 0.4x(1 - 0.000005x) - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$. If $y = 0$, then $dx/dt = 0.4x(1 - 0.000005x)$, so $dx/dt = 0 \Leftrightarrow x = 0$ or $x = 200,000$, which shows that the insect population increases logistically with a carrying capacity of 200,000. Since $dx/dt > 0$ for $0 < x < 200,000$ and $dx/dt < 0$ for $x > 200,000$, we expect the insect population to stabilize at 200,000.

(b) x and y are constant $\Rightarrow x' = 0$ and $y' = 0 \Rightarrow$

$$\begin{cases} 0 = 0.4x(1 - 0.000005x) - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{cases} \Rightarrow \begin{cases} 0 = 0.4x[(1 - 0.000005x) - 0.005y] \\ 0 = y(-0.2 + 0.000008x) \end{cases}$$

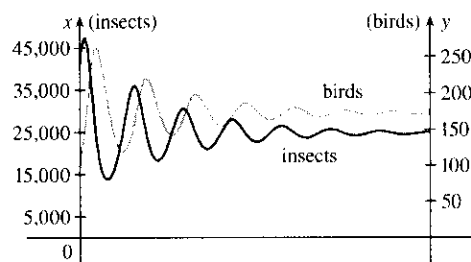
The second equation is true if $y = 0$ or $x = \frac{0.2}{0.000008} = 25,000$. If $y = 0$ in the first equation, then either $x = 0$ or $x = \frac{1}{0.000005} = 200,000$. If $x = 25,000$, then $0 = 0.4(25,000)[(1 - 0.000005 \cdot 25,000) - 0.005y] \Rightarrow 0 = 10,000[(1 - 0.125) - 0.005y] \Rightarrow 0 = 8750 - 50y \Rightarrow y = 175$.

Case (i): $y = 0, x = 0$: Zero populations

Case (ii): $y = 0, x = 200,000$: In the absence of birds, the insect population is always 200,000.

Case (iii): $x = 25,000, y = 175$: The predator/prey interaction balances and the populations are stable.

- (c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.



26. First note that, in this question, “weighs” is used in the informal sense, so what we really require is Barbara’s mass m in kg as a function of t . Barbara’s net intake of calories per day at time t (measured in days) is $c(t) = 1600 - 850 - 15m(t) = 750 - 15m(t)$, where $m(t)$ is her mass at time t . We are given that $m(0) = 60$ kg and $\frac{dm}{dt} = \frac{c(t)}{10,000}$, so $\frac{dm}{dt} = \frac{750 - 15m}{10,000} = \frac{150 - 3m}{2000} = \frac{-3(m - 50)}{2000}$ with $m(0) = 60$. From $\int \frac{dm}{m - 50} = \int \frac{-3 dt}{2000}$, we get $\ln |m - 50| = -\frac{3}{2000}t + C$. Since $m(0) = 60$, $C = \ln 10$. Now $\ln \frac{|m - 50|}{10} = -\frac{3t}{2000}$, so $|m - 50| = 10e^{-3t/2000}$. The quantity $m - 50$ is continuous, initially positive, and the right-hand side is never zero. Thus, $m - 50$ is positive for all t , and $m(t) = 50 + 10e^{-3t/2000}$ kg. As $t \rightarrow \infty$, $m(t) \rightarrow 50$ kg. Thus, Barbara’s mass gradually settles down to 50 kg.

27. (a) $\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. Setting $z = \frac{dy}{dx}$, we get $\frac{dz}{dx} = k\sqrt{1 + z^2} \Rightarrow \frac{dz}{\sqrt{1 + z^2}} = k dx$. Using Formula 25 gives $\ln(z + \sqrt{1 + z^2}) = kx + c \Rightarrow z + \sqrt{1 + z^2} = Ce^{kx}$ (where $C = e^c$) $\Rightarrow \sqrt{1 + z^2} = Ce^{kx} - z \Rightarrow 1 + z^2 = C^2e^{2kx} - 2Ce^{kx}z + z^2 \Rightarrow 2Ce^{kx}z = C^2e^{2kx} - 1 \Rightarrow z = \frac{C}{2}e^{kx} - \frac{1}{2C}e^{-kx}$. Now $\frac{dy}{dx} = \frac{C}{2}e^{kx} - \frac{1}{2C}e^{-kx} \Rightarrow y = \frac{C}{2k}e^{kx} + \frac{1}{2Ck}e^{-kx} + C'$. From the diagram in the text, we see that $y(0) = a$ and $y(\pm b) = h$. $a = y(0) = \frac{C}{2k} + \frac{1}{2Ck} + C' \Rightarrow C' = a - \frac{C}{2k} - \frac{1}{2Ck} \Rightarrow y = \frac{C}{2k}(e^{kx} - 1) + \frac{1}{2Ck}(e^{-kx} - 1) + a$. From $h = y(\pm b)$, we find $h = \frac{C}{2k}(e^{kb} - 1) + \frac{1}{2Ck}(e^{-kb} - 1) + a$ and $h = \frac{C}{2k}(e^{-kb} - 1) + \frac{1}{2Ck}(e^{kb} - 1) + a$. Subtracting the second equation from the first, we get $0 = \frac{C}{k} \frac{e^{kb} - e^{-kb}}{2} - \frac{1}{Ck} \frac{e^{kb} - e^{-kb}}{2} = \frac{1}{k} \left(C - \frac{1}{C}\right) \sinh kb$. Now $k > 0$ and $b > 0$, so $\sinh kb > 0$ and $C = \pm 1$. If $C = 1$, then $y = \frac{1}{2k}(e^{kx} - 1) + \frac{1}{2k}(e^{-kx} - 1) + a = \frac{1}{k} \frac{e^{kx} + e^{-kx}}{2} - \frac{1}{k} + a = a + \frac{1}{k}(\cosh kx - 1)$. If $C = -1$, then $y = -\frac{1}{2k}(e^{kx} - 1) - \frac{1}{2k}(e^{-kx} - 1) + a = -\frac{1}{k} \frac{e^{kx} + e^{-kx}}{2} + \frac{1}{k} + a = a - \frac{1}{k}(\cosh kx - 1)$. Since $k > 0$, $\cosh kx \geq 1$, and $y \geq a$, we conclude that $C = 1$ and $y = a + \frac{1}{k}(\cosh kx - 1)$, where $h = y(b) = a + \frac{1}{k}(\cosh kb - 1)$. Since $\cosh(kb) = \cosh(-kb)$, there is no further information to extract from the condition that $y(b) = y(-b)$. However, we could replace a with the expression $h - \frac{1}{k}(\cosh kb - 1)$, obtaining $y = h + \frac{1}{k}(\cosh kx - \cosh kb)$. It would be better still to keep a in the expression for y , and use the expression for h to solve for k in terms of a , b , and h . That would enable us to express y in terms of x and the given parameters a , b , and h . Sadly, it is not possible to solve for k in closed form. That would have to be done by numerical methods when specific parameter values are given.

- (b) The length of the cable is

$$\begin{aligned} L &= \int_{-b}^b \sqrt{1 + (dy/dx)^2} dx = \int_{-b}^b \sqrt{1 + \sinh^2 kx} dx = \int_{-b}^b \cosh kx dx = 2 \int_0^b \cosh kx dx \\ &= 2 \left[(1/k) \sinh kx \right]_0^b = (2/k) \sinh kb \end{aligned}$$

□ PROBLEMS PLUS

1. We use the Fundamental Theorem of Calculus to differentiate the given equation:

$$[f(x)]^2 = 100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\} dt \Rightarrow 2f(x)f'(x) = [f(x)]^2 + [f'(x)]^2 \Rightarrow$$

$[f(x)]^2 + [f'(x)]^2 - 2f(x)f'(x) = 0 \Rightarrow [f(x) - f'(x)]^2 = 0 \Leftrightarrow f(x) = f'(x)$. We can solve this as a separable equation, or else use Theorem 10.4.2 with $k = 1$, which says that the solutions are $f(x) = Ce^x$. Now $[f(0)]^2 = 100$, so $f(0) = C = \pm 10$, and hence $f(x) = \pm 10e^x$ are the only functions satisfying the given equation.

2. $(fg)' = f'g'$, where $f(x) = e^{x^2} \Rightarrow (e^{x^2}g)' = 2xe^{x^2}g'$. Since the student's mistake did not affect the answer,

$$(e^{x^2}g)' = e^{x^2}g' + 2xe^{x^2}g = 2xe^{x^2}g'. \text{ So } (2x-1)g' = 2xg, \text{ or } \frac{g'}{g} = \frac{2x}{2x-1} = 1 + \frac{1}{2x-1} \Rightarrow$$

$$\ln|g(x)| = x + \frac{1}{2} \ln(2x-1) + C \Rightarrow g(x) = Ae^x \sqrt{2x-1}.$$

3. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x)[f(h) - 1]}{h}$ [since $f(x+h) = f(x)f(h)$]
 $= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} = f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h - 0} = f(x)f'(0) = f(x)$

Therefore, $f'(x) = f(x)$ for all x and from Theorem 10.4.2 we get $f(x) = Ae^x$. Now $f(0) = 1 \Rightarrow A = 1 \Rightarrow f(x) = e^x$.

4. $\left(\int f(x) dx\right) \left(\int \frac{dx}{f(x)}\right) = -1 \Rightarrow \int \frac{dx}{f(x)} = \frac{-1}{\int f(x) dx} \Rightarrow \frac{1}{f(x)} = \frac{f'(x)}{[\int f(x) dx]^2}$ [after differentiating] $\Rightarrow$
 $\int f(x) dx = \pm f'(x)$ [after taking square roots] $\Rightarrow f(x) = \pm f'(x)$ [after differentiating again] $\Rightarrow$
 $y = Ae^x$ or $y = Ae^{-x}$ by Theorem 10.4.2. Therefore, $f(x) = Ae^x$ or $f(x) = Ae^{-x}$, for all nonzero constants A , are the functions satisfying the original equation.

5. Let $y(t)$ denote the temperature of the peach pie t minutes after 5:00 P.M. and R the temperature of the room.

Newton's Law of Cooling gives us $dy/dt = k(y - R)$. Solving for y we get $\frac{dy}{y - R} = k dt \Rightarrow$

$\ln|y - R| = kt + C \Rightarrow |y - R| = e^{kt+C} \Rightarrow y - R = \pm e^{kt} \cdot e^C \Rightarrow y = Me^{kt} + R$, where M is a nonzero constant. We are given temperatures at three times.

$$y(0) = 100 \Rightarrow 100 = M + R \Rightarrow R = 100 - M$$

$$y(10) = 80 \Rightarrow 80 = Me^{10k} + R \quad (1)$$

$$y(20) = 65 \Rightarrow 65 = Me^{20k} + R \quad (2)$$

Substituting $100 - M$ for R in (1) and (2) gives us

$$-20 = Me^{10k} - M \quad (3) \quad \text{and} \quad -35 = Me^{20k} - M \quad (4)$$

Dividing (3) by (4) gives us $\frac{-20}{-35} = \frac{M(e^{10k} - 1)}{M(e^{20k} - 1)} \Rightarrow \frac{4}{7} = \frac{e^{10k} - 1}{e^{20k} - 1} \Rightarrow 4e^{20k} - 4 = 7e^{10k} - 7 \Rightarrow$

$4e^{20k} - 7e^{10k} + 3 = 0$. This is a quadratic equation in e^{10k} . $(4e^{10k} - 3)(e^{10k} - 1) = 0 \Rightarrow e^{10k} = \frac{3}{4}$ or 1
 $\Rightarrow 10k = \ln \frac{3}{4}$ or $\ln 1 \Rightarrow k = \frac{1}{10} \ln \frac{3}{4}$ since k is a nonzero constant of proportionality. Substituting $\frac{3}{4}$ for e^{10k}
 in (3) gives us $-20 = M \cdot \frac{3}{4} - M \Rightarrow -20 = -\frac{1}{4}M \Rightarrow M = 80$. Now $R = 100 - M$ so $R = 20^\circ\text{C}$.

6. Let b be the number of hours before noon that it began to snow, t the time measured in hours after noon, and $x = x(t)$ = distance traveled by the plow at time t . Then dx/dt = speed of plow. Since the snow falls steadily, the height at time t is $h(t) = k(t + b)$, where k is a constant. We are given that the rate of removal is constant, say R

(in m^3/h). If the width of the path is w , then $R = \text{height} \times \text{width} \times \text{speed} = h(t) \times w \times \frac{dx}{dt} = k(t + b)w \frac{dx}{dt}$.

Thus, $\frac{dx}{dt} = \frac{C}{t + b}$, where $C = \frac{R}{kw}$ is a constant. This is a separable equation. $\int dx = C \int \frac{dt}{t + b} \Rightarrow$

$$x(t) = C \ln(t + b) + K.$$

Put $t = 0$: $0 = C \ln b + K \Rightarrow K = -C \ln b$, so $x(t) = C \ln(t + b) - C \ln b = C \ln(1 + t/b)$.

Put $t = 1$: $6000 = C \ln(1 + 1/b)$ [$x = 6$ km].

Put $t = 2$: $9000 = C \ln(1 + 2/b)$ [$x = (6 + 3)$ km].

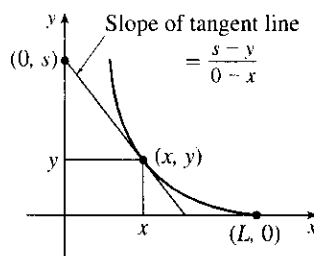
Solve for b : $\frac{\ln(1 + 1/b)}{6000} = \frac{\ln(1 + 2/b)}{9000} \Rightarrow 3 \ln\left(1 + \frac{1}{b}\right) = 2 \ln\left(1 + \frac{2}{b}\right) \Rightarrow \left(1 + \frac{1}{b}\right)^3 = \left(1 + \frac{2}{b}\right)^2$

$$\Rightarrow 1 + \frac{3}{b} + \frac{3}{b^2} + \frac{1}{b^3} = 1 + \frac{4}{b} + \frac{4}{b^2} \Rightarrow \frac{1}{b} + \frac{1}{b^2} - \frac{1}{b^3} = 0 \Rightarrow b^2 + b - 1 = 0 \Rightarrow b = \frac{-1 \pm \sqrt{5}}{2}.$$

But $b > 0$, so $b = \frac{-1 + \sqrt{5}}{2} \approx 0.618$ h ≈ 37 min. The snow began to fall $\frac{\sqrt{5}-1}{2}$ hours before noon; that is, at about 11:23 A.M.

7. (a) While running from $(L, 0)$ to (x, y) , the dog travels a distance

$s = \int_x^L \sqrt{1 + (dy/dx)^2} dx = - \int_L^x \sqrt{1 + (dy/dx)^2} dx$, so $\frac{ds}{dx} = -\sqrt{1 + (dy/dx)^2}$. The dog and rabbit run at the same speed, so the rabbit's position when the dog has traveled a distance s is $(0, s)$. Since the dog runs straight for the rabbit, $\frac{dy}{dx} = \frac{s - y}{0 - x}$ (see the figure).



Thus, $s = y - x \frac{dy}{dx} \Rightarrow \frac{ds}{dx} = \frac{dy}{dx} - \left(x \frac{d^2y}{dx^2} + 1 \frac{dy}{dx}\right) = -x \frac{d^2y}{dx^2}$. Equating the two expressions for $\frac{ds}{dx}$

gives us $x \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, as claimed.

(b) Letting $z = \frac{dy}{dx}$, we obtain the differential equation $x \frac{dz}{dx} = \sqrt{1+z^2}$, or $\frac{dz}{\sqrt{1+z^2}} = \frac{dx}{x}$. Integrating:

$$\ln x = \int \frac{dz}{\sqrt{1+z^2}} \stackrel{25}{=} \ln(z + \sqrt{1+z^2}) + C. \text{ When } x = L, z = dy/dx = 0, \text{ so } \ln L = \ln 1 + C.$$

Therefore, $C = \ln L$, so $\ln x = \ln(\sqrt{1+z^2} + z) + \ln L = \ln[L(\sqrt{1+z^2} + z)] \Rightarrow$

$$x = L(\sqrt{1+z^2} + z) \Rightarrow \sqrt{1+z^2} = \frac{x}{L} - z \Rightarrow 1 + z^2 = \left(\frac{x}{L}\right)^2 - \frac{2xz}{L} + z^2 \Rightarrow$$

$$\left(\frac{x}{L}\right)^2 - 2z\left(\frac{x}{L}\right) - 1 = 0 \Rightarrow z = \frac{(x/L)^2 - 1}{2(x/L)} = \frac{x^2 - L^2}{2Lx} = \frac{x}{2L} - \frac{L}{2x} \text{ [for } x > 0]. \text{ Since } z = \frac{dy}{dx},$$

$$y = \frac{x^2}{4L} - \frac{L}{2} \ln x + C_1. \text{ Since } y = 0 \text{ when } x = L, 0 = \frac{L}{4} - \frac{L}{2} \ln L + C_1 \Rightarrow C_1 = \frac{L}{2} \ln L - \frac{L}{4}. \text{ Thus,}$$

$$y = \frac{x^2}{4L} - \frac{L}{2} \ln x + \frac{L}{2} \ln L - \frac{L}{4} = \frac{x^2 - L^2}{4L} - \frac{L}{2} \ln\left(\frac{x}{L}\right).$$

(c) As $x \rightarrow 0^+$, $y \rightarrow \infty$, so the dog never catches the rabbit.

8. (a) If the dog runs twice as fast as the rabbit, then the rabbit's position when the dog has traveled a distance s is $(0, s/2)$. Since the dog runs straight toward the rabbit, the tangent line to the dog's path has slope

$$\frac{dy}{dx} = \frac{s/2 - y}{0 - x}. \text{ Thus, } s = 2y - 2x \frac{dy}{dx} \Rightarrow \frac{ds}{dx} = 2 \frac{dy}{dx} - \left(2x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx}\right) = -2x \frac{d^2y}{dx^2}.$$

$$\text{From Problem 7(a), } \frac{ds}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ so } 2x \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

$$\text{Letting } z = \frac{dy}{dx}, \text{ we obtain the differential equation } 2x \frac{dz}{dx} = \sqrt{1+z^2}, \text{ or } \frac{2dz}{\sqrt{1+z^2}} = \frac{dx}{x}.$$

$$\text{Integrating, we get } \ln x = \int \frac{2dz}{\sqrt{1+z^2}} = 2 \ln(\sqrt{1+z^2} + z) + C. \text{ [See Problem 7(b).]}$$

When $x = L$, $z = dy/dx = 0$, so $\ln L = 2 \ln 1 + C = C$. Thus,

$$\ln x = 2 \ln(\sqrt{1+z^2} + z) + \ln L = \ln\left(L(\sqrt{1+z^2} + z)^2\right) \Rightarrow x = L(\sqrt{1+z^2} + z)^2 \Rightarrow$$

$$\sqrt{1+z^2} = \sqrt{\frac{x}{L}} - z \Rightarrow 1 + z^2 = \frac{x}{L} - 2\sqrt{\frac{x}{L}}z + z^2 \Rightarrow 2\sqrt{\frac{x}{L}}z = \frac{x}{L} - 1 \Rightarrow$$

$$\frac{dy}{dx} = z = \frac{1}{2} \sqrt{\frac{x}{L}} - \frac{1}{2\sqrt{x/L}} = \frac{1}{2\sqrt{L}} x^{1/2} - \frac{\sqrt{L}}{2} x^{-1/2} \Rightarrow y = \frac{1}{3\sqrt{L}} x^{3/2} - \sqrt{L} x^{1/2} + C_1. \text{ When}$$

$$x = L, y = 0, \text{ so } 0 = \frac{1}{3\sqrt{L}} L^{3/2} - \sqrt{L} L^{1/2} + C_1 = \frac{L}{3} - L + C_1 = C_1 - \frac{2}{3}L. \text{ Therefore, } C_1 = \frac{2}{3}L \text{ and}$$

$$y = \frac{x^{3/2}}{3\sqrt{L}} - \sqrt{L} x^{1/2} + \frac{2}{3}L. \text{ As } x \rightarrow 0, y \rightarrow \frac{2}{3}L, \text{ so the dog catches the rabbit when the rabbit is at } (0, \frac{2}{3}L).$$

(At that point, the dog has traveled a distance of $\frac{4}{3}L$, twice as far as the rabbit has run.)

- (b) As in the solutions to part (a) and Problem 7, we get $z = \frac{dy}{dx} = \frac{x^2}{2L^2} - \frac{L^2}{2x^2}$ and hence $y = \frac{x^3}{6L^2} + \frac{L^2}{2x} - \frac{2}{3}L$.

We want to minimize the distance D from the dog at (x, y) to the rabbit at $(0, 2s)$. Now $s = \frac{1}{2}y - \frac{1}{2}x \frac{dy}{dx} \Rightarrow$

$$2s = y - xz \Rightarrow y - 2s = xz = x \left(\frac{x^2}{2L^2} - \frac{L^2}{2x^2} \right) = \frac{x^3}{2L^2} - \frac{L^2}{2x}, \text{ so}$$

$$\begin{aligned} D &= \sqrt{(x-0)^2 + (y-2s)^2} = \sqrt{x^2 + \left(\frac{x^3}{2L^2} - \frac{L^2}{2x} \right)^2} \\ &= \sqrt{\frac{x^6}{4L^4} + \frac{x^2}{2} + \frac{L^4}{4x^2}} = \sqrt{\left(\frac{x^3}{2L^2} + \frac{L^2}{2x} \right)^2} = \frac{x^3}{2L^2} + \frac{L^2}{2x} \end{aligned}$$

$$D' = 0 \Leftrightarrow \frac{3x^2}{2L^2} - \frac{L^2}{2x^2} = 0 \Leftrightarrow \frac{3x^2}{2L^2} = \frac{L^2}{2x^2} \Leftrightarrow x^4 = \frac{L^4}{3} \Leftrightarrow x = \frac{L}{\sqrt[4]{3}}, x > 0, L > 0.$$

Since $D''(x) = \frac{3x}{L^2} + \frac{L^2}{x^3} > 0$ for all $x > 0$, we know that

$D\left(\frac{L}{\sqrt[4]{3}}\right) = \frac{\left(L \cdot 3^{-1/4}\right)^3}{2L^2} + \frac{L^2}{2(L \cdot 3^{-1/4})} = \frac{2L}{3^{3/4}}$ is the minimum value of D , that is, the closest the dog gets to the rabbit. The positions at this distance are

$$\text{Dog: } (x, y) = \left(\frac{L}{\sqrt[4]{3}}, \left(\frac{5}{3^{7/4}} - \frac{2}{3} \right) L \right) = \left(\frac{L}{\sqrt[4]{3}}, \frac{5\sqrt[4]{3} - 6}{9} L \right)$$

$$\text{Rabbit: } (0, 2s) = \left(0, \frac{8\sqrt[4]{3}L}{9} - \frac{2L}{3} \right) = \left(0, \frac{8\sqrt[4]{3} - 6}{9} L \right)$$

9. (a) We are given that $V = \frac{1}{3}\pi r^2 h$, $dV/dt = 60,000\pi \text{ ft}^3/\text{h}$, and $r = 1.5h = \frac{3}{2}h$. So $V = \frac{1}{3}\pi \left(\frac{3}{2}h\right)^2 h = \frac{3}{4}\pi h^3$

$$\Rightarrow \frac{dV}{dt} = \frac{3}{4}\pi \cdot 3h^2 \frac{dh}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}. \text{ Therefore, } \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2} = \frac{240,000\pi}{9\pi h^2} = \frac{80,000}{3h^2} \quad (*) \Rightarrow$$

$\int 3h^2 dh = \int 80,000 dt \Rightarrow h^3 = 80,000t + C$. When $t = 0$, $h = 60$. Thus, $C = 60^3 = 216,000$, so $h^3 = 80,000t + 216,000$. Let $h = 100$. Then $100^3 = 1,000,000 = 80,000t + 216,000 \Rightarrow 80,000t = 784,000 \Rightarrow t = 9.8$, so the time required is 9.8 hours.

- (b) The floor area of the silo is $F = \pi \cdot 200^2 = 40,000\pi \text{ ft}^2$, and the area of the base of the pile is

$$A = \pi r^2 = \pi \left(\frac{3}{2}h\right)^2 = \frac{9\pi}{4}h^2. \text{ So the area of the floor which is not covered when } h = 60 \text{ is}$$

$$F - A = 40,000\pi - 8100\pi = 31,900\pi \approx 100,217 \text{ ft}^2. \text{ Now } A = \frac{9\pi}{4}h^2 \Rightarrow dA/dt = \frac{9\pi}{4} \cdot 2h (dh/dt),$$

and from (*) in part (a) we know that when $h = 60$, $dh/dt = \frac{80,000}{3(60)^2} = \frac{200}{27} \text{ ft/h}$. Therefore,

$$dA/dt = \frac{9\pi}{4}(2)(60)\left(\frac{200}{27}\right) = 2000\pi \approx 6283 \text{ ft}^2/\text{h}.$$

- (c) At $h = 90 \text{ ft}$, $dV/dt = 60,000\pi - 20,000\pi = 40,000\pi \text{ ft}^3/\text{h}$. From (*) in part (a),

$$\frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2} = \frac{4(40,000\pi)}{9\pi h^2} = \frac{160,000}{9h^2} \Rightarrow \int 9h^2 dh = \int 160,000 dt \Rightarrow 3h^3 = 160,000t + C.$$

When $t = 0$, $h = 90$; therefore, $C = 3 \cdot 729,000 = 2,187,000$. So $3h^3 = 160,000t + 2,187,000$. At the top,

$$h = 100 \Rightarrow 3(100)^3 = 160,000t + 2,187,000 \Rightarrow t = \frac{813,000}{160,000} \approx 5.1. \text{ The pile reaches the top after about 5.1 h.}$$

10. Let $P(a, b)$ be any first-quadrant point on the curve $y = f(x)$. The tangent line at P has equation

$y - b = f'(a)(x - a)$, or equivalently, $y = mx + b - ma$, where $m = f'(a)$. If $Q(0, c)$ is the y -intercept, then

$c = b - am$. If $R(k, 0)$ is the x -intercept, then $k = \frac{am - b}{m} = a - \frac{b}{m}$. Since the tangent line is bisected at P , we

know that $|PQ| = |PR|$; that is,

$$\sqrt{(a-0)^2 + [b - (b - am)]^2} = \sqrt{[a - (a - b/m)]^2 + (b-0)^2}$$

Squaring and simplifying gives us $a^2 + a^2 m^2 = b^2/m^2 + b^2 \Rightarrow a^2 m^2 + a^2 m^4 = b^2 + b^2 m^2 \Rightarrow a^2 m^4 + (a^2 - b^2)m^2 - b^2 = 0 \Rightarrow (a^2 m^2 - b^2)(m^2 + 1) = 0 \Rightarrow m^2 = b^2/a^2$. Since m is the slope of the line from a positive y -intercept to a positive x -intercept, m must be negative. Since a and b are positive, we have $m = -b/a$, so we will solve the equivalent differential equation $\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x} \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln y = -\ln x + C \quad [x, y > 0] \Rightarrow y = e^{-\ln x + C} = e^{\ln x^{-1}} \cdot e^C = x^{-1} \cdot A \Rightarrow y = A/x$. Since the point $(3, 2)$ is on the curve, $3 = A/2 \Rightarrow A = 6$ and the curve is $y = 6/x$ with $x > 0$.

11. Let $P(a, b)$ be any point on the curve. If m is the slope of the tangent line at P , then $m = y'(a)$, and an equation of the normal line at P is $y - b = -\frac{1}{m}(x - a)$, or equivalently, $y = -\frac{1}{m}x + b + \frac{a}{m}$.

The y -intercept is always 6, so $b + \frac{a}{m} = 6 \Rightarrow \frac{a}{m} = 6 - b \Rightarrow m = \frac{a}{6 - b}$.

We will solve the equivalent differential equation $\frac{dy}{dx} = \frac{x}{6 - y} \Rightarrow (6 - y) dy = x dx \Rightarrow$

$\int (6 - y) dy = \int x dx \Rightarrow 6y - \frac{1}{2}y^2 = \frac{1}{2}x^2 + C \Rightarrow 12y - y^2 = x^2 + K$. Since $(3, 2)$ is on the curve, $12(2) - 2^2 = 3^2 + K \Rightarrow K = 11$. So the curve is given by $12y - y^2 = x^2 + 11 \Rightarrow x^2 + y^2 - 12y + 36 = -11 + 36 \Rightarrow x^2 + (y - 6)^2 = 25$, a circle with center $(0, 6)$ and radius 5.

12. Suppose C is a curve with the required property and let $P = (x_0, y_0)$ be a point on C . The equation of the normal line to C at P is $y - y_0 = -\frac{1}{y'_0}(x - x_0)$, where y'_0 is the value of $\frac{dy}{dx}$ at $x = x_0$. This equation makes sense only if $y'_0 \neq 0$. If $y'_0 = 0$, then the normal line at P is $x = x_0$, which does not intersect the y -axis at all unless $x_0 = 0$. So let's assume that $y'_0 \neq 0$. Then the normal line to C at P intersects the x -axis at $(x_0 + y_0 y'_0, 0)$, and it intersects the y -axis at $(0, y_0 + x_0/y'_0)$. The condition on C implies that

$$[\text{distance from } P(x_0, y_0) \text{ to } (0, y_0 + x_0/y'_0)] = [\text{distance from } (0, y_0 + x_0/y'_0) \text{ to } (x_0 + y_0 y'_0, 0)]$$

$$\sqrt{(0 - x_0)^2 + (y_0 + x_0/y'_0 - y_0)^2} = \sqrt{(x_0 + y_0 y'_0 - 0)^2 + [0 - (y_0 + x_0/y'_0)]^2}$$

Squaring both sides, we get $x_0^2 + x_0^2/(y'_0)^2 = (x_0 + y_0 y'_0)^2 + (y_0 + x_0/y'_0)^2$ or

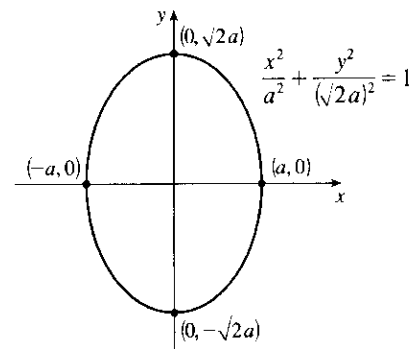
$x_0^2 + \frac{x_0^2}{(y'_0)^2} = x_0^2 + 2x_0 y_0 y'_0 + y_0^2 (y'_0)^2 + y_0^2 + 2\frac{x_0 y_0}{y'_0} + \frac{x_0^2}{(y'_0)^2}$. Subtracting $x_0^2 + \frac{x_0^2}{(y'_0)^2}$ from both sides and multiplying by y'_0 , we get

$$0 = y_0^2 y'_0 + y_0^2 (y'_0)^3 + 2x_0 y_0 [1 + (y'_0)^2] = y_0 \{ y_0 y'_0 + y_0 (y'_0)^3 + 2x_0 [1 + (y'_0)^2] \}$$

$$= y_0 \{ y_0 y'_0 [1 + (y'_0)^2] + 2x_0 [1 + (y'_0)^2] \} = y_0 (y_0 y'_0 + 2x_0) [1 + (y'_0)^2]$$

Since $1 + (y'_0)^2 \geq 1 > 0$, we conclude that $y_0(y_0 y'_0 + 2x_0) = 0$. Now P is an arbitrary point on C for which $y'_0 \neq 0$. Thus, we have shown that $y(yy' + 2x) = 0$ for points (x, y) along C where $y' \neq 0$. One solution of this equation is $y = 0$, but that curve (the x -axis) doesn't satisfy the condition required of C , since its normal lines at points for $x \neq 0$ don't intersect the y -axis. Thus, we can focus our attention on points of C where $y \neq 0$, and conclude that $yy' + 2x = 0$ at points of C where $y \neq 0$ and $y' \neq 0$. Integrating both sides of $yy' + 2x = 0$, we get $\frac{1}{2}y^2 + x^2 = c$. Clearly $c > 0$ (since $y \neq 0$), so we can write $c = a^2$, where $a = \sqrt{c} > 0$. Thus, $\frac{1}{2}y^2 + x^2 = a^2$ and

$x^2/a^2 + y^2/(\sqrt{2}a)^2 = 1$. This shows that C is (part of) the ellipse centered at $(0, 0)$ with semimajor axis $\sqrt{2}a$ in the y -direction and semiminor axis a in the x -direction. The points of C where $y = 0$ or $y' = 0$ are the vertices $(0, \pm\sqrt{2}a)$ and $(\pm a, 0)$. At these points, the condition on C is satisfied in a degenerate way. [When $P = (\pm a, 0)$, the normal line at P is the x -axis, so *all* the points of the normal line can be viewed as points of intersection with the x -axis. The intersection with the y -axis at $(0, 0)$ is midway between $(a, 0)$ and $(-a, 0)$; one of these points is P , and the other can be regarded as an intersection of the normal line with the x -axis. Similarly, when $P = (0, \pm\sqrt{2}a)$, the normal line is the y -axis, and the point $(0, \pm\sqrt{2}a/2)$, which can be regarded as an intersection of the normal line with the y -axis, is midway between P and $(0, 0)$, the intersection with the x -axis.]

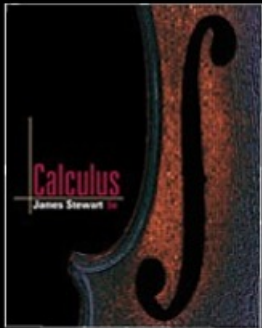


Conversely, if C is part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{2a^2} = 1$ for some $a > 0$, then the normal line at a point (x_0, y_0) of C (other than the four vertices) has equation $y - y_0 = \frac{y_0}{2x_0}(x - x_0)$. Its intersections with the coordinate axes are $(0, \frac{y_0}{2})$ and $(-x_0, 0)$. $\left[\text{distance from } (x_0, y_0) \text{ to } (0, \frac{y_0}{2})\right]^2 = x_0^2 + \frac{y_0^2}{4}$ and $\left[\text{distance from } (0, \frac{y_0}{2}) \text{ to } (-x_0, 0)\right]^2 = x_0^2 + \frac{y_0^2}{4}$, so the required condition is met at points other than the four vertices. As we have explained, if we are willing to interpret the condition broadly, then it can be viewed as holding even at the four vertices.

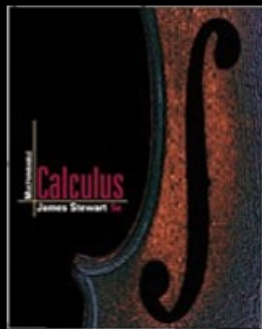
Another method: Let $P(x_0, y_0)$ be a point on the curve. Since the midpoint of the line segment determined by the normal line from (x_0, y_0) to its intersection with the x -axis has x -coordinate 0, the x -coordinate of the point of intersection with the x -axis must be $-x_0$. Hence, the normal line has slope $\frac{y_0 - 0}{x_0 - (-x_0)} = \frac{y_0}{2x_0}$. So the tangent line has slope $-\frac{2x_0}{y_0}$. This gives the differential equation $y' = -\frac{2x}{y} \Rightarrow y dy = -2x dx \Rightarrow \int y dy = \int (-2x) dx \Rightarrow \frac{1}{2}y^2 = -x^2 + C \Rightarrow x^2 + \frac{1}{2}y^2 = C \ (C > 0)$.

Chapter 11

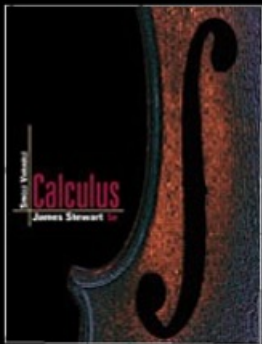
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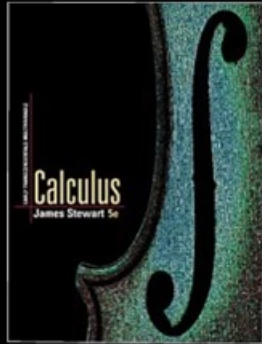
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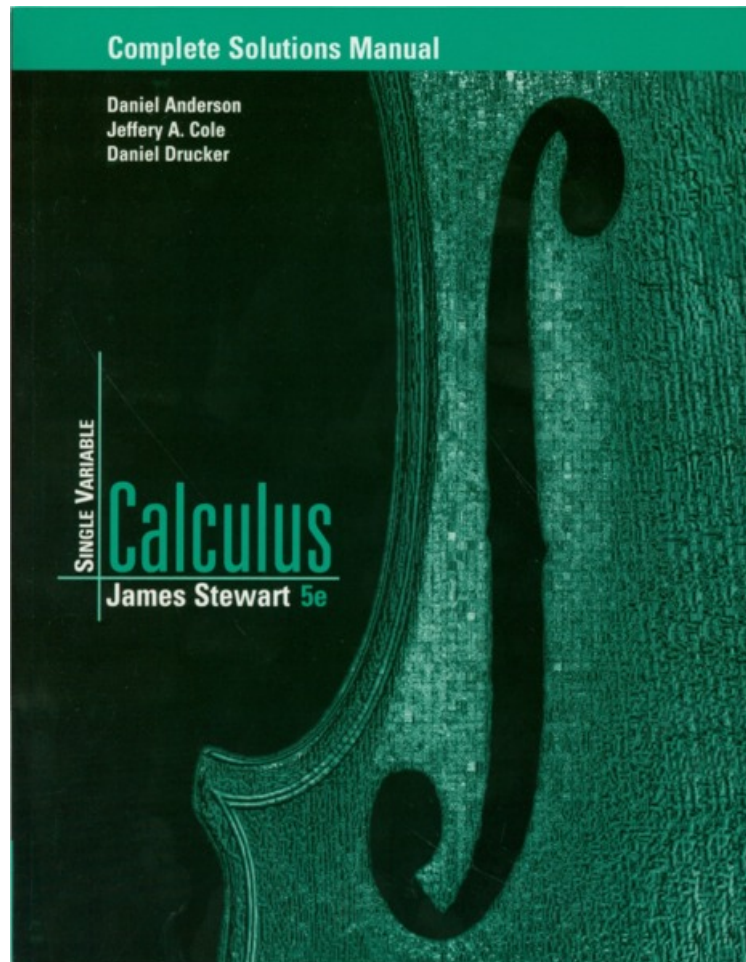
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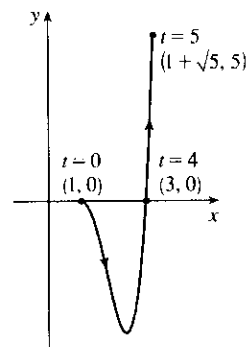


11 □ PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 Curves Defined by Parametric Equations

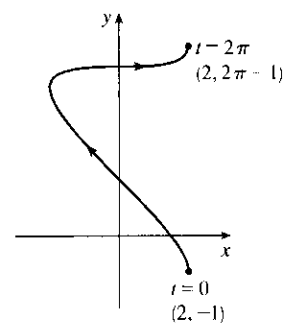
1. $x = 1 + \sqrt{t}$, $y = t^2 - 4t$, $0 \leq t \leq 5$

t	0	1	2	3	4	5
x	1	2	$1 + \sqrt{2}$	$1 + \sqrt{3}$	3	$1 + \sqrt{5}$
y	0	-3	2.41	2.73	0	3.24



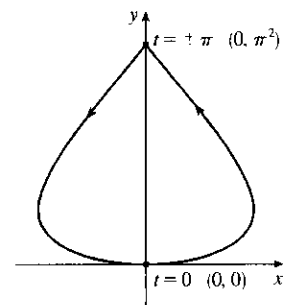
2. $x = 2 \cos t$, $y = t - \cos t$, $0 \leq t \leq 2\pi$

t	0	$\pi/2$	π	$3\pi/2$	2π
x	2	0	-2	0	2
y	-1	$\pi/2$	$\pi + 1$	$3\pi/2$	$2\pi - 1$
	1.57	4.14	4.71	5.28	



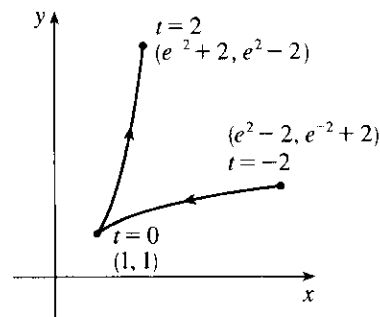
3. $x = 5 \sin t$, $y = t^2$, $-\pi \leq t \leq \pi$

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	0	-5	0	5	0
y	π^2	$\pi^2/4$	0	$\pi^2/4$	π^2
	9.87	2.47		2.47	9.87



4. $x = e^{-t} + t$, $y = e^t - t$, $-2 \leq t \leq 2$

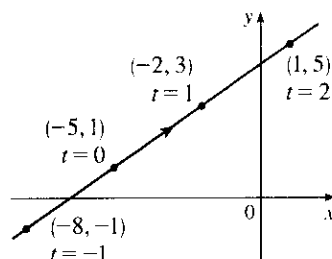
t	-2	-1	0	1	2
x	$e^2 - 2$	$e - 1$	1	$e^{-1} + 1$	$e^{-2} + 2$
	5.39	1.72		1.37	2.14
y	$e^{-2} + 2$	$e^{-1} + 1$	1	$e - 1$	$e^2 - 2$
	2.14	1.37		1.72	5.39



5. $x = 3t - 5$, $y = 2t + 1$

(a)	t	-2	-1	0	1	2	3	4
	x	-11	-8	-5	-2	1	4	7
	y	-3	-1	1	3	5	7	9

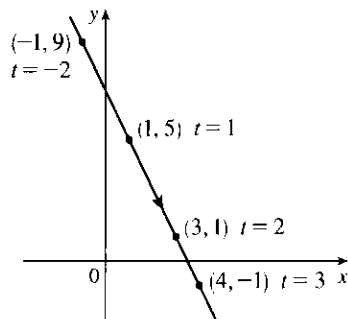
(b) $x = 3t - 5 \Rightarrow 3t = x + 5 \Rightarrow t = \frac{1}{3}(x + 5) \Rightarrow$
 $y = 2 \cdot \frac{1}{3}(x + 5) + 1$, so $y = \frac{2}{3}x + \frac{13}{3}$.



6. $x = 1 + t$, $y = 5 - 2t$, $-2 \leq t \leq 3$

(a)	t	-2	-1	0	1	2	3
	x	-1	0	1	2	3	4
	y	9	7	5	3	1	-1

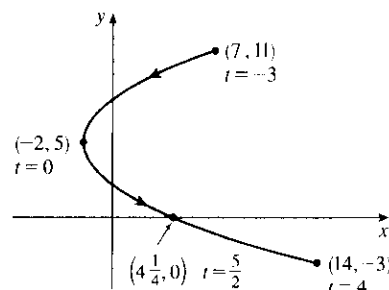
(b) $x = 1 + t \Rightarrow t = x - 1 \Rightarrow y = 5 - 2(x - 1)$,
 so $y = -2x + 7$, $-1 \leq x \leq 4$.



7. $x = t^2 - 2$, $y = 5 - 2t$, $-3 \leq t \leq 4$

(a)	t	-3	-2	-1	0	1	2	3	4
	x	7	2	-1	-2	-1	2	7	14
	y	11	9	7	5	3	1	-1	-3

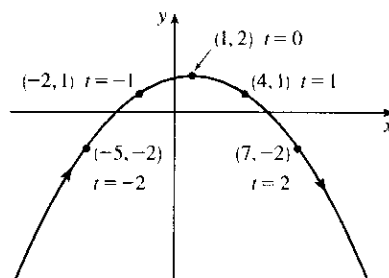
(b) $y = 5 - 2t \Rightarrow 2t = 5 - y \Rightarrow t = \frac{1}{2}(5 - y) \Rightarrow$
 $x = \left[\frac{1}{2}(5 - y)\right]^2 - 2$, so $x = \frac{1}{4}(5 - y)^2 - 2$,
 $-3 \leq y \leq 11$.



8. $x = 1 + 3t$, $y = 2 - t^2$

(a)	t	-3	-2	-1	0	1	2	3
	x	-8	-5	-2	1	4	7	10
	y	-7	-2	1	2	1	-2	-7

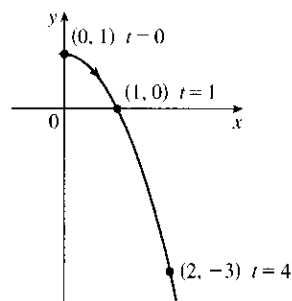
(b) $x = 1 + 3t \Rightarrow t = \frac{1}{3}(x - 1) \Rightarrow$
 $y = 2 - \left[\frac{1}{3}(x - 1)\right]^2$, so $y = -\frac{1}{9}(x - 1)^2 + 2$.



9. (a) $x = \sqrt{t}$, $y = 1 - t$

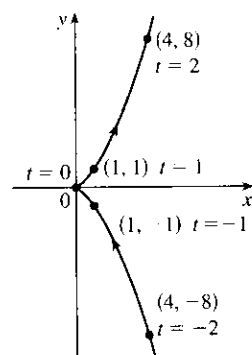
t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3

(b) $x = \sqrt{t} \Rightarrow t = x^2 \Rightarrow y = 1 - t = 1 - x^2$.
 Since $t \geq 0$, $x \geq 0$.



10. (a) $x = t^2, y = t^3$

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

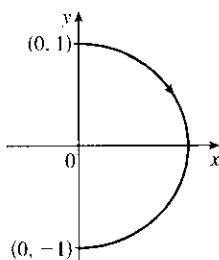


(b) $y = t^3 \Rightarrow t = \sqrt[3]{y} \Rightarrow x = t^2 = (\sqrt[3]{y})^2 = y^{2/3}$
 $t \in \mathbb{R}, y \in \mathbb{R}, x \geq 0.$

11. (a) $x = \sin \theta, y = \cos \theta, 0 \leq \theta \leq \pi.$

$$x^2 + y^2 = \sin^2 \theta + \cos^2 \theta = 1. \text{ Since } 0 \leq \theta \leq \pi, \text{ we have } \sin \theta \geq 0, \text{ so } x \geq 0.$$

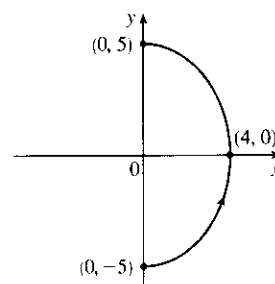
(b)



12. (a) $x = 4 \cos \theta, y = 5 \sin \theta, -\pi/2 \leq \theta \leq \pi/2.$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1, \text{ which is an ellipse with } x\text{-intercepts } (\pm 4, 0) \text{ and } y\text{-intercepts } (0, \pm 5). \text{ We obtain the portion of the ellipse with } x \geq 0 \text{ since } 4 \cos \theta \geq 0 \text{ for } -\pi/2 \leq \theta \leq \pi/2.$$

(b)

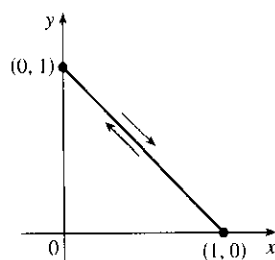


13. (a) $x = \sin^2 \theta, y = \cos^2 \theta.$

$$x + y = \sin^2 \theta + \cos^2 \theta = 1, 0 \leq x \leq 1.$$

Note that the curve is at $(0, 1)$ whenever $\theta = \pi n$ and is at $(1, 0)$ whenever $\theta = \frac{\pi}{2}n$ for every integer n .

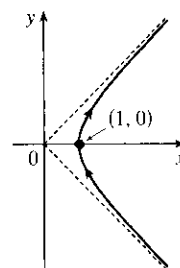
(b)



14. (a) $x = \sec \theta, y = \tan \theta, -\pi/2 < \theta < \pi/2.$

$$x^2 - y^2 = \sec^2 \theta - \tan^2 \theta = 1, x \geq 1, \text{ or } x = \sqrt{y^2 + 1}.$$

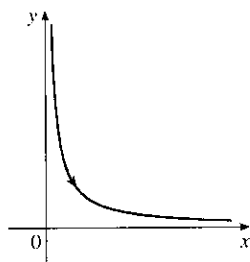
(b)



15. (a) $x = e^t, y = e^{-t}$.

$$y = 1/e^t = 1/x, x > 0$$

(b)

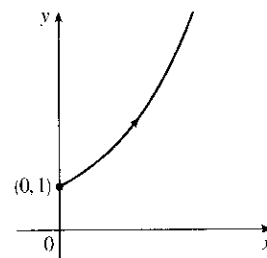


16. (a) $x = \ln t, y = \sqrt{t}, t \geq 1$.

$$x = \ln t \Rightarrow t = e^x \Rightarrow$$

$$y = \sqrt{t} = e^{x/2}, x \geq 0.$$

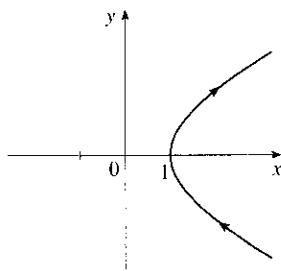
(b)



17. (a) $x = \cosh t, y = \sinh t$,

$$x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1, x \geq 1$$

(b)

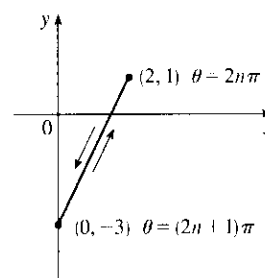


18. (a) $x = 1 + \cos \theta \Rightarrow \cos \theta = x - 1$.

$$y = 2 \cos \theta - 1 = 2(x - 1) - 1 = 2x - 3,$$

$$0 \leq x \leq 2.$$

(b)



19. $x^2 + y^2 = \cos^2 \pi t + \sin^2 \pi t = 1, 1 \leq t \leq 2$, so the particle moves counterclockwise along the circle $x^2 + y^2 = 1$ from $(-1, 0)$ to $(1, 0)$, along the lower half of the circle.

20. $(x - 2)^2 + (y - 3)^2 = \cos^2 t + \sin^2 t = 1$, so the motion takes place on a unit circle centered at $(2, 3)$. As t goes from 0 to 2π , the particle makes one complete counterclockwise rotation around the circle, starting and ending at $(3, 3)$.

21. $(\frac{1}{2}x)^2 + (\frac{1}{3}y)^2 = \sin^2 t + \cos^2 t = 1$, so the particle moves once clockwise along the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$, starting and ending at $(0, 3)$.

22. $x = \cos^2 t = y^2$, so the particle moves along the parabola $x = y^2$. As t goes from 0 to 4π , the particle moves from $(1, 1)$ down to $(1, -1)$ (at $t = \pi$), back up to $(1, 1)$ again (at $t = 2\pi$), and then repeats this entire cycle between $t = 2\pi$ and $t = 4\pi$.

23. We must have $1 \leq x \leq 4$ and $2 \leq y \leq 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.

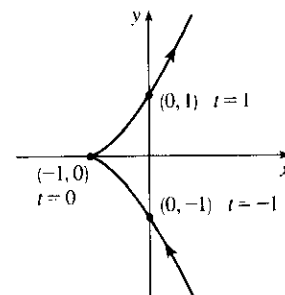
24. (a) From the first graph, we have $1 \leq x \leq 2$. From the second graph, we have $-1 \leq y \leq 1$. The only choice that satisfies either of those conditions is III.

(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.

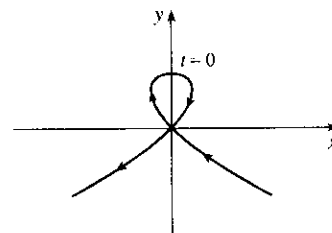
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \leq y \leq 2$. Choice IV satisfies these conditions.

(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

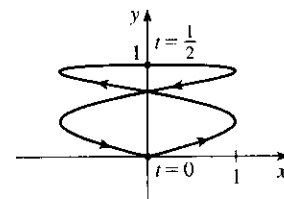
25. When $t = -1$, $(x, y) = (0, -1)$. As t increases to 0, x decreases to -1 and y increases to 0. As t increases from 0 to 1, x increases to 0 and y increases to 1. As t increases beyond 1, both x and y increase. For $t < -1$, x is positive and decreasing and y is negative and increasing. We could achieve greater accuracy by estimating x - and y -values for selected values of t from the given graphs and plotting the corresponding points.



26. For $t < -1$, x is positive and decreasing, while y is negative and increasing (these points are in Quadrant IV). When $t = -1$, $(x, y) = (0, 0)$ and, as t increases from -1 to 0, x becomes negative and y increases from 0 to 1. At $t = 0$, $(x, y) = (0, 1)$ and, as t increases from 0 to 1, y decreases from 1 to 0 and x is positive. At $t = 1$, $(x, y) = (0, 0)$ again, so the loop is completed. For $t > 1$, x and y both become large negative. This enables us to draw a rough sketch. We could achieve greater accuracy by estimating x - and y -values for selected values of t from the given graphs and plotting the corresponding points.

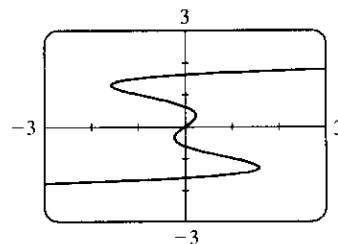


27. When $t = 0$ we see that $x = 0$ and $y = 0$, so the curve starts at the origin. As t increases from 0 to $\frac{1}{2}$, the graphs show that y increases from 0 to 1 while x increases from 0 to 1, decreases to 0 and to -1 , then increases back to 0, so we arrive at the point $(0, 1)$. Similarly, as t increases from $\frac{1}{2}$ to 1, y decreases from 1 to 0 while x repeats its pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating x - and y -values for selected values of t from the given graphs and plotting the corresponding points.

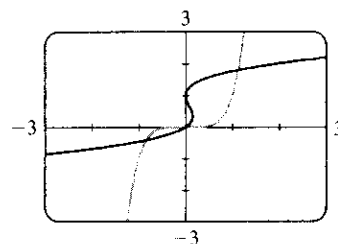


28. (a) Note that as $t \rightarrow -\infty$, we have $x \rightarrow -\infty$ and $y \rightarrow \infty$, whereas when $t \rightarrow \infty$, both x and $y \rightarrow \infty$. This description fits only IV. [But also note that $x(t)$ increases, then decreases, then increases again.]
- (b) Note that as $t \rightarrow \pm\infty$, $y \rightarrow -\infty$. This is only the case with VI.
- (c) If $t = 0$, then $(x, y) = (\sin 0, \sin 0) = (0, 0)$. Also, $|x| = |\sin 3t| \leq 1$ for all t , and $|y| = |\sin 4t| \leq 1$ for all t . The only graph which includes the point $(0, 0)$ and which has $|x| \leq 1$ and $|y| \leq 1$, is V.
- (d) Note that as $t \rightarrow -\infty$, both x and $y \rightarrow -\infty$, and as $t \rightarrow \infty$, both x and $y \rightarrow \infty$. This description fits only III. (Also note that, since $\sin 2t$ and $\sin 3t$ lie between -1 and 1 , the curve never strays very far from the line $y = x$.)
- (e) Note that both $x(t)$ and $y(t)$ are periodic with period 2π and satisfy $|x| \leq 1$ and $|y| \leq 1$. Now the only y -intercepts occur when $x = \sin(t + \sin t) = 0 \Leftrightarrow t = 0$ or π . So there should be two y -intercepts: $y(0) = \cos 1 \approx 0.54$ and $y(\pi) = \cos(\pi - 1) \approx -0.54$. Similarly, there should be two x -intercepts: $x(\frac{\pi}{2}) = \sin(\frac{\pi}{2} + 1) \approx 0.54$ and $x(\frac{3\pi}{2}) = \sin(\frac{3\pi}{2} - 1) \approx -0.54$. The only curve with these x - and y -intercepts is I.
- (f) Note that $x(t)$ is periodic with period 2π , so the only y -intercepts occur when $x = \cos t = 0 \Leftrightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Also, the graph is symmetric about the x -axis, since $y(-t) = \sin(-t + \sin 5(-t)) = \sin(-t - \sin 5t) = -\sin(t + \sin 5t) = -y(t)$, and $x(-t) = \cos(-t) = \cos t = x(t)$. The only graph which has only two y -intercepts, and is symmetric about the x -axis, is II.

29. As in Example 5, we let $y = t$ and $x = t - 3t^3 + t^5$ and use a t -interval of $[-2\pi, 2\pi]$.



30. We use $x_1 = t$, $y_1 = t^5$ and $x_2 = t(t-1)^2$, $y_2 = t$ with $-2\pi \leq t \leq 2\pi$. There are 3 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant III is approximately $(-0.8, -0.4)$ and the point in quadrant I is approximately $(1.1, 1.8)$.

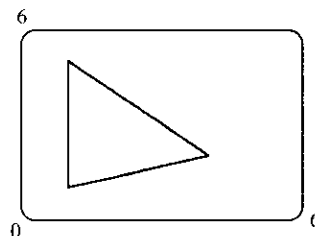


31. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \leq t \leq 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2 . For every value of t , x and y satisfy the relation $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$; if we call that common value t , then the given parametric equations yield the point (x, y) ; and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \leq t \leq 1$.

32. For the side of the triangle from A to B , use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. Hence, the equations are $x = x_1 + (x_2 - x_1)t = 1 + (4 - 1)t = 1 + 3t$, $y = y_1 + (y_2 - y_1)t = 1 + (2 - 1)t = 1 + t$. Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \leq t \leq 1$ gives us the side of the triangle from A to B . Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.



33. The circle $x^2 + y^2 = 4$ can be represented parametrically by $x = 2 \cos t$, $y = 2 \sin t$; $0 \leq t \leq 2\pi$. The circle $x^2 + (y - 1)^2 = 4$ can be represented by $x = 2 \cos t$, $y = 1 + 2 \sin t$; $0 \leq t \leq 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

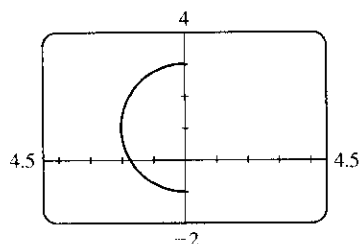
(a) To get a clockwise orientation, we could change the equations to $x = 2 \cos t$, $y = 1 - 2 \sin t$, $0 \leq t \leq 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2 \cos t$, $y = 1 + 2 \sin t$ with the domain expanded to $0 \leq t \leq 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2 \cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2 \cos t$, $y = 1 + 2 \sin t$; $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$.

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2 \sin t$, $y = 1 + 2 \cos t$, $0 \leq t \leq \pi$.

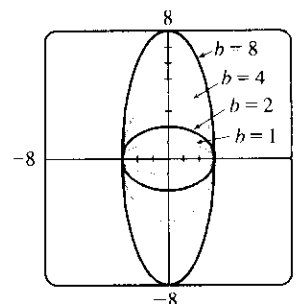
34.



35. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a \sin t$ and $y = b \cos t$ with $0 \leq t \leq 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3 \sin t$ and $y = b \cos t$ for $b \in \{1, 2, 4, 8\}$.

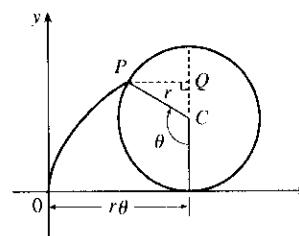
(c) As b increases, the ellipse stretches vertically.



36. The possible parametrizations of the curve $y = x^3$ include

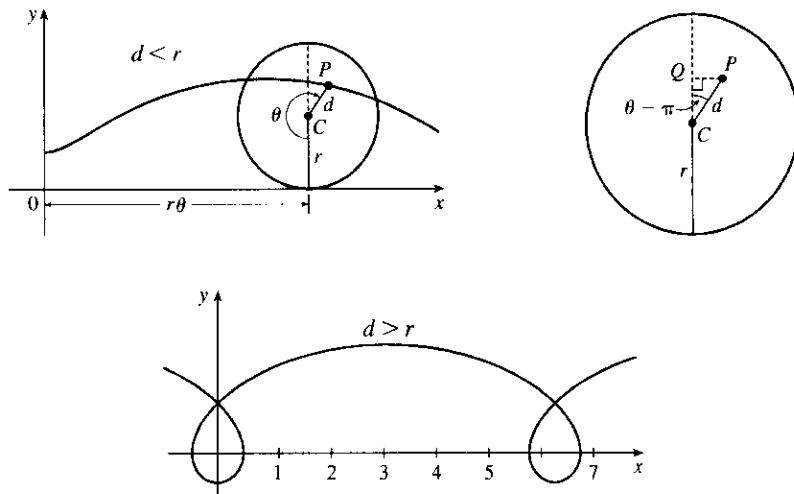
- (1) $x = t$, $y = t^3$, $t \in \mathbb{R}$
- (2) $x = -t$, $y = -t^3$, $t \in \mathbb{R}$
- (3) $x = t + 1$, $y = (t + 1)^3$, $t \in \mathbb{R}$

37. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 6, and Q has coordinates $(r\theta, r + r \cos(\pi - \theta)) = (r\theta, r(1 - \cos \theta))$ [since $\cos(\pi - \alpha) = \cos \pi \cos \alpha + \sin \pi \sin \alpha = -\cos \alpha$], so P has coordinates $(r\theta - r \sin(\pi - \theta), r(1 - \cos \theta)) = (r(\theta - \sin \theta), r(1 - \cos \theta))$ [since $\sin(\pi - \alpha) = \sin \pi \cos \alpha - \cos \pi \sin \alpha = \sin \alpha$]. Again we have the parametric equations $x = r(\theta - \sin \theta)$, $y = r(1 - \cos \theta)$.

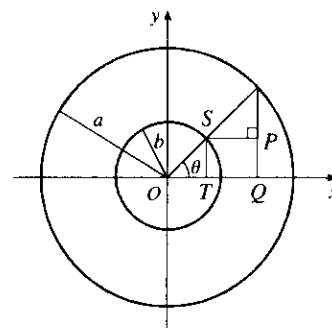


38. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 6, C has coordinates $(r\theta, r)$.

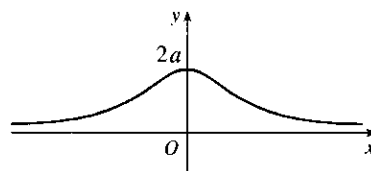
Now Q (in the second diagram) has coordinates $(r\theta, r + d \cos(\theta - \pi)) = (r\theta, r - d \cos \theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d \sin(\theta - \pi), r - d \cos \theta)$. That is, P has coordinates (x, y) , where $x = r\theta - d \sin \theta$ and $y = r - d \cos \theta$. When $d = r$, these equations agree with those of the cycloid.



39. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a \cos \theta$ and $y = |ST| = b \sin \theta$. Thus, the parametric equations are $x = a \cos \theta$ and $y = b \sin \theta$. To eliminate θ we rearrange: $\sin \theta = y/b \Rightarrow \sin^2 \theta = (y/b)^2$ and $\cos \theta = x/a \Rightarrow \cos^2 \theta = (x/a)^2$. Adding the two equations: $\sin^2 \theta + \cos^2 \theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



40. A has coordinates $(a \cos \theta, a \sin \theta)$. Since OA is perpendicular to AB , $\triangle OAB$ is a right triangle and B has coordinates $(a \sec \theta, 0)$. It follows that P has coordinates $(a \sec \theta, b \sin \theta)$. Thus, the parametric equations are $x = a \sec \theta$, $y = b \sin \theta$.
41. $C = (2a \cot \theta, 2a)$, so the x -coordinate of P is $x = 2a \cot \theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a \sin \theta$ and $A = ((2a \sin \theta) \cos \theta, (2a \sin \theta) \sin \theta)$. Thus, the y -coordinate of P is $y = 2a \sin^2 \theta$.



42. Let θ be the angle of inclination of segment OP . Then

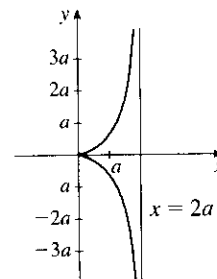
$$|OB| = \frac{2a}{\cos \theta}. \text{ Let } C = (2a, 0). \text{ Then by use of right triangle}$$

OAC we see that $|OA| = 2a \cos \theta$. Now

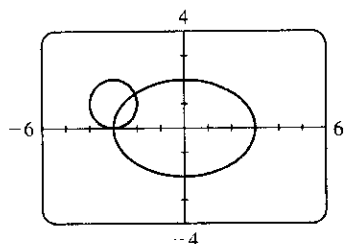
$$\begin{aligned} |OP| &= |AB| = |OB| - |OA| = 2a \left(\frac{1}{\cos \theta} - \cos \theta \right) \\ &= 2a \frac{1 - \cos^2 \theta}{\cos \theta} = 2a \frac{\sin^2 \theta}{\cos \theta} = 2a \sin \theta \tan \theta \end{aligned}$$

So P has coordinates $x = 2a \sin \theta \tan \theta \cdot \cos \theta = 2a \sin^2 \theta$ and

$$y = 2a \sin \theta \tan \theta \cdot \sin \theta = 2a \sin^3 \theta \tan \theta.$$



43. (a)



There are 2 points of intersection:

$(-3, 0)$ and approximately $(-2.1, 1.4)$.

- (b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t . So solve the equations:

$$3 \sin t = -3 + \cos t \quad (1)$$

$$2 \cos t = 1 + \sin t \quad (2)$$

From (2), $\sin t = 2 \cos t - 1$. Substituting into (1), we get $3(2 \cos t - 1) = -3 + \cos t \Rightarrow$

$$5 \cos t = 0 \quad (*) \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}. \text{ We check that } t = \frac{3\pi}{2} \text{ satisfies (1) and (2) but } t = \frac{\pi}{2} \text{ does}$$

not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t . If we do so, we see that the only value of t for which both pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

- (c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but

there are no collision points, since $(*)$ in part (b) becomes $5 \cos t = 6 \Rightarrow \cos t = \frac{6}{5} > 1$.

44. (a) If $\alpha = 30^\circ$ and $v_0 = 500$ m/s, then the equations become $x = (500 \cos 30^\circ)t = 250\sqrt{3}t$ and

$$y = (500 \sin 30^\circ)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2. \text{ } y = 0 \text{ when } t = 0 \text{ (when the gun is fired) and again when}$$

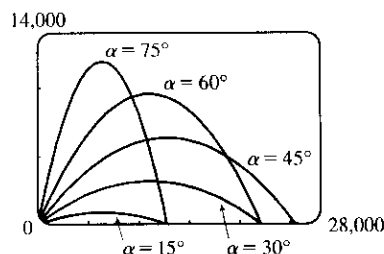
$t = \frac{250}{4.9} \approx 51$ s. Then $x = (250\sqrt{3})\left(\frac{250}{4.9}\right) \approx 22,092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t . To find the maximum y -value, we will complete the square:

$$y = -4.9\left(t^2 - \frac{250}{4.9}t\right) = -4.9\left[t^2 - \frac{250}{4.9}t + \left(\frac{125}{4.9}\right)^2\right] + \frac{125^2}{4.9} = -4.9\left(t - \frac{125}{4.9}\right)^2 + \frac{125^2}{4.9} \leq \frac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)



As α ($0^\circ < \alpha < 90^\circ$) increases up to 45° , the projectile attains a greater height and a greater range. As α increases past 45° , the projectile attains a greater height, but its range decreases.

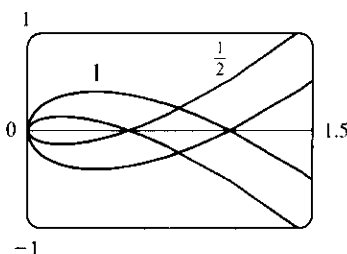
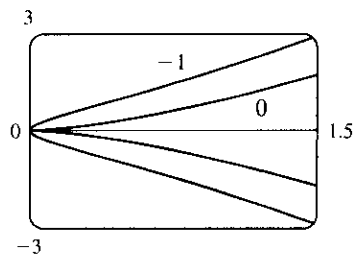
$$(c) \quad x = (v_0 \cos \alpha)t \Rightarrow t = \frac{x}{v_0 \cos \alpha}.$$

$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = (v_0 \sin \alpha) \frac{x}{v_0 \cos \alpha} - \frac{g}{2} \left(\frac{x}{v_0 \cos \alpha} \right)^2 = (\tan \alpha)x - \left(\frac{g}{2v_0^2 \cos^2 \alpha} \right)x^2,$$

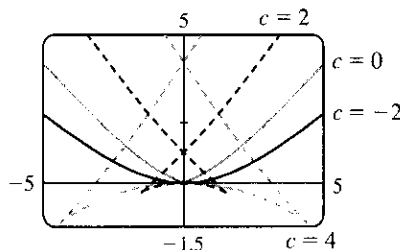
which is the equation of a parabola (quadratic in x).

45. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \leq t \leq \pi$.

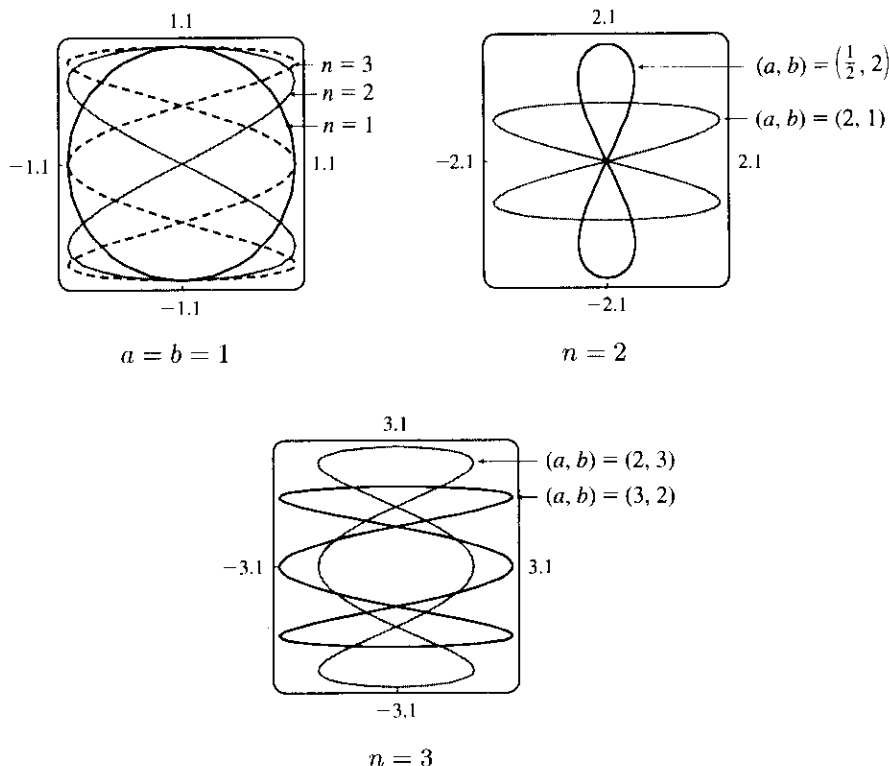
Note that all the members of the family are symmetric about the x -axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.



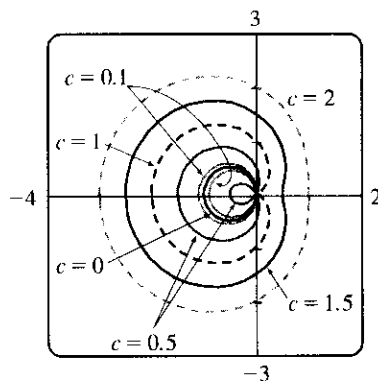
46. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \leq t \leq \pi$. Note that all the members of the family are symmetric about the y -axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x -axis. The size of the "swallowtail" increases as c increases.



47. Note that all the Lissajous figures are symmetric about the x -axis. The parameters a and b simply stretch the graph in the x - and y -directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x -axis. In general, the figures have $n - 1$ points of intersection, all of which are on the y -axis, and a total of n closed loops.



48. We use $-\pi \leq t \leq \pi$ in the viewing rectangle $[-4, 2] \times [-3, 3]$. We first observe that for $c = 0$, we obtain a circle with center $(-\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. As the value of c increases, there is a larger outer loop and a smaller inner loop until $c = 1$, when we obtain a curve with a dent (called a **cardioid**). As c increases, we get a curve with a dimple (called a **limaçon**) until $c = 2$. For $c > 2$, we have convex limaçons. For negative values of c , we obtain the same graphs as for positive c , but with different values of t corresponding to the points on the curve.



LABORATORY PROJECT Running Circles Around Circles

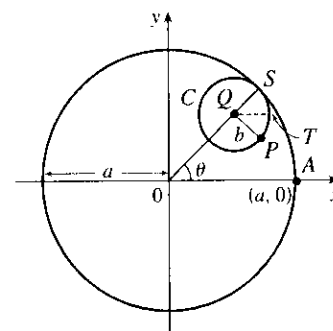
1. The center Q of the smaller circle has coordinates

$((a - b)\cos \theta, (a - b)\sin \theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.) Thus, $\angle PQS = \frac{a}{b}\theta$ and $\angle PQT = \frac{a}{b}\theta - \theta$, so P has coordinates

$$x = (a - b)\cos \theta + b\cos(\angle PQT) = (a - b)\cos \theta + b\cos\left(\frac{a - b}{b}\theta\right)$$

and

$$y = (a - b)\sin \theta - b\sin(\angle PQT) = (a - b)\sin \theta - b\sin\left(\frac{a - b}{b}\theta\right)$$

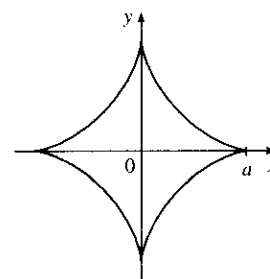


2. With $b = 1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a = 4$. Let $a = 4$ and $b = 1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain

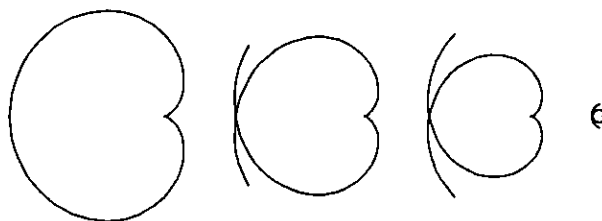
$$x = 3\cos \theta + \cos 3\theta = 3\cos \theta + (4\cos^3 \theta - 3\cos \theta) = 4\cos^3 \theta$$

and

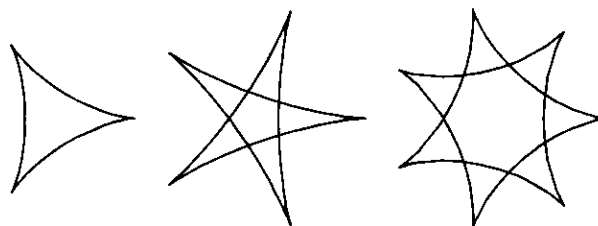
$$y = 3\sin \theta - \sin 3\theta = 3\sin \theta - (3\sin \theta - 4\sin^3 \theta) = 4\sin^3 \theta$$



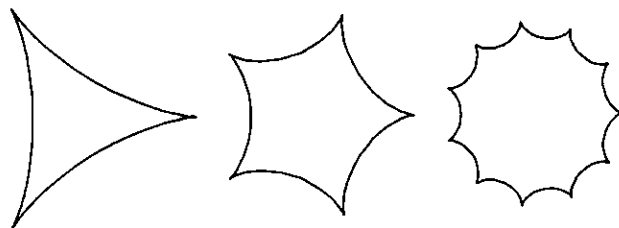
3. The following graphs are obtained with $b = 1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ and $\frac{1}{10}$ with $-2\pi \leq \theta \leq 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.



Letting $d = 2$ and $n = 3, 5,$ and 7 with $-2\pi \leq \theta \leq 2\pi$ gives us the following:



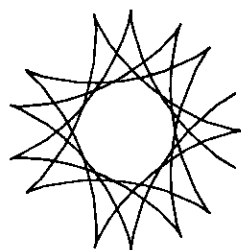
So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form). When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}$, $\frac{5}{4}$, and $\frac{11}{10}$.



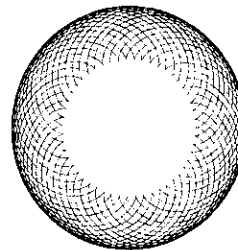
4. If $b = 1$, the equations for the hypocycloid are

$$x = (a - 1) \cos \theta + \cos((a - 1)\theta) \quad y = (a - 1) \sin \theta - \sin((a - 1)\theta)$$

which is a hypocycloid of a cusps (from Problem 2). In general, if $a > 1$, we get a figure with cusps on the “outside ring” and if $a < 1$, the cusps are on the “inside ring”. In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ , every point on the washer would eventually be arbitrarily close to a point on the curve.



$$a = \sqrt{2} \\ -10\pi \leq \theta \leq 10\pi$$



$$a = e - 2 \\ 0 \leq \theta \leq 446$$

5. The center Q of the smaller circle has coordinates $((a + b) \cos \theta, (a + b) \sin \theta)$. Arc

PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \frac{a\theta}{b}$, $\angle PQR = \pi - \frac{a\theta}{b}$, and

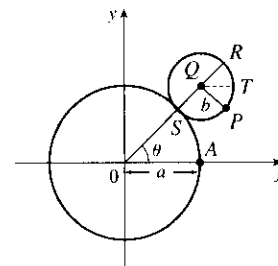
$\angle PQT = \pi - \frac{a\theta}{b} - \theta = \pi - \left(\frac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.

Thus, the coordinates of P are

$$x = (a + b) \cos \theta + b \cos\left(\pi - \frac{a+b}{b}\theta\right) = (a + b) \cos \theta - b \cos\left(\frac{a+b}{b}\theta\right)$$

and

$$y = (a + b) \sin \theta - b \sin\left(\pi - \frac{a+b}{b}\theta\right) = (a + b) \sin \theta - b \sin\left(\frac{a+b}{b}\theta\right).$$

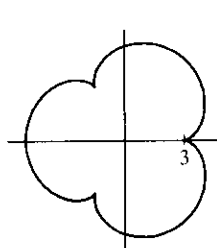


6. Let $b = 1$ and the equations become

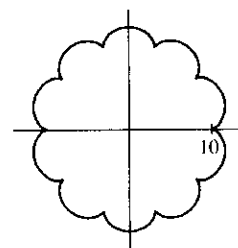
$$x = (a + 1) \cos \theta - \cos((a + 1)\theta)$$

$$y = (a + 1) \sin \theta - \sin((a + 1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an “ a -leafed clover”, with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

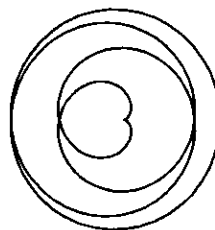


$$a = 3, -2\pi \leq \theta \leq 2\pi$$

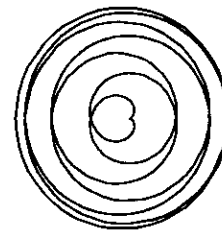


$$a = 10, -2\pi \leq \theta \leq 2\pi$$

If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \leq \theta \leq d\pi$ to be a closed curve traced exactly once.

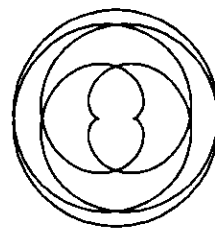


$$a = \frac{1}{4}, -4\pi \leq \theta \leq 4\pi$$

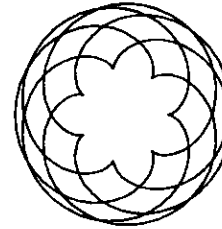


$$a = \frac{1}{7}, -7\pi \leq \theta \leq 7\pi$$

Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n -pointed star in the middle.

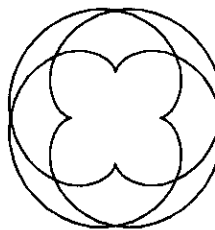


$$a = \frac{2}{5}, -5\pi \leq \theta \leq 5\pi$$

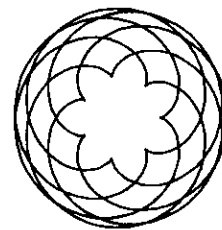


$$a = \frac{7}{5}, -5\pi \leq \theta \leq 5\pi$$

Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.

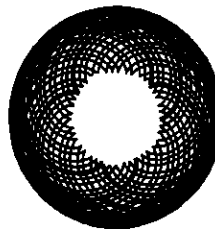


$$a = \frac{4}{3}, -3\pi \leq \theta \leq 3\pi$$

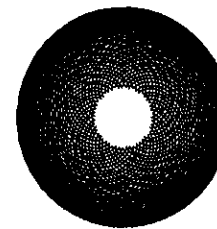


$$a = \frac{7}{6}, -6\pi \leq \theta \leq 6\pi$$

If a is irrational, we get washers that increase in size as a increases.



$$a = \sqrt{2}, 0 \leq \theta \leq 200$$



$$a = c - 2, 0 \leq \theta \leq 446$$

11.2 Calculus with Parametric Curves

$$1. x = t - t^3, y = 2 - 5t \Rightarrow \frac{dy}{dt} = -5, \frac{dx}{dt} = 1 - 3t^2, \text{ and } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-5}{1 - 3t^2} \text{ or } \frac{5}{3t^2 - 1}.$$

$$2. x = te^t, y = t + e^t \Rightarrow \frac{dy}{dt} = 1 + e^t, \frac{dx}{dt} = te^t + e^t, \text{ and } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + e^t}{te^t + e^t}.$$

$$3. x = t^4 + 1, y = t^3 + t; t = -1. \quad \frac{dy}{dt} = 3t^2 + 1, \frac{dx}{dt} = 4t^3, \text{ and } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 1}{4t^3}.$$

When $t = -1$, $(x, y) = (2, -2)$ and $dy/dx = \frac{4}{-4} = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - (-2) = (-1)(x - 2)$, or $y = -x$.

$$4. x = 2t^2 + 1, y = \frac{1}{3}t^3 - t; t = 3. \quad \frac{dy}{dt} = t^2 - 1, \frac{dx}{dt} = 4t, \text{ and } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2 - 1}{4t}. \text{ When } t = 3,$$

$$(x, y) = (19, 6) \text{ and } dy/dx = \frac{8}{12} = \frac{2}{3}, \text{ so an equation of the tangent line is } y - 6 = \frac{2}{3}(x - 19), \text{ or } y = \frac{2}{3}x - \frac{20}{3}.$$

$$5. x = e^{\sqrt{t}}, y = t - \ln t^2; t = 1. \quad \frac{dy}{dt} = 1 - \frac{2t}{t^2} = 1 - \frac{2}{t}, \frac{dx}{dt} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}, \text{ and}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 2/t}{e^{\sqrt{t}}/(2\sqrt{t})} \cdot \frac{2t}{2t} = \frac{2t - 4}{\sqrt{t}e^{\sqrt{t}}}. \text{ When } t = 1, (x, y) = (e, 1) \text{ and } \frac{dy}{dx} = -\frac{2}{e}, \text{ so an equation of the}$$

$$\text{tangent line is } y - 1 = -\frac{2}{e}(x - e), \text{ or } y = -\frac{2}{e}x + 3.$$

$$6. x = \cos \theta + \sin 2\theta, y = \sin \theta + \cos 2\theta; \theta = 0. \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta - 2 \sin 2\theta}{-\sin \theta + 2 \cos 2\theta}. \text{ When } \theta = 0,$$

$$(x, y) = (1, 1) \text{ and } dy/dx = \frac{1}{2}, \text{ so an equation of the tangent to the curve is } y - 1 = \frac{1}{2}(x - 1), \text{ or } y = \frac{1}{2}x + \frac{1}{2}.$$

$$7. (a) x = e^t, y = (t - 1)^2; (1, 1). \quad \frac{dy}{dt} = 2(t - 1), \frac{dx}{dt} = e^t, \text{ and } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(t - 1)}{e^t}.$$

At $(1, 1)$, $t = 0$ and $\frac{dy}{dx} = -2$, so an equation of the tangent is $y - 1 = -2(x - 1)$, or $y = -2x + 3$.

$$(b) x = e^t \Rightarrow t = \ln x, \text{ so } y = (t - 1)^2 = (\ln x - 1)^2 \text{ and } \frac{dy}{dx} = 2(\ln x - 1) \left(\frac{1}{x} \right). \text{ When } x = 1,$$

$$\frac{dy}{dx} = 2(-1)(1) = -2, \text{ so an equation of the tangent is } y = -2x + 3, \text{ as in part (a).}$$

$$8. (a) x = \tan \theta, y = \sec \theta; (1, \sqrt{2}). \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sec \theta \tan \theta}{\sec^2 \theta} = \frac{\tan \theta}{\sec \theta} = \sin \theta.$$

When $(x, y) = (1, \sqrt{2})$, $\theta = \frac{\pi}{4}$ (or $\frac{\pi}{4} + 2\pi n$ for some integer n), so $dy/dx = \sin \frac{\pi}{4} = \sqrt{2}/2$. Thus, an equation of the tangent to the curve is $y - \sqrt{2} = (\sqrt{2}/2)(x - 1)$, or $y = (\sqrt{2}/2)x + (\sqrt{2}/2)$.

$$(b) \tan^2 \theta + 1 = \sec^2 \theta \Rightarrow x^2 + 1 = y^2, \text{ so } \frac{d}{dx}(x^2 + 1) = \frac{d}{dx}(y^2) \Rightarrow 2x = 2y \frac{dy}{dx}.$$

When $(x, y) = (1, \sqrt{2})$, $\frac{dy}{dx} = \frac{x}{y} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, so an equation of the tangent is $y - \sqrt{2} = (\sqrt{2}/2)(x - 1)$, as in part (a).

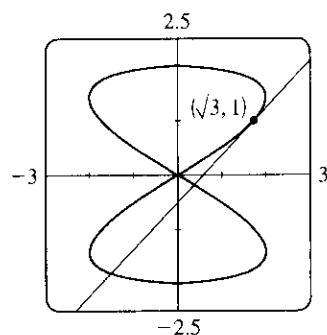
9. $x = 2 \sin 2t, y = 2 \sin t; (\sqrt{3}, 1)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{2 \cdot 2 \cos 2t} = \frac{\cos t}{2 \cos 2t}. \text{ The point } (\sqrt{3}, 1) \text{ corresponds}$$

to $t = \frac{\pi}{6}$, so the slope of the tangent at that point is

$$\frac{\cos \frac{\pi}{6}}{2 \cos \frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{2 \cdot \frac{1}{2}} = \frac{\sqrt{3}}{2}. \text{ An equation of the tangent is therefore}$$

$$(y - 1) = \frac{\sqrt{3}}{2}(x - \sqrt{3}), \text{ or } y = \frac{\sqrt{3}}{2}x - \frac{1}{2}.$$



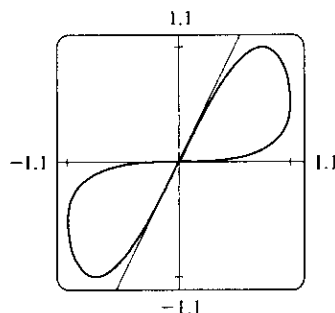
10. $x = \sin t, y = \sin(t + \sin t); (0, 0)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t + \sin t)(1 + \cos t)}{\cos t} = (\sec t + 1) \cos(t + \sin t)$$

Note that there are two tangents at the point $(0, 0)$, since both $t = 0$ and $t = \pi$ correspond to the origin. The tangent corresponding to $t = 0$ has slope $(\sec 0 + 1) \cos(0 + \sin 0) = 2 \cos 0 = 2$, and its equation is

$y = 2x$. The tangent corresponding to $t = \pi$ has slope

$(\sec \pi + 1) \cos(\pi + \sin \pi) = 0$, so it is the x -axis.



11. $x = 4 + t^2, y = t^2 + t^3 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d(dy/dx)/dt}{dx/dt} = \frac{(d/dt)(1 + \frac{3}{2}t)}{2t} = \frac{3/2}{2t} = \frac{3}{4t}. \text{ The curve is CU when } \frac{d^2y}{dx^2} > 0, \text{ that is,}$$

when $t > 0$.

12. $x = t^3 - 12t, y = t^2 - 1 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 12} \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{(3t^2 - 12) \cdot 2 - 2t(6t)}{(3t^2 - 12)^2}}{3t^2 - 12} = \frac{-6t^2 - 24}{(3t^2 - 12)^3} = \frac{-6(t^2 + 4)}{3^3(t^2 - 4)^3} = \frac{-2(t^2 + 4)}{9(t^2 - 4)^3}. \text{ Thus, the curve is}$$

CU when $t^2 - 4 < 0 \Rightarrow |t| < 2 \Rightarrow -2 < t < 2$.

13. $x = t - e^t, y = t + e^{-t} \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - e^{-t}}{1 - e^t} = \frac{1 - \frac{1}{e^t}}{1 - e^t} = \frac{\frac{e^t - 1}{e^t}}{1 - e^t} = -e^{-t} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt}(-e^{-t})}{dx/dt} = \frac{e^{-t}}{1 - e^t}.$$

The curve is CU when $e^t < 1$ [since $e^{-t} > 0$] $\Rightarrow t < 0$.

14. $x = t + \ln t, y = t - \ln t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 1/t}{1 + 1/t} = \frac{t - 1}{t + 1} = 1 - \frac{2}{t + 1} \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(1 - \frac{2}{t + 1} \right)}{1 + 1/t} = \frac{2/(t + 1)^2}{(t + 1)/t} = \frac{2t}{(t + 1)^3}, \text{ so the curve is CU for all } t \text{ in its domain, that}$$

is, $t > 0$ [$t < -1$ not in domain].

15. $x = 2 \sin t$, $y = 3 \cos t$, $0 < t < 2\pi$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3 \sin t}{2 \cos t} = -\frac{3}{2} \tan t, \text{ so } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{3}{2} \sec^2 t}{2 \cos t} = -\frac{3}{4} \sec^3 t.$$

The curve is CU when $\sec^3 t < 0 \Rightarrow \sec t < 0 \Rightarrow \cos t < 0 \Rightarrow \frac{\pi}{2} < t < \frac{3\pi}{2}$.

16. $x = \cos 2t$, $y = \cos t$, $0 < t < \pi$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{-2 \sin 2t} = \frac{\sin t}{2 \cdot 2 \sin t \cos t} = \frac{1}{4 \cos t} = \frac{1}{4} \sec t$, so

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{1}{4} \sec t \tan t}{-4 \sin t \cos t} = -\frac{1}{16} \sec^3 t. \text{ The curve is CU when } \sec^3 t < 0 \Rightarrow \sec t < 0 \Rightarrow \cos t < 0 \Rightarrow \frac{\pi}{2} < t < \pi.$$

17. $x = 10 - t^2$, $y = t^3 - 12t$.

$$dy/dt = 3t^2 - 12 = 3(t+2)(t-2), \text{ so}$$

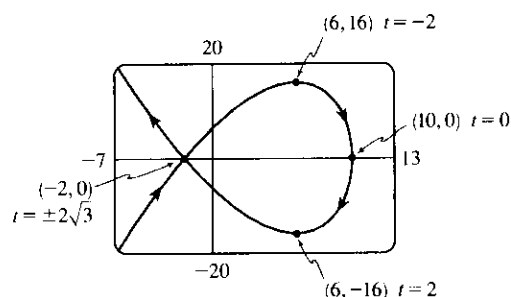
$$dy/dt = 0 \Leftrightarrow t = \pm 2 \Leftrightarrow$$

$$(x, y) = (6, \pm 16). \quad dx/dt = -2t, \text{ so } dx/dt = 0$$

$$\Leftrightarrow t = 0 \Leftrightarrow (x, y) = (10, 0). \text{ The curve has}$$

horizontal tangents at $(6, \pm 16)$ and a vertical

tangent at $(10, 0)$.



18. $x = 2t^3 + 3t^2 - 12t$, $y = 2t^3 + 3t^2 + 1$.

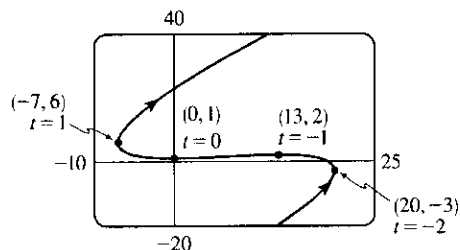
$$dy/dt = 6t^2 + 6t = 6t(t+1), \text{ so } dy/dt = 0 \Leftrightarrow$$

$$t = 0 \text{ or } -1 \Leftrightarrow (x, y) = (0, 1) \text{ or } (13, 2).$$

$$dx/dt = 6t^2 + 6t - 12 = 6(t+2)(t-1), \text{ so}$$

$$dx/dt = 0 \Leftrightarrow t = -2 \text{ or } 1 \Leftrightarrow$$

$(x, y) = (20, -3)$ or $(-7, 6)$. The curve has horizontal tangents at $(0, 1)$ and $(13, 2)$, and vertical tangents at $(20, -3)$ and $(-7, 6)$.



19. $x = 2 \cos \theta$, $y = \sin 2\theta$.

$$dy/d\theta = 2 \cos 2\theta, \text{ so } dy/d\theta = 0 \Leftrightarrow$$

$$2\theta = \frac{\pi}{2} + n\pi \quad (n \text{ an integer}) \Leftrightarrow \theta = \frac{\pi}{4} + \frac{\pi}{2}n$$

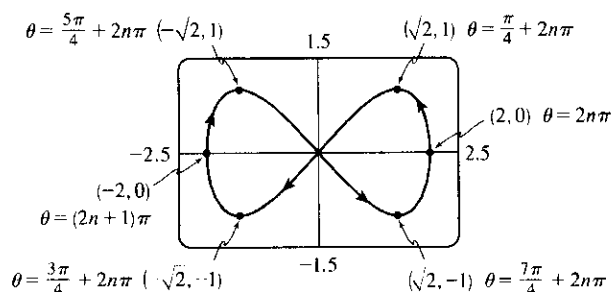
$$\Leftrightarrow (x, y) = (\pm\sqrt{2}, \pm 1). \text{ Also,}$$

$$dx/d\theta = -2 \sin \theta, \text{ so } dx/d\theta = 0 \Leftrightarrow \theta = n\pi$$

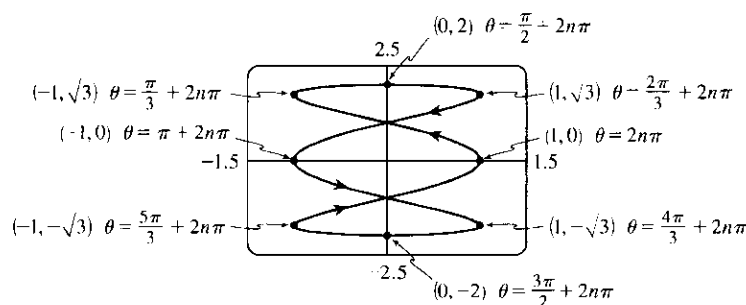
$$\Leftrightarrow (x, y) = (\pm 2, 0). \text{ The curve has horizontal}$$

tangents at $(\pm\sqrt{2}, \pm 1)$ (four points), and vertical

tangents at $(\pm 2, 0)$.



20. $x = \cos 3\theta$, $y = 2 \sin \theta$. $dy/d\theta = 2 \cos \theta$, so $dy/d\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + n\pi$ (n an integer) $\Leftrightarrow (x, y) = (0, \pm 2)$. Also, $dx/d\theta = -3 \sin 3\theta$, so $dx/d\theta = 0 \Leftrightarrow 3\theta = n\pi \Leftrightarrow \theta = \frac{\pi}{3}n \Leftrightarrow (x, y) = (\pm 1, 0)$ or $(\pm 1, \pm\sqrt{3})$. The curve has horizontal tangents at $(0, \pm 2)$, and vertical tangents at $(\pm 1, 0)$, $(\pm 1, -\sqrt{3})$ and $(\pm 1, \sqrt{3})$.

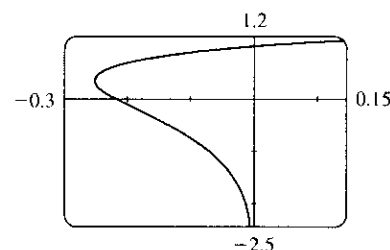


21. From the graph, it appears that the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$ is about $(-0.25, 0.36)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is,

$$0 = dx/dt = 4t^3 - 2t \Leftrightarrow 2t(2t^2 - 1) = 0 \Leftrightarrow$$

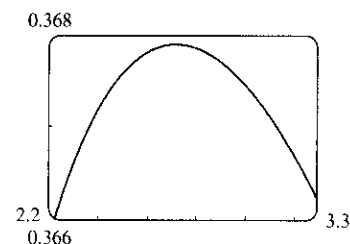
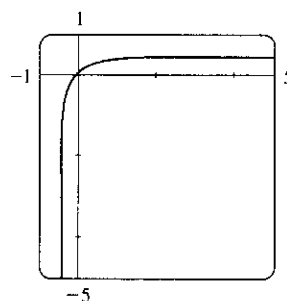
$$2t(\sqrt{2}t + 1)(\sqrt{2}t - 1) = 0 \Leftrightarrow t = 0 \text{ or } \pm \frac{1}{\sqrt{2}}. \text{ The negative and}$$

0 roots are inadmissible since $y(t)$ is only defined for $t > 0$, so the leftmost point must be



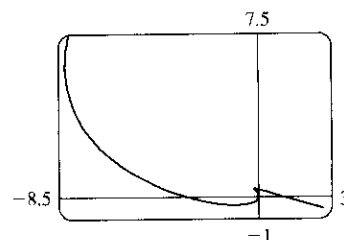
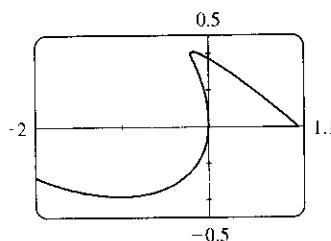
$$\left(x\left(\frac{1}{\sqrt{2}}\right), y\left(\frac{1}{\sqrt{2}}\right)\right) = \left(\left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{\sqrt{2}}\right)^2, \frac{1}{\sqrt{2}} + \ln \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{4}, \frac{1}{\sqrt{2}} - \frac{1}{2} \ln 2\right)$$

22. The curve is symmetric about the line $y = -x$ since replacing t with $-t$ has the effect of replacing (x, y) with $(-y, -x)$, so if we can find the highest point (x_h, y_h) , then the leftmost point is $(x_l, y_l) = (-y_h, -x_h)$. After carefully zooming in, we estimate that the highest point on the curve $x = te^t$, $y = te^{-t}$ is about $(2.7, 0.37)$.



To find the exact coordinates of the highest point, we find the value of t for which the curve has a horizontal tangent, that is, $dy/dt = 0 \Leftrightarrow t(-e^{-t}) + e^{-t} = 0 \Leftrightarrow (1 - t)e^{-t} = 0 \Leftrightarrow t = 1$. This corresponds to the point $(x(1), y(1)) = (e, 1/e)$. To find the leftmost point, we find the value of t for which $0 = dx/dt = te^t + e^t \Leftrightarrow (1 + t)e^t = 0 \Leftrightarrow t = -1$. This corresponds to the point $(x(-1), y(-1)) = (-1/e, -e)$. As $t \rightarrow -\infty$, $x(t) = te^t \rightarrow 0^-$ by l'Hospital's Rule and $y(t) = te^{-t} \rightarrow -\infty$, so the y -axis is an asymptote. As $t \rightarrow \infty$, $x(t) \rightarrow \infty$ and $y(t) \rightarrow 0^+$, so the x -axis is the other asymptote. The asymptotes can also be determined from the graph, if we use a larger t -interval.

23. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$. We estimate that the curve has horizontal tangents at about

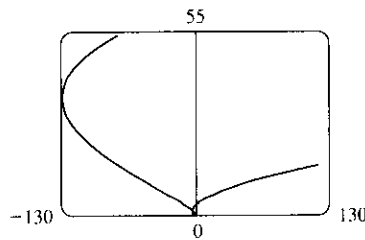
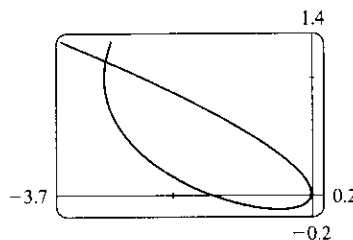


$(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{4t^3 - 6t^2 - 4t}. \text{ The horizontal tangents occur when } dy/dt = 3t^2 - 1 = 0 \Leftrightarrow t = \pm \frac{1}{\sqrt{3}}, \text{ so}$$

both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt = 2t(2t^2 - 3t - 2) = 0 \Leftrightarrow 2t(2t + 1)(t - 2) = 0 \Leftrightarrow t = 0, -\frac{1}{2} \text{ or } 2$. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t -interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

24. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.

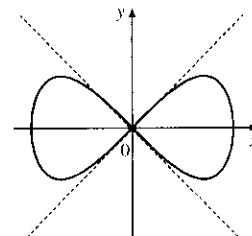


We calculate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t - 1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \Leftrightarrow$

$t = \frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where

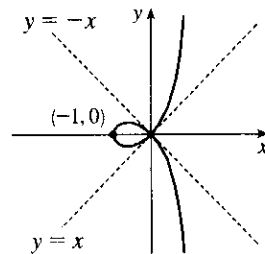
$dx/dt = 4t^3 + 12t^2 - 16t = 0 \Leftrightarrow 4t(t^2 + 3t - 4) = 0 \Leftrightarrow 4t(t + 4)(t - 1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.

25. $x = \cos t$, $y = \sin t \cos t$. $\frac{dx}{dt} = -\sin t$,
 $\frac{dy}{dt} = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \Leftrightarrow \cos t = 0 \Leftrightarrow$
 t is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = -1$, so
 $\frac{dy}{dx} = 1$. When $t = \frac{3\pi}{2}$, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -1$. So $\frac{dy}{dx} = -1$. Thus,
 $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



26. $x = 1 - 2\cos^2 t = -\cos 2t$, $y = (\tan t)(1 - 2\cos^2 t) = -(\tan t)\cos 2t$. To find a point where the curve crosses itself, we look for two values of t that give the same point (x, y) . Call these values t_1 and t_2 . Then $\cos^2 t_1 = \cos^2 t_2$ (from the equation for x) and either $\tan t_1 = \tan t_2$ or $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$ (from the equation for y). We can satisfy $\cos^2 t_1 = \cos^2 t_2$ and $\tan t_1 = \tan t_2$ by choosing t_1 arbitrarily and taking $t_2 = t_1 + \pi$, so evidently the whole curve is retraced every time t traverses an interval of length π . Thus, we can restrict our attention to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. If $t_2 = -t_1$, then $\cos^2 t_2 = \cos^2 t_1$, but $\tan t_2 = -\tan t_1$. This suggests that

we try to satisfy the condition $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$. Taking $t_1 = \frac{\pi}{4}$ and $t_2 = -\frac{\pi}{4}$ gives $(x, y) = (0, 0)$ for both values of t . $\frac{dx}{dt} = 2 \sin 2t$, and $\frac{dy}{dt} = 2 \sin 2t \tan t - \cos 2t \sec^2 t$. When $t = \frac{\pi}{4}$, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = 1$. When $t = -\frac{\pi}{4}$, $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = -1$. Thus, the equations of the two tangents at $(0, 0)$ are $y = x$ and $y = -x$.



27. (a) $x = r\theta - d \sin \theta$, $y = r - d \cos \theta$; $\frac{dx}{d\theta} = r - d \cos \theta$, $\frac{dy}{d\theta} = d \sin \theta$. So $\frac{dy}{dx} = \frac{d \sin \theta}{r - d \cos \theta}$.

(b) If $0 < d < r$, then $|d \cos \theta| \leq d < r$, so $r - d \cos \theta \geq r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

28. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

(a) $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$, $\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$, so $\frac{dy}{dx} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$.

(b) The tangent is horizontal $\Leftrightarrow dy/dx = 0 \Leftrightarrow \tan \theta = 0 \Leftrightarrow \theta = n\pi \Leftrightarrow (x, y) = (\pm a, 0)$. The tangent is vertical $\Leftrightarrow \cos \theta = 0 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{2} \Leftrightarrow (x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1 \Leftrightarrow \tan \theta = \pm 1 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{4} \Leftrightarrow (x, y) = \left(\pm \frac{\sqrt{2}}{4}a, \pm \frac{\sqrt{2}}{4}a\right)$ (All sign choices are valid.)

29. The line with parametric equations $x = -7t$, $y = 12t - 5$ is $y = 12\left(-\frac{1}{7}x\right) - 5$, which has slope $-\frac{12}{7}$. The curve $x = t^3 + 4t$, $y = 6t^2$ has slope $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t}{3t^2 + 4}$. This equals $-\frac{12}{7} \Leftrightarrow 3t^2 + 4 = -7t \Leftrightarrow (3t + 4)(t + 1) = 0 \Leftrightarrow t = -1$ or $t = -\frac{4}{3} \Leftrightarrow (x, y) = (-5, 6)$ or $\left(-\frac{208}{27}, \frac{32}{3}\right)$.

30. $x = 3t^2 + 1$, $y = 2t^3 + 1$, $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$, so $\frac{dy}{dx} = \frac{6t^2}{6t} = t$ (even where $t = 0$).

So at the point corresponding to parameter value t , an equation of the tangent line is

$y - (2t^3 + 1) = t[x - (3t^2 + 1)]$. If this line is to pass through $(4, 3)$, we must have

$$3 - (2t^3 + 1) = t[4 - (3t^2 + 1)] \Leftrightarrow 2t^3 - 2 = 3t^3 - 3t \Leftrightarrow t^3 - 3t + 2 = 0 \Leftrightarrow (t - 1)^2(t + 2) = 0$$

$\Leftrightarrow t = 1$ or -2 . Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

31. By symmetry of the ellipse about the x - and y -axes,

$$\begin{aligned} A &= 4 \int_0^a y \, dx = 4 \int_{\pi/2}^0 b \sin \theta (-a \sin \theta) \, d\theta = 4ab \int_0^{\pi/2} \sin^2 \theta \, d\theta = 4ab \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta) \, d\theta \\ &= 2ab \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = 2ab \left(\frac{\pi}{2} \right) = \pi ab \end{aligned}$$

32. $t + 1/t = 2.5 \Leftrightarrow t = \frac{1}{2}$ or 2 , and for $\frac{1}{2} < t < 2$, we have $t + 1/t < 2.5$. $x = -\frac{3}{2}$ when $t = \frac{1}{2}$ and $x = \frac{3}{2}$ when $t = 2$.

$$\begin{aligned} A &= \int_{-3/2}^{3/2} (2.5 - y) \, dx = \int_{1/2}^2 \left(\frac{5}{2} - t - 1/t \right) (1 + 1/t^2) \, dt \quad [x = t - 1/t, \, dx = (1 + 1/t^2) \, dt] \\ &= \int_{1/2}^2 \left(-t + \frac{5}{2} - 2t^{-1} + \frac{5}{2}t^{-2} - t^{-3} \right) \, dt = \left[-\frac{t^2}{2} + \frac{5t}{2} - 2 \ln |t| - \frac{5}{2t} + \frac{1}{2t^2} \right]_{1/2}^2 \\ &= \left(-2 + 5 - 2 \ln 2 - \frac{5}{4} + \frac{1}{8} \right) - \left(-\frac{1}{8} + \frac{5}{4} + 2 \ln 2 - 5 + 2 \right) = \frac{15}{4} - 4 \ln 2 \end{aligned}$$

$$33. A = \int_0^1 (y-1) dx = \int_{\pi/2}^0 (e^t - 1)(-\sin t) dt = \int_0^{\pi/2} (e^t \sin t - \sin t) dt \stackrel{98}{=} \left[\frac{1}{2} e^t (\sin t - \cos t) + \cos t \right]_0^{\pi/2} \\ = \frac{1}{2} (e^{\pi/2} - 1)$$

$$34. \text{ By symmetry, } A = 4 \int_0^a y dx = 4 \int_{\pi/2}^0 a \sin^3 \theta (-3a \cos^2 \theta \sin \theta) d\theta = 12a^2 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta. \text{ Now}$$

$$\int \sin^4 \theta \cos^2 \theta d\theta = \int \sin^2 \theta \left(\frac{1}{4} \sin^2 2\theta \right) d\theta = \frac{1}{8} \int (1 - \cos 2\theta) \sin^2 2\theta d\theta \\ = \frac{1}{8} \int \left[\frac{1}{2} (1 - \cos 4\theta) - \sin^2 2\theta \cos 2\theta \right] d\theta = \frac{1}{16} \theta - \frac{1}{64} \sin 4\theta - \frac{1}{48} \sin^3 2\theta + C$$

$$\text{so } \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta = \left[\frac{1}{16} \theta - \frac{1}{64} \sin 4\theta - \frac{1}{48} \sin^3 2\theta \right]_0^{\pi/2} = \frac{\pi}{32}. \text{ Thus, } A = 12a^2 \left(\frac{\pi}{32} \right) = \frac{3}{8} \pi a^2.$$

$$35. A = \int_0^{2\pi} y dx = \int_0^{2\pi} (r - d \cos \theta)(r - d \cos \theta) d\theta = \int_0^{2\pi} (r^2 - 2dr \cos \theta + d^2 \cos^2 \theta) d\theta \\ = \left[r^2 \theta - 2dr \sin \theta + \frac{1}{2} d^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} = 2\pi r^2 + \pi d^2$$

36. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x -axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt{3} \leq t \leq 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t dt$, we find that

$$\text{area} = 2 \int_0^3 y dx = 2 \int_0^{-\sqrt{3}} (t^3 - 3t) 2t dt = 2 \int_0^{-\sqrt{3}} (2t^4 - 6t^2) dt = 2 \left[\frac{2}{5} t^5 - 2t^3 \right]_0^{-\sqrt{3}} \\ = 2 \left[\frac{2}{5} (-3^{1/2})^5 - 2(-3^{1/2})^3 \right] = 2 \left[\frac{2}{5} (-9\sqrt{3}) - 2(-3\sqrt{3}) \right] = \frac{24}{5} \sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\text{volume} = \pi \int_0^3 y^2 dx = \pi \int_0^{-\sqrt{3}} (t^3 - 3t)^2 2t dt = 2\pi \int_0^{-\sqrt{3}} (t^6 - 6t^4 + 9t^2) t dt \\ = 2\pi \left[\frac{1}{8} t^8 - t^6 + \frac{9}{4} t^4 \right]_0^{-\sqrt{3}} = 2\pi \left[\frac{1}{8} (-3^{1/2})^8 - (-3^{1/2})^6 + \frac{9}{4} (-3^{1/2})^4 \right] \\ = 2\pi \left[\frac{81}{8} - 27 + \frac{81}{4} \right] = \frac{27}{4} \pi$$

(c) By symmetry, the y -coordinate of the centroid is 0. To find the x -coordinate, we note that it is the same as the x -coordinate of the centroid of the top half of $\mathcal{R}$, the area of which is $\frac{1}{2} \cdot \frac{24}{5} \sqrt{3} = \frac{12}{5} \sqrt{3}$. So, using

Formula 9.3.8 with $A = \frac{12}{5} \sqrt{3}$, we get

$$\bar{x} = \frac{5}{12\sqrt{3}} \int_0^3 xy dx = \frac{5}{12\sqrt{3}} \int_0^{-\sqrt{3}} t^2 (t^3 - 3t) 2t dt = \frac{5}{6\sqrt{3}} \left[\frac{1}{7} t^7 - \frac{3}{5} t^5 \right]_0^{-\sqrt{3}} \\ = \frac{5}{6\sqrt{3}} \left[\frac{1}{7} (-3^{1/2})^7 - \frac{3}{5} (-3^{1/2})^5 \right] = \frac{5}{6\sqrt{3}} \left[-\frac{27}{7} \sqrt{3} + \frac{27}{5} \sqrt{3} \right] = \frac{9}{7}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\frac{9}{7}, 0 \right)$.

$$37. x = t - t^2, y = \frac{4}{3} t^{3/2}, 1 \leq t \leq 2. dx/dt = 1 - 2t \text{ and } dy/dt = 2t^{1/2}, \text{ so} \\ (dx/dt)^2 + (dy/dt)^2 = (1 - 2t)^2 + (2t^{1/2})^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2. \text{ Thus,} \\ L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_1^2 \sqrt{1 + 4t^2} dt.$$

$$38. x = 1 + e^t, y = t^2, -3 \leq t \leq 3. dx/dt = e^t \text{ and } dy/dt = 2t, \text{ so } (dx/dt)^2 + (dy/dt)^2 = e^{2t} + 4t^2. \text{ Thus,} \\ L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_{-3}^3 \sqrt{e^{2t} + 4t^2} dt.$$

$$39. x = t + \cos t, y = t - \sin t, 0 \leq t \leq 2\pi. dx/dt = 1 - \sin t \text{ and } dy/dt = 1 - \cos t, \text{ so} \\ (dx/dt)^2 + (dy/dt)^2 = (1 - \sin t)^2 + (1 - \cos t)^2 = (1 - 2\sin t + \sin^2 t) + (1 - 2\cos t + \cos^2 t) \\ = 3 - 2\sin t - 2\cos t$$

$$\text{Thus, } L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^{2\pi} \sqrt{3 - 2\sin t - 2\cos t} dt.$$

40. $x = \ln t, y = \sqrt{t+1}, 1 \leq t \leq 5. \frac{dx}{dt} = \frac{1}{t} \text{ and } \frac{dy}{dt} = \frac{1}{2\sqrt{t+1}}, \text{ so}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{t^2} + \frac{1}{4(t+1)} = \frac{t^2 + 4t + 4}{4t^2(t+1)}. \text{ Thus,}$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_1^5 \sqrt{\frac{t^2 + 4t + 4}{4t^2(t+1)}} dt = \int_1^5 \sqrt{\frac{(t+2)^2}{(2t)^2(t+1)}} dt = \int_1^5 \frac{t+2}{2t\sqrt{t+1}} dt.$$

41. $x = 1 + 3t^2, y = 4 + 2t^3, 0 \leq t \leq 1. dx/dt = 6t \text{ and } dy/dt = 6t^2, \text{ so } (dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4.$

$$\begin{aligned} \text{Thus, } L &= \int_0^1 \sqrt{36t^2 + 36t^4} dt = \int_0^1 6t \sqrt{1 + t^2} dt = 6 \int_1^2 \sqrt{u} \left(\frac{1}{2} du\right) \quad [u = 1 + t^2, du = 2t dt] \\ &= 3 \left[\frac{2}{3} u^{3/2} \right]_1^2 = 2(2^{3/2} - 1) = 2(2\sqrt{2} - 1) \end{aligned}$$

42. $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta), 0 \leq \theta \leq \pi.$

$$\begin{aligned} (dx/d\theta)^2 + (dy/d\theta)^2 &= a^2 [(-\sin \theta + \theta \cos \theta + \sin \theta)^2 + (\cos \theta + \theta \sin \theta - \cos \theta)^2] \\ &= a^2 \theta^2 (\cos^2 \theta + \sin^2 \theta) = (a\theta)^2 \end{aligned}$$

$$\text{Thus, } L = \int_0^\pi a\theta d\theta = a \left[\frac{1}{2} \theta^2 \right]_0^\pi = \frac{1}{2} \pi^2 a.$$

43. $x = \frac{t}{1+t}, y = \ln(1+t), 0 \leq t \leq 2. \frac{dx}{dt} = \frac{(1+t) \cdot 1 - t \cdot 1}{(1+t)^2} = \frac{1}{(1+t)^2} \text{ and } \frac{dy}{dt} = \frac{1}{1+t}, \text{ so}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+t)^4} + \frac{1}{(1+t)^2} = \frac{1}{(1+t)^4} [1 + (1+t)^2] = \frac{t^2 + 2t + 2}{(1+t)^4}. \text{ Thus,}$$

$$\begin{aligned} L &= \int_0^2 \frac{\sqrt{t^2 + 2t + 2}}{(1+t)^2} dt = \int_1^3 \frac{\sqrt{u^2 + 1}}{u^2} du \quad [u = t + 1, du = dt] \stackrel{24}{=} \left[-\frac{\sqrt{u^2 + 1}}{u} + \ln(u + \sqrt{u^2 + 1}) \right]_1^3 \\ &= -\frac{\sqrt{10}}{3} + \ln(3 + \sqrt{10}) + \sqrt{2} - \ln(1 + \sqrt{2}) \end{aligned}$$

44. $x = e^t + e^{-t}, y = 5 - 2t, 0 \leq t \leq 3. dx/dt = e^t - e^{-t} \text{ and } dy/dt = -2, \text{ so}$

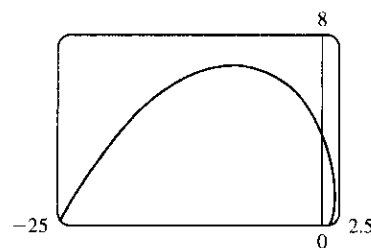
$$(dx/dt)^2 + (dy/dt)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2. \text{ Thus,}$$

$$L = \int_0^3 (e^t + e^{-t}) dt = [e^t - e^{-t}]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}.$$

45. $x = e^t \cos t, y = e^t \sin t, 0 \leq t \leq \pi.$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2 \\ &= (e^t)^2 (\cos^2 t - 2 \cos t \sin t + \sin^2 t) \\ &\quad + (e^t)^2 (\sin^2 t + 2 \sin t \cos t + \cos^2 t) \\ &= e^{2t} (2 \cos^2 t + 2 \sin^2 t) = 2e^{2t} \end{aligned}$$

$$\text{Thus, } L = \int_0^\pi \sqrt{2e^{2t}} dt = \int_0^\pi \sqrt{2} e^t dt = \sqrt{2} [e^t]_0^\pi = \sqrt{2}(e^\pi - 1).$$

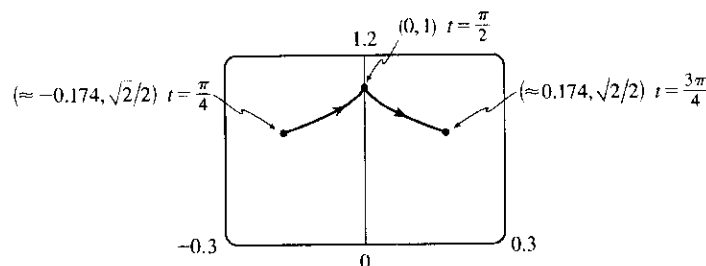


46. $x = \cos t + \ln(\tan \frac{1}{2}t), y = \sin t, \pi/4 \leq t \leq 3\pi/4.$

$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2} \sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2 \sin(t/2) \cos(t/2)} = -\sin t + \frac{1}{\sin t}$$

and $\frac{dy}{dt} = \cos t$, so $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t - 2 + \frac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t$. Thus,

$$\begin{aligned} L &= \int_{\pi/4}^{3\pi/4} |\cot t| dt = 2 \int_{\pi/4}^{\pi/2} \cot t dt = 2[\ln |\sin t|]_{\pi/4}^{\pi/2} = 2\left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right) \\ &= 2\left(0 + \ln \sqrt{2}\right) = 2\left(\frac{1}{2} \ln 2\right) = \ln 2. \end{aligned}$$

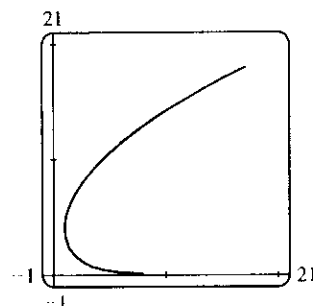


47. $x = e^t - t$, $y = 4e^{t/2}$, $-8 \leq t \leq 3$.

$$\begin{aligned} (dx/dt)^2 + (dy/dt)^2 &= (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t \\ &= e^{2t} + 2e^t + 1 = (e^t + 1)^2 \end{aligned}$$

Thus,

$$\begin{aligned} L &= \int_{-8}^3 \sqrt{(e^t + 1)^2} dt = \int_{-8}^3 (e^t + 1) dt = [e^t + t]_{-8}^3 \\ &= (e^3 + 3) - (e^{-8} - 8) = e^3 - e^{-8} + 11. \end{aligned}$$

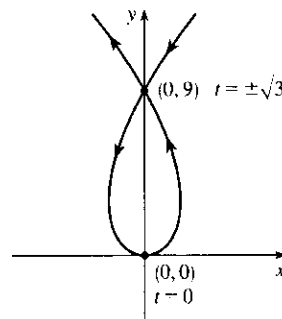


48. $x = 3t - t^3$, $y = 3t^2$. $dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2 \text{ and the length of}$$

the loop is given by

$$\begin{aligned} L &= \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2) dt = 2 \int_0^{\sqrt{3}} (3 + 3t^2) dt = 2[3t + t^3]_0^{\sqrt{3}} \\ &= 2(3\sqrt{3} + 3\sqrt{3}) = 12\sqrt{3}. \end{aligned}$$



49. $x = t - e^t$, $y = t + e^t$, $-6 \leq t \leq 6$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - e^t)^2 + (1 + e^t)^2 = (1 - 2e^t + e^{2t}) + (1 + 2e^t + e^{2t}) = 2 + 2e^{2t}, \text{ so}$$

$$L = \int_{-6}^6 \sqrt{2 + 2e^{2t}} dt. \text{ Set } f(t) = \sqrt{2 + 2e^{2t}}. \text{ Then by Simpson's Rule with } n = 6 \text{ and } \Delta t = \frac{6 - (-6)}{6} = 2,$$

$$\text{we get } L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053.$$

50. $x = 2a \cot \theta \Rightarrow dx/dt = -2a \csc^2 \theta$ and $y = 2a \sin^2 \theta \Rightarrow dy/dt = 4a \sin \theta \cos \theta = 2a \sin 2\theta$.

$$\text{So } L = \int_{\pi/4}^{\pi/2} \sqrt{4a^2 \csc^4 \theta + 4a^2 \sin^2 2\theta} d\theta = 2a \int_{\pi/4}^{\pi/2} \sqrt{\csc^4 \theta + \sin^2 2\theta} d\theta. \text{ Using Simpson's}$$

$$\text{Rule with } n = 4, \Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}, \text{ and } f(\theta) = \sqrt{\csc^4 \theta + \sin^2 2\theta}, \text{ we get}$$

$$L \approx 2a \cdot S_4 = (2a) \frac{\pi}{16 \cdot 3} \left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right) \right] \approx 2.2605a.$$

51. $x = \sin^2 t, y = \cos^2 t, 0 \leq t \leq 3\pi.$

$$(dx/dt)^2 + (dy/dt)^2 = (2 \sin t \cos t)^2 + (-2 \cos t \sin t)^2 = 8 \sin^2 t \cos^2 t = 2 \sin^2 2t \Rightarrow$$

$$\begin{aligned} \text{Distance} &= \int_0^{3\pi} \sqrt{2} |\sin 2t| dt = 6 \sqrt{2} \int_0^{\pi/2} \sin 2t dt \quad [\text{by symmetry}] = -3 \sqrt{2} \left[\cos 2t \right]_0^{\pi/2} \\ &= -3 \sqrt{2} (-1 - 1) = 6 \sqrt{2} \end{aligned}$$

The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \geq 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$.

Thus, $L = \int_0^{\pi/2} \sin 2t dt = \sqrt{2}$, as above.

52. $x = \cos^2 t, y = \cos t, 0 \leq t \leq 4\pi. \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2 \cos t \sin t)^2 + (-\sin t)^2 = \sin^2 t (4 \cos^2 t + 1)$

$$\begin{aligned} \text{Distance} &= \int_0^{4\pi} |\sin t| \sqrt{4 \cos^2 t + 1} dt = 4 \int_0^{\pi} \sin t \sqrt{4 \cos^2 t + 1} dt \\ &= -4 \int_1^{-1} \sqrt{4u^2 + 1} du \quad [u = \cos t, du = -\sin t dt] = 4 \int_{-1}^1 \sqrt{4u^2 + 1} du = 8 \int_0^1 \sqrt{4u^2 + 1} du \\ &= 8 \int_0^{\tan^{-1} 2} \sec \theta \cdot \frac{1}{2} \sec^2 \theta d\theta \quad [2u = \tan \theta, 2 du = \sec^2 \theta d\theta] \\ &= 4 \int_0^{\tan^{-1} 2} \sec^3 \theta d\theta \stackrel{71}{=} [2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta|]_0^{\tan^{-1} 2} = 4\sqrt{5} + 2 \ln(\sqrt{5} + 2) \end{aligned}$$

Thus, $L = \int_0^{\pi} |\sin t| \sqrt{4 \cos^2 t + 1} dt = \sqrt{5} + \frac{1}{2} \ln(\sqrt{5} + 2).$

53. $x = a \sin \theta, y = b \cos \theta, 0 \leq \theta \leq 2\pi.$

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (a \cos \theta)^2 + (-b \sin \theta)^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta = a^2 (1 - \sin^2 \theta) + b^2 \sin^2 \theta \\ &= a^2 - (a^2 - b^2) \sin^2 \theta = a^2 - c^2 \sin^2 \theta = a^2 \left(1 - \frac{c^2}{a^2} \sin^2 \theta\right) = a^2 (1 - e^2 \sin^2 \theta) \end{aligned}$$

$$\text{So } L = 4 \int_0^{\pi/2} \sqrt{a^2 (1 - e^2 \sin^2 \theta)} d\theta \quad [\text{by symmetry}] = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta.$$

54. $x = a \cos^3 \theta, y = a \sin^3 \theta.$

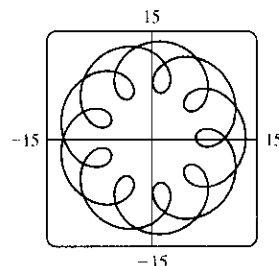
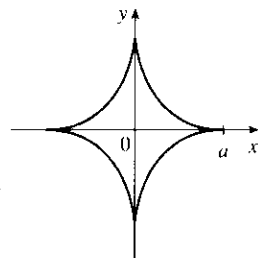
$$\begin{aligned} (dx/d\theta)^2 + (dy/d\theta)^2 &= (-3a \cos^2 \theta \sin \theta)^2 + (3a \sin^2 \theta \cos \theta)^2 \\ &= 9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta \\ &= 9a^2 \sin^2 \theta \cos^2 \theta (\cos^2 \theta + \sin^2 \theta) = 9a^2 \sin^2 \theta \cos^2 \theta. \end{aligned}$$

The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \leq \theta \leq \pi/2$. Thus,

$$\begin{aligned} L &= 4 \int_0^{\pi/2} 3a \sin \theta \cos \theta d\theta \quad [\text{since } a > 0 \text{ and } \sin \theta \text{ and } \cos \theta \text{ are positive for } 0 \leq \theta \leq \pi/2] \\ &= 12a \left[\frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} = 12a \left(\frac{1}{2} - 0 \right) = 6a. \end{aligned}$$

55. (a) Notice that $0 \leq t \leq 2\pi$ does not give the complete curve

because $x(0) \neq x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π .



(b) We use the CAS to find the derivatives dx/dt and dy/dt , and then use Formula 1 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$ as $88E(2\sqrt{2}i)$, where $E(x)$ is the elliptic integral $\int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}} dt$ and i is the imaginary number $\sqrt{-1}$. Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2), t=0..4*Pi))`; to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11 \int_0^{4\pi} \sqrt{-4 \cos t \cos(\frac{11t}{2}) - 4 \sin t \sin(\frac{11t}{2}) + 5} dt$.

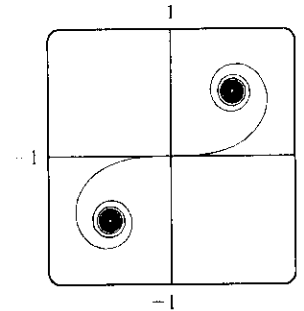
56. (a) It appears that as $t \rightarrow \infty$, $(x, y) \rightarrow (\frac{1}{2}, \frac{1}{2})$, and

as $t \rightarrow -\infty$, $(x, y) \rightarrow (-\frac{1}{2}, -\frac{1}{2})$.

- (b) By the Fundamental Theorem of Calculus,

$$dx/dt = \cos(\frac{\pi}{2}t^2) \text{ and } dy/dt = \sin(\frac{\pi}{2}t^2), \text{ so}$$

by Formula 6, the length of the curve from the origin to the point with parameter value t is



$$\begin{aligned} L &= \int_0^t \sqrt{(dx/du)^2 + (dy/du)^2} du = \int_0^t \sqrt{\cos^2(\frac{\pi}{2}u^2) + \sin^2(\frac{\pi}{2}u^2)} du \\ &= \int_0^t 1 du = t \quad [\text{or } -t \text{ if } t < 0] \end{aligned}$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

57. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \leq t \leq 2$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - 2t)^2 + (2t^{1/2})^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2$,
so $S = \int_1^2 2\pi y ds = \int_1^2 2\pi \cdot \frac{4}{3}t^{3/2} \sqrt{1 + 4t^2} dt = \int_1^2 \frac{8\pi}{3}t^{3/2} \sqrt{1 + 4t^2} dt$.

58. $x = \sin^2 t$, $y = \sin 3t$, $0 \leq t \leq \frac{\pi}{3}$. $dx/dt = 2 \sin t \cos t = \sin 2t$ and $dy/dt = 3 \cos 3t$, so
 $(dx/dt)^2 + (dy/dt)^2 = \sin^2 2t + 9 \cos^2 3t$ and $S = \int 2\pi y ds = \int_0^{\pi/3} 2\pi \sin 3t \sqrt{\sin^2 2t + 9 \cos^2 3t} dt$.

59. $x = t^3$, $y = t^2$, $0 \leq t \leq 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3t^2)^2 + (2t)^2 = 9t^4 + 4t^2$.

$$\begin{aligned} S &= \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2} dt = 2\pi \int_0^1 t^2 \sqrt{t^2(9t^2 + 4)} dt \\ &= 2\pi \int_4^{13} \left(\frac{u-4}{9}\right) \sqrt{u} \left(\frac{1}{18} du\right) \left[\begin{array}{l} u = 9t^2 + 4, \quad t^2 = (u-4)/9 \\ du = 18t dt, \text{ so } t dt = \frac{1}{18} du \end{array} \right] = \frac{2\pi}{9 \cdot 18} \int_4^{13} (u^{3/2} - 4u^{1/2}) du \\ &= \frac{\pi}{81} \left[\frac{2}{5} u^{5/2} - \frac{8}{3} u^{3/2} \right]_4^{13} = \frac{\pi}{81} \cdot \frac{2}{15} [3u^{5/2} - 20u^{3/2}]_4^{13} \\ &= \frac{2\pi}{1215} \left[(3 \cdot 13^2 \sqrt{13} - 20 \cdot 13 \sqrt{13}) - (3 \cdot 32 - 20 \cdot 8) \right] \\ &= \frac{2\pi}{1215} (247 \sqrt{13} + 64) \end{aligned}$$

60. $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = 9(1 + 2t^2 + t^4) = [3(1 + t^2)]^2$.
 $S = \int_0^1 2\pi \cdot 3t^2 \cdot 3(1 + t^2) dt = 18\pi \int_0^1 (t^2 + t^4) dt = 18\pi \left[\frac{1}{3}t^3 + \frac{1}{5}t^5 \right]_0^1 = \frac{48}{5}\pi$

61. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $0 \leq \theta \leq \frac{\pi}{2}$.

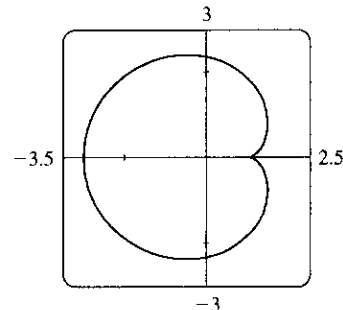
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-3a \cos^2 \theta \sin \theta)^2 + (3a \sin^2 \theta \cos \theta)^2 = 9a^2 \sin^2 \theta \cos^2 \theta.$$

$$S = \int_0^{\pi/2} 2\pi \cdot a \sin^3 \theta \cdot 3a \sin \theta \cos \theta d\theta = 6\pi a^2 \int_0^{\pi/2} \sin^4 \theta \cos \theta d\theta = \frac{6}{5}\pi a^2 [\sin^5 \theta]_0^{\pi/2} = \frac{6}{5}\pi a^2$$

62. $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-2 \sin \theta + 2 \sin 2\theta)^2 + (2 \cos \theta - 2 \cos 2\theta)^2$
 $= 4[(\sin^2 \theta - 2 \sin \theta \sin 2\theta + \sin^2 2\theta) + (\cos^2 \theta - 2 \cos \theta \cos 2\theta + \cos^2 2\theta)]$
 $= 4[1 + 1 - 2(\cos 2\theta \cos \theta + \sin 2\theta \sin \theta)] = 8[1 - \cos(2\theta - \theta)] = 8(1 - \cos \theta)$

We plot the graph with parameter interval $[0, 2\pi]$, and see that we should only integrate between 0 and π . (If the interval $[0, 2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y = 2 \sin \theta - \sin 2\theta = 2 \sin \theta(1 - \cos \theta)$. So

$$\begin{aligned} S &= \int_0^\pi 2\pi \cdot 2 \sin \theta(1 - \cos \theta) 2\sqrt{2} \sqrt{1 - \cos \theta} d\theta \\ &= 8\sqrt{2}\pi \int_0^\pi (1 - \cos \theta)^{3/2} \sin \theta d\theta \\ &= 8\sqrt{2}\pi \int_0^2 \sqrt{u^3} du \quad [\text{where } u = 1 - \cos \theta, du = \sin \theta d\theta] \\ &= 8\sqrt{2}\pi \left[\left(\frac{2}{5}\right) u^{5/2} \right]_0^2 = \frac{16}{5}\sqrt{2}\pi(2^{5/2}) = \frac{128}{5}\pi \end{aligned}$$



63. $x = t + t^3$, $y = t - \frac{1}{t^2}$, $1 \leq t \leq 2$. $\frac{dx}{dt} = 1 + 3t^2$ and $\frac{dy}{dt} = 1 + \frac{2}{t^3}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 + 3t^2)^2 + \left(1 + \frac{2}{t^3}\right)^2 \text{ and}$$

$$S = \int 2\pi y ds = \int_1^2 2\pi \left(t - \frac{1}{t^2}\right) \sqrt{(1 + 3t^2)^2 + \left(1 + \frac{2}{t^3}\right)^2} dt \approx 59.101.$$

64. $S = \int_{\pi/4}^{\pi/2} 2\pi \cdot 2a \sin^2 \theta \sqrt{\csc^4 \theta + \sin^2 2\theta} d\theta = 4\pi a \int_{\pi/4}^{\pi/2} \sin^2 \theta \sqrt{\csc^4 \theta + \sin^2 2\theta} d\theta.$

Using Simpson's Rule with $n = 4$, $\Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}$, and $f(\theta) = \sin^2 \theta \sqrt{\csc^4 \theta + \sin^2 2\theta}$, we get

$$S \approx (4\pi a) \frac{\pi}{16 \cdot 3} \left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right) \right] \approx 11.0893a.$$

65. $x = 3t^2$, $y = 2t^3$, $0 \leq t \leq 5 \Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \Rightarrow$

$$\begin{aligned} S &= \int_0^5 2\pi x \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^5 2\pi(3t^2)6t \sqrt{1 + t^2} dt = 18\pi \int_0^5 t^2 \sqrt{1 + t^2} 2t dt \\ &= 18\pi \int_1^{26} (u - 1) \sqrt{u} du \quad [\text{where } u = 1 + t^2, du = 2t dt] = 18\pi \int_1^{26} (u^{3/2} - u^{1/2}) du \\ &= 18\pi \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^{26} = 18\pi \left[\left(\frac{2}{5} \cdot 676 \sqrt{26} - \frac{2}{3} \cdot 26 \sqrt{26} \right) - \left(\frac{2}{5} - \frac{2}{3} \right) \right] \\ &= \frac{24}{5}\pi(949\sqrt{26} + 1) \end{aligned}$$

66. $x = e^t - t$, $y = 4e^{t/2}$, $0 \leq t \leq 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2$.

$$S = \int_0^1 2\pi(e^t - t) \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} dt = \int_0^1 2\pi(e^t - t)(e^t + 1) dt$$

$$= 2\pi \left[\frac{1}{2}e^{2t} + e^t - (t-1)e^t - \frac{1}{2}t^2 \right]_0^1 = \pi(e^2 + 2e - 6)$$

67. If f' is continuous and $f'(t) \neq 0$ for $a \leq t \leq b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \Rightarrow f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

68. By Formula 9.2.5 with $y = F(x)$, $S = \int_a^b 2\pi F(x) \sqrt{1 + [F'(x)]^2} dx$. But by Formula 11.2.2,

$$1 + [F'(x)]^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy/dt}{dx/dt}\right)^2 = \frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}.$$

Using the Substitution Rule with $x = x(t)$, where $a = x(\alpha)$ and $b = x(\beta)$, we have (since $dx = \frac{dx}{dt} dt$)

$$S = \int_{\alpha}^{\beta} 2\pi F(x(t)) \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}} \frac{dx}{dt} dt = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

which is Formula 11.2.7.

69. (a) $\phi = \tan^{-1}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d\phi}{dt} = \frac{d}{dt} \tan^{-1}\left(\frac{dy}{dx}\right) = \frac{1}{1 + (dy/dx)^2} \left[\frac{d}{dt} \left(\frac{dy}{dx} \right) \right]$. But $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} \Rightarrow$

$$\frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{\dot{y}}{\dot{x}} \right) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} \Rightarrow \frac{d\phi}{dt} = \frac{1}{1 + (\dot{y}/\dot{x})^2} \left(\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} \right) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}.$$

Using the Chain Rule, and the fact that $s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \Rightarrow$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = (\dot{x}^2 + \dot{y}^2)^{1/2}, \text{ we have that}$$

$$\frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} \right) \frac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \text{ So}$$

$$\kappa = \left| \frac{d\phi}{ds} \right| = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \right| = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

(b) $x = x$ and $y = f(x) \Rightarrow \dot{x} = 1$, $\ddot{x} = 0$ and $\dot{y} = \frac{dy}{dx}$, $\ddot{y} = \frac{d^2y}{dx^2}$.

$$\text{So } \kappa = \frac{|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)|}{[1 + (dy/dx)^2]^{3/2}} = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}.$$

70. (a) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \frac{d^2y}{dx^2} = 2$. So $\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$, and at $(1, 1)$,

$$\kappa = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}}.$$

- (b) $\kappa' = \frac{d\kappa}{dx} = -3(1+4x^2)^{-5/2}(8x) = 0 \Leftrightarrow x = 0 \Rightarrow y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and $\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

71. $x = \theta - \sin \theta \Rightarrow \dot{x} = 1 - \cos \theta \Rightarrow \ddot{x} = \sin \theta$, and $y = 1 - \cos \theta \Rightarrow \dot{y} = \sin \theta \Rightarrow \ddot{y} = \cos \theta$.

Therefore, $\kappa = \frac{|\cos \theta - \cos^2 \theta - \sin^2 \theta|}{[(1 - \cos \theta)^2 + \sin^2 \theta]^{3/2}} = \frac{|\cos \theta - (\cos^2 \theta + \sin^2 \theta)|}{(1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)^{3/2}} = \frac{|\cos \theta - 1|}{(2 - 2 \cos \theta)^{3/2}}$. The top

of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section 11.2, the tangent is horizontal when $\theta = (2n - 1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ :

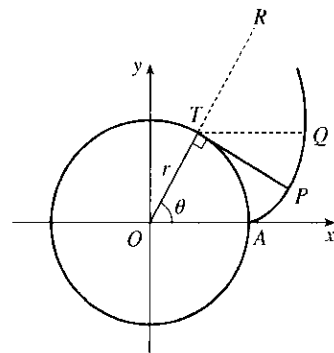
$$\kappa = \frac{|\cos \pi - 1|}{(2 - 2 \cos \pi)^{3/2}} = \frac{|-1 - 1|}{[2 - 2(-1)]^{3/2}} = \frac{1}{4}.$$

72. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and $v, w \neq 0$. For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve $x = a + (c - a)t$, $y = b + (d - b)t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$, and $\kappa = \frac{|v \cdot 0 - w \cdot 0|}{(v^2 + w^2)^{3/2}} = 0$.

(b) Parametric equations for a circle of radius r are $x = r \cos \theta$ and $y = r \sin \theta$. We can take the center to be the origin. So $\dot{x} = -r \sin \theta \Rightarrow \ddot{x} = -r \cos \theta$ and $\dot{y} = r \cos \theta \Rightarrow \ddot{y} = -r \sin \theta$. Therefore,

$$\kappa = \frac{|r^2 \sin^2 \theta + r^2 \cos^2 \theta|}{(r^2 \sin^2 \theta + r^2 \cos^2 \theta)^{3/2}} = \frac{r^2}{r^3} = \frac{1}{r}. \text{ And so for any } \theta \text{ (and thus any point), } \kappa = \frac{1}{r}.$$

73. The coordinates of T are $(r \cos \theta, r \sin \theta)$. Since TP was unwound from arc TA , TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r \cos \theta + r\theta \cos(\frac{1}{2}\pi - \theta) = r(\cos \theta + \theta \sin \theta)$, $y = r \sin \theta - r\theta \sin(\frac{1}{2}\pi - \theta) = r(\sin \theta - \theta \cos \theta)$.



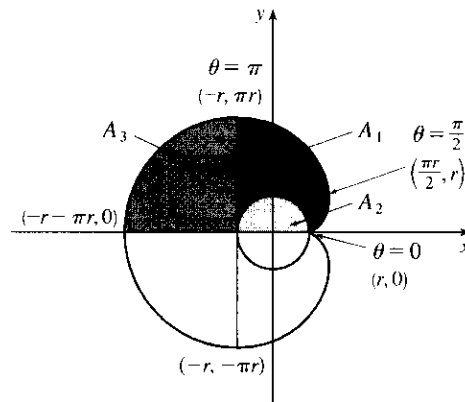
74. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 73 corresponding to the range $0 \leq \theta \leq \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr , arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x -axis of the initial involute path. (This corresponds to the range $-\pi \leq \theta \leq 0$.)

Referring to the figure, we see that the total grazing area is

$2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr ,

so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then

subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1 .



To find $A_1 + A_2$, first note that the rightmost point of the involute is $(\frac{\pi}{2}r, r)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $(\frac{\pi}{2}r, r)$.] The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y dx - \int_{\theta=0}^{\pi/2} y dx = \int_{\theta=\pi}^0 y dx$. Now $y dx = r(\sin \theta - \theta \cos \theta)r\theta \cos \theta d\theta = r^2(\theta \sin \theta \cos \theta - \theta^2 \cos^2 \theta)d\theta$. Integrate: $(1/r^2) \int y dx = -\theta \cos^2 \theta - \frac{1}{2}(\theta^2 - 1) \sin \theta \cos \theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$\begin{aligned} A_1 + A_2 &= r^2 \left[-\theta \cos^2 \theta - \frac{1}{2}(\theta^2 - 1) \sin \theta \cos \theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta \right]_{\pi}^0 = r^2 \left[0 - \left(-\pi - \frac{\pi^3}{6} + \frac{\pi}{2} \right) \right] \\ &= r^2 \left(\frac{\pi}{2} + \frac{\pi^3}{6} \right) \end{aligned}$$

Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2) = \frac{5}{6}\pi^3 r^2$.

LABORATORY PROJECT Bezier Curves

1. The parametric equations for a cubic Bézier curve are

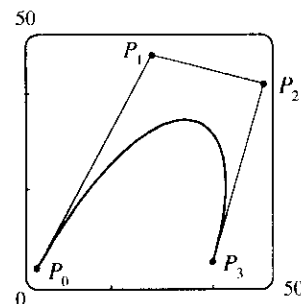
$$\begin{aligned} x &= x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3 \\ y &= y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3 \end{aligned}$$

where $0 \leq t \leq 1$. We are given the points $P_0(x_0, y_0) = (4, 1)$, $P_1(x_1, y_1) = (28, 48)$, $P_2(x_2, y_2) = (50, 42)$, and $P_3(x_3, y_3) = (40, 5)$. The curve is then given by

$$\begin{aligned} x(t) &= 4(1-t)^3 + 3 \cdot 28t(1-t)^2 + 3 \cdot 50t^2(1-t) + 40t^3 \\ y(t) &= 1(1-t)^3 + 3 \cdot 48t(1-t)^2 + 3 \cdot 42t^2(1-t) + 5t^3 \end{aligned}$$

where $0 \leq t \leq 1$. The line segments are of the form $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$:

$$\begin{array}{lll} P_0P_1 & x = 4 + 24t, & y = 1 + 47t \\ P_1P_2 & x = 28 + 22t, & y = 48 - 6t \\ P_2P_3 & x = 50 - 10t, & y = 42 - 37t \end{array}$$



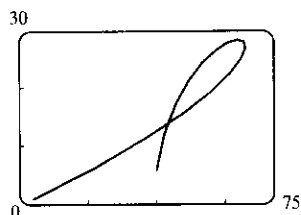
2. It suffices to show that the slope of the tangent at P_0 is the same as that of line segment P_0P_1 , namely $\frac{y_1 - y_0}{x_1 - x_0}$. We

calculate the slope of the tangent to the Bézier curve:

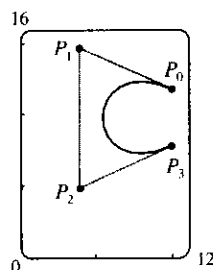
$$\frac{dy/dt}{dx/dt} = \frac{-3y_0(1-t)^2 + 3y_1[-2t(1-t) + (1-t)^2] + 3y_2[-t^2 + (2t)(1-t)] + 3y_3t^2}{-3x_0(1-t)^2 + 3x_1[-2t(1-t) + (1-t)^2] + 3x_2[-t^2 + (2t)(1-t)] + 3x_3t^2}$$

At point P_0 , $t = 0$, so the slope of the tangent is $\frac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \frac{y_1 - y_0}{x_1 - x_0}$. So the tangent to the curve at P_0 passes through P_1 . Similarly, the slope of the tangent at point P_3 (where $t = 1$) is $\frac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \frac{y_3 - y_2}{x_3 - x_2}$, which is also the slope of line P_2P_3 .

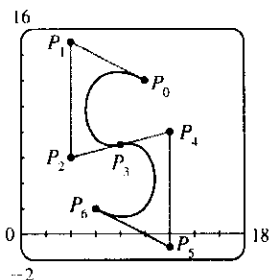
3. It seems that if P_1 were to the right of P_2 , a loop would appear. We try setting $P_1 = (110, 30)$, and the resulting curve does indeed have a loop.



4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try $P_0(10, 12)$, $P_1(4, 15)$, $P_2(4, 5)$, and $P_3(10, 8)$, and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the x - and y -scales the same so as not to distort the figure (this is called a “constrained projection” in Maple.)



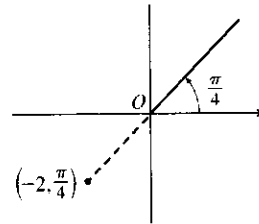
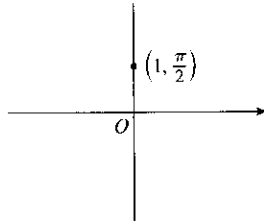
5. We use the same P_0 and P_1 as in Problem 4, and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4, 6)$ and P_3 down and to the left, to $(8, 7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4 , P_5 , and P_6 by rotating points P_2 , P_1 , and P_0 about the center of the letter (point P_3). The points are therefore $P_4(12, 8)$, $P_5(12, -1)$, and $P_6(6, 2)$.



11.3 Polar Coordinates

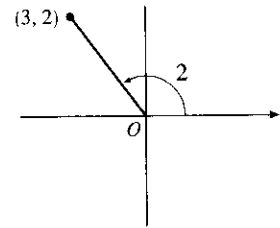
1. (a) By adding 2π to $\frac{\pi}{2}$, we obtain the (b) $(-2, \frac{\pi}{4})$

point $(1, \frac{5\pi}{2})$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $(-1, \frac{3\pi}{2})$ is a point that satisfies the $r < 0$ requirement.



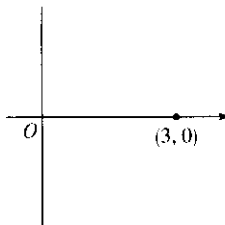
$$(2, \frac{5\pi}{4}), (-2, \frac{9\pi}{4})$$

- (c) $(3, 2)$



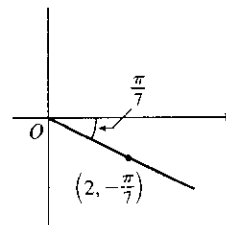
$$(3, 2 + 2\pi), (-3, 2 + \pi)$$

2. (a) $(3, 0)$



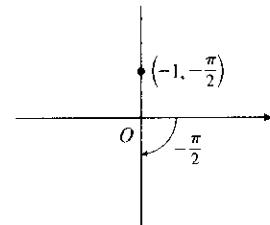
$$(3, 2\pi), (-3, \pi)$$

- (b) $(2, -\frac{\pi}{7})$



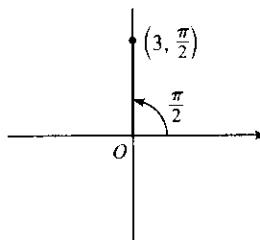
$$(2, \frac{13\pi}{7}), (-2, \frac{6\pi}{7})$$

- (c) $(-1, -\frac{\pi}{2})$



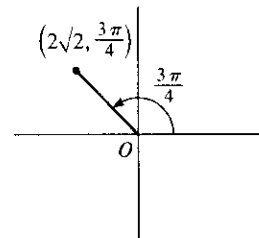
$$(1, \frac{\pi}{2}), (-1, \frac{3\pi}{2})$$

3. (a)



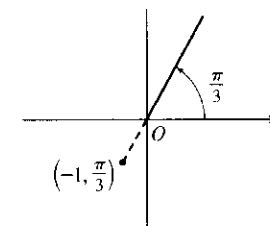
$x = 3 \cos \frac{\pi}{2} = 3(0) = 0$ and
 $y = 3 \sin \frac{\pi}{2} = 3(1) = 3$ give us
 the Cartesian coordinates $(0, 3)$.

- (b)



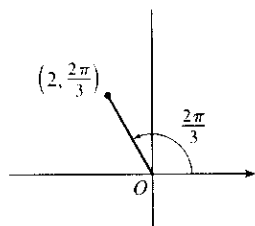
$$\begin{aligned} x &= 2\sqrt{2} \cos \frac{3\pi}{4} \\ &= 2\sqrt{2} \left(-\frac{1}{\sqrt{2}}\right) = -2 \text{ and} \\ y &= 2\sqrt{2} \sin \frac{3\pi}{4} = 2\sqrt{2} \left(\frac{1}{\sqrt{2}}\right) = 2 \\ &\text{give us } (-2, 2). \end{aligned}$$

- (c)



$$\begin{aligned} x &= -1 \cos \frac{\pi}{3} = -\frac{1}{2} \text{ and} \\ y &= -1 \sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2} \text{ give} \\ &\text{us } \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \end{aligned}$$

4. (a)

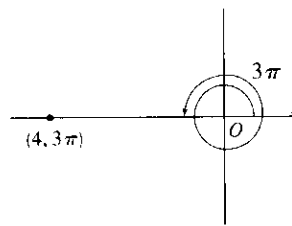


$$x = 2 \cos \frac{2\pi}{3} = -1 \text{ and}$$

$$y = 2 \sin \frac{2\pi}{3} = \sqrt{3} \text{ give}$$

$$\text{us } (-1, \sqrt{3}).$$

(b)

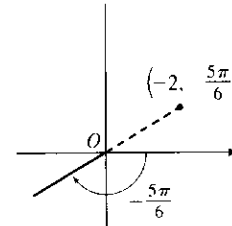


$$x = 4 \cos 3\pi = -4 \text{ and}$$

$$y = 4 \sin 3\pi = 0 \text{ give}$$

$$\text{us } (-4, 0).$$

(c)



$$x = -2 \cos\left(-\frac{5\pi}{6}\right) = \sqrt{3}$$

$$\text{and } y = -2 \sin\left(-\frac{5\pi}{6}\right) = 1$$

$$\text{give us } (\sqrt{3}, 1).$$

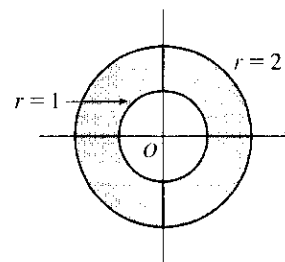
5. (a) $x = 1$ and $y = 1 \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$. Since $(1, 1)$ is in the first quadrant, the polar coordinates are (i) $(\sqrt{2}, \frac{\pi}{4})$ and (ii) $(-\sqrt{2}, \frac{5\pi}{4})$.

- (b) $x = 2\sqrt{3}$ and $y = -2 \Rightarrow r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$ and
- $$\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}.$$
- Since $(2\sqrt{3}, -2)$ is in the fourth quadrant and $0 \leq \theta \leq 2\pi$, the polar coordinates are (i) $(4, \frac{11\pi}{6})$ and (ii) $(-4, \frac{5\pi}{6})$.

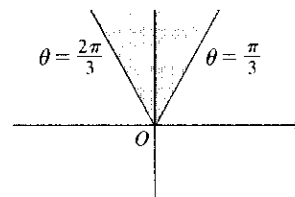
6. (a) $(x, y) = (-1, -\sqrt{3})$, $r = \sqrt{1 + 3} = 2$, $\tan \theta = y/x = \sqrt{3}$ and (x, y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$. The polar coordinates are (i) $(2, \frac{4\pi}{3})$ and (ii) $(-2, \frac{\pi}{3})$.

- (b) $(x, y) = (-2, 3)$, $r = \sqrt{4 + 9} = \sqrt{13}$, $\tan \theta = y/x = -\frac{3}{2}$ and (x, y) is in the second quadrant, so
- $$\theta = \tan^{-1}\left(-\frac{3}{2}\right) + \pi.$$
- The polar coordinates are (i) $(\sqrt{13}, \theta)$ and (ii) $(-\sqrt{13}, \theta + \pi)$.

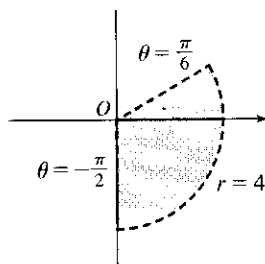
7. The curves $r = 1$ and $r = 2$ represent circles with center O and radii 1 and 2. The region in the plane satisfying $1 \leq r \leq 2$ consists of both circles and the shaded region between them in the figure.



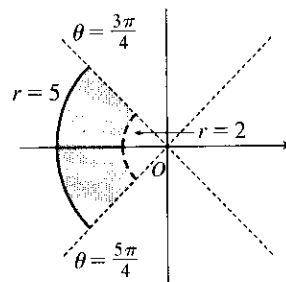
8. $r \geq 0$, $\pi/3 \leq \theta \leq 2\pi/3$



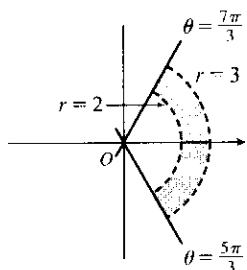
9. The region satisfying $0 \leq r < 4$ and $-\pi/2 \leq \theta < \pi/6$ does not include the circle $r = 4$ nor the line $\theta = \pi/6$.



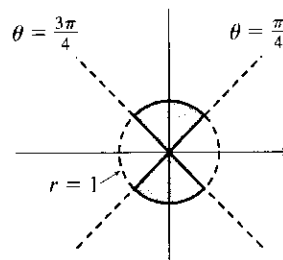
10. $2 < r \leq 5$, $3\pi/4 < \theta < 5\pi/4$



11. $2 < r < 3$, $\frac{5\pi}{3} \leq \theta \leq \frac{7\pi}{3}$



12. $-1 \leq r \leq 1$, $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$



13. $(r, \theta) = (1, \frac{\pi}{6}) \Rightarrow x = 1 \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $y = 1 \sin \frac{\pi}{6} = \frac{1}{2}$.

$(r, \theta) = (3, \frac{3\pi}{4}) \Rightarrow x = 3 \cos \frac{3\pi}{4} = -\frac{3\sqrt{2}}{2}$ and $y = 3 \sin \frac{3\pi}{4} = \frac{3\sqrt{2}}{2}$. The distance between them is

$$\begin{aligned} \sqrt{\left[\frac{\sqrt{3}}{2} - \left(-\frac{3\sqrt{2}}{2}\right)\right]^2 + \left(\frac{1}{2} - \frac{3\sqrt{2}}{2}\right)^2} &= \sqrt{\frac{1}{4}(\sqrt{3} + 3\sqrt{2})^2 + \frac{1}{4}(1 - 3\sqrt{2})^2} \\ &= \sqrt{\frac{1}{4}[(3 + 6\sqrt{6} + 18) + (1 - 6\sqrt{2} + 18)]} = \frac{1}{2}\sqrt{40 + 6\sqrt{6} - 6\sqrt{2}} \end{aligned}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1 \cos \theta_1, r_1 \sin \theta_1)$ and $(r_2 \cos \theta_2, r_2 \sin \theta_2)$, respectively. The square of the distance between them is

$$\begin{aligned} &(r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2 \\ &= (r_2^2 \cos^2 \theta_2 - 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1) + (r_2^2 \sin^2 \theta_2 - 2r_1 r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1) \\ &= r_1^2 (\sin^2 \theta_1 + \cos^2 \theta_1) + r_2^2 (\sin^2 \theta_2 + \cos^2 \theta_2) - 2r_1 r_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\ &= r_1^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2, \end{aligned}$$

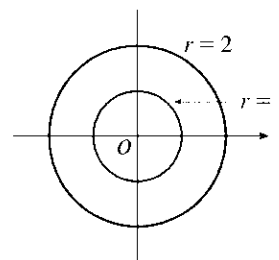
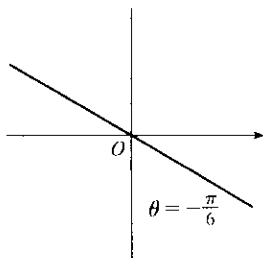
so the distance between them is $\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r = 2 \Leftrightarrow \sqrt{x^2 + y^2} = 2 \Leftrightarrow x^2 + y^2 = 4$, a circle of radius 2 centered at the origin.

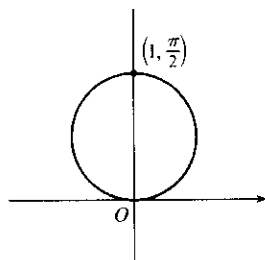
16. $r \cos \theta = 1 \Leftrightarrow x = 1$, a vertical line.

17. $r = 3 \sin \theta \Rightarrow r^2 = 3r \sin \theta \Leftrightarrow x^2 + y^2 = 3y \Leftrightarrow x^2 + (y - \frac{3}{2})^2 = (\frac{3}{2})^2$, a circle of radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$. The first two equations are actually equivalent since $r^2 = 3r \sin \theta \Rightarrow r(r - 3 \sin \theta) = 0 \Rightarrow r = 0$ or $r = 3 \sin \theta$. But $r = 3 \sin \theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the single equation $r = 3 \sin \theta$ is equivalent to the compound condition ($r = 0$ or $r = 3 \sin \theta$).

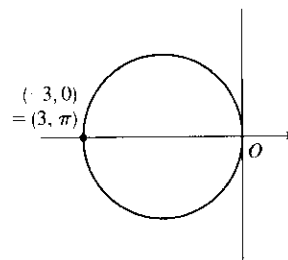
18. $r = 2 \sin \theta + 2 \cos \theta \Rightarrow r^2 = 2r \sin \theta + 2r \cos \theta \Leftrightarrow x^2 + y^2 = 2y + 2x \Leftrightarrow (x^2 - 2x + 1) + (y^2 - 2y + 1) = 2 \Leftrightarrow (x - 1)^2 + (y - 1)^2 = 2$. The first implication is reversible since $r^2 = 2r \sin \theta + 2r \cos \theta \Rightarrow r = 0$ or $r = 2 \sin \theta + 2 \cos \theta$, but the curve $r = 2 \sin \theta + 2 \cos \theta$ passes through the pole ($r = 0$) when $\theta = -\frac{\pi}{4}$, so $r = 2 \sin \theta + 2 \cos \theta$ includes the single point of $r = 0$. The curve is a circle of radius $\sqrt{2}$, centered at $(1, 1)$.
19. $r = \csc \theta \Leftrightarrow r = \frac{1}{\sin \theta} \Leftrightarrow r \sin \theta = 1 \Leftrightarrow y = 1$, a horizontal line 1 unit above the x -axis.
20. $r = \tan \theta \sec \theta = \frac{\sin \theta}{\cos^2 \theta} \Rightarrow r \cos^2 \theta = \sin \theta \Leftrightarrow (r \cos \theta)^2 = r \sin \theta \Leftrightarrow x^2 = y$, a parabola with vertex at the origin opening upward. The first implication is reversible since $\cos \theta = 0$ would imply $\sin \theta = r \cos^2 \theta = 0$, contradicting the fact that $\cos^2 \theta + \sin^2 \theta = 1$.
21. $x = 3 \Leftrightarrow r \cos \theta = 3 \Leftrightarrow r = 3 / \cos \theta \Leftrightarrow r = 3 \sec \theta$.
22. $x^2 + y^2 = 9 \Leftrightarrow r^2 = 9 \Leftrightarrow r = 3$. [$r = -3$ gives the same curve.]
23. $x = -y^2 \Leftrightarrow r \cos \theta = -r^2 \sin^2 \theta \Leftrightarrow \cos \theta = -r \sin^2 \theta \Leftrightarrow r = -\frac{\cos \theta}{\sin^2 \theta} = -\cot \theta \csc \theta$.
24. $x + y = 9 \Leftrightarrow r \cos \theta + r \sin \theta = 9 \Leftrightarrow r = 9 / (\cos \theta + \sin \theta)$.
25. $x^2 + y^2 = 2cx \Leftrightarrow r^2 = 2cr \cos \theta \Leftrightarrow r^2 - 2cr \cos \theta = 0 \Leftrightarrow r(r - 2c \cos \theta) = 0 \Leftrightarrow r = 0$ or $r = 2c \cos \theta$. $r = 0$ is included in $r = 2c \cos \theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c \cos \theta$.
26. $x^2 - y^2 = 1 \Leftrightarrow (r \cos \theta)^2 - (r \sin \theta)^2 = 1 \Leftrightarrow r^2(\cos^2 \theta - \sin^2 \theta) = 1 \Leftrightarrow r^2 \cos 2\theta = 1 \Rightarrow r^2 = \sec 2\theta$
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $\tan \theta = y/x \Rightarrow y = (\tan \frac{\pi}{6})x = \frac{1}{\sqrt{3}}x$ is slightly more difficult to derive.
(b) The easier description here is the Cartesian equation $x = 3$.
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x - 2)^2 + (y - 3)^2 = 5^2$.
(b) This circle is more easily given in polar coordinates: $r = 4$. The Cartesian equation is also simple: $x^2 + y^2 = 16$.
29. $\theta = -\pi/6$
30. $r^2 - 3r + 2 = 0 \Leftrightarrow (r - 1)(r - 2) = 0 \Leftrightarrow r = 1$ or $r = 2$



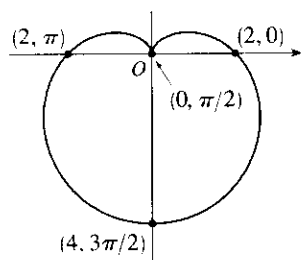
31. $r = \sin \theta \Leftrightarrow r^2 = r \sin \theta \Leftrightarrow x^2 + y^2 = y$
 $\Leftrightarrow x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$. The reasoning here
 is the same as in Exercise 17. This is a circle of
 radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.



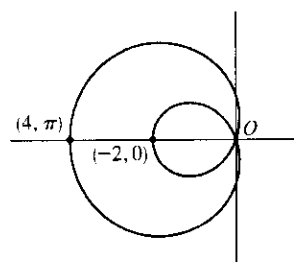
32. $r = -3 \cos \theta \Leftrightarrow r^2 = -3r \cos \theta \Leftrightarrow$
 $x^2 + y^2 = -3x \Leftrightarrow (x + \frac{3}{2})^2 + y^2 = (\frac{3}{2})^2$.
 This curve is a circle of radius $\frac{3}{2}$ centered
 at $(-\frac{3}{2}, 0)$.



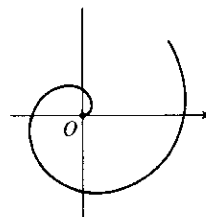
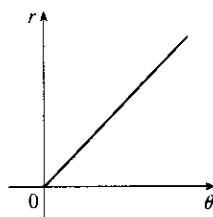
33. $r = 2(1 - \sin \theta)$. This curve is a cardioid.



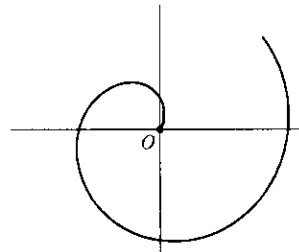
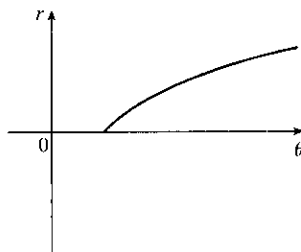
34. $r = 1 - 3 \cos \theta$. This is a limaçon.



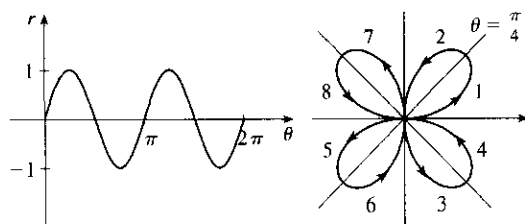
35. $r = \theta, \theta \geq 0$



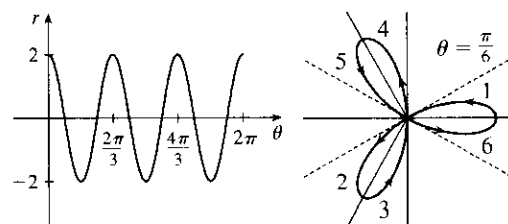
36. $r = \ln \theta, \theta \geq 1$



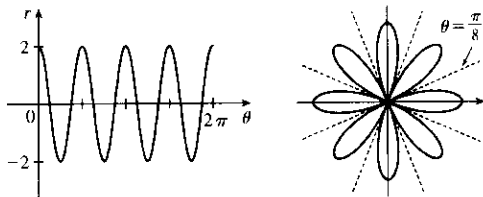
37. $r = \sin 2\theta$



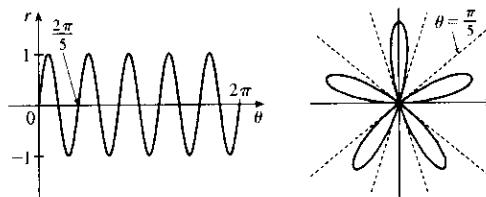
38. $r = 2 \cos 3\theta$



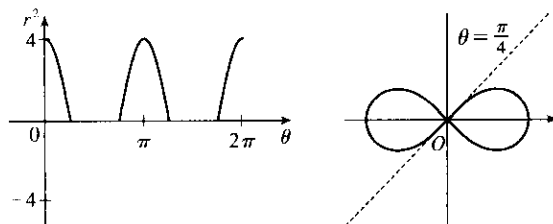
39. $r = 2 \cos 4\theta$



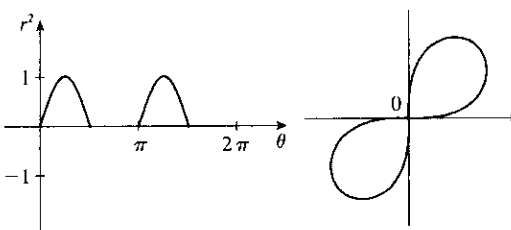
40. $r = \sin 5\theta$



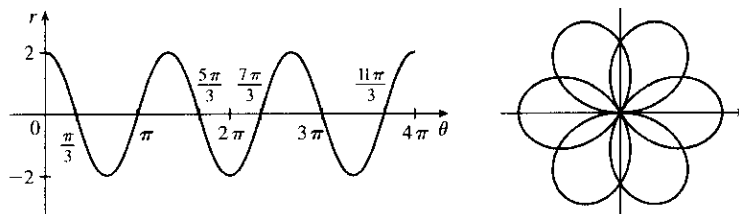
41. $r^2 = 4 \cos 2\theta$



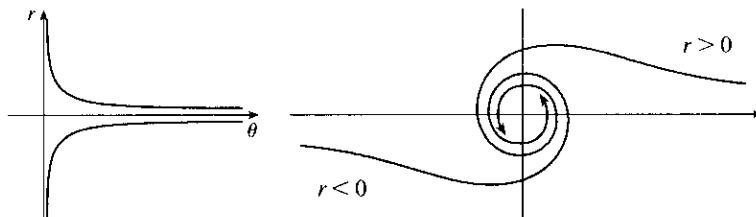
42. $r^2 = \sin 2\theta$



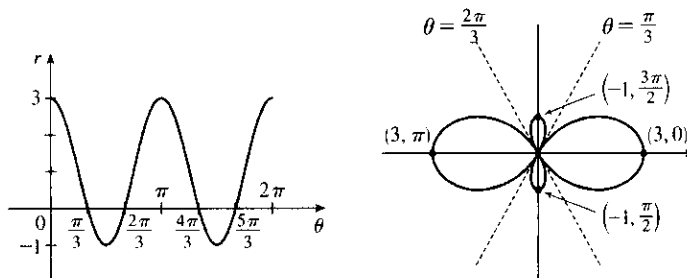
43. $r = 2 \cos(\frac{3}{2}\theta)$



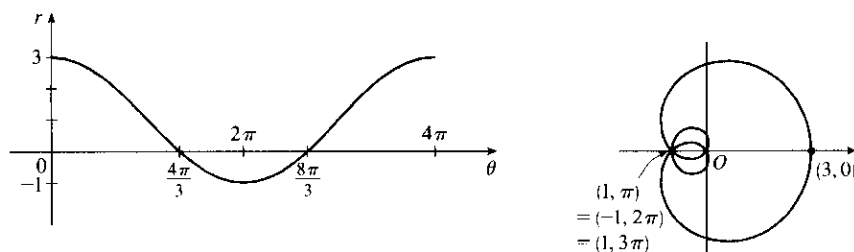
44. $r^2\theta = 1 \Leftrightarrow r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



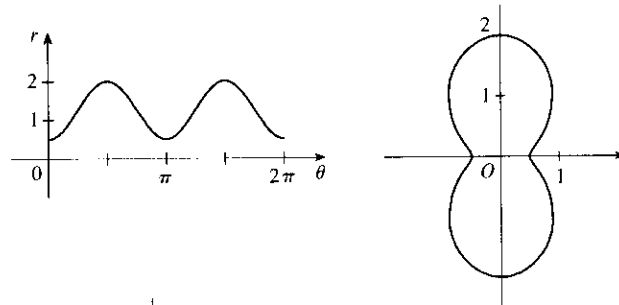
45. $r = 1 + 2 \cos 2\theta$



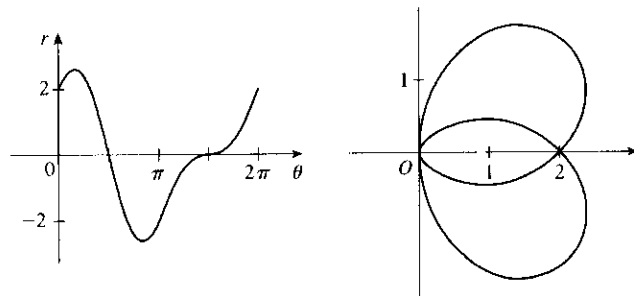
46. $r = 1 + 2 \cos(\theta/2)$



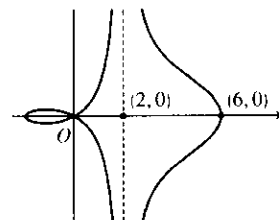
47. For $\theta = 0, \pi$, and 2π , r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \leq \theta \leq \pi$ and $\pi \leq \theta \leq 2\pi$.



48.



49. $x = r \cos \theta = (4 + 2 \sec \theta) \cos \theta = 4 \cos \theta + 2$. Now, $r \rightarrow \infty \Rightarrow (4 + 2 \sec \theta) \rightarrow \infty \Rightarrow \theta \rightarrow (\frac{\pi}{2})^-$ or $\theta \rightarrow (\frac{3\pi}{2})^+$ (since we need only consider $0 \leq \theta < 2\pi$), so $\lim_{r \rightarrow \infty} x = \lim_{\theta \rightarrow \pi/2^-} (4 \cos \theta + 2) = 2$. Also, $r \rightarrow -\infty \Rightarrow (4 + 2 \sec \theta) \rightarrow -\infty \Rightarrow \theta \rightarrow (\frac{\pi}{2})^+$ or $\theta \rightarrow (\frac{3\pi}{2})^-$, so $\lim_{r \rightarrow -\infty} x = \lim_{\theta \rightarrow \pi/2^+} (4 \cos \theta + 2) = 2$. Therefore, $\lim_{r \rightarrow \pm\infty} x = 2 \Rightarrow x = 2$ is a vertical asymptote.



50. $y = r \sin \theta = 2 \sin \theta - \csc \theta \sin \theta = 2 \sin \theta - 1$.

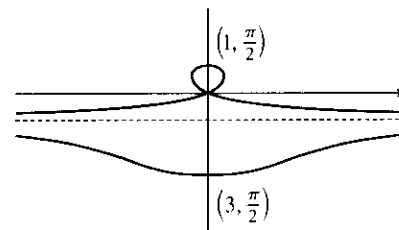
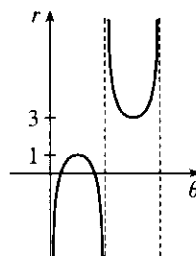
$r \rightarrow \infty \Rightarrow (2 - \csc \theta) \rightarrow \infty \Rightarrow \csc \theta \rightarrow -\infty \Rightarrow \theta \rightarrow \pi^+$ (since we need only consider $0 \leq \theta < 2\pi$) and so

$\lim_{r \rightarrow \infty} y = \lim_{\theta \rightarrow \pi^+} 2 \sin \theta - 1 = -1$. Also

$r \rightarrow -\infty \Rightarrow (2 - \csc \theta) \rightarrow -\infty \Rightarrow \csc \theta \rightarrow \infty$

$\Rightarrow \theta \rightarrow \pi^-$ and so

$\lim_{r \rightarrow -\infty} y = \lim_{\theta \rightarrow \pi^-} 2 \sin \theta - 1 = -1$. Therefore $\lim_{r \rightarrow \pm\infty} y = -1 \Rightarrow y = -1$ is a horizontal asymptote.



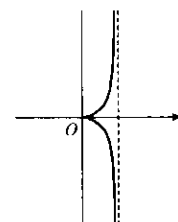
51. To show that $x = 1$ is an asymptote we must prove $\lim_{r \rightarrow \pm\infty} x = 1$.

$x = r \cos \theta = (\sin \theta \tan \theta) \cos \theta = \sin^2 \theta$. Now, $r \rightarrow \infty \Rightarrow \sin \theta \tan \theta \rightarrow \infty$

$\Rightarrow \theta \rightarrow (\frac{\pi}{2})^-$, so $\lim_{r \rightarrow \infty} x = \lim_{\theta \rightarrow \pi/2^-} \sin^2 \theta = 1$. Also, $r \rightarrow -\infty \Rightarrow$

$\sin \theta \tan \theta \rightarrow -\infty \Rightarrow \theta \rightarrow (\frac{\pi}{2})^+$, so $\lim_{r \rightarrow -\infty} x = \lim_{\theta \rightarrow \pi/2^+} \sin^2 \theta = 1$.

Therefore, $\lim_{r \rightarrow \pm\infty} x = 1 \Rightarrow x = 1$ is a vertical asymptote. Also notice that $x = \sin^2 \theta \geq 0$ for all θ , and



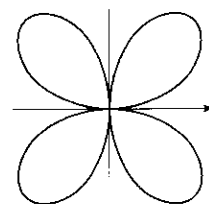
$x = \sin^2 \theta \leq 1$ for all θ . And $x \neq 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \leq x < 1$.

52. The equation is $(x^2 + y^2)^3 = 4x^2y^2$, but using polar coordinates we know that

$x^2 + y^2 = r^2$ and $x = r \cos \theta$ and $y = r \sin \theta$. Substituting into the given

$$\text{equation: } r^6 = 4r^2 \cos^2 \theta r^2 \sin^2 \theta \Rightarrow r^2 = 4 \cos^2 \theta \sin^2 \theta \Rightarrow$$

$$r = \pm 2 \cos \theta \sin \theta = \pm \sin 2\theta. \quad r = \pm \sin 2\theta \text{ is sketched at right.}$$



53. (a) We see that the curve crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r -values), so we solve the equation of the limaçon for $r = 0 \Leftrightarrow c \sin \theta = -1 \Leftrightarrow \sin \theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.
- (b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c -values $\frac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum: $y = r \sin \theta = \sin \theta + c \sin^2 \theta \Rightarrow \frac{dy}{d\theta} = \cos \theta + 2c \sin \theta \cos \theta = \cos \theta + c \sin 2\theta \Rightarrow$
- $$\frac{d^2y}{d\theta^2} = -\sin \theta + 2c \cos 2\theta. \text{ At } \theta = \frac{3\pi}{2}, \text{ this is equal to } -(-1) + 2c(-1) = 1 - 2c, \text{ which is negative only for } c > \frac{1}{2}.$$
- A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.
54. (a) $r = \sin(\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π .
- (b) $r = \sin(\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π .
- (c) $r = \sec(3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}$, and so on.
- (d) $r = \theta \sin \theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π . This graph is unbounded, and each time θ moves through an interval of 2π , the same basic shape is repeated (because of the periodic $\sin \theta$ factor) but it gets larger each time (since θ increases each time we go around.)
- (e) $r = 1 + 4 \cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.
- (f) $r = 1/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \rightarrow 0$ as $\theta \rightarrow \infty$.

55. $r = 2 \sin \theta \Rightarrow x = r \cos \theta = 2 \sin \theta \cos \theta = \sin 2\theta, y = r \sin \theta = 2 \sin^2 \theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot 2 \sin \theta \cos \theta}{\cos 2\theta \cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \tan\left(2 \cdot \frac{\pi}{6}\right) = \tan \frac{\pi}{3} = \sqrt{3}.$$

$$56. r = 2 - \sin \theta \Rightarrow x = r \cos \theta = (2 - \sin \theta) \cos \theta, y = r \sin \theta = (2 - \sin \theta) \sin \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2 - \sin \theta) \cos \theta + \sin \theta(-\cos \theta)}{(2 - \sin \theta)(-\sin \theta) + \cos \theta(-\cos \theta)} = \frac{2 \cos \theta - 2 \sin \theta \cos \theta}{-2 \sin \theta + \sin^2 \theta - \cos^2 \theta} = \frac{2 \cos \theta - \sin 2\theta}{-2 \sin \theta - \cos 2\theta}$$

$$\text{When } \theta = \frac{\pi}{3}, \frac{dy}{dx} = \frac{2(1/2) - (\sqrt{3}/2)}{-2(\sqrt{3}/2) - (-1/2)} = \frac{1 - \sqrt{3}/2}{-\sqrt{3} + 1/2} \cdot \frac{2}{2} = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}.$$

$$57. r = 1/\theta \Rightarrow x = r \cos \theta = (\cos \theta)/\theta, y = r \sin \theta = (\sin \theta)/\theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin \theta(-1/\theta^2) + (1/\theta) \cos \theta}{\cos \theta(-1/\theta^2) - (1/\theta) \sin \theta} \cdot \frac{\theta^2}{\theta^2} = \frac{-\sin \theta + \theta \cos \theta}{-\cos \theta - \theta \sin \theta}$$

$$\text{When } \theta = \pi, \frac{dy}{dx} = \frac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \frac{-\pi}{1} = -\pi.$$

$$58. r = \ln \theta \Rightarrow x = r \cos \theta = \ln \theta \cos \theta, y = r \sin \theta = \ln \theta \sin \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin \theta(1/\theta) + \ln \theta \cos \theta}{\cos \theta(1/\theta) - \ln \theta \sin \theta} \cdot \frac{\theta}{\theta} = \frac{\sin \theta + \theta \ln \theta \cos \theta}{\cos \theta - \theta \ln \theta \sin \theta}$$

$$\text{When } \theta = e, \frac{dy}{dx} = \frac{\sin e + e \ln e \cos e}{\cos e - e \ln e \sin e} = \frac{\sin e + e \cos e}{\cos e - e \sin e}.$$

$$59. r = 1 + \cos \theta \Rightarrow x = r \cos \theta = \cos \theta + \cos^2 \theta, y = r \sin \theta = \sin \theta + \sin \theta \cos \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + \cos^2 \theta - \sin^2 \theta}{-\sin \theta - 2 \cos \theta \sin \theta} = \frac{\cos \theta + \cos 2\theta}{-\sin \theta - \sin 2\theta}$$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{-\frac{1}{2} - \frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{-\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)} = -1.$$

$$60. r = \sin 3\theta \Rightarrow x = r \cos \theta = \sin 3\theta \cos \theta, y = r \sin \theta = \sin 3\theta \sin \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3 \cos 3\theta \sin \theta + \sin 3\theta \cos \theta}{3 \cos 3\theta \cos \theta - \sin 3\theta \sin \theta}$$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{3(0)(1/2) + 1(\sqrt{3}/2)}{3(0)(\sqrt{3}/2) - 1(1/2)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}.$$

$$61. r = 3 \cos \theta \Rightarrow x = r \cos \theta = 3 \cos \theta \cos \theta, y = r \sin \theta = 3 \cos \theta \sin \theta \Rightarrow$$

$$dy/d\theta = -3 \sin^2 \theta + 3 \cos^2 \theta = 3 \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}. \text{ So the tangent is}$$

$$\text{horizontal at } \left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right) \text{ and } \left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right) \left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]. dx/d\theta = -6 \sin \theta \cos \theta = -3 \sin 2\theta = 0 \Rightarrow$$

$$2\theta = 0 \text{ or } \pi \Leftrightarrow \theta = 0 \text{ or } \frac{\pi}{2}. \text{ So the tangent is vertical at } (3, 0) \text{ and } (0, \frac{\pi}{2}).$$

$$62. y = r \sin \theta = \cos \theta \sin \theta + \sin^2 \theta = \frac{1}{2} \sin 2\theta + \sin^2 \theta \Rightarrow dy/d\theta = \cos 2\theta + \sin 2\theta = 0 \Rightarrow \tan 2\theta = -1$$

$$\Rightarrow 2\theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4} \Leftrightarrow \theta = \frac{3\pi}{8} \text{ or } \frac{7\pi}{8} \Rightarrow \text{horizontal tangents at } \left(\cos \frac{3\pi}{8} + \sin \frac{3\pi}{8}, \frac{3\pi}{8}\right) \text{ and}$$

$$\left(\cos \frac{7\pi}{8} + \sin \frac{7\pi}{8}, \frac{7\pi}{8}\right). x = r \cos \theta = \cos^2 \theta + \cos \theta \sin \theta \Rightarrow dx/d\theta = -\sin 2\theta + \cos 2\theta = 0 \Rightarrow$$

$$\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \Leftrightarrow \theta = \frac{\pi}{8} \text{ or } \frac{5\pi}{8} \Rightarrow \text{vertical tangents at } \left(\cos \frac{\pi}{8} + \sin \frac{\pi}{8}, \frac{\pi}{8}\right) \text{ and}$$

$$\left(\cos \frac{5\pi}{8} + \sin \frac{5\pi}{8}, \frac{5\pi}{8}\right).$$

Note: These expressions can be simplified using trigonometric identities. For example,

$$\cos \frac{\pi}{8} + \sin \frac{\pi}{8} = \frac{1}{2} \sqrt{4 + 2\sqrt{2}}.$$

63. $r = 1 + \cos \theta \Rightarrow x = r \cos \theta = \cos \theta(1 + \cos \theta), y = r \sin \theta = \sin \theta(1 + \cos \theta) \Rightarrow$
 $dy/d\theta = (1 + \cos \theta) \cos \theta - \sin^2 \theta = 2 \cos^2 \theta + \cos \theta - 1 = (2 \cos \theta - 1)(\cos \theta + 1) = 0 \Rightarrow \cos \theta = \frac{1}{2} \text{ or } -1 \Rightarrow \theta = \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3} \Rightarrow \text{horizontal tangent at } (\frac{3}{2}, \frac{\pi}{3}), (0, \pi) \text{ [the pole]}, \text{ and } (\frac{3}{2}, \frac{5\pi}{3}).$
 $dx/d\theta = -(1 + \cos \theta) \sin \theta - \cos \theta \sin \theta = -\sin \theta(1 + 2 \cos \theta) = 0 \Rightarrow \sin \theta = 0 \text{ or } \cos \theta = -\frac{1}{2} \Rightarrow$
 $\theta = 0, \pi, \frac{2\pi}{3}, \text{ or } \frac{4\pi}{3} \Rightarrow \text{vertical tangent at } (2, 0), (\frac{1}{2}, \frac{2\pi}{3}), \text{ and } (\frac{1}{2}, \frac{4\pi}{3}).$ Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\lim_{\theta \rightarrow \pi} \frac{dy/d\theta}{dx/d\theta} = 0$.

64. $\frac{dy}{d\theta} = e^\theta \sin \theta + e^\theta \cos \theta = e^\theta(\sin \theta + \cos \theta) = 0 \Rightarrow \sin \theta = -\cos \theta \Rightarrow \tan \theta = -1 \Rightarrow$
 $\theta = -\frac{1}{4}\pi + n\pi \text{ (} n \text{ any integer)} \Rightarrow \text{horizontal tangents at } (e^{\pi(n-1/4)}, \pi(n - \frac{1}{4})).$
 $\frac{dx}{d\theta} = e^\theta \cos \theta - e^\theta \sin \theta = e^\theta(\cos \theta - \sin \theta) = 0 \Rightarrow \sin \theta = \cos \theta \Rightarrow \tan \theta = 1 \Rightarrow$
 $\theta = \frac{1}{4}\pi + n\pi \text{ (} n \text{ any integer)} \Rightarrow \text{vertical tangents at } (e^{\pi(n+1/4)}, \pi(n + \frac{1}{4})).$

65. $r = \cos 2\theta \Rightarrow x = r \cos \theta = \cos 2\theta \cos \theta, y = r \sin \theta = \cos 2\theta \sin \theta \Rightarrow$

$$\begin{aligned} dy/d\theta &= -2 \sin 2\theta \sin \theta + \cos 2\theta \cos \theta = -4 \sin^2 \theta \cos \theta + (\cos^3 \theta - \sin^2 \theta \cos \theta) \\ &= \cos \theta(\cos^2 \theta - 5 \sin^2 \theta) = \cos \theta(1 - 6 \sin^2 \theta) = 0 \Rightarrow \end{aligned}$$

$$\cos \theta = 0 \text{ or } \sin \theta = \pm \frac{1}{\sqrt{6}} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \alpha, \pi - \alpha, \pi + \alpha, \text{ or } 2\pi - \alpha \text{ (where } \alpha = \sin^{-1} \frac{1}{\sqrt{6}}).$$

So the tangent is horizontal at $(-1, \frac{\pi}{2}), (-1, \frac{3\pi}{2}), (\frac{2}{3}, \alpha), (\frac{2}{3}, \pi - \alpha), (\frac{2}{3}, \pi + \alpha),$ and $(\frac{2}{3}, 2\pi - \alpha).$

$$\begin{aligned} dx/d\theta &= -2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta = -4 \sin \theta \cos^2 \theta - (2 \cos^2 \theta - 1) \sin \theta \\ &= \sin \theta(1 - 6 \cos^2 \theta) = 0 \Rightarrow \end{aligned}$$

$$\sin \theta = 0 \text{ or } \cos \theta = \pm \frac{1}{\sqrt{6}} \Rightarrow \theta = 0, \pi, \beta, \pi - \beta, \pi + \beta, \text{ or } 2\pi - \beta \text{ (where } \beta = \cos^{-1} \frac{1}{\sqrt{6}}).$$

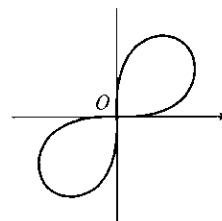
So the tangent is vertical at $(1, 0), (1, \pi), (-\frac{2}{3}, \beta), (-\frac{2}{3}, \pi - \beta), (-\frac{2}{3}, \pi + \beta),$ and $(-\frac{2}{3}, 2\pi - \beta).$

66. By differentiating implicitly, $r^2 = \sin 2\theta \Rightarrow 2r(dr/d\theta) = 2 \cos 2\theta \Rightarrow$

$dr/d\theta = (1/r) \cos 2\theta$, so

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{1}{r} \cos 2\theta \sin \theta + r \cos \theta = \frac{1}{r}(\cos 2\theta \sin \theta + r^2 \cos \theta) \\ &= \frac{1}{r}(\cos 2\theta \sin \theta + \sin 2\theta \cos \theta) = \frac{1}{r} \sin 3\theta \end{aligned}$$

This is 0 when $\sin 3\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$ (restricting θ to the domain of the lemniscate), so there are horizontal tangents at $(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3}), (\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3})$ and $(0, 0)$. Similarly, $dx/d\theta = (1/r) \cos 3\theta = 0$ when $\theta = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$, so there are vertical tangents at $(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6})$ and $(\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6})$ [and $(0, 0)$].

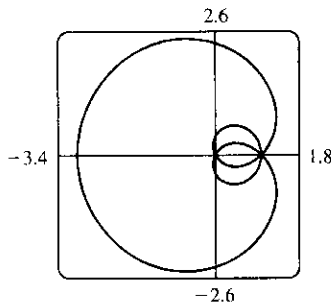


67. $r = a \sin \theta + b \cos \theta \Rightarrow r^2 = ar \sin \theta + br \cos \theta \Rightarrow x^2 + y^2 = ay + bx \Rightarrow$
 $x^2 - bx + (\frac{1}{2}b)^2 + y^2 - ay + (\frac{1}{2}a)^2 = (\frac{1}{2}b)^2 + (\frac{1}{2}a)^2 \Rightarrow (x - \frac{1}{2}b)^2 + (y - \frac{1}{2}a)^2 = \frac{1}{4}(a^2 + b^2),$ and this is a circle with center $(\frac{1}{2}b, \frac{1}{2}a)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

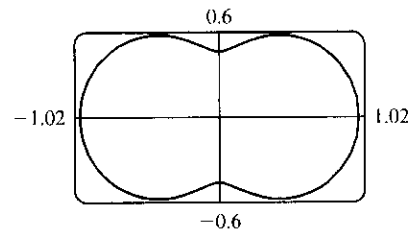
68. These curves are circles which intersect at the origin and at $(\frac{1}{\sqrt{2}}a, \frac{\pi}{4})$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle ($r = a \sin \theta$), $dy/d\theta = a \cos \theta \sin \theta + a \sin \theta \cos \theta = a \sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a \cos^2 \theta - a \sin^2 \theta = a \cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle ($r = a \cos \theta$), $dy/d\theta = a \cos 2\theta = 0$ and $dx/d\theta = -a \sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

Note for Exercises 69–74: Maple is able to plot polar curves using the `polarplot` command, or using the `coords=polar` option in a regular `plot` command. In Mathematica, use `PolarPlot`. In Derive, change to `Polar` under Options State. If your graphing device cannot plot polar equations, you must convert to parametric equations. For example, in Exercise 69, $x = r \cos \theta = [1 + 2 \sin(\theta/2)] \cos \theta$, $y = r \sin \theta = [1 + 2 \sin(\theta/2)] \sin \theta$.

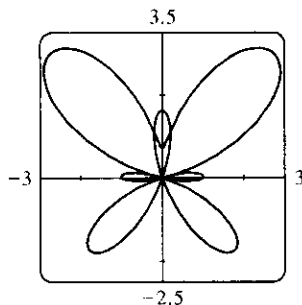
69. $r = 1 + 2 \sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.



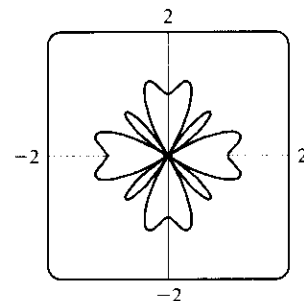
70. $r = \sqrt{1 - 0.8 \sin^2 \theta}$. The parameter interval is $[0, 2\pi]$.



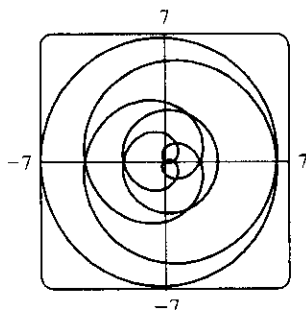
71. $r = e^{\sin \theta} - 2 \cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



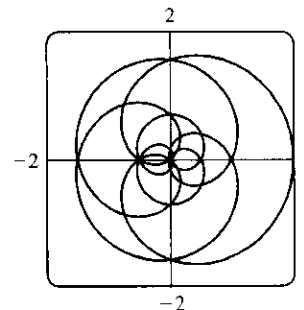
72. $r = \sin^2(4\theta) + \cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



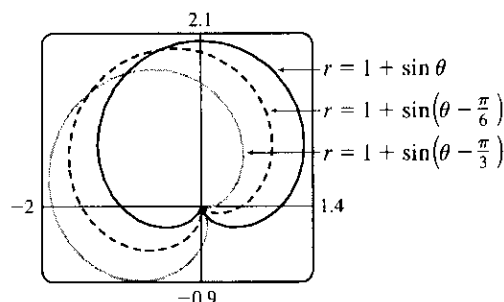
73. $r = 2 - 5 \sin(\theta/6)$. The parameter interval is $[-6\pi, 6\pi]$.



74. $r = \cos(\theta/2) + \cos(\theta/3)$. The parameter interval is $[-6\pi, 6\pi]$.

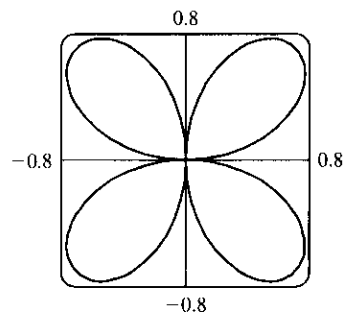


75.



It appears that the graph of $r = 1 + \sin(\theta - \frac{\pi}{6})$ is the same shape as the graph of $r = 1 + \sin \theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin(\theta - \frac{\pi}{3})$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.

76.



From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $dy/d\theta = 0$. $y = r \sin \theta = \sin \theta \sin 2\theta \Rightarrow$

$$\begin{aligned} dy/d\theta &= 2 \sin \theta \cos 2\theta + \cos \theta \sin 2\theta \\ &= 2 \sin \theta (2 \cos^2 \theta - 1) + \cos \theta (2 \sin \theta \cos \theta) \\ &= 2 \sin \theta (3 \cos^2 \theta - 1) \end{aligned}$$

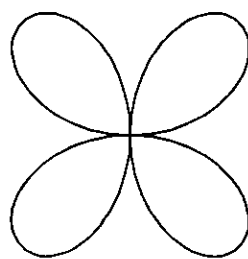
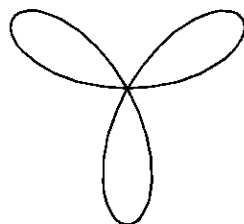
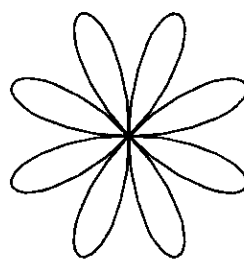
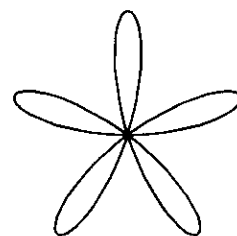
In the first quadrant, this is 0 when $\cos \theta = \frac{1}{\sqrt{3}} \Leftrightarrow \sin \theta = \sqrt{\frac{2}{3}} \Leftrightarrow$

$$y = 2 \sin^2 \theta \cos \theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{9} \approx 0.77.$$

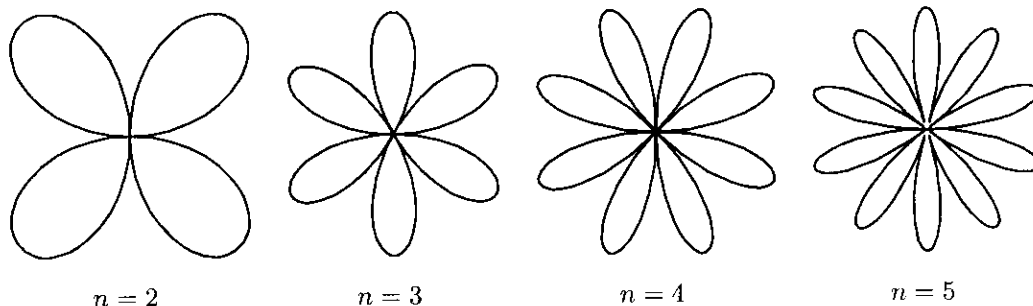
77. (a) $r = \sin n\theta$. From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n .

This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

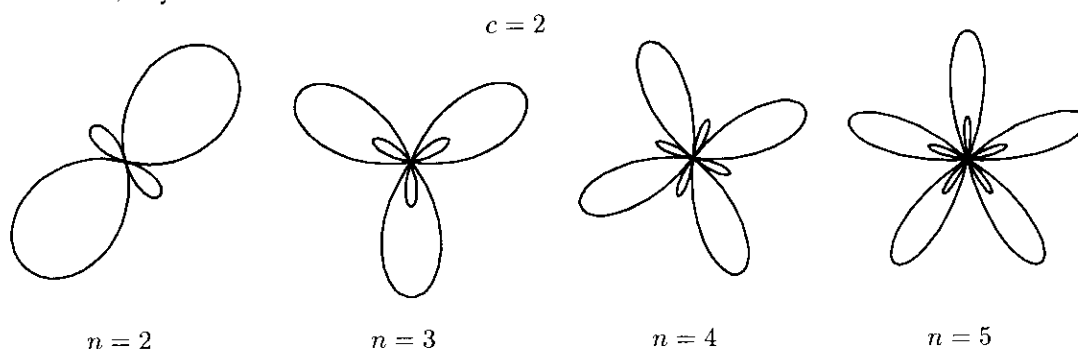
$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta \cos n\pi + \cos n\theta \sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

 $n = 2$  $n = 3$  $n = 4$  $n = 5$

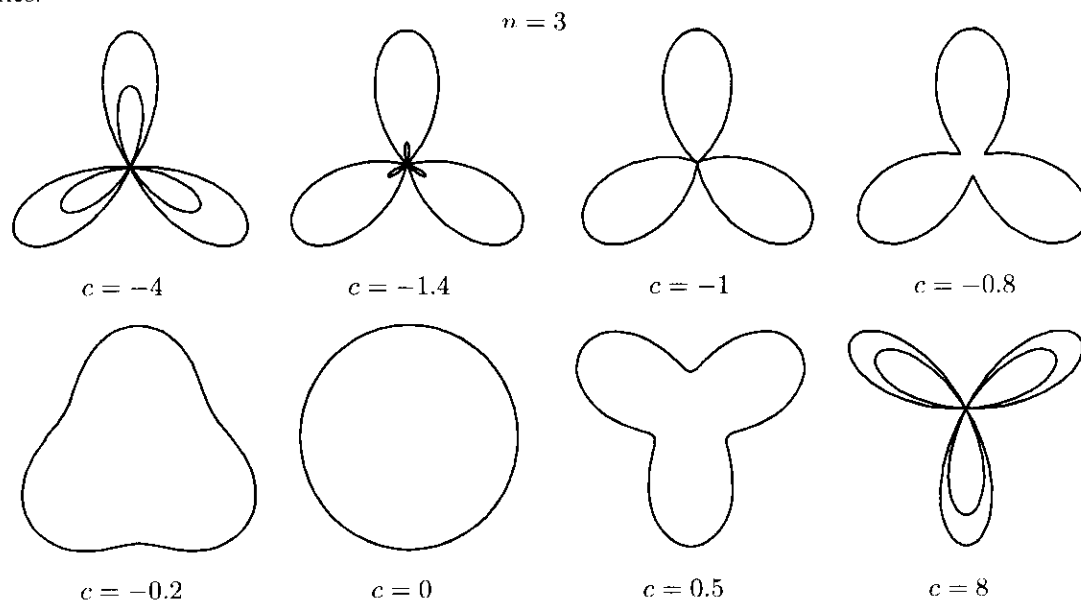
(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.



78. $r = 1 + c \sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 77: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

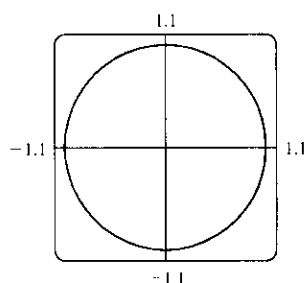
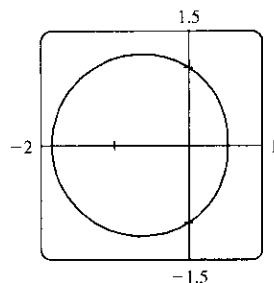
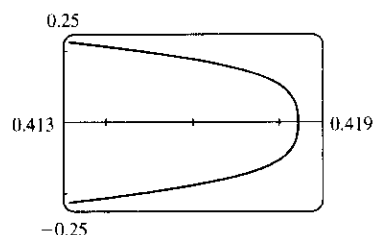
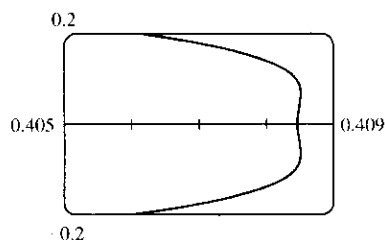
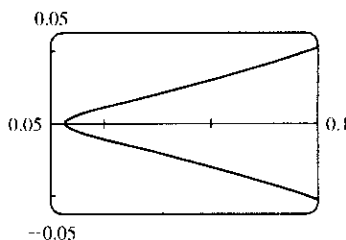
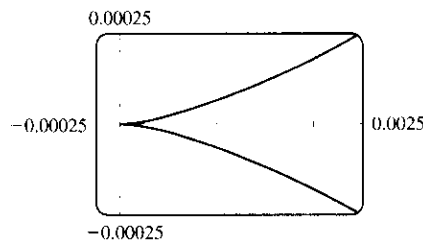
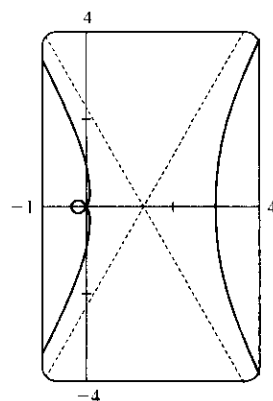
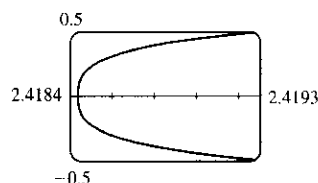
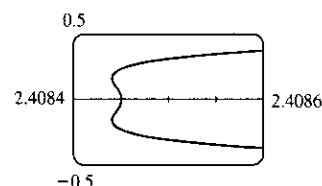
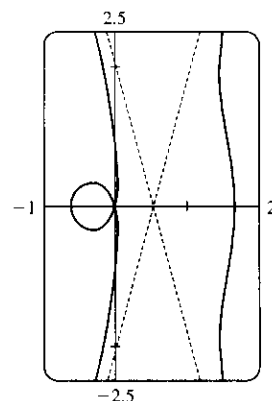


Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c) \sin n\theta = 1 + c \sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π . As $c \rightarrow \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.



79. $r = \frac{1 - a \cos \theta}{1 + a \cos \theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$.

As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ -ranges) seems to appear at $a \approx 0.42$ (the actual value is $\sqrt{2} - 1$). As $a \rightarrow 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \rightarrow \infty$ as $1 + a \cos \theta \rightarrow 0 \Leftrightarrow \theta \rightarrow \pm \cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ (actually, $\sqrt{2} + 1$). As a increases, the dimple grows more and more pronounced. If $a < 0$, we get the same graph as we do for the corresponding positive a -value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 78.

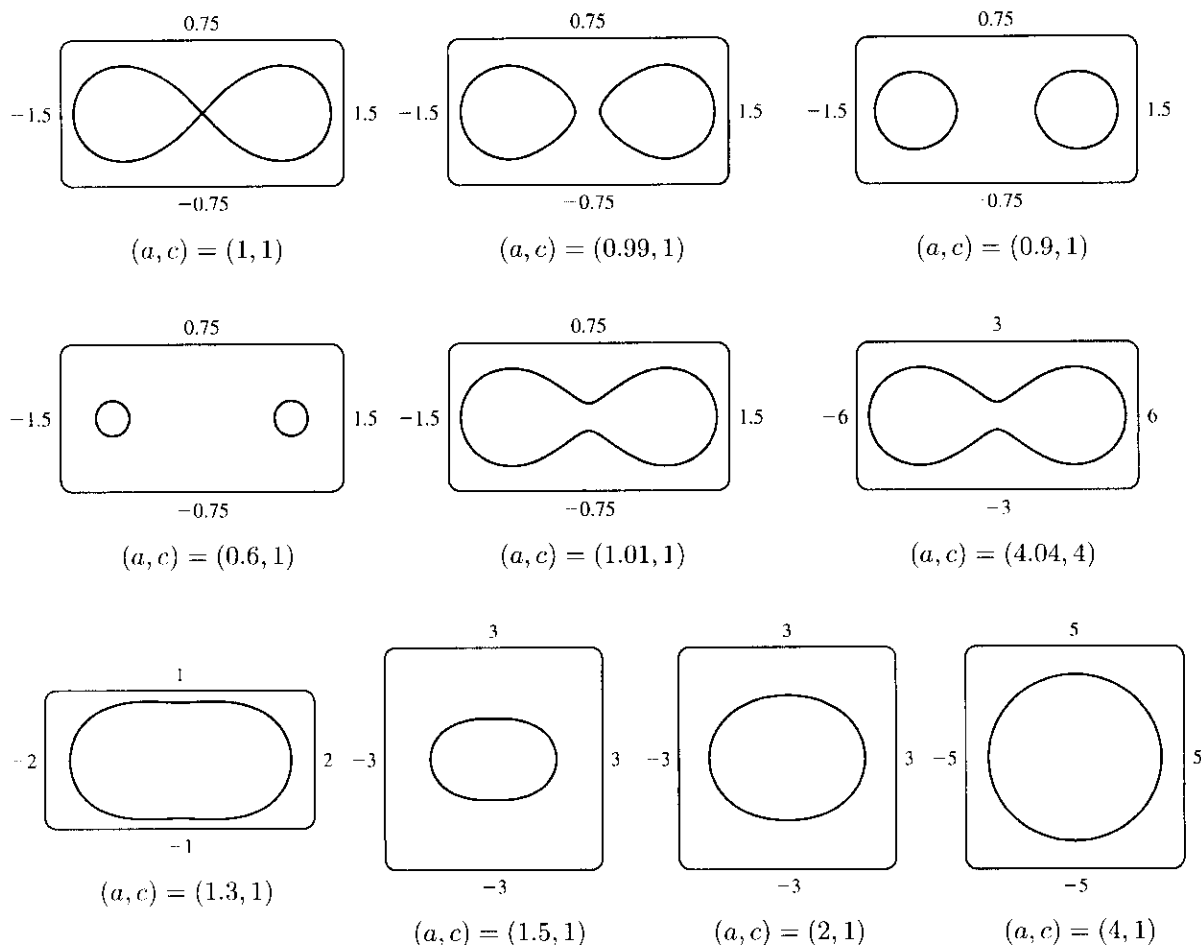
 $a = 0$  $a = 0.3$  $a = 0.41, |\theta| \leq 0.5$  $a = 0.42, |\theta| \leq 0.5$  $a = 0.9, |\theta| \leq 0.5$  $a = 1, |\theta| \leq 0.1$  $a = 2$  $a = 2.41, |\theta - \pi| \leq 0.2$  $a = 2.42, |\theta - \pi| \leq 0.2$  $a = 4$

80. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ , a , and c . We note that the given equation is a quadratic in r^2 , so we use the quadratic formula and find that

$$\begin{aligned} r^2 &= \frac{2c^2 \cos 2\theta \pm \sqrt{4c^4 \cos^2 2\theta - 4(c^4 - a^4)}}{2} \\ &= c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta} \end{aligned}$$

so $r = \pm \sqrt{c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π .

We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at $a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \rightarrow \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a , while the size of the graph varies as c and a jointly increase.



$$\begin{aligned}
 81. \quad \tan \psi &= \tan(\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta} = \frac{\frac{dy}{dx} - \tan \theta}{1 + \frac{dy}{dx} \tan \theta} = \frac{\frac{dy/d\theta}{dx/d\theta} - \tan \theta}{1 + \frac{dy/d\theta}{dx/d\theta} \tan \theta} \\
 &= \frac{\frac{dy}{d\theta} - \frac{dx}{d\theta} \tan \theta}{\frac{dx}{d\theta} + \frac{dy}{d\theta} \tan \theta} = \frac{\left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right) - \tan \theta \left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right)}{\left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right) + \tan \theta \left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right)} \\
 &= \frac{r \cos \theta + r \cdot \frac{\sin^2 \theta}{\cos \theta}}{\frac{dr}{d\theta} \cos \theta + \frac{dr}{d\theta} \cdot \frac{\sin^2 \theta}{\cos \theta}} = \frac{r \cos^2 \theta + r \sin^2 \theta}{\frac{dr}{d\theta} \cos^2 \theta + \frac{dr}{d\theta} \sin^2 \theta} = \frac{r}{dr/d\theta}
 \end{aligned}$$

82. (a) $r = e^\theta \Rightarrow dr/d\theta = e^\theta$, so by

Exercise 81, $\tan \psi = r/e^\theta = 1 \Rightarrow$

$\psi = \arctan 1 = \frac{\pi}{4}$.

(b) The Cartesian equation of the tangent line at $(1, 0)$ is

$y = x - 1$, and that of the tangent line at $(0, e^{\pi/2})$

is $y = e^{\pi/2} - x$.

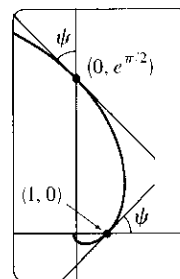
(c) Let a be the tangent of the angle between

the tangent and radial lines, that is,

$a = \tan \psi$. Then, by Exercise 81,

$a = \frac{r}{dr/d\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{a} r \Rightarrow$

$r = C e^{\theta/a}$ (by Theorem 10.4.2).



11.4 Areas and Lengths in Polar Coordinates

1. $r = \sqrt{\theta}$, $0 \leq \theta \leq \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = \int_0^{\pi/4} \frac{1}{2} (\sqrt{\theta})^2 d\theta = \int_0^{\pi/4} \frac{1}{2} \theta d\theta = \left[\frac{1}{4} \theta^2\right]_0^{\pi/4} = \frac{1}{64} \pi^2$

2. $r = e^{\theta/2}$, $\pi \leq \theta \leq 2\pi$. $A = \int_\pi^{2\pi} \frac{1}{2} (e^{\theta/2})^2 d\theta = \int_\pi^{2\pi} \frac{1}{2} e^\theta d\theta = \frac{1}{2} [e^\theta]_\pi^{2\pi} = \frac{1}{2} (e^{2\pi} - e^\pi)$

3. $r = \sin \theta$, $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$.

$$\begin{aligned}
 A &= \int_{\pi/3}^{2\pi/3} \frac{1}{2} \sin^2 \theta d\theta = \frac{1}{4} \int_{\pi/3}^{2\pi/3} (1 - \cos 2\theta) d\theta = \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right]_{\pi/3}^{2\pi/3} \\
 &= \frac{1}{4} \left[\frac{2\pi}{3} - \frac{1}{2} \sin \frac{4\pi}{3} - \frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} \right] = \frac{1}{4} \left[\frac{2\pi}{3} - \frac{1}{2} \left(-\frac{\sqrt{3}}{2} \right) - \frac{\pi}{3} + \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right) \right] = \frac{1}{4} \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{\pi}{12} + \frac{\sqrt{3}}{8}
 \end{aligned}$$

4. $r = \sqrt{\sin \theta}$, $0 \leq \theta \leq \pi$. $A = \int_0^\pi \frac{1}{2} (\sqrt{\sin \theta})^2 d\theta = \int_0^\pi \frac{1}{2} \sin \theta d\theta = \left[-\frac{1}{2} \cos \theta \right]_0^\pi = \frac{1}{2} + \frac{1}{2} = 1$

5. $r = \theta$, $0 \leq \theta \leq \pi$. $A = \int_0^\pi \frac{1}{2} \theta^2 d\theta = \left[\frac{1}{6} \theta^3 \right]_0^\pi = \frac{1}{6} \pi^3$

6. $r = 1 + \sin \theta$, $\frac{\pi}{2} \leq \theta \leq \pi$.

$$\begin{aligned}
 A &= \int_{\pi/2}^\pi \frac{1}{2} (1 + \sin \theta)^2 d\theta = \frac{1}{2} \int_{\pi/2}^\pi (1 + 2 \sin \theta + \sin^2 \theta) d\theta = \frac{1}{2} \int_{\pi/2}^\pi \left[1 + 2 \sin \theta + \frac{1}{2} (1 - \cos 2\theta) \right] d\theta \\
 &= \frac{1}{2} \left[\theta - 2 \cos \theta + \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_{\pi/2}^\pi = \frac{1}{2} \left[\pi + 2 + \frac{\pi}{2} - 0 - \left(\frac{\pi}{2} - 0 + \frac{\pi}{4} - 0 \right) \right] = \frac{1}{2} \left(\frac{3\pi}{4} + 2 \right) = \frac{3\pi}{8} + 1
 \end{aligned}$$

7. $r = 4 + 3 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

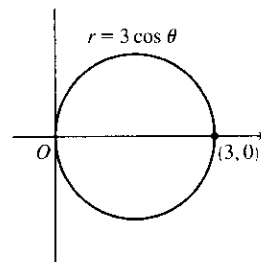
$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2} (4 + 3 \sin \theta)^2 d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 24 \sin \theta + 9 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 9 \sin^2 \theta) d\theta \quad [\text{by Theorem 5.5.6(b)}] \\ &= \frac{1}{2} \cdot 2 \int_0^{\pi/2} [16 + 9 \cdot \frac{1}{2} (1 - \cos 2\theta)] d\theta \quad [\text{by Theorem 5.5.6(a)}] \\ &= \int_0^{\pi/2} \left(\frac{41}{2} - \frac{9}{2} \cos 2\theta \right) d\theta = \left[\frac{41}{2} \theta - \frac{9}{4} \sin 2\theta \right]_0^{\pi/2} = \left(\frac{41\pi}{4} - 0 \right) - (0 - 0) = \frac{41\pi}{4} \end{aligned}$$

8. $r = \sin 4\theta$, $0 \leq \theta \leq \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2} \sin^2 4\theta d\theta = \int_0^{\pi/4} \frac{1}{4} (1 - \cos 8\theta) d\theta = \left[\frac{1}{4} \theta - \frac{1}{32} \sin 8\theta \right]_0^{\pi/4} = \frac{\pi}{16}$

9. The area above the polar axis is bounded by $r = 3 \cos \theta$ for

$\theta = 0$ to $\theta = \pi/2$ (not π). By symmetry,

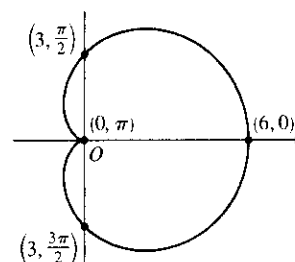
$$\begin{aligned} A &= 2 \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} (3 \cos \theta)^2 d\theta \\ &= 3^2 \int_0^{\pi/2} \cos^2 \theta d\theta = 9 \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \frac{9}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{9}{2} \left[\left(\frac{\pi}{2} + 0 \right) - (0 + 0) \right] = \frac{9\pi}{4}. \end{aligned}$$



Also, note that this is a circle with radius $\frac{3}{2}$, so its area is $\pi \left(\frac{3}{2} \right)^2 = \frac{9\pi}{4}$.

10. $A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} [3(1 + \cos \theta)]^2 d\theta$

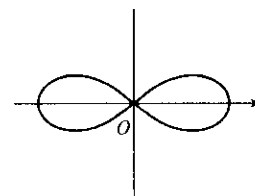
$$\begin{aligned} &= \frac{9}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{9}{2} \int_0^{2\pi} \left[1 + 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta) \right] d\theta \\ &= \frac{9}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{27}{2} \pi \end{aligned}$$



11. The curve $r^2 = 4 \cos 2\theta$ goes through the pole when

$\theta = \pi/4$, so we'll find the area for $0 \leq \theta \leq \pi/4$ and multiply it by 4.

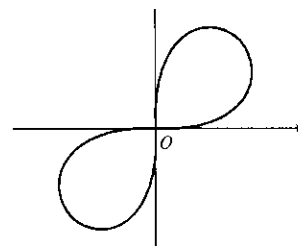
$$\begin{aligned} A &= 4 \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = 2 \int_0^{\pi/4} (4 \cos 2\theta) d\theta \\ &= 8 \int_0^{\pi/4} \cos 2\theta d\theta = 4 [\sin 2\theta]_0^{\pi/4} = 4(1 - 0) = 4 \end{aligned}$$



12. The curve $r^2 = \sin 2\theta$ goes through the pole when $\theta = \pi/2$,

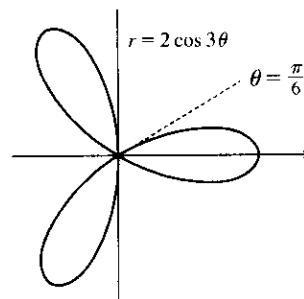
so we'll find the area for $0 \leq \theta \leq \pi/2$ and multiply it by 2.

$$\begin{aligned} A &= 2 \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \sin 2\theta d\theta = -\frac{1}{2} [\cos 2\theta]_0^{\pi/2} \\ &= -\frac{1}{2} (-1 - 1) = -\frac{1}{2} (-2) = 1 \end{aligned}$$



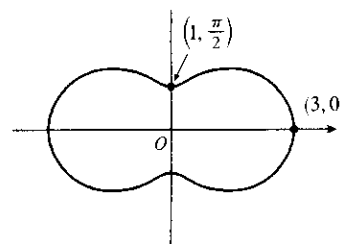
13. One-sixth of the area lies above the polar axis and is bounded by the curve $r = 2 \cos 3\theta$ for $\theta = 0$ to $\theta = \pi/6$.

$$\begin{aligned} A &= 6 \int_0^{\pi/6} \frac{1}{2} (2 \cos 3\theta)^2 d\theta = 12 \int_0^{\pi/6} \cos^2 3\theta d\theta \\ &= \frac{12}{2} \int_0^{\pi/6} (1 + \cos 6\theta) d\theta \\ &= 6 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = 6 \left(\frac{\pi}{6} \right) = \pi \end{aligned}$$



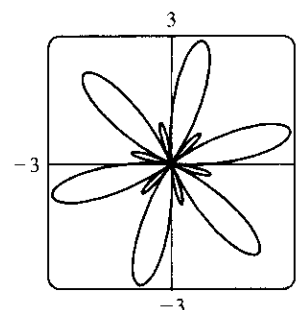
14. $A = \int_0^{2\pi} \frac{1}{2} (2 + \cos 2\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \cos 2\theta + \cos^2 2\theta) d\theta$

$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \left(4 + 4 \cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta \right) d\theta \\ &= \frac{1}{2} \left[\frac{9}{2} \theta + 2 \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{2\pi} \\ &= \frac{1}{2} (9\pi) = \frac{9\pi}{2} \end{aligned}$$



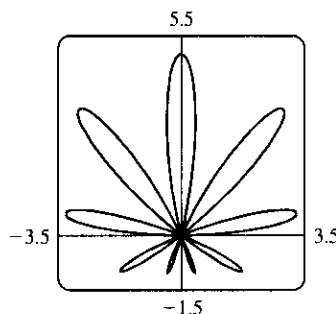
15. $A = \int_0^{2\pi} \frac{1}{2} (1 + 2 \sin 6\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 4 \sin 6\theta + 4 \sin^2 6\theta) d\theta$

$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \left[1 + 4 \sin 6\theta + 4 \cdot \frac{1}{2} (1 - \cos 12\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (3 + 4 \sin 6\theta - 2 \cos 12\theta) d\theta \\ &= \frac{1}{2} \left[3\theta - \frac{2}{3} \cos 6\theta - \frac{1}{6} \sin 12\theta \right]_0^{2\pi} \\ &= \frac{1}{2} \left[(6\pi - \frac{2}{3} - 0) - (0 - \frac{2}{3} - 0) \right] = 3\pi. \end{aligned}$$



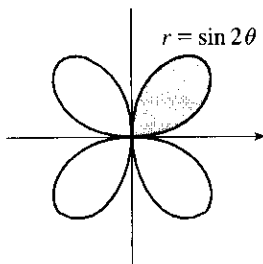
16. $A = \int_0^{\pi/2} \frac{1}{2} (2 \sin \theta + 3 \sin 9\theta)^2 d\theta = 2 \int_0^{\pi/2} \frac{1}{2} (2 \sin \theta + 3 \sin 9\theta)^2 d\theta$

$$\begin{aligned} &= \int_0^{\pi/2} (4 \sin^2 \theta + 12 \sin \theta \sin 9\theta + 9 \sin^2 9\theta) d\theta \\ &= \int_0^{\pi/2} \left[2(1 - \cos 2\theta) + 12 \cdot \frac{1}{2} (\cos(\theta - 9\theta) - \cos(\theta + 9\theta)) + \frac{9}{2} (1 - \cos 18\theta) \right] d\theta \\ &\quad \text{[integration by parts could be used for } \int \sin \theta \sin 9\theta d\theta \text{]} \\ &= \int_0^{\pi/2} \left(2 - 2 \cos 2\theta + 6 \cos 8\theta - 6 \cos 10\theta + \frac{9}{2} - \frac{9}{2} \cos 18\theta \right) d\theta \\ &= \left[\frac{13}{2} \theta - \sin 2\theta + \frac{3}{4} \sin 8\theta - \frac{3}{5} \sin 10\theta - \frac{1}{4} \sin 18\theta \right]_0^{\pi/2} = \frac{13}{4} \pi \end{aligned}$$

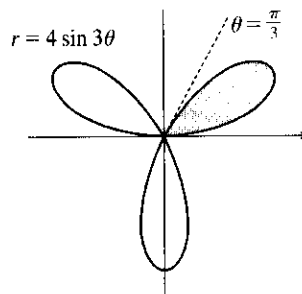


17. The shaded loop is traced out from $\theta = 0$ to $\theta = \pi/2$.

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) d\theta \\ &= \frac{1}{4} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} = \frac{1}{4} \left(\frac{\pi}{2} \right) = \frac{\pi}{8} \end{aligned}$$



$$\begin{aligned} 18. A &= \int_0^{\pi/3} \frac{1}{2} (4 \sin 3\theta)^2 d\theta = 8 \int_0^{\pi/3} \sin^2 3\theta d\theta \\ &= 4 \int_0^{\pi/3} (1 - \cos 6\theta) d\theta \\ &= 4 \left[\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/3} = \frac{4\pi}{3} \end{aligned}$$

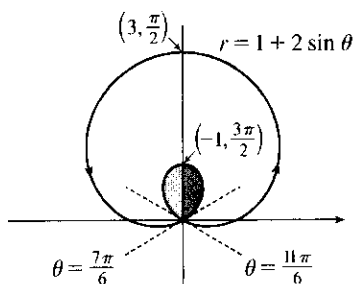
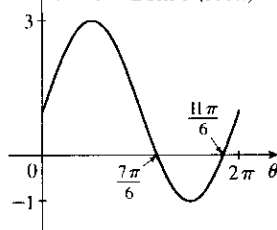


19. $r = 0 \Rightarrow 3 \cos 5\theta = 0 \Rightarrow 5\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{10}$.

$$A = \int_{-\pi/10}^{\pi/10} \frac{1}{2} (3 \cos 5\theta)^2 d\theta = \int_0^{\pi/10} 9 \cos^2 5\theta d\theta = \frac{9}{2} \int_0^{\pi/10} (1 + \cos 10\theta) d\theta = \frac{9}{2} \left[\theta + \frac{1}{10} \sin 10\theta \right]_0^{\pi/10} = \frac{9\pi}{20}$$

20. $A = 2 \int_0^{\pi/8} \frac{1}{2} (2 \cos 4\theta)^2 d\theta = 2 \int_0^{\pi/8} (1 + \cos 8\theta) d\theta = 2 \left[\theta + \frac{1}{8} \sin 8\theta \right]_0^{\pi/8} = \frac{\pi}{4}$

21. $r = 1 + 2 \sin \theta$ (rect.)



This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

$$\begin{aligned} A &= 2 \int_{7\pi/6}^{11\pi/6} \frac{1}{2} (1 + 2 \sin \theta)^2 d\theta = \int_{7\pi/6}^{11\pi/6} (1 + 4 \sin \theta + 4 \sin^2 \theta) d\theta \\ &= \int_{7\pi/6}^{11\pi/6} \left[1 + 4 \sin \theta + 4 \cdot \frac{1}{2} (1 - \cos 2\theta) \right] d\theta = \left[\theta - 4 \cos \theta + 2\theta - \sin 2\theta \right]_{7\pi/6}^{11\pi/6} \\ &= \left(\frac{9\pi}{2} \right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2} \right) = \pi - \frac{3\sqrt{3}}{2} \end{aligned}$$

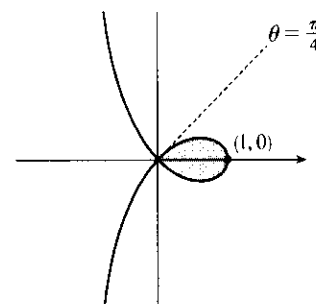
22. To determine when the strophoid $r = 2 \cos \theta - \sec \theta$ passes

through the pole, we solve $r = 0 \Rightarrow 2 \cos \theta - \frac{1}{\cos \theta} = 0 \Rightarrow$

$$2 \cos^2 \theta - 1 = 0 \Rightarrow \cos^2 \theta = \frac{1}{2} \Rightarrow \cos \theta = \pm \frac{1}{\sqrt{2}} \Rightarrow$$

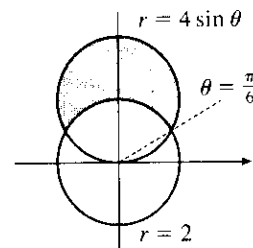
$$\theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4} \text{ for } 0 \leq \theta \leq \pi \text{ with } \theta \neq \frac{\pi}{2}.$$

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \frac{1}{2} (2 \cos \theta - \sec \theta)^2 d\theta = \int_0^{\pi/4} (4 \cos^2 \theta - 4 + \sec^2 \theta) d\theta \\ &= \int_0^{\pi/4} \left[4 \cdot \frac{1}{2} (1 + \cos 2\theta) - 4 + \sec^2 \theta \right] d\theta = \int_0^{\pi/4} (-2 + 2 \cos 2\theta + \sec^2 \theta) d\theta \\ &= [-2\theta + \sin 2\theta + \tan \theta]_0^{\pi/4} = \left(-\frac{\pi}{2} + 1 + 1 \right) - 0 = 2 - \frac{\pi}{2} \end{aligned}$$



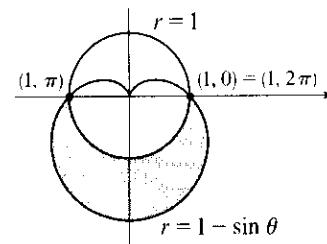
23. $4 \sin \theta = 2 \Leftrightarrow \sin \theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$ (for $0 \leq \theta \leq 2\pi$). We'll subtract the unshaded area from the shaded area for $\pi/6 \leq \theta \leq \pi/2$ and double that value.

$$\begin{aligned} A &= 2 \int_{\pi/6}^{\pi/2} \frac{1}{2} (4 \sin \theta)^2 d\theta - 2 \int_{\pi/6}^{\pi/2} \frac{1}{2} (2)^2 d\theta = 2 \int_{\pi/6}^{\pi/2} \frac{1}{2} [(4 \sin \theta)^2 - 2^2] d\theta \\ &= \int_{\pi/6}^{\pi/2} (16 \sin^2 \theta - 4) d\theta = \int_{\pi/6}^{\pi/2} [8(1 - \cos 2\theta) - 4] d\theta \\ &= \int_{\pi/6}^{\pi/2} (4 - 8 \cos 2\theta) d\theta = [4\theta - 4 \sin 2\theta]_{\pi/6}^{\pi/2} \\ &= (2\pi - 0) - \left(\frac{2\pi}{3} - 4 \cdot \frac{\sqrt{3}}{2} \right) = \frac{4}{3}\pi + 2\sqrt{3} \end{aligned}$$



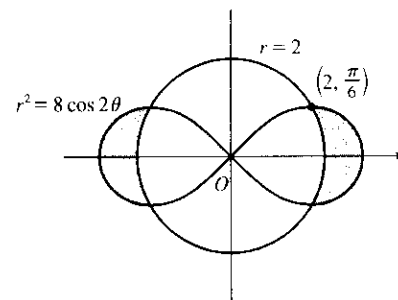
24. $1 - \sin \theta = 1 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0 \text{ or } \pi \Rightarrow$

$$\begin{aligned} A &= \int_{\pi}^{2\pi} \frac{1}{2} [(1 - \sin \theta)^2 - 1] d\theta = \frac{1}{2} \int_{\pi}^{2\pi} (\sin^2 \theta - 2 \sin \theta) d\theta \\ &= \frac{1}{4} \int_{\pi}^{2\pi} (1 - \cos 2\theta - 4 \sin \theta) d\theta = \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta + 4 \cos \theta \right]_{\pi}^{2\pi} \\ &= \frac{1}{4}\pi + 2 \end{aligned}$$



25. To find the area inside the lemniscate $r^2 = 8 \cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8 \cos 2\theta$ and $r = 2$; i.e., when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \leq \pi$, $\cos 2\theta = \frac{1}{2} \Leftrightarrow 2\theta = \pm\pi/3 \text{ or } \pm 5\pi/3 \Leftrightarrow \theta = \pm\pi/6 \text{ or } \pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

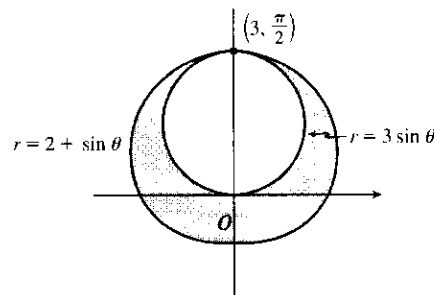
$$\begin{aligned} A &= 4 \int_0^{\pi/6} \left[\frac{1}{2} (8 \cos 2\theta) - \frac{1}{2} (2)^2 \right] d\theta = 8 \int_0^{\pi/6} (2 \cos 2\theta - 1) d\theta \\ &= 8 \left[\sin 2\theta - \theta \right]_0^{\pi/6} = 8 \left(\sqrt{3}/2 - \pi/6 \right) = 4\sqrt{3} - 4\pi/3 \end{aligned}$$



26. To find the shaded area A , we'll find the area A_1 inside the curve $r = 2 + \sin \theta$ and subtract $\pi \left(\frac{3}{2} \right)^2$ since $r = 3 \sin \theta$ is a circle with radius $\frac{3}{2}$.

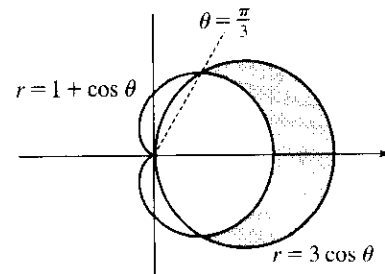
$$\begin{aligned} A_1 &= \int_0^{2\pi} \frac{1}{2} (2 + \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin \theta + \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[4 + 4 \sin \theta + \frac{1}{2} (1 - \cos 2\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(\frac{9}{2} + 4 \sin \theta - \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \frac{1}{2} \left[\frac{9}{2} \theta - 4 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{1}{2} [(9\pi - 4) - (-4)] = \frac{9\pi}{2} \end{aligned}$$

$$\text{So } A = A_1 - \frac{9\pi}{4} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}.$$



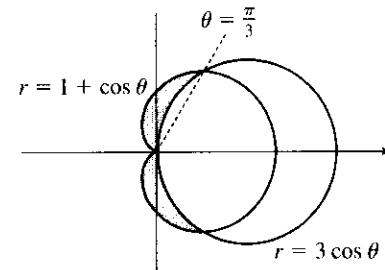
$$27. 3 \cos \theta = 1 + \cos \theta \Leftrightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}.$$

$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2} [(3 \cos \theta)^2 - (1 + \cos \theta)^2] d\theta \\ &= \int_0^{\pi/3} (8 \cos^2 \theta - 2 \cos \theta - 1) d\theta \\ &= \int_0^{\pi/3} [4(1 + \cos 2\theta) - 2 \cos \theta - 1] d\theta \\ &= \int_0^{\pi/3} (3 + 4 \cos 2\theta - 2 \cos \theta) d\theta \\ &= [3\theta + 2 \sin 2\theta - 2 \sin \theta]_0^{\pi/3} \\ &= \pi + \sqrt{3} - \sqrt{3} = \pi \end{aligned}$$



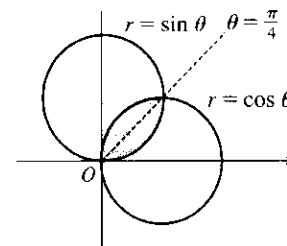
28. Note that $r = 1 + \cos \theta$ goes through the pole when $\theta = \pi$,
but $r = 3 \cos \theta$ goes through the pole when $\theta = \pi/2$.

$$\begin{aligned} A &= 2 \int_{\pi/3}^{\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta - 2 \int_{\pi/3}^{\pi/2} \frac{1}{2} (3 \cos \theta)^2 d\theta \\ &= \int_{\pi/3}^{\pi} [1 + 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta - \frac{9}{2} \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= [\theta + 2 \sin \theta + \frac{1}{2}(\theta + \frac{1}{2} \sin 2\theta)]_{\pi/3}^{\pi} - \frac{9}{2} [\theta + \frac{1}{2} \sin 2\theta]_{\pi/3}^{\pi/2} \\ &= (\pi - \frac{9}{8}\sqrt{3}) - \frac{9}{2}(\frac{\pi}{6} - \frac{1}{4}\sqrt{3}) = \frac{\pi}{4} \end{aligned}$$



$$29. A = 2 \int_0^{\pi/4} \frac{1}{2} \sin^2 \theta d\theta = \int_0^{\pi/4} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

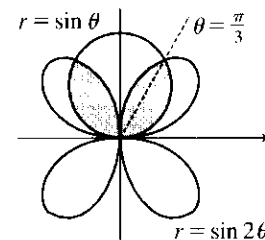
$$\begin{aligned} &= \frac{1}{2} [\theta - \frac{1}{2} \sin 2\theta]_0^{\pi/4} = \frac{1}{2} [(\frac{\pi}{4} - \frac{1}{2} \cdot 1) - (0 - 0)] \\ &= \frac{1}{8} \pi - \frac{1}{4} \end{aligned}$$



30. $r = \sin 2\theta$ takes on both positive and negative values.

$\sin \theta = \pm \sin 2\theta = \pm 2 \sin \theta \cos \theta \Rightarrow \sin \theta (1 \pm 2 \cos \theta) = 0$. From the figure we can see that the intersections occur where $\cos \theta = \pm \frac{1}{2}$, or $\theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$$\begin{aligned} A &= 2 \left[\int_0^{\pi/3} \frac{1}{2} \sin^2 \theta d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} \sin^2 2\theta d\theta \right] \\ &= \int_0^{\pi/3} \frac{1}{2} (1 - \cos 2\theta) d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) d\theta \\ &= \frac{1}{2} [\theta - \frac{1}{2} \sin 2\theta]_0^{\pi/3} + \frac{1}{2} [\theta - \frac{1}{4} \sin 4\theta]_{\pi/3}^{\pi/2} = \frac{4\pi - 3\sqrt{3}}{16} \end{aligned}$$

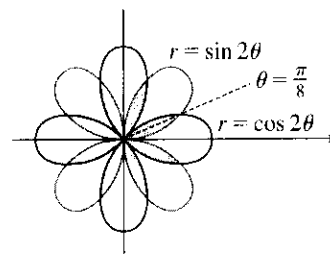


$$31. \sin 2\theta = \cos 2\theta \Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = 1 \Rightarrow \tan 2\theta = 1 \Rightarrow$$

$$2\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{8} \Rightarrow$$

$$A = 8 \cdot 2 \int_0^{\pi/8} \frac{1}{2} \sin^2 2\theta \, d\theta = 8 \int_0^{\pi/8} \frac{1}{2} (1 - \cos 4\theta) \, d\theta$$

$$= 4 \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/8} = 4 \left(\frac{\pi}{8} - \frac{1}{4} \cdot 1 \right) = \frac{1}{2} \pi - 1$$



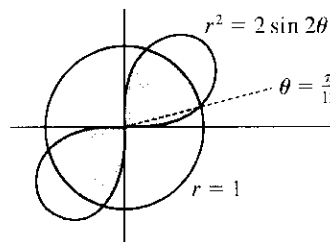
$$32. 2 \sin 2\theta = 1^2 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}.$$

$$A = 4 \left[\int_0^{\pi/12} \frac{1}{2} \cdot 2 \sin 2\theta \, d\theta + \int_{\pi/12}^{\pi/4} \frac{1}{2} (1^2) \, d\theta \right]$$

$$= [-2 \cos 2\theta]_0^{\pi/12} + [2\theta]_{\pi/12}^{\pi/4}$$

$$= -2 \left(\frac{\sqrt{3}}{2} - 1 \right) + 2 \left(\frac{1}{4} \pi - \frac{1}{12} \pi \right)$$

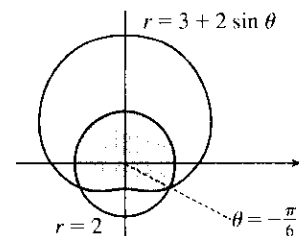
$$= 2 - \sqrt{3} + \frac{\pi}{3}$$



$$33. A = 2 \left[\int_{-\pi/2}^{\pi/6} \frac{1}{2} (3 + 2 \sin \theta)^2 \, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2} 2^2 \, d\theta \right]$$

$$= \int_{-\pi/2}^{\pi/6} (9 + 12 \sin \theta + 4 \sin^2 \theta) \, d\theta + [4\theta]_{\pi/6}^{\pi/2}$$

$$= [9\theta - 12 \cos \theta + 2\theta - \sin 2\theta]_{-\pi/2}^{\pi/6} + \frac{8\pi}{3} = \frac{19\pi}{3} - \frac{11\sqrt{3}}{2}$$



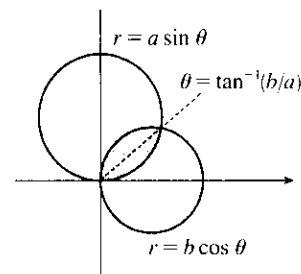
$$34. \text{ Let } \alpha = \tan^{-1}(b/a). \text{ Then}$$

$$A = \int_0^\alpha \frac{1}{2} (a \sin \theta)^2 \, d\theta + \int_\alpha^{\pi/2} \frac{1}{2} (b \cos \theta)^2 \, d\theta$$

$$= \frac{1}{4} a^2 \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^\alpha + \frac{1}{4} b^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_\alpha^{\pi/2}$$

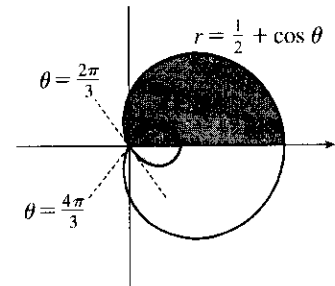
$$= \frac{1}{4} \alpha (a^2 - b^2) + \frac{1}{8} \pi b^2 - \frac{1}{4} (a^2 + b^2) (\sin \alpha \cos \alpha)$$

$$= \frac{1}{4} (a^2 - b^2) \tan^{-1}(b/a) + \frac{1}{8} \pi b^2 - \frac{1}{4} ab$$



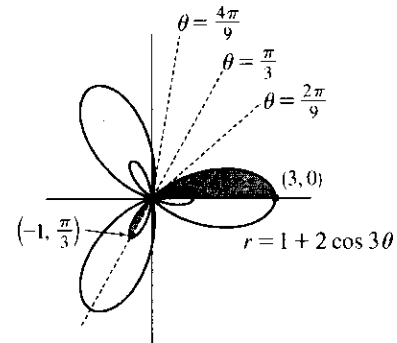
35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.

$$\begin{aligned} A &= 2 \left[\int_0^{2\pi/3} \frac{1}{2} \left(\frac{1}{2} + \cos \theta \right)^2 d\theta - \int_{2\pi/3}^{\pi} \frac{1}{2} \left(\frac{1}{2} + \cos \theta \right)^2 d\theta \right] \\ &= \int_0^{2\pi/3} \left(\frac{1}{4} + \cos \theta + \cos^2 \theta \right) d\theta - \int_{2\pi/3}^{\pi} \left(\frac{1}{4} + \cos \theta + \cos^2 \theta \right) d\theta \\ &= \int_0^{2\pi/3} \left[\frac{1}{4} + \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &\quad - \int_{2\pi/3}^{\pi} \left[\frac{1}{4} + \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &= \left[\frac{\theta}{4} + \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{2\pi/3}^{\pi} \\ &= \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right) - \left(\frac{\pi}{4} + \frac{\pi}{2} \right) + \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right) \\ &= \frac{\pi}{4} + \frac{3}{4}\sqrt{3} = \frac{1}{4}(\pi + 3\sqrt{3}) \end{aligned}$$



36. $r = 0 \Rightarrow 1 + 2\cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (for $0 \leq 3\theta \leq 2\pi$) $\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.

$$A = 2 \left[\int_0^{2\pi/9} \frac{1}{2} (1 + 2\cos 3\theta)^2 d\theta - \int_{2\pi/9}^{\pi/3} \frac{1}{2} (1 + 2\cos 3\theta)^2 d\theta \right]$$



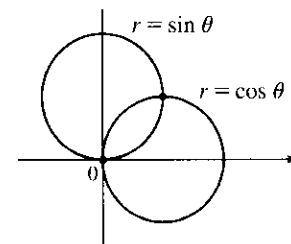
Now

$$\begin{aligned} r^2 &= (1 + 2\cos 3\theta)^2 = 1 + 4\cos 3\theta + 4\cos^2 3\theta = 1 + 4\cos 3\theta + 4 \cdot \frac{1}{2}(1 + \cos 6\theta) \\ &= 1 + 4\cos 3\theta + 2 + 2\cos 6\theta = 3 + 4\cos 3\theta + 2\cos 6\theta \end{aligned}$$

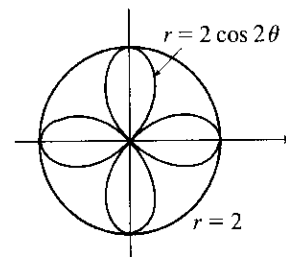
and $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

$$\begin{aligned} A &= \left[3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta \right]_0^{2\pi/9} - \left[3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta \right]_{2\pi/9}^{\pi/3} \\ &= \left[\left(\frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) - 0 \right] - \left[(\pi + 0 + 0) - \left(\frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) \right] \\ &= \frac{4\pi}{3} + \frac{4}{3}\sqrt{3} - \frac{1}{3}\sqrt{3} - \pi = \frac{\pi}{3} + \sqrt{3} \end{aligned}$$

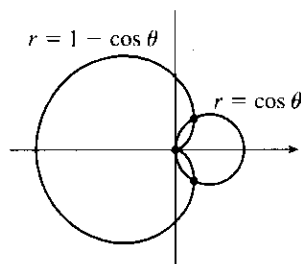
37. The two circles intersect at the pole since $(0, 0)$ satisfies the first equation and $(0, \frac{\pi}{2})$ the second. The other intersection point $(\frac{1}{\sqrt{2}}, \frac{\pi}{4})$ occurs where $\sin \theta = \cos \theta$.



38. $2 \cos 2\theta = \pm 2 \Rightarrow \cos 2\theta = \pm 1 \Rightarrow \theta = 0, \frac{\pi}{2}, \pi, \text{ or } \frac{3\pi}{2}$, so the points are $(2, 0)$, $(2, \frac{\pi}{2})$, $(2, \pi)$, and $(2, \frac{3\pi}{2})$.

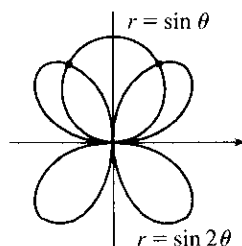


39. The curves intersect at the pole since $(0, \frac{\pi}{2})$ satisfies $r = \cos \theta$ and $(0, 0)$ satisfies $r = 1 - \cos \theta$. Now $\cos \theta = 1 - \cos \theta \Rightarrow 2 \cos \theta = 1 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \Rightarrow$ the other intersection points are $(\frac{1}{2}, \frac{\pi}{3})$ and $(\frac{1}{2}, \frac{5\pi}{3})$.

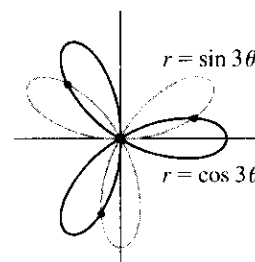


41. The pole is a point of intersection.

$$\begin{aligned} \sin \theta &= \sin 2\theta = 2 \sin \theta \cos \theta \Leftrightarrow \\ \sin \theta (1 - 2 \cos \theta) &= 0 \Leftrightarrow \sin \theta = 0 \text{ or } \\ \cos \theta &= \frac{1}{2} \Rightarrow \theta = 0, \pi, \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right) \\ \text{and } \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right) &\text{ (by symmetry) are the other} \\ \text{intersection points.} \end{aligned}$$

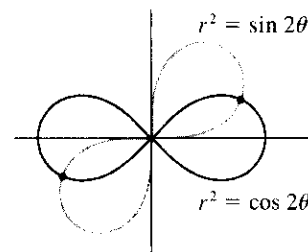


40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1 \Rightarrow 3\theta = \frac{\pi}{4} + n\pi$ (n any integer) $\Rightarrow \theta = \frac{\pi}{12} + \frac{\pi}{3}n \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \text{ or } \frac{3\pi}{4}$, so the three remaining intersection points are $(\frac{1}{\sqrt{2}}, \frac{\pi}{12})$, $(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12})$, and $(\frac{1}{\sqrt{2}}, \frac{3\pi}{4})$.

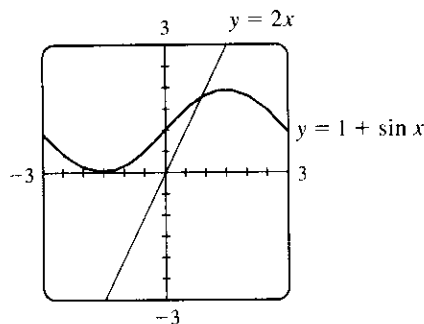
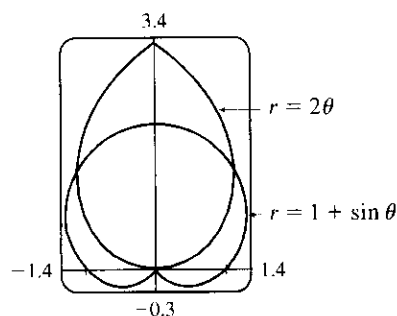


42. Clearly the pole is a point of intersection.

$$\begin{aligned} \sin 2\theta &= \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow \\ 2\theta &= \frac{\pi}{4} + 2n\pi \text{ (since } \sin 2\theta \text{ and } \cos 2\theta \text{ must be} \\ \text{positive in the equations)} &\Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \\ \theta &= \frac{\pi}{8} \text{ or } \frac{9\pi}{8}. \text{ So the curves also intersect at} \\ \left(\frac{1}{\sqrt{2}}, \frac{\pi}{8}\right) &\text{ and } \left(\frac{1}{\sqrt{2}}, \frac{9\pi}{8}\right). \end{aligned}$$



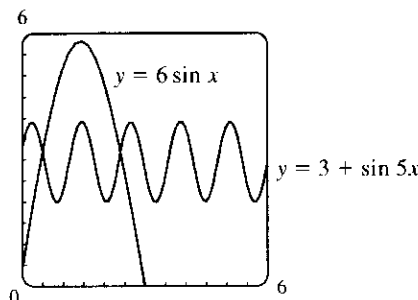
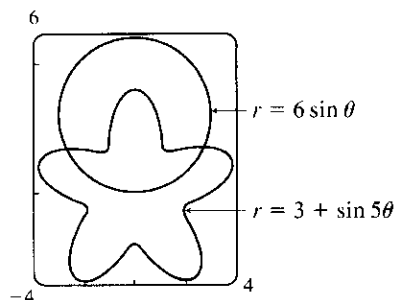
43.



From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the θ -values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$\begin{aligned} A &= 2 \int_0^\alpha \frac{1}{2} (2\theta)^2 d\theta + 2 \int_\alpha^{\pi/2} \frac{1}{2} (1 + \sin \theta)^2 d\theta = \int_0^\alpha 4\theta^2 d\theta + \int_\alpha^{\pi/2} [1 + 2\sin \theta + \frac{1}{2}(1 - \cos 2\theta)] d\theta \\ &= \left[\frac{4}{3}\theta^3 \right]_0^\alpha + \left[\theta - 2\cos \theta + \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right) \right]_\alpha^{\pi/2} \\ &= \frac{4}{3}\alpha^3 + \left[\left(\frac{\pi}{2} + \frac{\pi}{4} \right) - \left(\alpha - 2\cos \alpha + \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha \right) \right] \approx 3.4645 \end{aligned}$$

44.



From the first graph, it appears that the θ -values of the points of intersection are $\alpha \approx 0.57504 \approx 0.58$ and $\pi - \alpha \approx 2.57$. (These values may be more easily estimated by plotting $y = 3 + \sin 5x$ and $y = 6 \sin x$ in rectangular coordinates; see the second graph.) By symmetry, the total area enclosed in both curves is

$$\begin{aligned} A &= 2 \int_0^\alpha \frac{1}{2} (6 \sin \theta)^2 d\theta + 2 \int_\alpha^{\pi/2} \frac{1}{2} (3 + \sin 5\theta)^2 d\theta = \int_0^\alpha 36 \sin^2 \theta d\theta + \int_\alpha^{\pi/2} (9 + 6 \sin 5\theta + \sin^2 5\theta) d\theta \\ &= \int_0^\alpha 36 \cdot \frac{1}{2} (1 - \cos 2\theta) d\theta + \int_\alpha^{\pi/2} \left[9 + 6 \sin 5\theta + \frac{1}{2} (1 - \cos 10\theta) \right] d\theta \\ &= \left[36 \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right) \right]_0^\alpha + \left[9\theta - \frac{6}{5}\cos 5\theta + \left(\frac{1}{2}\theta - \frac{1}{20}\sin 10\theta \right) \right]_\alpha^{\pi/2} \approx 10.41 \end{aligned}$$

$$\begin{aligned} 45. L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{\pi/3} \sqrt{(3 \sin \theta)^2 + (3 \cos \theta)^2} d\theta = \int_0^{\pi/3} \sqrt{9(\sin^2 \theta + \cos^2 \theta)} d\theta \\ &= 3 \int_0^{\pi/3} d\theta = 3[\theta]_0^{\pi/3} = 3\left(\frac{\pi}{3}\right) = \pi. \end{aligned}$$

As a check, note that the circumference of a circle with radius $\frac{3}{2}$ is $2\pi\left(\frac{3}{2}\right) = 3\pi$, and since $\theta = 0$ to $\theta = \pi$ traces out $\frac{1}{3}$ of the circle (from $\theta = 0$ to $\theta = \pi$), $\frac{1}{3}(3\pi) = \pi$.

$$46. L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2} d\theta = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta = \int_0^{2\pi} \sqrt{5e^{4\theta}} d\theta$$

$$= \sqrt{5} \int_0^{2\pi} e^{2\theta} d\theta = \frac{\sqrt{5}}{2} [e^{2\theta}]_0^{2\pi} = \frac{\sqrt{5}}{2} (e^{4\pi} - 1)$$

$$47. L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2 + 4)} d\theta = \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta$$

Now let $u = \theta^2 + 4$, so that $du = 2\theta d\theta$ [$\theta d\theta = \frac{1}{2} du$] and

$$\int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta = \int_4^{4\pi^2 + 4} \frac{1}{2} \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2} \right]_4^{4\pi^2 + 4} = \frac{1}{3} \left[4^{3/2} (\pi^2 + 1)^{3/2} - 4^{3/2} \right]$$

$$= \frac{8}{3} \left[(\pi^2 + 1)^{3/2} - 1 \right]$$

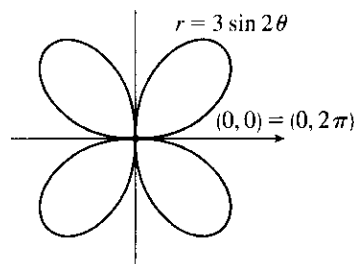
$$48. L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\theta^2 + 1} d\theta \stackrel{21}{=} \left[\frac{\theta}{2} \sqrt{\theta^2 + 1} + \frac{1}{2} \ln(\theta + \sqrt{\theta^2 + 1}) \right]_0^{2\pi}$$

$$= \pi \sqrt{4\pi^2 + 1} + \frac{1}{2} \ln(2\pi + \sqrt{4\pi^2 + 1})$$

49. The curve $r = 3 \sin 2\theta$ is completely traced with $0 \leq \theta \leq 2\pi$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (3 \sin 2\theta)^2 + (6 \cos 2\theta)^2 \Rightarrow$$

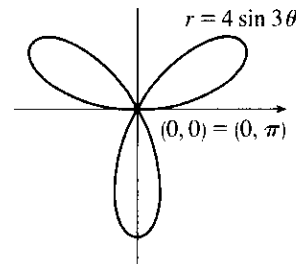
$$L = \int_0^{2\pi} \sqrt{9 \sin^2 2\theta + 36 \cos^2 2\theta} d\theta \approx 29.0653$$



50. The curve $r = 4 \sin 3\theta$ is completely traced with $0 \leq \theta \leq \pi$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (4 \sin 3\theta)^2 + (12 \cos 3\theta)^2 \Rightarrow$$

$$L = \int_0^{\pi} \sqrt{16 \sin^2 3\theta + 144 \cos^2 3\theta} d\theta \approx 26.7298$$

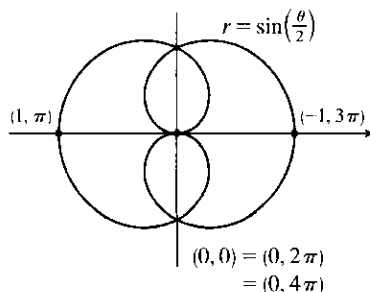


51. The curve $r = \sin(\frac{\theta}{2})$ is completely traced with $0 \leq \theta \leq 4\pi$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2\left(\frac{\theta}{2}\right) + \left[\frac{1}{2} \cos\left(\frac{\theta}{2}\right)\right]^2 \Rightarrow$$

$$L = \int_0^{4\pi} \sqrt{\sin^2\left(\frac{\theta}{2}\right) + \frac{1}{4} \cos^2\left(\frac{\theta}{2}\right)} d\theta$$

$$\approx 9.6884$$

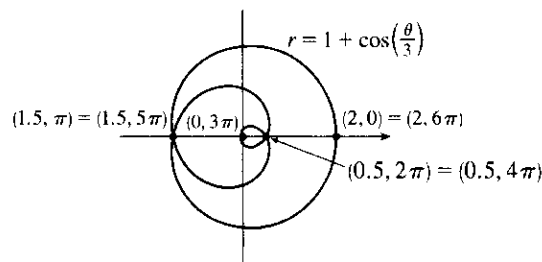


52. The curve $r = 1 + \cos(\frac{\theta}{3})$ is completely traced with $0 \leq \theta \leq 6\pi$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[1 + \cos\left(\frac{\theta}{3}\right)\right]^2 + \left[-\frac{1}{3} \sin\left(\frac{\theta}{3}\right)\right]^2 \Rightarrow$$

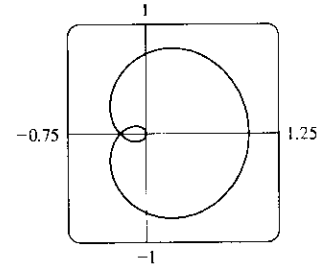
$$L = \int_0^{6\pi} \sqrt{\left[1 + \cos\left(\frac{\theta}{3}\right)\right]^2 + \frac{1}{9} \sin^2\left(\frac{\theta}{3}\right)} d\theta$$

$$\approx 19.6676$$



53. The curve $r = \cos^4(\theta/4)$ is completely traced with $0 \leq \theta \leq 4\pi$.

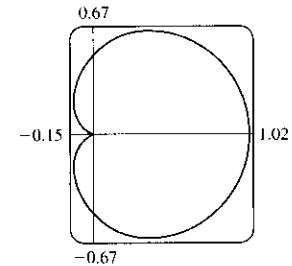
$$\begin{aligned} r^2 + (dr/d\theta)^2 &= [\cos^4(\theta/4)]^2 + [4\cos^3(\theta/4) \cdot (-\sin(\theta/4)) \cdot \frac{1}{4}]^2 \\ &= \cos^8(\theta/4) + \cos^6(\theta/4) \sin^2(\theta/4) \\ &= \cos^6(\theta/4) [\cos^2(\theta/4) + \sin^2(\theta/4)] \\ &= \cos^6(\theta/4) \end{aligned}$$



$$\begin{aligned} L &= \int_0^{4\pi} \sqrt{\cos^6(\theta/4)} d\theta = \int_0^{4\pi} |\cos^3(\theta/4)| d\theta \\ &= 2 \int_0^{2\pi} \cos^3(\theta/4) d\theta \quad [\text{since } \cos^3(\theta/4) \geq 0 \text{ for } 0 \leq \theta \leq 2\pi] = 8 \int_0^{\pi/2} \cos^3 u du \quad [u = \frac{1}{4}\theta] \\ &\stackrel{68}{=} 8 \left[\frac{1}{3}(2 + \cos^2 u) \sin u \right]_0^{\pi/2} = \frac{8}{3} [(2 \cdot 1) - (3 \cdot 0)] = \frac{16}{3} \end{aligned}$$

54. The curve $r = \cos^2(\theta/2)$ is completely traced with $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} r^2 + (dr/d\theta)^2 &= [\cos^2(\theta/2)]^2 + [2\cos(\theta/2) \cdot (-\sin(\theta/2)) \cdot \frac{1}{2}]^2 \\ &= \cos^4(\theta/2) + \cos^2(\theta/2) \sin^2(\theta/2) \\ &= \cos^2(\theta/2) [\cos^2(\theta/2) + \sin^2(\theta/2)] \\ &= \cos^2(\theta/2) \end{aligned}$$



$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\cos^2(\theta/2)} d\theta = \int_0^{2\pi} |\cos(\theta/2)| d\theta = 2 \int_0^{\pi} \cos(\theta/2) d\theta \quad [\text{since } \cos(\theta/2) \geq 0 \text{ for } 0 \leq \theta \leq \pi] \\ &= 4 \int_0^{\pi/2} \cos u du \quad [u = \frac{1}{2}\theta] = 4[\sin u]_0^{\pi/2} = 4(1 - 0) = 4 \end{aligned}$$

55. (a) From (11.2.7),

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta \\ &= \int_a^b 2\pi y \sqrt{r^2 + (dr/d\theta)^2} d\theta \quad [\text{from the derivation of Equation 11.4.5}] \\ &= \int_a^b 2\pi r \sin \theta \sqrt{r^2 + (dr/d\theta)^2} d\theta \end{aligned}$$

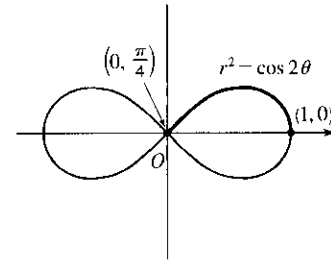
- (b) The curve $r^2 = \cos 2\theta$ goes through the pole when

$$\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}. \text{ We'll rotate the curve}$$

from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and double this value to obtain the total surface

$$\text{area generated. } r^2 = \cos 2\theta \Rightarrow 2r \frac{dr}{d\theta} = -2 \sin 2\theta \Rightarrow$$

$$\left(\frac{dr}{d\theta} \right)^2 = \frac{\sin^2 2\theta}{r^2} = \frac{\sin^2 2\theta}{\cos 2\theta}.$$



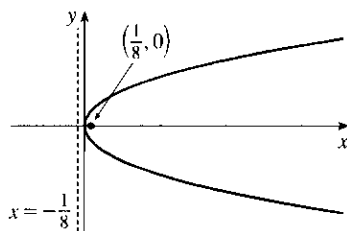
$$\begin{aligned} S &= 2 \int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \sin \theta \sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta} d\theta \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \sin \theta \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} d\theta = 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \sin \theta \frac{1}{\sqrt{\cos 2\theta}} d\theta \\ &= 4\pi \int_0^{\pi/4} \sin \theta d\theta = 4\pi [-\cos \theta]_0^{\pi/4} = -4\pi \left(\frac{\sqrt{2}}{2} - 1 \right) = 2\pi (2 - \sqrt{2}) \end{aligned}$$

56. (a) Rotation around $\theta = \frac{\pi}{2}$ is the same as rotation around the y -axis, that is, $S = \int_a^b 2\pi x \, ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt$ for a parametric equation, and for the special case of a polar equation, $x = r \cos \theta$ and $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta = \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$ [see the derivation of Equation 11.4.5]. Therefore, for a polar equation rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r \cos \theta \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$.
- (b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ to obtain the total surface area.

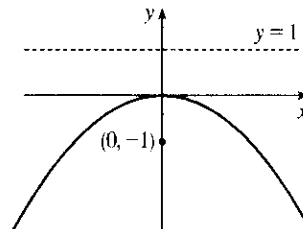
$$\begin{aligned} S &= 2 \int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \cos \theta \sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta} \, d\theta \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \cos \theta \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} \, d\theta \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \cos \theta \frac{1}{\sqrt{\cos 2\theta}} \, d\theta \\ &= 4\pi \int_0^{\pi/4} \cos \theta \, d\theta = 4\pi [\sin \theta]_0^{\pi/4} = 4\pi \left(\frac{\sqrt{2}}{2} - 0 \right) = 2\sqrt{2}\pi \end{aligned}$$

11.5 Conic Sections

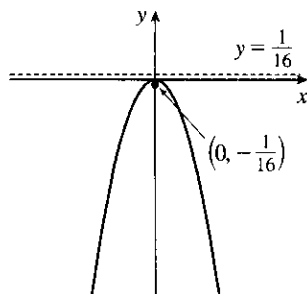
1. $x = 2y^2 \Rightarrow y^2 = \frac{1}{2}x$. $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(0, 0)$, the focus is $(\frac{1}{8}, 0)$, and the directrix is $x = -\frac{1}{8}$.



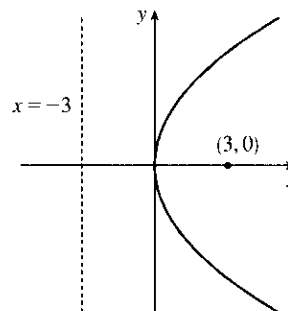
2. $4y + x^2 = 0 \Rightarrow x^2 = -4y$. $4p = -4$, so $p = -1$. The vertex is $(0, 0)$, the focus is $(0, -1)$, and the directrix is $y = 1$.



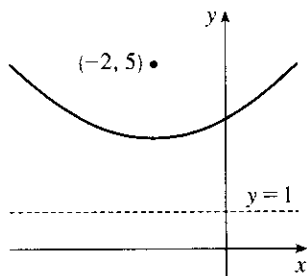
3. $4x^2 = -y \Rightarrow x^2 = -\frac{1}{4}y$. $4p = -\frac{1}{4}$, so $p = -\frac{1}{16}$. The vertex is $(0, 0)$, the focus is $(0, -\frac{1}{16})$, and the directrix is $y = \frac{1}{16}$.



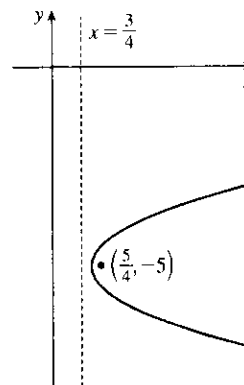
4. $y^2 = 12x$. $4p = 12$, so $p = 3$. The vertex is $(0, 0)$, the focus is $(3, 0)$, and the directrix is $x = -3$.



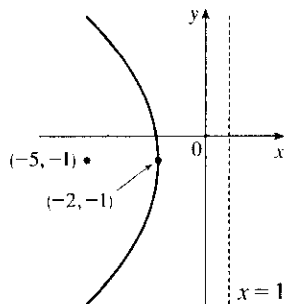
5. $(x + 2)^2 = 8(y - 3)$. $4p = 8$, so $p = 2$. The vertex is $(-2, 3)$, the focus is $(-2, 5)$, and the directrix is $y = 1$.



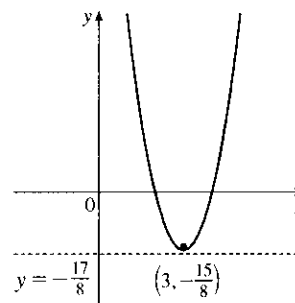
6. $x - 1 = (y + 5)^2$. $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(1, -5)$, the focus is $(\frac{5}{4}, -5)$, and the directrix is $x = \frac{3}{4}$.



7. $y^2 + 2y + 12x + 25 = 0 \Rightarrow$
 $y^2 + 2y + 1 = -12x - 24 \Rightarrow$
 $(y + 1)^2 = -12(x + 2)$. $4p = -12$, so $p = -3$.
 The vertex is $(-2, -1)$, the focus is $(-5, -1)$, and the directrix is $x = 1$.



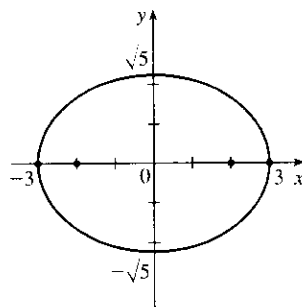
8. $y + 12x - 2x^2 = 16 \Rightarrow$
 $2x^2 - 12x = y - 16 \Rightarrow$
 $2(x^2 - 6x + 9) = y - 16 + 18 \Rightarrow$
 $2(x - 3)^2 = y + 2 \Rightarrow (x - 3)^2 = \frac{1}{2}(y + 2)$.
 $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(3, -2)$, the focus is $(3, -\frac{15}{8})$, and the directrix is $y = -\frac{17}{8}$.



9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1, 1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $(-\frac{1}{4}, 0)$ while the directrix is $x = \frac{1}{4}$.
10. The vertex is $(2, -2)$, so the equation is of the form $(x - 2)^2 = 4p(y + 2)$, where $p > 0$. The point $(0, 0)$ is on the parabola, so $4 = 4p(2)$ and $4p = 2$. Thus, an equation is $(x - 2)^2 = 2(y + 2)$. $4p = 2$, so $p = \frac{1}{2}$ and the focus is $(2, -\frac{3}{2})$ while the directrix is $y = -\frac{5}{2}$.

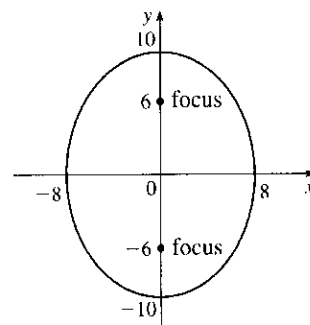
$$11. \frac{x^2}{9} + \frac{y^2}{5} = 1 \Rightarrow a = \sqrt{9} = 3, b = \sqrt{5},$$

$c = \sqrt{a^2 - b^2} = \sqrt{9 - 5} = 2$. The ellipse is centered at $(0, 0)$, with vertices at $(\pm 3, 0)$. The foci are $(\pm 2, 0)$.



$$12. \frac{x^2}{64} + \frac{y^2}{100} = 1 \Rightarrow a = \sqrt{100} = 10,$$

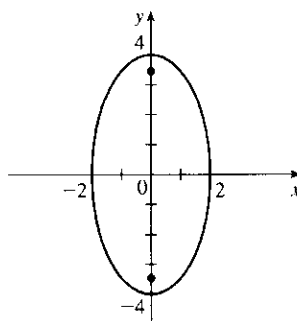
$b = \sqrt{64} = 8, c = \sqrt{a^2 - b^2} = \sqrt{100 - 64} = 6$. The ellipse is centered at $(0, 0)$, with vertices at $(0, \pm 10)$. The foci are $(0, \pm 6)$.



$$13. 4x^2 + y^2 = 16 \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1 \Rightarrow$$

$$a = \sqrt{16} = 4, b = \sqrt{4} = 2,$$

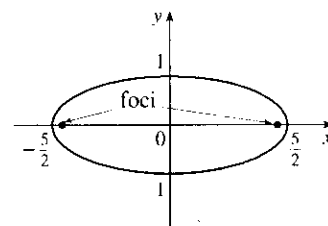
$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = 2\sqrt{3}$. The ellipse is centered at $(0, 0)$, with vertices at $(0, \pm 4)$. The foci are $(0, \pm 2\sqrt{3})$.



$$14. 4x^2 + 25y^2 = 25 \Rightarrow \frac{x^2}{25/4} + \frac{y^2}{1} = 1 \Rightarrow$$

$$a = \sqrt{\frac{25}{4}} = \frac{5}{2}, b = \sqrt{1} = 1,$$

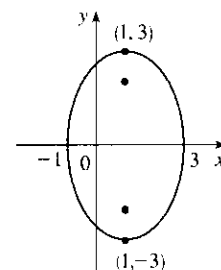
$c = \sqrt{a^2 - b^2} = \sqrt{\frac{25}{4} - 1} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$. The ellipse is centered at $(0, 0)$, with vertices at $(\pm \frac{5}{2}, 0)$. The foci are $(\pm \frac{\sqrt{21}}{2}, 0)$.



$$15. 9x^2 - 18x + 4y^2 = 27 \Leftrightarrow 9(x^2 - 2x + 1) + 4y^2 = 27 + 9 \Leftrightarrow$$

$$9(x - 1)^2 + 4y^2 = 36 \Leftrightarrow \frac{(x - 1)^2}{4} + \frac{y^2}{9} = 1 \Rightarrow a = 3, b = 2,$$

$c = \sqrt{5} \Rightarrow$ center $(1, 0)$, vertices $(1, \pm 3)$, foci $(1, \pm \sqrt{5})$



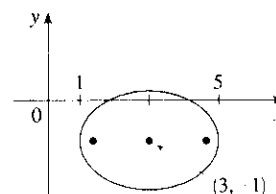
$$16. x^2 - 6x + 2y^2 + 4y = -7 \Leftrightarrow$$

$$x^2 - 6x + 9 + 2(y^2 + 2y + 1) = -7 + 9 + 2 \Leftrightarrow$$

$$(x - 3)^2 + 2(y + 1)^2 = 4 \Leftrightarrow$$

$$\frac{(x - 3)^2}{4} + \frac{(y + 1)^2}{2} = 1 \Rightarrow a = 2, b = \sqrt{2} = c \Rightarrow \text{center}$$

$(3, -1)$, vertices $(1, -1)$ and $(5, -1)$, foci $(3 \pm \sqrt{2}, -1)$

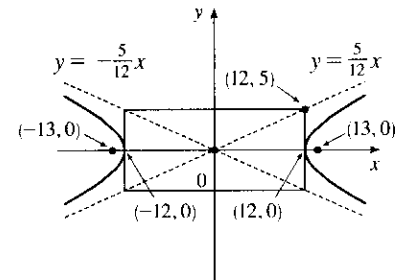


17. The center is $(0, 0)$, $a = 3$, and $b = 2$, so an equation is $\frac{x^2}{4} + \frac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $(0, \pm\sqrt{5})$.

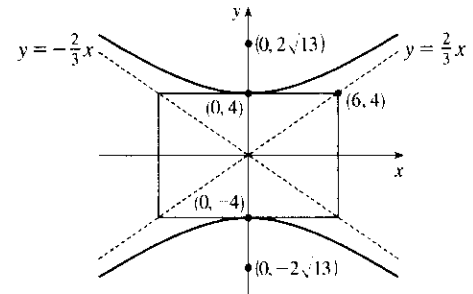
18. The ellipse is centered at $(2, 1)$, with $a = 3$ and $b = 2$. An equation is $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$.
 $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $(2 \pm \sqrt{5}, 1)$.

19. $\frac{x^2}{144} - \frac{y^2}{25} = 1 \Rightarrow a = 12, b = 5, c = \sqrt{144 + 25} = 13 \Rightarrow$
 center $(0, 0)$, vertices $(\pm 12, 0)$, foci $(\pm 13, 0)$,
 asymptotes $y = \pm \frac{5}{12}x$.

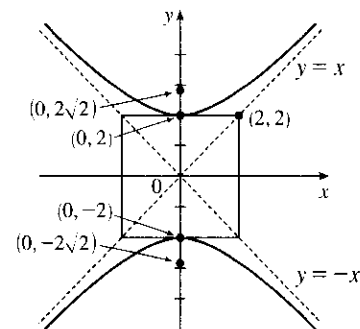
Note: It is helpful to draw a $2a$ -by- $2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



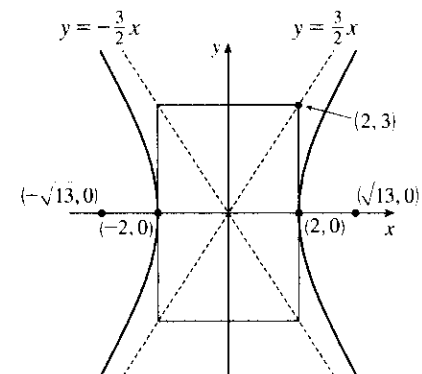
20. $\frac{y^2}{16} - \frac{x^2}{36} = 1 \Rightarrow a = 4, b = 6,$
 $c = \sqrt{a^2 + b^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$. The center is $(0, 0)$,
 the vertices are $(0, \pm 4)$, the foci are $(0, \pm 2\sqrt{13})$, and the
 asymptotes are the lines $y = \pm \frac{a}{b}x = \pm \frac{2}{3}x$.



21. $y^2 - x^2 = 4 \Leftrightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow a = \sqrt{4} = 2 = b,$
 $c = \sqrt{4 + 4} = 2\sqrt{2} \Rightarrow$ center $(0, 0)$, vertices $(0, \pm 2)$,
 foci $(0, \pm 2\sqrt{2})$, asymptotes $y = \pm x$



22. $9x^2 - 4y^2 = 36 \Leftrightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow a = \sqrt{4} = 2,$
 $b = \sqrt{9} = 3, c = \sqrt{4 + 9} = \sqrt{13} \Rightarrow$ center $(0, 0)$,
 vertices $(\pm 2, 0)$, foci $(\pm \sqrt{13}, 0)$, asymptotes $y = \pm \frac{3}{2}x$



$$23. 2y^2 - 4y - 3x^2 + 12x = -8 \Leftrightarrow$$

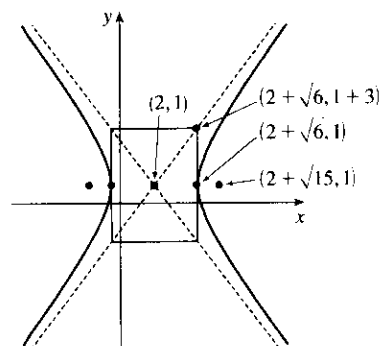
$$2(y^2 - 2y + 1) - 3(x^2 - 4x + 4) = -8 + 2 - 12 \Leftrightarrow$$

$$2(y - 1)^2 - 3(x - 2)^2 = -18 \Leftrightarrow \frac{(x - 2)^2}{6} - \frac{(y - 1)^2}{9} = 1$$

$$\Rightarrow a = \sqrt{6}, b = 3, c = \sqrt{15} \Rightarrow \text{center } (2, 1), \text{ vertices}$$

$$(2 \pm \sqrt{6}, 1), \text{ foci } (2 \pm \sqrt{15}, 1), \text{ asymptotes } y - 1 = \pm \frac{3}{\sqrt{6}}(x - 2)$$

$$\text{or } y - 1 = \pm \frac{\sqrt{6}}{2}(x - 2)$$



$$24. 16x^2 + 64x - 9y^2 - 90y = 305 \Leftrightarrow$$

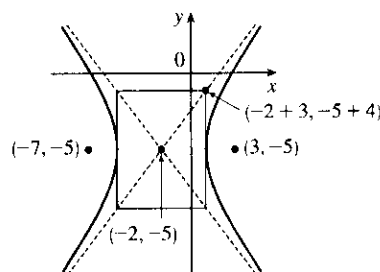
$$16(x^2 + 4x + 4) - 9(y^2 + 10y + 25) = 305 + 64 - 225 \Leftrightarrow$$

$$16(x + 2)^2 - 9(y + 5)^2 = 144 \Leftrightarrow \frac{(x + 2)^2}{9} - \frac{(y + 5)^2}{16} = 1$$

$$\Rightarrow a = 3, b = 4, c = 5 \Rightarrow \text{center } (-2, -5), \text{ vertices } (-5, -5)$$

$$\text{and } (1, -5), \text{ foci } (-7, -5) \text{ and } (3, -5), \text{ asymptotes}$$

$$y + 5 = \pm \frac{4}{3}(x + 2)$$



$$25. x^2 = y + 1 \Leftrightarrow x^2 = 1(y + 1). \text{ This is an equation of a parabola with } 4p = 1, \text{ so } p = \frac{1}{4}. \text{ The vertex is } (0, -1) \text{ and the focus is } (0, -\frac{3}{4}).$$

$$26. x^2 = y^2 + 1 \Leftrightarrow x^2 - y^2 = 1. \text{ This is an equation of a hyperbola with vertices } (\pm 1, 0). \text{ The foci are at } (\pm\sqrt{1+1}, 0) = (\pm\sqrt{2}, 0).$$

$$27. x^2 = 4y - 2y^2 \Leftrightarrow x^2 + 2y^2 - 4y = 0 \Leftrightarrow x^2 + 2(y^2 - 2y + 1) = 2 \Leftrightarrow x^2 + 2(y - 1)^2 = 2 \Leftrightarrow \frac{x^2}{2} + \frac{(y - 1)^2}{1} = 1. \text{ This is an equation of an ellipse with vertices at } (\pm\sqrt{2}, 1). \text{ The foci are at } (\pm\sqrt{2-1}, 1) = (\pm 1, 1).$$

$$28. y^2 - 8y = 6x - 16 \Leftrightarrow y^2 - 8y + 16 = 6x \Leftrightarrow (y - 4)^2 = 6x. \text{ This is an equation of a parabola with } 4p = 6, \text{ so } p = \frac{3}{2}. \text{ The vertex is } (0, 4) \text{ and the focus is } (\frac{3}{2}, 4).$$

$$29. y^2 + 2y = 4x^2 + 3 \Leftrightarrow y^2 + 2y + 1 = 4x^2 + 4 \Leftrightarrow (y + 1)^2 - 4x^2 = 4 \Leftrightarrow \frac{(y + 1)^2}{4} - x^2 = 1. \text{ This is an equation of a hyperbola with vertices } (0, -1 \pm 2) = (0, 1) \text{ and } (0, -3). \text{ The foci are at } (0, -1 \pm \sqrt{4+1}) = (0, -1 \pm \sqrt{5}).$$

$$30. 4x^2 + 4x + y^2 = 0 \Leftrightarrow 4(x^2 + x + \frac{1}{4}) + y^2 = 1 \Leftrightarrow 4(x + \frac{1}{2})^2 + y^2 = 1 \Leftrightarrow \frac{(x + \frac{1}{2})^2}{1/4} + y^2 = 1. \text{ This is an equation of an ellipse with vertices } (-\frac{1}{2}, 0 \pm 1) = (-\frac{1}{2}, \pm 1). \text{ The foci are at } (-\frac{1}{2}, 0 \pm \sqrt{1 - \frac{1}{4}}) = (-\frac{1}{2}, \pm \sqrt{3}/2).$$

$$31. \text{ The parabola with vertex } (0, 0) \text{ and focus } (0, -2) \text{ opens downward and has } p = -2, \text{ so its equation is } x^2 = 4py = -8y.$$

$$32. \text{ The parabola with vertex } (1, 0) \text{ and directrix } x = -5 \text{ opens to the right and has } p = 6, \text{ so its equation is } y^2 = 4p(x - 1) = 24(x - 1).$$

33. The distance from the focus $(-4, 0)$ to the directrix $x = 2$ is $2 - (-4) = 6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1, 0)$. Since the focus is to the left of the vertex, $p = -3$. An equation is $y^2 = 4p(x + 1) \Rightarrow y^2 = -12(x + 1)$.
34. The distance from the focus $(3, 6)$ to the vertex $(3, 2)$ is $6 - 2 = 4$. Since the focus is above the vertex, $p = 4$. An equation is $(x - 3)^2 = 4p(y - 2) \Rightarrow (x - 3)^2 = 16(y - 2)$.
35. The parabola must have equation $y^2 = 4px$, so $(-4)^2 = 4p(1) \Rightarrow p = 4 \Rightarrow y^2 = 16x$.
36. Vertical axis $\Rightarrow (x - h)^2 = 4p(y - k)$. Substituting $(-2, 3)$ and $(0, 3)$ gives $(-2 - h)^2 = 4p(3 - k)$ and $(-h)^2 = 4p(3 - k) \Rightarrow (-2 - h)^2 = (-h)^2 \Rightarrow 4 + 4h + h^2 = h^2 \Rightarrow h = -1 \Rightarrow 1 = 4p(3 - k)$. Substituting $(1, 9)$ gives $[1 - (-1)]^2 = 4p(9 - k) \Rightarrow 4 = 4p(9 - k)$. Solving for p from these equations gives $p = \frac{1}{4(3 - k)} = \frac{1}{9 - k} \Rightarrow 4(3 - k) = 9 - k \Rightarrow k = 1 \Rightarrow p = \frac{1}{8} \Rightarrow (x + 1)^2 = \frac{1}{2}(y - 1) \Rightarrow 2x^2 + 4x - y + 3 = 0$.
37. The ellipse with foci $(\pm 2, 0)$ and vertices $(\pm 5, 0)$ has center $(0, 0)$ and a horizontal major axis, with $a = 5$ and $c = 2$, so $b = \sqrt{a^2 - c^2} = \sqrt{21}$. An equation is $\frac{x^2}{25} + \frac{y^2}{21} = 1$.
38. The ellipse with foci $(0, \pm 5)$ and vertices $(0, \pm 13)$ has center $(0, 0)$ and a vertical major axis, with $c = 5$ and $a = 13$, so $b = \sqrt{a^2 - c^2} = 12$. An equation is $\frac{x^2}{144} + \frac{y^2}{169} = 1$.
39. Since the vertices are $(0, 0)$ and $(0, 8)$, the ellipse has center $(0, 4)$ with a vertical axis and $a = 4$. The foci at $(0, 2)$ and $(0, 6)$ are 2 units from the center, so $c = 2$ and $b = \sqrt{a^2 - c^2} = \sqrt{4^2 - 2^2} = \sqrt{12}$. An equation is $\frac{(x - 0)^2}{b^2} + \frac{(y - 4)^2}{a^2} = 1 \Rightarrow \frac{x^2}{12} + \frac{(y - 4)^2}{16} = 1$.
40. Since the foci are $(0, -1)$ and $(8, -1)$, the ellipse has center $(4, -1)$ with a horizontal axis and $c = 4$. The vertex $(9, -1)$ is 5 units from the center, so $a = 5$ and $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 4^2} = \sqrt{9}$. An equation is $\frac{(x - 4)^2}{a^2} + \frac{(y + 1)^2}{b^2} = 1 \Rightarrow \frac{(x - 4)^2}{25} + \frac{(y + 1)^2}{9} = 1$.
41. Center $(2, 2)$, $c = 2$, $a = 3 \Rightarrow b = \sqrt{5} \Rightarrow \frac{1}{9}(x - 2)^2 + \frac{1}{5}(y - 2)^2 = 1$
42. Center $(0, 0)$, $c = 2$, major axis horizontal $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $b^2 = a^2 - c^2 = a^2 - 4$. Since the ellipse passes through $(2, 1)$, we have $2a = |PF_1| + |PF_2| = \sqrt{17} + 1 \Rightarrow a^2 = \frac{9 + \sqrt{17}}{2}$ and $b^2 = \frac{1 + \sqrt{17}}{2}$, so the ellipse has equation $\frac{2x^2}{9 + \sqrt{17}} + \frac{2y^2}{1 + \sqrt{17}} = 1$.
43. Center $(0, 0)$, vertical axis, $c = 3$, $a = 1 \Rightarrow b = \sqrt{8} = 2\sqrt{2} \Rightarrow y^2 - \frac{1}{8}x^2 = 1$
44. Center $(0, 0)$, horizontal axis, $c = 6$, $a = 4 \Rightarrow b = 2\sqrt{5} \Rightarrow \frac{1}{16}x^2 - \frac{1}{20}y^2 = 1$
45. Center $(4, 3)$, horizontal axis, $c = 3$, $a = 2 \Rightarrow b = \sqrt{5} \Rightarrow \frac{1}{4}(x - 4)^2 - \frac{1}{5}(y - 3)^2 = 1$
46. Center $(2, 3)$, vertical axis, $c = 5$, $a = 3 \Rightarrow b = 4 \Rightarrow \frac{1}{9}(y - 3)^2 - \frac{1}{16}(x - 2)^2 = 1$
47. Center $(0, 0)$, horizontal axis, $a = 3$, $\frac{b}{a} = 2 \Rightarrow b = 6 \Rightarrow \frac{1}{9}x^2 - \frac{1}{36}y^2 = 1$
48. Center $(4, 2)$, horizontal axis, asymptotes $y - 2 = \pm(x - 4) \Rightarrow c = 2$, $b/a = 1 \Rightarrow a = b \Rightarrow c^2 = 4 = a^2 + b^2 = 2a^2 \Rightarrow a^2 = 2 \Rightarrow \frac{1}{2}(x - 4)^2 - \frac{1}{2}(y - 2)^2 = 1$

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a - c$ from it) while the farthest point is the other vertex (at a distance of $a + c$). So for this lunar orbit,

$$(a - c) + (a + c) = 2a = (1728 + 110) + (1728 + 314), \text{ or } a = 1940; \text{ and}$$

$$(a + c) - (a - c) = 2c = 314 - 110, \text{ or } c = 102. \text{ Thus, } b^2 = a^2 - c^2 = 3,753,196, \text{ and the equation is}$$

$$\frac{x^2}{3,763,600} + \frac{y^2}{3,753,196} = 1.$$

50. (a) Choose V to be the origin, with x -axis through V and F . Then F is $(p, 0)$, A is $(p, 5)$, so substituting A into the equation $y^2 = 4px$ gives $25 = 4p^2$ so $p = \frac{5}{2}$ and $y^2 = 10x$.

$$(b) x = 11 \Rightarrow y = \sqrt{110} \Rightarrow |CD| = 2\sqrt{110}$$

51. (a) Set up the coordinate system so that A is $(-200, 0)$ and B is $(200, 0)$.

$$|PA| - |PB| = (1200)(980) = 1,176,000 \text{ ft} = \frac{2450}{11} \text{ mi} = 2a \Rightarrow a = \frac{1225}{11}, \text{ and } c = 200 \text{ so}$$

$$b^2 = c^2 - a^2 = \frac{3,339,375}{121} \Rightarrow \frac{121x^2}{1,500,625} - \frac{121y^2}{3,339,375} = 1.$$

$$(b) \text{ Due north of } B \Rightarrow x = 200 \Rightarrow \frac{(121)(200)^2}{1,500,625} - \frac{121y^2}{3,339,375} = 1 \Rightarrow y = \frac{133,575}{539} \approx 248 \text{ mi}$$

52. $|PF_1| - |PF_2| = \pm 2a \Leftrightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \Leftrightarrow$

$$\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a \Leftrightarrow (x+c)^2 + y^2 = (x-c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2}$$

$$\Leftrightarrow 4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2} \Leftrightarrow c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2) \Leftrightarrow$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \Leftrightarrow b^2x^2 - a^2y^2 = a^2b^2 \text{ (where } b^2 = c^2 - a^2) \Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

53. The function whose graph is the upper branch of this hyperbola is concave upward. The

$$\text{function is } y = f(x) = a\sqrt{1 + \frac{x^2}{b^2}} = \frac{a}{b}\sqrt{b^2 + x^2}, \text{ so } y' = \frac{a}{b}x(b^2 + x^2)^{-1/2} \text{ and}$$

$$y'' = \frac{a}{b}[(b^2 + x^2)^{-1/2} - x^2(b^2 + x^2)^{-3/2}] = ab(b^2 + x^2)^{-3/2} > 0 \text{ for all } x, \text{ and so } f \text{ is concave upward.}$$

54. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1, 1)$ and $(-1, -1)$ in the distance formula (first equation of that derivation) so

$$\sqrt{(x-1)^2 + (y-1)^2} + \sqrt{(x+1)^2 + (y+1)^2} = 4 \text{ will lead (after moving the second term to the right, squaring, and simplifying) to } 2\sqrt{(x+1)^2 + (y+1)^2} = x + y + 4, \text{ which, after squaring and simplifying again, leads to } 3x^2 - 2xy + 3y^2 = 8.$$

55. (a) If $k > 16$, then $k - 16 > 0$, and $\frac{x^2}{k} + \frac{y^2}{k-16} = 1$ is an *ellipse* since it is the sum of two squares on the left side.

$$(b) \text{ If } 0 < k < 16, \text{ then } k - 16 < 0, \text{ and } \frac{x^2}{k} + \frac{y^2}{k-16} = 1 \text{ is a } \textit{hyperbola} \text{ since it is the difference of two squares on the left side.}$$

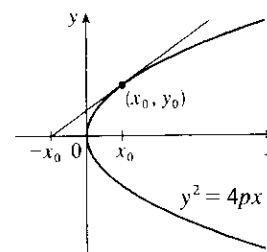
$$(c) \text{ If } k < 0, \text{ then } k - 16 < 0, \text{ and there is } \textit{no curve} \text{ since the left side is the sum of two negative terms, which cannot equal 1.}$$

$$(d) \text{ In case (a), } a^2 = k, b^2 = k - 16, \text{ and } c^2 = a^2 - b^2 = 16, \text{ so the foci are at } (\pm 4, 0). \text{ In case (b), } k - 16 < 0, \text{ so } a^2 = k, b^2 = 16 - k, \text{ and } c^2 = a^2 + b^2 = 16, \text{ and so again the foci are at } (\pm 4, 0).$$

56. (a) $y^2 = 4px \Rightarrow 2yy' = 4p \Rightarrow y' = \frac{2p}{y}$, so the tangent line is

$$y - y_0 = \frac{2p}{y_0}(x - x_0) \Rightarrow yy_0 - y_0^2 = 2p(x - x_0) \Leftrightarrow$$

$$yy_0 - 4px_0 = 2px - 2px_0 \Rightarrow yy_0 = 2p(x + x_0).$$



- (b) The x -intercept is $-x_0$.

57. Use the parametrization $x = 2 \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$ to get

$$L = 4 \int_0^{\pi/2} \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = 4 \int_0^{\pi/2} \sqrt{4 \sin^2 t + \cos^2 t} dt = 4 \int_0^{\pi/2} \sqrt{3 \sin^2 t + 1} dt$$

Using Simpson's Rule with $n = 10$, $\Delta t = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$, and $f(t) = \sqrt{3 \sin^2 t + 1}$, we get

$$L \approx \frac{4}{3} \left(\frac{\pi}{20} \right) \left[f(0) + 4f\left(\frac{\pi}{20}\right) + 2f\left(\frac{2\pi}{20}\right) + \cdots + 2f\left(\frac{8\pi}{20}\right) + 4f\left(\frac{9\pi}{20}\right) + f\left(\frac{\pi}{2}\right) \right] \approx 9.69$$

58. The length of the major axis is $2a$, so $a = \frac{1}{2}(1.18 \times 10^{10}) = 5.9 \times 10^9$. The length of the minor axis is $2b$, so

$b = \frac{1}{2}(1.14 \times 10^{10}) = 5.7 \times 10^9$. An equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or converting into parametric equations, $x = a \cos \theta$ and $y = b \sin \theta$. So

$$L = 4 \int_0^{\pi/2} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta = 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta$$

Using Simpson's Rule with $n = 10$, $\Delta \theta = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$, and $f(\theta) = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$, we get

$$L \approx 4 \cdot S_{10}$$

$$= 4 \cdot \frac{\pi}{20 \cdot 3} \left[f(0) + 4f\left(\frac{\pi}{20}\right) + 2f\left(\frac{2\pi}{20}\right) + \cdots + 2f\left(\frac{8\pi}{20}\right) + 4f\left(\frac{9\pi}{20}\right) + f\left(\frac{\pi}{2}\right) \right]$$

$$\approx 3.64 \times 10^{10} \text{ km}$$

59. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow y' = -\frac{b^2 x}{a^2 y}$ ($y \neq 0$). Thus, the slope of the tangent line at P is $-\frac{b^2 x_1}{a^2 y_1}$. The slope of $F_1 P$ is $\frac{y_1}{x_1 + c}$ and of $F_2 P$ is $\frac{y_1}{x_1 - c}$. By the formula from Problems Plus, we have

$$\tan \alpha = \frac{\frac{y_1}{x_1 + c} + \frac{b^2 x_1}{a^2 y_1}}{1 - \frac{b^2 x_1 y_1}{a^2 y_1 (x_1 + c)}} = \frac{a^2 y_1^2 + b^2 x_1 (x_1 + c)}{a^2 y_1 (x_1 + c) - b^2 x_1 y_1} = \frac{a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 + a^2 c y_1} \left[\begin{array}{l} \text{using } b^2 x_1^2 + a^2 y_1^2 = a^2 b^2 \\ \text{and } a^2 - b^2 = c^2 \end{array} \right]$$

$$= \frac{b^2 (c x_1 + a^2)}{c y_1 (c x_1 + a^2)} = \frac{b^2}{c y_1}$$

and

$$\tan \beta = \frac{-\frac{y_1}{x_1 - c} - \frac{b^2 x_1}{a^2 y_1}}{1 - \frac{b^2 x_1 y_1}{a^2 y_1 (x_1 - c)}} = \frac{-a^2 y_1^2 - b^2 x_1 (x_1 - c)}{a^2 y_1 (x_1 - c) - b^2 x_1 y_1} = \frac{-a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 - a^2 c y_1} = \frac{b^2 (c x_1 - a^2)}{c y_1 (c x_1 - a^2)} = \frac{b^2}{c y_1}$$

So $\alpha = \beta$.

60. The slopes of the line segments F_1P and F_2P are $\frac{y_1}{x_1 + c}$ and $\frac{y_1}{x_1 - c}$, where P is (x_1, y_1) . Differentiating implicitly, $\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{b^2x}{a^2y} \Rightarrow$ the slope of the tangent at P is $\frac{b^2x_1}{a^2y_1}$, so by the formula from Problems Plus,

$$\begin{aligned}\tan \alpha &= \frac{\frac{b^2x_1}{a^2y_1} - \frac{y_1}{x_1 + c}}{1 + \frac{b^2x_1y_1}{a^2y_1(x_1 + c)}} = \frac{b^2x_1(x_1 + c) - a^2y_1^2}{a^2y_1(x_1 + c) + b^2x_1y_1} \\ &= \frac{b^2(cx_1 + a^2)}{cy_1(cx_1 + a^2)} \left[\begin{array}{l} \text{using } x_1^2/a^2 - y_1^2/b^2 = 1 \\ \text{and } a^2 + b^2 = c^2 \end{array} \right] = \frac{b^2}{cy_1}\end{aligned}$$

and

$$\tan \beta = \frac{-\frac{b^2x_1}{a^2y_1} + \frac{y_1}{x_1 - c}}{1 + \frac{b^2x_1y_1}{a^2y_1(x_1 - c)}} = \frac{-b^2x_1(x_1 - c) + a^2y_1^2}{a^2y_1(x_1 - c) + b^2x_1y_1} = \frac{b^2(cx_1 - a^2)}{cy_1(cx_1 - a^2)} = \frac{b^2}{cy_1}$$

So $\alpha = \beta$.

11.6 Conic Sections in Polar Coordinates

1. The directrix $y = 6$ is above the focus at the origin, so we use the form with “ $+e \sin \theta$ ” in the denominator. (See

Theorem 6 and Figure 2.) $r = \frac{ed}{1 + e \sin \theta} = \frac{\frac{7}{4} \cdot 6}{1 + \frac{7}{4} \sin \theta} = \frac{42}{4 + 7 \sin \theta}$

2. The directrix $x = 4$ is to the right of the focus at the origin, so we use the form with “ $+e \cos \theta$ ” in the denominator.

$e = 1$ for a parabola, so an equation is $r = \frac{ed}{1 + e \cos \theta} = \frac{1 \cdot 4}{1 + 1 \cos \theta} = \frac{4}{1 + \cos \theta}$

3. The directrix $x = -5$ is to the left of the focus at the origin, so we use the form with “ $-e \cos \theta$ ” in the

denominator. $r = \frac{ed}{1 - e \cos \theta} = \frac{\frac{3}{4} \cdot 5}{1 - \frac{3}{4} \cos \theta} = \frac{15}{4 - 3 \cos \theta}$

4. The directrix $y = -2$ is below the focus at the origin, so we use the form with “ $-e \sin \theta$ ” in the denominator.

$$r = \frac{ed}{1 - e \sin \theta} = \frac{2 \cdot 2}{1 - 2 \sin \theta} = \frac{4}{1 - 2 \sin \theta}$$

5. The vertex $(4, 3\pi/2)$ is 4 units below the focus at the origin, so the directrix is 8 units below the focus ($d = 8$), and we use the form with “ $-e \sin \theta$ ” in the denominator. $e = 1$ for a parabola, so an equation is

$$r = \frac{cd}{1 - e \sin \theta} = \frac{1(8)}{1 - 1 \sin \theta} = \frac{8}{1 - \sin \theta}.$$

6. The vertex $P(1, \pi/2)$ is 1 unit above the focus F at the origin, so $|PF| = 1$ and we use the form with “ $+e \sin \theta$ ” in the denominator. The distance from the focus to the directrix l is d , so

$$e = \frac{|PF|}{|Pl|} \Rightarrow 0.8 = \frac{1}{d-1} \Rightarrow 0.8d - 0.8 = 1 \Rightarrow 0.8d = 1.8 \Rightarrow d = 2.25.$$

$$\text{An equation is } r = \frac{ed}{1 + e \sin \theta} = \frac{0.8(2.25)}{1 + 0.8 \sin \theta} \cdot \frac{5}{5} = \frac{9}{5 + 4 \sin \theta}.$$

7. The directrix $r = 4 \sec \theta$ (equivalent to $r \cos \theta = 4$ or $x = 4$) is to the right of the focus at the origin, so we will use the form with “ $+e \cos \theta$ ” in the denominator. The distance from the focus to the directrix is $d = 4$, so an equation is

$$r = \frac{ed}{1 + e \cos \theta} = \frac{0.5(4)}{1 + 0.5 \cos \theta} \cdot \frac{2}{2} = \frac{4}{2 + \cos \theta}.$$

8. The directrix $r = -6 \csc \theta$ (equivalent to $r \sin \theta = -6$ or $y = -6$) is below the focus at the origin, so we will use the form with “ $-e \sin \theta$ ” in the denominator. The distance from the focus to the directrix is $d = 6$, so an equation is

$$r = \frac{ed}{1 - e \sin \theta} = \frac{3(6)}{1 - 3 \sin \theta} = \frac{18}{1 - 3 \sin \theta}.$$

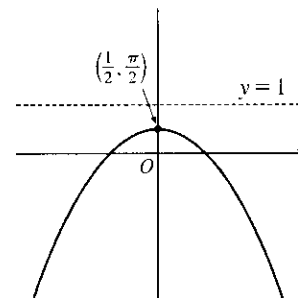
9. $r = \frac{1}{1 + \sin \theta} = \frac{ed}{1 + e \sin \theta}$, where $d = e = 1$.

(a) Eccentricity = $e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since “ $+e \sin \theta$ ” appears in the denominator, the directrix is above the focus at the origin. $d = |Fl| = 1$, so an equation of the directrix is $y = 1$.

(d) The vertex is at $(\frac{1}{2}, \frac{\pi}{2})$, midway between the focus and the directrix.



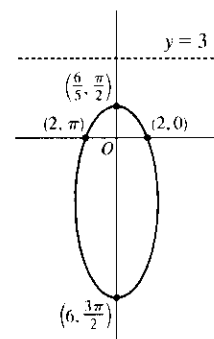
10. $r = \frac{6}{3 + 2 \sin \theta} = \frac{2}{1 + \frac{2}{3} \sin \theta} = \frac{\frac{2}{3} \cdot 3}{1 + \frac{2}{3} \sin \theta}$

(a) $e = \frac{2}{3}$

(b) Ellipse

(c) $y = 3$

(d) Vertices $(\frac{6}{5}, \frac{\pi}{2})$ and $(6, \frac{3\pi}{2})$; center $(\frac{12}{5}, \frac{3\pi}{2})$



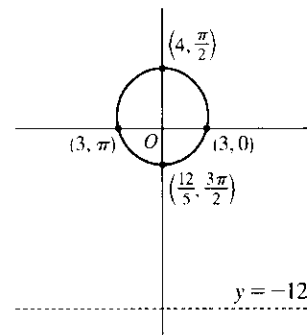
11. $r = \frac{12}{4 - \sin \theta} \cdot \frac{1/4}{1/4} = \frac{3}{1 - \frac{1}{4} \sin \theta}$, where $e = \frac{1}{4}$ and $ed = 3 \Rightarrow d = 12$.

(a) Eccentricity = $e = \frac{1}{4}$

(b) Since $e = \frac{1}{4} < 1$, the conic is an ellipse.

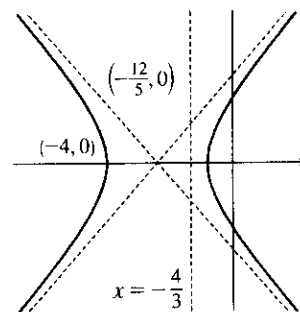
(c) Since “ $-e \sin \theta$ ” appears in the denominator, the directrix is below the focus at the origin. $d = |Fl| = 12$, so an equation of the directrix is $y = -12$.

(d) The vertices are $(4, \frac{\pi}{2})$ and $(\frac{12}{5}, \frac{3\pi}{2})$, so the center is midway between them, that is, $(\frac{4}{5}, \frac{\pi}{2})$.



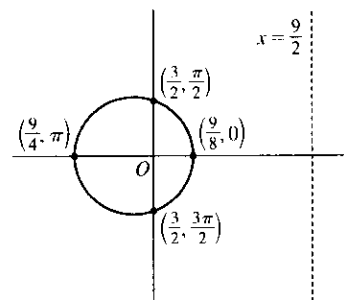
$$12. r = \frac{4}{2 - 3 \cos \theta} = \frac{2}{1 - \frac{3}{2} \cos \theta} = \frac{\frac{3}{2} \cdot \frac{4}{3}}{1 - \frac{3}{2} \cos \theta}$$

- (a) $e = \frac{3}{2}$
 (b) Hyperbola
 (c) $x = -\frac{4}{3}$
 (d) The vertices are $(-4, 0)$ and $(\frac{4}{5}, \pi) = (-\frac{4}{5}, 0)$, so the center is $(-\frac{12}{5}, 0)$. The asymptotes are parallel to $\theta = \pm \cos^{-1} \frac{2}{3}$. [Their slopes are $\pm \tan(\cos^{-1} \frac{2}{3}) = \pm \frac{\sqrt{5}}{2}$.]



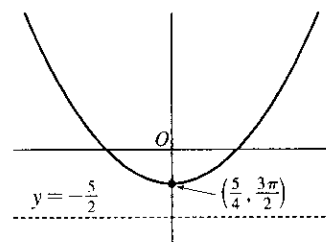
$$13. r = \frac{9}{6 + 2 \cos \theta} \cdot \frac{1/6}{1/6} = \frac{3/2}{1 + \frac{1}{3} \cos \theta}, \text{ where } e = \frac{1}{3} \text{ and } ed = \frac{3}{2} \Rightarrow d = \frac{9}{2}.$$

- (a) Eccentricity = $e = \frac{1}{3}$
 (b) Since $e = \frac{1}{3} < 1$, the conic is an ellipse.
 (c) Since “ $+e \cos \theta$ ” appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{9}{2}$, so an equation of the directrix is $x = \frac{9}{2}$.
 (d) The vertices are $(\frac{9}{8}, 0)$ and $(\frac{9}{4}, \pi)$, so the center is midway between them, that is, $(\frac{9}{16}, \pi)$.



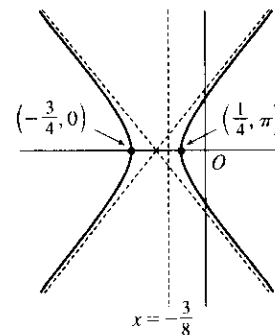
$$14. r = \frac{5}{2 - 2 \sin \theta} = \frac{\frac{5}{2}}{1 - \sin \theta}$$

- (a) $e = 1$
 (b) Parabola
 (c) $y = -\frac{5}{2}$
 (d) The focus is $(0, 0)$, so the vertex is $(\frac{5}{4}, \frac{3\pi}{2})$ and the parabola opens up.



$$15. r = \frac{3}{4 - 8 \cos \theta} \cdot \frac{1/4}{1/4} = \frac{3/4}{1 - 2 \cos \theta}, \text{ where } e = 2 \text{ and } ed = \frac{3}{4} \Rightarrow d = \frac{3}{8}.$$

- (a) Eccentricity = $e = 2$
 (b) Since $e = 2 > 1$, the conic is a hyperbola.
 (c) Since “ $-e \cos \theta$ ” appears in the denominator, the directrix is to the left of the focus at the origin. $d = |Fl| = \frac{3}{8}$, so an equation of the directrix is $x = -\frac{3}{8}$.
 (d) The vertices are $(-\frac{3}{4}, 0)$ and $(\frac{1}{4}, \pi)$, so the center is midway between them, that is, $(\frac{1}{2}, \pi)$.



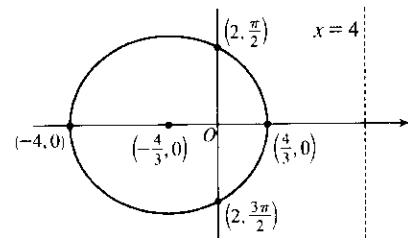
$$16. r = \frac{4}{2 + \cos \theta} = \frac{2}{1 + \frac{1}{2} \cos \theta} = \frac{\frac{1}{2} \cdot 4}{1 + \frac{1}{2} \cos \theta}$$

$$(a) e = \frac{1}{2}$$

(b) Ellipse

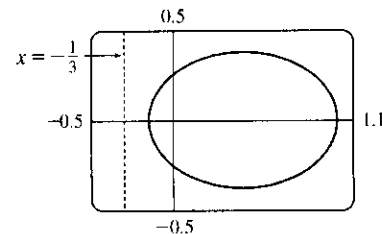
$$(c) x = 4$$

(d) The vertices are $(\frac{4}{3}, 0)$ and $(4, \pi) = (-4, 0)$, so the center is $(-\frac{4}{3}, 0)$.

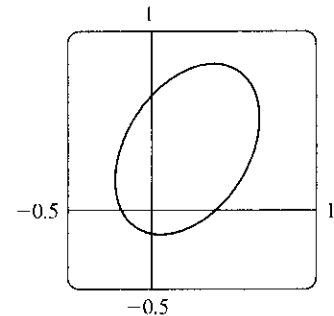


$$17. (a) \text{ The equation is } r = \frac{1}{4 - 3 \cos \theta} = \frac{1/4}{1 - \frac{3}{4} \cos \theta}, \text{ so } e = \frac{3}{4} \text{ and}$$

$ed = \frac{1}{4} \Rightarrow d = \frac{1}{3}$. The conic is an ellipse, and the equation of its directrix is $x = r \cos \theta = -\frac{1}{3} \Rightarrow r = -\frac{1}{3 \cos \theta}$. We must be careful in our choice of parameter values in this equation ($-1 \leq \theta \leq 1$ works well).



(b) The equation is obtained by replacing θ with $\theta - \frac{\pi}{3}$ in the equation of the original conic (see Example 4), so $r = \frac{1}{4 - 3 \cos(\theta - \frac{\pi}{3})}$.

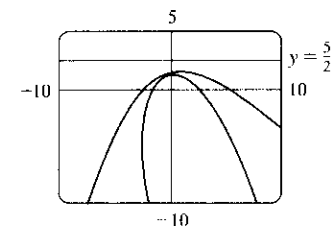


$$18. r = \frac{5}{2 + 2 \sin \theta} = \frac{5/2}{1 + \sin \theta}, \text{ so } e = 1 \text{ and } d = \frac{5}{2}. \text{ The equation of the}$$

directrix is $y = r \sin \theta = \frac{5}{2} \Rightarrow r = \frac{5}{2 \sin \theta}$. If the parabola is rotated about its focus (the origin) through $\frac{\pi}{6}$, its equation is the same as that of the original, with θ replaced by $\theta - \frac{\pi}{6}$ (see Example 4), so

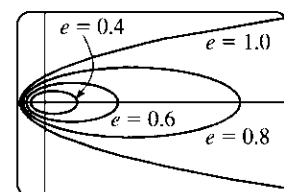
$$r = \frac{5}{2 + 2 \sin(\theta - \pi/6)}. \text{ In graphing each of these curves, we must be}$$

careful to select parameter ranges which prevent the denominator from vanishing while still showing enough of the curve.

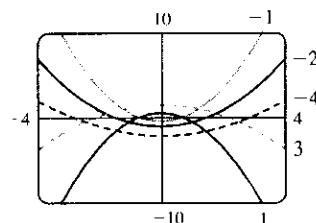


19. For $e < 1$ the curve is an ellipse. It is nearly circular when e is close to 0.

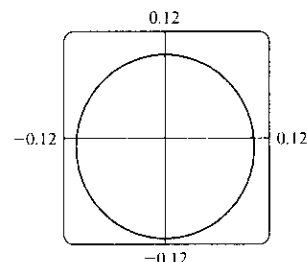
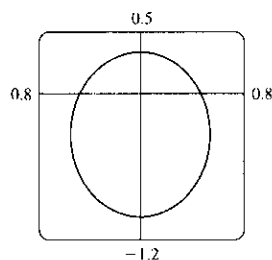
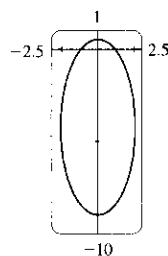
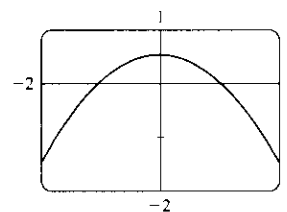
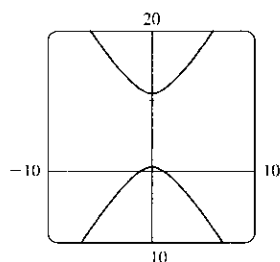
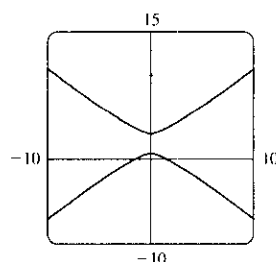
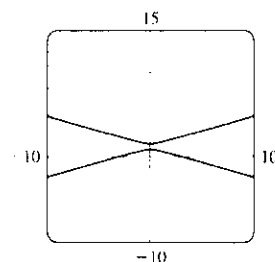
As e increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At $e = 1$, the curve becomes a parabola with focus at the origin.



20. (a) The value of d does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d < 0$ it opens upward, for $d > 0$ it opens downward).

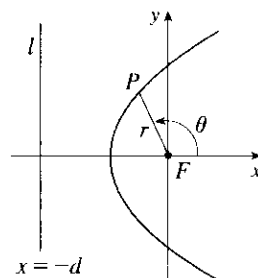


- (b) We consider only positive values of e . When $0 < e < 1$, the conic is an ellipse. As $e \rightarrow 0^+$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e = 1$ it turns into a parabola. For $e > 1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.

 $e = 0.1$  $e = 0.5$  $e = 0.9$  $e = 1$  $e = 1.1$  $e = 1.5$  $e = 10$

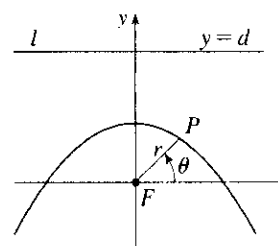
$$21. |PF| = e|Pl| \Rightarrow r = e[d - r \cos(\pi - \theta)] = e(d + r \cos \theta) \Rightarrow$$

$$r(1 - e \cos \theta) = ed \Rightarrow r = \frac{ed}{1 - e \cos \theta}$$



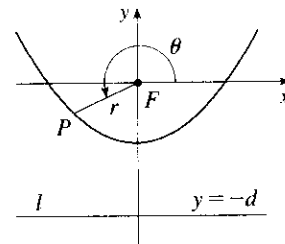
$$22. |PF| = e|Pl| \Rightarrow r = e[d - r \sin \theta] \Rightarrow r(1 + e \sin \theta) = ed \Rightarrow$$

$$r = \frac{ed}{1 + e \sin \theta}$$



$$23. |PF| = e|Pl| \Rightarrow r = e[d - r \sin(\theta - \pi)] = e(d + r \sin \theta) \Rightarrow$$

$$r(1 - e \sin \theta) = ed \Rightarrow r = \frac{ed}{1 - e \sin \theta}$$



$$24. \text{ The parabolas intersect at the two points where } \frac{c}{1 + \cos \theta} = \frac{d}{1 - \cos \theta} \Rightarrow \cos \theta = \frac{c - d}{c + d} \Rightarrow r = \frac{c + d}{2}.$$

For the first parabola, $\frac{dr}{d\theta} = \frac{c \sin \theta}{(1 + \cos \theta)^2}$, so

$$\frac{dy}{dx} = \frac{(dr/d\theta) \sin \theta + r \cos \theta}{(dr/d\theta) \cos \theta - r \sin \theta} = \frac{c \sin^2 \theta + c \cos \theta (1 + \cos \theta)}{c \sin \theta \cos \theta - c \sin \theta (1 + \cos \theta)} = \frac{1 + \cos \theta}{-\sin \theta}$$

and similarly for the second, $\frac{dy}{dx} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$. Since the product of these slopes is -1 , the parabolas intersect at right angles.

$$25. (a) \text{ If the directrix is } x = -d, \text{ then } r = \frac{ed}{1 - e \cos \theta} \text{ [see Figure 2(b)], and, from (4), } a^2 = \frac{e^2 d^2}{(1 - e^2)^2} \Rightarrow$$

$$ed = a(1 - e^2). \text{ Therefore, } r = \frac{a(1 - e^2)}{1 - e \cos \theta}.$$

$$(b) e = 0.017 \text{ and the major axis} = 2a = 2.99 \times 10^8 \Rightarrow a = 1.495 \times 10^8.$$

$$\text{Therefore } r = \frac{1.495 \times 10^8 [1 - (0.017)^2]}{1 - 0.017 \cos \theta} \approx \frac{1.49 \times 10^8}{1 - 0.017 \cos \theta}.$$

$$26. (a) \text{ The Sun is at point } F \text{ in Figure 1 so that perihelion is in the positive } x\text{-direction and aphelion is in the negative } x\text{-direction. At perihelion, } \theta = 0, \text{ so } r = \frac{a(1 - e^2)}{1 + e \cos 0} = \frac{a(1 - e)(1 + e)}{1 + e} = a(1 - e).$$

$$\text{At aphelion, } \theta = \pi, \text{ so } r = \frac{a(1 - e^2)}{1 + e \cos \pi} = \frac{a(1 - e)(1 + e)}{1 - e} = a(1 + e).$$

$$(b) \text{ At perihelion, } r = a(1 - e) \approx (1.495 \times 10^8)(1 - 0.017) \approx 1.47 \times 10^8 \text{ km.}$$

$$\text{At aphelion, } r = a(1 + e) \approx (1.495 \times 10^8)(1 + 0.017) \approx 1.52 \times 10^8 \text{ km.}$$

$$27. \text{ Here } 2a = \text{length of major axis} = 36.18 \text{ AU} \Rightarrow a = 18.09 \text{ AU and } e = 0.97. \text{ By Exercise 25(a), the equation of the orbit is } r = \frac{18.09[1 - (0.97)^2]}{1 - 0.97 \cos \theta} \approx \frac{1.07}{1 - 0.97 \cos \theta}. \text{ By Exercise 26(a), the maximum distance from the comet to the sun is } 18.09(1 + 0.97) \approx 35.64 \text{ AU or about 3.314 billion miles.}$$

$$28. \text{ Here } 2a = \text{length of major axis} = 356.5 \text{ AU} \Rightarrow a = 178.25 \text{ AU and } e = 0.9951. \text{ By Exercise 25(a), the equation of the orbit is } r = \frac{178.25[1 - (0.9951)^2]}{1 - 0.9951 \cos \theta} \approx \frac{1.7426}{1 - 0.9951 \cos \theta}. \text{ By Exercise 26(a), the minimum distance from the comet to the sun is } 178.25(1 - 0.9951) \approx 0.8734 \text{ AU or about 81 million miles.}$$

$$29. \text{ The minimum distance is at perihelion, where}$$

$$4.6 \times 10^7 = r = a(1 - e) = a(1 - 0.206) = a(0.794) \Rightarrow a = 4.6 \times 10^7 / 0.794. \text{ So the maximum distance, which is at aphelion, is } r = a(1 + e) = (4.6 \times 10^7 / 0.794)(1.206) \approx 7.0 \times 10^7 \text{ km.}$$

30. At perihelion, $r = a(1 - e) = 4.43 \times 10^9$, and at aphelion, $r = a(1 + e) = 7.37 \times 10^9$. Adding, we get $2a = 11.80 \times 10^9$, so $a = 5.90 \times 10^9$ km. Therefore $1 + e = a(1 + e)/a = \frac{7.37}{5.90} \approx 1.249$ and $e \approx 0.249$.
31. From Exercise 29, we have $e = 0.206$ and $a(1 - e) = 4.6 \times 10^7$ km. Thus, $a = 4.6 \times 10^7 / 0.794$. From

Exercise 25, we can write the equation of Mercury's orbit as $r = a \frac{1 - e^2}{1 - e \cos \theta}$. So since

$$\frac{dr}{d\theta} = \frac{-a(1 - e^2)e \sin \theta}{(1 - e \cos \theta)^2} \Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2(1 - e^2)^2}{(1 - e \cos \theta)^2} + \frac{a^2(1 - e^2)^2 e^2 \sin^2 \theta}{(1 - e \cos \theta)^4} = \frac{a^2(1 - e^2)^2}{(1 - e \cos \theta)^4} (1 - 2e \cos \theta + e^2)$$

the length of the orbit is

$$L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = a(1 - e^2) \int_0^{2\pi} \frac{\sqrt{1 + e^2 - 2e \cos \theta}}{(1 - e \cos \theta)^2} d\theta \approx 3.6 \times 10^8 \text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a \approx 3.6 \times 10^8$ km.

11 Review

CONCEPT CHECK

- (a) A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are continuous functions of a variable t .

(b) Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t , either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y . It may be easier to graph that equation than to work with the original formulas for x and y in terms of t .
- (a) You can find $\frac{dy}{dx}$ as a function of t by calculating $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (if $dx/dt \neq 0$).

(b) Calculate the area as $\int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt$ [or $\int_\beta^\alpha g(t)f'(t)dt$ if the leftmost point is $(f(\beta), g(\beta))$ rather than $(f(\alpha), g(\alpha))$].
- (a) $L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$

(b) $S = \int_\alpha^\beta 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_\alpha^\beta 2\pi g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$
- (a) See Figure 5 in Section 11.3.

(b) $x = r \cos \theta$, $y = r \sin \theta$

(c) To find a polar representation (r, θ) with $r \geq 0$ and $0 \leq \theta < 2\pi$, first calculate $r = \sqrt{x^2 + y^2}$. Then θ is specified by $\cos \theta = x/r$ and $\sin \theta = y/r$.

5. (a) Calculate $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(y)}{\frac{d}{d\theta}(x)} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{\left(\frac{dr}{d\theta}\right) \sin \theta + r \cos \theta}{\left(\frac{dr}{d\theta}\right) \cos \theta - r \sin \theta}$, where $r = f(\theta)$.
- (b) Calculate $A = \int_a^b \frac{1}{2} r^2 d\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$
- (c) $L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$
6. (a) A parabola is a set of points in a plane whose distances from a fixed point F (the focus) and a fixed line l (the directrix) are equal.
- (b) $x^2 = 4py$; $y^2 = 4px$
7. (a) An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.
- (b) $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$.
8. (a) A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant. This difference should be interpreted as the larger distance minus the smaller distance.
- (b) $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$
- (c) $y = \pm \frac{\sqrt{c^2 - a^2}}{a} x$
9. (a) If a conic section has focus F and corresponding directrix l , then the eccentricity e is the fixed ratio $|PF|/|Pl|$ for points P of the conic section.
- (b) $e < 1$ for an ellipse; $e > 1$ for a hyperbola; $e = 1$ for a parabola.
- (c) $x = d$: $r = \frac{ed}{1 + e \cos \theta}$; $x = -d$: $r = \frac{ed}{1 - e \cos \theta}$; $y = d$: $r = \frac{ed}{1 + e \sin \theta}$; $y = -d$: $r = \frac{ed}{1 - e \sin \theta}$.

TRUE-FALSE QUIZ

1. False. Consider the curve defined by $x = f(t) = (t - 1)^3$ and $y = g(t) = (t - 1)^2$. Then $g'(t) = 2(t - 1)$, so $g'(1) = 0$, but its graph has a *vertical tangent* when $t = 1$. *Note:* The statement is true if $f'(1) \neq 0$ when $g'(1) = 0$.
2. False. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$.
3. False. For example, if $f(t) = \cos t$ and $g(t) = \sin t$ for $0 \leq t \leq 4\pi$, then the curve is a circle of radius 1, hence its length is 2π , but

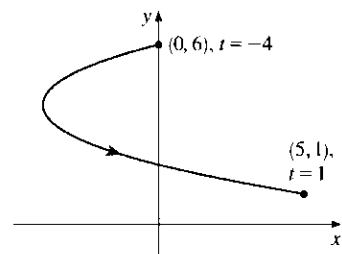
$$\begin{aligned} \int_0^{4\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + (\cos t)^2} dt \\ &= \int_0^{4\pi} 1 dt = 4\pi, \end{aligned}$$

since as t increases from 0 to 4π , the circle is traversed twice.

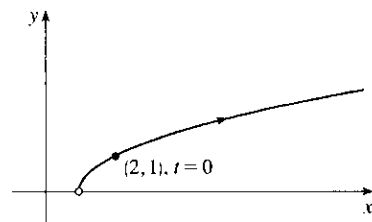
4. False. If $(r, \theta) = (1, \pi)$, then $(x, y) = (-1, 0)$, so $\tan^{-1}(y/x) = \tan^{-1} 0 = 0 \neq \theta$. The statement is true for points in quadrants I and IV.
5. True. The curve $r = 1 - \sin 2\theta$ is unchanged if we rotate it through 180° about O because $1 - \sin 2(\theta + \pi) = 1 - \sin(2\theta + 2\pi) = 1 - \sin 2\theta$. So it's unchanged if we replace r by $-r$. (See the discussion after Example 8 in Section 11.3.) In other words, it's the same curve as $r = -(1 - \sin 2\theta) = \sin 2\theta - 1$.
6. True. The polar equation $r = 2$, the Cartesian equation $x^2 + y^2 = 4$, and the parametric equations $x = 2 \sin 3t$, $y = 2 \cos 3t$ ($0 \leq t \leq 2\pi$) all describe the circle of radius 2 centered at the origin.
7. False. The first pair of equations yields the portion of the parabola $y = x^2$ with $x \geq 0$, whereas the second pair of equations traces out the whole parabola $y = x^2$.
8. True. $y^2 = 2y + 3x \Leftrightarrow (y-1)^2 = 3x+1 = 3(x+\frac{1}{3}) = 4(\frac{3}{4})(x+\frac{1}{3})$, which is the equation of a parabola with vertex $(-\frac{1}{3}, 1)$ and focus $(-\frac{1}{3} + \frac{3}{4}, 1)$, opening to the right.
9. True. By rotating and translating the parabola, we can assume it has an equation of the form $y = cx^2$, where $c > 0$. The tangent at the point (a, ca^2) is the line $y - ca^2 = 2ca(x - a)$; i.e., $y = 2cax - ca^2$. This tangent meets the parabola at the points (x, cx^2) where $cx^2 = 2cax - ca^2$. This equation is equivalent to $x^2 = 2ax - a^2$ (since $c > 0$). But $x^2 = 2ax - a^2 \Leftrightarrow x^2 - 2ax + a^2 = 0 \Leftrightarrow (x - a)^2 = 0 \Leftrightarrow x = a \Leftrightarrow (x, cx^2) = (a, ca^2)$. This shows that each tangent meets the parabola at exactly one point.
10. True. Consider a hyperbola with focus at the origin, oriented so that its polar equation is $r = \frac{ed}{1 + e \cos \theta}$, where $e > 1$. The directrix is $x = d$, but along the hyperbola we have
- $$x = r \cos \theta = \frac{ed \cos \theta}{1 + e \cos \theta} = d \left(\frac{e \cos \theta}{1 + e \cos \theta} \right) \neq d.$$

EXERCISES

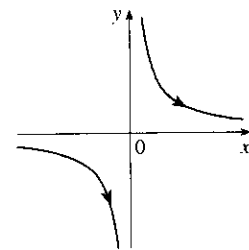
1. $x = t^2 + 4t$, $y = 2 - t$, $-4 \leq t \leq 1$. $t = 2 - y$, so
- $$x = (2 - y)^2 + 4(2 - y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12 \Leftrightarrow$$
- $$x + 4 = y^2 - 8y + 16 = (y - 4)^2. \text{ This is part of a parabola with vertex } (-4, 4), \text{ opening to the right.}$$



2. $x = 1 + e^{2t}$, $y = e^t$.
- $$x = 1 + e^{2t} = 1 + (e^t)^2 = 1 + y^2, y > 0.$$

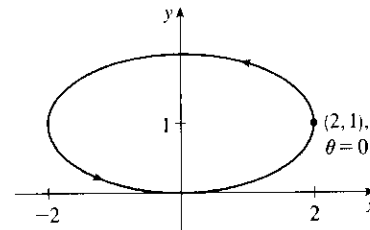


3. $x = \tan \theta$, $y = \cot \theta$, $y = 1/\tan \theta = 1/x$. The whole curve is traced out as θ ranges over the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ [or any open interval of the form $(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi)$, where n is an integer].



4. $x = 2 \cos \theta$, $y = 1 + \sin \theta$, $\cos^2 \theta + \sin^2 \theta = 1 \Rightarrow$

$(\frac{x}{2})^2 + (y - 1)^2 = 1 \Rightarrow \frac{x^2}{4} + (y - 1)^2 = 1$. This is an ellipse, centered at $(0, 1)$, with semimajor axis of length 2 and semiminor axis of length 1.



5. Three different sets of parametric equations for the curve $y = \sqrt{x}$ are

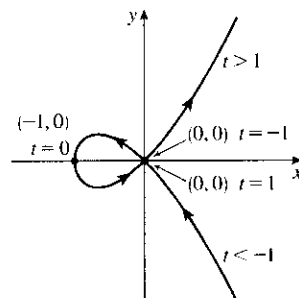
(i) $x = t$, $y = \sqrt{t}$, $t \geq 0$

(ii) $x = t^4$, $y = t^2$

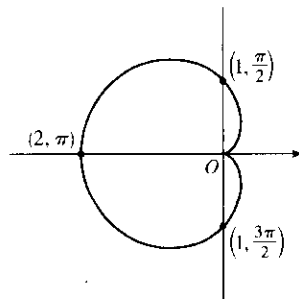
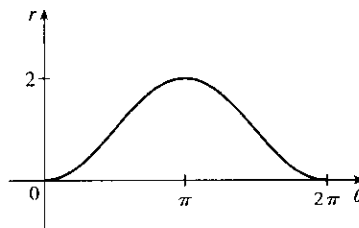
(iii) $x = \tan^2 t$, $y = \tan t$, $0 \leq t < \pi/2$

There are many other sets of equations that also give this curve.

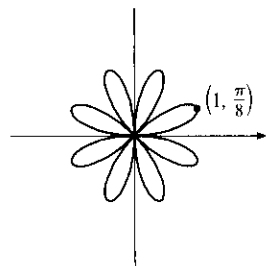
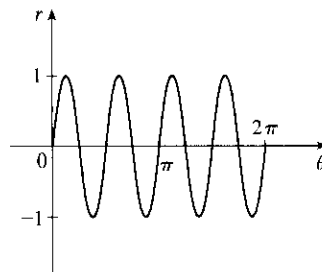
6. For $t < -1$, $x > 0$ and $y < 0$ with x decreasing and y increasing. When $t = -1$, $(x, y) = (0, 0)$. When $-1 < t < 0$, we have $-1 < x < 0$ and $0 < y < 1/2$. When $t = 0$, $(x, y) = (-1, 0)$. When $0 < t < 1$, $-1 < x < 0$ and $-\frac{1}{2} < y < 0$. When $t = 1$, $(x, y) = (0, 0)$ again. When $t > 1$, both x and y are positive and increasing.



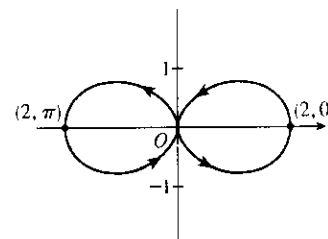
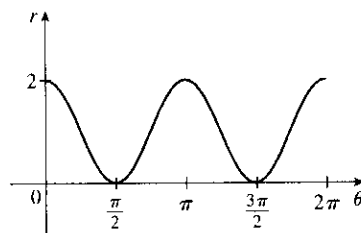
7. $r = 1 - \cos \theta$. This cardioid is symmetric about the polar axis.



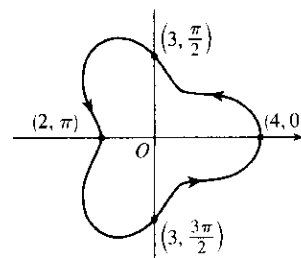
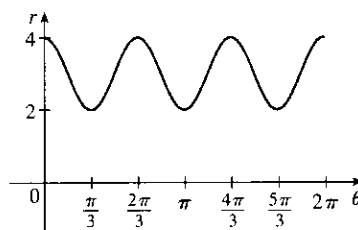
8. $r = \sin 4\theta$. This is an eight-leafed rose.



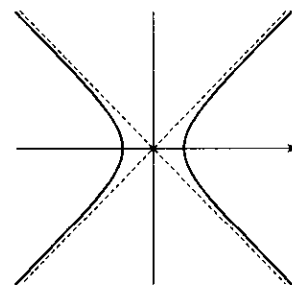
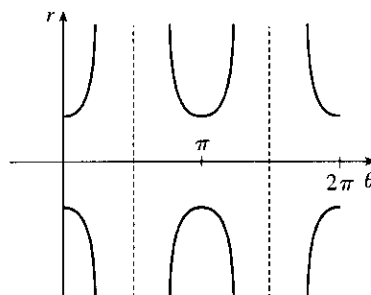
9. $r = 1 + \cos 2\theta$. The curve is symmetric about the pole and both the horizontal and vertical axes.



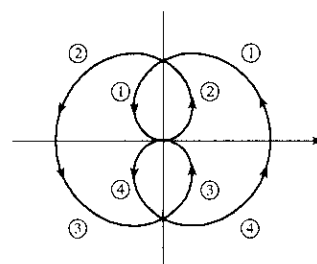
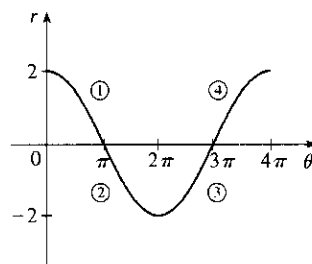
10. $r = 3 + \cos 3\theta$. The curve is symmetric about the horizontal axis.



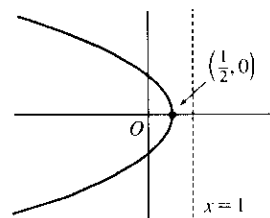
11. $r^2 = \sec 2\theta \Rightarrow$
 $r^2 \cos 2\theta = 1 \Rightarrow$
 $r^2 (\cos^2 \theta - \sin^2 \theta) = 1 \Rightarrow$
 $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1 \Rightarrow$
 $x^2 - y^2 = 1$, a hyperbola



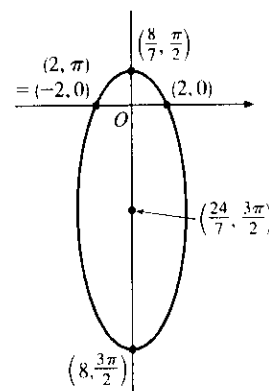
12. $r = 2 \cos (\theta/2)$. The curve is symmetric about the pole and both the horizontal and vertical axes.



13. $r = \frac{1}{1 + \cos \theta} \Rightarrow e = 1 \Rightarrow$ parabola; $d = 1 \Rightarrow$ directrix $x = 1$
 and vertex $(\frac{1}{2}, 0)$; y -intercepts are $(1, \frac{\pi}{2})$ and $(1, \frac{3\pi}{2})$.



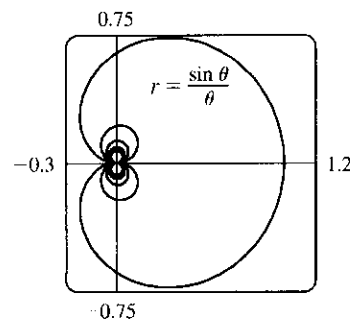
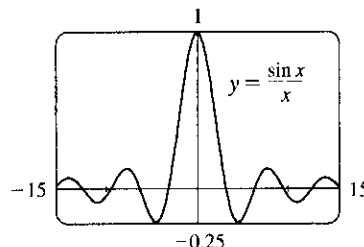
14. $r = \frac{8}{4 + 3 \sin \theta} = \frac{\frac{3}{4} \cdot \frac{8}{3}}{1 + \frac{3}{4} \sin \theta}$. This is an ellipse with focus at the pole, eccentricity $\frac{3}{4}$, and directrix $y = \frac{8}{3}$. The center is $(\frac{24}{7}, \frac{3\pi}{2})$.



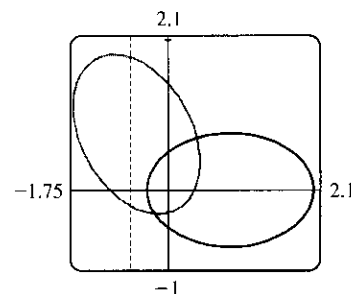
15. $x + y = 2 \Leftrightarrow r \cos \theta + r \sin \theta = 2 \Leftrightarrow r(\cos \theta + \sin \theta) = 2 \Leftrightarrow r = \frac{2}{\cos \theta + \sin \theta}$

16. $x^2 + y^2 = 2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$. ($r = -\sqrt{2}$ gives the same curve.)

17. $r = (\sin \theta)/\theta$. As $\theta \rightarrow \pm\infty$, $r \rightarrow 0$. As $\theta \rightarrow 0$, $r \rightarrow 1$. In the first figure, there are an infinite number of x -intercepts at $x = \pi n$, n a nonzero integer. These correspond to pole points in the second figure.



18. $r = \frac{2}{4 - 3 \cos \theta} = \frac{1/2}{1 - \frac{3}{4} \cos \theta} \Rightarrow e = \frac{3}{4}$ and $d = \frac{2}{3}$. The equation of the directrix is $x = -\frac{2}{3} \Rightarrow r = -2/(3 \cos \theta)$. To obtain the equation of the rotated ellipse, we replace θ in the original equation with $\theta - \frac{2\pi}{3}$, and get $r = \frac{2}{4 - 3 \cos(\theta - \frac{2\pi}{3})}$.



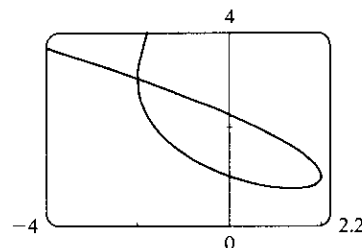
19. $x = \ln t$, $y = 1 + t^2$; $t = 1$. $\frac{dy}{dt} = 2t$ and $\frac{dx}{dt} = \frac{1}{t}$, so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$. When $t = 1$, $(x, y) = (0, 2)$ and $dy/dx = 2$.
20. $x = t^3 + 6t + 1$, $y = 2t - t^2$; $t = -1$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 - 2t}{3t^2 + 6}$. When $t = -1$, $(x, y) = (-6, -3)$ and $dy/dx = 4/9$.
21. $r = e^{-\theta} \Rightarrow y = r \sin \theta = e^{-\theta} \sin \theta$ and $x = r \cos \theta = e^{-\theta} \cos \theta \Rightarrow$
 $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{-e^{-\theta} \sin \theta + e^{-\theta} \cos \theta}{-e^{-\theta} \cos \theta - e^{-\theta} \sin \theta} \cdot \frac{-e^{\theta}}{-e^{\theta}} = \frac{\sin \theta - \cos \theta}{\cos \theta + \sin \theta}$. When $\theta = \pi$,
 $\frac{dy}{dx} = \frac{0 - (-1)}{-1 + 0} = \frac{1}{-1} = -1$.

22. $r = 3 + \cos 3\theta \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{-3 \sin 3\theta \sin \theta + (3 + \cos 3\theta) \cos \theta}{-3 \sin 3\theta \cos \theta - (3 + \cos 3\theta) \sin \theta}$. When $\theta = \pi/2$, $\frac{dy}{dx} = \frac{(-3)(-1)(1) + (3+0) \cdot 0}{(-3)(-1)(0) - (3+0) \cdot 1} = \frac{3}{-3} = -1$.

23. $x = t \cos t$, $y = t \sin t$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t \cos t + \sin t}{-t \sin t + \cos t}$. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt}$, where $\frac{d}{dt}(\frac{dy}{dx}) = \frac{(-t \sin t + \cos t)(-t \sin t + 2 \cos t) - (t \cos t + \sin t)(-t \cos t - 2 \sin t)}{(-t \sin t + \cos t)^2}$
 $= \frac{t^2 + 2}{(-t \sin t + \cos t)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{t^2 + 2}{(-t \sin t + \cos t)^3}$.

24. $x = 1 + t^2$, $y = t - t^3$. $\frac{dy}{dt} = 1 - 3t^2$ and $\frac{dx}{dt} = 2t$, so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{2t} = \frac{1}{2}t^{-1} - \frac{3}{2}t$.
 $\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{-\frac{1}{2}t^{-2} - \frac{3}{2}}{2t} = -\frac{1}{4}t^{-3} - \frac{3}{4}t^{-1} = -\frac{1}{4t^3}(1 + 3t^2) = -\frac{3t^2 + 1}{4t^3}$.

25. We graph the curve $x = t^3 - 3t$, $y = t^2 + t + 1$ for $-2.2 \leq t \leq 1.2$. By zooming in or using a cursor, we find that the lowest point is about $(1.4, 0.75)$. To find the exact values, we find the t -value at which $dy/dt = 2t + 1 = 0 \Leftrightarrow t = -\frac{1}{2} \Leftrightarrow (x, y) = (\frac{11}{8}, \frac{3}{4})$.



26. We estimate the coordinates of the point of intersection to be $(-2, 3)$. In fact this is exact, since both $t = -2$ and $t = 1$ give the point $(-2, 3)$. So the area enclosed by the loop is

$$\int_{t=-2}^{t=1} y \, dx = \int_{-2}^1 (t^2 + t + 1)(3t^2 - 3) \, dt = \int_{-2}^1 (3t^4 + 3t^3 - 3t - 3) \, dt$$

$$= \left[\frac{3}{5}t^5 + \frac{3}{4}t^4 - \frac{3}{2}t^2 - 3t \right]_{-2}^1 = \left(\frac{3}{5} + \frac{3}{4} - \frac{3}{2} - 3 \right) - \left[-\frac{96}{5} + 12 - 6 - (-6) \right] = \frac{81}{20}$$

27. $x = 2a \cos t - a \cos 2t \Rightarrow \frac{dx}{dt} = -2a \sin t + 2a \sin 2t = 2a \sin t(2 \cos t - 1) = 0 \Leftrightarrow \sin t = 0$ or $\cos t = \frac{1}{2} \Rightarrow t = 0, \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3}$.
 $y = 2a \sin t - a \sin 2t \Rightarrow \frac{dy}{dt} = 2a \cos t - 2a \cos 2t = 2a(1 + \cos t - 2 \cos^2 t) = 2a(1 - \cos t)(1 + 2 \cos t) = 0 \Rightarrow t = 0, \frac{2\pi}{3}, \text{ or } \frac{4\pi}{3}$.

Thus the graph has vertical tangents where

$t = \frac{\pi}{3}, \pi$ and $\frac{5\pi}{3}$, and horizontal tangents where

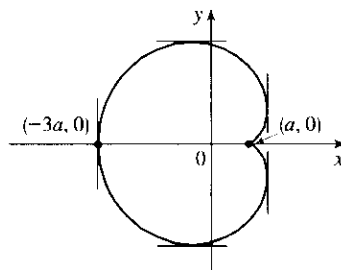
$t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. To determine what the slope is

where $t = 0$, we use l'Hospital's Rule to evaluate

$$\lim_{t \rightarrow 0} \frac{dy/dt}{dx/dt} = 0, \text{ so there is a horizontal tangent}$$

there.

t	x	y
0	a	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
π	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



28. From Exercise 27, $x = 2a \cos t - a \cos 2t$, $y = 2a \sin t - a \sin 2t \Rightarrow$

$$\begin{aligned} A &= 2 \int_{\pi}^0 (2a \sin t - a \sin 2t)(-2a \sin t + 2a \sin 2t) dt = 4a^2 \int_0^{\pi} (2 \sin^2 t + \sin^2 2t - 3 \sin t \sin 2t) dt \\ &= 4a^2 \int_0^{\pi} \left[(1 - \cos 2t) + \frac{1}{2} (1 - \cos 4t) - 6 \sin^2 t \cos t \right] dt = 4a^2 \left[t - \frac{1}{2} \sin 2t + \frac{1}{2} t - \frac{1}{8} \sin 4t - 2 \sin^3 t \right]_0^{\pi} \\ &= 4a^2 \left(\frac{3}{2} \right) \pi = 6\pi a^2 \end{aligned}$$

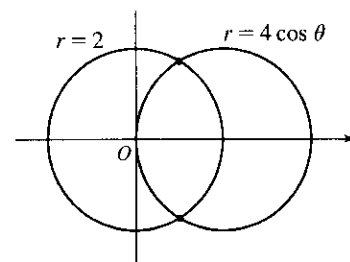
29. The curve $r^2 = 9 \cos 5\theta$ has 10 "petals." For instance, for $-\frac{\pi}{10} \leq \theta \leq \frac{\pi}{10}$, there are two petals, one with $r > 0$ and one with $r < 0$.

$$A = 10 \int_{-\pi/10}^{\pi/10} \frac{1}{2} r^2 d\theta = 5 \int_{-\pi/10}^{\pi/10} 9 \cos 5\theta d\theta = 5 \cdot 9 \cdot 2 \int_0^{\pi/10} \cos 5\theta d\theta = 18 [\sin 5\theta]_0^{\pi/10} = 18$$

30. $r = 1 - 3 \sin \theta$. The inner loop is traced out as θ goes from $\alpha = \sin^{-1} \frac{1}{3}$ to $\pi - \alpha$, so

$$\begin{aligned} A &= \int_{\alpha}^{\pi-\alpha} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\pi/2} (1 - 3 \sin \theta)^2 d\theta = \int_{\alpha}^{\pi/2} [1 - 6 \sin \theta + \frac{9}{2} (1 - \cos 2\theta)] d\theta \\ &= \left[\frac{11}{2} \theta + 6 \cos \theta - \frac{9}{4} \sin 2\theta \right]_{\alpha}^{\pi/2} = \frac{11}{4} \pi - \frac{11}{2} \sin^{-1} \frac{1}{3} - 3\sqrt{2} \end{aligned}$$

31. The curves intersect when $4 \cos \theta = 2 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$
for $-\pi \leq \theta \leq \pi$. The points of intersection are $(2, \frac{\pi}{3})$ and $(2, -\frac{\pi}{3})$.

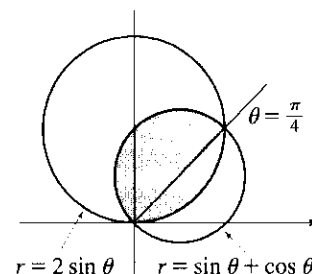


32. The two curves clearly both contain the pole. For other points of intersection, $\cot \theta = 2 \cos(\theta + 2n\pi)$ or $-2 \cos(\theta + \pi + 2n\pi)$, both of which reduce to $\cot \theta = 2 \cos \theta \Leftrightarrow \cos \theta = 2 \sin \theta \cos \theta \Leftrightarrow \cos \theta(1 - 2 \sin \theta) = 0 \Rightarrow \cos \theta = 0$ or $\sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2} \Rightarrow$ intersection points are $(0, \frac{\pi}{2})$, $(\sqrt{3}, \frac{\pi}{6})$, and $(\sqrt{3}, \frac{11\pi}{6})$.

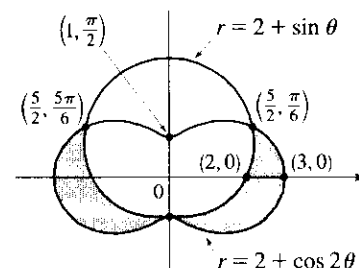
33. The curves intersect where $2 \sin \theta = \sin \theta + \cos \theta \Rightarrow$

$\sin \theta = \cos \theta \Rightarrow \theta = \frac{\pi}{4}$, and also at the origin (at which $\theta = \frac{3\pi}{4}$ on the second curve).

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} (2 \sin \theta)^2 d\theta + \int_{\pi/4}^{3\pi/4} \frac{1}{2} (\sin \theta + \cos \theta)^2 d\theta \\ &= \int_0^{\pi/4} (1 - \cos 2\theta) d\theta + \frac{1}{2} \int_{\pi/4}^{3\pi/4} (1 + \sin 2\theta) d\theta \\ &= \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/4} + \left[\frac{1}{2} \theta - \frac{1}{4} \cos 2\theta \right]_{\pi/4}^{3\pi/4} = \frac{1}{2} (\pi - 1) \end{aligned}$$



34. $A = 2 \int_{-\pi/2}^{\pi/6} \frac{1}{2} [(2 + \cos 2\theta)^2 - (2 + \sin \theta)^2] d\theta$
 $= \int_{-\pi/2}^{\pi/6} [4 \cos 2\theta + \cos^2 2\theta - 4 \sin \theta - \sin^2 \theta] d\theta$
 $= [2 \sin 2\theta + \frac{1}{2} \theta + \frac{1}{8} \sin 4\theta + 4 \cos \theta - \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta]_{-\pi/2}^{\pi/6}$
 $= \frac{51}{16} \sqrt{3}$



35. $x = 3t^2, y = 2t^3.$

$$\begin{aligned} L &= \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2} dt = \int_0^2 \sqrt{36t^2 + 36t^4} dt \\ &= \int_0^2 \sqrt{36t^2} \sqrt{1+t^2} dt = \int_0^2 6|t| \sqrt{1+t^2} dt = 6 \int_0^2 t \sqrt{1+t^2} dt \\ &= 6 \int_1^5 u^{1/2} \left(\frac{1}{2} du\right) \quad [u = 1+t^2, du = 2t dt] \\ &= 6 \cdot \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2}\right]_1^5 = 2(5^{3/2} - 1) = 2(5\sqrt{5} - 1) \end{aligned}$$

36. $x = 2 + 3t, y = \cosh 3t \Rightarrow (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3 \sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9 \cosh^2 3t$, so

$$L = \int_0^1 \sqrt{9 \cosh^2 3t} dt = \int_0^1 |3 \cosh 3t| dt = \int_0^1 3 \cosh 3t dt = [\sinh 3t]_0^1 = \sinh 3 - \sinh 0 = \sinh 3.$$

37. $L = \int_{\pi}^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_{\pi}^{2\pi} \sqrt{(1/\theta)^2 + (-1/\theta^2)^2} d\theta = \int_{\pi}^{2\pi} \frac{\sqrt{\theta^2 + 1}}{\theta^2} d\theta$

$$\begin{aligned} &\stackrel{24}{=} \left[-\frac{\sqrt{\theta^2 + 1}}{\theta} + \ln(\theta + \sqrt{\theta^2 + 1}) \right]_{\pi}^{2\pi} = \frac{\sqrt{\pi^2 + 1}}{\pi} - \frac{\sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right) \\ &= \frac{2\sqrt{\pi^2 + 1} - \sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right) \end{aligned}$$

38. $L = \int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{\pi} \sqrt{\sin^6(\frac{1}{3}\theta) + \sin^4(\frac{1}{3}\theta) \cos^2(\frac{1}{3}\theta)} d\theta$

$$= \int_0^{\pi} \sin^2(\frac{1}{3}\theta) d\theta = \left[\frac{1}{2}\left(\theta - \frac{3}{2}\sin\left(\frac{2}{3}\theta\right)\right)\right]_0^{\pi} = \frac{1}{2}\pi - \frac{3}{8}\sqrt{3}$$

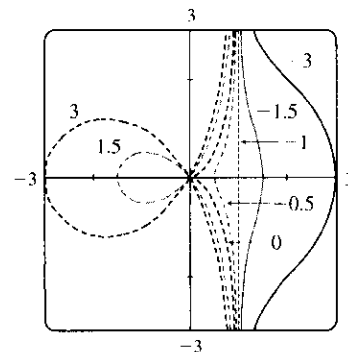
39. $x = 4\sqrt{t}, y = \frac{t^3}{3} + \frac{1}{2t^2}, 1 \leq t \leq 4 \Rightarrow$

$$\begin{aligned} S &= \int_1^4 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_1^4 2\pi \left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right) \sqrt{(2/\sqrt{t})^2 + (t^2 - t^{-3})^2} dt \\ &= 2\pi \int_1^4 \left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right) \sqrt{(t^2 + t^{-3})^2} dt = 2\pi \int_1^4 \left(\frac{1}{3}t^5 + \frac{5}{6} + \frac{1}{2}t^{-5}\right) dt = 2\pi \left[\frac{1}{18}t^6 + \frac{5}{6}t - \frac{1}{8}t^{-4}\right]_1^4 = \frac{471,295}{1024}\pi \end{aligned}$$

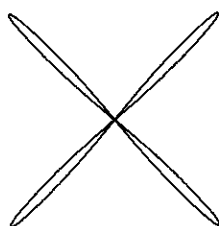
40. $x = 2 + 3t, y = \cosh 3t \Rightarrow (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3 \sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9 \cosh^2 3t$, so

$$\begin{aligned} S &= \int_0^1 2\pi y ds = \int_0^1 2\pi \cosh 3t \sqrt{9 \cosh^2 3t} dt = \int_0^1 2\pi \cosh 3t |3 \cosh 3t| dt \\ &= \int_0^1 2\pi \cosh 3t \cdot 3 \cosh 3t dt = 6\pi \int_0^1 \cosh^2 3t dt = 6\pi \int_0^1 \frac{1}{2}(1 + \cosh 6t) dt \\ &= 3\pi \left[t + \frac{1}{6} \sinh 6t\right]_0^1 = 3\pi \left(1 + \frac{1}{6} \sinh 6\right) = 3\pi + \frac{\pi}{2} \sinh 6 \end{aligned}$$

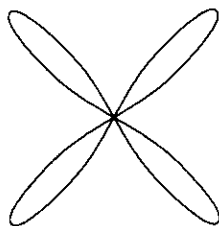
41. For all c except -1 , the curve is asymptotic to the line $x = 1$. For $c < -1$, the curve bulges to the right near $y = 0$. As c increases, the bulge becomes smaller, until at $c = -1$ the curve is the straight line $x = 1$. As c continues to increase, the curve bulges to the left, until at $c = 0$ there is a cusp at the origin. For $c > 0$, there is a loop to the left of the origin, whose size and roundness increase as c increases. Note that the x -intercept of the curve is always $-c$.



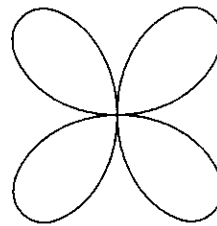
42. For a close to 0, the graph consists of four thin petals. As a increases, the petals get fatter, until as $a \rightarrow \infty$, each petal occupies almost its entire quarter-circle.



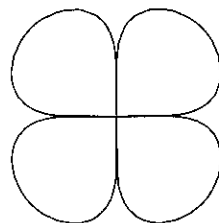
$a = 0.01$



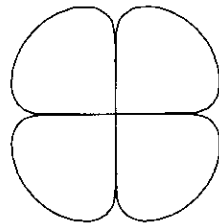
$a = 0.1$



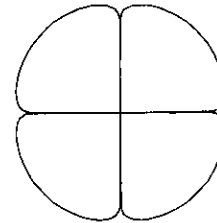
$a = 1$



$a = 5$



$a = 10$

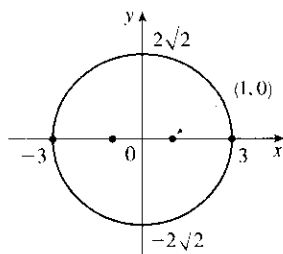


$a = 25$

43. $\frac{x^2}{9} + \frac{y^2}{8} = 1$ is an ellipse with center $(0, 0)$.

$$a = 3, b = 2\sqrt{2}, c = 1 \Rightarrow$$

foci $(\pm 1, 0)$, vertices $(\pm 3, 0)$.

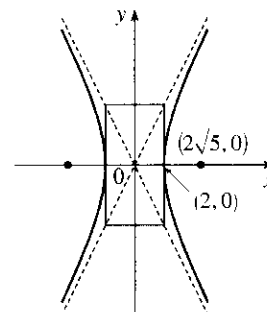


44. $\frac{x^2}{4} - \frac{y^2}{16} = 1$ is a hyperbola with center $(0, 0)$.

vertices $(\pm 2, 0)$, $a = 2$, $b = 4$.

$c = \sqrt{16 + 4} = 2\sqrt{5}$, foci $(\pm 2\sqrt{5}, 0)$ and

asymptotes $y = \pm 2x$.



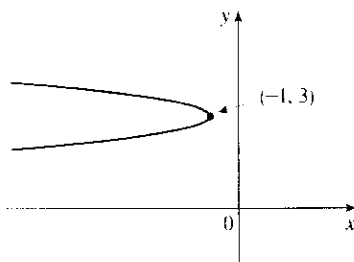
45. $6y^2 + x - 36y + 55 = 0 \Leftrightarrow$

$$6(y^2 - 6y + 9) = -(x + 1) \Leftrightarrow$$

$(y - 3)^2 = -\frac{1}{6}(x + 1)$, a parabola with vertex

$(-1, 3)$, opening to the left, $p = -\frac{1}{24} \Rightarrow$ focus

$(-\frac{25}{24}, 3)$ and directrix $x = -\frac{23}{24}$.



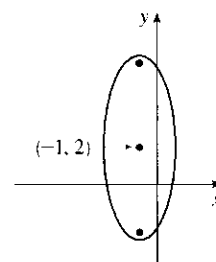
46. $25(x + 1)^2 + 4(y - 2)^2 = 100 \Leftrightarrow$

$\frac{1}{4}(x + 1)^2 + \frac{1}{25}(y - 2)^2 = 1$ is an ellipse centered

at $(-1, 2)$ with foci on the line $x = -1$, vertices

$(-1, 7)$ and $(-1, -3)$; $a = 5$, $b = 2 \Rightarrow$

$c = \sqrt{21} \Rightarrow$ foci $(-1, 2 \pm \sqrt{21})$.



47. The parabola opens upward with vertex $(0, 4)$ [midway between the focus $(0, 6)$ and the directrix $y = 2$] and $p = 2$, so its equation is $(x - 0)^2 = 4 \cdot 2(y - 4) \Leftrightarrow x^2 = 8(y - 4)$.

48. Center is $(0, 0)$, and $c = 5$, $a = 2 \Rightarrow b = \sqrt{21}$; foci on y -axis $\Rightarrow$ equation of the hyperbola is $\frac{y^2}{4} - \frac{x^2}{21} = 1$.

49. The hyperbola has center $(0, 0)$ and foci on the x -axis. $c = 3$ and $b/a = \frac{1}{2}$ (from the asymptotes)
 $\Rightarrow 9 = c^2 = a^2 + b^2 = (2b)^2 + b^2 = 5b^2 \Rightarrow b = \frac{3}{\sqrt{5}} \Rightarrow a = \frac{6}{\sqrt{5}} \Rightarrow$ an equation of the hyperbola is
 $\frac{x^2}{36/5} - \frac{y^2}{9/5} = 1 \Leftrightarrow 5x^2 - 20y^2 = 36$.

50. Center is $(3, 0)$, and $a = \frac{8}{2} = 4$, $c = 2 \Leftrightarrow b = \sqrt{4^2 - 2^2} = \sqrt{12} \Rightarrow$ an equation of the ellipse is
 $\frac{(x - 3)^2}{12} + \frac{y^2}{16} = 1$.

51. $x^2 = -(y - 100)$ has its vertex at $(0, 100)$, so one of the vertices of the ellipse is $(0, 100)$. Another form of the equation of a parabola is $x^2 = 4p(y - 100)$ so $4p(y - 100) = -(y - 100) \Rightarrow 4p = -1 \Rightarrow p = -\frac{1}{4}$. Therefore the shared focus is found at $(0, \frac{399}{4})$ so $2c = \frac{399}{4} - 0 \Rightarrow c = \frac{399}{8}$ and the center of the ellipse is $(0, \frac{399}{8})$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \frac{401^2 - 399^2}{8^2} = 25$. So the equation of the ellipse is
 $\frac{x^2}{b^2} + \frac{(y - \frac{399}{8})^2}{a^2} = 1 \Rightarrow \frac{x^2}{25} + \frac{(y - \frac{399}{8})^2}{(\frac{401}{8})^2} = 1$ or $\frac{x^2}{25} + \frac{(8y - 399)^2}{160,801} = 1$.

52. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2} \frac{x}{y}$. Therefore $\frac{dy}{dx} = m \Leftrightarrow y = -\frac{b^2}{a^2} \frac{x}{m}$.

Combining this condition with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we find that $x = \pm \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}$. In other words, the two points on

the ellipse where the tangent has slope m are $\left(\pm \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2 m^2 + b^2}} \right)$. The tangent lines at

these points have the equations $y \pm \frac{b^2}{\sqrt{a^2 m^2 + b^2}} = m \left(x \mp \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}} \right)$ or

$$y = mx \mp \frac{a^2 m^2}{\sqrt{a^2 m^2 + b^2}} \mp \frac{b^2}{\sqrt{a^2 m^2 + b^2}} = mx \mp \sqrt{a^2 m^2 + b^2}.$$

53. Directrix $x = 4 \Rightarrow d = 4$, so $e = \frac{1}{3} \Rightarrow r = \frac{ed}{1 + e \cos \theta} = \frac{4}{3 + \cos \theta}$.

54. See the end of the proof of Theorem 11.6.1. If $e > 1$, then $1 - e^2 < 0$ and Equations 11.6.4 become

$$a^2 = \frac{e^2 d^2}{(e^2 - 1)^2} \text{ and } b^2 = \frac{e^2 d^2}{e^2 - 1}, \text{ so } \frac{b^2}{a^2} = e^2 - 1. \text{ The asymptotes } y = \pm \frac{b}{a} x \text{ have slopes } \pm \frac{b}{a} = \pm \sqrt{e^2 - 1}, \text{ so}$$

the angles they make with the polar axis are $\pm \tan^{-1} [\sqrt{e^2 - 1}] = \cos^{-1} (\pm 1/e)$.

55. In polar coordinates, an equation for the circle is $r = 2a \sin \theta$. Thus, the coordinates of Q are

$$x = r \cos \theta = 2a \sin \theta \cos \theta \text{ and } y = r \sin \theta = 2a \sin^2 \theta. \text{ The coordinates of } R \text{ are } x = 2a \cot \theta \text{ and } y = 2a.$$

Since P is the midpoint of QR , we use the midpoint formula to get $x = a(\sin \theta \cos \theta + \cot \theta)$ and

$$y = a(1 + \sin^2 \theta).$$

□ PROBLEMS PLUS

1. $x = \int_1^t \frac{\cos u}{u} du$, $y = \int_1^t \frac{\sin u}{u} du$, so by FTC1, we have $\frac{dx}{dt} = \frac{\cos t}{t}$ and $\frac{dy}{dt} = \frac{\sin t}{t}$. Vertical tangent lines

occur when $\frac{dx}{dt} = 0 \Leftrightarrow \cos t = 0$. The parameter value corresponding to $(x, y) = (0, 0)$ is $t = 1$, so the

nearest vertical tangent occurs when $t = \frac{\pi}{2}$. Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}} dt = \int_1^{\pi/2} \frac{dt}{t} = [\ln t]_1^{\pi/2} = \ln \frac{\pi}{2}$$

2. (a) The curve $x^4 + y^4 = x^2 + y^2$ is symmetric about both axes and about the line $y = x$ (since interchanging x and y does not change the equation) so we need only consider $y \geq x \geq 0$ to begin with. Implicit differentiation

gives $4x^3 + 4y^3y' = 2x + 2yy' \Rightarrow y' = \frac{x(1 - 2x^2)}{y(2y^2 - 1)} \Rightarrow y' = 0$ when $x = 0$ and when $x = \pm \frac{1}{\sqrt{2}}$. If

$x = 0$, then $y^4 = y^2 \Rightarrow y^2(y^2 - 1) = 0 \Rightarrow y = 0$ or ± 1 . The point $(0, 0)$ can't be a highest or lowest point because it is isolated. If $-1 < x < 1$ and $-1 < y < 1$, then $x^4 < x^2$ and $y^4 < y^2 \Rightarrow$

$x^4 + y^4 < x^2 + y^2$, except for $(0, 0)$.] If $x = \frac{1}{\sqrt{2}}$, then $x^2 = \frac{1}{2}$, $x^4 = \frac{1}{4}$, so $\frac{1}{4} + y^4 = \frac{1}{2} + y^2 \Rightarrow$

$4y^4 - 4y^2 - 1 = 0 \Rightarrow y^2 = \frac{4 \pm \sqrt{16+16}}{8} = \frac{1 \pm \sqrt{2}}{2}$. But $y^2 > 0$, so $y^2 = \frac{1 + \sqrt{2}}{2} \Rightarrow$

$y = \pm \sqrt{\frac{1}{2}(1 + \sqrt{2})}$. Near the point $(0, 1)$, the denominator of y' is positive and the numerator changes from

negative to positive as x increases through 0, so $(0, 1)$ is a local minimum point. At $\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1 + \sqrt{2}}{2}}\right)$,

y' changes from positive to negative, so that point gives a maximum. By symmetry, the highest points on the

curve are $\left(\pm \frac{1}{\sqrt{2}}, \sqrt{\frac{1 + \sqrt{2}}{2}}\right)$ and the lowest points are $\left(\pm \frac{1}{\sqrt{2}}, -\sqrt{\frac{1 + \sqrt{2}}{2}}\right)$.

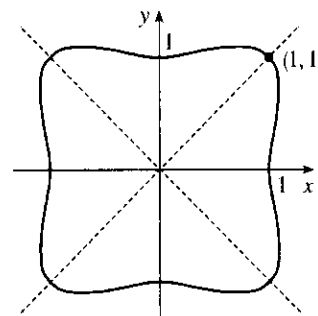
(b) We use the information from part (a), together with symmetry with respect to the axes and the lines $y = \pm x$, to sketch the curve.

(c) In polar coordinates, $x^4 + y^4 = x^2 + y^2$ becomes

$r^4 \cos^4 \theta + r^4 \sin^4 \theta = r^2$ or $r^2 = 1/(\cos^4 \theta + \sin^4 \theta)$. By the

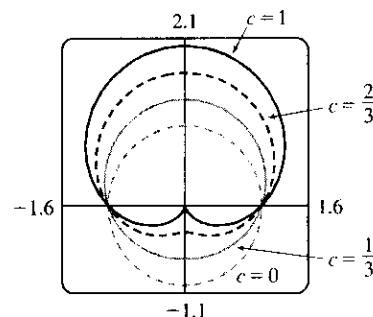
symmetry shown in part (b), the area enclosed by the curve is

$$A = 8 \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = 4 \int_0^{\pi/4} \frac{d\theta}{\cos^4 \theta + \sin^4 \theta} \stackrel{\text{CAS}}{=} \sqrt{2}\pi.$$

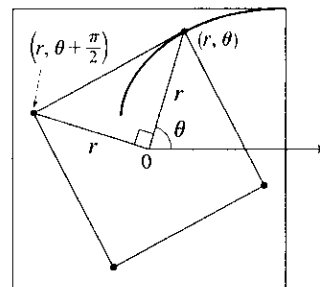


3. In terms of x and y , we have $x = r \cos \theta = (1 + c \sin \theta) \cos \theta = \cos \theta + c \sin \theta \cos \theta = \cos \theta + \frac{1}{2}c \sin 2\theta$ and $y = r \sin \theta = (1 + c \sin \theta) \sin \theta = \sin \theta + c \sin^2 \theta$. Now $-1 \leq \sin \theta \leq 1 \Rightarrow -1 \leq \sin \theta + c \sin^2 \theta \leq 1 + c \leq 2$, so $-1 \leq y \leq 2$. Furthermore, $y = 2$ when $c = 1$ and $\theta = \frac{\pi}{2}$, while $y = -1$ for $c = 0$ and $\theta = \frac{3\pi}{2}$. Therefore, we need a viewing rectangle with $-1 \leq y \leq 2$.

To find the x -values, look at the equation $x = \cos \theta + \frac{1}{2}c \sin 2\theta$ and use the fact that $\sin 2\theta \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$ and $\sin 2\theta \leq 0$ for $-\frac{\pi}{2} \leq \theta \leq 0$. [Because $r = 1 + c \sin \theta$ is symmetric about the y -axis, we only need to consider $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.] So for $-\frac{\pi}{2} \leq \theta \leq 0$, x has a maximum value when $c = 0$ and then $x = \cos \theta$ has a maximum value of 1 at $\theta = 0$. Thus, the maximum value of x must occur on $[0, \frac{\pi}{2}]$ with $c = 1$. Then $x = \cos \theta + \frac{1}{2} \sin 2\theta \Rightarrow \frac{dx}{d\theta} = -\sin \theta + \cos 2\theta = -\sin \theta + 1 - 2\sin^2 \theta \Rightarrow \frac{dx}{d\theta} = -(2\sin \theta - 1)(\sin \theta + 1) = 0$ when $\sin \theta = -1$ or $\frac{1}{2}$ (but $\sin \theta \neq -1$ for $0 \leq \theta \leq \frac{\pi}{2}$). If $\sin \theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ and $x = \cos \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} = \frac{3}{4}\sqrt{3}$. Thus, the maximum value of x is $\frac{3}{4}\sqrt{3}$, and, by symmetry, the minimum value is $-\frac{3}{4}\sqrt{3}$. Therefore, the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c \sin \theta$, where $0 \leq c \leq 1$, is $[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}] \times [-1, 2]$.



4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are (r, θ) , then the polar coordinates of the bug that it is crawling toward must be $(r, \theta + \frac{\pi}{2})$. (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be $\frac{\pi}{2}$.) The Cartesian coordinates of the first bug are $(r \cos \theta, r \sin \theta)$ and for the second bug we have $x = r \cos(\theta + \frac{\pi}{2}) = -r \sin \theta$, $y = r \sin(\theta + \frac{\pi}{2}) = r \cos \theta$. So the slope of the line joining the bugs is $\frac{r \cos \theta - r \sin \theta}{-r \sin \theta - r \cos \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$. This must be equal to the slope of the tangent line at (r, θ) , so by Equation 11.3.3 we have $\frac{(dr/d\theta) \sin \theta + r \cos \theta}{(dr/d\theta) \cos \theta - r \sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$. Solving for $\frac{dr}{d\theta}$, we get $\frac{dr}{d\theta} \sin^2 \theta + \frac{dr}{d\theta} \sin \theta \cos \theta + r \sin \theta \cos \theta + r \cos^2 \theta = \frac{dr}{d\theta} \sin \theta \cos \theta - \frac{dr}{d\theta} \cos^2 \theta - r \sin^2 \theta + r \sin \theta \cos \theta \Rightarrow \frac{dr}{d\theta} (\sin^2 \theta + \cos^2 \theta) + r (\cos^2 \theta + \sin^2 \theta) = 0 \Rightarrow \frac{dr}{d\theta} = -r$. Solving this differential equation as a separable equation (as in Section 10.3), or using Theorem 10.4.2 with $k = -1$, we get $r = Ce^{-\theta}$. To determine C we use the fact that, at its starting position, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\sqrt{2}}a$, so $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \Rightarrow C = \frac{1}{\sqrt{2}}ae^{\pi/4}$. Therefore, a polar equation of the bug's path is $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$ or $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$.



(b) The distance traveled by this bug is $L = \int_{\pi/4}^{\infty} \sqrt{r^2 + (dr/d\theta)^2} d\theta$, where $\frac{dr}{d\theta} = \frac{a}{\sqrt{2}} e^{\pi/4} (-e^{-\theta})$ and so

$$r^2 + (dr/d\theta)^2 = \frac{1}{2} a^2 e^{\pi/2} e^{-2\theta} + \frac{1}{2} a^2 e^{\pi/2} e^{-2\theta} = a^2 e^{\pi/2} e^{-2\theta}$$

Thus

$$\begin{aligned} L &= \int_{\pi/4}^{\infty} a e^{\pi/4} e^{-\theta} d\theta = a e^{\pi/4} \lim_{t \rightarrow \infty} \int_{\pi/4}^t e^{-\theta} d\theta = a e^{\pi/4} \lim_{t \rightarrow \infty} [-e^{-\theta}]_{\pi/4}^t \\ &= a e^{\pi/4} \lim_{t \rightarrow \infty} [e^{-\pi/4} - e^{-t}] = a e^{\pi/4} e^{-\pi/4} = a \end{aligned}$$

5. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\frac{3t_1}{1+t_1^3} = a$ and $\frac{3t_1^2}{1+t_1^3} = b$. If

$t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \neq 0$, then the point corresponding to $t = \frac{1}{t_1}$ is

given by $x = \frac{3(1/t_1)}{1+(1/t_1)^3} = \frac{3t_1^2}{t_1^3+1} = b$, $y = \frac{3(1/t_1)^2}{1+(1/t_1)^3} = \frac{3t_1}{t_1^3+1} = a$. So (b, a) also lies on the curve.

[Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when

$$\frac{3t}{1+t^3} = \frac{3t^2}{1+t^3} \Rightarrow t = t^2 \Rightarrow t = 0 \text{ or } 1, \text{ so the points are } (0, 0) \text{ and } \left(\frac{3}{2}, \frac{3}{2}\right).$$

$$(b) \frac{dy}{dt} = \frac{(1+t^3)(6t) - 3t^2(3t^2)}{(1+t^3)^2} = \frac{6t - 3t^4}{(1+t^3)^2} = 0 \text{ when } 6t - 3t^4 = 3t(2 - t^3) = 0 \Rightarrow t = 0 \text{ or } t = \sqrt[3]{2},$$

so there are horizontal tangents at $(0, 0)$ and $(\sqrt[3]{2}, \sqrt[3]{4})$. Using the symmetry from part (a), we see that there are vertical tangents at $(0, 0)$ and $(\sqrt[3]{4}, \sqrt[3]{2})$.

- (c) Notice that as $t \rightarrow -1^+$, we have $x \rightarrow -\infty$ and $y \rightarrow \infty$. As $t \rightarrow -1^-$, we have $x \rightarrow \infty$ and $y \rightarrow -\infty$. Also

$$y - (-x - 1) = y + x + 1 = \frac{3t + 3t^2 + (1+t^3)}{1+t^3} = \frac{(t+1)^3}{1+t^3} = \frac{(t+1)^2}{t^2 - t + 1} \rightarrow 0 \text{ as } t \rightarrow -1. \text{ So}$$

$y = -x - 1$ is a slant asymptote.

$$(d) \frac{dx}{dt} = \frac{(1+t^3)(3) - 3t(3t^2)}{(1+t^3)^2} = \frac{3 - 6t^3}{(1+t^3)^2} \text{ and from part (b) we have } \frac{dy}{dt} = \frac{6t - 3t^4}{(1+t^3)^2}.$$

So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t(2-t^3)}{1-2t^3}$. Also

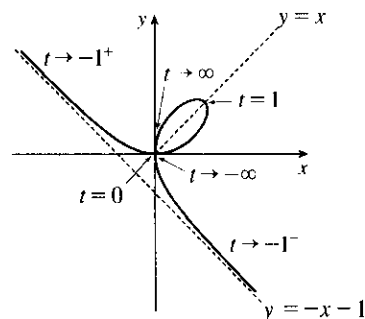
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{dx/dt} = \frac{2(1+t^3)^4}{3(1-2t^3)^3} > 0 \Leftrightarrow t < \frac{1}{\sqrt[3]{2}}. \text{ So}$$

the curve is concave upward there and has a minimum point

at $(0, 0)$ and a maximum point at $(\sqrt[3]{2}, \sqrt[3]{4})$. Using this

together with the information from parts (a), (b), and (c), we

sketch the curve.



$$(e) \ x^3 + y^3 = \left(\frac{3t}{1+t^3} \right)^3 + \left(\frac{3t^2}{1+t^3} \right)^3 = \frac{27t^3 + 27t^6}{(1+t^3)^3} = \frac{27t^3(1+t^3)}{(1+t^3)^3} = \frac{27t^3}{(1+t^3)^2} \text{ and}$$

$$3xy = 3 \left(\frac{3t}{1+t^3} \right) \left(\frac{3t^2}{1+t^3} \right) = \frac{27t^3}{(1+t^3)^2}, \text{ so } x^3 + y^3 = 3xy.$$

(f) We start with the equation from part (e) and substitute $x = r \cos \theta$, $y = r \sin \theta$. Then $x^3 + y^3 = 3xy \Rightarrow$

$$r^3 \cos^3 \theta + r^3 \sin^3 \theta = 3r^2 \cos \theta \sin \theta. \text{ For } r \neq 0, \text{ this gives } r = \frac{3 \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}. \text{ Dividing numerator and}$$

$$\text{denominator by } \cos^3 \theta, \text{ we obtain } r = \frac{3 \left(\frac{1}{\cos \theta} \right) \frac{\sin \theta}{\cos \theta}}{1 + \frac{\sin^3 \theta}{\cos^3 \theta}} = \frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}.$$

(g) The loop corresponds to $\theta \in (0, \frac{\pi}{2})$, so its area is

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{r^2}{2} d\theta = \frac{1}{2} \int_0^{\pi/2} \left(\frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta} \right)^2 d\theta = \frac{9}{2} \int_0^{\pi/2} \frac{\sec^2 \theta \tan^2 \theta}{(1 + \tan^3 \theta)^2} d\theta \\ &= \frac{9}{2} \int_0^\infty \frac{u^2 du}{(1 + u^3)^2} \quad [\text{let } u = \tan \theta] = \lim_{b \rightarrow \infty} \frac{9}{2} \left[-\frac{1}{3} (1 + u^3)^{-1} \right]_0^b = \frac{3}{2} \end{aligned}$$

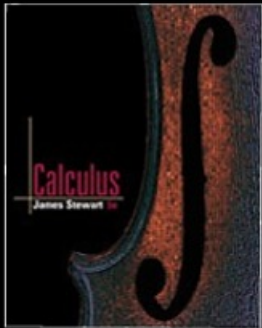
(h) By symmetry, the area between the folium and the line $y = -x - 1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since

$$y = -x - 1 \Rightarrow r \sin \theta = -r \cos \theta - 1 \Rightarrow r = -\frac{1}{\sin \theta + \cos \theta}, \text{ the area in the fourth quadrant is}$$

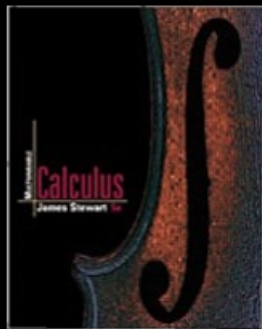
$$\frac{1}{2} \int_{-\pi/2}^{-\pi/4} \left[\left(-\frac{1}{\sin \theta + \cos \theta} \right)^2 - \left(\frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta} \right)^2 \right] d\theta \stackrel{\text{CAS}}{=} \frac{1}{2}. \text{ Therefore, the total area is } \frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}.$$

Chapter 12

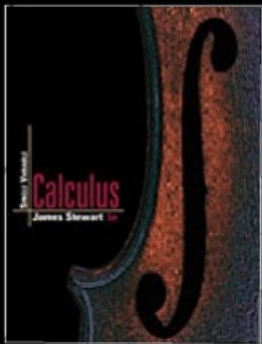
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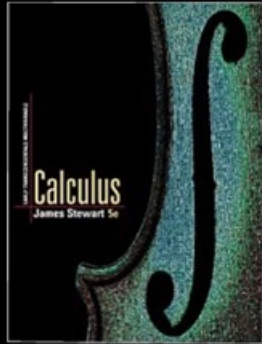
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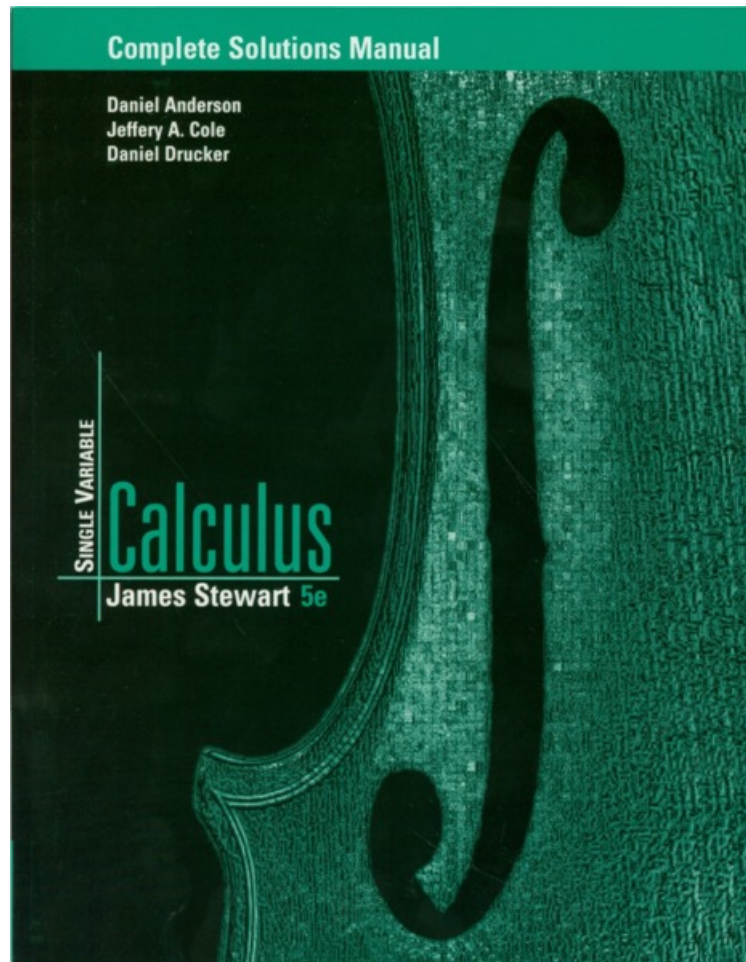
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12 INFINITE SEQUENCES AND SERIES

12.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.
(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.
2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n \rightarrow \infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$
(b) A divergent sequence is a sequence for which $\lim_{n \rightarrow \infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$
3. $a_n = 1 - (0.2)^n$, so the sequence is $\{0.8, 0.96, 0.992, 0.9984, 0.99968, \dots\}$.
4. $a_n = \frac{n+1}{3n-1}$, so the sequence is $\left\{\frac{2}{2}, \frac{3}{5}, \frac{4}{8}, \frac{5}{11}, \frac{6}{14}, \dots\right\} = \left\{1, \frac{3}{5}, \frac{1}{2}, \frac{5}{11}, \frac{3}{7}, \dots\right\}$.
5. $a_n = \frac{3(-1)^n}{n!}$, so the sequence is $\left\{\frac{-3}{1}, \frac{3}{2}, \frac{-3}{6}, \frac{3}{24}, \frac{-3}{120}, \dots\right\} = \left\{-3, \frac{3}{2}, -\frac{1}{2}, \frac{1}{8}, -\frac{1}{40}, \dots\right\}$.
6. $a_n = 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)$, so the sequence is
 $\{2, 2 \cdot 4, 2 \cdot 4 \cdot 6, 2 \cdot 4 \cdot 6 \cdot 8, 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10, \dots\} = \{2, 8, 48, 384, 3840, \dots\}$.
7. $a_1 = 3$, $a_{n+1} = 2a_n - 1$. Each term is defined in terms of the preceding term.
 $a_2 = 2a_1 - 1 = 2(3) - 1 = 5$. $a_3 = 2a_2 - 1 = 2(5) - 1 = 9$. $a_4 = 2a_3 - 1 = 2(9) - 1 = 17$.
 $a_5 = 2a_4 - 1 = 2(17) - 1 = 33$. The sequence is $\{3, 5, 9, 17, 33, \dots\}$.
8. $a_1 = 4$, $a_{n+1} = \frac{a_n}{a_n - 1}$. Each term is defined in terms of the preceding term.
 $a_2 = \frac{a_1}{a_1 - 1} = \frac{4}{4 - 1} = \frac{4}{3}$. $a_3 = \frac{a_2}{a_2 - 1} = \frac{4/3}{4/3 - 1} = \frac{4/3}{1/3} = 4$. Since $a_3 = a_1$, we can see that the terms of the sequence will alternately equal 4 and $4/3$, so the sequence is $\{4, \frac{4}{3}, 4, \frac{4}{3}, 4, \dots\}$.
9. The numerators are all 1 and the denominators are powers of 2, so $a_n = \frac{1}{2^n}$.
10. The numerators are all 1 and the denominators are multiples of 2, so $a_n = \frac{1}{2n}$.
11. $\{2, 7, 12, 17, \dots\}$. Each term is larger than the preceding one by 5, so
 $a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.
12. $\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\}$. The numerator of the n th term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \frac{n}{(n+1)^2}$.
13. $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \dots\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = (-\frac{2}{3})^{n-1}$.
14. $\{5, 1, 5, 1, 5, 1, \dots\}$. The average of 5 and 1 is 3, so we can think of the sequence as alternately adding 2 and -2 to 3. Thus, $a_n = 3 + (-1)^{n+1} \cdot 2$.
15. $a_n = n(n-1)$. $a_n \rightarrow \infty$ as $n \rightarrow \infty$, so the sequence diverges.

16. $a_n = \frac{n+1}{3n-1} = \frac{1+1/n}{3-1/n}$, so $a_n \rightarrow \frac{1+0}{3-0} = \frac{1}{3}$ as $n \rightarrow \infty$. Converges

17. $a_n = \frac{3+5n^2}{n+n^2} = \frac{(3+5n^2)/n^2}{(n+n^2)/n^2} = \frac{5+3/n^2}{1+1/n}$, so $a_n \rightarrow \frac{5+0}{1+0} = 5$ as $n \rightarrow \infty$. Converges

18. $a_n = \frac{\sqrt{n}}{1+\sqrt{n}} = \frac{1}{1/\sqrt{n}+1}$, so $a_n \rightarrow \frac{1}{0+1} = 1$ as $n \rightarrow \infty$. Converges

19. $a_n = \frac{2^n}{3^{n+1}} = \frac{1}{3} \left(\frac{2}{3}\right)^n$, so $\lim_{n \rightarrow \infty} a_n = \frac{1}{3} \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot 0 = 0$ by (8) with $r = \frac{2}{3}$. Converges

20. $a_n = \frac{n}{1+\sqrt{n}} = \frac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as $n \rightarrow \infty$, so $a_n \rightarrow \infty$ as $n \rightarrow \infty$ and the sequence diverges.

21. $a_n = \frac{(-1)^{n-1}n}{n^2+1} = \frac{(-1)^{n-1}}{n+1/n}$, so $0 \leq |a_n| = \frac{1}{n+1/n} \leq \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$, so $a_n \rightarrow 0$ by the Squeeze Theorem and Theorem 6. Converges

22. $a_n = \frac{(-1)^n n^3}{n^3+2n^2+1}$. Now $|a_n| = \frac{n^3}{n^3+2n^2+1} = \frac{1}{1+\frac{2}{n}+\frac{1}{n^3}} \rightarrow 1$ as $n \rightarrow \infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence $a_1, a_3, a_5, \dots$ converges to -1 and the sequence $a_2, a_4, a_6, \dots$ converges to $+1$. This shows that the given sequence diverges since its terms don't approach a single real number.

23. $a_n = \cos(n/2)$. This sequence diverges since the terms don't approach any particular real number as $n \rightarrow \infty$. The terms take on values between -1 and 1 .

24. $a_n = \cos(2/n)$. As $n \rightarrow \infty$, $2/n \rightarrow 0$, so $\cos(2/n) \rightarrow \cos 0 = 1$. Converges

25. $a_n = \frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \frac{1}{(2n+1)(2n)} \rightarrow 0$ as $n \rightarrow \infty$. Converges

26. $2n \rightarrow \infty$ as $n \rightarrow \infty$, so since $\lim_{x \rightarrow \infty} \arctan x = \frac{\pi}{2}$, we have $\lim_{n \rightarrow \infty} \arctan 2n = \frac{\pi}{2}$. Converges

27. $a_n = \frac{e^n + e^{-n}}{e^{2n} - 1} \cdot \frac{e^{-n}}{e^{-n}} = \frac{1 + e^{-2n}}{e^n - e^{-n}} \rightarrow \frac{1+0}{e^n - 0} \rightarrow 0$ as $n \rightarrow \infty$. Converges

28. $a_n = \frac{\ln n}{\ln 2n} = \frac{\ln n}{\ln 2 + \ln n} = \frac{1}{\frac{\ln 2}{\ln n} + 1} \rightarrow \frac{1}{0+1} \rightarrow 1$ as $n \rightarrow \infty$. Converges

29. $a_n = n^2 e^{-n} = \frac{n^2}{e^n}$. Since $\lim_{x \rightarrow \infty} \frac{x^2}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2x}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$, it follows from Theorem 3 that $\lim_{n \rightarrow \infty} a_n = 0$. Converges

30. $a_n = n \cos n\pi = n(-1)^n$. Since $|a_n| = n \rightarrow \infty$ as $n \rightarrow \infty$, the given sequence diverges.

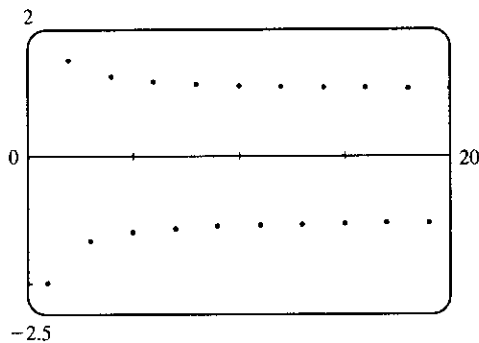
31. $0 \leq \frac{\cos^2 n}{2^n} \leq \frac{1}{2^n}$ [since $0 \leq \cos^2 n \leq 1$], so since $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$, $\left\{ \frac{\cos^2 n}{2^n} \right\}$ converges to 0 by the Squeeze Theorem.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right) \rightarrow \ln(1) = 0$ as $n \rightarrow \infty$. Converges

33. $a_n = n \sin(1/n) = \frac{\sin(1/n)}{1/n}$. Since $\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t}$ [where $t = 1/x$] = 1, it follows from Theorem 3 that $\{a_n\}$ converges to 1.

34. $a_n = \sqrt{n} - \sqrt{n^2 - 1} = \sqrt{n^2 \cdot \frac{1}{n}} - \sqrt{n^2 \left(1 - \frac{1}{n^2}\right)} = n \left(\frac{1}{\sqrt{n}} - \sqrt{1 - \frac{1}{n^2}} \right) \rightarrow n(0 - 1) \rightarrow -n$ as $n \rightarrow \infty$, so $a_n \rightarrow -\infty$ as $n \rightarrow \infty$. Diverges
35. $a_n = \left(1 + \frac{2}{n}\right)^{1/n} \Rightarrow \ln a_n = \frac{1}{n} \ln \left(1 + \frac{2}{n}\right)$. As $n \rightarrow \infty$, $\frac{1}{n} \rightarrow 0$ and $\ln \left(1 + \frac{2}{n}\right) \rightarrow 0$, so $\ln a_n \rightarrow 0$. Thus, $a_n \rightarrow e^0 = 1$ as $n \rightarrow \infty$. Converges
36. $a_n = \frac{\sin 2n}{1 + \sqrt{n}}$. $|a_n| \leq \frac{1}{1 + \sqrt{n}}$ and $\lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{n}} = 0$, so $\frac{-1}{1 + \sqrt{n}} \leq a_n \leq \frac{1}{1 + \sqrt{n}} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ by the Squeeze Theorem. Converges
37. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.
38. $\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots\}$. $a_{2n-1} = \frac{1}{n}$ and $a_{2n} = \frac{1}{n+2}$ for all positive integers n . $\lim_{n \rightarrow \infty} a_n = 0$ since $\lim_{n \rightarrow \infty} a_{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ and $\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} \frac{1}{n+2} = 0$. For n sufficiently large, a_n can be made as close to 0 as we like. Converges
39. $a_n = \frac{n!}{2^n} = \frac{1}{2} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdots \frac{(n-1)}{2} \cdot \frac{n}{2} \geq \frac{1}{2} \cdot \frac{n}{2}$ [for $n > 1$] $= \frac{n}{4} \rightarrow \infty$ as $n \rightarrow \infty$, so $\{a_n\}$ diverges.
40. $0 < |a_n| = \frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdots \frac{3}{(n-1)} \cdot \frac{3}{n} \leq \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{n}$ [for $n > 2$] $= \frac{27}{2n} \rightarrow 0$ as $n \rightarrow \infty$, so by the Squeeze Theorem and Theorem 6, $\{(-3)^n/n\}$ converges to 0.

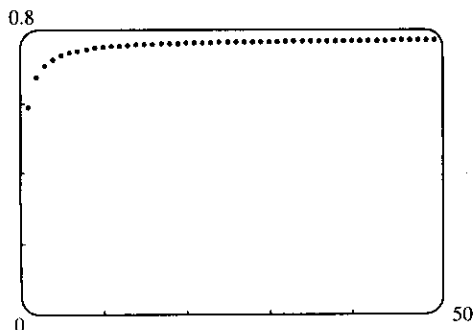
41.



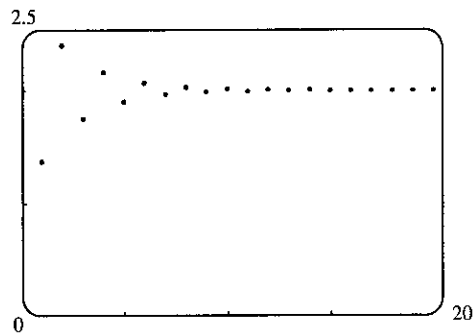
From the graph, we see that the sequence

$\left\{(-1)^n \frac{n+1}{n}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

43.



42.



From the graph, it appears that the sequence

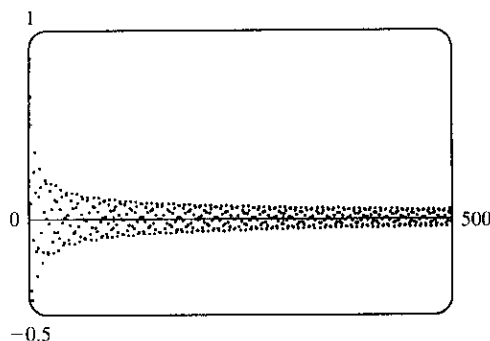
$\left\{2 + \left(-\frac{2}{\pi}\right)^n\right\}$ converges to 2 by (6), and hence $\left\{2 + \left(-\frac{2}{\pi}\right)^n\right\}$ converges to $2 + 0 = 2$.

From the graph, it appears that the sequence converges to about 0.78.

$$\lim_{n \rightarrow \infty} \frac{2n}{2n+1} = \lim_{n \rightarrow \infty} \frac{2}{2 + 1/n} = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \arctan\left(\frac{2n}{2n+1}\right) = \arctan 1 = \frac{\pi}{4}.$$

44.

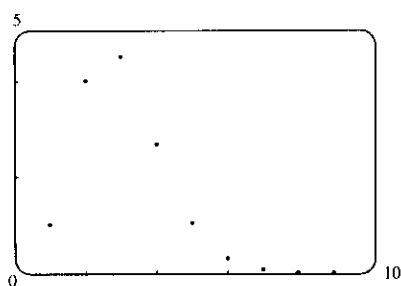


From the graph, it appears that the sequence converges (slowly) to 0.

$$0 \leq \frac{|\sin n|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ so by the}$$

Squeeze Theorem and Theorem 6, $\left\{ \frac{\sin n}{\sqrt{n}} \right\}$ converges to 0.

45.

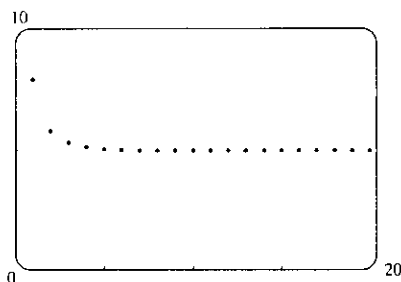


From the graph, it appears that the sequence converges to 0.

$$\begin{aligned} 0 < a_n &= \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdots \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} \\ &\leq \frac{n^2}{(n-1)(n-2)(n-3)} \quad [\text{for } n \geq 4] \\ &= \frac{1/n}{(1-1/n)(1-2/n)(1-3/n)} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

So by the Squeeze Theorem, $\{n^3/n!\}$ converges to 0.

46.



From the graph, it appears that the sequence converges to 5.

$$\begin{aligned} 5 &= \sqrt[n]{5^n} \leq \sqrt[n]{3^n + 5^n} \leq \sqrt[n]{5^n + 5^n} = \sqrt[n]{2} \sqrt[n]{5^n} \\ &= \sqrt[n]{2} \cdot 5 \rightarrow 5 \text{ as } n \rightarrow \infty \quad \left[\lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1 \right] \end{aligned}$$

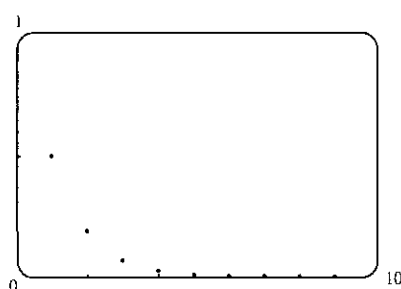
Hence, $a_n \rightarrow 5$ by the Squeeze Theorem.

Alternate Solution: Let $y = (3^n + 5^n)^{1/n}$. Then

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \frac{\ln(3^x + 5^x)}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5 \end{aligned}$$

so $\lim_{x \rightarrow \infty} y = e^{\ln 5} = 5$, and so $\{\sqrt[n]{3^n + 5^n}\}$ converges to 5.

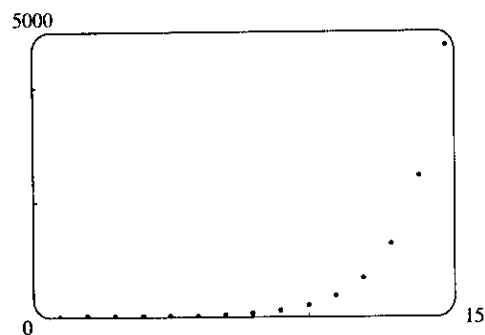
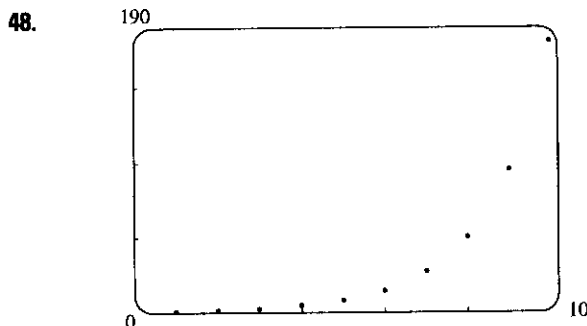
47.



From the graph, it appears that the sequence approaches 0.

$$\begin{aligned} 0 < a_n &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n} \\ &\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdots (1) = \frac{1}{2n} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

So by the Squeeze Theorem, $\left\{ \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n} \right\}$ converges to 0.



From the graphs, it seems that the sequence diverges. $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$. We first prove by induction

that $a_n \geq \left(\frac{3}{2}\right)^{n-1}$ for all n . This is clearly true for $n = 1$, so let $P(n)$ be the statement that the above is true for

n . We must show it is then true for $n + 1$. $a_{n+1} = a_n \cdot \frac{2n+1}{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{2n+1}{n+1}$ (induction hypothesis).

But $\frac{2n+1}{n+1} \geq \frac{3}{2}$ [since $2(2n+1) \geq 3(n+1) \Leftrightarrow 4n+2 \geq 3n+3 \Leftrightarrow n \geq 1$], and so we get that

$a_{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^n$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{\left(\frac{3}{2}\right)^{n-1}\right\}$

diverges (by (8)), so does the given sequence $\{a_n\}$.

49. (a) $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060, a_2 = 1123.60, a_3 = 1191.02, a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim_{n \rightarrow \infty} a_n = 1000 \lim_{n \rightarrow \infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

50. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40,

20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1 .

51. If $|r| \geq 1$, then $\{r^n\}$ diverges by (8), so $\{nr^n\}$ diverges also, since $|nr^n| = n|r^n| \geq |r^n|$. If $|r| < 1$ then

$$\lim_{x \rightarrow \infty} xr^x = \lim_{x \rightarrow \infty} \frac{x}{r^{-x}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{(-\ln r)r^{-x}} = \lim_{x \rightarrow \infty} \frac{r^x}{-\ln r} = 0, \text{ so } \lim_{n \rightarrow \infty} nr^n = 0, \text{ and hence } \{nr^n\} \text{ converges}$$

whenever $|r| < 1$.

52. (a) Let $\lim_{n \rightarrow \infty} a_n = L$. By Definition 1, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n + 1 > N \Leftrightarrow n > N - 1$. It follows that

$$\lim_{n \rightarrow \infty} a_{n+1} = L \text{ and so } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}.$$

(b) If $L = \lim_{n \rightarrow \infty} a_n$ then $\lim_{n \rightarrow \infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow$

$$L = \frac{-1+\sqrt{5}}{2} \text{ (since } L \text{ has to be non-negative if it exists).}$$

53. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L . L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

54. $a_n = 1/5^n$ defines a decreasing geometric sequence since $a_{n+1} = \frac{1}{5}a_n < a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

55. $a_n = \frac{1}{2n+3}$ is decreasing since $a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5} < \frac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

56. $a_n = \frac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \frac{2x-3}{3x+4}$,

$$f'(x) = \frac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \frac{17}{(3x+4)^2} > 0.$$
 The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$, and $a_n < \frac{2n-3}{3n} < \frac{2n}{3n} = \frac{2}{3}$ for $n \geq 1$.

57. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are 0, -1, 0, 1, 0, -1, 0, 1, ... In fact, the sequence consists of the terms 0, -1, 0, 1 repeated over and over again in that order. The sequence is bounded since $|a_n| \leq 1$ for all $n \geq 1$.

58. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$.
 $[f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0 \text{ for } x > 1.]$ The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

59. $a_n = \frac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \frac{x}{x^2+1}$,

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0 \text{ for } x \geq 1.$$
 The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

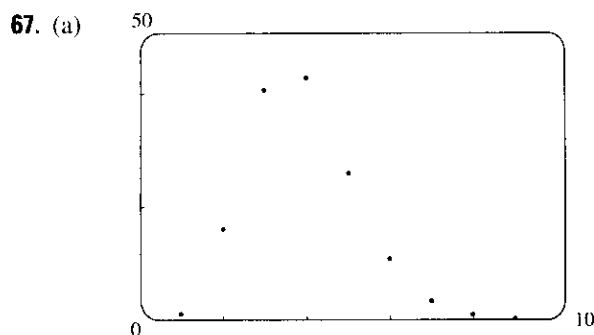
60. $a_n = n + \frac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \frac{1}{x}$ is increasing for $x > 1$.
 $[g'(x) = 1 - 1/x^2 > 0 \text{ for } x > 1.]$ The sequence is unbounded since $a_n \rightarrow \infty$ as $n \rightarrow \infty$. (It is, however, bounded below by $a_1 = 2$.)

61. $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, ..., so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 2^{1-(1/2^n)} = 2^1 = 2$.

Alternate solution: Let $L = \lim_{n \rightarrow \infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and

increasing.) Then L must satisfy $L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0$. $L \neq 0$ since the sequence increases, so $L = 2$.

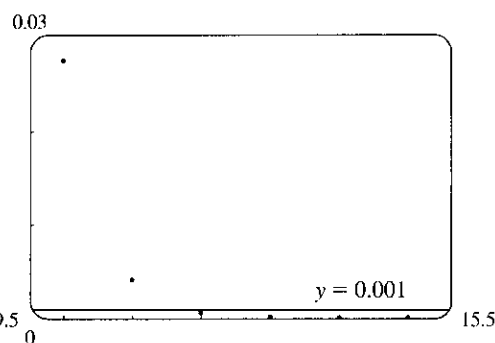
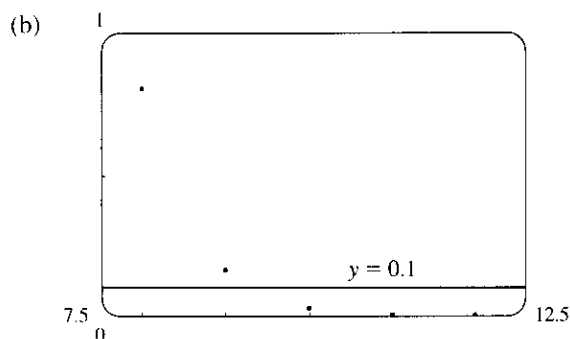
62. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \Leftrightarrow \sqrt{2+a_{n+1}} \geq \sqrt{2+a_n} \Leftrightarrow 2+a_{n+1} \geq 2+a_n \Leftrightarrow a_{n+1} \geq a_n$, which is the induction hypothesis. $a_{n+1} \leq 3 \Leftrightarrow \sqrt{2+a_n} \leq 3 \Leftrightarrow 2+a_n \leq 9 \Leftrightarrow a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n , and so $a_1 \leq a_n \leq 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim_{n \rightarrow \infty} a_n$ exists.
- (b) If $L = \lim_{n \rightarrow \infty} a_n$, then $\lim_{n \rightarrow \infty} a_{n+1} = L$ also, so $L = \sqrt{2+L} \Rightarrow L^2 = 2+L \Leftrightarrow L^2 - L - 2 = 0 \Leftrightarrow (L+1)(L-2) = 0 \Leftrightarrow L = 2$ (since L can't be negative).
63. We show by induction that $\{a_n\}$ is increasing and bounded above by 3.
- Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1} > a_n \Rightarrow \frac{1}{a_{n+1}} < \frac{1}{a_n} \Rightarrow -\frac{1}{a_{n+1}} > -\frac{1}{a_n}$.
- Now $a_{n+2} = 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n} = a_{n+1} \Leftrightarrow P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem.
- If $L = \lim_{n \rightarrow \infty} a_n$, then $\lim_{n \rightarrow \infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3 + \sqrt{5}}{2}$.
64. We use induction. Let P_n be the statement that $0 < a_{n+1} \leq a_n \leq 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \leq a_n \Rightarrow -a_{n+1} \geq -a_n \Rightarrow 3 - a_{n+1} \geq 3 - a_n \Rightarrow a_{n+2} = \frac{1}{3 - a_{n+1}} \leq \frac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ (since $3 - a_{n+1}$ is positive) and $a_{n+1} \leq 2$ by the induction hypothesis, so P_{n+1} is true.
- To find the limit, we use the fact that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} \Rightarrow L = \frac{1}{3-L} \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$. But $L \leq 2$, so we must have $L = \frac{3 - \sqrt{5}}{2}$.
65. (a) Let a_n be the number of rabbit pairs in the n th month. Clearly $a_1 = 1 = a_2$. In the n th month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.
- (b) $a_n = \frac{f_{n+1}}{f_n} \Rightarrow a_{n-1} = \frac{f_n}{f_{n-1}} = \frac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \frac{f_{n-2}}{f_{n-1}} = 1 + \frac{1}{f_{n-1}/f_{n-2}} = 1 + \frac{1}{a_{n-2}}$. If $L = \lim_{n \rightarrow \infty} a_n$, then $L = \lim_{n \rightarrow \infty} a_{n-1}$ and $L = \lim_{n \rightarrow \infty} a_{n-2}$, so L must satisfy $L = 1 + \frac{1}{L} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1 + \sqrt{5}}{2}$ (since L must be positive).
66. (a) If f is continuous, then $f(L) = f\left(\lim_{n \rightarrow \infty} a_n\right) = \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} a_{n+1} = L$ by Exercise 52(a).
- (b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.



From the graph, it appears that the

sequence $\left\{\frac{n^5}{n!}\right\}$ converges to 0, that is,

$$\lim_{n \rightarrow \infty} \frac{n^5}{n!} = 0.$$



From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon = 0.1$ is 9, since $n^5/n! < 0.1$ whenever $n \geq 10$, but $9^5/9! > 0.1$. From the second graph, it seems that for $\varepsilon = 0.001$, the smallest possible value for N is 11.

68. Let $\varepsilon > 0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n > N$ then $n > \ln(\varepsilon)/\ln|r| \Rightarrow n \ln|r| < \ln \varepsilon$ [since $|r| < 1 \Rightarrow \ln|r| < 0] \Rightarrow \ln(|r|^n) < \ln \varepsilon \Rightarrow |r|^n < \varepsilon \Rightarrow |r^n - 0| < \varepsilon$, and so by Definition 1, $\lim_{n \rightarrow \infty} r^n = 0$.

69. If $\lim_{n \rightarrow \infty} |a_n| = 0$ then $\lim_{n \rightarrow \infty} -|a_n| = 0$, and since $-|a_n| \leq a_n \leq |a_n|$, we have that $\lim_{n \rightarrow \infty} a_n = 0$ by the Squeeze Theorem.

70. (a)
$$\frac{b^{n+1} - a^{n+1}}{b - a} = b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n$$

$$< b^n + b^{n-1}b + b^{n-2}b^2 + b^{n-3}b^3 + \cdots + bb^{n-1} + b^n = (n+1)b^n$$

(b) Since $b - a > 0$, we have $b^{n+1} - a^{n+1} < (n+1)b^n(b-a) \Rightarrow b^{n+1} - (n+1)b^n(b-a) < a^{n+1} \Rightarrow b^n[(n+1)a - nb] < a^{n+1}$.

(c) With this substitution, $(n+1)a - nb = 1$, and so $b^n = \left(1 + \frac{1}{n}\right)^n < a^{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1 + \frac{1}{2n}\right)^n \left(\frac{1}{2}\right) < 1 \Rightarrow \left(1 + \frac{1}{2n}\right)^n < 2 \Rightarrow \left(1 + \frac{1}{2n}\right)^{2n} < 4$.

(e) $a_n < a_{2n}$ since $\{a_n\}$ is increasing, so $a_n < a_{2n} < 4$.

(f) Since $\{a_n\}$ is increasing and bounded above by 4, $a_1 \leq a_n \leq 4$, and so $\{a_n\}$ is bounded and monotonic, and hence has a limit by Theorem 11.

71. (a) First we show that $a > a_1 > b_1 > b$.

$$a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0 \quad (\text{since } a > b) \Rightarrow a_1 > b_1. \text{ Also}$$

$$a - a_1 = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) > 0 \text{ and } b - b_1 = b - \sqrt{ab} = \sqrt{b}(\sqrt{b} - \sqrt{a}) < 0, \text{ so } a > a_1 > b_1 > b.$$

In the same way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is, $a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$a_{k+2} - b_{k+2} = \frac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \frac{1}{2}(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1})$$

$$= \frac{1}{2}(\sqrt{a_{k+1}} - \sqrt{b_{k+1}})^2 > 0$$

$$a_{k+1} - a_{k+2} = a_{k+1} - \frac{1}{2}(a_{k+1} + b_{k+1}) = \frac{1}{2}(a_{k+1} - b_{k+1}) > 0$$

$$\text{and } b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}) < 0 \Rightarrow$$

$a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$, so the assertion is true for $n = k + 1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

$$\begin{aligned} \text{(c) Let } \lim_{n \rightarrow \infty} a_n = \alpha \text{ and } \lim_{n \rightarrow \infty} b_n = \beta. \text{ Then } \lim_{n \rightarrow \infty} a_{n+1} &= \lim_{n \rightarrow \infty} \frac{a_n + b_n}{2} \Rightarrow \alpha = \frac{\alpha + \beta}{2} \Rightarrow \\ 2\alpha &= \alpha + \beta \Rightarrow \alpha = \beta. \end{aligned}$$

72. (a) Let $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim_{n \rightarrow \infty} a_{2n+1} = L$, there exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then $n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim_{n \rightarrow \infty} a_n = L$.

$$\text{(b) } a_1 = 1, a_2 = 1 + \frac{1}{1+1} = \frac{3}{2} = 1.5, a_3 = 1 + \frac{1}{5/2} = \frac{7}{5} = 1.4, a_4 = 1 + \frac{1}{12/5} = \frac{17}{12} = 1.41\bar{6},$$

$$a_5 = 1 + \frac{1}{29/12} = \frac{41}{29} \approx 1.413793, a_6 = 1 + \frac{1}{70/29} = \frac{99}{70} \approx 1.414286, a_7 = 1 + \frac{1}{169/70} = \frac{239}{169} \approx 1.414201,$$

$$a_8 = 1 + \frac{1}{408/169} = \frac{577}{408} \approx 1.414216. \text{ Notice that } a_1 < a_3 < a_5 < a_7 \text{ and } a_2 > a_4 > a_6 > a_8. \text{ It appears}$$

that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2} > a_{2n}$ and

$a_{2n-1} < a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2} > a_{2k}$. Then $1 + a_{2k-2} > 1 + a_{2k} \Rightarrow$

$$\frac{1}{1 + a_{2k-2}} < \frac{1}{1 + a_{2k}} \Rightarrow 1 + \frac{1}{1 + a_{2k-2}} < 1 + \frac{1}{1 + a_{2k}} \Rightarrow a_{2k-1} < a_{2k+1} \Rightarrow$$

$$1 + a_{2k-1} < 1 + a_{2k+1} \Rightarrow \frac{1}{1 + a_{2k-1}} > \frac{1}{1 + a_{2k+1}} \Rightarrow 1 + \frac{1}{1 + a_{2k-1}} > 1 + \frac{1}{1 + a_{2k+1}} \Rightarrow$$

$a_{2k} > a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2, so both $\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and are therefore convergent by Theorem 11. Let $\lim_{n \rightarrow \infty} a_{2n} = L$. Then $\lim_{n \rightarrow \infty} a_{2n+2} = L$ also. We have

$$a_{n+2} = 1 + \frac{1}{1 + 1 + 1/(1 + a_n)} = 1 + \frac{1}{(3 + 2a_n)/(1 + a_n)} = \frac{4 + 3a_n}{3 + 2a_n}, \text{ so } a_{2n+2} = \frac{4 + 3a_{2n}}{3 + 2a_{2n}}. \text{ Taking}$$

$$\text{limits of both sides, we get } L = \frac{4 + 3L}{3 + 2L} \Rightarrow 3L + 2L^2 = 4 + 3L \Rightarrow L^2 = 2 \Rightarrow L = \sqrt{2} \text{ (since}$$

$L > 0$). Thus, $\lim_{n \rightarrow \infty} a_{2n} = \sqrt{2}$. Similarly we find that $\lim_{n \rightarrow \infty} a_{2n+1} = \sqrt{2}$. So, by part (a), $\lim_{n \rightarrow \infty} a_n = \sqrt{2}$.

$$73. \text{ (a) Suppose } \{p_n\} \text{ converges to } p. \text{ Then } p_{n+1} = \frac{bp_n}{a + p_n} \Rightarrow \lim_{n \rightarrow \infty} p_{n+1} = \frac{b \lim_{n \rightarrow \infty} p_n}{a + \lim_{n \rightarrow \infty} p_n} \Rightarrow$$

$$p = \frac{bp}{a + p} \Rightarrow p^2 + ap = bp \Rightarrow p(p + a - b) = 0 \Rightarrow p = 0 \text{ or } p = b - a.$$

$$\text{(b) } p_{n+1} = \frac{bp_n}{a + p_n} = \frac{\frac{b}{a}p_n}{1 + \frac{p_n}{a}} < \frac{b}{a}p_n \text{ since } 1 + \frac{p_n}{a} > 1.$$

(c) By part (b), $p_1 < \left(\frac{b}{a}\right)p_0$, $p_2 < \left(\frac{b}{a}\right)p_1 < \left(\frac{b}{a}\right)^2 p_0$, $p_3 < \left(\frac{b}{a}\right)p_2 < \left(\frac{b}{a}\right)^3 p_0$, etc. In general,

$$p_n < \left(\frac{b}{a}\right)^n p_0, \text{ so } \lim_{n \rightarrow \infty} p_n \leq \lim_{n \rightarrow \infty} \left(\frac{b}{a}\right)^n \cdot p_0 = 0 \text{ since } b < a. [\text{By result 8, } \lim_{n \rightarrow \infty} r^n = 0 \text{ if } -1 < r < 1.]$$

$$\text{Here } r = \frac{b}{a} \in (0, 1).]$$

(d) Let $a < b$. We first show, by induction, that if $p_0 < b - a$, then $p_n < b - a$ and $p_{n+1} > p_n$.

$$\text{For } n = 0, \text{ we have } p_1 - p_0 = \frac{bp_0}{a + p_0} - p_0 = \frac{p_0(b - a - p_0)}{a + p_0} > 0 \text{ since } p_0 < b - a. \text{ So } p_1 > p_0.$$

Now we suppose the assertion is true for $n = k$, that is, $p_k < b - a$ and $p_{k+1} > p_k$. Then

$$b - a - p_{k+1} = b - a - \frac{bp_k}{a + p_k} = \frac{a(b - a) + bp_k - ap_k - bp_k}{a + p_k} = \frac{a(b - a - p_k)}{a + p_k} > 0 \text{ because } p_k < b - a.$$

$$\text{So } p_{k+1} < b - a. \text{ And } p_{k+2} - p_{k+1} = \frac{bp_{k+1}}{a + p_{k+1}} - p_{k+1} = \frac{p_{k+1}(b - a - p_{k+1})}{a + p_{k+1}} > 0 \text{ since } p_{k+1} < b - a.$$

Therefore, $p_{k+2} > p_{k+1}$. Thus, the assertion is true for $n = k + 1$. It is therefore true for all n by mathematical induction. A similar proof by induction shows that if $p_0 > b - a$, then $p_n > b - a$ and $\{p_n\}$ is decreasing.

In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem. It then follows from part (a) that $\lim_{n \rightarrow \infty} p_n = b - a$.

LABORATORY PROJECT Logistic Sequences

1. To write such a program in Maple it is best to calculate all the points first and then graph them. One possible sequence of commands [taking $p_0 = \frac{1}{2}$ and $k = 1.5$ for the difference equation] is

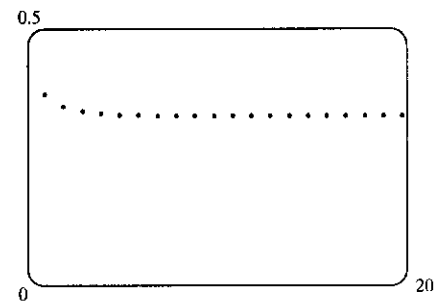
```
t:='t'; p(0):=1/2; k:=1.5;
for j from 1 to 20 do p(j):=k*p(j-1)*(1-p(j-1)) od;
plot([seq([t,p(t)] t=0..20)], t=0..20, p=0..0.5, style=point);
```

In Mathematica, we can use the following program:

```
p[0]=1/2
k=1.5
p[j_]:=k*p[j-1]*(1-p[j-1])
P=Table[p[t],{t,20}]
ListPlot[P]
```

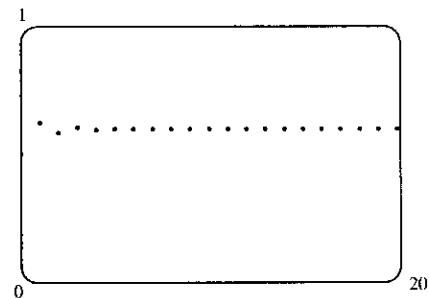
With $p_0 = \frac{1}{2}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.5	7	0.3338465076	14	0.3333373303
1	0.375	8	0.3335895255	15	0.3333353318
2	0.3515625	9	0.3334613309	16	0.3333343326
3	0.3419494629	10	0.3333973076	17	0.3333338329
4	0.3375300416	11	0.3333653143	18	0.3333335831
5	0.3354052689	12	0.3333493223	19	0.3333334582
6	0.3343628617	13	0.3333413274	20	0.3333333958



With $p_0 = \frac{1}{2}$ and $k = 2.5$:

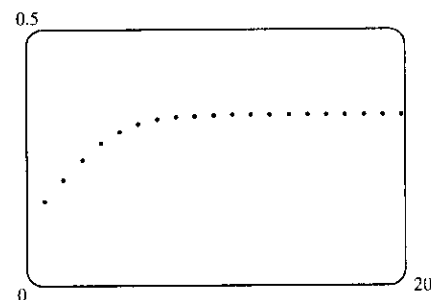
n	p_n	n	p_n	n	p_n
0	0.5	7	0.6004164790	14	0.5999967417
1	0.625	8	0.5997913269	15	0.6000016291
2	0.5859375	9	0.6001042277	16	0.5999991854
3	0.6065368651	10	0.5999478590	17	0.6000004073
4	0.5966247409	11	0.6000260637	18	0.5999997964
5	0.6016591486	12	0.5999869664	19	0.6000001018
6	0.5991635437	13	0.6000065164	20	0.5999999491



Both of these sequences seem to converge (the first to about $\frac{1}{3}$, the second to about 0.60).

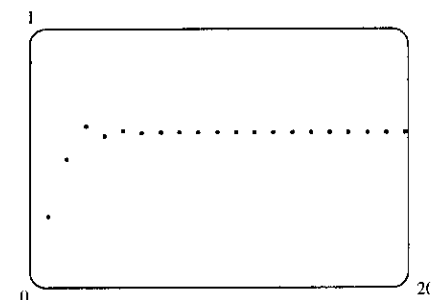
With $p_0 = \frac{7}{8}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.3239166554	14	0.3332554829
1	0.1640625	8	0.3284919837	15	0.3332943990
2	0.2057189941	9	0.3308775005	16	0.3333138639
3	0.2450980344	10	0.3320963702	17	0.3333235980
4	0.2775374819	11	0.3327125567	18	0.3333284655
5	0.3007656421	12	0.3330223670	19	0.3333308994
6	0.3154585059	13	0.3331777051	20	0.3333321164



With $p_0 = \frac{7}{8}$ and $k = 2.5$:

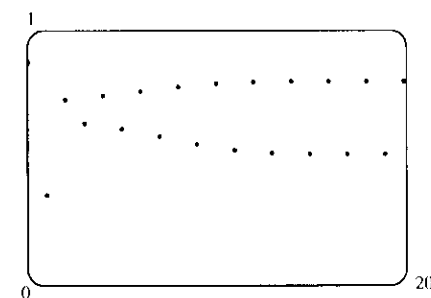
n	p_n	n	p_n	n	p_n
0	0.875	7	0.6016572368	14	0.5999869815
1	0.2734375	8	0.5991645155	15	0.6000065088
2	0.4966735840	9	0.6004159972	16	0.5999967455
3	0.6249723374	10	0.5997915688	17	0.6000016272
4	0.5859547872	11	0.6001041070	18	0.5999991864
5	0.6065294364	12	0.5999479194	19	0.6000004068
6	0.5966286980	13	0.6000260335	20	0.5999997966



The limit of the sequence seems to depend on k , but not on p_0 .

2. With $p_0 = \frac{7}{8}$ and $k = 3.2$:

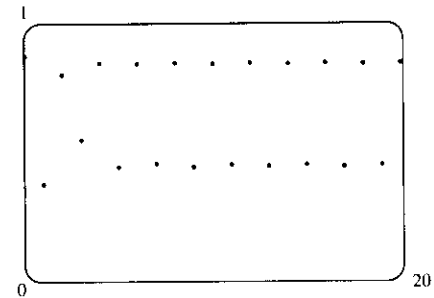
n	p_n	n	p_n	n	p_n
0	0.875	7	0.5830728495	14	0.7990633827
1	0.35	8	0.7779164854	15	0.5137954979
2	0.728	9	0.5528397669	16	0.7993909896
3	0.6336512	10	0.7910654689	17	0.5131681132
4	0.7428395416	11	0.5288988570	18	0.7994451225
5	0.6112926626	12	0.7973275394	19	0.5130643795
6	0.7603646184	13	0.5171082698	20	0.7994538304



It seems that eventually the terms fluctuate between two values (about 0.5 and 0.8 in this case).

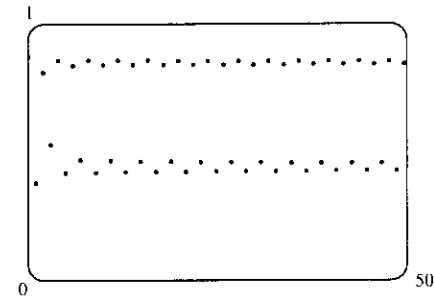
3. With $p_0 = \frac{7}{8}$ and $k = 3.42$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4523028596	14	0.8442074951
1	0.3740625	8	0.8472194412	15	0.4498025048
2	0.8007579316	9	0.4426802161	16	0.8463823232
3	0.5456427596	10	0.8437633929	17	0.4446659586
4	0.8478752457	11	0.4508474156	18	0.8445284520
5	0.4411212220	12	0.8467373602	19	0.4490464985
6	0.8431438501	13	0.4438243545	20	0.8461207931



With $p_0 = \frac{7}{8}$ and $k = 3.45$:

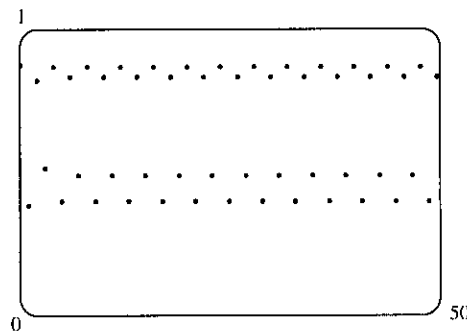
n	p_n	n	p_n	n	p_n
0	0.875	7	0.4670259170	14	0.8403376122
1	0.37734375	8	0.8587488490	15	0.4628875685
2	0.8105962830	9	0.4184824586	16	0.8577482026
3	0.5296783241	10	0.8395743720	17	0.4209559716
4	0.8594612299	11	0.4646778983	18	0.8409445432
5	0.4167173034	12	0.8581956045	19	0.4614610237
6	0.8385707740	13	0.4198508858	20	0.8573758782



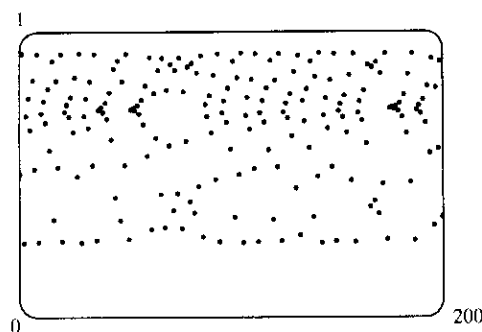
From the graphs above, it seems that for k between 3.4 and 3.5, the terms eventually fluctuate between four values.

In the graph below, the pattern followed by the terms is 0.395, 0.832, 0.487, 0.869, 0.395, ... Note that even for $k = 3.42$ (as in the first graph), there are four distinct “branches; even after 1000 terms, the first and third terms in the pattern differ by about 2×10^{-9} , while the first and fifth terms differ by only 2×10^{-10} .

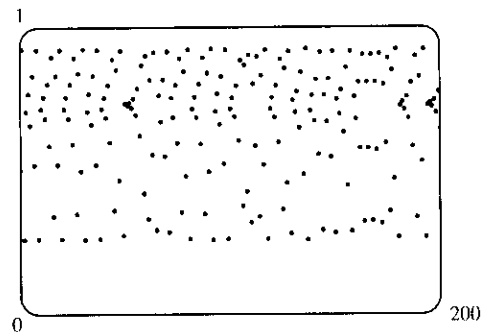
With $p_0 = \frac{7}{8}$ and $k = 3.48$:



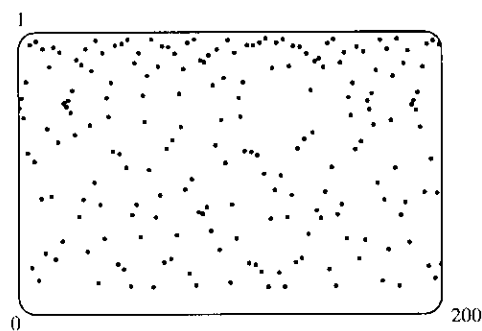
4.



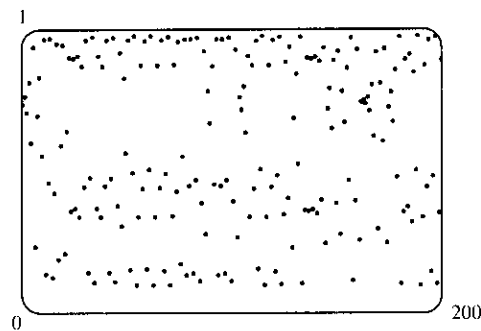
$$p_0 = 0.5, k = 3.7$$



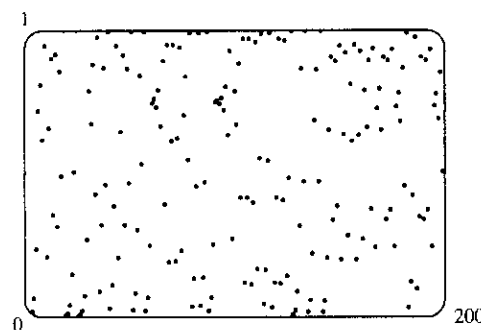
$$p_0 = 0.501, k = 3.7$$



$$p_0 = 0.75, k = 3.9$$



$$p_0 = 0.749, k = 3.9$$



$$p_0 = 0.5, k = 3.999$$

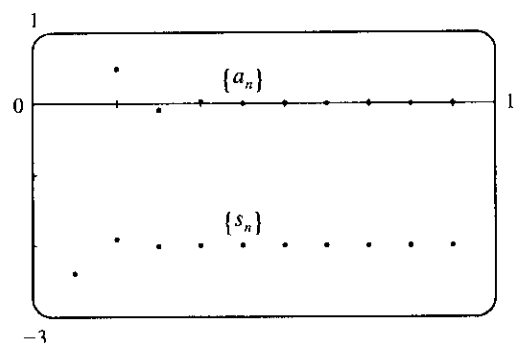
From the graphs, it seems that if p_0 is changed by 0.001, the whole graph changes completely. (Note, however, that this might be partially due to accumulated round-off error in the CAS. These graphs were generated by Maple with 100-digit accuracy, and different degrees of accuracy give different graphs.) There seem to be some fleeting patterns in these graphs, but on the whole they are certainly very chaotic. As k increases, the graph spreads out vertically, with more extreme values close to 0 or 1.

12.2 Series

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
 (b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.
2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n \rightarrow \infty} s_n = 5$, where s_n is the n th partial sum, that is, $\sum_{i=1}^n a_i$.

3.

n	s_n
1	-2.40000
2	-1.92000
3	-2.01600
4	-1.99680
5	-2.00064
6	-1.99987
7	-2.00003
8	-1.99999
9	-2.00000
10	-2.00000



From the graph and the table, it seems that the series converges to -2 . In fact, it is a geometric series with $a = -2.4$ and $r = -\frac{1}{5}$, so its sum is

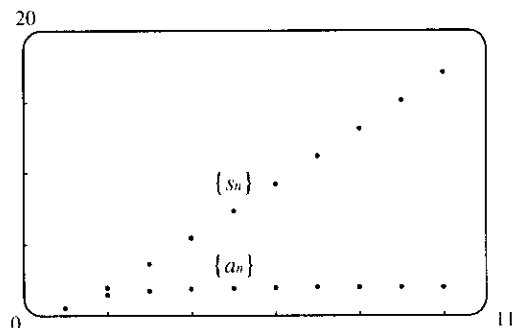
$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-2.4}{1 - (-\frac{1}{5})} = \frac{-2.4}{1.2} = -2. \text{ Note that the dot corresponding to}$$

$n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

TI-86 Note: To graph $\{a_n\}$ and $\{s_n\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under GRAPH, MORE, FORMT (F3).) Now under $E(t) =$ make the assignments: $xt1=t$, $yt1=12/(-5)^t$, $xt2=t$, $yt2=\text{sum seq}(yt1, t, 1, t, 1)$. (sum and seq are under LIST, OPS (F5), MORE.) Under WIND use 1, 10, 1, 0, 10, 1, -3, 1, 1 to obtain a graph similar to the one above. Then use TRACE (F4) to see the values.

4.

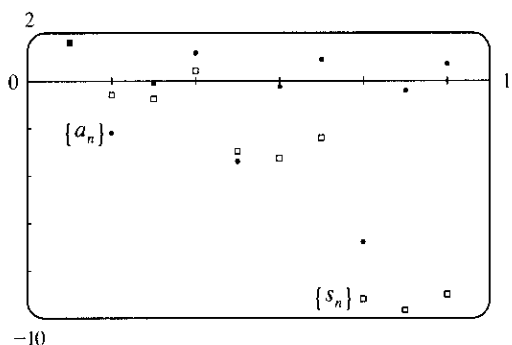
n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462



The series $\sum_{n=1}^{\infty} \frac{2n^2 - 1}{n^2 + 1}$ diverges, since its terms do not approach 0.

5.

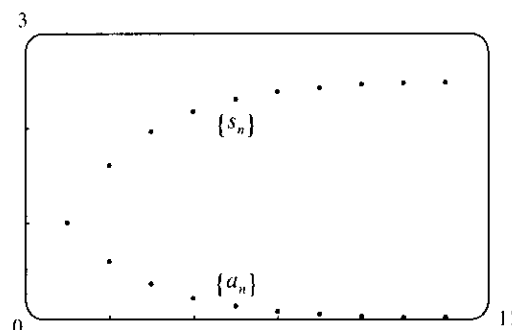
n	s_n
1	1.55741
2	-0.62763
3	-0.77018
4	0.38764
5	-2.99287
6	-3.28388
7	-2.41243
8	-9.21214
9	-9.66446
10	-9.01610



The series $\sum_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

6.

n	s_n
1	1.00000
2	1.60000
3	1.96000
4	2.17600
5	2.30560
6	2.38336
7	2.43002
8	2.45801
9	2.47481
10	2.48488



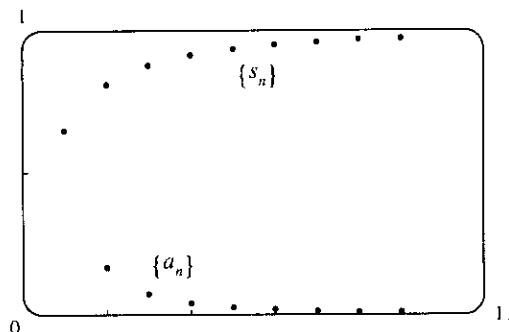
From the graph and the table, it seems that the series converges to 2.5.

In fact, it is a geometric series with $a = 1$ and $r = 0.6$, so its sum is

$$\sum_{n=1}^{\infty} (0.6)^{n-1} = \frac{1}{1 - 0.6} = \frac{1}{2/5} = 2.5.$$

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



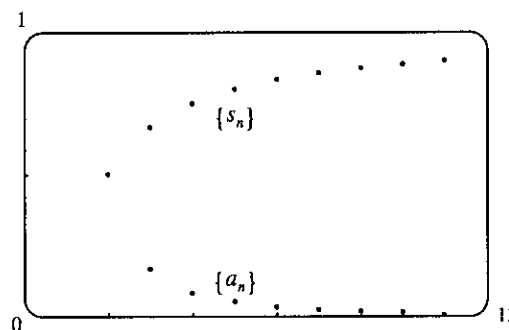
From the graph, it seems that the series converges to 1. To find the sum, we write

$$\begin{aligned}
 s_n &= \sum_{i=1}^n \left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}} \right) \\
 &= \left(1 - \frac{1}{2^{1.5}} \right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}} \right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}} \right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}} \right) = 1 - \frac{1}{(n+1)^{1.5}}
 \end{aligned}$$

So the sum is $\lim_{n \rightarrow \infty} s_n = 1 - 0 = 1$.

8.

n	s_n
2	0.50000
3	0.66667
4	0.75000
5	0.80000
6	0.83333
7	0.85714
8	0.87500
9	0.88889
10	0.90000
11	0.90909
100	0.99000



From the graph and the table, it seems that the series converges to 1. To find the sum, we write

$$\begin{aligned}
 s_n &= \sum_{i=2}^n \frac{1}{i(i-1)} = \sum_{i=2}^n \left(\frac{1}{i-1} - \frac{1}{i} \right) \quad [\text{partial fractions}] \\
 &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1 - \frac{1}{n},
 \end{aligned}$$

and so the sum is $\lim_{n \rightarrow \infty} s_n = 1 - 0 = 1$.

9. (a) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the sequence $\{a_n\}$ is convergent by (12.1.1).
 (b) Since $\lim_{n \rightarrow \infty} a_n = \frac{2}{3} \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).
10. (a) Both $\sum_{i=1}^n a_i$ and $\sum_{j=1}^n a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the n th partial sum.
 (b) $\sum_{i=1}^n a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$.
11. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ is a geometric series with first term $a = 3$ and common ratio $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{1/3} = 9$.
12. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ is a geometric series with $r = -2$. Since $|r| = 2 > 1$, the series diverges.
13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).
14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.
15. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.
16. $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.
17. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.
18. $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$ is a geometric series with ratio $r = \frac{1}{\sqrt{2}}$. Since $|r| = \frac{1}{\sqrt{2}} < 1$, the series converges. Its sum is $\frac{1}{1-1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}(\sqrt{2}+1) = 2 + \sqrt{2}$.
19. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n$ is a geometric series with ratio $r = \frac{\pi}{3}$. Since $|r| > 1$, the series diverges.
20. $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = 3 \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n$ is a geometric series with first term $3(e/3) = e$ and ratio $r = \frac{e}{3}$. Since $|r| < 1$, the series converges. Its sum is $\frac{e}{1-e/3} = \frac{3e}{3-e}$.
21. $\sum_{n=1}^{\infty} \frac{n}{n+5}$ diverges since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

22. $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. [If $\sum_{n=1}^{\infty} \frac{3}{n}$ were to converge, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Using partial fractions, the partial sums are

$$\begin{aligned} s_n &= \sum_{i=2}^n \frac{2}{(i-1)(i+1)} = \sum_{i=2}^n \left(\frac{1}{i-1} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1} \right) + \left(\frac{1}{n-2} - \frac{1}{n} \right) \end{aligned}$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

$$\text{Thus, } \sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n} \right) = \frac{3}{2}.$$

24. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{n^2 + 2n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2 + 2n} \right) = 1 \neq 0.$$

25. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

26. Converges. $s_n = \sum_{i=1}^n \frac{2}{i^2 + 4i + 3} = \sum_{i=1}^n \left(\frac{1}{i+1} - \frac{1}{i+3} \right)$ (using partial fractions). The latter sum is

$$\left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2} \right) + \left(\frac{1}{n+1} - \frac{1}{n+3} \right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\text{(telescoping series). Thus, } \sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3} = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

27. Converges. $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n} \right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n + \left(\frac{1}{3} \right)^n \right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

28. $\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n$ [difference of two convergent geometric series]
 $= \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}.$

29. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt[n]{2} = \lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1 \neq 0.$$

30. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln \left(\frac{n}{2n+5} \right) = \lim_{n \rightarrow \infty} \ln \left(\frac{1}{2+5/n} \right) = \ln \frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is $\frac{\cos 1}{1 - \cos 1} \approx 1.175343$.

33. The first series is a telescoping sum:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3}{n(n+3)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \end{aligned}$$

The second series is geometric with first term $\frac{5}{4}$ and ratio $\frac{1}{4}$: $\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{5/4}{1 - 1/4} = \frac{5}{3}$. Thus,

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{5}{4^n} \right) = \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \frac{5}{4^n} \text{ [sum of two convergent series]} = \frac{11}{6} + \frac{5}{3} = \frac{7}{2}.$$

34. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$ diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (If it converged, then $\frac{1}{2} \cdot 2 \sum_{n=1}^{\infty} \frac{1}{n}$ would also

converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.) If the given

series converges, then the difference $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right) - \sum_{n=1}^{\infty} \frac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty} \frac{3}{5^n}$ is a convergent

geometric series) and equal $\sum_{n=1}^{\infty} \frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges, so the given series must also diverge.

35. $0.\overline{2} = \frac{2}{10} + \frac{2}{10^2} + \cdots$ is a geometric series with $a = \frac{2}{10}$ and $r = \frac{1}{10}$. It converges to $\frac{a}{1-r} = \frac{2/10}{1-1/10} = \frac{2}{9}$.

36. $0.\overline{73} = \frac{73}{10^2} + \frac{73}{10^4} + \cdots = \frac{73/10^2}{1-1/10^2} = \frac{73/100}{99/100} = \frac{73}{99}$

37. $3.\overline{417} = 3 + \frac{417}{10^3} + \frac{417}{10^6} + \cdots = 3 + \frac{417/10^3}{1-1/10^3} = 3 + \frac{417}{999} = \frac{3414}{999} = \frac{1138}{333}$

38. $6.2\overline{54} = 6.2 + \frac{54}{10^3} + \frac{54}{10^5} + \cdots = 6.2 + \frac{54/10^3}{1-1/10^2} = \frac{62}{10} + \frac{54}{990} = \frac{6192}{990} = \frac{344}{55}$

n th partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^n \left(-\frac{1/4}{4k+1} + \frac{1/4}{4k-3} \right) = \frac{1}{4} \sum_{k=1}^n \left(\frac{1}{4k-3} - \frac{1}{4k+1} \right) \\ &= \frac{1}{4} \left[\left(1 - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{9} \right) + \left(\frac{1}{9} - \frac{1}{13} \right) + \cdots + \left(\frac{1}{4n-3} - \frac{1}{4n+1} \right) \right] = \frac{1}{4} \left(1 - \frac{1}{4n+1} \right) \end{aligned}$$

The series converges to $\lim_{n \rightarrow \infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using

`sum(f, 1..infinity)`; (in Maple), `Sum[f, {n, 1, Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and `Simplify` (in Derive).

48. See Exercise 47 for specific CAS commands. $\frac{n^2 + 3n + 1}{(n^2 + n)^2} = \frac{1}{n^2} + \frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1}$. So the n th partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^n \left(\frac{1}{k^2} + \frac{1}{k} - \frac{1}{(k+1)^2} - \frac{1}{k+1} \right) \\ &= \left(1 + 1 - \frac{1}{2^2} - \frac{1}{2} \right) + \left(\frac{1}{2^2} + \frac{1}{2} - \frac{1}{3^2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n^2} + \frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1} \right) \\ &= 1 + 1 - \frac{1}{(n+1)^2} - \frac{1}{n+1} \end{aligned}$$

The series converges to $\lim_{n \rightarrow \infty} s_n = 2$.

49. For $n = 1$, $a_1 = 0$ since $s_1 = 0$. For $n > 1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n - (n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

$$\text{Also, } \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1 + 1/n}{1 + 1/n} = 1.$$

50. $a_1 = s_1 = 3 - \frac{1}{2} = \frac{5}{2}$. For $n \neq 1$,

$$a_n = s_n - s_{n-1} = \left(3 - n2^{-n} \right) - \left[3 - (n-1)2^{-(n-1)} \right] = -\frac{n}{2^n} + \frac{n-1}{2^{n-1}} \cdot \frac{2}{2} = \frac{2(n-1)}{2^n} - \frac{n}{2^n} = \frac{n-2}{2^n}$$

$$\text{Also, } \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(3 - \frac{n}{2^n} \right) = 3 \text{ because } \lim_{x \rightarrow \infty} \frac{x}{2^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{2^x \ln 2} = 0.$$

51. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \frac{D(1-c^n)}{1-c} \text{ by (3).}$$

$$\begin{aligned} \text{(b) } \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{D(1-c^n)}{1-c} = \frac{D}{1-c} \lim_{n \rightarrow \infty} (1-c^n) = \frac{D}{1-c} \quad (\text{since } 0 < c < 1 \Rightarrow \lim_{n \rightarrow \infty} c^n = 0) \\ &= \frac{D}{s} \quad (\text{since } c + s = 1) = kD \quad (\text{since } k = 1/s) \end{aligned}$$

If $c = 0.8$, then $s = 1 - c = 0.2$ and the multiplier is $k = 1/s = 5$.

52. (a) Initially, the ball falls a distance H , then rebounds a distance rH , falls rH , rebounds r^2H , falls r^2H , etc. The total distance it travels is

$$\begin{aligned} H + 2rH + 2r^2H + 2r^3H + \cdots &= H(1 + 2r + 2r^2 + 2r^3 + \cdots) \\ &= H[1 + 2r(1 + r + r^2 + \cdots)] = H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters} \end{aligned}$$

- (b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\begin{aligned} \sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}}r + 2\sqrt{\frac{2H}{g}}r^2 + 2\sqrt{\frac{2H}{g}}r^3 + \cdots &= \sqrt{\frac{2H}{g}}[1 + 2\sqrt{r} + 2\sqrt{r^2} + 2\sqrt{r^3} + \cdots] \\ &= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}[1 + \sqrt{r} + \sqrt{r^2} + \cdots]\right) = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+\sqrt{r}}{1-\sqrt{r}} \end{aligned}$$

- (c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h . All these results follow from the formulas for vertical motion with gravitational acceleration $-g$: $\frac{d^2y}{dt^2} = -g \Rightarrow v = \frac{dy}{dt} = v_0 - gt \Rightarrow y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
...	...	...	...	...	...

The total travel time in seconds is

$$\begin{aligned} \sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots &= \sqrt{\frac{2H}{g}}(1 + 2k + 2k^2 + 2k^3 + \cdots) \\ &= \sqrt{\frac{2H}{g}}[1 + 2k(1 + k + k^2 + \cdots)] = \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+k}{1-k} \end{aligned}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $|(1+c)^{-1}| < 1 \Leftrightarrow |1+c| > 1 \Leftrightarrow 1+c > 1 \text{ or } 1+c < -1 \Leftrightarrow c > 0 \text{ or } c < -2$. We calculate

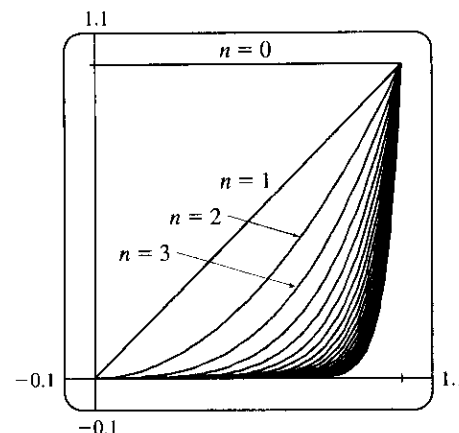
$$\text{the sum of the series and set it equal to 2: } \frac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \Leftrightarrow \left(\frac{1}{1+c}\right)^2 = 2 - 2\left(\frac{1}{1+c}\right) \Leftrightarrow$$

$1 = 2(1+c)^2 - 2(1+c) \Leftrightarrow 2c^2 + 2c - 1 = 0 \Leftrightarrow c = \frac{-2 \pm \sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

54. The area between $y = x^{n-1}$ and $y = x^n$ for $0 \leq x \leq 1$ is

$$\begin{aligned} \int_0^1 (x^{n-1} - x^n) dx &= \left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n} - \frac{1}{n+1} \\ &= \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)} \end{aligned}$$

We can see from the diagram that as $n \rightarrow \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1. So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.



55. Let d_n be the diameter of C_n . We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean

Theorem, we can write $1^2 + (1 - \frac{1}{2}d_1)^2 = (1 + \frac{1}{2}d_1)^2 \Leftrightarrow$

$$1 = (1 + \frac{1}{2}d_1)^2 - (1 - \frac{1}{2}d_1)^2 = 2d_1 \text{ (difference of squares)}$$

$$\Rightarrow d_1 = \frac{1}{2}. \text{ Similarly,}$$

$$\begin{aligned} 1 &= (1 + \frac{1}{2}d_2)^2 - (1 - d_1 - \frac{1}{2}d_2)^2 = 2d_2 + 2d_1 - d_1^2 - d_1d_2 \\ &= (2 - d_1)(d_1 + d_2) \Leftrightarrow \end{aligned}$$

$$d_2 = \frac{1}{2 - d_1} - d_1 = \frac{(1 - d_1)^2}{2 - d_1}, \quad 1 = (1 + \frac{1}{2}d_3)^2 - (1 - d_1 - d_2 - \frac{1}{2}d_3)^2 \Leftrightarrow d_3 = \frac{[1 - (d_1 + d_2)]^2}{2 - (d_1 + d_2)}, \text{ and}$$

in general, $d_{n+1} = \frac{(1 - \sum_{i=1}^n d_i)^2}{2 - \sum_{i=1}^n d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they

are $\frac{1}{6} = \frac{1}{2 \cdot 3}$ and $\frac{1}{12} = \frac{1}{3 \cdot 4}$ respectively, so we suspect that in general, $d_n = \frac{1}{n(n+1)}$. To prove this, we use

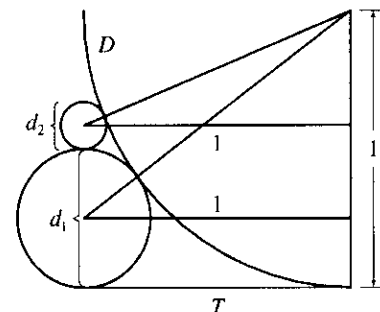
induction: Assume that for all $k \leq n$, $d_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Then

$$\sum_{i=1}^n d_i = 1 - \frac{1}{n+1} = \frac{n}{n+1} \text{ (telescoping sum). Substituting this into our formula for } d_{n+1}, \text{ we get}$$

$$d_{n+1} = \frac{\left[1 - \frac{n}{n+1}\right]^2}{2 - \left(\frac{n}{n+1}\right)} = \frac{\frac{1}{(n+1)^2}}{\frac{n+2}{n+1}} = \frac{1}{(n+1)(n+2)}, \text{ and the induction is complete.}$$

Now, we observe that the partial sums $\sum_{i=1}^n d_i$ of the diameters of the circles approach 1 as $n \rightarrow \infty$; that is,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \text{ which is what we wanted to prove.}$$



56. $|CD| = b \sin \theta$, $|DE| = |CD| \sin \theta = b \sin^2 \theta$, $|EF| = |DE| \sin \theta = b \sin^3 \theta$, ... Therefore,

$$|CD| + |DE| + |EF| + |FG| + \dots = b \sum_{n=1}^{\infty} \sin^n \theta = b \left(\frac{\sin \theta}{1 - \sin \theta} \right) \text{ since this is a geometric series with}$$

$r = \sin \theta$ and $|\sin \theta| < 1$ (because $0 < \theta < \frac{\pi}{2}$).

57. The series $1 - 1 + 1 - 1 + 1 - 1 + \dots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \dots$.

58. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$ by Theorem 6, so $\lim_{n \rightarrow \infty} \frac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^{\infty} ca_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n ca_i = \lim_{n \rightarrow \infty} c \sum_{i=1}^n a_i = c \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i = c \sum_{n=1}^{\infty} a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum (1/c)(ca_n) = \sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum (a_n + b_n)$ converges. Then $\sum (a_n + b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum [(a_n + b_n) - a_n]$ would also be convergent. But $\sum [(a_n + b_n) - a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

62. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum (-n)$, which both diverge, yet $\sum (a_n + b_n) = \sum 0$, which converges with sum 0.

63. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n . Also, the sequence $\{s_n\}$ is bounded since $s_n \leq 1000$ for all n . So by Theorem 12.1.11, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

$$\begin{aligned} 64. \text{ (a) RHS} &= \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{(f_{n-1} + f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}} \\ &= \frac{1}{f_{n-1} f_{n+1}} = \text{LHS} \end{aligned}$$

$$\begin{aligned} \text{(b) } \sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} &= \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) \quad [\text{from part (a)}] \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5} \right) + \dots \right. \\ &\quad \left. + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}} \right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1 \cdot 1} = 1 \text{ because } f_n \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} &= \sum_{n=2}^{\infty} \left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_n f_{n+1}} \right) \quad (\text{as above}) \\
 &= \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right) \\
 &= \lim_{n \rightarrow \infty} \left[\left(\frac{1}{f_1} - \frac{1}{f_3} \right) + \left(\frac{1}{f_2} - \frac{1}{f_4} \right) + \left(\frac{1}{f_3} - \frac{1}{f_5} \right) + \left(\frac{1}{f_4} - \frac{1}{f_6} \right) + \cdots \right. \\
 &\quad \left. + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right) \right] \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}} \right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \rightarrow \infty \text{ as } n \rightarrow \infty.
 \end{aligned}$$

65. (a) At the first step, only the interval $(\frac{1}{3}, \frac{2}{3})$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, which have a total length of $2 \cdot (\frac{1}{3})^2$. At the third step, we remove 2^2 intervals, each of length $(\frac{1}{3})^3$. In general, at the n th step we remove 2^{n-1} intervals, each of length $(\frac{1}{3})^n$, for a length of $2^{n-1} \cdot (\frac{1}{3})^n = \frac{1}{3}(\frac{2}{3})^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty} \frac{1}{3}(\frac{2}{3})^{n-1} = \frac{1/3}{1-2/3} = 1$ (geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$). Notice that at the n th step, the leftmost interval that is removed is $((\frac{1}{3})^n, (\frac{2}{3})^n)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $(1 - (\frac{2}{3})^n, 1 - (\frac{1}{3})^n)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9},$ and $\frac{8}{9}$.

- (b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot (\frac{1}{9})^2$; at the third step, $(8)^2 \cdot (\frac{1}{9})^3$. In general, the area removed at the n th step is $(8)^{n-1}(\frac{1}{9})^n = \frac{1}{9}(\frac{8}{9})^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{8}{9} \right)^{n-1} = \frac{1/9}{1-8/9} = 1.$$

66. (a)

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650
a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}, \frac{8}{3}, 2, 3, 667,$ and 334 . Note that the limits appear to be “weighted” more toward a_2 . In general, we guess that the limit is $\frac{a_1 + 2a_2}{3}$.

$$\begin{aligned}
 \text{(b) } a_{n+1} - a_n &= \frac{1}{2}(a_n + a_{n-1}) - a_n = -\frac{1}{2}(a_n - a_{n-1}) = -\frac{1}{2}\left[\frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1}\right] \\
 &= -\frac{1}{2}\left[-\frac{1}{2}(a_{n-1} - a_{n-2})\right] = \cdots = \left(-\frac{1}{2}\right)^{n-1}(a_2 - a_1)
 \end{aligned}$$

Note that we have used the formula $a_k = \frac{1}{2}(a_{k-1} + a_{k-2})$ a total of $n - 1$ times in this calculation, once for each k between 3 and $n + 1$. Now we can write

$$\begin{aligned}
 a_n &= a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) \\
 &= a_1 + \sum_{k=1}^{n-1} (a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1} \left(-\frac{1}{2}\right)^{k-1} (a_2 - a_1)
 \end{aligned}$$

and so

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n &= a_1 + (a_2 - a_1) \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^{k-1} = a_1 + (a_2 - a_1) \left[\frac{1}{1 - (-1/2)} \right] \\
 &= a_1 + \frac{2}{3}(a_2 - a_1) = \frac{a_1 + 2a_2}{3}
 \end{aligned}$$

67. (a) For $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$, $s_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{2}{1 \cdot 2 \cdot 3} = \frac{5}{6}$, $s_3 = \frac{5}{6} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{23}{24}$,
 $s_4 = \frac{23}{24} + \frac{4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n = \frac{(n+1)! - 1}{(n+1)!}$.

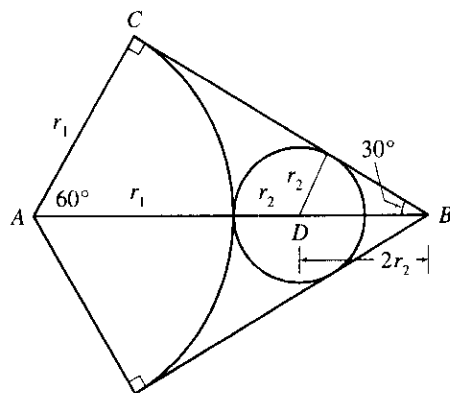
(b) For $n = 1$, $s_1 = \frac{1}{2} = \frac{2! - 1}{2!}$, so the formula holds for $n = 1$. Assume $s_k = \frac{(k+1)! - 1}{(k+1)!}$. Then

$$\begin{aligned}
 s_{k+1} &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+1)!(k+2)} \\
 &= \frac{(k+2)! - (k+2) + k+1}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!}
 \end{aligned}$$

Thus, the formula is true for $n = k + 1$. So by induction, the guess is correct.

(c) $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{(n+1)! - 1}{(n+1)!} = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{(n+1)!} \right] = 1$ and so $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

68.



Let r_1 = radius of the large circle, r_2 = radius of next circle, and so on. From the figure we have $\angle BAC = 60^\circ$ and $\cos 60^\circ = r_1 / |AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$.

Therefore, $2r_1 = r_1 + r_2 + 2r_2 = r_1 + 3r_2 \Rightarrow$

$r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3}r_n$, so the total area is

$$\begin{aligned}
 A &= \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots \\
 &= \pi r_1^2 + 3\pi r_2^2 \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots \right) \\
 &= \pi r_1^2 + 3\pi r_2^2 \cdot \frac{1}{1 - 1/9} = \pi r_1^2 + \frac{27}{8}\pi r_2^2
 \end{aligned}$$

Since the sides of the triangle have length 1, $|BC| = \frac{1}{2}$ and $\tan 30^\circ = \frac{r_1}{1/2}$. Thus, $r_1 = \frac{\tan 30^\circ}{2} = \frac{1}{2\sqrt{3}} \Rightarrow$

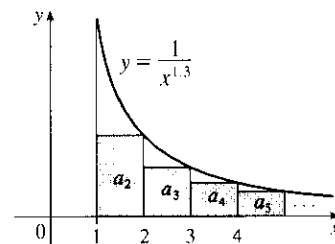
$r_2 = \frac{1}{6\sqrt{3}}$, so $A = \pi \left(\frac{1}{2\sqrt{3}} \right)^2 + \frac{27\pi}{8} \left(\frac{1}{6\sqrt{3}} \right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

12.3 The Integral Test and Estimates of Sums

1. The picture shows that $a_2 = \frac{1}{2^{1.3}} < \int_1^2 \frac{1}{x^{1.3}} dx$,

$$a_3 = \frac{1}{3^{1.3}} < \int_2^3 \frac{1}{x^{1.3}} dx, \text{ and so on, so } \sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}} dx. \text{ The}$$

integral converges by (8.8.2) with $p = 1.3 > 1$, so the series converges.



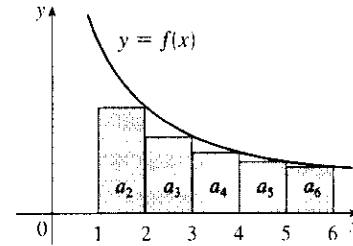
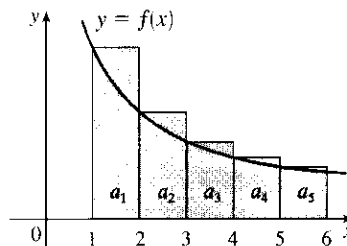
2. From the first figure, we see that

$$\int_1^6 f(x) dx < \sum_{i=1}^5 a_i. \text{ From the second}$$

figure, we see that

$$\sum_{i=2}^6 a_i < \int_1^6 f(x) dx. \text{ Thus, we have}$$

$$\sum_{i=2}^6 a_i < \int_1^6 f(x) dx < \sum_{i=1}^5 a_i.$$



3. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{x^4} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-4} dx = \lim_{t \rightarrow \infty} \left[\frac{x^{-3}}{-3} \right]_1^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{3t^3} + \frac{1}{3} \right) = \frac{1}{3}. \text{ Since this improper integral is}$$

convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent by the Integral Test.

4. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-1/4} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-1/4} dx = \lim_{t \rightarrow \infty} \left[\frac{4}{3} x^{3/4} \right]_1^t = \lim_{t \rightarrow \infty} \left(\frac{4}{3} t^{3/4} - \frac{4}{3} \right) = \infty, \text{ so } \sum_{n=1}^{\infty} 1/\sqrt[4]{n} \text{ diverges.}$$

5. The function $f(x) = 1/(3x+1)$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{dx}{3x+1} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{3x+1} = \lim_{b \rightarrow \infty} \left[\frac{1}{3} \ln(3x+1) \right]_1^b = \lim_{b \rightarrow \infty} \left[\frac{1}{3} \ln(3b+1) - \frac{1}{3} \ln 4 \right] = \infty$$

so the improper integral diverges, and so does the series $\sum_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x) = e^{-x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_1^b = \lim_{b \rightarrow \infty} (-e^{-b} + e^{-1}) = e^{-1}, \text{ so } \sum_{n=1}^{\infty} e^{-n} \text{ converges. Note:}$$

This is a geometric series, with first term $a = e^{-1}$ and ratio $r = e^{-1}$. Since $|r| < 1$, the series converges to $e^{-1}/(1 - e^{-1}) = 1/(e - 1)$.

7. $f(x) = xe^{-x}$ is continuous and positive on $[1, \infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) < 0$ for $x > 1$, so f is decreasing on $[1, \infty)$. Thus, the Integral Test applies.

$$\begin{aligned} \int_1^{\infty} xe^{-x} dx &= \lim_{b \rightarrow \infty} \int_1^b xe^{-x} dx = \lim_{b \rightarrow \infty} [-xe^{-x} - e^{-x}]_1^b \quad (\text{by parts}) \\ &= \lim_{b \rightarrow \infty} [-be^{-b} - e^{-b} + e^{-1} + e^{-1}] = 2/e \end{aligned}$$

since $\lim_{b \rightarrow \infty} be^{-b} = \lim_{b \rightarrow \infty} (b/e^b) \stackrel{H}{=} \lim_{b \rightarrow \infty} (1/e^b) = 0$ and $\lim_{b \rightarrow \infty} e^{-b} = 0$. Thus, $\sum_{n=1}^{\infty} ne^{-n}$ converges.

8. The function $f(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_1^t \left(1 + \frac{1}{x+1}\right) dx = \lim_{t \rightarrow \infty} [x + \ln(x+1)]_1^t = \lim_{t \rightarrow \infty} (t + \ln(t+1) - 1 - \ln 2) = \infty$, so $\int_1^{\infty} \frac{x+2}{x+1} dx$ is divergent and the series $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$ is divergent. NOTE: $\lim_{n \rightarrow \infty} \frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

9. The series $\sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ is a p -series with $p = 0.85 \leq 1$, so it diverges by (1). Therefore, the series $\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ would have to converge (by Theorem 8(i) in Section 11.2).

10. $\sum_{n=1}^{\infty} n^{-1.4}$ and $\sum_{n=1}^{\infty} n^{-1.2}$ are p -series with $p > 1$, so they converge by (1). Thus, $\sum_{n=1}^{\infty} 3n^{-1.2}$ converges by Theorem 8(i) in Section 11.2. It follows from Theorem 8(ii) that the given series $\sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2})$ also converges.

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p -series with $p = 3 > 1$, so it converges by (1).

12. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p -series with $p = \frac{3}{2} > 1$, so it converges by (1).

13. $\sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3} = 5 \sum_{n=1}^{\infty} \frac{1}{n^3} - 2 \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ by Theorem 12.2.8, since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ both converge by (1) (with $p = 3 > 1$ and $p = \frac{5}{2} > 1$). Thus, $\sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3}$ converges.

14. The function $f(x) = \frac{5}{x-2}$ is continuous, positive, and decreasing on $[3, \infty)$, so we can apply the Integral Test.

$\int_3^{\infty} \frac{5}{x-2} dx = \lim_{t \rightarrow \infty} \int_3^t \frac{5}{x-2} dx = \lim_{t \rightarrow \infty} [5 \ln(x-2)]_3^t = \lim_{t \rightarrow \infty} [5 \ln(t-2) - 0] = \infty$, so the series $\sum_{n=3}^{\infty} \frac{5}{n-2}$ diverges.

15. The function $f(x) = \frac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can apply the Integral Test.

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2+4} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+4} dx = \lim_{t \rightarrow \infty} \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_1^t = \frac{1}{2} \lim_{t \rightarrow \infty} \left[\tan^{-1} \left(\frac{t}{2} \right) - \tan^{-1} \left(\frac{1}{2} \right) \right] \\ &= \frac{1}{2} \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{1}{2} \right) \right] \end{aligned}$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges.

16. The function $f(x) = \frac{3x+2}{x(x+1)} = \frac{2}{x} + \frac{1}{x+1}$ [by partial fractions] is continuous, positive, and decreasing on $[1, \infty)$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\begin{aligned}\int_1^\infty \frac{3x+2}{x(x+1)} dx &= \lim_{t \rightarrow \infty} \int_1^t \left[\frac{2}{x} + \frac{1}{x+1} \right] dx = \lim_{t \rightarrow \infty} [2 \ln x + \ln(x+1)]_1^t \\ &= \lim_{t \rightarrow \infty} [2 \ln t + \ln(t+1) - \ln 2] = \infty\end{aligned}$$

Thus, the series $\sum_{n=1}^\infty \frac{3n+2}{n(n+1)}$ diverges.

17. $f(x) = \frac{x}{x^2+1}$ is continuous and positive on $[1, \infty)$, and since

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \text{ for } x > 1, f \text{ is also decreasing. Using the Integral Test,}$$

$$\int_1^\infty \frac{x}{x^2+1} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{x}{x^2+1} dx = \lim_{t \rightarrow \infty} \left[\frac{\ln(x^2+1)}{2} \right]_1^t = \frac{1}{2} \lim_{t \rightarrow \infty} [\ln(t^2+1) - \ln 2] = \infty, \text{ so the series diverges.}$$

18. The function $f(x) = \frac{1}{x^2-4x+5} = \frac{1}{(x-2)^2+1}$ is continuous, positive, and decreasing on $[2, \infty)$, so the

$$\text{Integral Test applies. } \int_2^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_2^t f(x) dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{(x-2)^2+1} dx = \lim_{t \rightarrow \infty} [\tan^{-1}(x-2)]_2^t =$$

$$\lim_{t \rightarrow \infty} [\tan^{-1}(t-2) - \tan^{-1} 0] = \frac{\pi}{2} - 0 = \frac{\pi}{2}, \text{ so the series } \sum_{n=2}^\infty \frac{1}{n^2-4n+5} \text{ converges. Of course this means}$$

$$\text{that } \sum_{n=1}^\infty \frac{1}{n^2-4n+5} \text{ converges too.}$$

19. $f(x) = xe^{-x^2}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = e^{-x^2}(1-2x^2) < 0$ for $x > 1$, f is decreasing as well. Thus, we can use the Integral Test.

$$\int_1^\infty xe^{-x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2}e^{-x^2} \right]_1^t = 0 - \left(-\frac{1}{2}e^{-1} \right) = 1/(2e). \text{ Since the integral converges, the series converges.}$$

20. $f(x) = \frac{\ln x}{x^2}$ is continuous and positive for $x \geq 2$, and $f'(x) = \frac{1-2 \ln x}{x^3} < 0$ for $x \geq 2$, so f is decreasing.

$$\int_2^\infty \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_2^t \text{ [by parts] } \stackrel{H}{=} 1. \text{ Thus, } \sum_{n=1}^\infty \frac{\ln n}{n^2} = \sum_{n=2}^\infty \frac{\ln n}{n^2} \text{ converges by the Integral Test.}$$

21. $f(x) = \frac{1}{x \ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\frac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$,

$$\text{so we can use the Integral Test. } \int_2^\infty \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} [\ln(\ln x)]_2^t = \lim_{t \rightarrow \infty} [\ln(\ln t) - \ln(\ln 2)] = \infty, \text{ so the series diverges.}$$

22. The function $f(x) = \frac{x}{x^4 + 1}$ is positive, continuous, and decreasing on $[1, \infty)$. [Note that

$$f'(x) = \frac{x^4 + 1 - 4x^4}{(x^4 + 1)^2} = \frac{1 - 3x^4}{(x^4 + 1)^2} < 0 \text{ on } [1, \infty).] \text{ Thus, we can apply the Integral Test.}$$

$$\begin{aligned} \int_1^\infty \frac{x}{x^4 + 1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\frac{1}{2}(2x)}{1 + (x^2)^2} dx = \lim_{t \rightarrow \infty} \left[\frac{1}{2} \tan^{-1}(x^2) \right]_1^t = \frac{1}{2} \lim_{t \rightarrow \infty} [\tan^{-1}(t^2) - \tan^{-1} 1] \\ &= \frac{1}{2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{8} \end{aligned}$$

so the series $\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$ converges.

23. The function $f(x) = \frac{1}{x^3 + x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\begin{aligned} \int_1^\infty \frac{1}{x^3 + x} dx &= \lim_{t \rightarrow \infty} \int_1^t \left[\frac{1}{x} - \frac{x}{1 + x^2} \right] dx = \lim_{t \rightarrow \infty} \left[\ln x - \frac{1}{2} \ln(1 + x^2) \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left[\ln \frac{x}{\sqrt{1 + x^2}} \right]_1^t = \lim_{t \rightarrow \infty} \left(\ln \frac{t}{\sqrt{1 + t^2}} - \ln \frac{1}{\sqrt{2}} \right) \\ &= \lim_{t \rightarrow \infty} \left(\ln \frac{1}{\sqrt{1 + 1/t^2}} + \frac{1}{2} \ln 2 \right) = \frac{1}{2} \ln 2 \end{aligned}$$

so the series $\sum_{n=1}^{\infty} \frac{1}{n^3 + n}$ converges.

24. $f(x) = \frac{1}{x \ln x \ln(\ln x)}$ is positive and continuous on $[3, \infty)$, and is decreasing since x , $\ln x$, and $\ln(\ln x)$ are all increasing; so we can apply the Integral Test. $\int_3^\infty \frac{dx}{x \ln x \ln(\ln x)} = \lim_{t \rightarrow \infty} [\ln(\ln(\ln x))]_3^t = \infty$. The integral diverges, so $\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$ diverges.

25. We have already shown (in Exercise 21) that when $p = 1$ the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges, so assume that $p \neq 1$.

$f(x) = \frac{1}{x(\ln x)^p}$ is continuous and positive on $[2, \infty)$, and $f'(x) = -\frac{p + \ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p} dx = \lim_{t \rightarrow \infty} \left[\frac{(\ln x)^{1-p}}{1-p} \right]_2^t \quad (\text{for } p \neq 1) = \lim_{t \rightarrow \infty} \left[\frac{(\ln t)^{1-p}}{1-p} \right] - \frac{(\ln 2)^{1-p}}{1-p}$$

This limit exists whenever $1 - p < 0 \Leftrightarrow p > 1$, so the series converges for $p > 1$.

26. As in Exercise 24, we can apply the Integral Test. $\int_3^\infty \frac{dx}{x \ln x (\ln \ln x)^p} = \lim_{t \rightarrow \infty} \left[\frac{(\ln \ln x)^{-p+1}}{-p+1} \right]_3^t$ (for $p \neq 1$; if

$p = 1$ see Exercise 24) and $\lim_{t \rightarrow \infty} \frac{(\ln \ln t)^{-p+1}}{-p+1}$ exists whenever $-p+1 < 0 \Leftrightarrow p > 1$, so the series converges for $p > 1$.

27. Clearly the series cannot converge if $p \geq -\frac{1}{2}$, because then $\lim_{n \rightarrow \infty} n(1+n^2)^p \neq 0$. Also, if $p = -1$ the series diverges (see Exercise 17). So assume $p < -\frac{1}{2}$, $p \neq -1$. Then $f(x) = x(1+x^2)^p$ is continuous, positive, and eventually decreasing on $[1, \infty)$, and we can use the Integral Test.

$$\int_1^{\infty} x(1+x^2)^p dx = \lim_{t \rightarrow \infty} \left[\frac{1}{2} \cdot \frac{(1+x^2)^{p+1}}{p+1} \right]_1^t = \lim_{t \rightarrow \infty} \frac{1}{2} \cdot \frac{(1+t^2)^{p+1}}{p+1} - \frac{2^p}{p+1}.$$

This limit exists and is finite $\Leftrightarrow p+1 < 0 \Leftrightarrow p < -1$, so the series converges whenever $p < -1$.

28. If $p \leq 0$, $\lim_{n \rightarrow \infty} \frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \frac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$$\int_1^{\infty} \frac{\ln x}{x^p} dx = \lim_{t \rightarrow \infty} \left[\frac{x^{1-p} [(1-p) \ln x - 1]}{(1-p)^2} \right]_1^t \quad (\text{for } p \neq 1) = \frac{1}{(1-p)^2} \left[\lim_{t \rightarrow \infty} t^{1-p} [(1-p) \ln t - 1] + 1 \right],$$

which exists whenever $1-p < 0 \Leftrightarrow p > 1$. Since we have already done the case $p = 1$ in Exercise 25 (set $p = -1$ in that exercise), $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges $\Leftrightarrow p > 1$.

29. Since this is a p -series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of numbers x such that the series is convergent.

30. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral

Test applies. $\sum_{n=1}^{\infty} \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{3x^3} \right]_{10}^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{3t^3} + \frac{1}{3(10)^3} \right) = \frac{1}{3000}, \text{ so the error is at most } 0.000\bar{3}.$$

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^4} dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^4} dx \Rightarrow s_{10} + \frac{1}{3(11)^3} \leq s \leq s_{10} + \frac{1}{3(10)^3} \Rightarrow$
 $1.082037 + 0.000250 = 1.082287 \leq s \leq 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005 .

(c) $R_n \leq \int_n^{\infty} \frac{1}{x^4} dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \Rightarrow \frac{1}{3n^3} < \frac{1}{10^5} \Rightarrow 3n^3 > 10^5 \Rightarrow$
 $n > \sqrt[3]{(10)^5/3} \approx 32.2$, that is, for $n > 32$.

31. (a) $f(x) = \frac{1}{x^2}$ is positive and continuous and $f'(x) = -\frac{2}{x^3}$ is negative for $x > 0$, and so the Integral

Test applies. $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.549768$.

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_{10}^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + \frac{1}{10} \right) = \frac{1}{10}, \text{ so the error is at most } 0.1.$$

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^2} dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^2} dx \Rightarrow s_{10} + \frac{1}{11} \leq s \leq s_{10} + \frac{1}{10} \Rightarrow$
 $1.549768 + 0.090909 = 1.640677 \leq s \leq 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

(c) $R_n \leq \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n}$. So $R_n < 0.001$ if $\frac{1}{n} < \frac{1}{1000} \Leftrightarrow n > 1000$.

32. $f(x) = 1/x^5$ is positive and continuous and $f'(x) = -5/x^6$ is negative for $x > 0$, and so the Integral Test applies.

Using (3), $R_n \leq \int_n^\infty x^{-5} dx = \lim_{t \rightarrow \infty} \left[\frac{-1}{4x^4} \right]_n^t = \frac{1}{4n^4}$. If we take $n = 5$, then $s_5 \approx 1.036662$ and $R_5 \leq 0.0004$.

So $s \approx s_5 \approx 1.037$.

33. $f(x) = x^{-3/2}$ is positive and continuous and $f'(x) = -\frac{3}{2}x^{-5/2}$ is negative for $x > 0$, so the Integral Test applies. From the end of Example 6, we see that the error is at most half the length of the interval. From (3), the interval is $\left(s_n + \int_{n+1}^\infty f(x) dx, s_n + \int_n^\infty f(x) dx\right)$, so its length is $\int_n^\infty f(x) dx - \int_{n+1}^\infty f(x) dx = \int_n^{n+1} f(x) dx$. Thus, we need n such that

$$0.01 > \frac{1}{2} \int_n^{n+1} x^{-3/2} dx = \frac{1}{2} \left[\frac{-2}{\sqrt{x}} \right]_n^{n+1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

$\Leftrightarrow n > 13.08$ (use a graphing calculator to solve $1/\sqrt{x} - 1/\sqrt{x+1} < 0.01$). Again from the end of Example 6, we approximate s by the midpoint of this interval. In general, the midpoint is

$\frac{1}{2} \left[\left(s_n + \int_{n+1}^\infty f(x) dx\right) + \left(s_n + \int_n^\infty f(x) dx\right) \right] = s_n + \frac{1}{2} \left(\int_{n+1}^\infty f(x) dx + \int_n^\infty f(x) dx \right)$. So using $n = 14$,

we have $s \approx s_{14} + \frac{1}{2} \left(\int_{14}^\infty x^{-3/2} dx + \int_{15}^\infty x^{-3/2} dx \right) \approx 2.0872 + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} \approx 2.6127 \approx 2.61$. Any larger value of n will also work. For instance, $s \approx s_{30} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{31}} \approx 2.6124$.

34. $f(x) = \frac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3}$ is negative for $x > 1$, so the Integral Test

applies. Using (2), we need $0.01 > \int_n^\infty \frac{dx}{x(\ln x)^2} = \lim_{t \rightarrow \infty} \left[\frac{-1}{\ln x} \right]_n^t = \frac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to take this many terms, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.

35. $\sum_{n=1}^\infty n^{-1.001} = \sum_{n=1}^\infty \frac{1}{n^{1.001}}$ is a convergent p -series with $p = 1.001 > 1$. Using (2), we get

$$R_n \leq \int_n^\infty x^{-1.001} dx = \lim_{t \rightarrow \infty} \left[\frac{x^{-0.001}}{-0.001} \right]_n^t = -1000 \lim_{t \rightarrow \infty} \left[\frac{1}{x^{0.001}} \right]_n^t = -1000 \left(-\frac{1}{n^{0.001}} \right) = \frac{1000}{n^{0.001}}. \text{ We want}$$

$$R_n < 0.000000005 \Leftrightarrow \frac{1000}{n^{0.001}} < 5 \times 10^{-9} \Leftrightarrow n^{0.001} > \frac{1000}{5 \times 10^{-9}} \Leftrightarrow$$

$$n > (2 \times 10^{11})^{1000} = 2^{1000} \times 10^{11,000} \approx 1.07 \times 10^{301} \times 10^{11,000} = 1.07 \times 10^{11,301}.$$

36. (a) $f(x) = \left(\frac{\ln x}{x}\right)^2$ is continuous and positive for $x > 1$, and since $f'(x) = \frac{2 \ln x (1 - \ln x)}{x^3} < 0$ for $x > e$, we

can apply the Integral Test. Using a CAS, we get $\int_1^\infty \left(\frac{\ln x}{x}\right)^2 dx = 2$, so the series also converges.

(b) Since the Integral Test applies, the error in $s \approx s_n$ is $R_n \leq \int_n^\infty \left(\frac{\ln x}{x}\right)^2 dx = \frac{(\ln n)^2 + 2 \ln n + 2}{n}$.

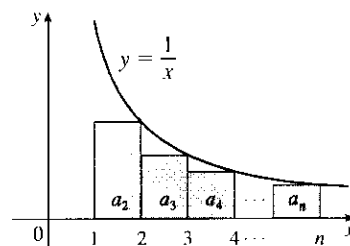
(c) By graphing the functions $y_1 = \frac{(\ln x)^2 + 2 \ln x + 2}{x}$ and $y_2 = 0.05$, we see that $y_1 < y_2$ for $n \geq 1373$.

(d) Using the CAS to sum the first 1373 terms, we get $s_{1373} \approx 1.94$.

37. (a) From the figure, $a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x) dx$, so with

$$f(x) = \frac{1}{x}, \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \leq \int_1^n \frac{1}{x} dx = \ln n. \text{ Thus,}$$

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \leq 1 + \ln n.$$



- (b) By part (a), $s_{10^6} \leq 1 + \ln 10^6 \approx 14.82 < 15$ and $s_{10^9} \leq 1 + \ln 10^9 \approx 21.72 < 22$.

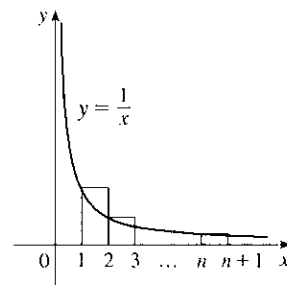
38. (a) The sum of the areas of the n rectangles in the graph to the right is

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}. \text{ Now } \int_1^{n+1} \frac{dx}{x} \text{ is less than this sum because}$$

the rectangles extend above the curve $y = 1/x$, so

$$\int_1^{n+1} \frac{1}{x} dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \text{ and since}$$

$$\ln n < \ln(n+1), 0 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n = t_n.$$



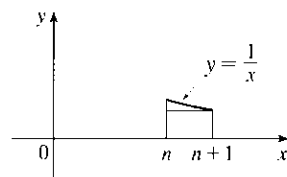
- (b) The area under $f(x) = 1/x$ between $x = n$ and $x = n+1$ is

$$\int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n, \text{ and this is clearly greater than the}$$

area of the inscribed rectangle in the figure to the right

$$\left[\text{which is } \frac{1}{n+1} \right], \text{ so } t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1} > 0,$$

and so $t_n > t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.



- (c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n . Thus, $0 < t_n \leq t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by Theorem 12.1.11.

39. $b^{\ln n} = (e^{\ln b})^{\ln n} = (e^{\ln n})^{\ln b} = n^{\ln b} = \frac{1}{n^{-\ln b}}$. This is a p -series, which converges for all b such that $-\ln b > 1$

$$\Leftrightarrow \ln b < -1 \Leftrightarrow b < e^{-1} \Leftrightarrow b < 1/e \text{ [with } b > 0].$$

12.4 The Comparison Tests

- (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)
 (b) If $a_n < b_n$ for all n , then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]
- (a) If $a_n > b_n$ for all n , then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]
 (b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. $\frac{1}{n^2 + n + 1} < \frac{1}{n^2}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p -series with $p = 2 > 1$.
4. $\frac{2}{n^3 + 4} < \frac{2}{n^3}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{2}{n^3 + 4}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{2}{n^3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges because it is a constant multiple of a convergent p -series ($p = 3 > 1$).
5. $\frac{5}{2 + 3^n} < \frac{5}{3^n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{5}{2 + 3^n}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{5}{3^n} = 5 \sum_{n=1}^{\infty} \frac{1}{3^n}$, which converges because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series with $r = \frac{1}{3}$ ($|r| < 1$).
6. $\frac{1}{n - \sqrt{n}} > \frac{1}{n}$ for all $n \geq 2$, so $\sum_{n=2}^{\infty} \frac{1}{n - \sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$.
7. $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.
8. $\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.
9. $\frac{\cos^2 n}{n^2 + 1} \leq \frac{1}{n^2 + 1} < \frac{1}{n^2}$, so the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2 + 1}$ converges by comparison with the p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ($p = 2 > 1$).
10. $\frac{n^2 - 1}{3n^4 + 1} < \frac{n^2}{3n^4 + 1} < \frac{n^2}{3n^4} = \frac{1}{3} \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p -series ($p = 2 > 1$). The terms of the given series are positive for $n > 1$, which is good enough.
11. If $a_n = \frac{n^2 + 1}{n^3 - 1}$ and $b_n = \frac{1}{n}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3 + n}{n^3 - 1} = \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{1 - 1/n^3} = 1$, so $\sum_{n=2}^{\infty} \frac{n^2 + 1}{n^3 - 1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$.
 Or: Since $a_n = \frac{n^2 + 1}{n^3 - 1} > \frac{n^2 + 1}{n^3} > \frac{n^2}{n^3} = \frac{1}{n} = b_n$, we could use the Comparison Test.
12. $\frac{1 + \sin n}{10^n} \leq \frac{2}{10^n}$ and $\sum_{n=0}^{\infty} \frac{2}{10^n} = 2 \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.
13. $\frac{n-1}{n4^n}$ is positive for $n > 1$ and $\frac{n-1}{n4^n} < \frac{n}{n4^n} = \frac{1}{4^n}$, so $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$.

14. $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p -series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ ($p = \frac{1}{2} \leq 1$).

15. $\frac{2 + (-1)^n}{n\sqrt{n}} \leq \frac{3}{n\sqrt{n}}$, and $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ ($p = \frac{3}{2} > 1$), so the given series converges by the Comparison Test.

16. $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ ($p = \frac{3}{2} > 1$).

17. Use the Limit Comparison Test with $a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{n}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 > 0. \text{ Since the harmonic series } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, so does } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}.$$

18. Use the Limit Comparison Test with $a_n = \frac{1}{2n+3}$ and $b_n = \frac{1}{n}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{2n+3} = \lim_{n \rightarrow \infty} \frac{1}{2+(3/n)} = \frac{1}{2} > 0. \text{ Since the harmonic series } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, so does } \sum_{n=1}^{\infty} \frac{1}{2n+3}.$$

19. $\frac{2^n}{1+3^n} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$. $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series ($|r| = \frac{2}{3} < 1$), so $\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges by the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \frac{1+2^n}{1+3^n}$ and $b_n = \frac{2^n}{3^n}$: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(1/2)^n + 1}{(1/3)^n + 1} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} < 1$), $\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ also converges.

21. Use the Limit Comparison Test with $a_n = \frac{1}{1+\sqrt{n}}$ and $b_n = \frac{1}{\sqrt{n}}$: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p -series ($p = \frac{1}{2} \leq 1$), $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

22. Use the Limit Comparison Test with $a_n = \frac{n+2}{(n+1)^3}$ and $b_n = \frac{1}{n^2}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2(n+2)}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0. \text{ Since } \sum_{n=3}^{\infty} \frac{1}{n^2} \text{ is a convergent (partial) } p\text{-series } (p = 2 > 1), \text{ the series } \sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3} \text{ also converges.}$$

23. Use the Limit Comparison Test with $a_n = \frac{5+2n}{(1+n^2)^2}$ and $b_n = \frac{1}{n^3}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n \rightarrow \infty} \frac{5n^3+2n^4}{(1+n^2)^2} \cdot \frac{1/n^4}{1/(n^2)^2} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0. \text{ Since } \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ is a}$$

convergent p -series ($p = 3 > 1$), the series $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ also converges.

24. If $a_n = \frac{n^2-5n}{n^3+n+1}$ and $b_n = \frac{1}{n}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3-5n^2}{n^3+n+1} = \lim_{n \rightarrow \infty} \frac{1-5/n}{1+1/n^2+1/n^3} = 1 > 0$, so

$\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. (Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \frac{1}{n}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n \rightarrow \infty} \frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1 > 0, \text{ so } \sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}} \text{ diverges by the}$$

Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

26. If $a_n = \frac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \frac{n}{\sqrt[3]{n^7}} = \frac{n}{n^{7/3}} = \frac{1}{n^{4/3}}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}} \cdot \frac{n^{-7/3}}{n^{-7/3}} = \lim_{n \rightarrow \infty} \frac{1+5/n}{[(n^7+n^2)/n^7]^{1/3}} \\ &= \lim_{n \rightarrow \infty} \frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1 > 0, \end{aligned}$$

so $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$.

27. Use the Limit Comparison Test with $a_n = \left(1 + \frac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = 1 > 0$.

Since $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n}$ is a convergent geometric series ($|r| = \frac{1}{e} < 1$), the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$ also converges.

28. Use the Limit Comparison Test with $a_n = \frac{2n^2+7n}{3^n(n^2+5n-1)}$ and $b_n = \frac{1}{3^n}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n^2+7n}{n^2+5n-1} = 2 > 0, \text{ and since } \sum_{n=1}^{\infty} b_n \text{ is a convergent geometric series } (|r| = \frac{1}{3} < 1),$$

$\sum_{n=1}^{\infty} \frac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2) \cdots (3)(2) \geq 2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 = 2^{n-1}$, so $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a convergent

geometric series ($|r| = \frac{1}{2} < 1$), so $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the Comparison Test.

30. $\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1)n}{n \cdot n \cdot n \cdots n \cdot n} \leq \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots 1$ for $n \geq 2$, so since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges ($p = 2 > 1$), $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges also by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges. (Note that we could also use l'Hospital's Rule to evaluate the limit:
- $$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\cos(1/x) \cdot (-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \cos \frac{1}{x} = \cos 0 = 1.)$$
32. Use the Limit Comparison Test with $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{n^{1+1/n}} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} = 1$ (since $\lim_{x \rightarrow \infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.
33. $\sum_{n=1}^{10} \frac{1}{n^4 + n^2} = \frac{1}{2} + \frac{1}{20} + \frac{1}{90} + \cdots + \frac{1}{10,100} \approx 0.567975$. Now $\frac{1}{n^4 + n^2} < \frac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is $R_{10} \leq T_{10} = \sum_{n=11}^{\infty} \frac{1}{n^4} \leq \int_{10}^{\infty} \frac{dx}{x^4} = \lim_{t \rightarrow \infty} \left[-\frac{x^{-3}}{3} \right]_{10}^t = \frac{1}{3000} = 0.000\bar{3}$.
34. $\sum_{n=1}^{10} \frac{1 + \cos n}{n^5} = 1 + \cos 1 + \frac{1 + \cos 2}{32} + \frac{1 + \cos 3}{243} + \cdots + \frac{1 + \cos 10}{100,000} \approx 1.55972$. Now $\frac{1 + \cos n}{n^5} \leq \frac{2}{n^5}$, so as in Example 5, $R_{10} \leq T_{10} \leq \int_{10}^{\infty} \frac{2}{x^5} dx = 2 \lim_{t \rightarrow \infty} \left[-\frac{1}{4} x^{-4} \right]_{10}^t = 0.00005$.
35. $\sum_{n=1}^{10} \frac{1}{1 + 2^n} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{1025} \approx 0.76352$. Now $\frac{1}{1 + 2^n} < \frac{1}{2^n}$, so the error is $R_{10} \leq T_{10} = \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1 - 1/2}$ (geometric series) ≈ 0.00098 .
36. $\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{1}{6} + \frac{2}{27} + \frac{3}{108} + \cdots + \frac{10}{649,539} \approx 0.283597$. Now $\frac{n}{(n+1)3^n} < \frac{n}{n \cdot 3^n} = \frac{1}{3^n}$, so the error is $R_{10} \leq T_{10} = \sum_{n=11}^{\infty} \frac{1}{3^n} = \frac{1/3^{11}}{1 - 1/3} \approx 0.0000085$.
37. Since $\frac{d_n}{10^n} \leq \frac{9}{10^n}$ for each n , and since $\sum_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1 d_2 d_3 \dots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.
38. Clearly, if $p < 0$ then the series diverges, since $\lim_{n \rightarrow \infty} \frac{1}{n^p \ln n} = \infty$. If $0 \leq p \leq 1$, then $n^p \ln n \leq n \ln n \Rightarrow \frac{1}{n^p \ln n} \geq \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise 12.3.21), so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum_{n=2}^{\infty} b_n$ converges, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.
39. Since $\sum a_n$ converges, $\lim_{n \rightarrow \infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \Rightarrow 0 \leq a_n < 1$ for all $n > N \Rightarrow 0 \leq a_n^2 \leq a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. (a) Since $\lim_{n \rightarrow \infty} (a_n/b_n) = 0$, there is a number $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.

(b) (i) If $a_n = \frac{\ln n}{n^3}$ and $b_n = \frac{1}{n^2}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$, so $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by part (a).

(ii) If $a_n = \frac{\ln n}{\sqrt{n}e^n}$ and $b_n = \frac{1}{e^n}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

Now $\sum b_n$ is a convergent geometric series with ratio $r = 1/e$ ($|r| < 1$), so $\sum a_n$ converges by part (a).

41. (a) Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, there is an integer N such that $\frac{a_n}{b_n} > 1$ whenever $n > N$. (Take $M = 1$ in Definition 12.1.5.) Then $a_n > b_n$ whenever $n > N$ and since $\sum b_n$ is divergent, $\sum a_n$ is also divergent by the Comparison Test.

(b) (i) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$ for $n \geq 2$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{x \rightarrow \infty} \frac{x}{\ln x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{1/x} = \lim_{x \rightarrow \infty} x = \infty, \text{ so by part (a), } \sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ is divergent.}$$

(ii) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \ln n = \lim_{x \rightarrow \infty} \ln x = \infty, \text{ so } \sum_{n=1}^{\infty} a_n \text{ diverges by part (a).}$$

42. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

43. $\lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} \frac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n}$. Since $\lim_{n \rightarrow \infty} na_n > 0$ we know that either both series converge or both series diverge, and we also know that $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges (p -series with $p = 1$). Therefore, $\sum a_n$ must be divergent.

44. First we observe that, by l'Hospital's Rule, $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then

$\lim_{n \rightarrow \infty} a_n = 0$ by Theorem 12.2.6. Therefore, $\lim_{n \rightarrow \infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln(1+a_n)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim_{n \rightarrow \infty} a_n = 0$ by Theorem 12.2.6, and

$\sum b_n = \sum \sin(a_n)$ is a series with positive terms (for large enough n). We have

$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{\sin(a_n)}{a_n} = 1 > 0$ by Theorem 3.5.2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n \rightarrow \infty$, so for some integer N , $a_n \leq 1$ for all $n \geq N$. But then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} a_n b_n \leq \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty} b_n$ converges. So $\sum a_n b_n$ converges by the Comparison Test.

12.5 Alternating Series

- (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges if $0 < b_{n+1} \leq b_n$ for all n and $\lim_{n \rightarrow \infty} b_n = 0$. (This is the Alternating Series Test.)

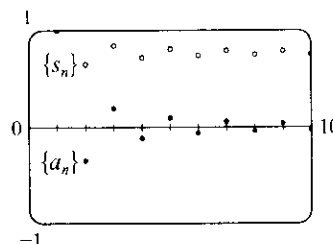
(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1} ; that is, $|R_n| \leq b_{n+1}$. (This is the Alternating Series Estimation Theorem.)
- $-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$. Here $a_n = (-1)^n \frac{n}{n+2}$. Since $\lim_{n \rightarrow \infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.
- $\frac{4}{7} - \frac{4}{8} + \frac{4}{9} - \frac{4}{10} + \frac{4}{11} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4}{n+6}$. Now $b_n = \frac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series converges by the Alternating Series Test.
- $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$. $b_n = \frac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$, so the series converges by the Alternating Series Test.
- $b_n = \frac{1}{\sqrt{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.
- $b_n = \frac{1}{3n-1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.
- $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{3-1/n}{2+1/n} = \frac{3}{2} \neq 0$. Since $\lim_{n \rightarrow \infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.
- $b_n = \frac{2n}{4n^2+1} > 0$, $\{b_n\}$ is decreasing [since $b_n - b_{n+1} = \frac{2n}{4n^2+1} - \frac{2n+2}{4n^2+8n+5} = \frac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)} > 0$ for $n \geq 1$], and $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{2/n}{4+1/n^2} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$ converges by the Alternating Series Test.
 Alternatively, to show that $\{b_n\}$ is decreasing, we could verify that $\frac{d}{dx} \left(\frac{2x}{4x^2+1} \right) < 0$ for $x \geq 1$.
- $b_n = \frac{1}{4n^2+1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

10. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{2+1/\sqrt{n}} = \frac{1}{2} \neq 0$. Since $\lim_{n \rightarrow \infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.
11. $b_n = \frac{n^2}{n^3+4} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since
- $$\left(\frac{x^2}{x^3+4} \right)' = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0 \text{ for } x > 2. \text{ Also,}$$
- $$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1/n}{1+4/n^3} = 0. \text{ Thus, the series } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4} \text{ converges by the Alternating Series Test.}$$
12. $b_n = \frac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since
- $$\left(\frac{e^{1/x}}{x} \right)' = \frac{x \cdot e^{1/x}(-1/x^2) - e^{1/x} \cdot 1}{x^2} = \frac{-e^{1/x}(1+x)}{x^3} < 0 \text{ for } x > 0. \text{ Also, } \lim_{n \rightarrow \infty} b_n = 0 \text{ since}$$
- $$\lim_{n \rightarrow \infty} e^{1/n} = 1. \text{ Thus, the series } \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n} \text{ converges by the Alternating Series Test.}$$
13. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$. $\lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{x \rightarrow \infty} \frac{x}{\ln x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.
14. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n} \right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n} \right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,
- $$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0, \text{ so the series converges by the Alternating Series Test.}$$
15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n \rightarrow \infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.
16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.
17. $\sum_{n=1}^{\infty} (-1)^n \sin \frac{\pi}{n}$. $b_n = \sin \frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin \frac{\pi}{n} \geq \sin \frac{\pi}{n+1}$, and $\lim_{n \rightarrow \infty} \sin \frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.
18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n \rightarrow \infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n \rightarrow \infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.
19. $\frac{n^n}{n!} = \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} \geq n \Rightarrow \lim_{n \rightarrow \infty} \frac{n^n}{n!} = \infty \Rightarrow \lim_{n \rightarrow \infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

20. $\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$ diverges by the Test for Divergence since $\lim_{n \rightarrow \infty} \left(\frac{n}{5}\right)^n = \infty \Rightarrow \lim_{n \rightarrow \infty} \left(-\frac{n}{5}\right)^n$ does not exist.

21.

n	a_n	s_n
1	1	1
2	-0.35355	0.64645
3	0.19245	0.83890
4	-0.125	0.71390
5	0.08944	0.80334
6	-0.06804	0.73530
7	0.05399	0.78929
8	-0.04419	0.74510
9	0.03704	0.78214
10	-0.03162	0.75051



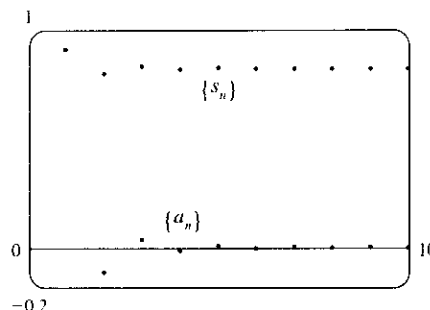
By the Alternating Series Estimation Theorem, the error in the

approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \approx 0.75051$ is

$|s - s_{10}| \leq b_{11} = 1/(11)^{3/2} \approx 0.0275$ (to four decimal places, rounded up).

22.

n	a_n	s_n
1	1	1
2	-0.125	0.875
3	0.03704	0.91204
4	-0.01563	0.89641
5	0.008	0.90441
6	-0.00463	0.89978
7	0.00292	0.90270
8	-0.00195	0.90074
9	0.00137	0.90212
10	-0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the

approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \approx 0.90112$ is

$|s - s_{10}| \leq b_{11} = 1/11^3 \approx 0.0007513$.

23. The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and

(ii) $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$, so the series is convergent. Now $b_{10} = \frac{1}{10^2} = 0.01$ and $b_{11} = \frac{1}{11^2} = \frac{1}{121} \approx 0.008 < 0.01$, so

by the Alternating Series Estimation Theorem, $n = 10$. (That is, since the 11th term is less than the desired error, we need to add the first 10 terms to get the sum to the desired accuracy.)

24. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^4} < \frac{1}{n^4}$ and

(ii) $\lim_{n \rightarrow \infty} \frac{1}{n^4} = 0$, so the series is convergent. Now $b_5 = 1/5^4 = 0.0016 > 0.001$ and

$b_6 = 1/6^4 \approx 0.00077 < 0.001$, so by the Alternating Series Estimation Theorem, $n = 5$.

25. The series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test

$$\text{because } b_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2 \cdot 2^n}{(n+1)n!} = \frac{2}{n+1} \cdot \frac{2^n}{n!} = \frac{2}{n+1} \cdot b_n \leq b_n \text{ and (ii)}$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{n!} = \frac{2}{n} \cdot \frac{2}{n-1} \cdots \frac{2}{2} \cdot \frac{2}{1} = 0, \text{ so the series is convergent. Now } b_7 = 2^7/7! \approx 0.025 > 0.01 \text{ and}$$

$b_8 = 2^8/8! \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. The series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{4^n}$ satisfies (i) of the Alternating Series Test because

$$b_{n+1} = \frac{n+1}{4^{n+1}} < \frac{n+3n}{4 \cdot 4^n} = \frac{4n}{4 \cdot 4^n} = \frac{n}{4^n} = b_n \text{ and (ii) } \lim_{n \rightarrow \infty} \frac{n}{4^n} = 0, \text{ so the series is convergent. Now}$$

$b_5 = 5/4^5 \approx 0.0049 > 0.002$ and $b_6 = 6/4^6 \approx 0.0015 < 0.002$, so by the Alternating Series Estimation Theorem, $n = 5$.

27. $b_7 = \frac{1}{7^5} = \frac{1}{16,807} \approx 0.0000595$, so

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx s_6 = \sum_{n=1}^6 \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} \approx 0.972080. \text{ Adding } b_7 \text{ to } s_6 \text{ does}$$

not change the fourth decimal place of s_6 , so the sum of the series, correct to four decimal places, is 0.9721.

28. $b_6 = \frac{6}{8^6} = \frac{6}{262,144} \approx 0.000023$, so

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n} \approx s_5 = \sum_{n=1}^5 \frac{(-1)^n n}{8^n} = -\frac{1}{8} + \frac{2}{64} - \frac{3}{512} + \frac{4}{4096} - \frac{5}{32,768} \approx -0.098785. \text{ Adding } b_6 \text{ to } s_5 \text{ does not}$$

change the fourth decimal place of s_5 , so the sum of the series, correct to four decimal places, is -0.0988 .

29. $b_7 = \frac{7^2}{10^7} = 0.0000049$, so

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n} \approx s_6 = \sum_{n=1}^6 \frac{(-1)^{n-1} n^2}{10^n} = \frac{1}{10} - \frac{4}{100} + \frac{9}{1000} - \frac{16}{10,000} + \frac{25}{100,000} - \frac{36}{1,000,000} = 0.067614.$$

Adding b_7 to s_6 does not change the fourth decimal place of s_6 , so the sum of the series, correct to four decimal places, is 0.0676.

30. $b_6 = \frac{1}{3^6 \cdot 6!} = \frac{1}{524,880} \approx 0.0000019$, so

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!} \approx s_5 = \sum_{n=1}^5 \frac{(-1)^n}{3^n n!} = -\frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \frac{1}{1944} - \frac{1}{29,160} \approx -0.283471. \text{ Adding } b_6 \text{ to } s_5 \text{ does not}$$

change the fourth decimal place of s_5 , so the sum of the series, correct to four decimal places, is -0.2835 .

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52}\right) + \left(\frac{1}{53} - \frac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.
32. If $p > 0$, $\frac{1}{(n+1)^p} \leq \frac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \leq 0$, $\lim_{n \rightarrow \infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus,
- $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \text{ converges } \Leftrightarrow p > 0.$$
33. Clearly $b_n = \frac{1}{n+p}$ is decreasing and eventually positive and $\lim_{n \rightarrow \infty} b_n = 0$ for any p . So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n+p \neq 0$ for $n \geq 1$, or p is not a negative integer.
34. Let $f(x) = \frac{(\ln x)^p}{x}$. Then $f'(x) = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p . Clearly $\lim_{n \rightarrow \infty} \frac{(\ln n)^p}{n} = 0$ if $p \leq 0$, and if $p > 0$ we can apply l'Hospital's Rule $\llbracket p+1 \rrbracket$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).
35. $\sum b_{2n} = \sum 1/(2n)^2$ clearly converges (by comparison with the p -series for $p = 2$). So suppose that $\sum (-1)^{n-1} b_n$ converges. Then by Theorem 12.2.8(ii), so does
- $$\sum [(-1)^{n-1} b_n + b_n] = 2 \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = 2 \sum \frac{1}{2n-1}.$$
- But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\sum (-1)^{n-1} b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.
36. (a) We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$. $P(1)$ is the statement $s_2 = h_2 - h_1$, which is true since $1 - \frac{1}{2} = \left(1 + \frac{1}{2}\right) - 1$. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.
- $$\begin{aligned} h_{2n+2} - h_{n+1} &= \left(h_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(h_n + \frac{1}{n+1}\right) = (h_{2n} - h_n) + \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} = s_{2n+2} \end{aligned}$$
- which is $P(n+1)$, and proves that $s_{2n} = h_{2n} - h_n$ for all n .
- (b) We know that $h_{2n} - \ln(2n) \rightarrow \gamma$ and $h_n - \ln n \rightarrow \gamma$ as $n \rightarrow \infty$. So
- $$s_{2n} = h_{2n} - h_n = [h_{2n} - \ln(2n)] - (h_n - \ln n) + [\ln(2n) - \ln n], \text{ and}$$
- $$\lim_{n \rightarrow \infty} s_{2n} = \gamma - \gamma + \lim_{n \rightarrow \infty} [\ln(2n) - \ln n] = \lim_{n \rightarrow \infty} (\ln 2 + \ln n - \ln n) = \ln 2.$$

12.6 Absolute Convergence and the Ratio and Root Tests

1. (a) Since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.
 (b) Since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).
 (c) Since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.
2. The series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has positive terms and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right] = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.
3. $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$. Using the Ratio Test, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{-10}{n+1} \right| = 0 < 1$, so the series is absolutely convergent.
4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n \rightarrow \infty} \frac{2^n}{n^4} = \infty$, so $\lim_{n \rightarrow \infty} (-1)^{n-1} \frac{2^n}{n^4}$ does not exist.
5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ is a divergent p -series ($p = \frac{1}{4} \leq 1$), so the given series is conditionally convergent.
6. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p -series ($p = 4 > 1$), so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is absolutely convergent.
7. $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{5+n} = \lim_{n \rightarrow \infty} \frac{1}{5/n+1} = 1$, so $\lim_{n \rightarrow \infty} a_n \neq 0$. Thus, the given series is divergent by the Test for Divergence.
8. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:
 $\lim_{n \rightarrow \infty} \frac{n/(n^2+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1$. But $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges by the Alternating Series Test:
 $\left\{ \frac{n}{n^2+1} \right\}$ has positive terms, is decreasing since $\left(\frac{x}{x^2+1} \right)' = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$, and
 $\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0$. Thus, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ is conditionally convergent.
9. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1/(2n+2)!}{1/(2n)!} = \lim_{n \rightarrow \infty} \frac{(2n)!}{(2n+2)!} = \lim_{n \rightarrow \infty} \frac{(2n)!}{(2n+2)(2n+1)(2n)!}$
 $= \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.
10. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!/e^{n+1}}{n!/e^n} \right| = \frac{1}{e} \lim_{n \rightarrow \infty} (n+1) = \infty$, so the series $\sum_{n=1}^{\infty} e^{-n} n!$ diverges by the Ratio Test.

11. Since $0 \leq \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3} = e \left(\frac{1}{n^3} \right)$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p -series ($p = 3 > 1$), $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ converges, and so $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent.
12. $\left| \frac{\sin 4n}{4^n} \right| \leq \frac{1}{4^n}$, so $\sum_{n=1}^{\infty} \left| \frac{\sin 4n}{4^n} \right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ ($|r| = \frac{1}{4} < 1$). Thus, $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is absolutely convergent.
13. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{(n+1)3^{n+1}}{4^n} \cdot \frac{4^{n-1}}{n \cdot 3^n} \right] = \lim_{n \rightarrow \infty} \left(\frac{3}{4} \cdot \frac{n+1}{n} \right) = \frac{3}{4} < 1$, so the series $\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$ is absolutely convergent by the Ratio Test.
14. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n} \right] = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right)^2 \cdot \frac{2}{n+1} \right] = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.
15. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{10^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{10^n} \right] = \lim_{n \rightarrow \infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2} \right) = \frac{5}{8} < 1$, so the series $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.
16. $n^{2/3} - 2 > 0$ for $n \geq 3$, so $\frac{3 - \cos n}{n^{2/3} - 2} > \frac{1}{n^{2/3} - 2} > \frac{1}{n^{2/3}}$ for $n \geq 3$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges ($p = \frac{2}{3} \leq 1$), so does $\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$ by the Comparison Test.
17. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$ and $\left\{ \frac{1}{\ln n} \right\}$ is decreasing. Now $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$, and since $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent (partial) harmonic series, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.
18. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges absolutely by the Ratio Test.
19. $\frac{|\cos(n\pi/3)|}{n!} \leq \frac{1}{n!}$ and $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges (use the Ratio Test or the result of Exercise 12.4.29), so the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

20. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0 < 1$, so the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

21. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left(\frac{n^n}{3^{1+3n}} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt[3]{3} \cdot 3} = \infty$, so the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

$$\begin{aligned} \text{Or: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{(n+1)^{n+1}}{3^{1+3(n+1)}} \cdot \frac{3^{1+3n}}{n^n} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{3^3} \cdot \left(\frac{n+1}{n} \right)^n (n+1) \right] \\ &= \frac{1}{27} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n \lim_{n \rightarrow \infty} (n+1) = \frac{1}{27} e \lim_{n \rightarrow \infty} (n+1) = \infty, \end{aligned}$$

so the series is divergent by the Ratio Test.

22. Since $\left\{ \frac{1}{n \ln n} \right\}$ is decreasing and $\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the Alternating Series

Test. Since $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test (Exercise 12.3.21), the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is conditionally convergent.

23. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{2 + 1/n^2} = \frac{1}{2} < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{n^2 + 1}{2n^2 + 1} \right)^n$ is absolutely convergent by the Root Test.

24. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{1}{\arctan n} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent by the Root Test.

25. Use the Ratio Test with the series $1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n-1)!} + \cdots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n-1)!}.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)[2(n+1)-1]}{[2(n+1)-1]!} \cdot \frac{(2n-1)!}{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{2n} = 0 < 1, \end{aligned}$$

so the given series is absolutely convergent and therefore convergent.

26. Use the Ratio Test with the series $\frac{2}{5} + \frac{2 \cdot 6}{5 \cdot 8} + \frac{2 \cdot 6 \cdot 10}{5 \cdot 8 \cdot 11} + \frac{2 \cdot 6 \cdot 10 \cdot 14}{5 \cdot 8 \cdot 11 \cdot 14} + \cdots = \sum_{n=1}^{\infty} \frac{2 \cdot 6 \cdot 10 \cdots (4n-2)}{5 \cdot 8 \cdot 11 \cdots (3n+2)}.$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 6 \cdot 10 \cdots (4n-2)[4(n+1)-2]}{5 \cdot 8 \cdot 11 \cdots (3n+2)[3(n+1)+2]} \cdot \frac{5 \cdot 8 \cdot 11 \cdots (3n+2)}{2 \cdot 6 \cdot 10 \cdots (4n-2)} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4n+2}{3n+5} = \frac{4}{3} > 1, \end{aligned}$$

so the given series is divergent.

$$27. \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{n!} = \sum_{n=1}^{\infty} \frac{(2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdots (2 \cdot n)}{n!} = \sum_{n=1}^{\infty} \frac{2^n n!}{n!} = \sum_{n=1}^{\infty} 2^n, \text{ which diverges by the}$$

Test for Divergence since $\lim_{n \rightarrow \infty} 2^n = \infty$.

$$28. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1}(n+1)!}{5 \cdot 8 \cdot 11 \cdots (3n+5)}}{\frac{2^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}} \right| = \lim_{n \rightarrow \infty} \frac{2(n+1)}{3n+5} = \frac{2}{3} < 1, \text{ so the series converges absolutely}$$

by the Ratio Test.

$$29. \text{ By the recursive definition, } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5n+1}{4n+3} \right| = \frac{5}{4} > 1, \text{ so the series diverges by the Ratio Test.}$$

$$30. \text{ By the recursive definition, } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 + \cos n}{\sqrt{n}} \right| = 0 < 1, \text{ so the series converges absolutely by the}$$

Ratio Test.

$$31. (a) \lim_{n \rightarrow \infty} \left| \frac{1/(n+1)^3}{1/n^3} \right| = \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^3} = 1. \text{ Inconclusive.}$$

$$(b) \lim_{n \rightarrow \infty} \left| \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2n} \right) = \frac{1}{2}. \text{ Conclusive (convergent).}$$

$$(c) \lim_{n \rightarrow \infty} \left| \frac{(-3)^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^{n-1}} \right| = 3 \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 3 \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+1/n}} = 3. \text{ Conclusive (divergent).}$$

$$(d) \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} \right| = \lim_{n \rightarrow \infty} \left[\sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(1+1/n)^2} \right] = 1. \text{ Inconclusive.}$$

32. We use the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^2}{[k(n+1)]!} \cdot \frac{(kn)!}{(n!)^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{[k(n+1)][k(n+1)-1] \cdots [kn+1]} \right|$$

Now if $k = 1$, then this is equal to $\lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(n+1)} \right| = \infty$, so the series diverges; if $k = 2$, the limit is

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| = \frac{1}{4} < 1, \text{ so the series converges, and if } k > 2, \text{ then the highest power of } n \text{ in the}$$

denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

$$33. (a) \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1, \text{ so by the Ratio}$$

Test the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x .

(b) Since the series of part (a) always converges, we must have $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ by Theorem 12.2.6.

$$\begin{aligned}
 34. (a) R_n &= a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots = a_{n+1} \left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots \right) \\
 &= a_{n+1} \left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}} \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}} \frac{a_{n+3}}{a_{n+2}} \frac{a_{n+2}}{a_{n+1}} + \cdots \right) \\
 &= a_{n+1} (1 + r_{n+1} + r_{n+2} r_{n+1} + r_{n+3} r_{n+2} r_{n+1} + \cdots) \quad (*) \\
 &\leq a_{n+1} (1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots) \quad [\text{since } \{r_n\} \text{ is decreasing}] = \frac{a_{n+1}}{1 - r_{n+1}}
 \end{aligned}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n \rightarrow L$ as $n \rightarrow \infty$, we have $r_n < L$ for all n . So, starting with equation (*),

$$R_n = a_{n+1} (1 + r_{n+1} + r_{n+2} r_{n+1} + r_{n+3} r_{n+2} r_{n+1} + \cdots) \leq a_{n+1} (1 + L + L^2 + L^3 + \cdots) = \frac{a_{n+1}}{1 - L}$$

$$35. (a) s_5 = \sum_{n=1}^5 \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} \approx 0.68854. \text{ Now the ratios}$$

$$r_n = \frac{a_{n+1}}{a_n} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)} \text{ form an increasing sequence, since}$$

$$r_{n+1} - r_n = \frac{n+1}{2(n+2)} - \frac{n}{2(n+1)} = \frac{(n+1)^2 - n(n+2)}{2(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)} > 0. \text{ So by Exercise 34(b),}$$

$$\text{the error in using } s_5 \text{ is } R_5 \leq \frac{a_6}{1 - \lim_{n \rightarrow \infty} r_n} = \frac{1/(6 \cdot 2^6)}{1 - 1/2} = \frac{1}{192} \approx 0.00521.$$

(b) The error in using s_n as an approximation to the sum is $R_n = \frac{a_{n+1}}{1 - \frac{1}{2}} = \frac{2}{(n+1)2^{n+1}}$. We want

$$R_n < 0.00005 \Leftrightarrow \frac{1}{(n+1)2^n} < 0.00005 \Leftrightarrow (n+1)2^n > 20,000. \text{ To find such an } n \text{ we can use trial}$$

and error or a graph. We calculate $(11+1)2^{11} = 24,576$, so $s_{11} = \sum_{n=1}^{11} \frac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

$$36. s_{10} = \sum_{n=1}^{10} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{10}{1024} \approx 1.988. \text{ The ratios}$$

$$r_n = \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2} \left(1 + \frac{1}{n} \right) \text{ form a decreasing sequence, and}$$

$$r_{11} = \frac{11+1}{2(11)} = \frac{12}{22} = \frac{6}{11} < 1, \text{ so by Exercise 34(a), the error in using } s_{10} \text{ to approximate the sum of the series}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \text{ is } R_{10} \leq \frac{a_{11}}{1 - r_{11}} = \frac{\frac{11}{2048}}{1 - \frac{6}{11}} = \frac{121}{10,240} \approx 0.0118.$$

37. Summing the inequalities $-|a_i| \leq a_i \leq |a_i|$ for $i = 1, 2, \dots, n$, we get $-\sum_{i=1}^n |a_i| \leq \sum_{i=1}^n a_i \leq \sum_{i=1}^n |a_i|$

$$\Rightarrow -\lim_{n \rightarrow \infty} \sum_{i=1}^n |a_i| \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n |a_i| \Rightarrow -\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n| \Rightarrow$$

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

38. (a) Following the hint, we get that $|a_n| < r^n$ for $n \geq N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges ($0 < r < 1$), the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- (b) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \geq N$, so $|a_n| > 1$ for $n \geq N$. Thus, $\lim_{n \rightarrow \infty} a_n \neq 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.
39. (a) Since $\sum a_n$ is absolutely convergent, and since $|a_n^+| \leq |a_n|$ and $|a_n^-| \leq |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent. (Or use Theorem 12.2.8.)
- (b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum (a_n^+ - \frac{1}{2}a_n)$ by Theorem 12.2.8. But $\sum (a_n^+ - \frac{1}{2}a_n) = \sum [\frac{1}{2}(a_n + |a_n|) - \frac{1}{2}a_n] = \frac{1}{2} \sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.
40. Let $\sum b_n$ be the rearranged series constructed in the hint. [This series can be constructed by virtue of the result of Exercise 39(b).] This series will have partial sums s_n that oscillate in value back and forth across r . Since $\lim_{n \rightarrow \infty} a_n = 0$ (by Theorem 12.2.6), and since the size of the oscillations $|s_n - r|$ is always less than $|a_n|$ because of the way $\sum b_n$ was constructed, we have that $\sum b_n = \lim_{n \rightarrow \infty} s_n = r$.

12.7 Strategy for Testing Series

- $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1 - 1/n^2}{1 + 1/n} = 1 \neq 0$, so the series $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + 1}$ diverges by the Test for Divergence.
- If $a_n = \frac{n-1}{n^2+n}$ and $b_n = \frac{1}{n}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2 - n}{n^2 + n} = \lim_{n \rightarrow \infty} \frac{1 - 1/n}{1 + 1/n} = 1$, so the series $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.
- $\frac{1}{n^2+n} < \frac{1}{n^2}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p -series that converges because $p = 2 > 1$.
- Let $b_n = \frac{n-1}{n^2+n}$. Then $b_1 = 0$, and $b_2 = b_3 = \frac{1}{6}$, but $b_n > b_{n+1}$ for $n \geq 3$ since $\left(\frac{x-1}{x^2+x} \right)' = \frac{(x^2+x) - (x-1)(2x+1)}{(x^2+x)^2} = \frac{-x^2+2x+1}{(x^2+x)^2} = \frac{2-(x-1)^2}{(x^2+x)^2} < 0$ for $x \geq 3$. Thus,

$\{b_n \mid n \geq 3\}$ is decreasing and $\lim_{n \rightarrow \infty} b_n = 0$, so $\sum_{n=3}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ converges by the Alternating Series Test.

Hence, the full series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ also converges.

$$5. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \frac{2^{3n}}{(-3)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3} \right| = \lim_{n \rightarrow \infty} \frac{3}{2^3} = \frac{3}{8} < 1, \text{ so the series}$$

$$\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} \text{ is absolutely convergent by the Ratio Test.}$$

$$6. \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{3n}{1+8n} = \lim_{n \rightarrow \infty} \frac{3}{1/n+8} = \frac{3}{8} < 1, \text{ so } \sum_{n=1}^{\infty} \left(\frac{3n}{1+8n} \right)^n \text{ converges by the Root Test.}$$

7. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\int \frac{1}{x\sqrt{\ln x}} dx \left[\begin{array}{l} u = \ln x, \\ du = dx/x \end{array} \right] = \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t \rightarrow \infty} [2\sqrt{\ln x}]_2^t = \lim_{t \rightarrow \infty} (2\sqrt{\ln t} - 2\sqrt{\ln 2}) = \infty. \text{ Since the integral}$$

diverges, the given series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

$$8. \sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{2^k}{(k+1)(k+2)}. \text{ Using the Ratio Test, we get}$$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{2^{k+1}}{(k+2)(k+3)} \cdot \frac{(k+1)(k+2)}{2^k} \right| = \lim_{k \rightarrow \infty} \left(2 \cdot \frac{k+1}{k+3} \right) = 2 > 1, \text{ so the series diverges.}$$

Or: Use the Test for Divergence.

$$9. \sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}. \text{ Using the Ratio Test, we get}$$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right| = \lim_{k \rightarrow \infty} \left[\left(\frac{k+1}{k} \right)^2 \cdot \frac{1}{e} \right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1, \text{ so the series converges.}$$

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1, \infty)$, and $f'(x) = \frac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is

decreasing on $[1, \infty)$ as well, and we can apply the Integral Test. $\int_1^{\infty} x^2 e^{-x^3} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{3} e^{-x^3} \right]_1^t = \frac{1}{3e}$, so the

integral converges, and hence, the series converges.

11. $b_n = \frac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test.

12. Let $b_n = \frac{n}{n^2 + 25}$. Then $b_n > 0$, $\lim_{n \rightarrow \infty} b_n = 0$, and

$$b_n - b_{n+1} = \frac{n}{n^2 + 25} - \frac{n+1}{n^2 + 2n + 26} = \frac{n^2 + n - 25}{(n^2 + 25)(n^2 + 2n + 26)}, \text{ which is positive for } n \geq 5, \text{ so the}$$

sequence $\{b_n\}$ decreases from $n = 5$ on. Hence, the given series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 25}$ converges by the Alternating Series Test.

13. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} \right| = \lim_{n \rightarrow \infty} \left[\frac{3(n+1)^2}{(n+1)n^2} \right] = 3 \lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ converges by the Ratio Test.

14. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim_{n \rightarrow \infty} \sin n$ does not exist.

15. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots (3n+2)[3(n+1)+2]} \cdot \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{n!} \right|$
 $= \lim_{n \rightarrow \infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1$

so the series $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$ converges by the Ratio Test.

16. Using the Limit Comparison Test with $a_n = \frac{n^2 + 1}{n^3 + 1}$ and $b_n = \frac{1}{n}$, we have

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{n^2 + 1}{n^3 + 1} \cdot \frac{n}{1} \right) = \lim_{n \rightarrow \infty} \frac{n^3 + n}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{1 + 1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

17. $\lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n \rightarrow \infty} (-1)^n 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

18. $b_n = \frac{1}{\sqrt{n} - 1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim_{n \rightarrow \infty} b_n = 0$, so $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} - 1}$ converges by the Alternating Series Test.

19. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1/n}{1/(2\sqrt{n})} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

20. $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{k+6}{5^{k+1}} \cdot \frac{5^k}{k+5} \right| = \frac{1}{5} \lim_{k \rightarrow \infty} \frac{k+6}{k+5} = \frac{1}{5} < 1$, so the series $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$ converges by the Ratio Test.
21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n} \right)^n \cdot \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.
22. $\frac{\sqrt{n^2-1}}{n^3+2n^2+5} < \frac{n}{n^3+2n^2+5} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p -series $\sum_{n=1}^{\infty} 1/n^2$ ($p = 2 > 1$).
23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have
- $$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\tan(1/n)}{1/n} = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\sec^2(1/x) \cdot (-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \sec^2(1/x) = 1^2 = 1 > 0.$$
- Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.
24. $\frac{|\cos(n/2)|}{n^2+4n} < \frac{1}{n^2+4n} < \frac{1}{n^2}$ and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.
25. Use the Ratio Test. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim_{n \rightarrow \infty} \frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.
26. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(\frac{n^2+2n+2}{5^{n+1}} \cdot \frac{5^n}{n^2+1} \right) = \lim_{n \rightarrow \infty} \left(\frac{1+2/n+2/n^2}{1+1/n^2} \cdot \frac{1}{5} \right) = \frac{1}{5} < 1$, so $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ converges by the Ratio Test.
27. $\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^t$ (using integration by parts) $\stackrel{H}{=} 1$. So $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ converges by the Comparison Test.
28. Since $\left\{ \frac{1}{n} \right\}$ is a decreasing sequence, $e^{1/n} \leq e^{1/1} = e$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{e}{n^2}$ converges ($p = 2 > 1$), so $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)
29. $0 < \frac{\tan^{-1} n}{n^{3/2}} < \frac{\pi/2}{n^{3/2}} \cdot \sum_{n=1}^{\infty} \frac{\pi/2}{n^{3/2}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p -series ($p = \frac{3}{2} > 1$), so $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{3/2}}$ converges by the Comparison Test.

30. Let $f(x) = \frac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \frac{5-x}{2\sqrt{x}(x+5)^2} < 0$ for

$x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+5} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/2} + 5n^{-1/2}} = 0, \text{ so the series } \sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5} \text{ converges.}$$

31. $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{5^k}{3^k + 4^k} = [\text{divide by } 4^k] \lim_{k \rightarrow \infty} \frac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\lim_{k \rightarrow \infty} \left(\frac{3}{4}\right)^k = 0$ and $\lim_{k \rightarrow \infty} \left(\frac{5}{4}\right)^k = \infty$.

Thus, $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

32. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{2n}{n^2} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0$, so the series $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ converges by the Root Test.

33. Let $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1}{n\sqrt{n}}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by

limit comparison with the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ ($p = 3/2 > 1$).

34. $0 \leq n \cos^2 n \leq n$, so $\frac{1}{n + n \cos^2 n} \geq \frac{1}{n + n} = \frac{1}{2n}$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n + n \cos^2 n}$ diverges by comparison with

$\sum_{n=1}^{\infty} \frac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

35. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^{n^2/n} = \lim_{n \rightarrow \infty} \frac{1}{[(n+1)/n]^n} = \frac{1}{\lim_{n \rightarrow \infty} (1 + 1/n)^n} = \frac{1}{e} < 1$, so the series

$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = (e^{\ln \ln n})^{\ln n} = (e^{\ln n})^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \rightarrow \infty$ as $n \rightarrow \infty$, so $\ln \ln n > 2$ for

sufficiently large n . For these n we have $(\ln n)^{\ln n} > n^2$, so $\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges

($p = 2 > 1$), so does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} (2^{1/n} - 1) = 1 - 1 = 0 < 1$, so the series $\sum_{n=1}^{\infty} (2^{1/n} - 1)^n$ converges by the Root Test.

38. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^{1/n} - 1}{1/n}$
- $$= \lim_{x \rightarrow \infty} \frac{2^{1/x} - 1}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2^{1/x} \cdot \ln 2 \cdot (-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} (2^{1/x} \cdot \ln 2) = 1 \cdot \ln 2 = \ln 2 > 0.$$
- So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$.

Alternate Solution:

$$\sqrt[n]{2} - 1 = \frac{1}{2^{(n-1)/n} + 2^{(n-2)/n} + 2^{(n-3)/n} + \dots + 2^{1/n} + 1} \quad [\text{rationalize the numerator}] \geq \frac{1}{2n}.$$

and since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$ by the Comparison Test.

12.8 Power Series

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$, where x is a variable and the c_n 's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \dots$ is called a power series in $(x - a)$ or a power series centered at a or a power series about a , where a is a constant.

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, the radius of convergence is:

- (i) 0 if the series converges only when $x = a$
- (ii) ∞ if the series converges for all x , or
- (iii) a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

In most cases, R can be found by using the Ratio Test.

- (b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. If $a_n = \frac{x^n}{\sqrt{n}}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{\sqrt{n+1}/\sqrt{n}} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{\sqrt{1+1/n}} = |x|.$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll

check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because it is a p -series with

$p = \frac{1}{2} \leq 1$. When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-1, 1)$.

4. If $a_n = \frac{(-1)^n x^n}{n+1}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{1 + 1/(n+1)} = |x|$. By the Ratio Test, the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

5. If $a_n = \frac{(-1)^{n-1} x^n}{n^3}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-1)^{n-1} x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)xn^3}{(n+1)^3} \right|$
 $= \lim_{n \rightarrow \infty} \left[\left(\frac{n}{n+1} \right)^3 |x| \right] = 1^3 \cdot |x| = |x|$. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3}$ converges when $|x| < 1$,

so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test. When $x = -1$, the series

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^3} = -\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is a constant multiple of a convergent p -series ($p = 3 > 1$).

Thus, the interval of convergence is $I = [-1, 1]$.

6. $a_n = \sqrt{n}x^n$, so we need $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}|x|^{n+1}}{\sqrt{n}|x|^n} = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}}|x| = |x| < 1$ for convergence (by the Ratio Test), so $R = 1$. When $x = \pm 1$, $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \sqrt{n} = \infty$, so the series diverges by the Test for Divergence. Thus, $I = (-1, 1)$.

7. If $a_n = \frac{x^n}{n!}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for all real x . So, by the Ratio Test, $R = \infty$, and $I = (-\infty, \infty)$.

8. Here the Root Test is easier. If $a_n = n^n x^n$ then $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} n|x| = \infty$ if $x \neq 0$, so $R = 0$ and $I = \{0\}$.

9. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)4^{n+1}|x|^{n+1}}{n4^n|x|^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) 4|x| = 4|x|$. Now $4|x| < 1 \Leftrightarrow |x| < \frac{1}{4}$, so by the Ratio Test, $R = \frac{1}{4}$. When $x = \frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty} (-1)^n n$, and when $x = -\frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty} n$. Thus, $I = (-\frac{1}{4}, \frac{1}{4})$.

10. If $a_n = \frac{x^n}{n3^n}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{xn}{(n+1)3} \right| = \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{|x|}{3}$.

By the Ratio Test, the series converges when $\frac{|x|}{3} < 1 \Leftrightarrow |x| < 3$, so $R = 3$. When $x = -3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x = 3$, it is the harmonic series, which diverges. Thus, $I = (-3, 3)$.

11. $a_n = \frac{(-2)^n x^n}{\sqrt[n]{n}}$, so $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2^{n+1}|x|^{n+1}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{2^n|x|^n} = \lim_{n \rightarrow \infty} 2|x| \sqrt[n]{\frac{n}{n+1}} = 2|x|$, so by the

Ratio Test, the series converges when $2|x| < 1 \Leftrightarrow |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. When $x = -\frac{1}{2}$, we get the divergent

p -series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ ($p = \frac{1}{4} \leq 1$). When $x = \frac{1}{2}$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$, which converges by the Alternating Series Test. Thus, $I = (-\frac{1}{2}, \frac{1}{2}]$.

12. $a_n = \frac{x^n}{5^n n^5}$, so $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{5^{n+1} (n+1)^5} \cdot \frac{5^n n^5}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{5} \left(\frac{n}{n+1} \right)^5 = \frac{|x|}{5}$. By the Ratio Test, the series converges when $|x|/5 < 1 \Leftrightarrow |x| < 5$, so $R = 5$. When $x = -5$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x = 5$, we get the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ ($p = 5 > 1$). Thus, $I = [-5, 5]$.

13. If $a_n = (-1)^n \frac{x^n}{4^n \ln n}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{4^{n+1} \ln(n+1)} \cdot \frac{4^n \ln n}{x^n} \right| = \frac{|x|}{4} \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} = \frac{|x|}{4} \cdot 1 \quad (\text{by l'Hospital's Rule}) = \frac{|x|}{4}.$$

By the Ratio Test, the series converges when $\frac{|x|}{4} < 1 \Leftrightarrow |x| < 4$, so $R = 4$. When $x = -4$,

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}. \text{ Since } \ln n < n \text{ for } n \geq 2, \frac{1}{\ln n} > \frac{1}{n} \text{ and } \sum_{n=2}^{\infty} \frac{1}{n} \text{ is the}$$

divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$,

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}, \text{ which converges by the Alternating Series Test. Thus, } I = (-4, 4].$$

14. $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$, so $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}} = \lim_{n \rightarrow \infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0$. Thus, by the Ratio Test, the series converges for all real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

15. If $a_n = \sqrt{n} (x-1)^n$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1} |x-1|^{n+1}}{\sqrt{n} |x-1|^n} \right| = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} |x-1| = |x-1|$. By the Ratio Test, the series converges when $|x-1| < 1$ [so $R = 1$] $\Leftrightarrow -1 < x-1 < 1 \Leftrightarrow 0 < x < 2$. When $x = 0$, the series becomes $\sum_{n=0}^{\infty} (-1)^n \sqrt{n}$, which diverges by the Test for Divergence. When $x = 2$, the series becomes $\sum_{n=0}^{\infty} \sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I = (0, 2)$.

16. If $a_n = n^3 (x-5)^n$, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3 (x-5)^{n+1}}{n^3 (x-5)^n} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^3 |x-5| = |x-5|$. By the Ratio Test, the series converges when $|x-5| < 1 \Leftrightarrow -1 < x-5 < 1 \Leftrightarrow 4 < x < 6$. When $x = 4$, the series becomes $\sum_{n=0}^{\infty} (-1)^n n^3$, which diverges by the Test for Divergence. When $x = 6$, the series becomes $\sum_{n=0}^{\infty} n^3$, which also diverges by the Test for Divergence. Thus, $R = 1$ and $I = (4, 6)$.

17. If $a_n = (-1)^n \frac{(x+2)^n}{n2^n}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{|x+2|^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{|x+2|^n} \right] = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{|x+2|}{2} = \frac{|x+2|}{2}. \text{ By the Ratio Test, the}$$

series converges when $\frac{|x+2|}{2} < 1 \Leftrightarrow |x+2| < 2$ [so $R = 2$] $\Leftrightarrow -2 < x+2 < 2 \Leftrightarrow -4 < x < 0$.

When $x = -4$, the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent harmonic series.

When $x = 0$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-4, 0]$.

18. If $a_n = \frac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-2)^n(x+3)^n} \right| = \lim_{n \rightarrow \infty} \frac{2|x+3|}{\sqrt{1+1/n}} = 2|x+3| < 1 \Leftrightarrow$$

$|x+3| < \frac{1}{2}$ [so $R = \frac{1}{2}$] $\Leftrightarrow -\frac{7}{2} < x < -\frac{5}{2}$. When $x = -\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges

because it is a p -series with $p = \frac{1}{2} \leq 1$. When $x = -\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the

Alternating Series Test. Thus, $I = (-\frac{7}{2}, -\frac{5}{2}]$.

19. If $a_n = \frac{(x-2)^n}{n^n}$, then $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test).
 $R = \infty$ and $I = (-\infty, \infty)$.

20. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(3x-2)^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{|3x-2|}{3} \cdot \frac{1}{1+1/n} \right) = \frac{|3x-2|}{3} = |x - \frac{2}{3}|$, so by the Ratio Test, the series converges when $|x - \frac{2}{3}| < 1 \Leftrightarrow -\frac{1}{3} < x < \frac{5}{3}$. $R = 1$. When $x = -\frac{1}{3}$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the convergent alternating harmonic series. When $x = \frac{5}{3}$, the series becomes the divergent harmonic series. Thus, $I = [-\frac{1}{3}, \frac{5}{3})$.

21. $a_n = \frac{n}{b^n}(x-a)^n$, where $b > 0$.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)|x-a|^{n+1}}{b^{n+1}} \cdot \frac{b^n}{n|x-a|^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \frac{|x-a|}{b} = \frac{|x-a|}{b}.$$

By the Ratio Test, the series converges when $\frac{|x-a|}{b} < 1 \Leftrightarrow |x-a| < b$ [so $R = b$] $\Leftrightarrow$

$-b < x-a < b \Leftrightarrow a-b < x < a+b$. When $|x-a| = b$, $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} n = \infty$, so the series diverges.

Thus, $I = (a-b, a+b)$.

22. $a_n = \frac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)|x-4|^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{n|x-4|^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \frac{n^3+1}{n^3+3n^2+3n+2} |x-4| = |x-4|.$$

By the Ratio Test, the series converges when $|x - 4| < 1$ [so $R = 1$] $\Leftrightarrow -1 < x - 4 < 1 \Leftrightarrow 3 < x < 5$.

When $|x - 4| = 1$, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$, which converges by comparison with the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ($p = 2 > 1$). Thus, $I = [3, 5]$.

23. If $a_n = n!(2x - 1)^n$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n} \right| = \lim_{n \rightarrow \infty} (n+1)|2x-1| \rightarrow \infty$ as $n \rightarrow \infty$ for all $x \neq \frac{1}{2}$. Since the series diverges for all $x \neq \frac{1}{2}$, $R = 0$ and $I = \{\frac{1}{2}\}$.

24. $a_n = \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{n^2 x^n}{2^n n!} = \frac{n x^n}{2^n (n-1)!}$, so
 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)|x|^{n+1}}{2^{n+1} n!} \cdot \frac{2^n (n-1)!}{n|x|^n} = \lim_{n \rightarrow \infty} \frac{n+1}{n^2} \frac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for all real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

25. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{|4x+1|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|4x+1|^n} \right] = \lim_{n \rightarrow \infty} \frac{|4x+1|}{(1+1/n)^2} = |4x+1|$, so by the Ratio Test, the series converges when $|4x+1| < 1 \Leftrightarrow -1 < 4x+1 < 1 \Leftrightarrow -2 < 4x < 0 \Leftrightarrow -\frac{1}{2} < x < 0$, so $R = \frac{1}{4}$. When $x = -\frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x = 0$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p -series ($p = 2 > 1$). $I = [-\frac{1}{2}, 0]$.

26. If $a_n = \frac{(-1)^n (2x+3)^n}{n \ln n}$, then we need $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |2x+3| \lim_{n \rightarrow \infty} \frac{n \ln n}{(n+1) \ln(n+1)} = |2x+3| < 1$ for convergence, so $-2 < x < -1$ and $R = \frac{1}{2}$. When $x = -2$, $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$, which diverges (Integral Test), and when $x = -1$, $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$, which converges (Alternating Series Test), so $I = (-2, -1]$.

27. If $a_n = \frac{x^n}{(\ln n)^n}$, then $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{|x|}{\ln n} = 0 < 1$ for all x , so $R = \infty$ and $I = (-\infty, \infty)$ by the Root Test.

28. If $a_n = \frac{2 \cdot 4 \cdot 6 \cdots (2n) x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$, then we need $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} |x| \left(\frac{2n+2}{2n+1} \right) = |x| < 1$ for convergence, so $R = 1$. If $x = \pm 1$, $|a_n| = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1$ for all n since each integer in the numerator is larger than the corresponding one in the denominator, so $\sum a_n$ diverges in both cases by the Test for Divergence, and $I = (-1, 1)$.

29. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \leq 4$. In particular, it converges when $x = -2$; that is, $\sum_{n=0}^{\infty} c_n (-2)^n$ is convergent.
 (b) It does not follow that $\sum_{n=0}^{\infty} c_n (-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n / (n4^n)$.]

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \leq x < 4$ and diverges for at least $x \geq 6$ and $x < -6$. Therefore:

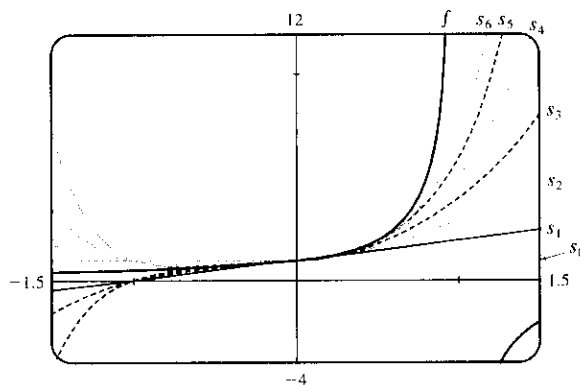
- (a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.
 (b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.
 (c) It converges when $x = -3$; that is, $\sum c_n (-3)^n$ is convergent.
 (d) It diverges when $x = -9$; that is, $\sum c_n (-9)^n = \sum (-1)^n c_n 9^n$ is divergent.

31. If $a_n = \frac{(n!)^k}{(kn)!} x^n$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{[(n+1)!]^k (kn)!}{(n!)^k [k(n+1)]!} |x| = \lim_{n \rightarrow \infty} \frac{(n+1)^k}{(kn+k)(kn+k-1) \cdots (kn+2)(kn+1)} |x| \\ &= \lim_{n \rightarrow \infty} \left[\frac{(n+1)}{(kn+1)} \frac{(n+1)}{(kn+2)} \cdots \frac{(n+1)}{(kn+k)} \right] |x| \\ &= \lim_{n \rightarrow \infty} \left[\frac{n+1}{kn+1} \right] \lim_{n \rightarrow \infty} \left[\frac{n+1}{kn+2} \right] \cdots \lim_{n \rightarrow \infty} \left[\frac{n+1}{kn+k} \right] |x| = \left(\frac{1}{k} \right)^k |x| < 1 \Leftrightarrow \end{aligned}$$

$|x| < k^k$ for convergence, and the radius of convergence is $R = k^k$.

32. The partial sums of the series $\sum_{n=0}^{\infty} x^n$ definitely do not converge to $f(x) = 1/(1-x)$ for $x \geq 1$, since f is undefined at $x = 1$ and negative on $(1, \infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x \leq -1$, since $0 < f(x) < 1$ on this interval, while the partial sums are either larger than 1 or less than 0. The partial sums seem to converge to f on $(-1, 1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x| < 1$, divergence for $|x| \geq 1$ (see Example 12.2.5).



33. (a) If $a_n = \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}}$, then

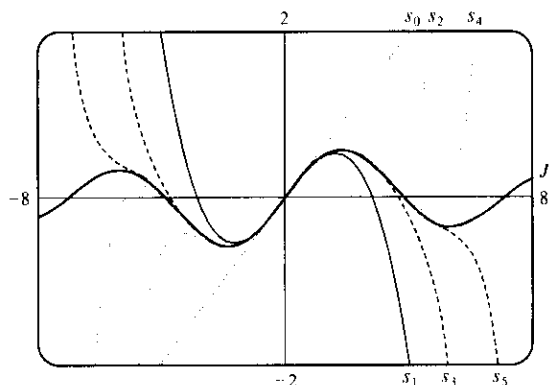
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(n+1)!(n+2)! 2^{2n+3}} \cdot \frac{n!(n+1)! 2^{2n+1}}{x^{2n+1}} \right| = \left(\frac{x}{2} \right)^2 \lim_{n \rightarrow \infty} \frac{1}{(n+1)(n+2)} = 0 \text{ for all } x. \text{ So } J_1(x) \text{ converges for all } x \text{ and its domain is } (-\infty, \infty).$$

- (b), (c) The initial terms of $J_1(x)$ up to $n = 5$ are

$$a_0 = \frac{x}{2}, a_1 = -\frac{x^3}{16}, a_2 = \frac{x^5}{384}, a_3 = -\frac{x^7}{18,432},$$

$$a_4 = \frac{x^9}{1,474,560}, \text{ and } a_5 = -\frac{x^{11}}{176,947,200}.$$

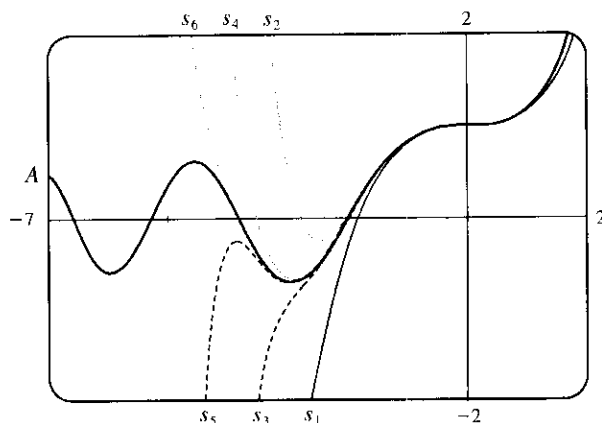
The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



34. (a) $A(x) = 1 + \sum_{n=1}^{\infty} a_n$, where $a_n = \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|^3 \lim_{n \rightarrow \infty} \frac{1}{(3n+2)(3n+3)} = 0 \text{ for all } x, \text{ so the domain is } \mathbb{R}.$$

(b), (c)



$s_0 = 1$ has been omitted from the graph. The partial sums seem to approximate $A(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.

To plot A , we must first define $A(x)$ for the CAS. Note that for $n \geq 1$, the denominator of a_n is

$$2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) \cdot 3n = \frac{(3n)!}{1 \cdot 4 \cdot 7 \cdots (3n-2)} = \frac{(3n)!}{\prod_{k=1}^n (3k-2)}, \text{ so } a_n = \frac{\prod_{k=1}^n (3k-2)}{(3n)!} x^{3n} \text{ and}$$

$$\text{thus } A(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^n (3k-2)}{(3n)!} x^{3n}. \text{ Both Maple and Mathematica are able to plot } A \text{ if we define it}$$

this way, and Derive is able to produce a similar graph using a suitable partial sum of $A(x)$.

Derive, Maple and Mathematica all have two initially known Airy functions, called `AI_SERIES(z, m)` and `BI_SERIES(z, m)` from `BESSEL.MTH` in Derive and `AiryAi` and `AiryBi` in Maple and Mathematica (just `Ai` and `Bi` in older versions of Maple). However, it is very difficult to solve for A in terms of the CAS's Airy

$$\text{functions, although in fact } A(x) = \frac{\sqrt{3} \text{AiryAi}(x) + \text{AiryBi}(x)}{\sqrt{3} \text{AiryAi}(0) + \text{AiryBi}(0)}.$$

$$\begin{aligned} 35. s_{2n-1} &= 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots + x^{2n-2} + 2x^{2n-1} \\ &= 1(1+2x) + x^2(1+2x) + x^4(1+2x) + \cdots + x^{2n-2}(1+2x) \\ &= (1+2x)(1+x^2+x^4+\cdots+x^{2n-2}) \\ &= (1+2x) \frac{1-x^{2n}}{1-x^2} \quad [\text{by (12.2.3)}] \quad \text{with } r = x^2 \rightarrow \frac{1+2x}{1-x^2} \text{ as } n \rightarrow \infty \quad [\text{by (12.2.4)}], \end{aligned}$$

when $|x| < 1$. Also $s_{2n} = s_{2n-1} + x^{2n} \rightarrow \frac{1+2x}{1-x^2}$ since $x^{2n} \rightarrow 0$ for $|x| < 1$. Therefore,

$s_n \rightarrow \frac{1+2x}{1-x^2}$ since s_{2n} and s_{2n-1} both approach $\frac{1+2x}{1-x^2}$ as $n \rightarrow \infty$. Thus, the interval of convergence is $(-1, 1)$

$$\text{and } f(x) = \frac{1+2x}{1-x^2}.$$

$$36. s_{4n-1} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_0x^4 + c_1x^5 + c_2x^6 + c_3x^7 + \cdots + c_3x^{4n-1}$$

$$= (c_0 + c_1x + c_2x^2 + c_3x^3)(1 + x^4 + x^8 + \cdots + x^{4n-4}) \rightarrow \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4} \text{ as } n \rightarrow \infty$$

[by (12.2.4) with $r = x^4$] for $|x^4| < 1 \Leftrightarrow |x| < 1$. Also $s_{4n}, s_{4n+1}, s_{4n+2}$ have the same limits (for example,

$s_{4n} = s_{4n-1} + c_0x^{4n}$ and $x^{4n} \rightarrow 0$ for $|x| < 1$). So if at least one of c_0, c_1, c_2 , and c_3 is nonzero, then the interval

of convergence is $(-1, 1)$ and $f(x) = \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$.

$$37. \text{ We use the Root Test on the series } \sum c_n x^n. \text{ We need } \lim_{n \rightarrow \infty} \sqrt[n]{|c_n x^n|} = |x| \lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} = c|x| < 1 \text{ for}$$

convergence, or $|x| < 1/c$, so $R = 1/c$.

38. Suppose $c_n \neq 0$. Applying the Ratio Test to the series $\sum c_n(x-a)^n$, we find that

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \lim_{n \rightarrow \infty} \frac{|x-a|}{|c_n/c_{n+1}|} (*) = \frac{|x-a|}{\lim_{n \rightarrow \infty} |c_n/c_{n+1}|} \text{ (if}$$

$\lim_{n \rightarrow \infty} |c_n/c_{n+1}| \neq 0$), so the series converges when $\frac{|x-a|}{\lim_{n \rightarrow \infty} |c_n/c_{n+1}|} < 1 \Leftrightarrow |x-a| < \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$. Thus,

$$R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|. \text{ If } \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = 0 \text{ and } |x-a| \neq 0, \text{ then } (*) \text{ shows that } L = \infty \text{ and so the series diverges,}$$

and hence, $R = 0$. Thus, in all cases, $R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$.

39. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 12.2.61, $\sum (c_n + d_n) x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n) x^n$ is 2.

40. Since $\sum c_n x^n$ converges whenever $|x| < R$, $\sum c_n x^{2n} = \sum c_n (x^2)^n$ converges whenever $|x^2| < R \Leftrightarrow |x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

12.9 Representations of Functions as Power Series

1. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ with $|-x| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \frac{3}{1-x^4} = 3\left(\frac{1}{1-x^4}\right) = 3(1+x^4+x^8+x^{12}+\cdots) = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty}3x^{4n}$ with $|x^4| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$. [Note that $3\sum_{n=0}^{\infty}(x^4)^n$ converges $\Leftrightarrow \sum_{n=0}^{\infty}(x^4)^n$ converges, so the appropriate condition (from equation (1)) is $|x^4| < 1$.]
5. Replacing x with x^3 in (1) gives $f(x) = \frac{1}{1-x^3} = \sum_{n=0}^{\infty}(x^3)^n = \sum_{n=0}^{\infty}x^{3n}$. The series converges when $|x^3| < 1 \Leftrightarrow |x|^3 < 1 \Leftrightarrow |x| < \sqrt[3]{1} \Leftrightarrow |x| < 1$. Thus, $R = 1$ and $I = (-1, 1)$.
6. $f(x) = \frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n}$. The series converges when $|-9x^2| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = (-\frac{1}{3}, \frac{1}{3})$.
7. $f(x) = \frac{1}{x-5} = -\frac{1}{5}\left(\frac{1}{1-x/5}\right) = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$ or equivalently, $-\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}x^n$. The series converges when $\left|\frac{x}{5}\right| < 1$; that is, when $|x| < 5$, so $I = (-5, 5)$.
8. $f(x) = \frac{x}{4x+1} = x \cdot \frac{1}{1-(-4x)} = x\sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x| < 1$; that is, when $|x| < \frac{1}{4}$, so $I = (-\frac{1}{4}, \frac{1}{4})$.
9. $f(x) = \frac{x}{9+x^2} = \frac{x}{9}\left[\frac{1}{1+(x/3)^2}\right] = \frac{x}{9}\left[\frac{1}{1-\{-(x/3)^2\}}\right] = \frac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n$
 $= \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{9^n} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{9^{n+1}}$. The geometric series $\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n$ converges when $\left|-\left(\frac{x}{3}\right)^2\right| < 1 \Leftrightarrow \frac{|x^2|}{9} < 1 \Leftrightarrow |x|^2 < 9 \Leftrightarrow |x| < 3$, so $R = 3$ and $I = (-3, 3)$.
10. $f(x) = \frac{x^2}{a^3-x^3} = \frac{x^2}{a^3} \cdot \frac{1}{1-x^3/a^3} = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n = \sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}$. The series converges when $|x^3/a^3| < 1 \Leftrightarrow |x^3| < |a^3| \Leftrightarrow |x| < |a|$, so $R = |a|$ and $I = (-|a|, |a|)$.
11. $f(x) = \frac{3}{x^2+x-2} = \frac{3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \Rightarrow 3 = A(x-1) + B(x+2)$. Taking $x = -2$, we get $A = -1$. Taking $x = 1$, we get $B = 1$. Thus,

$$\begin{aligned}\frac{3}{x^2+x-2} &= \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x/2} = -\sum_{n=0}^{\infty}x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\left[-1 - \frac{1}{2}\left(-\frac{1}{2}\right)^n\right]x^n = \sum_{n=0}^{\infty}\left[-1 + \left(-\frac{1}{2}\right)^{n+1}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]x^n\end{aligned}$$

We represented the given function as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

$$12. f(x) = \frac{7x-1}{3x^2+2x-1} = \frac{7x-1}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1} = \frac{1}{3x-1} + \frac{2}{x+1} = 2 \cdot \frac{1}{1-(-x)} - \frac{1}{1-3x}$$

$$= 2 \sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} [2(-1)^n - 3^n] x^n$$

The series $\sum (-x)^n$ converges for $x \in (-1, 1)$ and the series $\sum (3x)^n$ converges for $x \in (-\frac{1}{3}, \frac{1}{3})$, so their sum converges for $x \in (-\frac{1}{3}, \frac{1}{3}) = I$.

$$13. (a) f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx} \left(\frac{-1}{1+x} \right) = -\frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] \quad [\text{from Exercise 3}]$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} \quad [\text{from Theorem 2(i)}] = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \quad \text{with } R = 1.$$

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

$$(b) f(x) = \frac{1}{(1+x)^3} = -\frac{1}{2} \frac{d}{dx} \left[\frac{1}{(1+x)^2} \right] = -\frac{1}{2} \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n (n+1) x^n \right] \quad [\text{from part (a)}]$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) n x^{n-1} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n \quad \text{with } R = 1.$$

$$(c) f(x) = \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n \quad [\text{from part (b)}]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^{n+2}.$$

To write the power series with x^n rather than x^{n+2} , we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n)(n-1) x^n$.

$$14. (a) \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n \quad [\text{geometric series with } R = 1], \text{ so}$$

$$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad [C = 0 \text{ since } f(0) = \ln 1 = 0], \text{ with } R = 1$$

$$(b) f(x) = x \ln(1+x) = x \left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \right] \quad [\text{by part (a)}] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1}$$

with $R = 1$.

$$(c) f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^2)^n}{n} \quad [\text{by part (a)}] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} \quad \text{with } R = 1.$$

$$15. f(x) = \ln(5-x) = -\int \frac{dx}{5-x} = -\frac{1}{5} \int \frac{dx}{1-x/5}$$

$$= -\frac{1}{5} \int \left[\sum_{n=0}^{\infty} \left(\frac{x}{5} \right)^n \right] dx = C - \frac{1}{5} \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \Leftrightarrow |x| < 5$, so $R = 5$.

16. We know that $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$. Differentiating, we get $\frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=0}^{\infty} 2^{n+1} (n+1) x^n$, so

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2} \sum_{n=0}^{\infty} 2^{n+1} (n+1) x^n = \sum_{n=0}^{\infty} 2^n (n+1) x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2} (n-1) x^n,$$

with $R = \frac{1}{2}$.

17. $\frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n$ for $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$. Now

$$\frac{1}{(x-2)^2} = \frac{d}{dx} \left(\frac{1}{2-x} \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \right) = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1} = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n. \text{ So}$$

$$f(x) = \frac{x^3}{(x-2)^2} = x^3 \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^{n+3} \text{ or } \sum_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n \text{ for } |x| < 2. \text{ Thus, } R = 2 \text{ and}$$

$$I = (-2, 2).$$

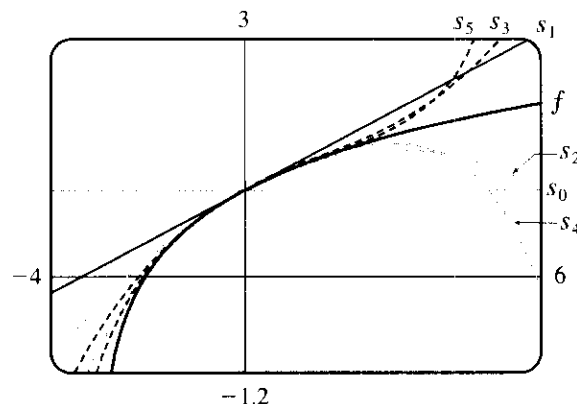
18. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1} \text{ for } \left|\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3,$$

so $R = 3$.

19. $f(x) = \ln(3+x) = \int \frac{dx}{3+x} = \frac{1}{3} \int \frac{dx}{1+x/3} = \frac{1}{3} \int \frac{dx}{1-(-x/3)} = \frac{1}{3} \int \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n dx$
 $= C + \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^n} x^{n+1} = \ln 3 + \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^{n-1}} x^n \quad [C = f(0) = \ln 3]$
 $= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n} x^n. \text{ The series converges when } |-x/3| < 1 \Leftrightarrow |x| < 3, \text{ so } R = 3.$

The terms of the series are $a_0 = \ln 3, a_1 = \frac{x}{3}, a_2 = -\frac{x^2}{18}, a_3 = \frac{x^3}{81}, a_4 = -\frac{x^4}{324}, a_5 = \frac{x^5}{1215}, \dots$

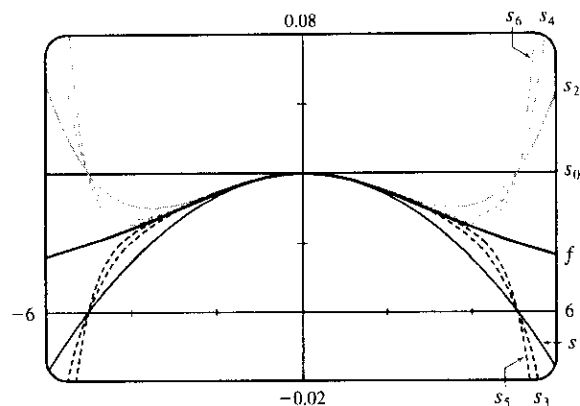


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3, 3)$.

$$20. f(x) = \frac{1}{x^2 + 25} = \frac{1}{25} \left(\frac{1}{1 + x^2/25} \right) = \frac{1}{25} \left(\frac{1}{1 - (-x^2/25)} \right) = \frac{1}{25} \sum_{n=0}^{\infty} \left(-\frac{x^2}{25} \right)^n = \frac{1}{25} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{5} \right)^{2n}.$$

The series converges when $|-x^2/25| < 1 \Leftrightarrow x^2 < 25 \Leftrightarrow |x| < 5$, so $R = 5$. The terms of the series are

$$a_0 = \frac{1}{25}, a_1 = -\frac{x^2}{625}, a_2 = \frac{x^4}{15,625}, \dots$$

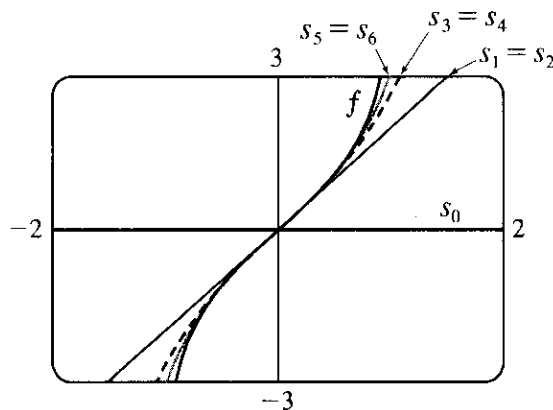


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5, 5)$.

$$\begin{aligned} 21. f(x) &= \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x} \\ &= \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n \right] dx \\ &= \int [(1-x+x^2-x^3+x^4-\dots) + (1+x+x^2+x^3+x^4+\dots)] dx \\ &= \int (2+2x^2+2x^4+\dots) dx = \int \sum_{n=0}^{\infty} 2x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} \end{aligned}$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then

$$f(x) = \pm 2 \sum_{n=0}^{\infty} \frac{1}{2n+1}, \text{ which both diverge by the Limit Comparison Test with } b_n = \frac{1}{n}.$$

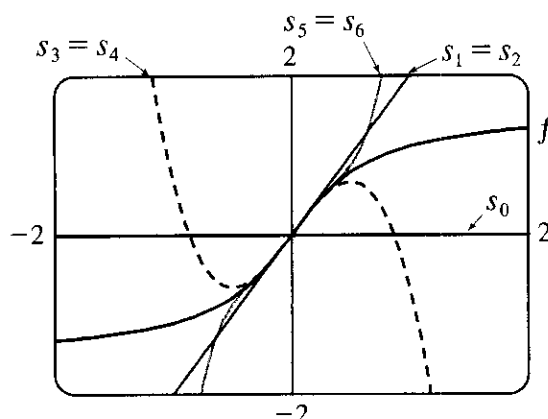


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

$$\begin{aligned}
 22. f(x) &= \tan^{-1}(2x) = 2 \int \frac{dx}{1+4x^2} = 2 \int \sum_{n=0}^{\infty} (-1)^n (4x^2)^n dx = 2 \int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} dx \\
 &= C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \quad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0].
 \end{aligned}$$

The series converges when $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm \frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and

$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $[-\frac{1}{2}, \frac{1}{2}]$.

$$23. \frac{t}{1-t^8} = t \cdot \frac{1}{1-t^8} = t \sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8} dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}. \text{ The series for } \frac{1}{1-t^8}$$

converges when $|t^8| < 1 \Leftrightarrow |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem

2, the series for $\int \frac{t}{1-t^8} dt$ also has $R = 1$.

$$24. \text{ By Example 6, } \ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \text{ for } |t| < 1, \text{ so } \frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \text{ and}$$

$$\int \frac{\ln(1-t)}{t} dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}. \text{ By Theorem 2, } R = 1.$$

$$25. \text{ By Example 7, } \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ with } R = 1, \text{ so}$$

$$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

$$\text{ and } \frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}, \text{ so}$$

$$\int \frac{x - \tan^{-1} x}{x^3} dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{4n^2-1}. \text{ By Theorem 2, } R = 1.$$

26. By Example 7, $\int \tan^{-1}(x^2) dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R = 1$.

$$27. \frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n} \Rightarrow$$

$$\int \frac{1}{1+x^5} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{5n} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+1}}{5n+1}. \text{ Thus,}$$

$$I = \int_0^{0.2} \frac{1}{1+x^5} dx = \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} - \dots \right]_0^{0.2} = 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \dots. \text{ The series is alternating, so}$$

if we use the first two terms, the error is at most $(0.2)^{11}/11 \approx 1.9 \times 10^{-9}$. So $I \approx 0.2 - (0.2)^6/6 \approx 0.199989$ to six decimal places.

28. From Example 6 we know $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, so

$$\ln(1+x^4) = \ln[1-(-x^4)] = -\sum_{n=1}^{\infty} \frac{(-x^4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n} \Rightarrow$$

$$\int \ln(1+x^4) dx = \int \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n} dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n+1}}{n(4n+1)}. \text{ Thus,}$$

$$I = \int_0^{0.4} \ln(1+x^4) dx = \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \dots \right]_0^{0.4} = \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \dots.$$

The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68 \approx 2.5 \times 10^{-9}$. So

$I \approx (0.4)^5/5 - (0.4)^9/18 + (0.4)^{13}/39 \approx 0.002034$ to six decimal places.

29. We substitute x^4 for x in Example 7, and find that

$$\begin{aligned} \int x^2 \tan^{-1}(x^4) dx &= \int x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{2n+1} dx \\ &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+6}}{2n+1} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+7}}{(2n+1)(8n+7)} \end{aligned}$$

$$\text{So } \int_0^{1/3} x^2 \tan^{-1}(x^4) dx = \left[\frac{x^7}{7} - \frac{x^{15}}{45} + \dots \right]_0^{1/3} = \frac{1}{7 \cdot 3^7} - \frac{1}{45 \cdot 3^{15}} + \dots. \text{ The series is alternating,}$$

so if we use only one term, the error is at most $1/(45 \cdot 3^{15}) \approx 1.5 \times 10^{-9}$. So

$$\int_0^{1/3} x^2 \tan^{-1}(x^4) dx \approx 1/(7 \cdot 3^7) \approx 0.000065 \text{ to six decimal places.}$$

$$\begin{aligned} 30. \int_0^{0.5} \frac{dx}{1+x^6} &= \int_0^{0.5} \sum_{n=0}^{\infty} (-1)^n x^{6n} dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{6n+1}}{6n+1} \right]_0^{0.5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)2^{6n+1}} \\ &= \frac{1}{2} - \frac{1}{7 \cdot 2^7} + \frac{1}{13 \cdot 2^{13}} - \frac{1}{19 \cdot 2^{19}} + \dots \end{aligned}$$

The series is alternating, so if we use only three terms, the error is at most $\frac{1}{19 \cdot 2^{19}} \approx 1.0 \times 10^{-7}$. So, to six

$$\text{decimal places, } \int_0^{0.5} \frac{dx}{1+x^6} \approx \frac{1}{2} - \frac{1}{7 \cdot 2^7} + \frac{1}{13 \cdot 2^{13}} \approx 0.498893.$$

31. Using the result of Example 6, $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, with $x = -0.1$, we have

$\ln 1.1 = \ln[1 - (-0.1)] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \dots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5} = 0.000002$. So

$$\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531.$$

32. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$ [the first term disappears], so

$$\begin{aligned} f''(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad [\text{substituting } n+1 \text{ for } n] \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \Rightarrow f''(x) + f(x) = 0. \end{aligned}$$

33. (a) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J'_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$, and $J''_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n-2}}{2^{2n}(n!)^2}$, so

$$\begin{aligned} x^2 J''_0(x) + x J'_0(x) + J_0(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2} [(n-1)!]^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^{-1} 2^2 n^2 x^{2n}}{2^{2n}(n!)^2} \\ &= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1) + 2n - 2^2 n^2}{2^{2n}(n!)^2} \right] x^{2n} = \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 - 2n + 2n - 4n^2}{2^{2n}(n!)^2} \right] x^{2n} = 0 \end{aligned}$$

$$\begin{aligned} \text{(b)} \int_0^1 J_0(x) dx &= \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \dots \right) dx \\ &= \left[x - \frac{x^3}{3 \cdot 4} + \frac{x^5}{5 \cdot 64} - \frac{x^7}{7 \cdot 2304} + \dots \right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16,128} + \dots \end{aligned}$$

Since $\frac{1}{16,128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three

decimal places, $\int_0^1 J_0(x) dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

$$34. (a) J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}}, \quad J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n!(n+1)! 2^{2n+1}}, \text{ and}$$

$$J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n-1}}{n!(n+1)! 2^{2n+1}}.$$

$$\begin{aligned} x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!(n+1)! 2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n+1}}{n!(n+1)! 2^{2n+1}} \\ &\quad + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(n+1)! 2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!(n+1)! 2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n+1}}{n!(n+1)! 2^{2n+1}} \\ &\quad - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(n-1)! n! 2^{2n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}} \quad \left[\begin{array}{l} \text{Replace } n \text{ with } n-1 \\ \text{in the third term} \end{array} \right] \\ &= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) - (n)(n+1)2^2 - 1}{n!(n+1)! 2^{2n+1}} \right] x^{2n+1} = 0 \end{aligned}$$

$$(b) J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} \Rightarrow$$

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n} (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(n+1) x^{2n+1}}{2^{2n+2} [(n+1)!]^2} \quad [\text{Replace } n \text{ with } n+1]$$

$$= - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} (n+1)! n!} \quad [\text{cancel 2 and } n+1; \text{ take } -1 \text{ outside sum}] = -J_1(x)$$

$$35. (a) f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

(b) By Theorem 10.4.2, the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.

Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.

$$36. \frac{|\sin nx|}{n^2} \leq \frac{1}{n^2}, \text{ so } \sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \text{ converges by the Comparison Test. } \frac{d}{dx} \left(\frac{\sin nx}{n^2} \right) = \frac{\cos nx}{n}, \text{ so when } x = 2k\pi$$

$$(k \text{ an integer}), \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos(2kn\pi)}{n} = \sum_{n=1}^{\infty} \frac{1}{n}, \text{ which diverges (harmonic series). } f_n''(x) = -\sin nx, \text{ so}$$

$$\sum_{n=1}^{\infty} f_n''(x) = - \sum_{n=1}^{\infty} \sin nx, \text{ which converges only if } \sin nx = 0, \text{ or } x = k\pi (k \text{ an integer}).$$

$$37. \text{ If } a_n = \frac{x^n}{n^2}, \text{ then by the Ratio Test, } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 = |x| < 1 \text{ for}$$

$$\text{convergence, so } R = 1. \text{ When } x = \pm 1, \sum_{n=1}^{\infty} \left| \frac{x^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which is a convergent } p\text{-series } (p = 2 > 1), \text{ so the}$$

interval of convergence for f is $[-1, 1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need

only check the endpoints. $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$, and this series diverges for $x = 1$ (harmonic series) and converges for $x = -1$ (Alternating Series Test), so the interval of convergence is $[-1, 1)$. $f''(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$, so its interval of convergence is $(-1, 1)$.

$$38. (a) \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \frac{d}{dx} \left[\sum_{n=0}^{\infty} x^n \right] = \frac{d}{dx} \left[\frac{1}{1-x} \right] = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}, |x| < 1.$$

$$(b) (i) \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x \left[\frac{1}{(1-x)^2} \right] \text{ [from part (a)]} = \frac{x}{(1-x)^2} \text{ for } |x| < 1.$$

$$(ii) \text{ Put } x = \frac{1}{2} \text{ in (i): } \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n \left(\frac{1}{2} \right)^n = \frac{1/2}{(1-1/2)^2} = 2.$$

$$(c) (i) \sum_{n=2}^{\infty} n(n-1)x^n = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2 \frac{d}{dx} \left[\sum_{n=1}^{\infty} nx^{n-1} \right] = x^2 \frac{d}{dx} \frac{1}{(1-x)^2} \\ = x^2 \frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3} \text{ for } |x| < 1.$$

$$(ii) \text{ Put } x = \frac{1}{2} \text{ in (i): } \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} = \sum_{n=2}^{\infty} n(n-1) \left(\frac{1}{2} \right)^n = \frac{2(1/2)^2}{(1-1/2)^3} = 4.$$

$$(iii) \text{ From (b)(ii) and (c)(ii), we have } \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2 - n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4 + 2 = 6.$$

39. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \frac{1}{\sqrt{3}}$, we have

$$\frac{\pi}{6} = \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\sqrt{3}} \right)^n \frac{1}{\sqrt{3}} \frac{1}{2n+1}, \text{ so}$$

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

$$40. (a) \int_0^{1/2} \frac{dx}{x^2 - x + 1} = \int_0^{1/2} \frac{dx}{(x - 1/2)^2 + 3/4} \left[\begin{array}{l} x - 1/2 = (\sqrt{3}/2)u, \quad u = (2/\sqrt{3})(x - 1/2) \\ dx = (\sqrt{3}/2) du \end{array} \right]$$

$$= \int_{-1/\sqrt{3}}^0 \frac{(\sqrt{3}/2) du}{(3/4)(u^2 + 1)} = \frac{2\sqrt{3}}{3} [\tan^{-1} u]_{-1/\sqrt{3}}^0 = \frac{2}{\sqrt{3}} \left[0 - \left(-\frac{\pi}{6} \right) \right] = \frac{\pi}{3\sqrt{3}}$$

$$(b) \frac{1}{x^3 + 1} = \frac{1}{(x+1)(x^2 - x + 1)} \Rightarrow \frac{1}{x^2 - x + 1} = (x+1) \left(\frac{1}{1+x^3} \right) = (x+1) \frac{1}{1 - (-x^3)}$$

$$= (x+1) \sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n x^{3n+1} + \sum_{n=0}^{\infty} (-1)^n x^{3n} \text{ for } |x| < 1 \Rightarrow \int \frac{dx}{x^2 - x + 1}$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} \text{ for } |x| < 1 \Rightarrow \int_0^{1/2} \frac{dx}{x^2 - x + 1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{4 \cdot 8^n(3n+2)} + \frac{1}{2 \cdot 8^n(3n+1)} \right] = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right)$$

$$\text{By part (a), this equals } \frac{\pi}{3\sqrt{3}}, \text{ so } \pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right).$$

12.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \frac{f^{(n)}(a)}{n!}$, so $b_8 = \frac{f^{(8)}(5)}{8!}$.
2. (a) Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \dots$. Comparing to the given series, $1.6 - 0.8(x-1) + \dots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

- (b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \dots$.

Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \dots$, we must have

$\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\vdots$	$\vdots$	$\vdots$

We use Equation 7 with $f(x) = \cos x$.

$$\begin{aligned}\cos x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\end{aligned}$$

If $a_n = \frac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x.$$

So $R = \infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2 \cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, so

$$\begin{aligned}\sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}\end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2^2 |x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$ (Ratio Test).

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
 (1+x)^{-3} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots \\
 &= 1 - 3x + \frac{4 \cdot 3}{2!}x^2 - \frac{5 \cdot 4 \cdot 3}{3!}x^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3}{4!}x^4 - \dots \\
 &= 1 - 3x + \frac{4 \cdot 3 \cdot 2}{2 \cdot 2!}x^2 - \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3!}x^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 4!}x^4 - \dots \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! x^n}{2(n!)} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x^n}{2}
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+3)(n+2)x^{n+1}}{2} \cdot \frac{2}{(n+2)(n+1)x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+3}{n+1} = |x| < 1 \text{ for convergence,}$$

so $R = 1$ (Ratio Test).

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
 \ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\
 &\quad + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\
 &= x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4 + \frac{24}{120}x^5 - \dots \\
 &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{1 + 1/n} = |x| < 1 \text{ for}$$

convergence, so $R = 1$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	$5^2 e^{5x}$	25
3	$5^3 e^{5x}$	125
4	$5^4 e^{5x}$	625
$\vdots$	$\vdots$	$\vdots$

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n.$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{5^{n+1} |x|^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n |x|^n} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{5|x|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.
 \end{aligned}$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
$\vdots$	$\vdots$	$\vdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}.$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n} \right] = \lim_{n \rightarrow \infty} \frac{|x|}{n} = 0 < 1 \text{ for}$$

all x , so $R = \infty$.

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \quad \text{so } \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Use the Ratio Test to find R . If $a_n = \frac{x^{2n+1}}{(2n+1)!}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= x^2 \cdot \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} = 0 < 1 \end{aligned}$$

for all x , so $R = \infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad \text{so } \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Use the Ratio Test to find R . If $a_n = \frac{x^{2n}}{(2n)!}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| \\ &= x^2 \cdot \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1 \end{aligned}$$

for all x , so $R = \infty$.

11.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1 + x + x^2$	7
1	$1 + 2x$	5
2	2	2
3	0	0
4	0	0
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) &= 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-2)^n \\ &= 7 + 5(x-2) + (x-2)^2 \end{aligned}$$

Since $a_n = 0$ for large n , $R = \infty$.

12.

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) = -1 + 3(x+1) - \frac{6}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$

$$= -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$$

Since $a_n = 0$ for large n , $R = \infty$.

13. Clearly, $f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$. If $a_n = \frac{e^3}{n!}(x-3)^n$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n} \right| = \lim_{n \rightarrow \infty} \frac{|x-3|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

14.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3 \cdot 2x^{-4}$	$-\frac{3 \cdot 2}{16}$
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n} \text{ for } n \geq 1, \text{ so } \ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n \cdot 2^n}.$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-2|}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{|x-2|}{2} < 1 \text{ for convergence, so } |x-2| < 2 \Rightarrow R = 2.$$

15.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$$\cos x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \dots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}.$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{|x-\pi|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x-\pi|^{2n}} \right] = \lim_{n \rightarrow \infty} \frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

16.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\sin x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!} \left(x - \frac{\pi}{2}\right)^k \\ &= 1 - \frac{(x - \pi/2)^2}{2!} + \frac{(x - \pi/2)^4}{4!} - \frac{(x - \pi/2)^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x - \pi/2)^{2n}}{(2n)!}\end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{|x - \pi/2|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x - \pi/2|^{2n}} \right] = \lim_{n \rightarrow \infty} \frac{|x - \pi/2|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2} \cdot \frac{1}{3^3}$
2	$\frac{3}{4}x^{-5/2}$	$-\frac{1}{2} \cdot \left(-\frac{3}{2}\right) \cdot \frac{1}{3^5}$
3	$-\frac{15}{8}x^{-7/2}$	$-\frac{1}{2} \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{5}{2}\right) \cdot \frac{1}{3^7}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\frac{1}{\sqrt{x}} &= \frac{1}{3} - \frac{1}{2 \cdot 3^3} (x-9) + \frac{3}{2^2 \cdot 3^5} \frac{(x-9)^2}{2!} - \frac{3 \cdot 5}{2^3 \cdot 3^7} \frac{(x-9)^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot 3^{2n+1} \cdot n!} (x-9)^n.\end{aligned}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)[2(n+1)-1] |x-9|^{n+1}}{2^{n+1} \cdot 3^{[2(n+1)+1]} \cdot (n+1)!} \cdot \frac{2^n \cdot 3^{2n+1} \cdot n!}{1 \cdot 3 \cdot 5 \cdots (2n-1) |x-9|^n} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{(2n+1) |x-9|}{2 \cdot 3^{2(n+1)}} \right] = \frac{1}{9} |x-9| < 1\end{aligned}$$

for convergence, so $|x-9| < 9$ and $R = 9$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}x^{-2} &= 1 - 2(x-1) + 6 \cdot \frac{(x-1)^2}{2!} - 24 \cdot \frac{(x-1)^3}{3!} + 120 \cdot \frac{(x-1)^4}{4!} - \cdots \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n.\end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+2) |x-1|^{n+1}}{(n+1) |x-1|^n} = \lim_{n \rightarrow \infty} \left[\frac{n+2}{n+1} \cdot |x-1| \right] = |x-1| < 1 \text{ for convergence, so } R = 1.$$

19. If $f(x) = \cos x$, then $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$. In each case, $|f^{(n+1)}(x)| \leq 1$, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1}$. Thus, $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$ by Equation 10. So $\lim_{n \rightarrow \infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 3 represents $\cos x$ for all x .
20. If $f(x) = \sin x$, then $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$. In each case, $|f^{(n+1)}(x)| \leq 1$, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \leq \frac{1}{(n+1)!} \left|x - \frac{\pi}{2}\right|^{n+1}$. Thus, $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$ by Equation 10. So $\lim_{n \rightarrow \infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 16 represents $\sin x$ for all x .
21. If $f(x) = \sinh x$, then for all n , $f^{(n+1)}(x) = \cosh x$ or $\sinh x$. Since $|\sinh x| < |\cosh x| = \cosh x$ for all x , we have $|f^{(n+1)}(x)| \leq \cosh x$ for all n . If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| \leq \cosh x \leq \cosh d$, so by Formula 9 with $a = 0$ and $M = \cosh d$, we have $|R_n(x)| \leq \frac{\cosh d}{(n+1)!} |x|^{n+1}$. It follows that $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$ for $|x| \leq d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\sinh x$ for all x .
22. If $f(x) = \cosh x$, then for all n , $f^{(n+1)}(x) = \cosh x$ or $\sinh x$. Since $|\sinh x| < |\cosh x| = \cosh x$ for all x , we have $|f^{(n+1)}(x)| \leq \cosh x$ for all n . If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| \leq \cosh x \leq \cosh d$, so by Formula 9 with $a = 0$ and $M = \cosh d$, we have $|R_n(x)| \leq \frac{\cosh d}{(n+1)!} |x|^{n+1}$. It follows that $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$ for $|x| \leq d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\cosh x$ for all x .
23. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}, R = \infty$
24. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f(x) = e^{-x/2} = \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n, R = \infty$
25. $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \Rightarrow f(x) = x \tan^{-1} x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1}, R = 1$
26. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \Rightarrow f(x) = \sin(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{8n+4}, R = \infty$
27. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f(x) = x^2 e^{-x} = x^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n!}, R = \infty$
28. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n} \Rightarrow$
 $f(x) = x \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n+1}, R = \infty$
29. $\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} \left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right] = \frac{1}{2} \left[1 - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right]$
 $= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}, R = \infty$

$$30. \cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2} \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right] = \frac{1}{2} \left[1 + 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} \right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}, R = \infty$$

Another method: Use $\cos^2 x = 1 - \sin^2 x$ and Exercise 29.

$$31. \frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \text{ and this series also gives the required value at } x = 0 \text{ (namely 1);}$$

$$R = \infty.$$

$$32. \frac{x - \sin x}{x^3} = \frac{1}{x^3} \left[x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right] = \frac{1}{x^3} \left[x - x - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right] = \frac{1}{x^3} \left[- \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \right]$$

$$= \frac{1}{x^3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+3)!} \text{ and this series also gives the required value at } x = 0 \text{ (namely } 1/6);$$

$$R = \infty.$$

33.

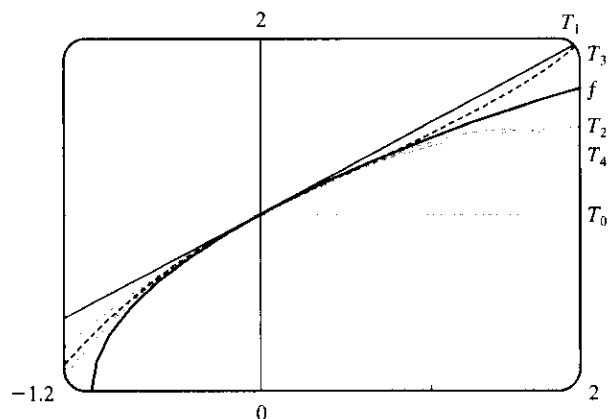
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$
$\vdots$	$\vdots$	$\vdots$

So $f^{(n)}(0) = \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}$ for $n \geq 2$, and

$$\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^n. \text{ If } a_n = \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^n,$$

$$\text{then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)x^{n+1}}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-3)x^n} \right|$$

$$= \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{2n-1}{n+1} = \frac{|x|}{2} \cdot 2 = |x| < 1 \text{ for convergence, so } R = 1.$$

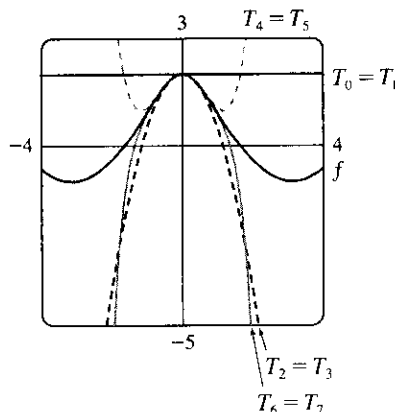


Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.

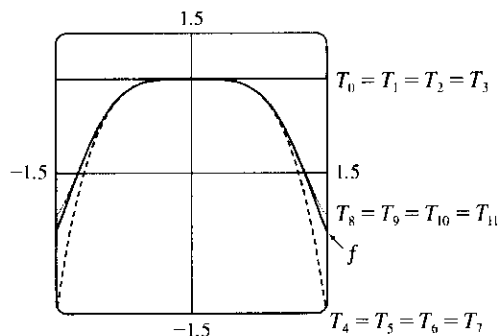
34. $e^x \stackrel{(11)}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$. Also, $\cos x \stackrel{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, so

$$f(x) = e^{-x^2} + \cos x = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n!} + \frac{1}{(2n)!} \right) x^{2n} = 2 - \frac{3}{2}x^2 + \frac{13}{24}x^4 - \frac{121}{720}x^6 + \dots$$

The series for e^x and $\cos x$ converge for all x , so the same is true of the series for $f(x)$; that is, $R = \infty$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.



35. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$, $R = \infty$

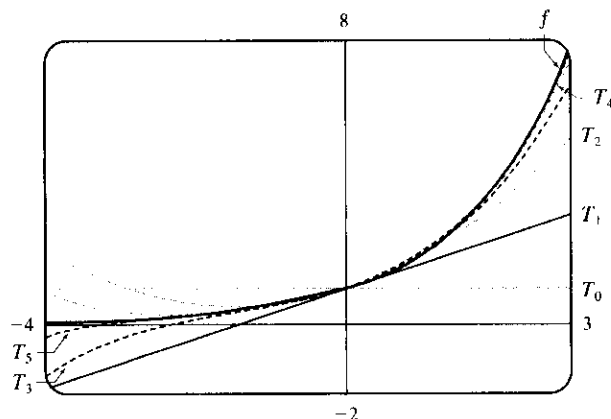


Notice that, as n increases, $T_n(x)$

becomes a better approximation to $f(x)$.

36. $2^x = (e^{\ln 2})^x$
 $= e^{x \ln 2}$
 $= \sum_{n=0}^{\infty} \frac{(x \ln 2)^n}{n!}$
 $= \sum_{n=0}^{\infty} \frac{(\ln 2)^n x^n}{n!}, \quad R = \infty.$

Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.



$$37. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ so}$$

$$e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{1}{2!}(0.2)^2 - \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 - \frac{1}{5!}(0.2)^5 + \frac{1}{6!}(0.2)^6 - \dots. \text{ But}$$

$$\frac{1}{6!}(0.2)^6 = 8.8 \times 10^{-8}, \text{ so by the Alternating Series Estimation Theorem, } e^{-0.2} \approx \sum_{n=0}^5 \frac{(-0.2)^n}{n!} \approx 0.81873,$$

correct to five decimal places.

$$38. 3^\circ = \frac{\pi}{60} \text{ radians and } \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \text{ so}$$

$$\sin \frac{\pi}{60} = \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \dots = \frac{\pi}{60} - \frac{\pi^3}{1,296,000} + \frac{\pi^5}{93,312,000,000} - \dots$$

$$\text{But } \frac{\pi^5}{93,312,000,000} < 10^{-8}, \text{ so by the Alternating Series Estimation Theorem,}$$

$$\sin \frac{\pi}{60} \approx \frac{\pi}{60} - \frac{\pi^3}{1,296,000} \approx 0.05234.$$

$$39. \cos x \stackrel{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \Rightarrow$$

$$x \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \Rightarrow \int x \cos(x^3) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}, \text{ with } R = \infty.$$

$$40. \frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, \text{ so}$$

$$\int \frac{\sin x}{x} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

41. Using the series from Exercise 33 and substituting x^3 for x , we get

$$\begin{aligned} \int \sqrt{x^3+1} dx &= \int \left[1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{3n} \right] dx \\ &= C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n! (3n+1)} x^{3n+1} \end{aligned}$$

$$42. e^x \stackrel{(11)}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \Rightarrow$$

$$\int \frac{e^x - 1}{x} dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}, \text{ with } R = \infty.$$

43. By Exercise 39, $\int x \cos(x^3) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, so $\int_0^1 x \cos(x^3) dx$
- $$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+2)(2n)!} = \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} - \frac{1}{20 \cdot 6!} + \cdots, \text{ but}$$
- $$\frac{1}{20 \cdot 6!} = \frac{1}{14,400} \approx 0.000069, \text{ so } \int_0^1 x \cos(x^3) dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440 \text{ (correct to three decimal places)}$$
- by the Alternating Series Estimation Theorem.

44. From the table of Maclaurin series in Section 12.10, we see that

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ for } x \text{ in } [-1, 1] \text{ and } \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all real numbers } x, \text{ so}$$

$$\tan^{-1}(x^3) + \sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!} \text{ for } x^3 \text{ in } [-1, 1] \Leftrightarrow x \text{ in } [-1, 1]. \text{ Thus,}$$

$$I = \int_0^{0.2} [\tan^{-1}(x^3) + \sin(x^3)] dx = \int_0^{0.2} \sum_{n=0}^{\infty} (-1)^n x^{6n+3} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) dx$$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{6n+4} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) \right]_0^{0.2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(0.2)^{6n+4}}{6n+4} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) = \frac{(0.2)^4}{4} (1+1) - \frac{(0.2)^{10}}{10} \left(\frac{1}{3} + \frac{1}{3!} \right) + \cdots$$

But $\frac{(0.2)^{10}}{10} \left(\frac{1}{3} + \frac{1}{3!} \right) = \frac{(0.2)^{10}}{20} = 5.12 \times 10^{-9}$, so by the Alternating Series Estimation Theorem,

$$I \approx \frac{(0.2)^4}{2} = 0.00080 \text{ (correct to five decimal places). [Actually, the value is } 0.0008000, \text{ correct to seven decimal places.]}$$

45. We first find a series representation for $f(x) = (1+x)^{-1/2}$, and then substitute.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
$\vdots$	$\vdots$	$\vdots$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{4} \left(\frac{x^2}{2!} \right) - \frac{15}{8} \left(\frac{x^3}{3!} \right) + \cdots \Rightarrow \frac{1}{\sqrt{1+x^3}} = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \cdots \Rightarrow$$

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} = \left[x - \frac{1}{8}x^4 + \frac{3}{56}x^7 - \frac{1}{32}x^{10} + \cdots \right]_0^{0.1} \approx (0.1) - \frac{1}{8}(0.1)^4, \text{ by the Alternating Series}$$

Estimation Theorem, since $\frac{3}{56}(0.1)^7 \approx 0.000000054 < 10^{-8}$, which is the maximum desired error. Therefore,

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} \approx 0.09998750.$$

$$46. \int_0^{0.5} x^2 e^{-x^2} dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)} \right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}} \text{ and since the}$$

term with $n = 2$ is $\frac{1}{1792} < 0.001$, we use $\sum_{n=0}^1 \frac{(-1)^n}{n!(2n+3)2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354$.

$$47. \lim_{x \rightarrow 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x \rightarrow 0} \frac{x - (x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots)}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \cdots \right) = \frac{1}{3}$$

since power series are continuous functions.

$$48. \lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \rightarrow 0} \frac{1 - (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots)}{1 + x - (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots)}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots}{-\frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 - \cdots}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^4 - \cdots}{-\frac{1}{2!} - \frac{1}{3!}x - \frac{1}{4!}x^2 - \frac{1}{5!}x^3 - \frac{1}{6!}x^4 - \cdots} = \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1$$

since power series are continuous functions.

$$49. \lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x \rightarrow 0} \frac{(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots) - x + \frac{1}{6}x^3}{x^5}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x \rightarrow 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots \right) = \frac{1}{5!} = \frac{1}{120}$$

since power series are continuous functions.

$$50. \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots) - x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \lim_{x \rightarrow 0} \left(\frac{1}{3} + \frac{2}{15}x^2 + \cdots \right) = \frac{1}{3}$$

since power series are continuous functions.

$$51. \text{ As in Example 8(a), we have } e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \text{ and we know that } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

from Equation 16. Therefore, $e^{-x^2} \cos x = (1 - x^2 + \frac{1}{2}x^4 - \cdots)(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots)$. Writing only the

terms with degree ≤ 4 , we get $e^{-x^2} \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^4 + \cdots = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$.

52.

$$\begin{array}{r}
 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots \\
 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \overline{) 1} \\
 \underline{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots} \\
 \frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots \\
 \underline{\frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots} \\
 \frac{5}{24}x^4 + \cdots \\
 \underline{\frac{5}{24}x^4 + \cdots} \\
 \cdots
 \end{array}$$

$\sec x = \frac{1}{\cos x} \stackrel{(16)}{=} \frac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots}$. From the long division above, $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots$.

53.

$$\begin{array}{r}
 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots \\
 x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \overline{) x} \\
 \underline{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots} \\
 \frac{1}{6}x^3 - \frac{1}{120}x^5 + \cdots \\
 \underline{\frac{1}{6}x^3 - \frac{1}{36}x^5 + \cdots} \\
 \frac{7}{360}x^5 + \cdots \\
 \underline{\frac{7}{360}x^5 + \cdots} \\
 \cdots
 \end{array}$$

$\frac{x}{\sin x} \stackrel{(15)}{=} \frac{x}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}$. From the long division above, $\frac{x}{\sin x} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots$.

54. From Example 6 in Section 12.9, we have $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$, $|x| < 1$. Therefore,

$$\begin{aligned}
 e^x \ln(1-x) &= \left(1 + x + \frac{1}{2}x^2 + \cdots\right) \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots\right) \\
 &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - x^2 - \frac{1}{2}x^3 - \frac{1}{2}x^3 - \cdots \\
 &= -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - \cdots, \quad |x| < 1
 \end{aligned}$$

$$55. \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}, \text{ by (11).}$$

$$56. \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \text{ by (16).}$$

$$57. \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \text{ by (15).}$$

$$58. \sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}, \text{ by (11).}$$

$$59. 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots = \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{3^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} - 1 = e^3 - 1, \text{ by (11).}$$

$$60. 1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!} = e^{-\ln 2} = (e^{\ln 2})^{-1} = 2^{-1} = \frac{1}{2}, \text{ by (11).}$$

$$61. \text{ Assume that } |f'''(x)| \leq M, \text{ so } f'''(x) \leq M \text{ for } a \leq x \leq a+d. \text{ Now } \int_a^x f'''(t) dt \leq \int_a^x M dt \Rightarrow$$

$$f''(x) - f''(a) \leq M(x-a) \Rightarrow f''(x) \leq f''(a) + M(x-a). \text{ Thus, } \int_a^x f''(t) dt \leq \int_a^x [f''(a) + M(t-a)] dt$$

$$\Rightarrow f'(x) - f'(a) \leq f''(a)(x-a) + \frac{1}{2}M(x-a)^2 \Rightarrow f'(x) \leq f'(a) + f''(a)(x-a) + \frac{1}{2}M(x-a)^2 \Rightarrow$$

$$\int_a^x f'(t) dt \leq \int_a^x [f'(a) + f''(a)(t-a) + \frac{1}{2}M(t-a)^2] dt \Rightarrow$$

$$f(x) - f(a) \leq f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}M(x-a)^3. \text{ So}$$

$$f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2 \leq \frac{1}{6}M(x-a)^3. \text{ But}$$

$$R_2(x) = f(x) - T_2(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2, \text{ so } R_2(x) \leq \frac{1}{6}M(x-a)^3.$$

$$\text{A similar argument using } f'''(x) \geq -M \text{ shows that } R_2(x) \geq -\frac{1}{6}M(x-a)^3. \text{ So } |R_2(x)| \leq \frac{1}{6}M|x-a|^3.$$

Although we have assumed that $x > a$, a similar calculation shows that this inequality is also true if $x < a$.

$$62. (a) f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \text{ so}$$

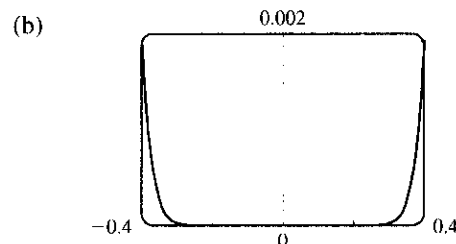
$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x} = \lim_{x \rightarrow 0} \frac{1/x}{e^{1/x^2}} = \lim_{x \rightarrow 0} \frac{x}{2e^{1/x^2}} = 0 \text{ (using l'Hospital's Rule and}$$

simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule

to show that $f''(0) = 0$, $f^{(3)}(0) = 0$, $\dots$, $f^{(n)}(0) = 0$, so that the Maclaurin series for f consists entirely of

zero terms. But since $f(x) \neq 0$ except for $x = 0$, we see that f cannot equal its Maclaurin series except

at $x = 0$.



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be “infinitely flat” at $x = 0$, since all of its derivatives are 0 there.

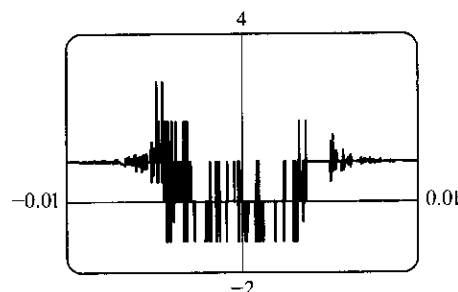
LABORATORY PROJECT An Elusive Limit

$$1. f(x) = \frac{n(x)}{d(x)} = \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$$

x	$f(x)$
1	1.1838
0.1	0.9821
0.01	2.0000
0.001	3.3333
0.0001	3.3333

The table of function values were obtained using Maple with 10 digits of precision. The results of this project will vary depending on the CAS and precision level. It appears that as $x \rightarrow 0^+$, $f(x) \rightarrow \frac{10}{3}$. Since f is an even function, we have $f(x) \rightarrow \frac{10}{3}$ as $x \rightarrow 0$.

2. The graph is inconclusive about the limit of f as $x \rightarrow 0$.



3. The limit has the indeterminate form $\frac{0}{0}$. Applying l'Hospital's Rule, we obtain the form $\frac{0}{0}$ six times. Finally, on the seventh application we obtain $\lim_{x \rightarrow 0} \frac{n^{(7)}(x)}{d^{(7)}(x)} = \frac{-168}{-168} = 1$.

$$\begin{aligned}
 4. \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{n(x)}{d(x)} \stackrel{\text{CAS}}{=} \lim_{x \rightarrow 0} \frac{-\frac{1}{30}x^7 - \frac{29}{756}x^9 + \dots}{-\frac{1}{30}x^7 + \frac{13}{756}x^9 + \dots} \\
 &= \lim_{x \rightarrow 0} \frac{(-\frac{1}{30}x^7 - \frac{29}{756}x^9 + \dots)/x^7}{(-\frac{1}{30}x^7 + \frac{13}{756}x^9 + \dots)/x^7} \\
 &= \lim_{x \rightarrow 0} \frac{-\frac{1}{30} - \frac{29}{756}x^2 + \dots}{-\frac{1}{30} + \frac{13}{756}x^2 + \dots} = \frac{-\frac{1}{30}}{-\frac{1}{30}} = 1
 \end{aligned}$$

Note that $n^{(7)}(x) = d^{(7)}(x) = -\frac{7!}{30} = -\frac{5040}{30} = -168$, which agrees with the result in Problem 3.

5. The limit command gives the result that $\lim_{x \rightarrow 0} f(x) = 1$.
6. The strange results (with only 10 digits of precision) must be due to the fact that the terms being subtracted in the numerator and denominator are very close in value when $|x|$ is small. Thus, the differences are imprecise (have few correct digits).

12.11 The Binomial Series

1. The general binomial series in (2) is

$$\begin{aligned}
 (1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots \\
 (1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} x^3 + \cdots \\
 &= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{2^4 \cdot 4!} + \cdots \\
 &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) x^n}{2^n \cdot n!} \text{ for } |x| < 1, \text{ so } R = 1
 \end{aligned}$$

2. $\frac{1}{(1+x)^4} = (1+x)^{-4} = \sum_{n=0}^{\infty} \binom{-4}{n} x^n$. The binomial coefficient is

$$\begin{aligned}
 \binom{-4}{n} &= \frac{(-4)(-5)(-6) \cdots (-4-n+1)}{n!} = \frac{(-4)(-5)(-6) \cdots [-(n+3)]}{n!} \\
 &= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n+1)(n+2)(n+3)}{2 \cdot 3 \cdot n!} = \frac{(-1)^n (n+1)(n+2)(n+3)}{6}
 \end{aligned}$$

$$\text{Thus, } \frac{1}{(1+x)^4} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n \text{ for } |x| < 1, \text{ so } R = 1.$$

3. $\frac{1}{(2+x)^3} = \frac{1}{[2(1+x/2)]^3} = \frac{1}{8} \left(1 + \frac{x}{2}\right)^{-3} = \frac{1}{8} \sum_{n=0}^{\infty} \binom{-3}{n} \left(\frac{x}{2}\right)^n$. The binomial coefficient is

$$\begin{aligned}
 \binom{-3}{n} &= \frac{(-3)(-4)(-5) \cdots (-3-n+1)}{n!} = \frac{(-3)(-4)(-5) \cdots [-(n+2)]}{n!} \\
 &= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n+1)(n+2)}{2 \cdot n!} = \frac{(-1)^n (n+1)(n+2)}{2}
 \end{aligned}$$

$$\text{Thus, } \frac{1}{(2+x)^3} = \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2) x^n}{2^{n+4}} \text{ for } \left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2, \text{ so } R = 2.$$

$$\begin{aligned}
 4. (1-x)^{2/3} &= \sum_{n=0}^{\infty} \binom{2/3}{n} (-x)^n \\
 &= 1 + \frac{2}{3}(-x) + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!} (-x)^2 + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!} (-x)^3 + \cdots \\
 &= 1 - \frac{2}{3}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (-1)^n \cdot 2 \cdot [1 \cdot 4 \cdot 7 \cdots (3n-5)]}{3^n \cdot n!} x^n \\
 &= 1 - \frac{2}{3}x - 2 \sum_{n=2}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-5)}{3^n \cdot n!} x^n
 \end{aligned}$$

$$\text{and } |-x| < 1 \Leftrightarrow |x| < 1, \text{ so } R = 1.$$

$$\begin{aligned}
 5. \sqrt[4]{1-8x} &= (1-8x)^{1/4} = \sum_{n=0}^{\infty} \binom{1/4}{n} (-8x)^n \\
 &= 1 + \frac{1}{4}(-8x) + \frac{\frac{1}{4}(-\frac{3}{4})}{2!}(-8x)^2 + \frac{\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})}{3!}(-8x)^3 + \cdots \\
 &= 1 - 2x + \sum_{n=2}^{\infty} \frac{(-1)^n (-1)^{n-1} \cdot 3 \cdot 7 \cdots (4n-5) 8^n}{4^n \cdot n!} x^n \\
 &= 1 - 2x - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdots (4n-5) 2^n}{n!} x^n
 \end{aligned}$$

and $|-8x| < 1 \Leftrightarrow |x| < \frac{1}{8}$, so $R = \frac{1}{8}$.

$$\begin{aligned}
 6. \frac{1}{\sqrt[5]{32-x}} &= \frac{1}{2\sqrt[5]{1-x/32}} = \frac{1}{2} \left(1 - \frac{x}{32}\right)^{-1/5} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/5}{n} \left(-\frac{x}{32}\right)^n = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/5}{n} \frac{(-1)^n x^n}{2^{5n}} \\
 &= \frac{1}{2} \left[1 + \binom{-1/5}{1} \left(-\frac{x}{2^5}\right) + \frac{\binom{-1/5}{2} (-\frac{6}{5}) x^2}{2! 2^{10}} + \frac{\binom{-1/5}{3} (-\frac{6}{5})(-\frac{11}{5})}{3!} \left(-\frac{x^3}{2^{15}}\right) + \cdots \right] \\
 &= \frac{1}{2} + \frac{1}{5 \cdot 2^6} x + \frac{1 \cdot 6}{5^2 \cdot 2! \cdot 2^{11}} x^2 + \frac{1 \cdot 6 \cdot 11}{5^3 \cdot 3! \cdot 2^{16}} x^3 + \cdots = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 6 \cdots (5n-4)}{5^n 2^{5n+1} n!} x^n
 \end{aligned}$$

The radius of convergence is 32.

7. We must write the binomial in the form $(1 + \text{expression})$, so we'll factor out a 4.

$$\begin{aligned}
 \frac{x}{\sqrt{4+x^2}} &= \frac{x}{\sqrt{4(1+x^2/4)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2} \left(1 + \frac{x^2}{4}\right)^{-1/2} = \frac{x}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x^2}{4}\right)^n \\
 &= \frac{x}{2} \left[1 + \binom{-1/2}{1} \frac{x^2}{4} + \frac{\binom{-1/2}{2} (-\frac{3}{2})}{2!} \left(\frac{x^2}{4}\right)^2 + \frac{\binom{-1/2}{3} (-\frac{3}{2})(-\frac{5}{2})}{3!} \left(\frac{x^2}{4}\right)^3 + \cdots \right] \\
 &= \frac{x}{2} + \frac{x}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot 4^n \cdot n!} x^{2n} \\
 &= \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{3n+1}} x^{2n+1} \text{ and } \frac{x^2}{4} < 1 \Leftrightarrow \frac{|x|}{2} < 1 \Leftrightarrow
 \end{aligned}$$

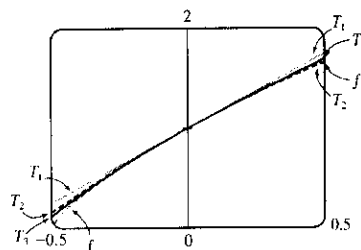
$|x| < 2$, so $R = 2$.

$$\begin{aligned}
 8. \frac{x^2}{\sqrt{2+x}} &= \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}} \left(1 + \frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x}{2}\right)^n \\
 &= \frac{x^2}{\sqrt{2}} \left[1 + \binom{-1/2}{1} \left(\frac{x}{2}\right) + \frac{\binom{-1/2}{2} (-\frac{3}{2})}{2!} \left(\frac{x}{2}\right)^2 + \frac{\binom{-1/2}{3} (-\frac{3}{2})(-\frac{5}{2})}{3!} \left(\frac{x}{2}\right)^3 + \cdots \right] \\
 &= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}} \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{2n}} x^n \\
 &= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{2n+1/2}} x^{n+2} \text{ and } \left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2, \text{ so } R = 2.
 \end{aligned}$$

$$\begin{aligned}
 9. (1+2x)^{3/4} &= 1 + \frac{3}{4}(2x) + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2!}(2x)^2 + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(2x)^3 + \cdots \\
 &= 1 + \frac{3}{2}x + 3 \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1 \cdot 5 \cdot 9 \cdots (4n-7)}{4^n \cdot n!} \cdot 2^n x^n \\
 &= 1 + \frac{3}{2}x + 3 \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1 \cdot 5 \cdot 9 \cdots (4n-7)}{2^n \cdot n!} x^n \text{ and } |2x| < 1 \Leftrightarrow |x| < \frac{1}{2}, \text{ so } R = \frac{1}{2}.
 \end{aligned}$$

The three Taylor polynomials are $T_1(x) = 1 + \frac{3}{2}x$, $T_2(x) = 1 + \frac{3}{2}x - \frac{3}{8}x^2$, and

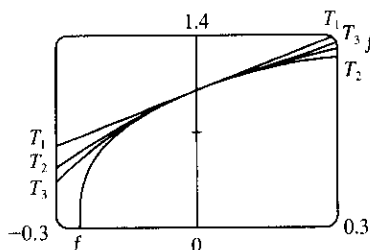
$$T_3(x) = 1 + \frac{3}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3.$$



$$\begin{aligned}
 10. \sqrt[3]{1+4x} &= (1+4x)^{1/3} \\
 &= 1 + \frac{1}{3}(4x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(4x)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(4x)^3 + \cdots \\
 &= 1 + \frac{4}{3}x + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{2 \cdot 5 \cdot 8 \cdots (3n-4)}{3^n \cdot n!} (4x)^n \text{ and } |4x| < 1 \Leftrightarrow |x| < \frac{1}{4}, \text{ so } R = \frac{1}{4}.
 \end{aligned}$$

The three Taylor polynomials are $T_1(x) = 1 + \frac{4}{3}x$, $T_2(x) = 1 + \frac{4}{3}x - \frac{16}{9}x^2$, and

$$T_3(x) = 1 + \frac{4}{3}x - \frac{16}{9}x^2 + \frac{320}{81}x^3.$$



$$\begin{aligned}
 11. (a) \frac{1}{\sqrt{1-x^2}} &= [1 + (-x^2)]^{-1/2} \\
 &= 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots \\
 &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} x^{2n}
 \end{aligned}$$

$$\begin{aligned}
 (b) \sin^{-1} x &= \int \frac{1}{\sqrt{1-x^2}} dx = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1} \\
 &= x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1} \quad \text{since } 0 = \sin^{-1} 0 = C.
 \end{aligned}$$

$$12. (a) (1+x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) x^{2n}}{2^n \cdot n!}$$

$$(b) \sinh^{-1} x = \int \frac{dx}{\sqrt{1+x^2}} = C + x + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) x^{2n+1}}{2^n \cdot n! (2n+1)}, \text{ but } C = 0 \text{ since}$$

$$\sinh^{-1} 0 = 0, \text{ so } \sinh^{-1} x = x + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) x^{2n+1}}{2^n \cdot n! (2n+1)}, R = 1.$$

$$13. (a) \sqrt[3]{1+x} = (1+x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^n$$

$$= 1 + \frac{1}{3}x + \frac{(\frac{1}{3})(-\frac{2}{3})}{2!}x^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{3} + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{2 \cdot 5 \cdot 8 \cdots (3n-4)}{3^n \cdot n!} x^n$$

(b) $\sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \cdots$. $\sqrt[3]{1.01} = \sqrt[3]{1+0.01}$, so let $x = 0.01$. The sum of the first two terms is then $1 + \frac{1}{3}(0.01) \approx 1.0033$. The third term is $\frac{1}{9}(0.01)^2 \approx 0.00001$, which does not affect the fourth decimal place of the sum, so we have $\sqrt[3]{1.01} \approx 1.0033$.

$$14. (a) 1/\sqrt[4]{1+x} = (1+x)^{-1/4} = \sum_{n=0}^{\infty} \binom{-1/4}{n} x^n$$

$$= 1 - \frac{1}{4}x + \frac{(-\frac{1}{4})(-\frac{5}{4})}{2!}x^2 + \frac{(-\frac{1}{4})(-\frac{5}{4})(-\frac{9}{4})}{3!}x^3 + \cdots$$

$$= 1 - \frac{1}{4}x + \sum_{n=2}^{\infty} (-1)^n \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{4^n \cdot n!} x^n$$

(b) $1/\sqrt[4]{1+x} = 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 - \cdots$. $1/\sqrt[4]{1.1} = 1/\sqrt[4]{1+0.1}$, so let $x = 0.1$. The sum of the first four terms is then $1 - \frac{1}{4}(0.1) + \frac{5}{32}(0.1)^2 - \frac{15}{128}(0.1)^3 \approx 0.976$. The fifth term is $\frac{195}{2048}(0.1)^4 \approx 0.0000095$, which does not affect the third decimal place of the sum, so we have $1/\sqrt[4]{1.1} \approx 0.976$. (Note that the third decimal place of the sum of the first three terms is affected by the fourth term, so we need to use more than three terms for the sum.)

$$15. (a) [1+(-x)]^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + \cdots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n,$$

$$\text{so } \frac{x}{(1-x)^2} = x \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} (n+1)x^{n+1} = \sum_{n=1}^{\infty} nx^n.$$

$$(b) \text{ With } x = \frac{1}{2} \text{ in part (a), we have } \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2.$$

$$\begin{aligned}
 16. (a) [1 + (-x)]^{-3} &= \sum_{n=0}^{\infty} \binom{-3}{n} (-x)^n \\
 &= 1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^3 + \dots \\
 &= 1 + \sum_{n=1}^{\infty} \frac{3 \cdot 4 \cdot 5 \cdots (n+2)}{n!} x^n = \sum_{n=0}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdots (n+2)}{2 \cdot n!} x^n \\
 &= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n \Rightarrow
 \end{aligned}$$

$$\begin{aligned}
 (x + x^2)[1 + (-x)]^{-3} &= x[1 + (-x)]^{-3} + x^2[1 + (-x)]^{-3} \\
 &= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^{n+1} + \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^{n+2} \\
 &= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n + \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^{n+1} \\
 &= x + \sum_{n=2}^{\infty} \frac{n(n+1)}{2} x^n + \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^n = x + \sum_{n=2}^{\infty} \left[\frac{n(n+1)}{2} + \frac{(n-1)n}{2} \right] x^n \\
 &= x + \sum_{n=2}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} n^2 x^n, -1 < x < 1
 \end{aligned}$$

(b) Setting $x = \frac{1}{2}$ in the last series above gives the required series, so $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2} + (\frac{1}{2})^2}{(1 - \frac{1}{2})^3} = \frac{\frac{3}{4}}{\frac{1}{8}} = 6$.

$$\begin{aligned}
 17. (a) (1 + x^2)^{1/2} &= 1 + \left(\frac{1}{2}\right)x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^2)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^2)^3 + \dots \\
 &= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n \cdot n!} x^{2n}
 \end{aligned}$$

(b) The coefficient of x^{10} (corresponding to $n = 5$) in the above Maclaurin series is $\frac{f^{(10)}(0)}{10!}$, so

$$\frac{f^{(10)}(0)}{10!} = \frac{(-1)^4 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2^5 \cdot 5!} \Rightarrow f^{(10)}(0) = 10! \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 \cdot 5!} \right) = 99,225.$$

$$\begin{aligned}
 18. (a) (1 + x^3)^{-1/2} &= \sum_{n=0}^{\infty} \binom{-1/2}{n} (x^3)^n \\
 &= 1 + \left(-\frac{1}{2}\right)(x^3) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(x^3)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(x^3)^3 + \dots \\
 &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} x^{3n}
 \end{aligned}$$

(b) The coefficient of x^9 (corresponding to $n = 3$) in the preceding series is

$$\frac{f^{(9)}(0)}{9!}, \text{ so } \frac{f^{(9)}(0)}{9!} = \frac{(-1)^3 1 \cdot 3 \cdot 5}{2^3 \cdot 3!} \Rightarrow f^{(9)}(0) = -\frac{9! \cdot 5}{8 \cdot 2} = -113,400.$$

19. (a) $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n \Rightarrow g'(x) = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1}$, so

$$\begin{aligned} (1+x)g'(x) &= (1+x) \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} + \sum_{n=1}^{\infty} \binom{k}{n} n x^n \\ &= \sum_{n=0}^{\infty} \binom{k}{n+1} (n+1) x^n + \sum_{n=0}^{\infty} \binom{k}{n} n x^n \quad \left[\begin{array}{l} \text{Replace } n \text{ with } n+1 \\ \text{in the first series} \end{array} \right] \\ &= \sum_{n=0}^{\infty} (n+1) \frac{k(k-1)(k-2) \cdots (k-n+1)(k-n)}{(n+1)!} x^n \\ &\quad + \sum_{n=0}^{\infty} \left[\binom{k}{n} \frac{k(k-1)(k-2) \cdots (k-n+1)}{n!} \right] x^n \\ &= \sum_{n=0}^{\infty} \frac{(n+1)k(k-1)(k-2) \cdots (k-n+1)}{(n+1)!} [(k-n) + n] x^n \\ &= k \sum_{n=0}^{\infty} \frac{k(k-1)(k-2) \cdots (k-n+1)}{n!} x^n = k \sum_{n=0}^{\infty} \binom{k}{n} x^n = kg(x) \end{aligned}$$

Thus, $g'(x) = \frac{kg(x)}{1+x}$.

(b) $h(x) = (1+x)^{-k} g(x) \Rightarrow$

$$h'(x) = -k(1+x)^{-k-1} g(x) + (1+x)^{-k} g'(x) \quad [\text{Product Rule}]$$

$$= -k(1+x)^{-k-1} g(x) + (1+x)^{-k} \frac{kg(x)}{1+x} \quad [\text{from part (a)}]$$

$$= -k(1+x)^{-k-1} g(x) + k(1+x)^{-k-1} g(x) = 0$$

(c) From part (b) we see that $h(x)$ must be constant for $x \in (-1, 1)$, so $h(x) = h(0) = 1$ for $x \in (-1, 1)$.

Thus, $h(x) = 1 = (1+x)^{-k} g(x) \Leftrightarrow g(x) = (1+x)^k$ for $x \in (-1, 1)$.

20. By Exercise 12.11.1, $\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n \cdot n!}$, so

$$(1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n \cdot n!} x^{2n} \text{ and}$$

$$\sqrt{1-e^2 \sin^2 \theta} = 1 - \frac{1}{2}e^2 \sin^2 \theta - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n \cdot n!} e^{2n} \sin^{2n} \theta. \text{ Thus,}$$

$$\begin{aligned}
 L &= 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta = 4a \int_0^{\pi/2} \left(1 - \frac{1}{2} e^2 \sin^2 \theta - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n \cdot n!} e^{2n} \sin^{2n} \theta \right) d\theta \\
 &= 4a \left[\frac{\pi}{2} - \frac{e^2}{2} S_1 - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} \left(\frac{e^2}{2} \right)^n S_n \right]
 \end{aligned}$$

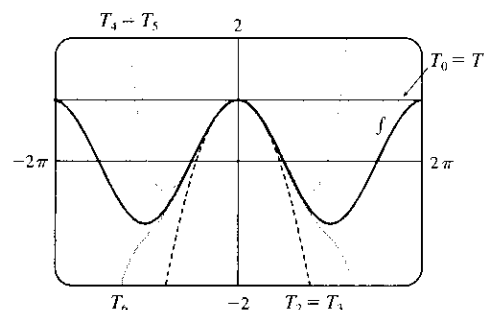
where $S_n = \int_0^{\pi/2} \sin^{2n} \theta d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}$ by Exercise 44 of 8.1.

$$\begin{aligned}
 L &= 4a \left(\frac{\pi}{2} \right) \left[1 - \frac{e^2}{2} \cdot \frac{1}{2} - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} \left(\frac{e^2}{2} \right)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \right] \\
 &= 2\pi a \left[1 - \frac{e^2}{4} - \sum_{n=2}^{\infty} \frac{e^{2n}}{2^n} \cdot \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-3)^2 (2n-1)}{n! \cdot 2^n \cdot n!} \right] \\
 &= 2\pi a \left[1 - \frac{e^2}{4} - \sum_{n=2}^{\infty} \frac{e^{2n}}{4^n} \left(\frac{1 \cdot 3 \cdots (2n-3)}{n!} \right)^2 (2n-1) \right] \\
 &= 2\pi a \left[1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256} - \cdots \right] = \frac{\pi a}{128} (256 - 64e^2 - 12e^4 - 5e^6 - \cdots)
 \end{aligned}$$

12.12 Applications of Taylor Polynomials

1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1 - \frac{1}{2}x^2$
3	$\sin x$	0	$1 - \frac{1}{2}x^2$
4	$\cos x$	1	$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$
5	$-\sin x$	0	$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$
6	$-\cos x$	-1	$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$



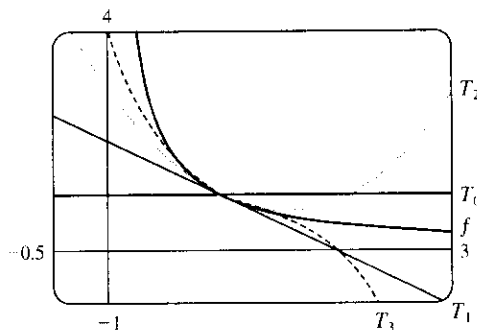
(b)

x	f	$T_0 = T_1$	$T_2 = T_3$	$T_4 = T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$T_n(x)$
0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1 - (x - 1) = 2 - x$
2	$2x^{-3}$	2	$1 - (x - 1) + (x - 1)^2 = x^2 - 3x + 3$
3	$-6x^{-4}$	-6	$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 = -x^3 + 4x^2 - 6x + 4$



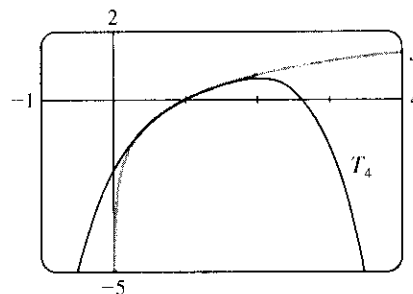
(b)

x	f	T_0	T_1	T_2	T_3
0.9	1.1	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

 (c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

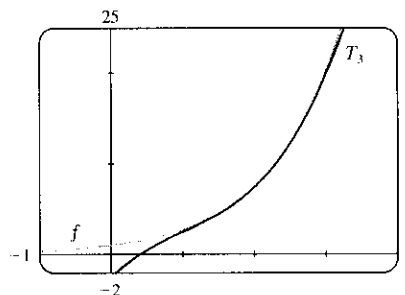
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x) = \sum_{n=0}^4 \frac{f^{(n)}(1)}{n!} (x-1)^n = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

4.

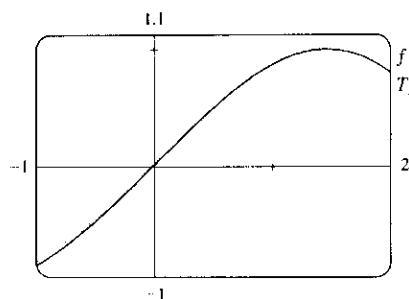
n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2



$$T_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(2)}{n!} (x-2)^n = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3$$

5.

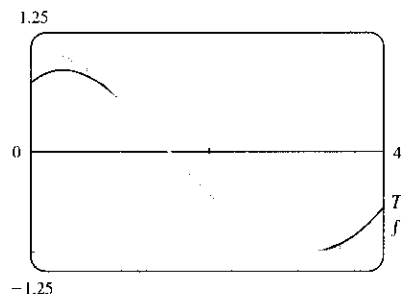
n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{6})$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(\frac{\pi}{6})}{n!} (x - \frac{\pi}{6})^n = \frac{1}{2} + \frac{\sqrt{3}}{2} (x - \frac{\pi}{6}) - \frac{1}{4} (x - \frac{\pi}{6})^2 - \frac{\sqrt{3}}{12} (x - \frac{\pi}{6})^3$$

6.

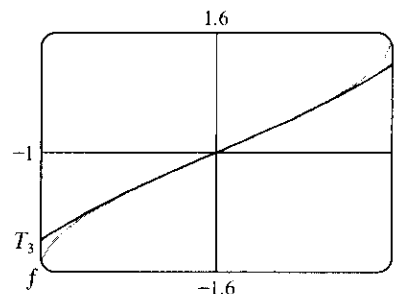
n	$f^{(n)}(x)$	$f^{(n)}(\frac{2\pi}{3})$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$



$$T_4(x) = \sum_{n=0}^4 \frac{f^{(n)}(\frac{2\pi}{3})}{n!} (x - \frac{2\pi}{3})^n = -\frac{1}{2} - \frac{\sqrt{3}}{2} (x - \frac{2\pi}{3}) + \frac{1}{4} (x - \frac{2\pi}{3})^2 + \frac{\sqrt{3}}{12} (x - \frac{2\pi}{3})^3 - \frac{1}{48} (x - \frac{2\pi}{3})^4$$

7.

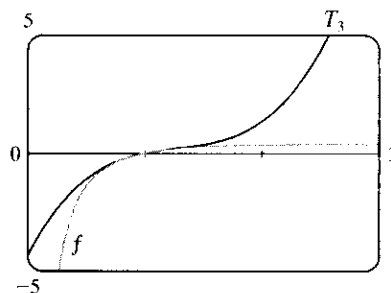
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arcsin x$	0
1	$1/\sqrt{1-x^2}$	1
2	$x/(1-x^2)^{3/2}$	0
3	$(2x^2+1)/(1-x^2)^{5/2}$	1



$$T_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(0)}{n!} x^n = x + \frac{x^3}{6}$$

8.

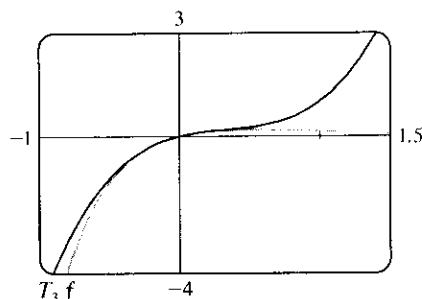
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(\ln x)/x$	0
1	$(1 - \ln x)/x^2$	1
2	$(-3 + 2 \ln x)/x^3$	-3
3	$(11 - 6 \ln x)/x^4$	11



$$T_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(1)}{n!} (x-1)^n = (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3$$

9.

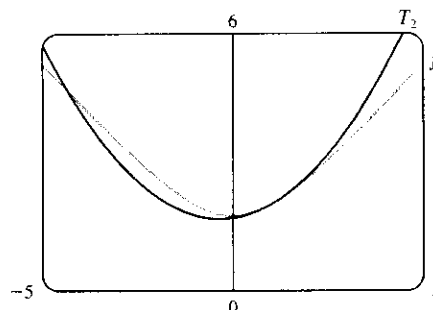
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(0)}{n!} x^n = \frac{0}{1} \cdot 1 + \frac{1}{1} x^1 + \frac{-4}{2} x^2 + \frac{12}{6} x^3 = x - 2x^2 + 2x^3$$

10.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(3+x^2)^{1/2}$	2
1	$x(3+x^2)^{-1/2}$	$\frac{1}{2}$
2	$3(3+x^2)^{-3/2}$	$\frac{3}{8}$

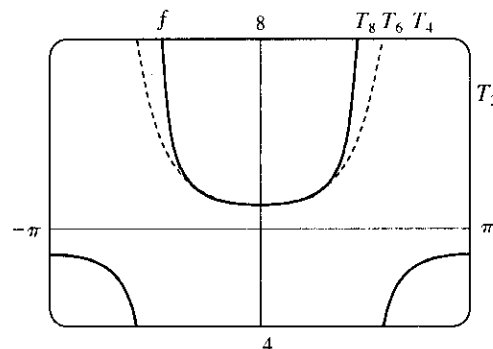


$$T_2(x) = \sum_{n=0}^2 \frac{f^{(n)}(1)}{n!} (x-1)^n = 2 + \frac{1}{2}(x-1) + \frac{3/8}{2}(x-1)^2 = 2 + \frac{1}{2}(x-1) + \frac{3}{16}(x-1)^2$$

11. In Maple, we can find the Taylor polynomials by the following method: first define $f := \sec(x)$; and then set $T_2 := \text{convert}(\text{taylor}(f, x=0, 3), \text{polynom})$; $T_4 := \text{convert}(\text{taylor}(f, x=0, 5), \text{polynom})$; etc. (The third argument in the `taylor` function is one more than the degree of the desired polynomial). We must convert to the type `polynom` because the output of the `taylor` function contains an error term which we do not want. In Mathematica, we use

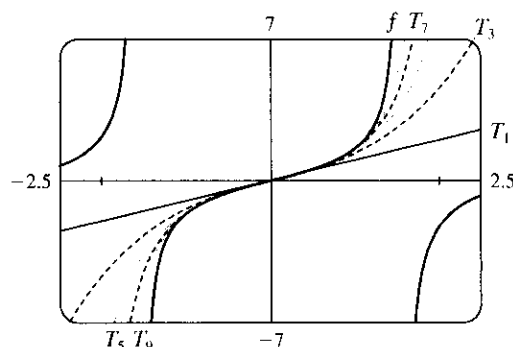
$T_n := \text{Normal}[\text{Series}[f, \{x, 0, n\}]]$, with $n=2, 4$, etc. Note that in Mathematica, the “degree” argument is the same as the degree of the desired polynomial. In Derive, author `sec x`, then enter `Calculus, Taylor, 8, 0`; and then simplify the expression. The eighth Taylor polynomial is

$$T_8(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8.$$



12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for $\tan x$ is

$$T_9(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9.$$



13.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	

$$(a) f(x) = \sqrt{x} \approx T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

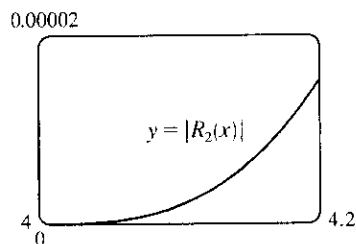
$$(b) |R_2(x)| \leq \frac{M}{3!} |x-4|^3, \text{ where } |f'''(x)| \leq M. \text{ Now } 4 \leq x \leq 4.2 \Rightarrow |x-4| \leq 0.2 \Rightarrow$$

$$|x-4|^3 \leq 0.008. \text{ Since } f'''(x) \text{ is decreasing on } [4, 4.2], \text{ we can take } M = |f'''(4)| = \frac{3}{8}4^{-5/2} = \frac{3}{256}, \text{ so}$$

$$|R_2(x)| \leq \frac{3/256}{6}(0.008) = \frac{0.008}{512} = 0.000015625.$$

(c) From the graph of $|R_2(x)| = |\sqrt{x} - T_2(x)|$, it seems

that the error is less than 1.52×10^{-5} on $[4, 4.2]$.



14.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	

$$(a) f(x) = x^{-2} \approx T_2(x)$$

$$= 1 - 2(x-1) + \frac{6}{2!}(x-1)^2$$

$$= 1 - 2(x-1) + 3(x-1)^2$$

$$(b) |R_2(x)| \leq \frac{M}{3!} |x-1|^3, \text{ where } |f'''(x)| \leq M. \text{ Now}$$

$$0.9 \leq x \leq 1.1 \Rightarrow |x-1| \leq 0.1 \Rightarrow$$

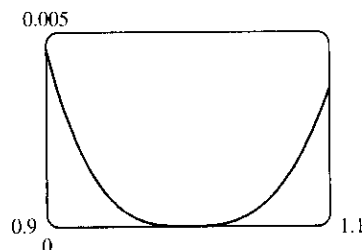
$$|x-1|^3 \leq 0.001. \text{ Since } f'''(x) \text{ is decreasing on}$$

$$[0.9, 1.1], \text{ we can take } M = |f'''(0.9)| = \frac{24}{(0.9)^5}, \text{ so}$$

$$|R_2(x)| \leq \frac{24/(0.9)^5}{6}(0.001) = \frac{0.004}{0.59049}$$

$$\approx 0.00677404$$

(c)



From the graph of $|R_2(x)| = |x^{-2} - T_2(x)|$, it seems that the error is less than 0.0046 on $[0.9, 1.1]$.

15.

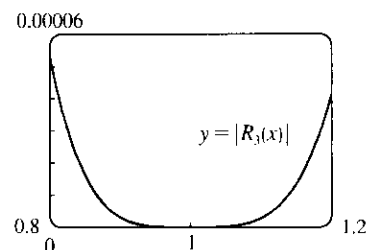
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

$$(a) f(x) = x^{2/3} \approx T_3(x) = 1 + \frac{2}{3}(x-1) - \frac{2/9}{2!}(x-1)^2 + \frac{8/27}{3!}(x-1)^3$$

$$= 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3$$

(b) $|R_3(x)| \leq \frac{M}{4!} |x-1|^4$, where $|f^{(4)}(x)| \leq M$. Now $0.8 \leq x \leq 1.2 \Rightarrow |x-1| \leq 0.2 \Rightarrow$
 $|x-1|^4 \leq 0.0016$. Since $|f^{(4)}(x)|$ is decreasing on $[0.8, 1.2]$, we can take $M = |f^{(4)}(0.8)| = \frac{56}{81}(0.8)^{-10/3}$,
 so $|R_3(x)| \leq \frac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016) \approx 0.000\,096\,97$.

(c) From the graph of $|R_3(x)| = |x^{2/3} - T_3(x)|$, it seems
 that the error is less than 0.000 053 3 on $[0.8, 1.2]$.



16.

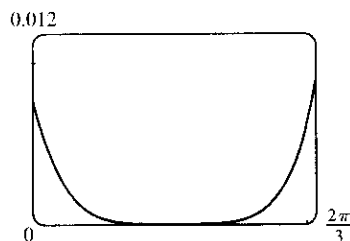
n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{3})$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
5	$-\sin x$	

(a) $f(x) = \cos x \approx T_4(x)$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4$$

(b) $|R_4(x)| \leq \frac{M}{5!} |x - \frac{\pi}{3}|^5$, where $|f^{(5)}(x)| \leq M$. Now $0 \leq x \leq \frac{2\pi}{3} \Rightarrow (x - \frac{\pi}{3})^5 \leq (\frac{\pi}{3})^5$, and letting
 $x = \frac{\pi}{2}$ gives $M = 1$, so $|R_4(x)| \leq \frac{1}{5!} (\frac{\pi}{3})^5 \approx 0.0105$.

(c)



From the graph of $|R_4(x)| = |\cos x - T_4(x)|$, it seems that the error is less than 0.01 on $[0, \frac{2\pi}{3}]$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2 \sec^2 x \tan x$	0
3	$4 \sec^2 x \tan^2 x + 2 \sec^4 x$	2
4	$8 \sec^2 x \tan^3 x + 16 \sec^4 x \tan x$	

(a) $f(x) = \tan x \approx T_3(x) = x + \frac{1}{3}x^3$

(b) $|R_3(x)| \leq \frac{M}{4!} |x|^4$, where $|f^{(4)}(x)| \leq M$. Now

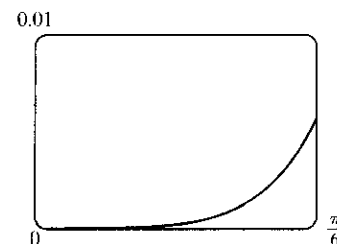
$$0 \leq x \leq \frac{\pi}{6} \Rightarrow x^4 \leq \left(\frac{\pi}{6}\right)^4, \text{ and letting } x = \frac{\pi}{6}$$

[since $f^{(4)}$ is increasing on $(0, \frac{\pi}{6})$] gives

$$|R_3(x)| \leq \frac{8\left(\frac{2}{\sqrt{3}}\right)^2 \left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4 \left(\frac{1}{\sqrt{3}}\right)}{4!} \left(\frac{\pi}{6}\right)^4$$

$$= \frac{4\sqrt{3}}{9} \left(\frac{\pi}{6}\right)^4 \approx 0.057859$$

(c)



From the graph of $|R_3(x)| = |\tan x - T_3(x)|$, it seems that the error is less than 0.006 on $[0, \pi/6]$.

18.

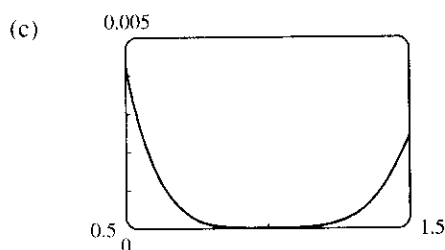
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x) = \ln(1+2x) \approx T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{4/9}{2!}(x-1)^2 + \frac{16/27}{3!}(x-1)^3$

(b) $|R_3(x)| \leq \frac{M}{4!} |x-1|^4$, where $|f^{(4)}(x)| \leq M$. Now $0.5 \leq x \leq 1.5 \Rightarrow -0.5 \leq x-1 \leq 0.5 \Rightarrow$

$$|x-1| \leq 0.5 \Rightarrow |x-1|^4 \leq \frac{1}{16}, \text{ and letting } x = 0.5 \text{ gives } M = 6, \text{ so}$$

$$|R_3(x)| \leq \frac{6}{4!} \cdot \frac{1}{16} = \frac{1}{64} = 0.015625.$$



From the graph of $|R_3(x)| = |\ln(1+2x) - T_3(x)|$, it seems that the error is less than 0.005 on $[0.5, 1.5]$.

19.

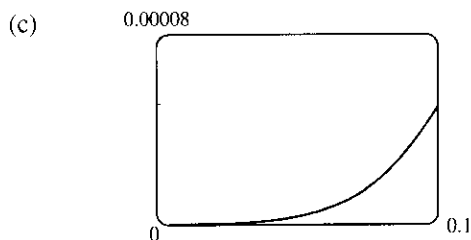
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2 + 4x^2)$	2
3	$e^{x^2}(12x + 8x^3)$	0
4	$e^{x^2}(12 + 48x^2 + 16x^4)$	

(a) $f(x) = e^{x^2} \approx T_3(x) = 1 + \frac{2}{2!}x^2 = 1 + x^2$

(b) $|R_3(x)| \leq \frac{M}{4!}|x|^4$, where $|f^{(4)}(x)| \leq M$.

Now $0 \leq x \leq 0.1 \Rightarrow x^4 \leq (0.1)^4$, and letting $x = 0.1$ gives

$$|R_3(x)| \leq \frac{e^{0.01}(12 + 0.48 + 0.0016)}{24}(0.1)^4 \approx 0.00006.$$



From the graph of $|R_3(x)| = |e^{x^2} - (1 + x^2)|$, it appears that the error is less than 0.000051 on $[0, 0.1]$.

20.

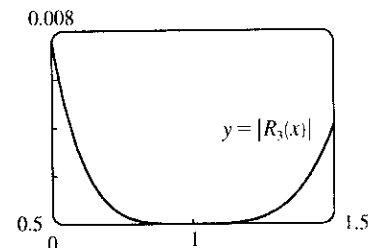
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x \ln x$	0
1	$\ln x + 1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	

(a) $f(x) = x \ln x \approx T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$

(b) $|R_3(x)| \leq \frac{M}{4!}|x-1|^4$, where $|f^{(4)}(x)| \leq M$. Now

$0.5 \leq x \leq 1.5 \Rightarrow |x-1| \leq \frac{1}{2} \Rightarrow |x-1|^4 \leq \frac{1}{16}$. Since $|f^{(4)}(x)|$ is decreasing on $[0.5, 1.5]$, we can take $M = |f^{(4)}(0.5)| = 2/(0.5)^3 = 16$, so $|R_3(x)| \leq \frac{16}{24}(1/16) = \frac{1}{24} = 0.041\bar{6}$.

- (c) From the graph of $|R_3(x)| = |x \ln x - T_3(x)|$, it seems that the error is less than 0.0076 on $[0.5, 1.5]$.



21.

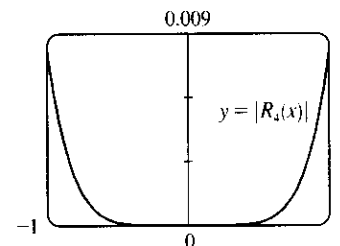
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x \sin x$	0
1	$\sin x + x \cos x$	0
2	$2 \cos x - x \sin x$	2
3	$-3 \sin x - x \cos x$	0
4	$-4 \cos x + x \sin x$	-4
5	$5 \sin x + x \cos x$	

(a) $f(x) = x \sin x \approx T_4(x) = \frac{2}{2!}(x-0)^2 + \frac{-4}{4!}(x-0)^4 = x^2 - \frac{1}{6}x^4$

- (b) $|R_4(x)| \leq \frac{M}{5!}|x|^5$, where $|f^{(5)}(x)| \leq M$. Now $-1 \leq x \leq 1 \Rightarrow |x| \leq 1$, and a graph of $f^{(5)}(x)$ shows that $|f^{(5)}(x)| \leq 5$ for $-1 \leq x \leq 1$. Thus, we can take $M = 5$ and get $|R_4(x)| \leq \frac{5}{5!} \cdot 1^5 = \frac{1}{24} = 0.041\bar{6}$.

- (c) From the graph of $|R_4(x)| = |x \sin x - T_4(x)|$, it

seems that the error is less than 0.0082 on $[-1, 1]$.



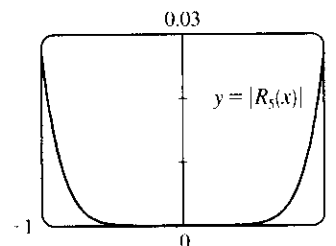
22.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh 2x$	0
1	$2 \cosh 2x$	2
2	$4 \sinh 2x$	0
3	$8 \cosh 2x$	8
4	$16 \sinh 2x$	0
5	$32 \cosh 2x$	32
6	$64 \sinh 2x$	

(a) $f(x) = \sinh 2x \approx T_5(x) = 2x + \frac{8}{3!}x^3 + \frac{32}{5!}x^5 = 2x + \frac{4}{3}x^3 + \frac{4}{15}x^5$

(b) $|R_5(x)| \leq \frac{M}{6!} |x|^6$, where $|f^{(6)}(x)| \leq M$. For x in $[-1, 1]$, we have $|x| \leq 1$. Since $f^{(6)}(x)$ is an increasing odd function on $[-1, 1]$, we see that $|f^{(6)}(x)| \leq f^{(6)}(1) = 64 \sinh 2 = 32(e^2 - e^{-2}) \approx 232.119$, so we can take $M = 232.12$ and get $|R_5(x)| \leq \frac{232.12}{720} \cdot 1^6 \approx 0.3224$.

(c) From the graph of $|R_5(x)| = |\sinh 2x - T_5(x)|$, it seems that the error is less than 0.027 on $[-1, 1]$.



23. From Exercise 5, $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{1}{4}(x - \frac{\pi}{6})^2 - \frac{\sqrt{3}}{12}(x - \frac{\pi}{6})^3 + R_3(x)$, where $|R_3(x)| \leq \frac{M}{4!} |x - \frac{\pi}{6}|^4$

with $|f^{(4)}(x)| = |\sin x| \leq M = 1$. Now $x = 35^\circ = (30^\circ + 5^\circ) = (\frac{\pi}{6} + \frac{\pi}{36})$ radians, so the error is

$$|R_3(\frac{\pi}{36})| \leq \frac{(\frac{\pi}{36})^4}{4!} < 0.000003. \text{ Therefore, to five decimal places,}$$

$$\sin 35^\circ \approx \frac{1}{2} + \frac{\sqrt{3}}{2}(\frac{\pi}{36}) - \frac{1}{4}(\frac{\pi}{36})^2 - \frac{\sqrt{3}}{12}(\frac{\pi}{36})^3 \approx 0.57358.$$

24. From Exercise 16, $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{4}(x - \frac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x - \frac{\pi}{3})^3 + \frac{1}{48}(x - \frac{\pi}{3})^4 + R_4(x)$. Now since

$$x = 69^\circ = (60^\circ + 9^\circ) = (\frac{\pi}{3} + \frac{\pi}{20}) \text{ radians, the error is } |R_4(x)| \leq \frac{(\frac{\pi}{20})^5}{5!} < 8 \times 10^{-7}. \text{ Therefore, to five}$$

$$\text{decimal places, } \cos 69^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2}(\frac{\pi}{20}) - \frac{1}{4}(\frac{\pi}{20})^2 + \frac{\sqrt{3}}{12}(\frac{\pi}{20})^3 + \frac{1}{48}(\frac{\pi}{20})^4 \approx 0.35837.$$

25. All derivatives of e^x are e^x , so $|R_n(x)| \leq \frac{e^x}{(n+1)!} |x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$,

$$R_n(0.1) \leq \frac{e^{0.1}}{(n+1)!} (0.1)^{n+1} < 0.00001, \text{ and by trial and error we find that } n = 3 \text{ satisfies this inequality since}$$

$R_3(0.1) < 0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2$, and 3, we can estimate $e^{0.1}$ to within 0.00001. (In fact, this sum is $1.1051\bar{6}$ and $e^{0.1} \approx 1.10517$.)

26. Example 6 in Section 12.9 gives the Maclaurin series for $\ln(1-x)$ as $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x| < 1$. Thus,

$$\ln 1.4 = \ln[1 - (-0.4)] = -\sum_{n=1}^{\infty} \frac{(-0.4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.4)^n}{n}. \text{ Since this is an alternating series, the error is}$$

less than the first neglected term by the Alternating Series Estimation Theorem, and we find that

$$|a_6| = (0.4)^6 / 6 \approx 0.0007 < 0.001. \text{ So we need the first five (non-zero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and } \ln 1.4 \approx 0.33647.)$$

27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$. By the Alternating

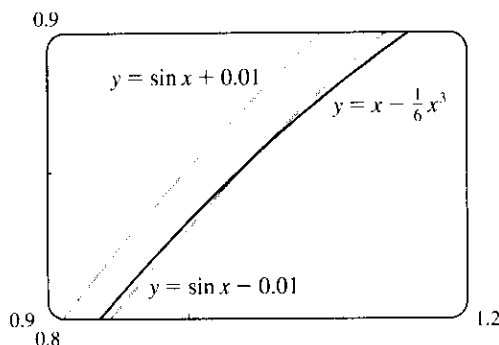
Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than

$$\left| \frac{1}{5!}x^5 \right| < 0.01 \Leftrightarrow |x^5| < 120(0.01) \Leftrightarrow$$

$$|x| < (1.2)^{1/5} \approx 1.037. \text{ The curves } y = x - \frac{1}{6}x^3 \text{ and}$$

$y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and

the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.037 < x < 1.037$.



28. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$. By the

Alternating Series Estimation Theorem, the error is less

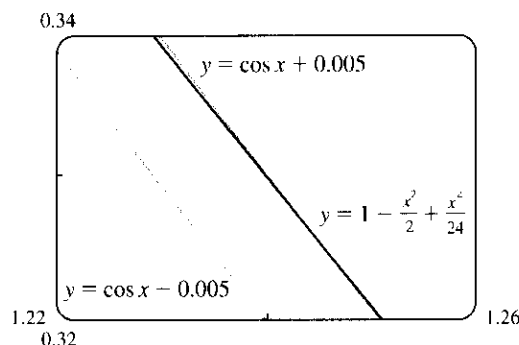
$$\text{than } \left| -\frac{1}{6!}x^6 \right| < 0.005 \Leftrightarrow x^6 < 720(0.005) \Leftrightarrow$$

$$|x| < (3.6)^{1/6} \approx 1.238. \text{ The curves}$$

$$y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \text{ and } y = \cos x + 0.005 \text{ intersect}$$

at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation

are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.



29. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is

$v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is

$$T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2. \text{ We estimate the distance travelled during the next second to be}$$

$s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be

$$140 \text{ m/s} \approx 313 \text{ mi/h!})$$

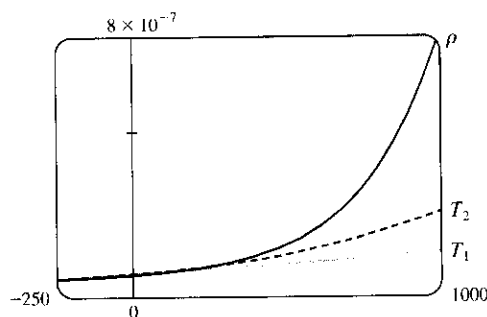
30. (a)

n	$\rho^{(n)}(t)$	$\rho^{(n)}(20)$
0	$\rho_{20}e^{\alpha(t-20)}$	ρ_{20}
1	$\alpha\rho_{20}e^{\alpha(t-20)}$	$\alpha\rho_{20}$
2	$\alpha^2\rho_{20}e^{\alpha(t-20)}$	$\alpha^2\rho_{20}$

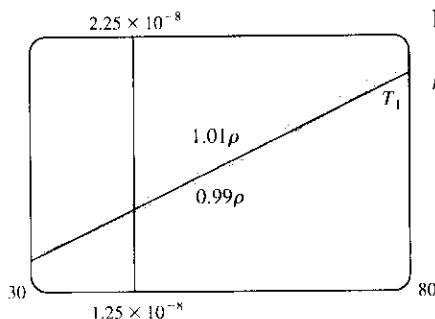
The linear approximation is $T_1(t) = \rho(20) + \rho'(20)(t - 20) = \rho_{20} [1 + \alpha(t - 20)]$. The quadratic approximation is

$$T_2(t) = \rho(20) + \rho'(20)(t - 20) + \frac{\rho''(20)}{2}(t - 20)^2 = \rho_{20} \left[1 + \alpha(t - 20) + \frac{1}{2}\alpha^2(t - 20)^2 \right]$$

(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t) \leq T_1(t) \leq 1.01\rho(t)$, for $-14^\circ\text{C} \leq t \leq 58^\circ\text{C}$.

$$31. E = \frac{q}{D^2} - \frac{q}{(D+d)^2} = \frac{q}{D^2} - \frac{q}{D^2(1+d/D)^2} = \frac{q}{D^2} \left[1 - \left(1 + \frac{d}{D} \right)^{-2} \right].$$

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$\begin{aligned} E &= \frac{q}{D^2} \left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2 \cdot 3}{2!} \left(\frac{d}{D}\right)^2 - \frac{2 \cdot 3 \cdot 4}{3!} \left(\frac{d}{D}\right)^3 + \dots \right) \right] \\ &= \frac{q}{D^2} \left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \dots \right] \approx \frac{q}{D^2} \cdot 2\left(\frac{d}{D}\right) = 2qd \cdot \frac{1}{D^3} \end{aligned}$$

when D is much larger than d ; that is, when P is far away from the dipole.

$$32. (a) \frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R} \left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o} \right) \quad (\text{Equation 1}) \text{ where}$$

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R) \cos \phi} \quad \text{and} \quad \ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R) \cos \phi} \quad (2)$$

Using $\cos \phi \approx 1$ gives

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)} = \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o - 2R^2} = \sqrt{s_o^2} = s_o$$

and similarly, $\ell_i = s_i$. Thus, Equation 1 becomes

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{1}{R} \left(\frac{n_2 s_i}{s_i} - \frac{n_1 s_o}{s_o} \right) \Rightarrow \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

(b) Using $\cos \phi \approx 1 - \frac{1}{2}\phi^2$ in (2) gives us

$$\begin{aligned}\ell_o &= \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)(1 - \frac{1}{2}\phi^2)} \\ &= \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o + Rs_o\phi^2 - 2R^2 + R^2\phi^2} = \sqrt{s_o^2 + Rs_o\phi^2 + R^2\phi^2}\end{aligned}$$

Anticipating that we will use the binomial series expansion $(1+x)^k \approx 1+kx$, we can write the last expression

for ℓ_o as $s_o \sqrt{1 + \phi^2 \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right)}$ and similarly, $\ell_i = s_i \sqrt{1 - \phi^2 \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right)}$. Thus, from Equation 1,

$$\begin{aligned}\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} &= \frac{1}{R} \left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o} \right) \Leftrightarrow n_1 \ell_o^{-1} + n_2 \ell_i^{-1} = \frac{n_2}{R} \cdot \frac{s_i}{\ell_i} - \frac{n_1}{R} \cdot \frac{s_o}{\ell_o} \Leftrightarrow \\ \frac{n_1}{s_o} \left[1 + \phi^2 \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) \right]^{-1/2} &+ \frac{n_2}{s_i} \left[1 - \phi^2 \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) \right]^{-1/2} \\ &= \frac{n_2}{R} \left[1 - \phi^2 \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) \right]^{-1/2} - \frac{n_1}{R} \left[1 + \phi^2 \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) \right]^{-1/2}\end{aligned}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_i^{-1} by the first two terms in their binomial series, we get

$$\begin{aligned}\frac{n_1}{s_o} \left[1 - \frac{1}{2}\phi^2 \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) \right] &+ \frac{n_2}{s_i} \left[1 + \frac{1}{2}\phi^2 \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) \right] \\ &= \frac{n_2}{R} \left[1 + \frac{1}{2}\phi^2 \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) \right] - \frac{n_1}{R} \left[1 - \frac{1}{2}\phi^2 \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) \right] \Leftrightarrow \\ \frac{n_1}{s_o} - \frac{n_1\phi^2}{2s_o} \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) &+ \frac{n_2}{s_i} + \frac{n_2\phi^2}{2s_i} \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) \\ &= \frac{n_2}{R} + \frac{n_2\phi^2}{2R} \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) - \frac{n_1}{R} + \frac{n_1\phi^2}{2R} \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) \Leftrightarrow \\ \frac{n_1}{s_o} + \frac{n_2}{s_i} &= \frac{n_2}{R} - \frac{n_1}{R} + \frac{n_1\phi^2}{2s_o} \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) + \frac{n_1\phi^2}{2R} \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) + \frac{n_2\phi^2}{2R} \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) - \frac{n_2\phi^2}{2s_i} \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) \\ &= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2}{2} \left(\frac{R}{s_o} + \frac{R^2}{s_o^2} \right) \left(\frac{1}{s_o} + \frac{1}{R} \right) + \frac{n_2\phi^2}{2} \left(\frac{R}{s_i} - \frac{R^2}{s_i^2} \right) \left(\frac{1}{R} - \frac{1}{s_i} \right) \\ &= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2 R^2}{2s_o} \left(\frac{1}{R} + \frac{1}{s_o} \right) \left(\frac{1}{R} + \frac{1}{s_o} \right) + \frac{n_2\phi^2 R^2}{2s_i} \left(\frac{1}{R} - \frac{1}{s_i} \right) \left(\frac{1}{R} - \frac{1}{s_i} \right) \\ &= \frac{n_2 - n_1}{R} + \phi^2 R^2 \left[\frac{n_1}{2s_o} \left(\frac{1}{R} + \frac{1}{s_o} \right)^2 + \frac{n_2}{2s_i} \left(\frac{1}{R} - \frac{1}{s_i} \right)^2 \right]\end{aligned}$$

From Figure 8, we see that $\sin \phi = h/R$. So if we approximate $\sin \phi$ with ϕ , we get $h = R\phi$ and $h^2 = \phi^2 R^2$ and hence, Equation 4, as desired.

33. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \rightarrow 1$ as $x \rightarrow \infty$. So we can approximate

$$\tanh(2\pi d/L) \approx 1, \text{ and so } v^2 \approx gL/(2\pi) \Leftrightarrow v \approx \sqrt{gL/(2\pi)}.$$

- (b) From the table, the first term in the Maclaurin series of $\tanh x$ is x , so if the water is shallow,

we can approximate $\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$, and so

$$v^2 \approx \frac{gL}{2\pi} \cdot \frac{2\pi d}{L} \Leftrightarrow v \approx \sqrt{gd}.$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2 \operatorname{sech}^2 x \tanh x$	0
3	$2 \operatorname{sech}^2 x (3 \tanh^2 x - 1)$	-2

- (c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation

$$\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L} \text{ is less than the first neglected term, which is } \frac{|f'''(0)|}{3!} \left(\frac{2\pi d}{L}\right)^3 = \frac{1}{3} \left(\frac{2\pi d}{L}\right)^3.$$

If $L > 10d$, then $\frac{1}{3} \left(\frac{2\pi d}{L}\right)^3 < \frac{1}{3} \left(2\pi \cdot \frac{1}{10}\right)^3 = \frac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less

$$\text{than } \frac{gL}{2\pi} \cdot \frac{\pi^3}{375} \approx 0.0132gL.$$

$$\begin{aligned} 34. (a) & 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} [1 + (-k^2 \sin^2 x)]^{-1/2} dx \\ &= 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \left[1 - \frac{1}{2}(-k^2 \sin^2 x) + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2!} (-k^2 \sin^2 x)^2 - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{3!} (-k^2 \sin^2 x)^3 + \dots \right] dx \\ &= 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \left[1 + \left(\frac{1}{2}\right)k^2 \sin^2 x + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)k^4 \sin^4 x + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)k^6 \sin^6 x + \dots \right] dx \\ &= 4 \sqrt{\frac{L}{g}} \left[\frac{\pi}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{2} \cdot \frac{\pi}{2}\right)k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)\left(\frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2}\right)k^4 \right. \\ &\quad \left. + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2}\right)k^6 + \dots \right] \\ &\quad \text{[split up the integral and use the result from Exercise 8.1.44]} \\ &= 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k^6 + \dots \right] \end{aligned}$$

- (b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

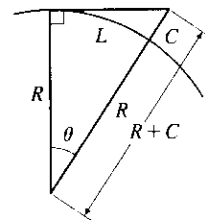
$$\begin{aligned} T &= 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k^6 + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}k^8 + \dots \right] \\ &\leq 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{1}{4}k^2 + \frac{1}{4}k^4 + \frac{1}{4}k^6 + \frac{1}{4}k^8 + \dots \right] \end{aligned}$$

The terms in brackets (after the first) form a geometric series with $a = \frac{1}{4}k^2$ and $r = k^2 = \sin^2(\frac{1}{2}\theta_0) < 1$.

$$\text{So } T \leq 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{k^2/4}{1 - k^2} \right] = 2\pi \sqrt{\frac{L}{g}} \frac{4 - 3k^2}{4 - 4k^2}.$$

- (c) We substitute $L = 1$, $g = 9.8$, and $k = \sin(10^\circ/2) \approx 0.08716$, and the inequality from part (b) becomes $2.01090 \leq T \leq 2.01093$, so $T \approx 2.0109$. The estimate $T \approx 2\pi\sqrt{L/g} \approx 2.0071$ differs by about 0.2%. If $\theta_0 = 42^\circ$, then $k \approx 0.35837$ and the inequality becomes $2.07153 \leq T \leq 2.08103$, so $T \approx 2.0763$. The one-term estimate is the same, and the discrepancy between the two estimates increases to about 3.4%.

35. (a) L is the length of the arc subtended by the angle θ , so $L = R\theta \Rightarrow \theta = L/R$. Now $\sec \theta = (R + C)/R \Rightarrow R \sec \theta = R + C \Rightarrow C = R \sec \theta - R = R \sec(L/R) - R$.



- (b) From Exercise 11, $\sec x \approx T_4(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. By part (a),

$$C \approx R \left[1 + \frac{1}{2} \left(\frac{L}{R} \right)^2 + \frac{5}{24} \left(\frac{L}{R} \right)^4 \right] - R = R + \frac{1}{2} R \cdot \frac{L^2}{R^2} + \frac{5}{24} R \cdot \frac{L^4}{R^4} - R = \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

- (c) Taking $L = 100$ km and $R = 6370$ km, the formula in part (a) says that

$$C = R \sec(L/R) - R = 6370 \sec(100/6370) - 6370 \approx 0.785\,009\,965\,44 \text{ km}$$

The formula in part (b) says that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3} = \frac{100^2}{2 \cdot 6370} + \frac{5 \cdot 100^4}{24 \cdot 6370^3} \approx 0.785\,009\,957\,36 \text{ km}.$$

The difference between these two results is only 0.000 000 008 08 km, or 0.000 008 08 m!

36. $T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0 \leq m \leq n$. Then

$$T_n^{(m)}(x) = m! \frac{f^{(m)}(a)}{m!}(x-a)^0 + (m+1)(m) \cdots (2) \frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1 + \cdots + n(n-1) \cdots (n-m+1) \frac{f^{(n)}(a)}{n!}(x-a)^{n-m}$$

For $x = a$, all terms in this sum except the first one are 0, so $T_n^{(m)}(a) = \frac{m! f^{(m)}(a)}{m!} = f^{(m)}(a)$.

37. Using $f(x) = T_n(x) + R_n(x)$ with $n = 1$ and $x = r$, we have $f(r) = T_1(r) + R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a . Because $a = x_n$, $f(r) = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. But r is a root of f , so $f(r) = 0$ and we have $0 = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. Taking the first two terms to the left side gives us

$$f'(x_n)(x_n - r) - f(x_n) = R_1(r). \text{ Dividing by } f'(x_n), \text{ we get } x_n - r - \frac{f(x_n)}{f'(x_n)} = \frac{R_1(r)}{f'(x_n)}.$$

By the formula for Newton's method, the left side of the preceding equation is $x_{n+1} - r$, so $|x_{n+1} - r| = \left| \frac{R_1(r)}{f'(x_n)} \right|$. Taylor's

Inequality gives us $|R_1(r)| \leq \frac{|f''(r)|}{2!} |r - x_n|^2$. Combining this inequality with the facts $|f''(x)| \leq M$ and

$$|f'(x)| \geq K \text{ gives us } |x_{n+1} - r| \leq \frac{M}{2K} |x_n - r|^2.$$

APPLIED PROJECT Radiation from the Stars

1. If we write $f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{a\lambda^{-5}}{e^{b/(\lambda T)} - 1}$, then as $\lambda \rightarrow 0^+$, it is of the form ∞/∞ , and as $\lambda \rightarrow \infty$ it is of the form $0/0$, so in either case we can use l'Hospital's Rule. First of all,

$$\lim_{\lambda \rightarrow \infty} f(\lambda) \stackrel{H}{=} \lim_{\lambda \rightarrow \infty} \frac{a(-5\lambda^{-6})}{-\frac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 5\frac{aT}{b} \lim_{\lambda \rightarrow \infty} \frac{\lambda^2 \lambda^{-6}}{e^{b/(\lambda T)}} = 5\frac{aT}{b} \lim_{\lambda \rightarrow \infty} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} = 0$$

Also,

$$\lim_{\lambda \rightarrow 0^+} f(\lambda) \stackrel{H}{=} 5\frac{aT}{b} \lim_{\lambda \rightarrow 0^+} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} \stackrel{H}{=} 5\frac{aT}{b} \lim_{\lambda \rightarrow 0^+} \frac{-4\lambda^{-5}}{-\frac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 20\frac{aT^2}{b^2} \lim_{\lambda \rightarrow 0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}}$$

This is still indeterminate, but note that each time we use l'Hospital's Rule, we gain a factor of λ in the numerator, as well as a constant factor, and the denominator is unchanged. So if we use l'Hospital's Rule three more times, the exponent of λ in the numerator will become 0. That is, for some $\{k_i\}$, all constant,

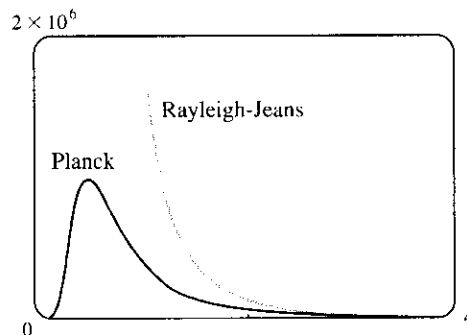
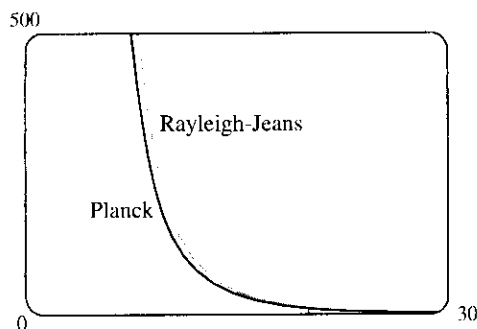
$$\lim_{\lambda \rightarrow 0^+} f(\lambda) \stackrel{H}{=} k_1 \lim_{\lambda \rightarrow 0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}} \stackrel{H}{=} k_2 \lim_{\lambda \rightarrow 0^+} \frac{\lambda^{-2}}{e^{b/(\lambda T)}} \stackrel{H}{=} k_3 \lim_{\lambda \rightarrow 0^+} \frac{\lambda^{-1}}{e^{b/(\lambda T)}} \stackrel{H}{=} k_4 \lim_{\lambda \rightarrow 0^+} \frac{1}{e^{b/(\lambda T)}} = 0$$

2. We expand the denominator of Planck's Law using the Taylor series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ with $x = \frac{hc}{\lambda kT}$, and use the fact that if λ is large, then all subsequent terms in the Taylor expansion are very small compared to the first one, so we can approximate using the Taylor polynomial T_1 :

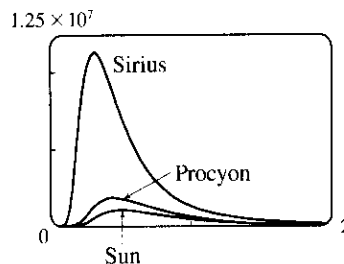
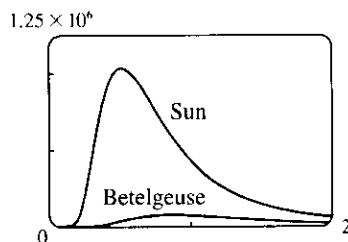
$$\begin{aligned} f(\lambda) &= \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{8\pi hc\lambda^{-5}}{\left[1 + \frac{hc}{\lambda kT} + \frac{1}{2!} \left(\frac{hc}{\lambda kT}\right)^2 + \frac{1}{3!} \left(\frac{hc}{\lambda kT}\right)^3 + \dots\right] - 1} \\ &\approx \frac{8\pi hc\lambda^{-5}}{\left(1 + \frac{hc}{\lambda kT}\right) - 1} = \frac{8\pi kT}{\lambda^4} \end{aligned}$$

which is the Rayleigh-Jeans Law.

3. To convert to μm , we substitute $\lambda/10^6$ for λ in both laws. The first figure shows that the two laws are similar for large λ . The second figure shows that the two laws are very different for short wavelengths (Planck's Law gives a maximum at $\lambda \approx 0.51 \mu\text{m}$; the Rayleigh-Jeans Law gives no minimum or maximum.).



4. From the graph in Problem 3, $f(\lambda)$ has a maximum under Planck's Law at $\lambda \approx 0.51 \mu\text{m}$.
- 5.



As T gets larger, the total area under the curve increases, as we would expect: the hotter the star, the more energy it emits. Also, as T increases, the λ -value of the maximum decreases, so the higher the temperature, the shorter the peak wavelength (and consequently the average wavelength) of light emitted. This is why Sirius is a blue star and Betelgeuse is a red star: most of Sirius's light is of a fairly short wavelength; that is, a higher frequency, toward the blue end of the spectrum, whereas most of Betelgeuse's light is of a lower frequency, toward the red end of the spectrum.

12 Review

CONCEPT CHECK

- (a) See Definition 12.1.1.

(b) See Definition 12.2.2.

(c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.
- (a) See Definition 12.1.10.

(b) A sequence is monotonic if it is either increasing or decreasing.

(c) By Theorem 12.1.11, every bounded, monotonic sequence is convergent.
- (a) See (4) in Section 12.2.

(b) The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.
- If $\sum a_n = 3$, then $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} s_n = 3$.
- (a) See the Test for Divergence on page 754.

(b) See the Integral Test on page 760.

(c) See the Comparison Test on page 767.

(d) See the Limit Comparison Test on page 768.

(e) See the Alternating Series Test on page 772.

(f) See the Ratio Test on page 778.

(g) See the Root Test on page 780.

6. (a) A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent.
 (b) If a series $\sum a_n$ is absolutely convergent, then it is convergent.
 (c) A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.
7. (a) Use (3) in Section 12.3.
 (b) See Example 5 in Section 12.4.
 (c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 774.
8. (a) $\sum_{n=0}^{\infty} c_n(x-a)^n$
 (b) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:
 (i) 0 if the series converges only when $x = a$
 (ii) ∞ if the series converges for all x , or
 (iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.
 (c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a-R$ and $a+R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.
9. (a), (b) See Theorem 12.9.2.
10. (a) $T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$
 (b) $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$
 (c) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ [$a = 0$ in part (b)]
 (d) See Theorem 12.10.8.
 (e) See Taylor's Inequality (12.10.9).
11. (a) – (e) See the table on page 803.
12. See the Binomial Series (12.11.2) for the expansion. The radius of convergence for the binomial series is 1.

TRUE-FALSE QUIZ

1. False. See Note 2 after Theorem 12.2.6.
2. False. The series $\sum_{n=1}^{\infty} n^{-\sin 1} = \sum_{n=1}^{\infty} \frac{1}{n^{\sin 1}}$ is a p -series with $p = \sin 1 \approx 0.84 \leq 1$, so the series diverges.
3. True. If $\lim_{n \rightarrow \infty} a_n = L$, then given any $\varepsilon > 0$, we can find a positive integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. If $n > N$, then $2n+1 > N$ and $|a_{2n+1} - L| < \varepsilon$. Thus, $\lim_{n \rightarrow \infty} a_{2n+1} = L$.
4. True by Theorem 12.8.3.
 Or: Use the Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.
5. False. For example, take $c_n = (-1)^n / (n6^n)$.

6. True by Theorem 12.8.3.
7. False, since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^3} \cdot \frac{n^3}{1} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^3}{(n+1)^3} \cdot \frac{1/n^3}{1/n^3} \right| = \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^3} = 1$.
8. True, since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)!} \cdot \frac{n!}{1} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1$.
9. False. See the note after Example 2 in Section 12.4.
10. True, since $\frac{1}{e} = e^{-1}$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.
11. True. See (8) in Section 12.1.
12. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 12.6.3.
13. True. By Theorem 12.10.5 the coefficient of x^3 is $\frac{f'''(0)}{3!} = \frac{1}{3} \Rightarrow f'''(0) = 2$.
Or: Use Theorem 12.9.2 to differentiate f three times.
14. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.
15. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.
16. True by the Monotonic Sequence Theorem, since $\{a_n\}$ is decreasing and $0 < a_n \leq a_1$ for all $n \Rightarrow \{a_n\}$ is bounded.
17. True by Theorem 12.6.3. [$\sum (-1)^n a_n$ is absolutely convergent and hence convergent.]
18. True. $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges (Ratio Test) $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ [Theorem 12.2.6].

EXERCISES

1. $\left\{ \frac{2+n^3}{1+2n^3} \right\}$ converges since $\lim_{n \rightarrow \infty} \frac{2+n^3}{1+2n^3} = \lim_{n \rightarrow \infty} \frac{2/n^3+1}{1/n^3+2} = \frac{1}{2}$.
2. $a_n = \frac{9^{n+1}}{10^n} = 9 \cdot \left(\frac{9}{10}\right)^n$, so $\lim_{n \rightarrow \infty} a_n = 9 \lim_{n \rightarrow \infty} \left(\frac{9}{10}\right)^n = 9 \cdot 0 = 0$ by (12.1.8).
3. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^3}{1+n^2} = \lim_{n \rightarrow \infty} \frac{n}{1/n^2+1} = \infty$, so the sequence diverges.
4. $a_n = \cos(n\pi/2)$, so $a_n = 0$ if n is odd and $a_n = \pm 1$ if n is even. As n increases, a_n keeps cycling through the values 0, 1, 0, -1, so the sequence $\{a_n\}$ is divergent.
5. $|a_n| = \left| \frac{n \sin n}{n^2+1} \right| \leq \frac{n}{n^2+1} < \frac{1}{n}$, so $|a_n| \rightarrow 0$ as $n \rightarrow \infty$. Thus, $\lim_{n \rightarrow \infty} a_n = 0$. The sequence $\{a_n\}$ is convergent.
6. $a_n = \frac{\ln n}{\sqrt{n}}$. Let $f(x) = \frac{\ln x}{\sqrt{x}}$ for $x > 0$. Then $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$.
Thus, by Theorem 3 in Section 12.1, $\{a_n\}$ converges and $\lim_{n \rightarrow \infty} a_n = 0$.

7. $\left\{ \left(1 + \frac{3}{n} \right)^{4n} \right\}$ is convergent. Let $y = \left(1 + \frac{3}{x} \right)^{4x}$. Then

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} 4x \ln \left(1 + \frac{3}{x} \right) = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{3}{x} \right)}{1/(4x)} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{1+3/x} \left(-\frac{3}{x^2} \right)}{-1/(4x^2)} = \lim_{x \rightarrow \infty} \frac{12}{1+3/x} = 12$$

$$\text{so } \lim_{x \rightarrow \infty} y = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n} \right)^{4n} = e^{12}.$$

Or: Use Exercise 7.7.54.

8. $\left\{ \frac{(-10)^n}{n!} \right\}$ converges, since $\frac{10^n}{n!} = \frac{10 \cdot 10 \cdot 10 \cdots 10}{1 \cdot 2 \cdot 3 \cdots n} \leq \frac{10 \cdot 10 \cdots 10}{11 \cdot 12 \cdots n} \leq 10^{10} \left(\frac{10}{11} \right)^{n-10} \rightarrow 0$ as $n \rightarrow \infty$,

$$\text{so } \lim_{n \rightarrow \infty} \frac{(-10)^n}{n!} = 0 \text{ (Squeeze Theorem). Or: Use (12.10.10).}$$

9. We use induction, hypothesizing that $a_{n-1} < a_n < 2$. Note first that $1 < a_2 = \frac{1}{3}(1+4) = \frac{5}{3} < 2$, so the hypothesis holds for $n = 2$. Now assume that $a_{k-1} < a_k < 2$. Then

$$a_k = \frac{1}{3}(a_{k-1} + 4) < \frac{1}{3}(a_k + 4) < \frac{1}{3}(2 + 4) = 2. \text{ So } a_k < a_{k+1} < 2, \text{ and the induction is complete. To find the limit of the sequence, we note that } L = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} \Rightarrow L = \frac{1}{3}(L + 4) \Rightarrow L = 2.$$

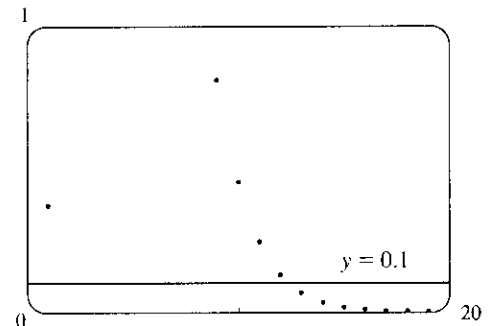
$$\begin{aligned} 10. \lim_{x \rightarrow \infty} \frac{x^4}{e^x} &\stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{4x^3}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{12x^2}{e^x} \\ &\stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{24x}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{24}{e^x} = 0 \end{aligned}$$

Then we conclude from Theorem 12.1.3 that $\lim_{n \rightarrow \infty} n^4 e^{-n} = 0$.

From the graph, it seems that $12^4 e^{-12} > 0.1$, but $n^4 e^{-n} < 0.1$

whenever $n > 12$. So the smallest value of N corresponding to

$\varepsilon = 0.1$ in the definition of the limit is $N = 12$.



11. $\frac{n}{n^3 + 1} < \frac{n}{n^3} = \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$ converges by the Comparison Test with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad (p = 2 > 1).$$

12. Let $a_n = \frac{n^2 + 1}{n^3 + 1}$ and $b_n = \frac{1}{n}$, so $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3 + n}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{1 + 1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ also diverges by the Limit Comparison Test.

13. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right] = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^3 \cdot \frac{1}{5} = \frac{1}{5} < 1$, so $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n = \frac{1}{\sqrt{n+1}}$. Then b_n is positive for $n \geq 1$, the sequence $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$
 converges by the Alternating Series Test.

15. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\begin{aligned} \int_2^\infty f(x) dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x\sqrt{\ln x}} dx \quad \left[\begin{array}{l} u = \ln x, \\ du = \frac{1}{x} dx \end{array} \right] = \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} u^{-1/2} du = \lim_{t \rightarrow \infty} [2\sqrt{u}]_{\ln 2}^{\ln t} \\ &= \lim_{t \rightarrow \infty} (2\sqrt{\ln t} - 2\sqrt{\ln 2}) = \infty, \text{ so the series } \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \text{ diverges.} \end{aligned}$$

16. $\lim_{n \rightarrow \infty} \frac{n}{3n+1} = \frac{1}{3}$, so $\lim_{n \rightarrow \infty} \ln\left(\frac{n}{3n+1}\right) = \ln \frac{1}{3} \neq 0$. Thus, the series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$ diverges by the Test for Divergence.

17. $|a_n| = \left| \frac{\cos 3n}{1 + (1.2)^n} \right| \leq \frac{1}{1 + (1.2)^n} < \frac{1}{(1.2)^n} = \left(\frac{5}{6}\right)^n$, so $\sum_{n=1}^{\infty} |a_n|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$ ($r = \frac{5}{6} < 1$). It follows that $\sum_{n=1}^{\infty} a_n$ converges (by Theorem 3 in Section 12.6).

18. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^{2n}}{(1+2n^2)^n} \right|} = \lim_{n \rightarrow \infty} \frac{n^2}{1+2n^2} = \lim_{n \rightarrow \infty} \frac{1}{1/n^2 + 2} = \frac{1}{2} < 1$, so $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$ converges by the Root Test.

19. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{5^{n+1}(n+1)!} \cdot \frac{5^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \lim_{n \rightarrow \infty} \frac{2n+1}{5(n+1)} = \frac{2}{5} < 1$, so the series converges by the Ratio Test.

20. $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{25}{9}\right)^n$. Now $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{25^{n+1}}{(n+1)^2 \cdot 9^{n+1}} \cdot \frac{n^2 \cdot 9^n}{25^n} = \lim_{n \rightarrow \infty} \frac{25n^2}{9(n+1)^2} = \frac{25}{9} > 1$, so the series diverges by the Ratio Test.

21. $b_n = \frac{\sqrt{n}}{n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$ converges by the Alternating Series Test.

22. Use the Limit Comparison Test with $a_n = \frac{\sqrt{n+1} - \sqrt{n-1}}{n} = \frac{2}{n(\sqrt{n+1} + \sqrt{n-1})}$ (rationalizing the numerator) and $b_n = \frac{1}{n^{3/2}}$. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2\sqrt{n}}{\sqrt{n+1} + \sqrt{n-1}} = 1$, so since $\sum_{n=1}^{\infty} b_n$ converges ($p = \frac{3}{2} > 1$), $\sum_{n=1}^{\infty} a_n$ converges also.

23. Consider the series of absolute values: $\sum_{n=1}^{\infty} n^{-1/3}$ is a p -series with $p = \frac{1}{3} \leq 1$ and is therefore divergent. But if we apply the Alternating Series Test, we see that $b_n = \frac{1}{\sqrt[3]{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$ converges. Thus, $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$ is conditionally convergent.

24. $\sum_{n=1}^{\infty} |(-1)^{n-1} n^{-3}| = \sum_{n=1}^{\infty} n^{-3}$ is a convergent p -series ($p = 3 > 1$). Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$ is absolutely convergent.

$$25. \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} (n+2) 3^{n+1}}{2^{2n+3}} \cdot \frac{2^{2n+1}}{(-1)^n (n+1) 3^n} \right| = \frac{n+2}{n+1} \cdot \frac{3}{4} = \frac{1+(2/n)}{1+(1/n)} \cdot \frac{3}{4} \rightarrow \frac{3}{4} < 1 \text{ as } n \rightarrow \infty, \text{ so}$$

by the Ratio Test, $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1) 3^n}{2^{2n+1}}$ is absolutely convergent.

$$26. \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/(2\sqrt{x})}{1/x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{2} = \infty. \text{ Therefore, } \lim_{n \rightarrow \infty} \frac{(-1)^n \sqrt{n}}{\ln n} \neq 0, \text{ so the given series is divergent by the Test for Divergence.}$$

$$27. \frac{2^{2n+1}}{5^n} = \frac{2^{2n} \cdot 2^1}{5^n} = \frac{(2^2)^n \cdot 2}{5^n} = 2 \left(\frac{4}{5} \right)^n, \text{ so } \sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n} = 2 \sum_{n=1}^{\infty} \left(\frac{4}{5} \right)^n \text{ is a geometric series with } a = \frac{8}{5} \text{ and } r = \frac{4}{5}. \text{ Since } |r| = \frac{4}{5} < 1, \text{ the series converges to } \frac{a}{1-r} = \frac{8/5}{1-4/5} = \frac{8/5}{1/5} = 8.$$

$$28. \sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \left[\frac{1}{3n} - \frac{1}{3(n+3)} \right] \text{ (partial fractions).}$$

$$s_n = \sum_{i=1}^n \left[\frac{1}{3i} - \frac{1}{3(i+3)} \right] = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - \frac{1}{3(n+1)} - \frac{1}{3(n+2)} - \frac{1}{3(n+3)} \text{ (telescoping sum), so}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \lim_{n \rightarrow \infty} s_n = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}.$$

$$29. \sum_{n=1}^{\infty} [\tan^{-1}(n+1) - \tan^{-1} n] = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} [(\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + \cdots + (\tan^{-1}(n+1) - \tan^{-1} n)] \\ = \lim_{n \rightarrow \infty} [\tan^{-1}(n+1) - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$30. \sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n} (2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \cdot \frac{\pi^n}{3^{2n}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \cdot \left(\frac{\sqrt{\pi}}{3} \right)^{2n} = \cos \left(\frac{\sqrt{\pi}}{3} \right) \text{ since}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x.$$

$$31. 1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{e^n}{n!} = \sum_{n=0}^{\infty} \frac{(-e)^n}{n!} = e^{-e} \text{ since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x.$$

$$32. 4.17\overline{326} = 4.17 + \frac{326}{10^5} + \frac{326}{10^8} + \cdots = 4.17 + \frac{326/10^5}{1 - 1/10^3} = \frac{417}{100} + \frac{326}{99,900} = \frac{416,909}{99,900}$$

$$33. \cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) \\ = \frac{1}{2} \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \right) \right] \\ = \frac{1}{2} \left(2 + 2 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^4}{4!} + \cdots \right) \\ = 1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} \frac{x^{2n}}{(2n)!} \\ \geq 1 + \frac{1}{2} x^2 \text{ for all } x$$

34. $\sum_{n=1}^{\infty} (\ln x)^n$ is a geometric series which converges whenever $|\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e$.

35. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16,807} - \frac{1}{32,768} + \cdots$

Since $b_8 = \frac{1}{8^5} = \frac{1}{32,768} < 0.000031$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx \sum_{n=1}^7 \frac{(-1)^{n+1}}{n^5} \approx 0.9721$.

36. (a) $s_5 = \sum_{n=1}^5 \frac{1}{n^6} = 1 + \frac{1}{2^6} + \cdots + \frac{1}{5^6} \approx 1.017305$. The series $\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges by the Integral Test, so we estimate the remainder R_5 with (12.3.2): $R_5 \leq \int_5^{\infty} \frac{dx}{x^6} = \left[-\frac{x^{-5}}{5} \right]_5^{\infty} = \frac{5^{-5}}{5} = 0.000064$. So the error is at most 0.000064.

(b) In general, $R_n \leq \int_n^{\infty} \frac{dx}{x^6} = \frac{1}{5n^5}$. If we take $n = 9$, then $s_9 \approx 1.01734$ and $R_9 \leq \frac{1}{5 \cdot 9^5} \approx 3.4 \times 10^{-6}$. So to five decimal places, $\sum_{n=1}^{\infty} \frac{1}{n^5} \approx \sum_{n=1}^9 \frac{1}{n^5} \approx 1.01734$.

Another Method: Use (12.3.3) instead of (12.3.2).

37. $\sum_{n=1}^{\infty} \frac{1}{2+5^n} \approx \sum_{n=1}^8 \frac{1}{2+5^n} \approx 0.18976224$. To estimate the error, note that $\frac{1}{2+5^n} < \frac{1}{5^n}$, so the remainder term is

$R_8 = \sum_{n=9}^{\infty} \frac{1}{2+5^n} < \sum_{n=9}^{\infty} \frac{1}{5^n} = \frac{1/5^9}{1-1/5} = 6.4 \times 10^{-7}$ (geometric series with $a = \frac{1}{5^9}$ and $r = \frac{1}{5}$).

38. (a) $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{[2(n+1)]!} \cdot \frac{(2n)!}{n^n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^n (n+1)^1}{(2n+2)(2n+1)n^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n \frac{1}{2(2n+1)}$

$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n \frac{1}{2(2n+1)} = e \cdot 0 = 0 < 1$

so the series converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\lim_{n \rightarrow \infty} a_n = 0$ by Theorem 12.2.6.

39. Use the Limit Comparison Test. $\lim_{n \rightarrow \infty} \left| \frac{\left(\frac{n+1}{n}\right)a_n}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 1 > 0$.

Since $\sum |a_n|$ is convergent, so is $\sum \left| \left(\frac{n+1}{n} \right) a_n \right|$, by the Limit Comparison Test.

40. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^2} \frac{|x|}{5} = \frac{|x|}{5}$, so by the Ratio Test,

$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$ converges when $\frac{|x|}{5} < 1 \Leftrightarrow |x| < 5$. $R = 5$. When $x = -5$, the series becomes the

convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $p = 2 > 1$. When $x = 5$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges by the

Alternating Series Test. Thus, $I = [-5, 5]$.

$$41. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{|x+2|^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{|x+2|^n} \right] = \lim_{n \rightarrow \infty} \left[\frac{n}{n+1} \frac{|x+2|}{4} \right] = \frac{|x+2|}{4} < 1 \Leftrightarrow |x+2| < 4,$$

so $R = 4$, $|x+2| < 4 \Leftrightarrow -4 < x+2 < 4 \Leftrightarrow -6 < x < 2$. If $x = -6$, then the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$

becomes $\sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series

Test. When $x = 2$, the series becomes the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Thus, $I = [-6, 2)$.

$$42. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}(x-2)^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{2^n(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{2}{n+3} |x-2| = 0 < 1, \text{ so the series}$$

$\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$ converges for all x . $R = \infty$ and $I = (-\infty, \infty)$.

$$43. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n(x-3)^n} \right| = 2|x-3| \lim_{n \rightarrow \infty} \sqrt{\frac{n+3}{n+4}} = 2|x-3| < 1 \Leftrightarrow$$

$|x-3| < \frac{1}{2}$, so $R = \frac{1}{2}$. $|x-3| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < x-3 < \frac{1}{2} \Leftrightarrow \frac{5}{2} < x < \frac{7}{2}$. For $x = \frac{7}{2}$, the series

$\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$ becomes $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} = \sum_{n=3}^{\infty} \frac{1}{n^{1/2}}$, which diverges ($p = \frac{1}{2} \leq 1$), but for $x = \frac{5}{2}$, we get

$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$, which is a convergent alternating series, so $I = [\frac{5}{2}, \frac{7}{2})$.

$$44. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!x^{n+1}}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!x^n} \right| = \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} |x| = 4|x|.$$

To converge, we must have $4|x| < 1 \Leftrightarrow |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

45.

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{6})$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} \sin x &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{6}\right)}{4!}\left(x - \frac{\pi}{6}\right)^4 + \cdots \\ &= \frac{1}{2} \left[1 - \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{6}\right)^4 - \cdots \right] + \frac{\sqrt{3}}{2} \left[\left(x - \frac{\pi}{6}\right) - \frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 + \cdots \right] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(x - \frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(x - \frac{\pi}{6}\right)^{2n+1} \end{aligned}$$

16.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\cos x &= f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{3}\right)}{4!}\left(x - \frac{\pi}{3}\right)^4 + \cdots \\ &= \frac{1}{2} \left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots \right] + \frac{\sqrt{3}}{2} \left[-\left(x - \frac{\pi}{3}\right) + \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 - \cdots \right] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(x - \frac{\pi}{3}\right)^{2n} + \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)!} \left(x - \frac{\pi}{3}\right)^{2n+1}\end{aligned}$$

$$47. \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \text{ for } |x| < 1 \Rightarrow \frac{x^2}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^{n+2} \text{ with } R = 1.$$

$$48. \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ with interval of convergence } [-1, 1], \text{ so}$$

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}, \text{ which converges when } x^2 \in [-1, 1] \Leftrightarrow x \in [-1, 1]. \text{ Therefore, } R = 1.$$

$$49. \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1 \Rightarrow \ln(1-x) = -\int \frac{dx}{1-x} = -\int \sum_{n=0}^{\infty} x^n dx = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

$$\ln(1-0) = C - 0 \Rightarrow C = 0 \Rightarrow \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{-x^n}{n} \text{ with } R = 1.$$

$$50. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \Rightarrow xe^{2x} = x \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}, R = \infty$$

$$51. \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!} \text{ for all } x, \text{ so the radius of convergence is } \infty.$$

$$52. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow 10^x = e^{(\ln 10)x} = \sum_{n=0}^{\infty} \frac{[(\ln 10)x]^n}{n!} = \sum_{n=0}^{\infty} \frac{(\ln 10)^n x^n}{n!}, R = \infty$$

$$\begin{aligned}
 53. f(x) &= 1 / \sqrt[4]{16-x} = 1 / \left(\sqrt[4]{16} \sqrt[4]{1 - \frac{1}{16}x} \right) = \frac{1}{2} \left(1 - \frac{1}{16}x \right)^{-1/4} \\
 &= \frac{1}{2} \left[1 + \left(-\frac{1}{4} \right) \left(-\frac{x}{16} \right) + \frac{\left(-\frac{1}{4} \right) \left(-\frac{5}{4} \right)}{2!} \left(-\frac{x}{16} \right)^2 + \frac{\left(-\frac{1}{4} \right) \left(-\frac{5}{4} \right) \left(-\frac{9}{4} \right)}{3!} \left(-\frac{x}{16} \right)^3 + \dots \right] \\
 &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{2 \cdot 4^n \cdot n! \cdot 16^n} x^n = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{2^{6n+1} n!} x^n
 \end{aligned}$$

$$\text{for } \left| -\frac{x}{16} \right| < 1 \quad \Leftrightarrow \quad |x| < 16, \text{ so } R = 16.$$

$$\begin{aligned}
 54. (1-3x)^{-5} &= \sum_{n=0}^{\infty} \binom{-5}{n} (-3x)^n = 1 + (-5)(-3x) + \frac{(-5)(-6)}{2!} (-3x)^2 + \frac{(-5)(-6)(-7)}{3!} (-3x)^3 + \dots \\
 &= 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 6 \cdot 7 \cdots (n+4) \cdot 3^n x^n}{n!} \quad \text{for } |-3x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{3}, \text{ so } R = \frac{1}{3}.
 \end{aligned}$$

$$55. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ so } \frac{e^x}{x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \text{ and } \int \frac{e^x}{x} dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}.$$

$$\begin{aligned}
 56. (1+x^4)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} (x^4)^n = 1 + \left(\frac{1}{2} \right) x^4 + \frac{\left(\frac{1}{2} \right) \left(-\frac{1}{2} \right)}{2!} (x^4)^2 + \frac{\left(\frac{1}{2} \right) \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right)}{3!} (x^4)^3 + \dots \\
 &= 1 + \frac{1}{2} x^4 - \frac{1}{8} x^8 + \frac{1}{16} x^{12} - \dots
 \end{aligned}$$

$$\text{so } \int_0^1 (1+x^4)^{1/2} dx = \left[x + \frac{1}{10} x^5 - \frac{1}{72} x^9 + \frac{1}{208} x^{13} - \dots \right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \dots$$

This is an alternating series, so by the Alternating Series Test, the error in the approximation

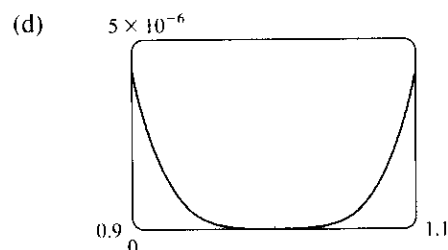
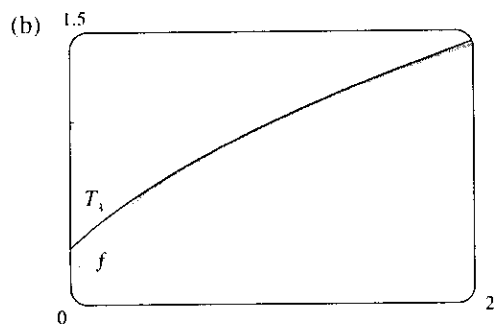
$$\int_0^1 (1+x^4)^{1/2} dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086 \text{ is less than } \frac{1}{208}, \text{ sufficient for the desired accuracy.}$$

Thus, correct to two decimal places, $\int_0^1 (1+x^4)^{1/2} dx \approx 1.09$.

57. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
 \sqrt{x} &\approx T_3(x) = 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 \\
 &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3
 \end{aligned}$$



(c) $|R_3(x)| \leq \frac{M}{4!} |x-1|^4$, where $|f^{(4)}(x)| \leq M$ with

$$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}. \text{ Now } 0.9 \leq x \leq 1.1 \Rightarrow$$

$$-0.1 \leq x-1 \leq 0.1 \Rightarrow (x-1)^4 \leq (0.1)^4,$$

and letting $x = 0.9$ gives $M = \frac{15}{16(0.9)^{7/2}}$, so

$$|R_3(x)| \leq \frac{15}{16(0.9)^{7/2}4!} (0.1)^4 \approx 0.000005648.$$

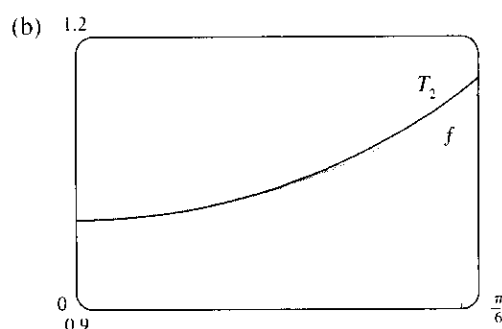
From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears

that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

58. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x \tan x$	0
2	$\sec x \tan^2 x + \sec^3 x$	1
3	$\sec x \tan^3 x + 5 \sec^3 x \tan x$	0
$\vdots$	$\vdots$	$\vdots$

$$\sec x \approx T_2(x) = 1 + \frac{1}{2}x^2$$



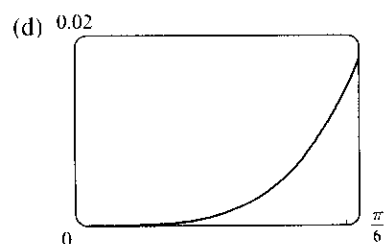
(c) $|R_2(x)| \leq \frac{M}{3!} |x|^3$, where $|f^{(3)}(x)| \leq M$ with

$$f^{(3)}(x) = \sec x \tan^3 x + 5 \sec^3 x \tan x. \text{ Now}$$

$$0 \leq x \leq \frac{\pi}{6} \Rightarrow x^3 \leq \left(\frac{\pi}{6}\right)^3, \text{ and letting } x = \frac{\pi}{6}$$

gives $M = \frac{14}{3}$, so

$$|R_2(x)| \leq \frac{14}{3 \cdot 6} \left(\frac{\pi}{6}\right)^3 \approx 0.111648.$$



From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the error is less than 0.02 on $[0, \frac{\pi}{6}]$.

59. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$, so $\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ and

$$\frac{\sin x - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \dots. \text{ Thus, } \lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \left(-\frac{1}{6} + \frac{x^2}{120} - \frac{x^4}{5040} + \dots \right) = -\frac{1}{6}.$$

60. (a) $F = \frac{mgR^2}{(R+h)^2} = \frac{mg}{(1+h/R)^2} = mg \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{h}{R}\right)^n$ (Binomial Series)

(b) We expand $F = mg [1 - 2(h/R) + 3(h/R)^2 - \dots]$.

This is an alternating series, so by the Alternating Series

Estimation Theorem, the error in the approximation $F = mg$ is less

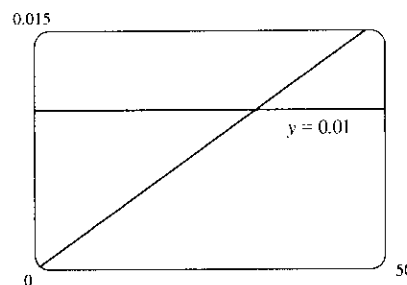
than $2mgh/R$, so for accuracy within 1% we want

$$\left| \frac{2mgh/R}{mgR^2/(R+h)^2} \right| < 0.01 \Leftrightarrow \frac{2h(R+h)^2}{R^3} < 0.01. \text{ This}$$

inequality would be difficult to solve for h , so we substitute

$R = 6,400$ km and plot both sides of the inequality. It appears that

the approximation is accurate to within 1% for $h < 31$ km.



61. $f(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow f(-x) = \sum_{n=0}^{\infty} c_n (-x)^n = \sum_{n=0}^{\infty} (-1)^n c_n x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \Rightarrow \sum_{n=0}^{\infty} (-1)^n c_n x^n = \sum_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 12.10.5), so $(-1)^n c_n = -c_n$. If n is even, then

$$(-1)^n = 1, \text{ so } c_n = -c_n \Rightarrow 2c_n = 0 \Rightarrow c_n = 0. \text{ Thus, all even coefficients are 0, that is,}$$

$$c_0 = c_2 = c_4 = \dots = 0.$$

(b) If f is even, then $f(-x) = f(x) \Rightarrow \sum_{n=0}^{\infty} (-1)^n c_n x^n = \sum_{n=0}^{\infty} c_n x^n \Rightarrow (-1)^n c_n = c_n$.

If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \Rightarrow 2c_n = 0 \Rightarrow c_n = 0$. Thus, all odd coefficients are 0, that is, $c_1 = c_3 = c_5 = \dots = 0$.

62. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f(x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$. By Theorem 12.10.6 with $a = 0$,

we also have $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. Comparing coefficients for $k = 2n$, we have $\frac{f^{(2n)}(0)}{(2n)!} = \frac{1}{n!} \Rightarrow$

$$f^{(2n)}(0) = \frac{(2n)!}{n!}.$$

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PROBLEMS PLUS

1. It would be far too much work to compute 15 derivatives of f . The key idea is to remember that $f^{(n)}(0)$ occurs in the coefficient of x^n in the Maclaurin series of f . We start with the Maclaurin series for $\sin$:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$. Then $\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \dots$, and so the coefficient of x^{15} is

$$\frac{f^{(15)}(0)}{15!} = \frac{1}{5!}. \text{ Therefore, } f^{(15)}(0) = \frac{15!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 = 10,897,286,400.$$

2. We use the problem-solving strategy of taking cases:

Case (i): If $|x| < 1$, then $0 \leq x^2 < 1$, so $\lim_{n \rightarrow \infty} x^{2n} = 0$ (see Example 9 in Section 12.1)

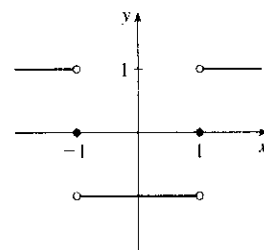
$$\text{and } f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n} - 1}{x^{2n} + 1} = \frac{0 - 1}{0 + 1} = -1.$$

Case (ii): If $|x| = 1$, that is, $x = \pm 1$, then $x^2 = 1$, so $f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n} - 1}{x^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{1 - 1}{1 + 1} = 0$.

Case (iii): If $|x| > 1$, then $x^2 > 1$, so $\lim_{n \rightarrow \infty} x^{2n} = \infty$ and

$$f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n} - 1}{x^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{1 - (1/x^{2n})}{1 + (1/x^{2n})} = \frac{1 - 0}{1 + 0} = 1.$$

$$\text{Thus, } f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ -1 & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$$



The graph shows that f is continuous everywhere except at $x = \pm 1$.

3. (a) From Formula 14a in Appendix D, with $x = y = \theta$, we get $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$, so $\cot 2\theta = \frac{1 - \tan^2 \theta}{2 \tan \theta} \Rightarrow$

$$2 \cot 2\theta = \frac{1 - \tan^2 \theta}{\tan \theta} = \cot \theta - \tan \theta. \text{ Replacing } \theta \text{ by } \frac{1}{2}x, \text{ we get } 2 \cot x = \cot \frac{1}{2}x - \tan \frac{1}{2}x, \text{ or}$$

$$\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2 \cot x$$

- (b) From part (a) with $\frac{x}{2^{n-1}}$ in place of x , $\tan \frac{x}{2^n} = \cot \frac{x}{2^n} - 2 \cot \frac{x}{2^{n-1}}$, so the n th partial sum of

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} \text{ is}$$

$$\begin{aligned} s_n &= \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \dots + \frac{\tan(x/2^n)}{2^n} \\ &= \left[\frac{\cot(x/2)}{2} - \cot x \right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2} \right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4} \right] + \dots \\ &\quad + \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot(x/2^{n-1})}{2^{n-1}} \right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \quad (\text{telescoping sum}) \end{aligned}$$

$$\text{Now } \frac{\cot(x/2^n)}{2^n} = \frac{\cos(x/2^n)}{2^n \sin(x/2^n)} = \frac{\cos(x/2^n)}{x} \cdot \frac{x/2^n}{\sin(x/2^n)} \rightarrow \frac{1}{x} \cdot 1 = \frac{1}{x} \text{ as } n \rightarrow \infty \text{ since } x/2^n \rightarrow 0$$

for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq k\pi$ where k is any integer, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(-\cot x + \frac{1}{2^n} \cot \frac{x}{2^n} \right) = -\cot x + \frac{1}{x}$$

If $x = 0$, then all terms in the series are 0, so the sum is 0.

4. $|AP_2|^2 = 2$, $|AP_3|^2 = 2 + 2^2$, $|AP_4|^2 = 2 + 2^2 + (2^2)^2$, $|AP_5|^2 = 2 + 2^2 + (2^2)^2 + (2^3)^2$, ... ,

$$\begin{aligned} |AP_n|^2 &= 2 + 2^2 + (2^2)^2 + \cdots + (2^{n-2})^2 \quad [\text{for } n \geq 3] = 2 + (4 + 4^2 + 4^3 + \cdots + 4^{n-2}) \\ &= 2 + \frac{4(4^{n-2} - 1)}{4 - 1} \quad [\text{finite geometric sum with } a = 4, r = 4] = \frac{6}{3} + \frac{4^{n-1} - 4}{3} = \frac{2}{3} + \frac{4^{n-1}}{3} \end{aligned}$$

$$\text{So } \tan \angle P_n A P_{n+1} = \frac{|P_n P_{n+1}|}{|AP_n|} = \frac{2^{n-1}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \frac{\sqrt{4^{n-1}}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \frac{1}{\sqrt{\frac{2}{3 \cdot 4^{n-1}} + \frac{1}{3}}} \rightarrow \sqrt{3} \text{ as } n \rightarrow \infty.$$

Thus, $\angle P_n A P_{n+1} \rightarrow \frac{\pi}{3}$ as $n \rightarrow \infty$.

5. (a) At each stage, each side is replaced by four shorter sides, each of

length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and ℓ_0

for the number of sides and the length of the side of the initial triangle, we generate the table at right. In general, we have

$s_n = 3 \cdot 4^n$ and $\ell_n = (\frac{1}{3})^n$, so the length of the perimeter at the n th stage of construction is $p_n = s_n \ell_n = 3 \cdot 4^n \cdot (\frac{1}{3})^n = 3 \cdot (\frac{4}{3})^n$.

$s_0 = 3$	$\ell_0 = 1$
$s_1 = 3 \cdot 4$	$\ell_1 = 1/3$
$s_2 = 3 \cdot 4^2$	$\ell_2 = 1/3^2$
$s_3 = 3 \cdot 4^3$	$\ell_3 = 1/3^3$
...	...

(b) $p_n = \frac{4^n}{3^{n-1}} = 4 \left(\frac{4}{3} \right)^{n-1}$. Since $\frac{4}{3} > 1$, $p_n \rightarrow \infty$ as $n \rightarrow \infty$.

- (c) The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage. Let a be the area of the original triangle. Then the area a_n of each of the small triangles added at stage n is $a_n = a \cdot \frac{1}{9^n} = \frac{a}{9^n}$. Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at the n th stage is

$$A_n = s_{n-1} \cdot a_n = 3 \cdot 4^{n-1} \cdot \frac{a}{9^n} = a \cdot \frac{4^{n-1}}{3^{2n-1}}. \text{ Then the total area enclosed by the snowflake curve is}$$

$$\begin{aligned} A &= a + A_1 + A_2 + A_3 + \cdots = a + a \cdot \frac{1}{3} + a \cdot \frac{4}{3^3} + a \cdot \frac{4^2}{3^5} + a \cdot \frac{4^3}{3^7} + \cdots. \text{ After the first term, this is a} \\ &\text{geometric series with common ratio } \frac{4}{9}, \text{ so } A = a + \frac{a/3}{1 - \frac{4}{9}} = a + \frac{a}{3} \cdot \frac{9}{5} = \frac{8a}{5}. \text{ But the area of the original} \\ &\text{equilateral triangle with side 1 is } a = \frac{1}{2} \cdot 1 \cdot \sin \frac{\pi}{3} = \frac{\sqrt{3}}{4}. \text{ So the area enclosed by the snowflake curve is} \\ &\frac{8}{5} \cdot \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}. \end{aligned}$$

6. Let the series $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$. Then every term in S is of the form $\frac{1}{2^m 3^n}$, $m, n \geq 0$, and furthermore each term occurs only once. So we can write

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m 3^n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m} \frac{1}{3^n} = \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} = 2 \cdot \frac{3}{2} = 3$$

7. (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 14b in Appendix D,

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x) \tan(\arctan y)} = \frac{x - y}{1 + xy}.$$

Now $\arctan x - \arctan y = a - b = \arctan(\tan(a - b)) = \arctan \frac{x - y}{1 + xy}$ since $-\frac{\pi}{2} < a - b < \frac{\pi}{2}$.

- (b) From part (a) we have

$$\arctan \frac{120}{119} - \arctan \frac{1}{239} = \arctan \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \arctan \frac{\frac{28,561}{28,441}}{\frac{28,561}{28,441}} = \arctan 1 = \frac{\pi}{4}.$$

- (c) Replacing y by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy}$. So

$$\begin{aligned} 4 \arctan \frac{1}{5} &= 2 \left(\arctan \frac{1}{5} + \arctan \frac{1}{5} \right) = 2 \arctan \frac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{5} \cdot \frac{1}{5}} = 2 \arctan \frac{5}{12} = \arctan \frac{5}{12} + \arctan \frac{5}{12} \\ &= \arctan \frac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{5}{12} \cdot \frac{5}{12}} = \arctan \frac{120}{119} \end{aligned}$$

Thus, from part (b), we have $4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

- (d) From Example 7 in Section 12.9 we have $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$, so

$$\arctan \frac{1}{5} = \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} - \frac{1}{11 \cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between s_5 and s_6 , that is, $0.197395560 < \arctan \frac{1}{5} < 0.197395562$.

- (e) From the series in part (d) we get $\arctan \frac{1}{239} = \frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \cdots$. The third term is less than

2.6×10^{-13} , so by the Alternating Series Estimation Theorem, we have, to nine decimal places,

$$\arctan \frac{1}{239} \approx s_2 \approx 0.004184076. \text{ Thus, } 0.004184075 < \arctan \frac{1}{239} < 0.004184077.$$

- (f) From part (c) we have $\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$, so from parts (d) and (e) we have

$$16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \Rightarrow$$

$$3.141592652 < \pi < 3.141592692. \text{ So, to 7 decimal places, } \pi \approx 3.1415927.$$

8. (a) Let $a = \operatorname{arccot} x$ and $b = \operatorname{arccot} y$ where $0 < a - b < \pi$. Then

$$\begin{aligned} \cot(a - b) &= \frac{1}{\tan(a - b)} = \frac{1 + \tan a \tan b}{\tan a - \tan b} = \frac{1 + \frac{1}{\cot a} \cdot \frac{1}{\cot b}}{\frac{1}{\cot a} - \frac{1}{\cot b}} \cdot \frac{\cot a \cot b}{\cot a \cot b} \\ &= \frac{\cot a \cot b + 1}{\cot b - \cot a}, \text{ so} \end{aligned}$$

$$\cot(a - b) = \frac{1 + \cot a \cot b}{\cot b - \cot a} = \frac{1 + \cot(\operatorname{arccot} x) \cot(\operatorname{arccot} y)}{\cot(\operatorname{arccot} y) - \cot(\operatorname{arccot} x)} = \frac{1 + xy}{y - x}.$$

Now $\operatorname{arccot} x - \operatorname{arccot} y = a - b = \operatorname{arccot}(\cot(a - b))$

$$= \operatorname{arccot} \frac{1 + xy}{y - x} \text{ since } 0 < a - b < \pi.$$

(b) Applying the identity in part (a) with $x = n$ and $y = n + 1$, we have

$$\operatorname{arccot}(n^2 + n + 1) = \operatorname{arccot}(1 + n(n + 1)) = \operatorname{arccot} \frac{1 + n(n + 1)}{(n + 1) - n} = \operatorname{arccot} n - \operatorname{arccot}(n + 1)$$

Thus, we have a telescoping series with n th partial sum

$$\begin{aligned} s_n &= [\operatorname{arccot} 0 - \operatorname{arccot} 1] + [\operatorname{arccot} 1 - \operatorname{arccot} 2] + \cdots + [\operatorname{arccot} n - \operatorname{arccot}(n + 1)] \\ &= \operatorname{arccot} 0 - \operatorname{arccot}(n + 1) \end{aligned}$$

$$\text{Thus, } \sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1) = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} [\operatorname{arccot} 0 - \operatorname{arccot}(n + 1)] = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

9. We start with the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \Rightarrow$$

$$\sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2} \text{ for } |x| < 1. \text{ Differentiate again:}$$

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \frac{x^2+x}{(1-x)^3} \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3 x^{n-1} = \frac{d}{dx} \frac{x^2+x}{(1-x)^3} = \frac{(1-x)^3(2x+1) - (x^2+x)3(1-x)^2(-1)}{(1-x)^6} = \frac{x^2+4x+1}{(1-x)^4} \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3 x^n = \frac{x^3+4x^2+x}{(1-x)^4}, |x| < 1. \text{ The radius of convergence is 1 because that is the radius of convergence for}$$

the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3 (\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

10. Let's first try the case $k = 1$: $a_0 + a_1 = 0 \Rightarrow a_1 = -a_0 \Rightarrow$

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_0 \sqrt{n} + a_1 \sqrt{n+1}) &= \lim_{n \rightarrow \infty} (a_0 \sqrt{n} - a_0 \sqrt{n+1}) = a_0 \lim_{n \rightarrow \infty} (\sqrt{n} - \sqrt{n+1}) \frac{\sqrt{n} + \sqrt{n+1}}{\sqrt{n} + \sqrt{n+1}} \\ &= a_0 \lim_{n \rightarrow \infty} \frac{-1}{\sqrt{n} + \sqrt{n+1}} = 0 \end{aligned}$$

In general we have $a_0 + a_1 + \cdots + a_k = 0 \Rightarrow a_k = -a_0 - a_1 - \cdots - a_{k-1} \Rightarrow$

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_0 \sqrt{n} + a_1 \sqrt{n+1} + a_2 \sqrt{n+2} + \cdots + a_k \sqrt{n+k}) \\ &= \lim_{n \rightarrow \infty} (a_0 \sqrt{n} + a_1 \sqrt{n+1} + \cdots + a_{k-1} \sqrt{n+k-1} - a_0 \sqrt{n+k} - a_1 \sqrt{n+k} - \cdots - a_{k-1} \sqrt{n+k}) \\ &= a_0 \lim_{n \rightarrow \infty} (\sqrt{n} - \sqrt{n+k}) + a_1 \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n+k}) + \cdots + a_{k-1} \lim_{n \rightarrow \infty} (\sqrt{n+k-1} - \sqrt{n+k}) \end{aligned}$$

Each of these limits is 0 by the same type of simplification as in the case $k = 1$. So we have

$$\lim_{n \rightarrow \infty} (a_0 \sqrt{n} + a_1 \sqrt{n+1} + a_2 \sqrt{n+2} + \cdots + a_k \sqrt{n+k}) = a_0(0) + a_1(0) + \cdots + a_{k-1}(0) = 0$$

$$\begin{aligned}
 11. \ln\left(1 - \frac{1}{n^2}\right) &= \ln\left(\frac{n^2 - 1}{n^2}\right) = \ln\frac{(n+1)(n-1)}{n^2} = \ln[(n+1)(n-1)] - \ln n^2 \\
 &= \ln(n+1) + \ln(n-1) - 2\ln n \\
 &= \ln(n-1) - \ln n - \ln n + \ln(n+1) \\
 &= \ln\frac{n-1}{n} - [\ln n - \ln(n+1)] = \ln\frac{n-1}{n} - \ln\frac{n}{n+1}.
 \end{aligned}$$

Let $s_k = \sum_{n=2}^k \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^k \left(\ln\frac{n-1}{n} - \ln\frac{n}{n+1}\right)$ for $k \geq 2$. Then

$$s_k = \left(\ln\frac{1}{2} - \ln\frac{2}{3}\right) + \left(\ln\frac{2}{3} - \ln\frac{3}{4}\right) + \cdots + \left(\ln\frac{k-1}{k} - \ln\frac{k}{k+1}\right) = \ln\frac{1}{2} - \ln\frac{k}{k+1}, \text{ so}$$

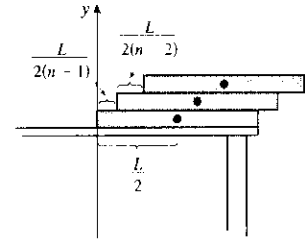
$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \lim_{k \rightarrow \infty} s_k = \lim_{k \rightarrow \infty} \left(\ln\frac{1}{2} - \ln\frac{k}{k+1}\right) = \ln\frac{1}{2} - \ln 1 = \ln 1 - \ln 2 - \ln 1 = -\ln 2.$$

12. Place the y -axis as shown and let the length of each book be L . We want to show that the center of mass of the system of n books lies above the table, that is, $\bar{x} < L$. The x -coordinates of the centers of mass of the books are

$$x_1 = \frac{L}{2}, x_2 = \frac{L}{2(n-1)} + \frac{L}{2}, x_3 = \frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2}, \text{ and so on.}$$

Each book has the same mass m , so if there are n books, then

$$\begin{aligned}
 \bar{x} &= \frac{mx_1 + mx_2 + \cdots + mx_n}{mn} = \frac{x_1 + x_2 + \cdots + x_n}{n} \\
 &= \frac{1}{n} \left[\frac{L}{2} + \left(\frac{L}{2(n-1)} + \frac{L}{2} \right) + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2} \right) + \cdots \right. \\
 &\quad \left. + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \cdots + \frac{L}{4} + \frac{L}{2} + \frac{L}{2} \right) \right] \\
 &= \frac{L}{n} \left[\frac{n-1}{2(n-1)} + \frac{n-2}{2(n-2)} + \cdots + \frac{2}{4} + \frac{1}{2} + \frac{n}{2} \right] = \frac{L}{n} \left[(n-1)\frac{1}{2} + \frac{n}{2} \right] = \frac{2n-1}{2n}L < L
 \end{aligned}$$



This shows that, no matter how many books are added according to the given scheme, the center of mass lies above the table. It remains to observe that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{1}{2} \sum (1/n)$ is divergent (harmonic series), so we can make the top book extend as far as we like beyond the edge of the table if we add enough books.

$$13. u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots, v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots, w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots.$$

Use the Ratio Test to show that the series for u , v , and w have positive radii of convergence (∞ in each case), so Theorem 12.9.2 applies, and hence, we may differentiate each of these series:

$$\frac{du}{dx} = \frac{3x^2}{3!} + \frac{6x^5}{6!} + \frac{9x^8}{9!} + \cdots = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots = w$$

Similarly, $\frac{dw}{dx} = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots = u$, and $\frac{dw}{dx} = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots = v$.

So $u' = w$, $v' = u$, and $w' = v$. Now differentiate the left hand side of the desired equation:

$$\begin{aligned}\frac{d}{dx}(u^3 + v^3 + w^3 - 3uvw) &= 3u^2u' + 3v^2v' + 3w^2w' - 3(u'vw + uv'w + uvw') \\ &= 3u^2w + 3v^2u + 3w^2v - 3(vw^2 + u^2w + uv^2) = 0 \Rightarrow\end{aligned}$$

$u^3 + v^3 + w^3 - 3uvw = C$. To find the value of the constant C , we put $x = 0$ in the last equation and get

$$1^3 + 0^3 + 0^3 - 3(1 \cdot 0 \cdot 0) = C \Rightarrow C = 1, \text{ so } u^3 + v^3 + w^3 - 3uvw = 1.$$

14. First notice that both series are absolutely convergent (p -series with $p > 1$.) Let the given expression be called x .

Then

$$\begin{aligned}x &= \frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \frac{1 + \left(2 \cdot \frac{1}{2^p} - \frac{1}{2^p}\right) + \frac{1}{3^p} + \left(2 \cdot \frac{1}{4^p} - \frac{1}{4^p}\right) + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} \\ &= \frac{\left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots\right) + \left(2 \cdot \frac{1}{2^p} + 2 \cdot \frac{1}{4^p} + 2 \cdot \frac{1}{6^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} \\ &= 1 + \frac{2\left(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \frac{1}{8^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + \frac{\frac{1}{2^{p-1}}\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + 2^{1-p}x\end{aligned}$$

$$\text{Therefore, } x = 1 + 2^{1-p}x \Leftrightarrow x - 2^{1-p}x = 1 \Leftrightarrow x(1 - 2^{1-p}) = 1 \Leftrightarrow x = \frac{1}{1 - 2^{1-p}}.$$

15. If L is the length of a side of the equilateral triangle, then the area is $A = \frac{1}{2}L \cdot \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$ and so $L^2 = \frac{4}{\sqrt{3}}A$.

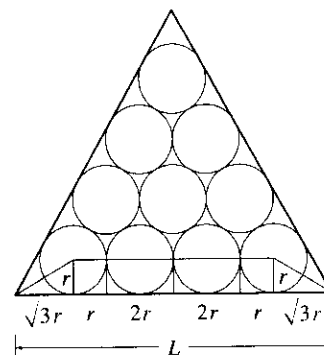
Let r be the radius of one of the circles. When there are n rows of circles, the figure shows that

$$L = \sqrt{3}r + r + (n-2)(2r) + r + \sqrt{3}r = r(2n-2+2\sqrt{3}), \text{ so } r = \frac{L}{2(n+\sqrt{3}-1)}.$$

The number of circles is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, and so the total area of the circles is

$$\begin{aligned}A_n &= \frac{n(n+1)}{2} \pi r^2 = \frac{n(n+1)}{2} \pi \frac{L^2}{4(n+\sqrt{3}-1)^2} = \frac{n(n+1)}{2} \pi \frac{4A/\sqrt{3}}{4(n+\sqrt{3}-1)^2} \\ &= \frac{n(n+1)}{(n+\sqrt{3}-1)^2} \frac{\pi A}{2\sqrt{3}} \Rightarrow\end{aligned}$$

$$\begin{aligned}\frac{A_n}{A} &= \frac{n(n+1)}{(n+\sqrt{3}-1)^2} \frac{\pi}{2\sqrt{3}} \\ &= \frac{1+1/n}{[1+(\sqrt{3}-1)/n]^2} \frac{\pi}{2\sqrt{3}} \rightarrow \frac{\pi}{2\sqrt{3}} \text{ as } n \rightarrow \infty\end{aligned}$$



16. Given $a_0 = a_1 = 1$ and $a_n = \frac{(n-1)(n-2)a_{n-1} - (n-3)a_{n-2}}{n(n-1)}$, we calculate the next few terms of the

$$\text{sequence: } a_2 = \frac{1 \cdot 0 \cdot a_1 - (-1)a_0}{2 \cdot 1} = \frac{1}{2}, a_3 = \frac{2 \cdot 1 \cdot a_2 - 0 \cdot a_1}{3 \cdot 2} = \frac{1}{6}, a_4 = \frac{3 \cdot 2 \cdot a_3 - 1 \cdot a_2}{4 \cdot 3} = \frac{1}{24}.$$

It seems that $a_n = \frac{1}{n!}$, so we try to prove this by induction. The first step is done, so assume $a_k = \frac{1}{k!}$ and

$a_{k-1} = \frac{1}{(k-1)!}$. Then

$$a_{k+1} = \frac{k(k-1)a_k - (k-2)a_{k-1}}{(k+1)k} = \frac{\frac{k(k-1)}{k!} - \frac{k-2}{(k-1)!}}{(k+1)k} = \frac{(k-1) - (k-2)}{[(k+1)(k)](k-1)!} = \frac{1}{(k+1)!}$$

and the induction is complete. Therefore, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 1/n! = e$.

17. As in Section 12.9 we have to integrate the function x^x by integrating series. Writing $x^x = (e^{\ln x})^x = e^{x \ln x}$ and

$$\text{using the Maclaurin series for } e^x, \text{ we have } x^x = (e^{\ln x})^x = e^{x \ln x} = \sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n (\ln x)^n}{n!}.$$

As with power series, we can integrate this series term-by-term:

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^n (\ln x)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\ln x)^n dx$$

We integrate by parts with $u = (\ln x)^n$, $dv = x^n dx$, so $du = \frac{n(\ln x)^{n-1}}{x} dx$ and $v = \frac{x^{n+1}}{n+1}$:

$$\begin{aligned}\int_0^1 x^n (\ln x)^n dx &= \lim_{t \rightarrow 0^+} \int_t^1 x^n (\ln x)^n dx = \lim_{t \rightarrow 0^+} \left[\frac{x^{n+1}}{n+1} (\ln x)^n \right]_t^1 - \lim_{t \rightarrow 0^+} \int_t^1 \frac{n}{n+1} x^n (\ln x)^{n-1} dx \\ &= 0 - \frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} dx\end{aligned}$$

(where l'Hospital's Rule was used to help evaluate the first limit).

Further integration by parts gives $\int_0^1 x^n (\ln x)^k dx = -\frac{k}{n+1} \int_0^1 x^n (\ln x)^{k-1} dx$ and, combining

$$\text{these steps, we get } \int_0^1 x^n (\ln x)^n dx = \frac{(-1)^n n!}{(n+1)^{n+1}} \int_0^1 x^n dx = \frac{(-1)^n n!}{(n+1)^{n+1}} \Rightarrow$$

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\ln x)^n dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}.$$

8. (a) Since P_n is defined as the midpoint of $P_{n-4}P_{n-3}$, $x_n = \frac{1}{2}(x_{n-4} + x_{n-3})$ for $n \geq 5$. So we prove by induction that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$. The case $n = 1$ is immediate, since $\frac{1}{2} \cdot 0 + 1 + 1 + 0 = 2$. Assume that the result holds for $n = k - 1$, that is, $\frac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2$. Then for $n = k$,

$$\begin{aligned}\frac{1}{2}x_k + x_{k+1} + x_{k+2} + x_{k+3} &= \frac{1}{2}x_k + x_{k+1} + x_{k+2} + \frac{1}{2}(x_{k+3-4} + x_{k+3-3}) \quad (\text{by above}) \\ &= \frac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2 \quad (\text{by the induction hypothesis})\end{aligned}$$

Similarly, for $n \geq 5$, $y_n = \frac{1}{2}(y_{n-4} + y_{n-3})$, so the same argument as above holds for y , with 2 replaced by

$$\frac{1}{2}y_1 + y_2 + y_3 + y_4 = \frac{1}{2} \cdot 1 + 1 + 0 + 0 = \frac{3}{2}. \text{ So } \frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2} \text{ for all } n.$$

- (b) $\lim_{n \rightarrow \infty} (\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}) = \frac{1}{2} \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} x_{n+1} + \lim_{n \rightarrow \infty} x_{n+2} + \lim_{n \rightarrow \infty} x_{n+3} = 2$. Since all the limits on the left hand side are the same, we get $\frac{7}{2} \lim_{n \rightarrow \infty} x_n = 2 \Rightarrow \lim_{n \rightarrow \infty} x_n = \frac{4}{7}$. In the same way,

$$\frac{7}{2} \lim_{n \rightarrow \infty} y_n = \frac{3}{2} \Rightarrow \lim_{n \rightarrow \infty} y_n = \frac{3}{7}, \text{ so } P = \left(\frac{4}{7}, \frac{3}{7}\right).$$

19. Let $f(x) = \sum_{m=0}^{\infty} c_m x^m$ and $g(x) = e^{f(x)} = \sum_{n=0}^{\infty} d_n x^n$. Then $g'(x) = \sum_{n=0}^{\infty} n d_n x^{n-1}$, so $n d_n$ occurs as the coefficient of x^{n-1} . But also

$$\begin{aligned}g'(x) &= e^{f(x)} f'(x) = \left(\sum_{n=0}^{\infty} d_n x^n\right) \left(\sum_{m=1}^{\infty} m c_m x^{m-1}\right) \\ &= (d_0 + d_1 x + d_2 x^2 + \cdots + d_{n-1} x^{n-1} + \cdots) (c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1} + \cdots)\end{aligned}$$

so the coefficient of x^{n-1} is $c_1 d_{n-1} + 2c_2 d_{n-2} + 3c_3 d_{n-3} + \cdots + n c_n d_0 = \sum_{i=1}^n i c_i d_{n-i}$. Therefore, $n d_n = \sum_{i=1}^n i c_i d_{n-i}$.

20. Suppose the base of the first right triangle has length a . Then by repeated use of the Pythagorean theorem, we find that the base of the second right triangle has length $\sqrt{1+a^2}$, the base of the third right triangle has length $\sqrt{2+a^2}$, and in general, the n th right triangle has base of length $\sqrt{n-1+a^2}$ and hypotenuse of length $\sqrt{n+a^2}$. Thus, $\theta_n = \tan^{-1}(1/\sqrt{n-1+a^2})$ and $\sum_{n=1}^{\infty} \theta_n = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{n-1+a^2}}\right) = \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{n+a^2}}\right)$. We wish to show that this series diverges.

First notice that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+a^2}}$ diverges by the Limit Comparison Test with the divergent

p -series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ($p = \frac{1}{2} \leq 1$) since

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n+a^2}}{1/\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+a^2}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+a^2}} = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+a^2/n}} = 1 > 0. \text{ Thus, } \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+a^2}} \text{ also}$$

diverges. Now $\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{n+a^2}}\right)$ diverges by the Limit Comparison Test with $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+a^2}}$ since

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\tan^{-1}(1/\sqrt{n+a^2})}{1/\sqrt{n+a^2}} &= \lim_{x \rightarrow \infty} \frac{\tan^{-1}(1/\sqrt{x+a^2})}{1/\sqrt{x+a^2}} = \lim_{y \rightarrow \infty} \frac{\tan^{-1}(1/y)}{1/y} \quad [y = \sqrt{x+a^2}] \\ &= \lim_{z \rightarrow 0^+} \frac{\tan^{-1} z}{z} \quad [z = 1/y] \stackrel{H}{=} \lim_{z \rightarrow 0^+} \frac{1/(1+z^2)}{1} = 1 > 0\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} \theta_n$ is a divergent series.

21. Call the series S . We group the terms according to the number of digits in their denominators:

$$S = \underbrace{\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{8} + \frac{1}{9}\right)}_{g_1} + \underbrace{\left(\frac{1}{11} + \cdots + \frac{1}{99}\right)}_{g_2} + \underbrace{\left(\frac{1}{111} + \cdots + \frac{1}{999}\right)}_{g_3} + \cdots$$

Now in the group g_n , since we have 9 choices for each of the n digits in the denominator, there are 9^n terms.

Furthermore, each term in g_n is less than $\frac{1}{10^{n-1}}$ [except for the first term in g_1]. So $g_n < 9^n \cdot \frac{1}{10^{n-1}} = 9\left(\frac{9}{10}\right)^{n-1}$.

Now $\sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1}$ is a geometric series with $a = 9$ and $r = \frac{9}{10} < 1$. Therefore, by the Comparison Test,

$$S = \sum_{n=1}^{\infty} g_n < \sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1} = \frac{9}{1 - 9/10} = 90.$$

22. (a) Let $f(x) = \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$. Then

$$x = (1 - x - x^2)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$\begin{aligned}x &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\ &\quad - c_0 x - c_1 x^2 - c_2 x^3 - c_3 x^4 - c_4 x^5 - \cdots \\ &\quad - c_0 x^2 - c_1 x^3 - c_2 x^4 - c_3 x^5 - \cdots\end{aligned}$$

$$x = c_0 + (c_1 - c_0)x + (c_2 - c_1 - c_0)x^2 + (c_3 - c_2 - c_1)x^3 + \cdots$$

Comparing coefficients of powers of x gives us $c_0 = 0$ and

$$c_1 - c_0 = 1 \quad \Rightarrow \quad c_1 = c_0 + 1 = 1$$

$$c_2 - c_1 - c_0 = 0 \quad \Rightarrow \quad c_2 = c_1 + c_0 = 1 + 0 = 1$$

$$c_3 - c_2 - c_1 = 0 \quad \Rightarrow \quad c_3 = c_2 + c_1 = 1 + 1 = 2$$

In general, we have $c_n = c_{n-1} + c_{n-2}$ for $n \geq 3$. Each c_n is equal to the n th Fibonacci number; that is,

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} f_n x^n$$

(b) Completing the square on $x^2 + x - 1$ gives us

$$\begin{aligned}\left(x^2 + x + \frac{1}{4}\right) - 1 - \frac{1}{4} &= \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} = \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 \\ &= \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) = \left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)\end{aligned}$$

So $\frac{x}{1-x-x^2} = \frac{-x}{x^2+x-1} = \frac{-x}{\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)}$. The factors in the denominator are linear,

so the partial fraction decomposition is

$$\frac{-x}{\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)} = \frac{A}{x + \frac{1+\sqrt{5}}{2}} + \frac{B}{x + \frac{1-\sqrt{5}}{2}}$$

$$-x = A\left(x + \frac{1-\sqrt{5}}{2}\right) + B\left(x + \frac{1+\sqrt{5}}{2}\right)$$

If $x = \frac{-1+\sqrt{5}}{2}$, then $-\frac{-1+\sqrt{5}}{2} = B\sqrt{5} \Rightarrow B = \frac{1-\sqrt{5}}{2\sqrt{5}}$.

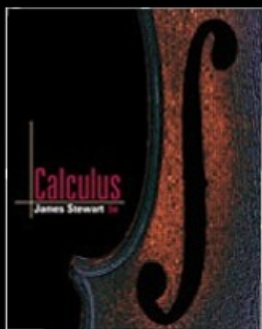
If $x = \frac{-1-\sqrt{5}}{2}$, then $-\frac{-1-\sqrt{5}}{2} = A(-\sqrt{5}) \Rightarrow A = \frac{1+\sqrt{5}}{-2\sqrt{5}}$. Thus,

$$\begin{aligned} \frac{x}{1-x-x^2} &= \frac{\frac{1+\sqrt{5}}{-2\sqrt{5}}}{x + \frac{1+\sqrt{5}}{2}} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{x + \frac{1-\sqrt{5}}{2}} \\ &= \frac{\frac{1+\sqrt{5}}{-2\sqrt{5}} \cdot \frac{2}{1+\sqrt{5}}}{x + \frac{1+\sqrt{5}}{2}} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}} \cdot \frac{2}{1-\sqrt{5}}}{x + \frac{1-\sqrt{5}}{2}} \\ &= \frac{-1/\sqrt{5}}{1 + \frac{2}{1+\sqrt{5}}x} + \frac{1/\sqrt{5}}{1 + \frac{2}{1-\sqrt{5}}x} \\ &= -\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(-\frac{2}{1+\sqrt{5}}x\right)^n + \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(-\frac{2}{1-\sqrt{5}}x\right)^n \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left[\left(\frac{-2}{1-\sqrt{5}}\right)^n - \left(\frac{-2}{1+\sqrt{5}}\right)^n \right] x^n \\ &= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left[\frac{(-2)^n (1+\sqrt{5})^n - (-2)^n (1-\sqrt{5})^n}{(1-\sqrt{5})^n (1+\sqrt{5})^n} \right] x^n \quad [\text{the } n=0 \text{ term is } 0] \\ &= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left[\frac{(-2)^n \left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right)}{(1-5)^n} \right] x^n \\ &= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left[\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n} \right] x^n \quad [(-4)^n = (-2)^n \cdot 2^n] \end{aligned}$$

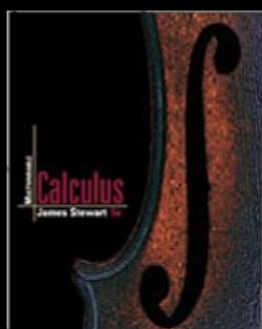
From part (a), this series must equal $\sum_{n=1}^{\infty} f_n x^n$, so $f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}$, which is an explicit formula for the n th Fibonacci number.

Chapter 13

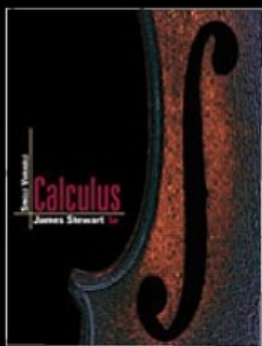
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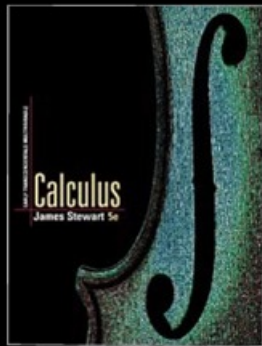
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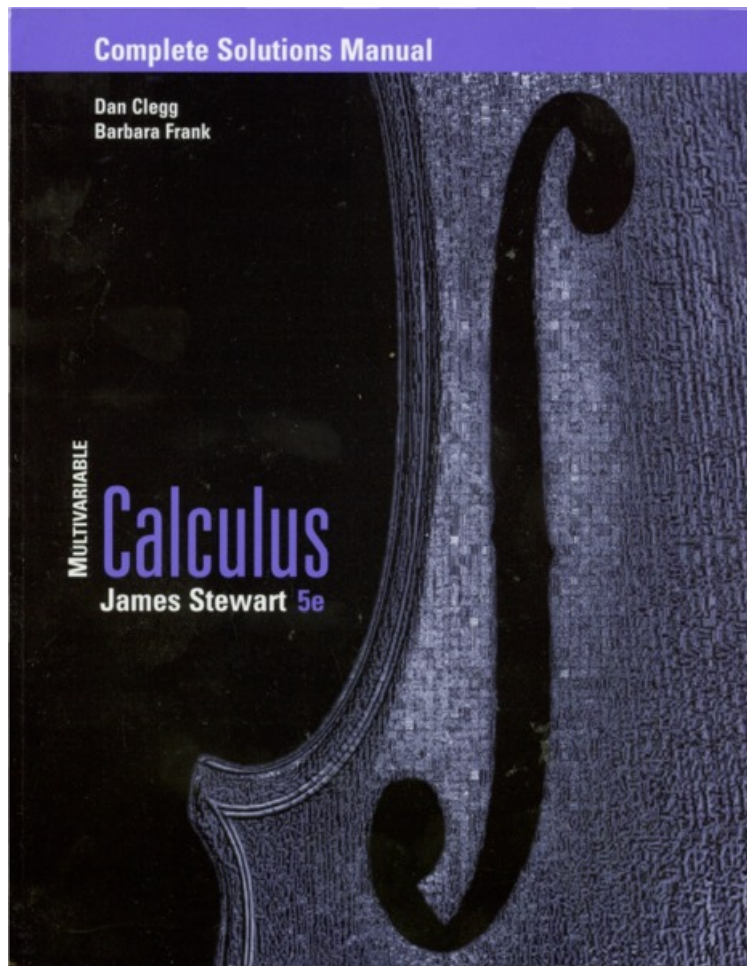
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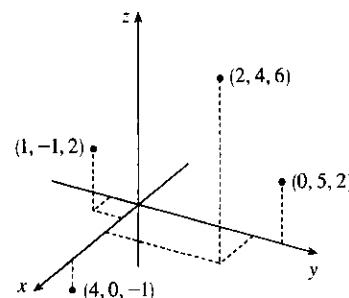
13.1 Three-Dimensional Coordinate Systems

ET 12.1

1. We start at the origin, which has coordinates $(0, 0, 0)$.

First we move 4 units along the positive x -axis, affecting only the x -coordinate, bringing us to the point $(4, 0, 0)$. We then move 3 units straight downward, in the negative z -direction. Thus only the z -coordinate is affected, and we arrive at $(4, 0, -3)$.

2.

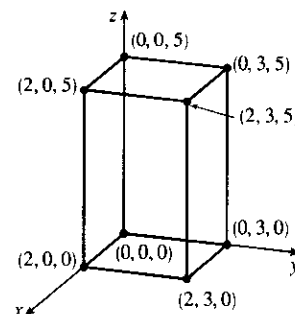


3. The distance from a point to the xz -plane is the absolute value of the y -coordinate of the point. $Q(-5, -1, 4)$ has the y -coordinate with the smallest absolute value, so Q is the point closest to the xz -plane. $R(0, 3, 8)$ must lie in the yz -plane since the distance from R to the yz -plane, given by the x -coordinate of R , is 0.

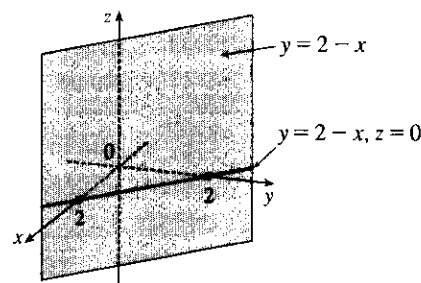
4. The projection of $(2, 3, 5)$ on the xy -plane is $(2, 3, 0)$; on the yz -plane, $(0, 3, 5)$; on the xz -plane, $(2, 0, 5)$.

The length of the diagonal of the box is the distance between the origin and $(2, 3, 5)$, given by

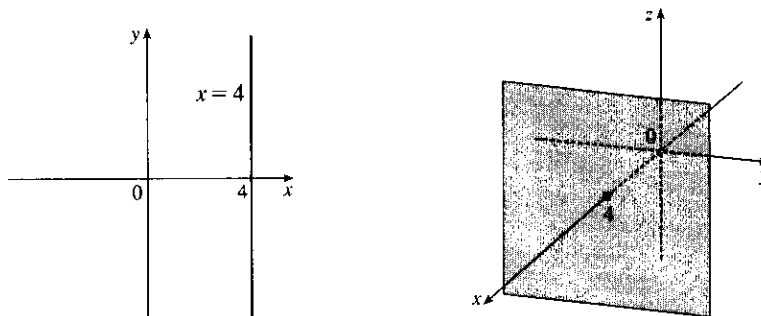
$$\sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} = \sqrt{38} \approx 6.16$$



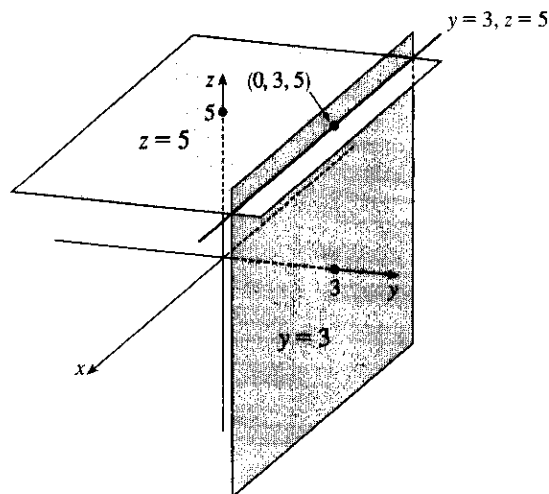
5. The equation $x + y = 2$ represents the set of all points in $\mathbb{R}^3$ whose x - and y -coordinates have a sum of 2, or equivalently where $y = 2 - x$. This is the set $\{(x, 2 - x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ which is a vertical plane that intersects the xy -plane in the line $y = 2 - x, z = 0$.



6. (a) In $\mathbb{R}^2$, the equation $x = 4$ represents a line parallel to the y -axis. In $\mathbb{R}^3$, the equation $x = 4$ represents the set $\{(x, y, z) \mid x = 4\}$, the set of all points whose x -coordinate is 4. This is the vertical plane that is parallel to the yz -plane and 4 units in front of it.



- (b) In $\mathbb{R}^3$, the equation $y = 3$ represents a vertical plane that is parallel to the xz -plane and 3 units to the right of it. The equation $z = 5$ represents a horizontal plane parallel to the xy -plane and 5 units above it. The pair of equations $y = 3, z = 5$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y = 3, z = 5$. This line can also be described as the set $\{(x, 3, 5) \mid x \in \mathbb{R}\}$, which is the set of all points in $\mathbb{R}^3$ whose x -coordinate may vary but whose y - and z -coordinates are fixed at 3 and 5, respectively. Thus the line is parallel to the x -axis and intersects the yz -plane in the point $(0, 3, 5)$.



7. We first find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{[1 - (-2)]^2 + (2 - 4)^2 + (-1 - 0)^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$$

$$|QR| = \sqrt{(-1 - 1)^2 + (1 - 2)^2 + [2 - (-1)]^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|PR| = \sqrt{[-1 - (-2)]^2 + (1 - 4)^2 + (2 - 0)^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$$

Since all three sides have the same length, PQR is an equilateral triangle.

8. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|AB| = \sqrt{(3-1)^2 + (4-2)^2 + [-2-(-3)]^2} = \sqrt{4+4+1} = 3$$

$$|BC| = \sqrt{(3-3)^2 + (-2-4)^2 + [1-(-2)]^2} = \sqrt{0+36+9} = \sqrt{45} = 3\sqrt{5}$$

$$|AC| = \sqrt{(3-1)^2 + (-2-2)^2 + [1-(-3)]^2} = \sqrt{4+16+16} = 6$$

Since the Pythagorean Theorem is satisfied by $|AB|^2 + |AC|^2 = |BC|^2$, ABC is a right triangle. ABC is not isosceles, as no two sides have the same length.

9. (a) First we find the distances between points:

$$|AB| = \sqrt{(7-5)^2 + (9-1)^2 + (-1-3)^2} = \sqrt{84} = 2\sqrt{21}$$

$$|BC| = \sqrt{(1-7)^2 + (-15-9)^2 + [11-(-1)]^2} = \sqrt{756} = 6\sqrt{21}$$

$$|AC| = \sqrt{(1-5)^2 + (-15-1)^2 + (11-3)^2} = \sqrt{336} = 4\sqrt{21}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $|AB| + |AC| = |BC|$, the three points lie on a straight line.

- (b) The distances between points are

$$|KL| = \sqrt{(1-0)^2 + (2-3)^2 + [-2-(-4)]^2} = \sqrt{6}$$

$$|LM| = \sqrt{(3-1)^2 + (0-2)^2 + [1-(-2)]^2} = \sqrt{17}$$

$$|KM| = \sqrt{(3-0)^2 + (0-3)^2 + [1-(-4)]^2} = \sqrt{43}$$

Since $\sqrt{6} + \sqrt{17} \neq \sqrt{43}$, the three points do not lie on a straight line.

10. (a) The distance from a point to the xy -plane is the absolute value of the z -coordinate of the point. Thus, the distance is $|-5| = 5$.
- (b) Similarly, the distance is the absolute value of the x -coordinate of the point: $|3| = 3$.
- (c) The distance is the absolute value of the y -coordinate of the point: $|7| = 7$.
- (d) The point on the x -axis closest to $(3, 7, -5)$ is the point $(3, 0, 0)$. (Approach the x -axis perpendicularly.) The distance from $(3, 7, -5)$ to the x -axis is the distance between these two points:
- $$\sqrt{(3-3)^2 + (7-0)^2 + (-5-0)^2} = \sqrt{74} \approx 8.60.$$
- (e) The point on the y -axis closest to $(3, 7, -5)$ is $(0, 7, 0)$. The distance between these points is
- $$\sqrt{(3-0)^2 + (7-7)^2 + (-5-0)^2} = \sqrt{34} \approx 5.83.$$
- (f) The point on the z -axis closest to $(3, 7, -5)$ is $(0, 0, -5)$. The distance between these points is
- $$\sqrt{(3-0)^2 + (7-0)^2 + [-5-(-5)]^2} = \sqrt{58} \approx 7.62.$$

11. An equation of the sphere with center $(1, -4, 3)$ and radius 5 is $(x - 1)^2 + [y - (-4)]^2 + (z - 3)^2 = 5^2$ or $(x - 1)^2 + (y + 4)^2 + (z - 3)^2 = 25$. The intersection of this sphere with the xz -plane is the set of points on the sphere whose y -coordinate is 0. Putting $y = 0$ into the equation, we have $(x - 1)^2 + 4^2 + (z - 3)^2 = 25$, $y = 0$ or $(x - 1)^2 + (z - 3)^2 = 9$, $y = 0$, which represents a circle in the xz -plane with center $(1, 0, 3)$ and radius 3.
12. An equation of the sphere with center $(6, 5, -2)$ and radius $\sqrt{7}$ is $(x - 6)^2 + (y - 5)^2 + [z - (-2)]^2 = (\sqrt{7})^2$ or $(x - 6)^2 + (y - 5)^2 + (z + 2)^2 = 7$. The intersection of this sphere with the xy -plane is the set of points on the sphere whose z -coordinate is 0. Putting $z = 0$ into the equation, we have $(x - 6)^2 + (y - 5)^2 = 3$, $z = 0$ which represents a circle in the xy -plane with center $(6, 5, 0)$ and radius $\sqrt{3}$. To find the intersection with the xz -plane, we set $y = 0$: $(x - 6)^2 + (z + 2)^2 = -18$. Since no points satisfy this equation, the sphere does not intersect the xz -plane. (Also note that the distance from the center of the sphere to the xz -plane is greater than the radius of the sphere.) Similarly, the sphere does not intersect the yz -plane since substituting $x = 0$ into the equation gives $(y - 5)^2 + (z + 2)^2 = -29$.
13. The radius of the sphere is the distance between $(4, 3, -1)$ and $(3, 8, 1)$:

$$r = \sqrt{(3 - 4)^2 + (8 - 3)^2 + [1 - (-1)]^2} = \sqrt{30}.$$
 Thus, an equation of the sphere is

$$(x - 3)^2 + (y - 8)^2 + (z - 1)^2 = 30.$$
14. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1, 2, 3)$: $r = \sqrt{(1 - 0)^2 + (2 - 0)^2 + (3 - 0)^2} = \sqrt{14}$. Then an equation of the sphere is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14.$$
15. Completing squares in the equation $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$ gives

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) + (z^2 - 2z + 1) = 11 + 9 + 4 + 1 \Rightarrow$$

$$(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 25$$
 which we recognize as an equation of a sphere with center $(3, -2, 1)$ and radius 5.
16. Completing squares in the equation gives

$$(x^2 - 4x + 4) + (y^2 + 2y + 1) + z^2 = 0 + 4 + 1 \Rightarrow (x - 2)^2 + (y + 1)^2 + z^2 = 5$$
 which we recognize as an equation of a sphere with center $(2, -1, 0)$ and radius $\sqrt{5}$.
17. Completing squares in the equation gives $(x^2 - x + \frac{1}{4}) + (y^2 - y + \frac{1}{4}) + (z^2 - z + \frac{1}{4}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \Rightarrow$

$$(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + (z - \frac{1}{2})^2 = \frac{3}{4}$$
 which we recognize as an equation of a sphere with center $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and radius $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.
18. Completing squares in the equation gives $4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4z^2 = 1 + 4 + 16 \Rightarrow$

$$4(x - 1)^2 + 4(y + 2)^2 + 4z^2 = 21 \Rightarrow (x - 1)^2 + (y + 2)^2 + z^2 = \frac{21}{4},$$
 which we recognize as an equation of a sphere with center $(1, -2, 0)$ and radius $\sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$.

19. (a) If the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$Q = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$, then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$.

We verify that this is the case:

$$\begin{aligned} |P_1P_2| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ |P_1Q| &= \sqrt{\left[\frac{1}{2}(x_1 + x_2) - x_1\right]^2 + \left[\frac{1}{2}(y_1 + y_2) - y_1\right]^2 + \left[\frac{1}{2}(z_1 + z_2) - z_1\right]^2} \\ &= \sqrt{\left(\frac{1}{2}x_2 - \frac{1}{2}x_1\right)^2 + \left(\frac{1}{2}y_2 - \frac{1}{2}y_1\right)^2 + \left(\frac{1}{2}z_2 - \frac{1}{2}z_1\right)^2} \\ &= \sqrt{\left(\frac{1}{2}\right)^2 [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]} \\ &= \frac{1}{2} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \frac{1}{2} |P_1P_2| \\ |QP_2| &= \sqrt{\left[x_2 - \frac{1}{2}(x_1 + x_2)\right]^2 + \left[y_2 - \frac{1}{2}(y_1 + y_2)\right]^2 + \left[z_2 - \frac{1}{2}(z_1 + z_2)\right]^2} \\ &= \sqrt{\left(\frac{1}{2}x_2 - \frac{1}{2}x_1\right)^2 + \left(\frac{1}{2}y_2 - \frac{1}{2}y_1\right)^2 + \left(\frac{1}{2}z_2 - \frac{1}{2}z_1\right)^2} \\ &= \sqrt{\left(\frac{1}{2}\right)^2 [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]} \\ &= \frac{1}{2} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \frac{1}{2} |P_1P_2| \end{aligned}$$

So Q is indeed the midpoint of P_1P_2 .

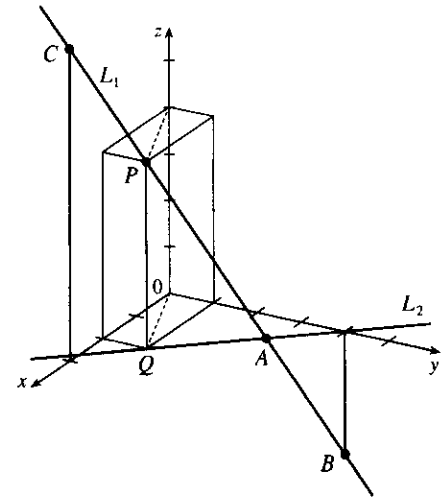
- (b) By part (a), the midpoints of sides AB , BC and CA are $P_1(-\frac{1}{2}, 1, 4)$, $P_2(1, \frac{1}{2}, 5)$ and $P_3(\frac{5}{2}, \frac{3}{2}, 4)$. (Recall that a median of a triangle is a line segment from a vertex to the midpoint of the opposite side.) Then the lengths of the medians are:

$$\begin{aligned} |AP_2| &= \sqrt{0^2 + \left(\frac{1}{2} - 2\right)^2 + (5 - 3)^2} = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2} \\ |BP_3| &= \sqrt{\left(\frac{5}{2} + 2\right)^2 + \left(\frac{3}{2}\right)^2 + (4 - 5)^2} = \sqrt{\frac{81}{4} + \frac{9}{4} + 1} = \sqrt{\frac{94}{4}} = \frac{1}{2}\sqrt{94} \\ |CP_1| &= \sqrt{\left(-\frac{1}{2} - 4\right)^2 + (1 - 1)^2 + (4 - 5)^2} = \sqrt{\frac{81}{4} + 1} = \frac{1}{2}\sqrt{85} \end{aligned}$$

20. By Exercise 19(a), the midpoint of the diameter (and thus the center of the sphere) is $C(3, 2, 7)$. The radius is half the diameter, so $r = \frac{1}{2} \sqrt{(4 - 2)^2 + (3 - 1)^2 + (10 - 4)^2} = \frac{1}{2} \sqrt{44} = \sqrt{11}$. Therefore an equation of the sphere is $(x - 3)^2 + (y - 2)^2 + (z - 7)^2 = 11$.
21. (a) Since the sphere touches the xy -plane, its radius is the distance from its center, $(2, -3, 6)$, to the xy -plane, namely 6. Therefore $r = 6$ and an equation of the sphere is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 6^2 = 36$.
- (b) The radius of this sphere is the distance from its center $(2, -3, 6)$ to the yz -plane, which is 2. Therefore, an equation is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 4$.
- (c) Here the radius is the distance from the center $(2, -3, 6)$ to the xz -plane, which is 3. Therefore, an equation is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 9$.

22. The largest sphere contained in the first octant must have a radius equal to the minimum distance from the center $(5, 4, 9)$ to any of the three coordinate planes. The shortest such distance is to the xz -plane, a distance of 4. Thus an equation of the sphere is $(x - 5)^2 + (y - 4)^2 + (z - 9)^2 = 16$.
23. The equation $y = -4$ represents a plane parallel to the xz -plane and 4 units to the left of it.
24. The equation $x = 10$ represents a plane parallel to the yz -plane and 10 units in front of it.
25. The inequality $x > 3$ represents a half-space consisting of all points in front of the plane $x = 3$.
26. The inequality $y \geq 0$ represents a half-space consisting of all points on or to the right of the xz -plane.
27. The inequality $0 \leq z \leq 6$ represents all points on or between the horizontal planes $z = 0$ (the xy -plane) and $z = 6$.
28. The equation $y = z$ represents a plane perpendicular to the yz -plane and intersecting the yz -plane in the line $y = z$, $x = 0$.
29. The inequality $x^2 + y^2 + z^2 > 1$ is equivalent to $\sqrt{x^2 + y^2 + z^2} > 1$, so the region consists of those points whose distance from the origin is greater than 1. This is the set of all points outside the sphere with radius 1 and center $(0, 0, 0)$.
30. The inequality $1 \leq x^2 + y^2 + z^2 \leq 25$ is equivalent to $1 \leq \sqrt{x^2 + y^2 + z^2} \leq 5$, so the region consists of those points whose distance from the origin is at least 1 and at most 5. This is the set of all points on or between the concentric spheres with radii 1 and 5 and center $(0, 0, 0)$.
31. Completing the square in z gives $x^2 + y^2 + (z^2 - 2z + 1) < 3 + 1$ or $x^2 + y^2 + (z - 1)^2 < 4$, which is equivalent to $\sqrt{x^2 + y^2 + (z - 1)^2} < 2$. Thus the region consists of those points whose distance from the point $(0, 0, 1)$ is less than 2. This is the set of all points inside the sphere with radius 2 and center $(0, 0, 1)$.
32. The equation $x^2 + y^2 = 1$ represents the set of all points in $\mathbb{R}^3$ where $x^2 + y^2 = 1$, a surface that intersects the xy -plane in the circle $x^2 + y^2 = 1$, $z = 0$. Since z can vary, the surface is a circular cylinder of radius 1. Thus, the equation represents the region consisting of all points on a circular cylinder of radius 1 with axis the z -axis.
33. Here $x^2 + z^2 \leq 9$ or equivalently $\sqrt{x^2 + z^2} \leq 3$ which describes the set of all points in $\mathbb{R}^3$ whose distance from the y -axis is at most 3. Thus, the inequality represents the region consisting of all points on or inside a circular cylinder of radius 3 with axis the y -axis.
34. The equation $xyz = 0$ is satisfied when any of x , y , or z is 0. Thus, the equation represents the region consisting of all points on the three coordinate planes $x = 0$, $y = 0$, and $z = 0$.
35. This describes all points with negative y -coordinates, that is, $y < 0$.
36. Because the box lies in the first quadrant, each point must comprise only nonnegative coordinates. So inequalities describing the region are $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$.
37. This describes a region all of whose points have a distance to the origin which is greater than r , but smaller than R . So inequalities describing the region are $r < \sqrt{x^2 + y^2 + z^2} < R$, or $r^2 < x^2 + y^2 + z^2 < R^2$.
38. The solid sphere itself is represented by $\sqrt{x^2 + y^2 + z^2} \leq 2$. Since we want only the upper hemisphere, we restrict the z -coordinate to nonnegative values. Then inequalities describing the region are $\sqrt{x^2 + y^2 + z^2} \leq 2$, $z \geq 0$, or $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$.

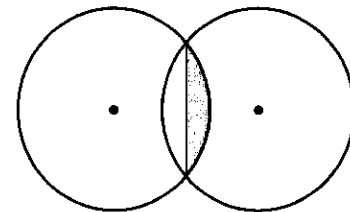
39. (a) To find the x - and y -coordinates of the point P , we project it onto L_2 and project the resulting point Q onto the x - and y -axes. To find the z -coordinate, we project P onto either the xz -plane or the yz -plane (using our knowledge of its x - or y -coordinate) and then project the resulting point onto the z -axis. (Or, we could draw a line parallel to QO from P to the z -axis.) The coordinates of P are $(2, 1, 4)$.



- (b) A is the intersection of L_1 and L_2 , B is directly below the y -intercept of L_2 , and C is directly above the x -intercept of L_2 .

40. Let $P = (x, y, z)$. Then $2|PB| = |PA| \Leftrightarrow 4|PB|^2 = |PA|^2 \Leftrightarrow 4((x-6)^2 + (y-2)^2 + (z+2)^2) = (x+1)^2 + (y-5)^2 + (z-3)^2 \Leftrightarrow 4(x^2 - 12x + 36) - x^2 - 2x + 4(y^2 - 4y + 4) - y^2 + 10y + 4(z^2 + 4z + 4) - z^2 + 6z = 35 \Leftrightarrow 3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = 35 - 144 - 16 - 16 \Leftrightarrow x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$. By completing the square three times we get $(x - \frac{25}{3})^2 + (y - 1)^2 + (z + \frac{11}{3})^2 = \frac{332}{9}$, which is an equation of a sphere with center $(\frac{25}{3}, 1, -\frac{11}{3})$ and radius $\frac{\sqrt{332}}{3}$.
41. We need to find a set of points $\{P(x, y, z) \mid |AP| = |BP|\}$. $\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2} \Rightarrow (x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2 \Rightarrow x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4 \Rightarrow 14x - 6y - 10z = 9$. Thus the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

42. Completing the square three times in the first equation gives $(x+2)^2 + (y-1)^2 + (z+2)^2 = 2^2$, a sphere with center $(-2, 1, 2)$ and radius 2. The second equation is that of a sphere with center $(0, 0, 0)$ and radius 2. The distance between the centers of the spheres is $\sqrt{(-2-0)^2 + (1-0)^2 + (-2-0)^2} = \sqrt{4+1+4} = 3$. Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h = 2 - \frac{3}{2} = \frac{1}{2}$. From Exercise 6.2.49 [ET 6.2.49], the volume of a cap of a sphere is $V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi(\frac{1}{2})^2(3 \cdot 2 - \frac{1}{2}) = \frac{11\pi}{24}$. So the total volume is $2 \cdot \frac{11\pi}{24} = \frac{11\pi}{12}$.

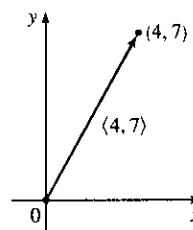


13.2 Vectors

ET 12.2

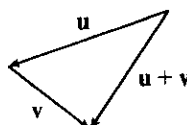
1. (a) The cost of a theater ticket is a scalar, because it has only magnitude.
- (b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.
- (c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.
- (d) The population of the world is a scalar, because it has only magnitude.

2. If the initial point of the vector $\langle 4, 7 \rangle$ is placed at the origin, then $\langle 4, 7 \rangle$ is the position vector of the point $(4, 7)$.

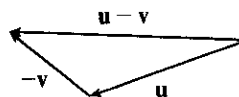


3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, and $\overrightarrow{EA} = \overrightarrow{CE}$.
4. (a) The initial point of $\overrightarrow{QR}$ is positioned at the terminal point of $\overrightarrow{PQ}$, so by the Triangle Law the sum $\overrightarrow{PQ} + \overrightarrow{QR}$ is the vector with initial point P and terminal point R , namely $\overrightarrow{PR}$.
- (b) By the Triangle Law, $\overrightarrow{RP} + \overrightarrow{PS}$ is the vector with initial point R and terminal point S , namely $\overrightarrow{RS}$.
- (c) First we consider $\overrightarrow{QS} - \overrightarrow{PS}$ as $\overrightarrow{QS} + (-\overrightarrow{PS})$. Then since $-\overrightarrow{PS}$ has the same length as $\overrightarrow{PS}$ but points in the opposite direction, we have $-\overrightarrow{PS} = \overrightarrow{SP}$ and so $\overrightarrow{QS} - \overrightarrow{PS} = \overrightarrow{QS} + \overrightarrow{SP} = \overrightarrow{QP}$.
- (d) We use the Triangle Law twice: $\overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ} = (\overrightarrow{RS} + \overrightarrow{SP}) + \overrightarrow{PQ} = \overrightarrow{RP} + \overrightarrow{PQ} = \overrightarrow{RQ}$

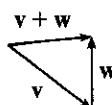
5. (a)



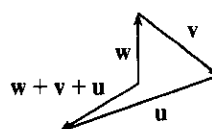
(b)



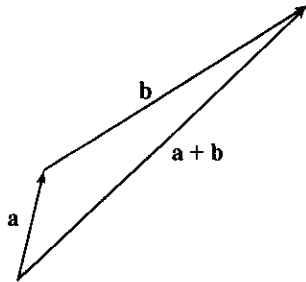
(c)



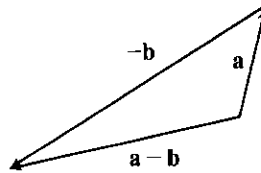
(d)



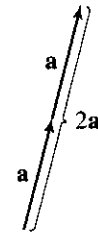
6. (a)



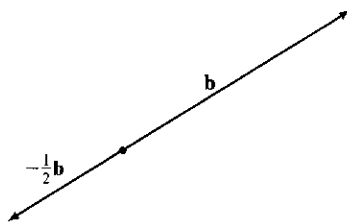
(b)



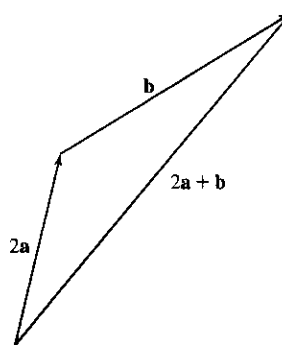
(c)



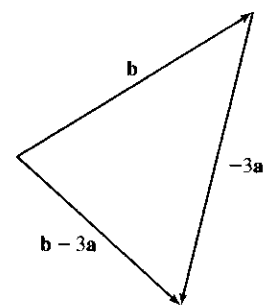
d)



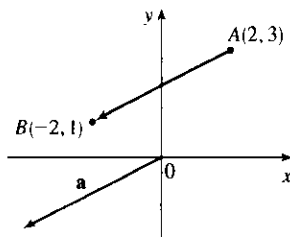
(e)



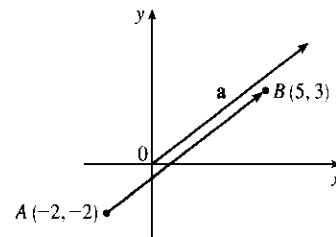
(f)



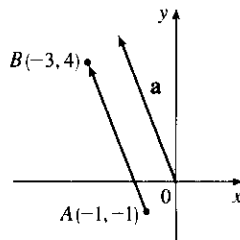
7. $\mathbf{a} = \langle -2 - 2, 1 - 3 \rangle = \langle -4, -2 \rangle$



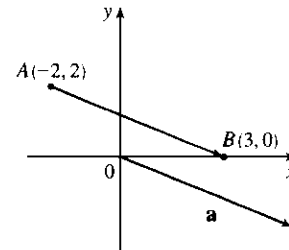
8. $\mathbf{a} = \langle 5 - (-2), 3 - (-2) \rangle = \langle 7, 5 \rangle$



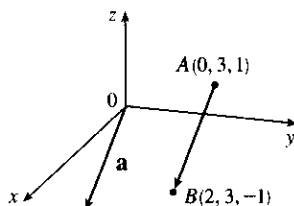
9. $\mathbf{a} = \langle -3 - (-1), 4 - (-1) \rangle = \langle -2, 5 \rangle$



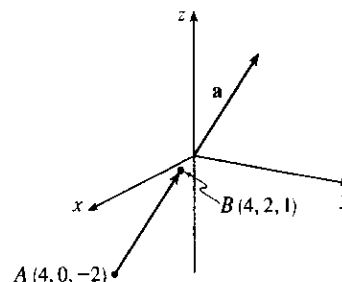
10. $\mathbf{a} = \langle 3 - (-2), 0 - 2 \rangle = \langle 5, -2 \rangle$



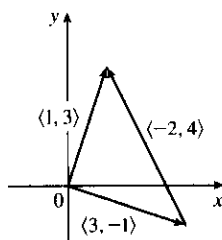
11. $\mathbf{a} = \langle 2 - 0, 3 - 3, -1 - 1 \rangle = \langle 2, 0, -2 \rangle$



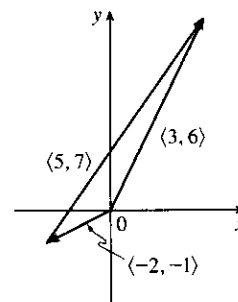
12. $\mathbf{a} = \langle 4 - 4, 2 - 0, 1 - (-2) \rangle = \langle 0, 2, 3 \rangle$



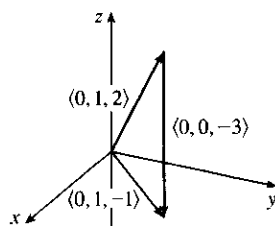
13. $\langle 3, -1 \rangle + \langle -2, 4 \rangle = \langle 3 + (-2), -1 + 4 \rangle = \langle 1, 3 \rangle$



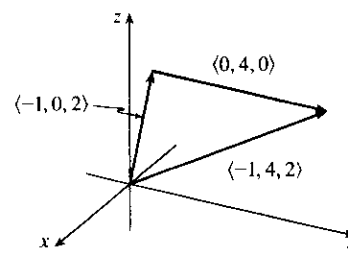
14. $\langle -2, -1 \rangle + \langle 5, 7 \rangle = \langle -2 + 5, -1 + 7 \rangle = \langle 3, 6 \rangle$



15. $\langle 0, 1, 2 \rangle + \langle 0, 0, -3 \rangle = \langle 0 + 0, 1 + 0, 2 + (-3) \rangle = \langle 0, 1, -1 \rangle$



16. $\langle -1, 0, 2 \rangle + \langle 0, 4, 0 \rangle = \langle -1 + 0, 0 + 4, 2 + 0 \rangle = \langle -1, 4, 2 \rangle$



17. $|\mathbf{a}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

$\mathbf{a} + \mathbf{b} = \langle -4 + 6, 3 + 2 \rangle = \langle 2, 5 \rangle$

$\mathbf{a} - \mathbf{b} = \langle -4 - 6, 3 - 2 \rangle = \langle -10, 1 \rangle$

$2\mathbf{a} = \langle 2(-4), 2(3) \rangle = \langle -8, 6 \rangle$

$3\mathbf{a} + 4\mathbf{b} = \langle -12, 9 \rangle + \langle 24, 8 \rangle = \langle 12, 17 \rangle$

18. $|\mathbf{a}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\mathbf{a} + \mathbf{b} = (2\mathbf{i} - 3\mathbf{j}) + (\mathbf{i} + 5\mathbf{j}) = 3\mathbf{i} + 2\mathbf{j}$

$\mathbf{a} - \mathbf{b} = (2\mathbf{i} - 3\mathbf{j}) - (\mathbf{i} + 5\mathbf{j}) = \mathbf{i} - 8\mathbf{j}$

$2\mathbf{a} = 2(2\mathbf{i} - 3\mathbf{j}) = 4\mathbf{i} - 6\mathbf{j}$

$$3\mathbf{a} + 4\mathbf{b} = 3(2\mathbf{i} - 3\mathbf{j}) + 4(\mathbf{i} + 5\mathbf{j})$$

$$= 6\mathbf{i} - 9\mathbf{j} + 4\mathbf{i} + 20\mathbf{j} = 10\mathbf{i} + 11\mathbf{j}$$

$$19. |\mathbf{a}| = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{49} = 7$$

$$\mathbf{a} + \mathbf{b} = \langle 6 + (-1), 2 + 5, 3 + (-2) \rangle = \langle 5, 7, 1 \rangle$$

$$\begin{aligned}\mathbf{a} - \mathbf{b} &= \langle 6 - (-1), 2 - 5, 3 - (-2) \rangle \\ &= \langle 7, -3, 5 \rangle\end{aligned}$$

$$2\mathbf{a} = \langle 2(6), 2(2), 2(3) \rangle = \langle 12, 4, 6 \rangle$$

$$\begin{aligned}3\mathbf{a} + 4\mathbf{b} &= \langle 18, 6, 9 \rangle + \langle -4, 20, -8 \rangle \\ &= \langle 14, 26, 1 \rangle\end{aligned}$$

$$20. |\mathbf{a}| = \sqrt{(-3)^2 + (-4)^2 + (-1)^2} = \sqrt{26}$$

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \langle -3 + 6, -4 + 2, -1 + (-3) \rangle \\ &= \langle 3, -2, -4 \rangle\end{aligned}$$

$$\begin{aligned}\mathbf{a} - \mathbf{b} &= \langle -3 - 6, -4 - 2, -1 - (-3) \rangle \\ &= \langle -9, -6, 2 \rangle\end{aligned}$$

$$2\mathbf{a} = \langle 2(-3), 2(-4), 2(-1) \rangle = \langle -6, -8, -2 \rangle$$

$$\begin{aligned}3\mathbf{a} + 4\mathbf{b} &= \langle -9, -12, -3 \rangle + \langle 24, 8, -12 \rangle \\ &= \langle 15, -4, -15 \rangle\end{aligned}$$

$$21. |\mathbf{a}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$\mathbf{a} + \mathbf{b} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

$$\mathbf{a} - \mathbf{b} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} - 3\mathbf{j} - \mathbf{k}$$

$$2\mathbf{a} = 2(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\begin{aligned}3\mathbf{a} + 4\mathbf{b} &= 3(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + 4(\mathbf{j} + 2\mathbf{k}) \\ &= 3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} + 4\mathbf{j} + 8\mathbf{k} \\ &= 3\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}\end{aligned}$$

$$22. |\mathbf{a}| = \sqrt{3^2 + 0^2 + (-2)^2} = \sqrt{13}$$

$$\mathbf{a} + \mathbf{b} = (3\mathbf{i} - 2\mathbf{k}) + (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$\mathbf{a} - \mathbf{b} = (3\mathbf{i} - 2\mathbf{k}) - (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

$$2\mathbf{a} = 2(3\mathbf{i} - 2\mathbf{k}) = 6\mathbf{i} - 4\mathbf{k}$$

$$\begin{aligned}3\mathbf{a} + 4\mathbf{b} &= 3(3\mathbf{i} - 2\mathbf{k}) + 4(\mathbf{i} - \mathbf{j} + \mathbf{k}) \\ &= 9\mathbf{i} - 6\mathbf{k} + 4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} \\ &= 13\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}\end{aligned}$$

$$23. |\langle 9, -5 \rangle| = \sqrt{9^2 + (-5)^2} = \sqrt{106}, \text{ so } \mathbf{u} = \frac{1}{\sqrt{106}} \langle 9, -5 \rangle = \left\langle \frac{9}{\sqrt{106}}, \frac{-5}{\sqrt{106}} \right\rangle.$$

$$24. |12\mathbf{i} - 5\mathbf{j}| = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13, \text{ so } \mathbf{u} = \frac{1}{13} (12\mathbf{i} - 5\mathbf{j}) = \frac{12}{13}\mathbf{i} - \frac{5}{13}\mathbf{j}.$$

$$25. \text{ The vector } 8\mathbf{i} - \mathbf{j} + 4\mathbf{k} \text{ has length } |8\mathbf{i} - \mathbf{j} + 4\mathbf{k}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{81} = 9, \text{ so by Equation 4 the unit vector with the same direction is } \frac{1}{9}(8\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = \frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}.$$

$$26. |\langle -2, 4, 2 \rangle| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{24} = 2\sqrt{6}, \text{ so a unit vector in the direction}$$

$$\text{of } \langle -2, 4, 2 \rangle \text{ is } \mathbf{u} = \frac{1}{2\sqrt{6}} \langle -2, 4, 2 \rangle. \text{ A vector in the same direction but with length 6 is}$$

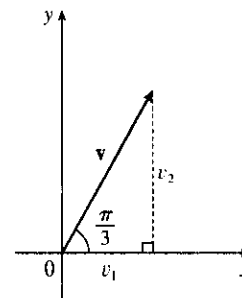
$$6\mathbf{u} = 6 \cdot \frac{1}{2\sqrt{6}} \langle -2, 4, 2 \rangle = \left\langle -\frac{6}{\sqrt{6}}, \frac{12}{\sqrt{6}}, \frac{6}{\sqrt{6}} \right\rangle \text{ or } \langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6} \rangle.$$

27. From the figure, we see that the x -component of $\mathbf{v}$ is

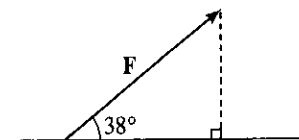
$$v_1 = |\mathbf{v}| \cos(\pi/3) = 4 \cdot \frac{1}{2} = 2 \text{ and the } y\text{-component is}$$

$$v_2 = |\mathbf{v}| \sin(\pi/3) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}. \text{ Thus}$$

$$\mathbf{v} = \langle v_1, v_2 \rangle = \langle 2, 2\sqrt{3} \rangle.$$



28. From the figure, we see that the horizontal component of the force $\mathbf{F}$ is $|\mathbf{F}| \cos 38^\circ = 50 \cos 38^\circ \approx 39.4$ N, and the vertical component is $|\mathbf{F}| \sin 38^\circ = 50 \sin 38^\circ \approx 30.8$ N.



29. $|\mathbf{F}_1| = 10$ lb and $|\mathbf{F}_2| = 12$ lb.

$$\begin{aligned}\mathbf{F}_1 &= -|\mathbf{F}_1| \cos 45^\circ \mathbf{i} + |\mathbf{F}_1| \sin 45^\circ \mathbf{j} = -10 \cos 45^\circ \mathbf{i} + 10 \sin 45^\circ \mathbf{j} \\ &= -5\sqrt{2} \mathbf{i} + 5\sqrt{2} \mathbf{j}\end{aligned}$$

$$\mathbf{F}_2 = |\mathbf{F}_2| \cos 30^\circ \mathbf{i} + |\mathbf{F}_2| \sin 30^\circ \mathbf{j} = 12 \cos 30^\circ \mathbf{i} + 12 \sin 30^\circ \mathbf{j} = 6\sqrt{3} \mathbf{i} + 6 \mathbf{j}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = (6\sqrt{3} - 5\sqrt{2}) \mathbf{i} + (6 + 5\sqrt{2}) \mathbf{j} \approx 3.32 \mathbf{i} + 13.07 \mathbf{j}$$

$$|\mathbf{F}| \approx \sqrt{(3.32)^2 + (13.07)^2} \approx 13.5 \text{ lb.} \quad \tan \theta = \frac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}} \Rightarrow \theta = \tan^{-1} \frac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}} \approx 76^\circ.$$

30. Set up the coordinate axes so that north is the positive y -direction, and east is the positive x -direction. The wind is blowing at 50 km/h from the direction $N45^\circ W$, so that its velocity vector is 50 km/h $S45^\circ E$, which can be written as $\mathbf{v}_{\text{wind}} = 50(\cos 45^\circ \mathbf{i} - \sin 45^\circ \mathbf{j})$. With respect to the still air, the velocity vector of the plane is 250 km/h $N60^\circ E$, or equivalently $\mathbf{v}_{\text{plane}} = 250(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$. The velocity of the plane relative to the ground is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_{\text{wind}} + \mathbf{v}_{\text{plane}} = (50 \cos 45^\circ + 250 \cos 30^\circ) \mathbf{i} + (-50 \sin 45^\circ + 250 \sin 30^\circ) \mathbf{j} \\ &= (25\sqrt{2} + 125\sqrt{3}) \mathbf{i} + (125 - 25\sqrt{2}) \mathbf{j} \approx 251.9 \mathbf{i} + 89.6 \mathbf{j}\end{aligned}$$

The ground speed is $|\mathbf{v}| \approx \sqrt{(251.9)^2 + (89.6)^2} \approx 267$ km/h. The angle the velocity vector makes with the x -axis is $\theta \approx \tan^{-1} \left(\frac{89.6}{251.9} \right) \approx 20^\circ$. Therefore, the true course of the plane is about $N(90 - 20)^\circ E = N70^\circ E$.

31. With respect to the water's surface, the woman's velocity is the vector sum of the velocity of the ship with respect to the water, and the woman's velocity with respect to the ship. If we let north be the positive y -direction, then $\mathbf{v} = \langle 0, 22 \rangle + \langle -3, 0 \rangle = \langle -3, 22 \rangle$. The woman's speed is $|\mathbf{v}| = \sqrt{9 + 484} \approx 22.2$ mi/h. The vector $\mathbf{v}$ makes an angle θ with the east, where $\theta = \tan^{-1} \left(\frac{22}{-3} \right) \approx 98^\circ$. Therefore, the woman's direction is about $N(98 - 90)^\circ W = N8^\circ W$.

32. Call the two tensile forces $\mathbf{T}_3$ and $\mathbf{T}_5$, corresponding to the ropes of length 3 m and 5 m. In terms of vertical and horizontal components,

$$\mathbf{T}_3 = -|\mathbf{T}_3| \cos 52^\circ \mathbf{i} + |\mathbf{T}_3| \sin 52^\circ \mathbf{j} \quad (1) \quad \text{and} \quad \mathbf{T}_5 = |\mathbf{T}_5| \cos 40^\circ \mathbf{i} + |\mathbf{T}_5| \sin 40^\circ \mathbf{j} \quad (2)$$

The resultant of these forces, $\mathbf{T}_3 + \mathbf{T}_5$, counterbalances the force of gravity acting on the decoration [which is $-5g \mathbf{j} \approx -5(9.8) \mathbf{j} = -49 \mathbf{j}$]. So $\mathbf{T}_3 + \mathbf{T}_5 = 49 \mathbf{j}$. Hence

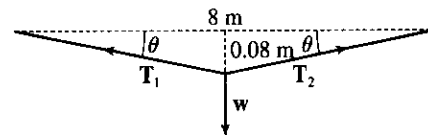
$$\mathbf{T}_3 + \mathbf{T}_5 = (-|\mathbf{T}_3| \cos 52^\circ + |\mathbf{T}_5| \cos 40^\circ) \mathbf{i} + (|\mathbf{T}_3| \sin 52^\circ + |\mathbf{T}_5| \sin 40^\circ) \mathbf{j} = 49 \mathbf{j}. \text{ Thus}$$

$$-|\mathbf{T}_3| \cos 52^\circ + |\mathbf{T}_5| \cos 40^\circ = 0 \text{ and } |\mathbf{T}_3| \sin 52^\circ + |\mathbf{T}_5| \sin 40^\circ = 49.$$

From the first of these two equations $|\mathbf{T}_3| = |\mathbf{T}_5| \frac{\cos 40^\circ}{\cos 52^\circ}$. Substituting this into the second equation gives

$$|\mathbf{T}_5| = \frac{49}{\cos 40^\circ \tan 52^\circ + \sin 40^\circ} \approx 30 \text{ N. Therefore, } |\mathbf{T}_3| = |\mathbf{T}_5| \frac{\cos 40^\circ}{\cos 52^\circ} \approx 38 \text{ N. Finally, from (1) and (2), } \mathbf{T}_3 \approx -23\mathbf{i} + 30\mathbf{j}, \text{ and } \mathbf{T}_5 \approx 23\mathbf{i} + 19\mathbf{j}.$$

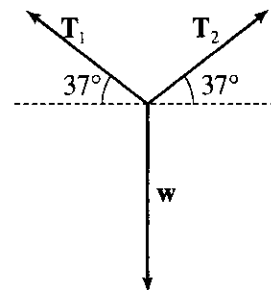
33. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ represent the tension vectors in each side of the clothesline as shown in the figure. $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal vertical components and opposite horizontal components, so we can write



$\mathbf{T}_1 = -a\mathbf{i} + b\mathbf{j}$ and $\mathbf{T}_2 = a\mathbf{i} + b\mathbf{j}$ ($a, b > 0$). By similar triangles, $\frac{b}{a} = \frac{0.08}{4} \Rightarrow a = 50b$. The force due to gravity acting on the shirt has magnitude $0.8g \approx (0.8)(9.8) = 7.84 \text{ N}$, hence we have $\mathbf{w} = -7.84\mathbf{j}$. The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensile forces counterbalances $\mathbf{w}$, so $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} \Rightarrow (-a\mathbf{i} + b\mathbf{j}) + (a\mathbf{i} + b\mathbf{j}) = 7.84\mathbf{j} \Rightarrow (-50b\mathbf{i} + b\mathbf{j}) + (50b\mathbf{i} + b\mathbf{j}) = 2b\mathbf{j} = 7.84\mathbf{j} \Rightarrow b = \frac{7.84}{2} = 3.92$ and $a = 50b = 196$. Thus the tensions are $\mathbf{T}_1 = -a\mathbf{i} + b\mathbf{j} = -196\mathbf{i} + 3.92\mathbf{j}$ and $\mathbf{T}_2 = a\mathbf{i} + b\mathbf{j} = 196\mathbf{i} + 3.92\mathbf{j}$.

Alternatively, we can find the value of θ and proceed as in Example 7.

34. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\mathbf{T}_1$, $\mathbf{T}_2$ in each end of the chain and the weight $\mathbf{w}$, as shown in the figure. We know $|\mathbf{T}_1| = |\mathbf{T}_2| = 25 \text{ N}$ so, in terms of vertical and horizontal components, we have



$$\mathbf{T}_1 = -25 \cos 37^\circ \mathbf{i} + 25 \sin 37^\circ \mathbf{j}$$

$$\mathbf{T}_2 = 25 \cos 37^\circ \mathbf{i} + 25 \sin 37^\circ \mathbf{j}$$

The resultant vector $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$, giving $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w}$. Since $\mathbf{w} = -|\mathbf{w}|\mathbf{j}$, we have $(-25 \cos 37^\circ \mathbf{i} + 25 \sin 37^\circ \mathbf{j}) + (25 \cos 37^\circ \mathbf{i} + 25 \sin 37^\circ \mathbf{j}) = |\mathbf{w}|\mathbf{j} \Rightarrow$

$50 \sin 37^\circ \mathbf{j} = |\mathbf{w}|\mathbf{j} \Rightarrow |\mathbf{w}| = 50 \sin 37^\circ \approx 30.1$. So the weight is 30.1 N, and since $w = mg$, the mass is $\frac{30.1}{9.8} \approx 3.07 \text{ kg}$.

35. By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA}$, but

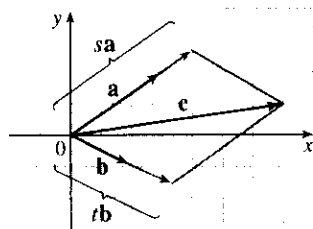
$$\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AC} + (-\overrightarrow{AC}) = \mathbf{0}. \text{ So } \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}.$$

36. $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{BA}$. $\mathbf{c} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{1}{3}\overrightarrow{AB} \Rightarrow \overrightarrow{AB} = 3\mathbf{c} - 3\mathbf{a}$.

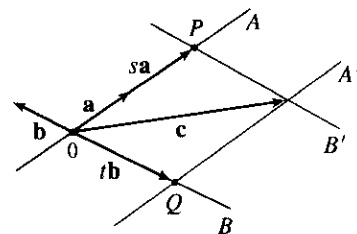
$$\mathbf{c} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BA} \Rightarrow \overrightarrow{BA} = \frac{3}{2}\mathbf{c} - \frac{3}{2}\mathbf{b}. \overrightarrow{BA} = -\overrightarrow{AB}, \text{ so } \frac{3}{2}\mathbf{c} - \frac{3}{2}\mathbf{b} = 3\mathbf{a} - 3\mathbf{c} \Leftrightarrow$$

$$\mathbf{c} + 2\mathbf{c} = 2\mathbf{a} + \mathbf{b} \Leftrightarrow \mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}.$$

37. (a), (b)

(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.(d) $\mathbf{c} = s\mathbf{a} + t\mathbf{b} \Leftrightarrow 7 = 3s + 2t$ and $1 = 2s - t$.Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

38. Draw $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ emanating from the origin. Extend $\mathbf{a}$ and $\mathbf{b}$ to form lines A and B , and draw lines A' and B' parallel to these two lines through the terminal point of $\mathbf{c}$.



Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that

$$\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{c}, \text{ so if } s = \frac{|\overrightarrow{OP}|}{|\mathbf{a}|} \text{ (or its negative, if } \mathbf{a} \text{ points in the direction opposite } \overrightarrow{OP}\text{)} \text{ and } t = \frac{|\overrightarrow{OQ}|}{|\mathbf{b}|}$$

(or its negative, as in the diagram), then $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$, as required.

Argument using components: Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$. We need $sa_1 + tb_1 = c_1$ and $sa_2 + tb_2 = c_2$.

Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t = \frac{c_2a_1 - c_1a_2}{b_2a_1 - b_1a_2}$. Similarly

$$s = \frac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}. \text{ Since } \mathbf{a} \neq \mathbf{0} \text{ and } \mathbf{b} \neq \mathbf{0} \text{ and } \mathbf{a} \text{ is not a scalar multiple of } \mathbf{b}, \text{ the denominator is not zero.}$$

39. $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the points (x, y, z) and (x_0, y_0, z_0) , so the set of points is a sphere with radius 1 and center (x_0, y_0, z_0) .

$$\text{Alternate method: } |\mathbf{r} - \mathbf{r}_0| = 1 \Leftrightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = 1 \Leftrightarrow$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1, \text{ which is the equation of a sphere with radius 1 and center } (x_0, y_0, z_0).$$

40. Let P_1 and P_2 be the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2|$ is the sum of the distances from (x, y) to P_1 and P_2 . Since this sum is constant, the set of points (x, y) represents an ellipse with foci P_1 and P_2 . The condition $k > |\mathbf{r}_1 - \mathbf{r}_2|$ assures us that the ellipse is not degenerate.

$$\begin{aligned} 41. \mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2 \rangle + (\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle) = \langle a_1, a_2 \rangle + \langle b_1 + c_1, b_2 + c_2 \rangle \\ &= \langle a_1 + b_1 + c_1, a_2 + b_2 + c_2 \rangle = \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2 \rangle \\ &= \langle a_1 + b_1, a_2 + b_2 \rangle + \langle c_1, c_2 \rangle = (\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle) + \langle c_1, c_2 \rangle \\ &= (\mathbf{a} + \mathbf{b}) + \mathbf{c} \end{aligned}$$

42. Algebraically:

$$\begin{aligned} c(\mathbf{a} + \mathbf{b}) &= c(\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle) = c\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\ &= \langle c(a_1 + b_1), c(a_2 + b_2), c(a_3 + b_3) \rangle = \langle ca_1 + cb_1, ca_2 + cb_2, ca_3 + cb_3 \rangle \\ &= \langle ca_1, ca_2, ca_3 \rangle + \langle cb_1, cb_2, cb_3 \rangle = c\mathbf{a} + c\mathbf{b} \end{aligned}$$

Geometrically:

According to the Triangle Law, if $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{QR}$, then

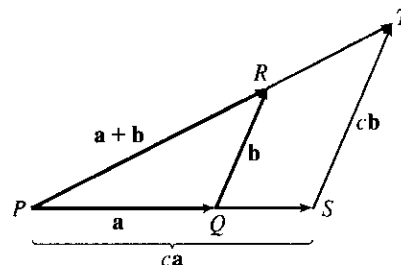
$\mathbf{a} + \mathbf{b} = \overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS} = c\mathbf{a}$

and $\overrightarrow{ST} = c\mathbf{b}$. (We have drawn the case where $c > 1$.) By the

Triangle Law, $\overrightarrow{PT} = c\mathbf{a} + c\mathbf{b}$. But triangle PQR and triangle PST are similar triangles because $c\mathbf{b}$ is parallel to $\mathbf{b}$.

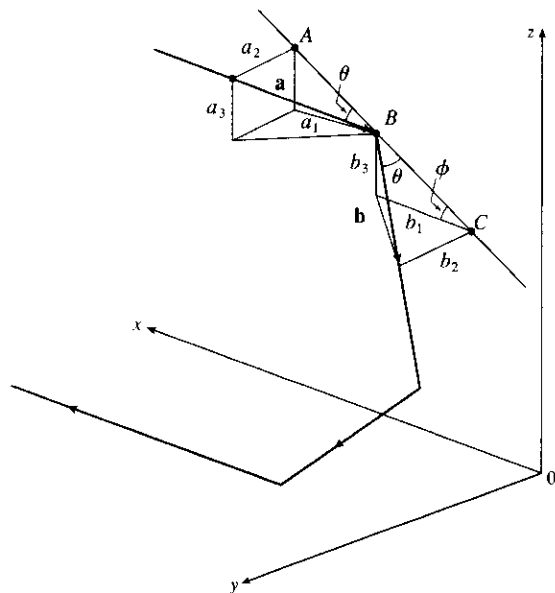
Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT} = c\overrightarrow{PR}$.

Thus, $c\mathbf{a} + c\mathbf{b} = c(\mathbf{a} + \mathbf{b})$.



43. Consider triangle ABC , where D and E are the midpoints of AB and BC . We know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (1) and $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ (2). However, $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into (2) gives $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$. Comparing this with (1) gives $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $|\overrightarrow{DE}| = \frac{1}{2}|\overrightarrow{AC}|$.

44.



The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Let $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, as in the diagram. We can let $|\mathbf{b}| = |\mathbf{a}|$, since only its direction is important. Then $\frac{|b_2|}{|\mathbf{b}|} = \sin \theta = \frac{|a_2|}{|\mathbf{a}|} \Rightarrow |b_2| = |a_2|$.

From the diagram $b_2\mathbf{j}$ and $a_2\mathbf{j}$ point in opposite directions, so $b_2 = -a_2$. $|\mathbf{AB}| = |\mathbf{BC}|$, so $|b_3| = \sin \phi |\mathbf{BC}| = \sin \phi |\mathbf{AB}| = |a_3|$, and $|b_1| = \cos \phi |\mathbf{BC}| = \cos \phi |\mathbf{AB}| = |a_1|$.

$b_3\mathbf{k}$ and $a_3\mathbf{k}$ have the same direction, as do $b_1\mathbf{i}$ and $a_1\mathbf{i}$, so $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\langle -a_1, -a_2, -a_3 \rangle = -\mathbf{a}$, which is parallel to $\mathbf{a}$.

13.3 The Dot Product

ET 12.3

1. (a) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ has no meaning.
 (b) $(\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.
 (c) Both $|\mathbf{a}|$ and $\mathbf{b} \cdot \mathbf{c}$ are scalars, so $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ is an ordinary product of real numbers, and has meaning.
 (d) Both $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$ are vectors, so the dot product $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ has meaning.
 (e) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and this expression has no meaning.
 (f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$ has no meaning.
2. Let the vectors be $\mathbf{a}$ and $\mathbf{b}$. Then by Theorem 3, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = (6)\left(\frac{1}{3}\right) \cos \frac{\pi}{4} = \frac{6}{3\sqrt{2}} = \sqrt{2}$.
3. $\mathbf{a} \cdot \mathbf{b} = \langle 4, -1 \rangle \cdot \langle 3, 6 \rangle = (4)(3) + (-1)(6) = 6$
4. $\mathbf{a} \cdot \mathbf{b} = \left\langle \frac{1}{2}, 4 \right\rangle \cdot \langle -8, -3 \rangle = \left(\frac{1}{2}\right)(-8) + (4)(-3) = -16$
5. $\mathbf{a} \cdot \mathbf{b} = \langle 5, 0, -2 \rangle \cdot \langle 3, -1, 10 \rangle = (5)(3) + (0)(-1) + (-2)(10) = -5$
6. $\mathbf{a} \cdot \mathbf{b} = \langle s, 2s, 3s \rangle \cdot \langle t, -t, 5t \rangle = (s)(t) + (2s)(-t) + (3s)(5t) = st - 2st + 15st = 14st$
7. $\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (5\mathbf{i} + 9\mathbf{k}) = (1)(5) + (-2)(0) + (3)(9) = 32$
8. $\mathbf{a} \cdot \mathbf{b} = (4\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) = (0)(2) + (4)(4) + (-3)(6) = -2$
9. Use Theorem 3: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = (12)(15) \cos \frac{\pi}{6} = 180 \cdot \frac{\sqrt{3}}{2} = 90\sqrt{3} \approx 155.9$
10. Use Theorem 3: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = (4)(10) \cos 120^\circ = 40\left(-\frac{1}{2}\right) = -20$
11. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos 60^\circ = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}| |\mathbf{w}| \cos 120^\circ = (1)(1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$.
12. $\mathbf{u}$ is a unit vector, so $\mathbf{w}$ is also a unit vector, and $|\mathbf{v}|$ can be determined by examining the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$. Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is 45° , we have $|\mathbf{v}| = |\mathbf{u}| \cos 45^\circ = \frac{\sqrt{2}}{2}$. Then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos 45^\circ = (1)\left(\frac{\sqrt{2}}{2}\right) \frac{\sqrt{2}}{2} = \frac{1}{2}$. Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, $\mathbf{u} \cdot \mathbf{w} = 0$.
13. (a) $\mathbf{i} \cdot \mathbf{j} = \langle 1, 0, 0 \rangle \cdot \langle 0, 1, 0 \rangle = (1)(0) + (0)(1) + (0)(0) = 0$. Similarly $\mathbf{j} \cdot \mathbf{k} = (0)(0) + (1)(0) + (0)(1) = 0$ and $\mathbf{k} \cdot \mathbf{i} = (0)(1) + (0)(0) + (1)(0) = 0$.
 Another method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos \frac{\pi}{2} = 0$.
 (b) By Property 1 of the dot product, $\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}|^2 = 1$.
14. The dot product $\mathbf{A} \cdot \mathbf{P}$ is

$$\langle a, b, c \rangle \cdot \langle 2, 1.5, 1 \rangle = a(2) + b(1.5) + c(1)$$

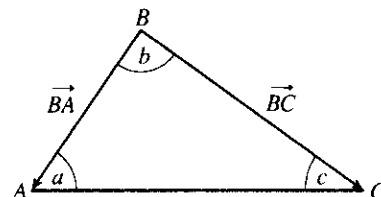
$$= (\text{number of hamburgers sold})(\text{price per hamburger})$$

$$+ (\text{number of hot dogs sold})(\text{price per hot dog})$$

$$+ (\text{number of soft drinks sold})(\text{price per soft drink})$$
 so it is equal to the vendor's total revenue for that day.

15. $|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5$, $|\mathbf{b}| = \sqrt{5^2 + 12^2} = 13$, and $\mathbf{a} \cdot \mathbf{b} = (3)(5) + (4)(12) = 63$. Using Corollary 6, we have $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{63}{5 \cdot 13} = \frac{63}{65}$. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{63}{65}\right) \approx 14^\circ$.
16. $|\mathbf{a}| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$, $|\mathbf{b}| = \sqrt{0^2 + 25} = 5$, and $\mathbf{a} \cdot \mathbf{b} = (\sqrt{3})(0) + (1)(5) = 5$. Using Corollary 6, we have $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{5}{2 \cdot 5} = \frac{1}{2}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$.
17. $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, $|\mathbf{b}| = \sqrt{4^2 + 0^2 + (-1)^2} = \sqrt{17}$, and $\mathbf{a} \cdot \mathbf{b} = (1)(4) + (2)(0) + (3)(-1) = 1$. Then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{1}{\sqrt{14} \cdot \sqrt{17}} = \frac{1}{\sqrt{238}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{238}}\right) \approx 86^\circ$.
18. $|\mathbf{a}| = \sqrt{6^2 + (-3)^2 + 2^2} = 7$, $|\mathbf{b}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, and $\mathbf{a} \cdot \mathbf{b} = (6)(2) + (-3)(1) + (2)(-2) = 5$. Then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{5}{7 \cdot 3} = \frac{5}{21}$ and $\theta = \cos^{-1}\left(\frac{5}{21}\right) \approx 76^\circ$.
19. $|\mathbf{a}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$, $|\mathbf{b}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$, and $\mathbf{a} \cdot \mathbf{b} = (0)(1) + (1)(2) + (1)(-3) = -1$. Then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-1}{\sqrt{2} \cdot \sqrt{14}} = \frac{-1}{2\sqrt{7}}$ and $\theta = \cos^{-1}\left(-\frac{1}{2\sqrt{7}}\right) \approx 101^\circ$.
20. $|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$, $|\mathbf{b}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$, and $\mathbf{a} \cdot \mathbf{b} = (2)(3) + (-1)(2) + (1)(-1) = 3$. Then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{3}{\sqrt{6} \cdot \sqrt{14}} = \frac{3}{2\sqrt{21}}$ and $\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) \approx 71^\circ$.
21. Let a , b , and c be the angles at vertices A , B , and C respectively.

Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



$$\text{Thus } \cos a = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{\langle 2, 6 \rangle \cdot \langle -2, 4 \rangle}{\sqrt{2^2 + 6^2} \sqrt{(-2)^2 + 4^2}} = \frac{1}{\sqrt{40} \sqrt{20}} (-4 + 24) = \frac{20}{\sqrt{800}} = \frac{\sqrt{2}}{2}$$

$$\text{and } a = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ. \text{ Similarly,}$$

$$\cos b = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} = \frac{\langle -2, -6 \rangle \cdot \langle -4, -2 \rangle}{\sqrt{4 + 36} \sqrt{16 + 4}} = \frac{1}{\sqrt{40} \sqrt{20}} (8 + 12) = \frac{20}{\sqrt{800}} = \frac{\sqrt{2}}{2} \text{ so}$$

$$b = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ \text{ and } c = 180^\circ - (45^\circ + 45^\circ) = 90^\circ.$$

Alternate solution: Apply the Law of Cosines three times as follows: $\cos a = \frac{|\overrightarrow{BC}|^2 - |\overrightarrow{AB}|^2 - |\overrightarrow{AC}|^2}{2 |\overrightarrow{AB}| |\overrightarrow{AC}|}$,

$$\cos b = \frac{|\overrightarrow{AC}|^2 - |\overrightarrow{AB}|^2 - |\overrightarrow{BC}|^2}{2 |\overrightarrow{AB}| |\overrightarrow{BC}|}, \text{ and } \cos c = \frac{|\overrightarrow{AB}|^2 - |\overrightarrow{AC}|^2 - |\overrightarrow{BC}|^2}{2 |\overrightarrow{AC}| |\overrightarrow{BC}|}.$$

22. As in Exercise 21, let d , e , and f be the angles at vertices D , E , and F . Then d is the angle between vectors $\overrightarrow{DE}$ and $\overrightarrow{DF}$, e is the angle between vectors $\overrightarrow{ED}$ and $\overrightarrow{EF}$, and f is the angle between vectors $\overrightarrow{FD}$ and $\overrightarrow{FE}$. Thus

$$\cos d = \frac{\overrightarrow{DE} \cdot \overrightarrow{DF}}{|\overrightarrow{DE}| |\overrightarrow{DF}|} = \frac{\langle -2, 3, 2 \rangle \cdot \langle 1, 1, -2 \rangle}{\sqrt{(-2)^2 + 3^2 + 2^2} \sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt{17} \sqrt{6}} (-2 + 3 - 4) = -\frac{3}{\sqrt{102}}$$

$$\text{and } d = \cos^{-1} \left(-\frac{3}{\sqrt{102}} \right) \approx 107^\circ. \text{ Similarly,}$$

$$\cos e = \frac{\overrightarrow{ED} \cdot \overrightarrow{EF}}{|\overrightarrow{ED}| |\overrightarrow{EF}|} = \frac{\langle 2, -3, -2 \rangle \cdot \langle 3, -2, -4 \rangle}{\sqrt{4 + 9 + 4} \sqrt{9 + 4 + 16}} = \frac{1}{\sqrt{17} \sqrt{29}} (6 + 6 + 8) = \frac{20}{\sqrt{493}} \text{ so}$$

$$e = \cos^{-1} \left(\frac{20}{\sqrt{493}} \right) \approx 26^\circ \text{ and } f \approx 180^\circ - (107^\circ + 26^\circ) = 47^\circ.$$

Alternate solution: Apply the Law of Cosines three times as follows: $\cos d = \frac{|\overrightarrow{EF}|^2 - |\overrightarrow{DE}|^2 - |\overrightarrow{DF}|^2}{2 |\overrightarrow{DE}| |\overrightarrow{DF}|},$

$$\cos e = \frac{|\overrightarrow{DF}|^2 - |\overrightarrow{DE}|^2 - |\overrightarrow{EF}|^2}{2 |\overrightarrow{DE}| |\overrightarrow{EF}|}, \text{ and } \cos f = \frac{|\overrightarrow{DE}|^2 - |\overrightarrow{DF}|^2 - |\overrightarrow{EF}|^2}{2 |\overrightarrow{DF}| |\overrightarrow{EF}|}.$$

23. (a) $\mathbf{a} \cdot \mathbf{b} = (-5)(6) + (3)(-8) + (7)(2) = -40 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

(b) $\mathbf{a} \cdot \mathbf{b} = (4)(-3) + (6)(2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(c) $\mathbf{a} \cdot \mathbf{b} = (-1)(3) + (2)(4) + (5)(-1) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(d) Because $\mathbf{a} = -\frac{2}{3}\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

24. (a) Because $\mathbf{u} = -\frac{3}{4}\mathbf{v}$, $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors (and thus not orthogonal).

(b) $\mathbf{u} \cdot \mathbf{v} = (1)(2) + (-1)(-1) + (2)(1) = 5 \neq 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. Also, $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, so $\mathbf{u}$ and $\mathbf{v}$ are not parallel.

(c) $\mathbf{u} \cdot \mathbf{v} = (a)(-b) + (b)(a) + (c)(0) = -ab + ab + 0 = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (and not parallel).

25. $\overrightarrow{QP} = \langle -1, -3, 2 \rangle$, $\overrightarrow{QR} = \langle 4, -2, -1 \rangle$, and $\overrightarrow{QP} \cdot \overrightarrow{QR} = -4 + 6 - 2 = 0$. Thus $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal, so the angle of the triangle at vertex Q is a right angle.

26. $\langle -6, b, 2 \rangle$ and $\langle b, b^2, b \rangle$ are orthogonal when $\langle -6, b, 2 \rangle \cdot \langle b, b^2, b \rangle = 0 \Leftrightarrow (-6)(b) + (b)(b^2) + (2)(b) = 0$
 $\Leftrightarrow b^3 - 4b = 0 \Leftrightarrow b(b+2)(b-2) = 0 \Leftrightarrow b = 0 \text{ or } b = \pm 2.$

27. Let $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ be a vector orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$. Then $\mathbf{a} \cdot (\mathbf{i} + \mathbf{j}) = 0 \Leftrightarrow a_1 + a_2 = 0$ and $\mathbf{a} \cdot (\mathbf{i} + \mathbf{k}) = 0 \Leftrightarrow a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm \frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}} \mathbf{i} - \frac{1}{\sqrt{3}} \mathbf{j} - \frac{1}{\sqrt{3}} \mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}} \mathbf{i} + \frac{1}{\sqrt{3}} \mathbf{j} + \frac{1}{\sqrt{3}} \mathbf{k}$ are two such unit vectors.

28. Let $\mathbf{u} = \langle a, b \rangle$ be a unit vector. By Theorem 3 we need $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos 60^\circ \Leftrightarrow 3a + 4b = (1)(5)\frac{1}{2} \Leftrightarrow b = \frac{5}{8} - \frac{3}{4}a$. Since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = \sqrt{a^2 + b^2} = 1 \Leftrightarrow a^2 + b^2 = 1 \Leftrightarrow a^2 + \left(\frac{5}{8} - \frac{3}{4}a\right)^2 = 1 \Leftrightarrow \frac{25}{16}a^2 - \frac{15}{16}a + \frac{25}{64} = 1 \Leftrightarrow 100a^2 - 60a - 39 = 0$. By the quadratic formula,
- $$a = \frac{-(-60) \pm \sqrt{(-60)^2 - 4(100)(-39)}}{2(100)} = \frac{60 \pm \sqrt{19,200}}{200} = \frac{3 \pm 4\sqrt{3}}{10}.$$
- If $a = \frac{3 + 4\sqrt{3}}{10}$ then
- $$b = \frac{5}{8} - \frac{3}{4}\left(\frac{3 + 4\sqrt{3}}{10}\right) = \frac{4 - 3\sqrt{3}}{10},$$
- and if $a = \frac{3 - 4\sqrt{3}}{10}$ then $b = \frac{5}{8} - \frac{3}{4}\left(\frac{3 - 4\sqrt{3}}{10}\right) = \frac{4 + 3\sqrt{3}}{10}$.
- Thus the two unit vectors are $\left\langle \frac{3 + 4\sqrt{3}}{10}, \frac{4 - 3\sqrt{3}}{10} \right\rangle \approx \langle 0.9928, -0.1196 \rangle$ and
- $$\left\langle \frac{3 - 4\sqrt{3}}{10}, \frac{4 + 3\sqrt{3}}{10} \right\rangle \approx \langle -0.3928, 0.9196 \rangle.$$
29. Since $|\langle 3, 4, 5 \rangle| = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$, using Equations 8 and 9 we have $\cos \alpha = \frac{3}{5\sqrt{2}}$, $\cos \beta = \frac{4}{5\sqrt{2}}$, and $\cos \gamma = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$. The direction angles are given by $\alpha = \cos^{-1}\left(\frac{3}{5\sqrt{2}}\right) \approx 65^\circ$, $\beta = \cos^{-1}\left(\frac{4}{5\sqrt{2}}\right) \approx 56^\circ$, and $\gamma = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$.
30. Since $|\langle 1, -2, -1 \rangle| = \sqrt{1 + 4 + 1} = \sqrt{6}$, using Equations 8 and 9 we have $\cos \alpha = \frac{1}{\sqrt{6}}$, $\cos \beta = \frac{-2}{\sqrt{6}}$, and $\cos \gamma = \frac{-1}{\sqrt{6}}$. The direction angles are given by $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$, $\beta = \cos^{-1}\left(\frac{-2}{\sqrt{6}}\right) \approx 145^\circ$, and $\gamma = \cos^{-1}\left(\frac{-1}{\sqrt{6}}\right) \approx 114^\circ$.
31. Since $|2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$, Equations 8 and 9 give $\cos \alpha = \frac{2}{7}$, $\cos \beta = \frac{3}{7}$, and $\cos \gamma = \frac{-6}{7}$, while $\alpha = \cos^{-1}\left(\frac{2}{7}\right) \approx 73^\circ$, $\beta = \cos^{-1}\left(\frac{3}{7}\right) \approx 65^\circ$, and $\gamma = \cos^{-1}\left(\frac{-6}{7}\right) \approx 149^\circ$.
32. Since $|2\mathbf{i} - \mathbf{j} + 2\mathbf{k}| = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$, Equations 8 and 9 give $\cos \alpha = \frac{2}{3}$, $\cos \beta = \frac{-1}{3}$, and $\cos \gamma = \frac{2}{3}$, while $\alpha = \gamma = \cos^{-1}\left(\frac{2}{3}\right) \approx 48^\circ$ and $\beta = \cos^{-1}\left(\frac{-1}{3}\right) \approx 109^\circ$.
33. $|\langle c, c, c \rangle| = \sqrt{c^2 + c^2 + c^2} = \sqrt{3}c$ (since $c > 0$), so $\cos \alpha = \cos \beta = \cos \gamma = \frac{c}{\sqrt{3}c} = \frac{1}{\sqrt{3}}$ and
- $$\alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ.$$
34. Since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$,
- $$\cos^2 \gamma = 1 - \cos^2 \alpha - \cos^2 \beta = 1 - \cos^2\left(\frac{\pi}{4}\right) - \cos^2\left(\frac{\pi}{3}\right) = 1 - \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$
- Thus $\cos \gamma = \pm \frac{1}{2}$ and $\gamma = \frac{\pi}{3}$ or $\gamma = \frac{2\pi}{3}$.
35. $|\mathbf{a}| = \sqrt{3^2 + (-4)^2} = 5$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{3 \cdot 5 + (-4) \cdot 0}{5} = 3$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = 3 \cdot \frac{1}{5} \langle 3, -4 \rangle = \left\langle \frac{9}{5}, -\frac{12}{5} \right\rangle$.
36. $|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{1(-4) + 2 \cdot 1}{\sqrt{5}} = -\frac{2}{\sqrt{5}}$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = -\frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \langle 1, 2 \rangle = \left\langle -\frac{2}{5}, -\frac{4}{5} \right\rangle$.

37. $|\mathbf{a}| = \sqrt{16 + 4 + 0} = 2\sqrt{5}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{1}{2\sqrt{5}}(4 + 2 + 0) = \frac{3}{\sqrt{5}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{5}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{\sqrt{5}} \cdot \frac{1}{2\sqrt{5}} \langle 4, 2, 0 \rangle = \frac{1}{5} \langle 6, 3, 0 \rangle = \langle \frac{6}{5}, \frac{3}{5}, 0 \rangle$.

38. $|\mathbf{a}| = \sqrt{1 + 4 + 4} = 3$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{-3 + (-6) + 8}{3} = -\frac{1}{3}$, while

the vector projection is $\text{proj}_{\mathbf{a}} \mathbf{b} = -\frac{1}{3} \frac{\mathbf{a}}{|\mathbf{a}|} = -\frac{1}{3} \cdot \frac{\langle -1, -2, 2 \rangle}{3} = \langle \frac{1}{9}, \frac{2}{9}, -\frac{2}{9} \rangle$.

39. $|\mathbf{a}| = \sqrt{1 + 0 + 1} = \sqrt{2}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{1}{\sqrt{2}}(1 + 0 + 0) = \frac{1}{\sqrt{2}}$ while

the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{1}{\sqrt{2}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}) = \frac{1}{2}(\mathbf{i} + \mathbf{k})$.

40. $|\mathbf{a}| = \sqrt{4 + 9 + 1} = \sqrt{14}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{2 - 18 - 2}{\sqrt{14}} = -\frac{18}{\sqrt{14}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\text{proj}_{\mathbf{a}} \mathbf{b} = -\frac{18}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = -\frac{18}{\sqrt{14}} \cdot \frac{2\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{14}} = -\frac{9}{7}(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

41. $(\text{orth}_{\mathbf{a}} \mathbf{b}) \cdot \mathbf{a} = (\mathbf{b} - \text{proj}_{\mathbf{a}} \mathbf{b}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - (\text{proj}_{\mathbf{a}} \mathbf{b}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} \cdot \mathbf{a}$

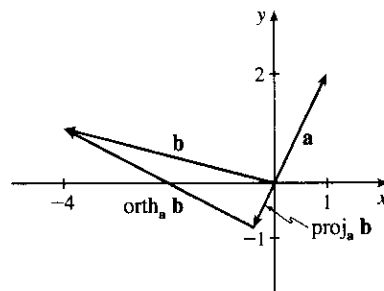
$$= \mathbf{b} \cdot \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} |\mathbf{a}|^2 = \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} = 0$$

So they are orthogonal by (7).

42. Using the formula in Exercise 41 and the result of Exercise 36,

we have

$$\begin{aligned} \text{orth}_{\mathbf{a}} \mathbf{b} &= \mathbf{b} - \text{proj}_{\mathbf{a}} \mathbf{b} = \langle -4, 1 \rangle - \left\langle -\frac{2}{5}, -\frac{4}{5} \right\rangle \\ &= \left\langle -\frac{18}{5}, \frac{9}{5} \right\rangle \end{aligned}$$



43. $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = 2 \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 2|\mathbf{a}| = 2\sqrt{10}$. If $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then we need $3b_1 + 0b_2 - 1b_3 = 2\sqrt{10}$.

One possible solution is obtained by taking $b_1 = 0, b_2 = 0, b_3 = -2\sqrt{10}$.

In general, $\mathbf{b} = \langle s, t, 3s - 2\sqrt{10} \rangle, s, t \in \mathbb{R}$.

44. (a) $\text{comp}_{\mathbf{a}} \mathbf{b} = \text{comp}_{\mathbf{b}} \mathbf{a} \Leftrightarrow \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|} \Leftrightarrow \frac{1}{|\mathbf{a}|} = \frac{1}{|\mathbf{b}|}$ or $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow |\mathbf{b}| = |\mathbf{a}|$ or $\mathbf{a} \cdot \mathbf{b} = 0$.

That is, if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or if they have the same length.

(b) $\text{proj}_{\mathbf{a}} \mathbf{b} = \text{proj}_{\mathbf{b}} \mathbf{a} \Leftrightarrow \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|^2} \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$ or $\frac{\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{b}}{|\mathbf{b}|^2}$. But $\frac{\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{b}}{|\mathbf{b}|^2} \Rightarrow$

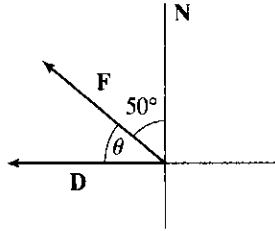
$$\frac{|\mathbf{a}|}{|\mathbf{a}|^2} = \frac{|\mathbf{b}|}{|\mathbf{b}|^2} \Rightarrow |\mathbf{a}| = |\mathbf{b}|. \text{ Substituting this into the previous equation gives } \mathbf{a} = \mathbf{b}.$$

So $\text{proj}_{\mathbf{a}} \mathbf{b} = \text{proj}_{\mathbf{b}} \mathbf{a} \Leftrightarrow \mathbf{a}$ and $\mathbf{b}$ are orthogonal, or they are equal.

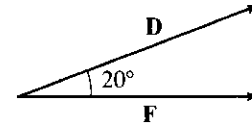
45. Here $\mathbf{D} = (4 - 2)\mathbf{i} + (9 - 3)\mathbf{j} + (15 - 0)\mathbf{k} = 2\mathbf{i} + 6\mathbf{j} + 15\mathbf{k}$ so by Equation 12 we have

$$W = \mathbf{F} \cdot \mathbf{D} = 20 + 108 - 90 = 38 \text{ joules.}$$

46. $W = |\mathbf{F}| |\mathbf{D}| \cos \theta = (20)(4) \cos 40^\circ$
 $\approx 61 \text{ ft-lb}$



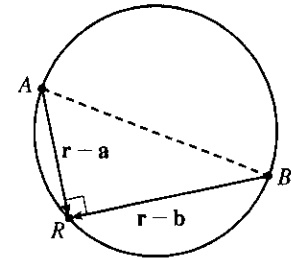
47. $W = |\mathbf{F}| |\mathbf{D}| \cos \theta = (25)(10) \cos 20^\circ$
 $\approx 235 \text{ ft-lb}$



48. Here $|\mathbf{D}| = 100 \text{ m}$, $|\mathbf{F}| = 50 \text{ N}$, and $\theta = 30^\circ$. Thus $W = |\mathbf{F}| |\mathbf{D}| \cos \theta = (50)(100) \left(\frac{\sqrt{3}}{2} \right) = 2500 \sqrt{3} \text{ joules}$.

49. First note that $\mathbf{n} = \langle a, b \rangle$ is perpendicular to the line, because if $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$ lie on the line, then $\mathbf{n} \cdot \overrightarrow{Q_1 Q_2} = aa_2 - aa_1 + bb_2 - bb_1 = 0$, since $aa_2 + bb_2 = -c = aa_1 + bb_1$ from the equation of the line. Let $P_2 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection of $\overrightarrow{P_1 P_2}$ onto $\mathbf{n}$. $\text{comp}_{\mathbf{n}} (\overrightarrow{P_1 P_2}) = \frac{|\mathbf{n} \cdot \langle x_2 - x_1, y_2 - y_1 \rangle|}{|\mathbf{n}|} = \frac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2 + b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ since $ax_2 + by_2 = -c$. The required distance is $\frac{|3 \cdot -2 + -4 \cdot 3 + 5|}{\sqrt{3^2 + 4^2}} = \frac{13}{5}$.

50. $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ implies that the vectors $\mathbf{r} - \mathbf{a}$ and $\mathbf{r} - \mathbf{b}$ are orthogonal. From the diagram (in which A, B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B . The center of this circle is the midpoint of AB , that is,



$\frac{1}{2}(\mathbf{a} + \mathbf{b}) = \left\langle \frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2), \frac{1}{2}(a_3 + b_3) \right\rangle$, and its radius is $\frac{1}{2}|\mathbf{a} - \mathbf{b}| = \frac{1}{2} \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$.

Or: Expand the given equation, substitute $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$ and complete the squares.

51. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1, 1, 1)$ has vector representation $\langle 1, 1, 1 \rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x -axis [that is, $\langle 1, 0, 0 \rangle$] is given by $\cos \theta = \frac{\langle 1, 1, 1 \rangle \cdot \langle 1, 0, 0 \rangle}{|\langle 1, 1, 1 \rangle| |\langle 1, 0, 0 \rangle|} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1} \left(\frac{1}{\sqrt{3}} \right) \approx 55^\circ$.

52. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then $\cos \theta = \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j})}{|\mathbf{i} + \mathbf{j} + \mathbf{k}| |\mathbf{i} + \mathbf{j}|} = \frac{1+1}{\sqrt{3} \sqrt{2}} = \frac{\sqrt{2}}{3} \Rightarrow \theta = \cos^{-1} \sqrt{\frac{2}{3}} \approx 35^\circ$.

53. Consider the H-C-H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1, 0, 0)$ and $(0, 1, 0)$ (or any H-C-H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\langle 1 - \frac{1}{2}, 0 - \frac{1}{2}, 0 - \frac{1}{2} \rangle = \langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rangle$ and $\langle 0 - \frac{1}{2}, 1 - \frac{1}{2}, 0 - \frac{1}{2} \rangle = \langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle$. The bond angle, θ , is therefore given by

$$\cos \theta = \frac{\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle}{|\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rangle| |\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle|} = \frac{-\frac{1}{4} - \frac{1}{4} + \frac{1}{4}}{\sqrt{\frac{3}{4}} \sqrt{\frac{3}{4}}} = -\frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109.5^\circ.$$

54. Let α be the angle between $\mathbf{a}$ and $\mathbf{c}$ and β be the angle between $\mathbf{c}$ and $\mathbf{b}$. We need to show that $\alpha = \beta$. Now

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}| |\mathbf{c}|} = \frac{\mathbf{a} \cdot |\mathbf{a}| \mathbf{b} + \mathbf{a} \cdot |\mathbf{b}| \mathbf{a}}{|\mathbf{a}| |\mathbf{c}|} = \frac{|\mathbf{a}| \mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2 |\mathbf{b}|}{|\mathbf{a}| |\mathbf{c}|} = \frac{\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}| |\mathbf{b}|}{|\mathbf{c}|}.$$
 Similarly,

$$\cos \beta = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}| |\mathbf{c}|} = \frac{|\mathbf{a}| |\mathbf{b}| + \mathbf{b} \cdot \mathbf{a}}{|\mathbf{c}|}.$$
 Thus $\cos \alpha = \cos \beta$. However $0^\circ \leq \alpha \leq 180^\circ$ and $0^\circ \leq \beta \leq 180^\circ$, so

$\alpha = \beta$ and $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

55. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

Property 2: $\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$
 $= b_1 a_1 + b_2 a_2 + b_3 a_3 = \langle b_1, b_2, b_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = \mathbf{b} \cdot \mathbf{a}$

Property 4: $(c\mathbf{a}) \cdot \mathbf{b} = \langle ca_1, ca_2, ca_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = (ca_1)b_1 + (ca_2)b_2 + (ca_3)b_3$
 $= c(a_1 b_1 + a_2 b_2 + a_3 b_3) = c(\mathbf{a} \cdot \mathbf{b}) = a_1(cb_1) + a_2(cb_2) + a_3(cb_3)$
 $= \langle a_1, a_2, a_3 \rangle \cdot \langle cb_1, cb_2, cb_3 \rangle = \mathbf{a} \cdot (c\mathbf{b})$

Property 5: $\mathbf{0} \cdot \mathbf{a} = \langle 0, 0, 0 \rangle \cdot \langle a_1, a_2, a_3 \rangle = (0)(a_1) + (0)(a_2) + (0)(a_3) = 0$

56. Let the figure be called quadrilateral $ABCD$. The diagonals can be represented by $\overrightarrow{AC}$ and $\overrightarrow{BD}$. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{BC} - \overrightarrow{AB}$ (Since opposite sides of the object are of the same length and parallel, $\overrightarrow{AB} = \overrightarrow{DC}$.) Thus

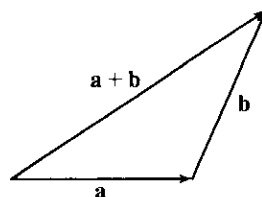
$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{BD} &= (\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} - \overrightarrow{AB}) = \overrightarrow{AB} \cdot (\overrightarrow{BC} - \overrightarrow{AB}) + \overrightarrow{BC} \cdot (\overrightarrow{BC} - \overrightarrow{AB}) \\ &= \overrightarrow{AB} \cdot \overrightarrow{BC} - |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 - \overrightarrow{AB} \cdot \overrightarrow{BC} = |\overrightarrow{BC}|^2 - |\overrightarrow{AB}|^2 \end{aligned}$$

But $|\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2$ because all sides of the quadrilateral are equal in length. Therefore $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$, and since both of these vectors are nonzero this tells us that the diagonals of the quadrilateral are perpendicular.

57. $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \cos \theta| = |\mathbf{a}| |\mathbf{b}| |\cos \theta|$. Since $|\cos \theta| \leq 1$, $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| |\cos \theta| \leq |\mathbf{a}| |\mathbf{b}|$.

Note: We have equality in the case of $\cos \theta = \pm 1$, so $\theta = 0$ or $\theta = \pi$, thus equality when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

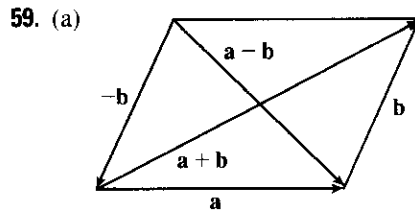
58. (a)



The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.

$$\begin{aligned} \text{(b) } |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a}) + 2(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 \\ &\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \quad [\text{by the Cauchy-Schwartz Inequality}] \\ &= (|\mathbf{a}| + |\mathbf{b}|)^2 \end{aligned}$$

Thus, taking the square root of both sides, $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

$$\begin{aligned} \text{(b) } |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 \text{ and} \\ |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2. \end{aligned}$$

Adding these two equations gives $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$.

13.4 The Cross Product

ET 12.4

$$\mathbf{1. a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \mathbf{k} = (2-0)\mathbf{i} - (1-0)\mathbf{j} + (3-0)\mathbf{k} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}}$$

Now $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle 2, -1, 3 \rangle \cdot \langle 1, 2, 0 \rangle = 2 - 2 + 0 = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle 2, -1, 3 \rangle \cdot \langle 0, 3, 1 \rangle = 0 - 3 + 3 = 0$, so $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned} \mathbf{2. a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 5 & 4 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 5 & 1 \\ -1 & 0 \end{vmatrix} \mathbf{k} \\ &= (2 - 0) \mathbf{i} - [10 - (-4)] \mathbf{j} + [0 - (-1)] \mathbf{k} = 2\mathbf{i} - 14\mathbf{j} + \mathbf{k} \end{aligned}$$

Now $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle 2, -14, 1 \rangle \cdot \langle 5, 1, 4 \rangle = 10 - 14 + 4 = 0$ and

Now $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle 2, -14, 1 \rangle \cdot \langle 5, 1, 4 \rangle = 10 - 14 + 4 = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle 2, -14, 1 \rangle \cdot \langle -1, 0, 2 \rangle = -2 + 0 + 2 = 0$, so $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned} \mathbf{3. a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= [2 - (-1)] \mathbf{i} - (4 - 0) \mathbf{j} + (2 - 0) \mathbf{k} = 3 \mathbf{i} - 4 \mathbf{j} + 2 \mathbf{k} \end{aligned}$$

Now $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 6 - 4 - 2 = 0$ and

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{j} + 2\mathbf{k}) = 0 - 4 + 4 = 0, \text{ so } \mathbf{a} \times \mathbf{b} \text{ is orthogonal to both } \mathbf{a} \text{ and } \mathbf{b}.$$

$$\begin{aligned} \mathbf{4. a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= (-1-1)\mathbf{i} - (1-1)\mathbf{j} + [1-(-1)]\mathbf{k} = -2\mathbf{i} + 2\mathbf{k} \end{aligned}$$

Now $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (-2\mathbf{i} + 2\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = -2 + 0 + 2 = 0$ and

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (-2\mathbf{i} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = -2 + 0 + 2 = 0, \text{ so } \mathbf{a} \times \mathbf{b} \text{ is orthogonal to both } \mathbf{a} \text{ and } \mathbf{b}.$$

$$\begin{aligned}
 5. \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{k} \\
 &= [-6 - (-8)]\mathbf{i} - (-9 - 4)\mathbf{j} + (-6 - 2)\mathbf{k} = 2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}
 \end{aligned}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = 6 + 26 - 32 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 2 - 26 + 24 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{b}$.

$$\begin{aligned}
 6. \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & e^t & e^{-t} \\ 2 & e^t & -e^{-t} \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & e^{-t} \\ 2 & -e^{-t} \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & e^t \\ 2 & e^t \end{vmatrix} \mathbf{k} \\
 &= (-1 - 1)\mathbf{i} - (-e^{-t} - 2e^{-t})\mathbf{j} + (e^t - 2e^t)\mathbf{k} = -2\mathbf{i} + 3e^{-t}\mathbf{j} - e^t\mathbf{k}
 \end{aligned}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (-2\mathbf{i} + 3e^{-t}\mathbf{j} - e^t\mathbf{k}) \cdot (\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}) = -2 + 3 - 1 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.

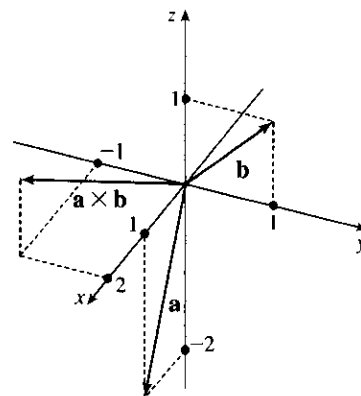
Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (-2\mathbf{i} + 3e^{-t}\mathbf{j} - e^t\mathbf{k}) \cdot (2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}) = -4 + 3 + 1 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{b}$.

$$\begin{aligned}
 7. \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} \mathbf{k} \\
 &= (3t^4 - 2t^4)\mathbf{i} - (3t^3 - t^3)\mathbf{j} + (2t^2 - t^2)\mathbf{k} = t^4\mathbf{i} - 2t^3\mathbf{j} + t^2\mathbf{k}
 \end{aligned}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle t, t^2, t^3 \rangle = t^5 - 2t^5 + t^5 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle = t^4 - 4t^4 + 3t^4 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{b}$.

$$\begin{aligned}
 8. \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
 &= 2\mathbf{i} - \mathbf{j} + \mathbf{k}
 \end{aligned}$$

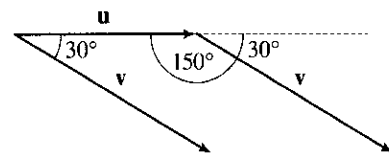


9. (a) Since $\mathbf{b} \times \mathbf{c}$ is a vector, the dot product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is meaningful and is a scalar.
- (b) $\mathbf{b} \cdot \mathbf{c}$ is a scalar, so $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.
- (c) Since $\mathbf{b} \times \mathbf{c}$ is a vector, the cross product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is meaningful and results in another vector.
- (d) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, so the cross product $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ is meaningless.
- (e) Since $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{c} \cdot \mathbf{d})$ are both scalars, the cross product $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ is meaningless.
- (f) $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are both vectors, so the dot product $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ is meaningful and is a scalar.
10. Using Theorem 6, we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = (5)(10)\sin 60^\circ = 25\sqrt{3}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed into the page.

11. If we sketch $\mathbf{u}$ and $\mathbf{v}$ starting from the same initial point, we see that the angle between them is 30° . Using Theorem 6, we have

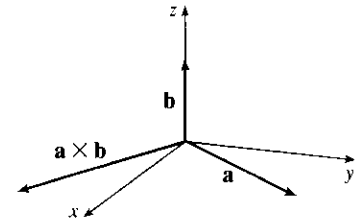
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin 30^\circ = (6)(8)\left(\frac{1}{2}\right) = 24$$

By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed into the page.



12. (a) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta = 3 \cdot 2 \cdot \sin \frac{\pi}{2} = 6$

(b) $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{k}$, so it lies in the xy -plane, and its z -coordinate is 0. By the right-hand rule, its y -component is negative and its x -component is positive.



$$13. \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \mathbf{k} = (6-1)\mathbf{i} - (3-0)\mathbf{j} + (1-0)\mathbf{k} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} = (1-6)\mathbf{i} - (0-3)\mathbf{j} + (0-1)\mathbf{k} = -5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Notice $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ here, as we know is always true by Theorem 8.

$$14. \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 0 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{k} = -4\mathbf{i} - 4\mathbf{j} \text{ so}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ -4 & -4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -4 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ -4 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ -4 & -4 \end{vmatrix} \mathbf{k} = 8\mathbf{i} - 8\mathbf{j} - 8\mathbf{k}.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} \mathbf{k} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \text{ so}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 4 \\ 0 & 0 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 0 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 4 \\ 0 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -2 \\ 0 & 0 \end{vmatrix} \mathbf{k} = 8\mathbf{i} - 8\mathbf{j}.$$

Thus $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

15. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$\langle 1, -1, 1 \rangle \times \langle 0, 4, 4 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} \mathbf{k} = -8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}.$$

So two unit vectors orthogonal to both are $\pm \frac{\langle -8, -4, 4 \rangle}{\sqrt{64 + 16 + 16}} = \pm \frac{\langle -8, -4, 4 \rangle}{4\sqrt{6}}$, that is, $\left\langle -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$

and $\left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$.

16. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \mathbf{k} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}. \text{ Thus, two unit vectors orthogonal to both}$$

are $\pm \frac{1}{\sqrt{6}} \langle 1, 1, -2 \rangle$, that is, $\left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle$ and $\left\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$.

17. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Then

$$\mathbf{0} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} = \mathbf{0},$$

$$\mathbf{a} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix} \mathbf{k} = \mathbf{0}.$$

18. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} &= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3 \\ &= (a_2 b_3 b_1 - a_3 b_2 b_1) - (a_1 b_3 b_2 - a_3 b_1 b_2) + (a_1 b_2 b_3 - a_2 b_1 b_3) = 0 \end{aligned}$$

$$\begin{aligned} 19. \mathbf{a} \times \mathbf{b} &= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \\ &= \langle (-1)(b_2 a_3 - b_3 a_2), (-1)(b_3 a_1 - b_1 a_3), (-1)(b_1 a_2 - b_2 a_1) \rangle \\ &= -\langle b_2 a_3 - b_3 a_2, b_3 a_1 - b_1 a_3, b_1 a_2 - b_2 a_1 \rangle = -\mathbf{b} \times \mathbf{a} \end{aligned}$$

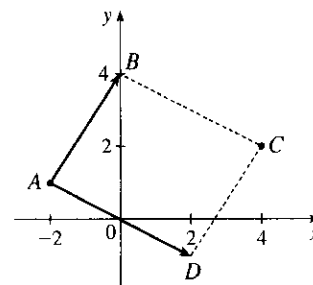
20. $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$, so

$$\begin{aligned} (c\mathbf{a}) \times \mathbf{b} &= \langle ca_2 b_3 - ca_3 b_2, ca_3 b_1 - ca_1 b_3, ca_1 b_2 - ca_2 b_1 \rangle \\ &= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle = c(\mathbf{a} \times \mathbf{b}) \\ &= \langle ca_2 b_3 - ca_3 b_2, ca_3 b_1 - ca_1 b_3, ca_1 b_2 - ca_2 b_1 \rangle \\ &= \langle a_2(cb_3) - a_3(cb_2), a_3(cb_1) - a_1(cb_3), a_1(cb_2) - a_2(cb_1) \rangle \\ &= \mathbf{a} \times c\mathbf{b} \end{aligned}$$

$$\begin{aligned} 21. \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle \\ &= \langle a_2 b_3 + a_2 c_3 - a_3 b_2 - a_3 c_2, a_3 b_1 + a_3 c_1 - a_1 b_3 - a_1 c_3, a_1 b_2 + a_1 c_2 - a_2 b_1 - a_2 c_1 \rangle \\ &= \langle (a_2 b_3 - a_3 b_2) + (a_2 c_3 - a_3 c_2), (a_3 b_1 - a_1 b_3) + (a_3 c_1 - a_1 c_3), \\ &\quad (a_1 b_2 - a_2 b_1) + (a_1 c_2 - a_2 c_1) \rangle \\ &= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle + \langle a_2 c_3 - a_3 c_2, a_3 c_1 - a_1 c_3, a_1 c_2 - a_2 c_1 \rangle \\ &= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) \end{aligned}$$

$$\begin{aligned}
 22. \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= -\mathbf{c} \times (\mathbf{a} + \mathbf{b}) && \text{by Property 1 of Theorem 8} \\
 &= -(\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}) && \text{by Property 3 of Theorem 8} \\
 &= -(-\mathbf{a} \times \mathbf{c} + (-\mathbf{b} \times \mathbf{c})) && \text{by Property 1 of Theorem 8} \\
 &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} && \text{by Property 2 of Theorem 8}
 \end{aligned}$$

23. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB} = \langle 2, 3 \rangle$ and $\overrightarrow{AD} = \langle 4, -2 \rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three-dimensional vector $\langle 2, 3, 0 \rangle$ (and similarly for $\overrightarrow{AD}$), and then the area of parallelogram $ABCD$ is



$$\left| \overrightarrow{AB} \times \overrightarrow{AD} \right| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} \right| = |(0)\mathbf{i} - (0)\mathbf{j} + (-4 - 12)\mathbf{k}| = |-16\mathbf{k}| = 16$$

24. The parallelogram is determined by the vectors $\overrightarrow{KL} = \langle 0, 1, 3 \rangle$ and $\overrightarrow{KN} = \langle 2, 5, 0 \rangle$, so the area of parallelogram $KLMN$ is

$$\left| \overrightarrow{KL} \times \overrightarrow{KN} \right| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} \right| = |(-15)\mathbf{i} - (-6)\mathbf{j} + (-2)\mathbf{k}| = |-15\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}| = \sqrt{265} \approx 16.28$$

25. (a) Because the plane through P , Q , and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ} = \langle -1, 2, 0 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(3) - (0)(0), (0)(-1) - (-1)(3), (-1)(0) - (2)(-1) \rangle = \langle 6, 3, 2 \rangle$$

Therefore, $\langle 6, 3, 2 \rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P , Q , and R .

- (b) Note that the area of the triangle determined by P , Q , and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is

$$\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = |\langle 6, 3, 2 \rangle| = \sqrt{36 + 9 + 4} = 7, \text{ so the area of the triangle is } \frac{1}{2}(7) = \frac{7}{2}.$$

26. (a) $\overrightarrow{PQ} = \langle -3, 2, -1 \rangle$ and $\overrightarrow{PR} = \langle 1, -1, 1 \rangle$, so a vector orthogonal to the plane through P , Q , and R is $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(1) - (-1)(-1), (-1)(1) - (-3)(1), (-3)(-1) - (2)(1) \rangle = \langle 1, 2, 1 \rangle$ (or any scalar multiple thereof).

- (b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = |\langle 1, 2, 1 \rangle| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}, \text{ so the area of triangle } PQR \text{ is } \frac{1}{2}\sqrt{6}.$$

27. (a) $\overrightarrow{PQ} = \langle 4, 3, -2 \rangle$ and $\overrightarrow{PR} = \langle 5, 5, 1 \rangle$, so a vector orthogonal to the plane through P , Q , and R is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (3)(1) - (-2)(5), (-2)(5) - (4)(1), (4)(5) - (3)(5) \rangle = \langle 13, -14, 5 \rangle$$

(or any scalar multiple thereof).

- (b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle 13, -14, 5 \rangle| = \sqrt{13^2 + (-14)^2 + 5^2} = \sqrt{390}, \text{ so the area of triangle } PQR \text{ is } \frac{1}{2}\sqrt{390}.$$

28. (a) $\overrightarrow{PQ} = \langle 1, 1, 3 \rangle$ and $\overrightarrow{PR} = \langle 3, 2, 5 \rangle$, so a vector orthogonal to the plane through P , Q , and R is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)(5) - (3)(2), (3)(3) - (1)(5), (1)(2) - (1)(3) \rangle = \langle -1, 4, -1 \rangle \text{ (or any scalar multiple thereof).}$$

- (b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -1, 4, -1 \rangle| = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}, \text{ so the area of triangle } PQR \text{ is}$$

$$\frac{1}{2} \cdot 3\sqrt{2} = \frac{3}{2}\sqrt{2}.$$

29. We know that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product, which is

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix} \\ &= 6(5 + 4) - 3(0 - 8) - (0 - 4) = 82 \end{aligned}$$

Thus the volume of the parallelepiped is 82 cubic units.

$$30. \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = -2 - 2 + 0 = -4.$$

So the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|-4| = 4$ cubic units.

31. $\mathbf{a} = \overrightarrow{PQ} = \langle 2, 1, 1 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 1, -1, 2 \rangle$, and $\mathbf{c} = \overrightarrow{PS} = \langle 0, -2, 3 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} = 2 \begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = 2 - 3 - 2 = -3,$$

so the volume of the parallelepiped is 3 cubic units.

32. $\mathbf{a} = \overrightarrow{PQ} = \langle 2, 3, 3 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle -1, -1, -1 \rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 6, -2, 2 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & 3 \\ -1 & -1 & -1 \\ 6 & -2 & 2 \end{vmatrix} = 2 \begin{vmatrix} -1 & -1 \\ -2 & 2 \end{vmatrix} - 3 \begin{vmatrix} -1 & -1 \\ 6 & 2 \end{vmatrix} + 3 \begin{vmatrix} -1 & -1 \\ 6 & -2 \end{vmatrix} = -8 - 12 + 24 = 4,$$

so the volume of the parallelepiped is 4 cubic units.

$$33. \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = 2 \begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ 7 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 7 & 3 \end{vmatrix} = -4 - 6 + 10 = 0, \text{ which says that the}$$

volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 0, and thus these three vectors are coplanar.

34. $\mathbf{a} = \overrightarrow{PQ} = \langle 1, 4, 5 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 2, -1, 1 \rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 5, 2, 7 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & 2 & 7 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 2 & 7 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix} + 5 \begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = -9 - 36 + 45 = 0,$$

so the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 0, which says that these vectors lie in the same plane. Therefore, their initial and terminal points P , Q , R and S also lie in the same plane.

35. The magnitude of the torque is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta = (0.18 \text{ m})(60 \text{ N}) \sin(70 + 10)^\circ = 10.8 \sin 80^\circ \approx 10.6 \text{ J}.$$

36. $|\mathbf{r}| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ ft. A line drawn from the point P to the point of application of the force makes an angle of $180^\circ - (45 + 30)^\circ = 105^\circ$ with the force vector. Therefore,

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta = (4\sqrt{2})(36) \sin 105^\circ \approx 197 \text{ ft}\cdot\text{lb}.$$

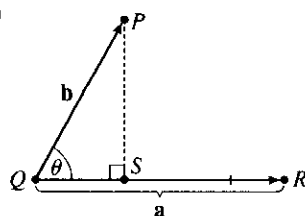
37. Using the notation of the text, $\mathbf{r} = \langle 0, 0.3, 0 \rangle$ and $\mathbf{F}$ has direction $\langle 0, 3, -4 \rangle$. The angle θ between them can be

$$\text{determined by } \cos \theta = \frac{\langle 0, 0.3, 0 \rangle \cdot \langle 0, 3, -4 \rangle}{|\langle 0, 0.3, 0 \rangle| |\langle 0, 3, -4 \rangle|} \Rightarrow \cos \theta = \frac{0.9}{(0.3)(5)} \Rightarrow \cos \theta = 0.6 \Rightarrow \theta \approx 53.1^\circ.$$

$$\text{Then } |\boldsymbol{\tau}| = |\mathbf{r}| |\mathbf{F}| \sin \theta \Rightarrow 100 = 0.3 |\mathbf{F}| \sin 53.1^\circ \Rightarrow |\mathbf{F}| \approx 417 \text{ N}.$$

38. Since $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$, $0 \leq \theta \leq \pi$, $|\mathbf{u} \times \mathbf{v}|$ achieves its maximum value for $\sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$, in which case $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| = 15$. The minimum value is zero, which occurs when $\sin \theta = 0 \Rightarrow \theta = 0$ or π , so when $\mathbf{u}$, $\mathbf{v}$ are parallel. Thus, when $\mathbf{u}$ points in the same direction as $\mathbf{v}$, so $\mathbf{u} = 3\mathbf{j}$, $|\mathbf{u} \times \mathbf{v}| = 0$. As $\mathbf{u}$ rotates counterclockwise, $\mathbf{u} \times \mathbf{v}$ is directed in the negative z -direction (by the right-hand rule) and the length increases until $\theta = \frac{\pi}{2}$, in which case $\mathbf{u} = -3\mathbf{i}$ and $|\mathbf{u} \times \mathbf{v}| = 15$. As $\mathbf{u}$ rotates to the negative y -axis, $\mathbf{u} \times \mathbf{v}$ remains pointed in the negative z -direction and the length of $\mathbf{u} \times \mathbf{v}$ decreases to 0, after which the direction of $\mathbf{u} \times \mathbf{v}$ reverses to point in the positive z -direction and $|\mathbf{u} \times \mathbf{v}|$ increases. When $\mathbf{u} = 3\mathbf{i}$ (so $\theta = \frac{\pi}{2}$), $|\mathbf{u} \times \mathbf{v}|$ again reaches its maximum of 15, after which $|\mathbf{u} \times \mathbf{v}|$ decreases to 0 as $\mathbf{u}$ rotates to the positive y -axis.

39. (a)



The distance between a point and a line is the length of the

perpendicular from the point to the line, here $|\overrightarrow{PS}| = d$. But referring

to triangle PQS , $d = |\overrightarrow{PS}| = |\overrightarrow{QP}| \sin \theta = |\mathbf{b}| \sin \theta$. But θ is the

angle between $\overrightarrow{QP} = \mathbf{b}$ and $\overrightarrow{QR} = \mathbf{a}$. Thus by Theorem 6,

$$\sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} \text{ and so } d = |\mathbf{b}| \sin \theta = \frac{|\mathbf{b}| |\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}.$$

- (b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, -2, -1 \rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 1, -5, -7 \rangle$. Then

$$\mathbf{a} \times \mathbf{b} = \langle (-2)(-7) - (-1)(-5), (-1)(1) - (-1)(-7), (-1)(-5) - (-2)(1) \rangle = \langle 9, -8, 7 \rangle. \text{ Thus the}$$

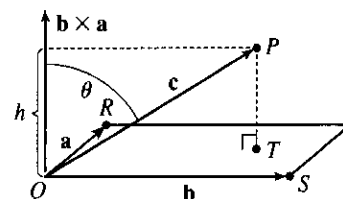
$$\text{distance is } d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{1}{\sqrt{6}} \sqrt{81 + 64 + 49} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}.$$

40. (a) The distance between a point and a plane is the length of the

perpendicular from the point to the plane, here $|\overrightarrow{TP}| = d$. But $\overrightarrow{TP}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (because $\mathbf{b} \times \mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{a}$) and

$$d = \left| \overrightarrow{TP} \right| = \text{the absolute value of the scalar projection of } \mathbf{c} \text{ along}$$

$\mathbf{b} \times \mathbf{a}$, which is $|\mathbf{c}| |\cos \theta|$. (Notice that this is the same setup as the development of the volume of a parallelepiped with $h = |\mathbf{c}| |\cos \theta|$). Thus $d = |\mathbf{c}| |\cos \theta| = h = V/A$ where $A = |\mathbf{a} \times \mathbf{b}|$, the area of the base.

$$\text{So finally } d = \frac{V}{A} = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|} = \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{|\mathbf{a} \times \mathbf{b}|} \text{ by Theorem 8 \#5.}$$


(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, 2, 0 \rangle$, $\mathbf{b} = \overrightarrow{QS} = \langle -1, 0, 3 \rangle$ and $\mathbf{c} = \overrightarrow{QP} = \langle 1, 1, 4 \rangle$. Then

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} + 0 = 17$$

$$\text{and} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} \mathbf{k} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\text{Thus } d = \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{|\mathbf{a} \times \mathbf{b}|} = \frac{17}{\sqrt{36+9+4}} = \frac{17}{7}.$$

$$\begin{aligned}
 41. (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= (\mathbf{a} - \mathbf{b}) \times \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{b} && \text{by Theorem 8 \#3} \\
 &= \mathbf{a} \times \mathbf{a} + (-\mathbf{b}) \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + (-\mathbf{b}) \times \mathbf{b} && \text{by Theorem 8 \#4} \\
 &= (\mathbf{a} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{b}) && \text{by Theorem 8 \#2 (with } c = -1) \\
 &= \mathbf{0} - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - \mathbf{0} && \text{by Example 2} \\
 &= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) && \text{by Theorem 8 \#1} \\
 &= 2(\mathbf{a} \times \mathbf{b})
 \end{aligned}$$

42. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, so $\mathbf{b} \times \mathbf{c} = \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 \rangle$
and

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1), \\ &\quad a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \rangle \\ &= \langle a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3, a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1, \\ &\quad a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \rangle \\ &= \langle (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1, (a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2, \\ &\quad (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 \rangle \end{aligned}$$

$$\begin{aligned}
 (*) &= \langle (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 + a_1b_1c_1 - a_1b_1c_1, \\
 &\quad (a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2 + a_2b_2c_2 - a_2b_2c_2, \\
 &\quad (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 + a_3b_3c_3 - a_3b_3c_3 \rangle \\
 &= \langle (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1, \\
 &\quad (a_1c_1 + a_2c_2 + a_3c_3)b_2 - (a_1b_1 + a_2b_2 + a_3b_3)c_2, \\
 &\quad (a_1c_1 + a_2c_2 + a_3c_3)b_3 - (a_1b_1 + a_2b_2 + a_3b_3)c_3 \rangle \\
 &= (a_1c_1 + a_2c_2 + a_3c_3) \langle b_1, b_2, b_3 \rangle - (a_1b_1 + a_2b_2 + a_3b_3) \langle c_1, c_2, c_3 \rangle \\
 &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
 \end{aligned}$$

(*) Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don't change the value of the component.

43. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$

$$\begin{aligned}
 &= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] + [(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] + [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}] \quad \text{by Exercise 42} \\
 &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \mathbf{0}
 \end{aligned}$$

44. Let $\mathbf{c} \times \mathbf{d} = \mathbf{v}$. Then

$$\begin{aligned}
 (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{v} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{v}) \quad \text{by Theorem 8 \#5} \\
 &= \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})] \\
 &= \mathbf{a} \cdot [(\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d}] \quad \text{by Exercise 42} \\
 &= (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \quad \text{by Properties 3 and 4 of the dot product} \\
 &= \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}
 \end{aligned}$$

45. (a) No. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$, which can happen if $\mathbf{b} \neq \mathbf{c}$.
For example, let $\mathbf{a} = \langle 1, 1, 1 \rangle$, $\mathbf{b} = \langle 1, 0, 0 \rangle$ and $\mathbf{c} = \langle 0, 1, 0 \rangle$.

(b) No. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ then $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, which implies that $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$, which of course can happen if $\mathbf{b} \neq \mathbf{c}$.

(c) Yes. Since $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$, by part (a). From part (b), $\mathbf{a}$ is also parallel to $\mathbf{b} - \mathbf{c}$.
Thus since $\mathbf{a} \neq \mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b} - \mathbf{c}$, we have $\mathbf{b} - \mathbf{c} = \mathbf{0}$, so $\mathbf{b} = \mathbf{c}$.

46. (a) $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_i$ if $i \neq j$ by the definition of $\mathbf{k}_i$ and Theorem 5.

$$\begin{aligned}
 (b) \mathbf{k}_1 \cdot \mathbf{v}_1 &= \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \cdot \mathbf{v}_1 = \frac{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} = 1 \\
 \mathbf{k}_2 \cdot \mathbf{v}_2 &= \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \cdot \mathbf{v}_2 = \frac{\mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{v}_1)}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} = \frac{(\mathbf{v}_2 \times \mathbf{v}_3) \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} = 1 \quad [\text{by Theorem 8 \#5}] \\
 \mathbf{k}_3 \cdot \mathbf{v}_3 &= \frac{(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} = \frac{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} = 1 \quad [\text{by Theorem 8 \#5}]
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } \mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) &= \mathbf{k}_1 \cdot \left(\frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \times \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \right) = \frac{\mathbf{k}_1}{[\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]^2} \cdot [(\mathbf{v}_3 \times \mathbf{v}_1) \times (\mathbf{v}_1 \times \mathbf{v}_2)] \\
 &= \frac{\mathbf{k}_1}{[\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]^2} \cdot ([(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2] \mathbf{v}_1 - [(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_1] \mathbf{v}_2) \quad [\text{by Exercise 42}]
 \end{aligned}$$

But $(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_1 = 0$ since $\mathbf{v}_3 \times \mathbf{v}_1$ is orthogonal to $\mathbf{v}_1$, and

$(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{v}_1) = (\mathbf{v}_2 \times \mathbf{v}_3) \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$. Thus

$$\begin{aligned}
 \mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) &= \frac{\mathbf{k}_1}{[\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]^2} \cdot [\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)] \mathbf{v}_1 = \frac{\mathbf{k}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \\
 &= \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \quad [\text{by part (b)}]
 \end{aligned}$$

DISCOVERY PROJECT The Geometry of a Tetrahedron

1. Set up a coordinate system so that vertex S is at the origin, $R = (0, y_1, 0)$, $Q = (x_2, y_2, 0)$, $P = (x_3, y_3, z_3)$.

Then $\overrightarrow{SR} = \langle 0, y_1, 0 \rangle$, $\overrightarrow{SQ} = \langle x_2, y_2, 0 \rangle$, $\overrightarrow{SP} = \langle x_3, y_3, z_3 \rangle$, $\overrightarrow{QR} = \langle -x_2, y_1 - y_2, 0 \rangle$, and $\overrightarrow{QP} = \langle x_3 - x_2, y_3 - y_2, z_3 \rangle$. Let

$$\begin{aligned}
 \mathbf{v}_S &= \overrightarrow{QR} \times \overrightarrow{QP} \\
 &= (y_1 z_3 - y_2 z_3) \mathbf{i} + x_2 z_3 \mathbf{j} + (-x_2 y_3 - x_3 y_1 + x_3 y_2 + x_2 y_1) \mathbf{k}
 \end{aligned}$$

Then $\mathbf{v}_S$ is an outward normal to the face opposite vertex S . Similarly,

$\mathbf{v}_R = \overrightarrow{SQ} \times \overrightarrow{SP} = y_2 z_3 \mathbf{i} - x_2 z_3 \mathbf{j} + (x_2 y_3 - x_3 y_2) \mathbf{k}$, $\mathbf{v}_Q = \overrightarrow{SP} \times \overrightarrow{SR} = -y_1 z_3 \mathbf{i} + x_3 y_1 \mathbf{k}$, and

$\mathbf{v}_P = \overrightarrow{SR} \times \overrightarrow{SQ} = -x_2 y_1 \mathbf{k} \Rightarrow \mathbf{v}_S + \mathbf{v}_R + \mathbf{v}_Q + \mathbf{v}_P = \mathbf{0}$. Now

$$\begin{aligned}
 |\mathbf{v}_S| &= \text{area of the parallelogram determined by } \overrightarrow{QR} \text{ and } \overrightarrow{QP} \\
 &= 2 (\text{area of triangle } RQP) \\
 &= 2 |\mathbf{v}_1|
 \end{aligned}$$

So $\mathbf{v}_S = 2\mathbf{v}_1$, and similarly $\mathbf{v}_R = 2\mathbf{v}_2$, $\mathbf{v}_Q = 2\mathbf{v}_3$, $\mathbf{v}_P = 2\mathbf{v}_4$. Thus $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$.

2. (a) Let $S = (x_0, y_0, z_0)$, $R = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, $P = (x_3, y_3, z_3)$ be the four vertices. Then

$$\begin{aligned}
 \text{Volume} &= \frac{1}{3} (\text{distance from } S \text{ to plane } RQP) \times (\text{area of triangle } RQP) \\
 &= \frac{1}{3} \frac{|\mathbf{N} \cdot \overrightarrow{SR}|}{|\mathbf{N}|} \cdot \frac{1}{2} |\overrightarrow{RQ} \times \overrightarrow{RP}|
 \end{aligned}$$

where $\mathbf{N}$ is a vector which is normal to the face RQP . Thus $\mathbf{N} = \overrightarrow{RQ} \times \overrightarrow{RP}$. Therefore

$$V = \left| \frac{1}{6} (\overrightarrow{RQ} \times \overrightarrow{RP}) \cdot \overrightarrow{SR} \right| = \frac{1}{6} \begin{vmatrix} x_0 - x_1 & y_0 - y_1 & z_0 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}.$$

$$\text{(b) Using the formula from part (a), } V = \frac{1}{6} \begin{vmatrix} 1-1 & 1-2 & 1-3 \\ 1-1 & 1-2 & 2-3 \\ 3-1 & -1-2 & 2-3 \end{vmatrix} = \frac{1}{6} |2(1-2)| = \frac{1}{3}.$$

3. We define a vector $\mathbf{v}_1$ to have length equal to the area of the face opposite vertex P , so we can say $|\mathbf{v}_1| = A$, and direction perpendicular to the face and pointing outward, as in Problem 1. Similarly, we define $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ so that $|\mathbf{v}_2| = B$, $|\mathbf{v}_3| = C$, and $|\mathbf{v}_4| = D$ and with the analogous directions. From Problem 1, we know $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0} \Rightarrow \mathbf{v}_4 = -(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \Rightarrow |\mathbf{v}_4| = |-(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)| = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| \Rightarrow |\mathbf{v}_4|^2 = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|^2 \Rightarrow$

$$\begin{aligned}\mathbf{v}_4 \cdot \mathbf{v}_4 &= (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \\ &= \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_2 \cdot \mathbf{v}_3 + \mathbf{v}_3 \cdot \mathbf{v}_1 + \mathbf{v}_3 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3\end{aligned}$$

Since the vertex S is trirectangular, we know the three faces meeting at S are mutually perpendicular, so the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are also mutually perpendicular. Therefore, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Thus we have $\mathbf{v}_4 \cdot \mathbf{v}_4 = \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3 \Rightarrow |\mathbf{v}_4|^2 = |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 + |\mathbf{v}_3|^2 \Rightarrow D^2 = A^2 + B^2 + C^2$.

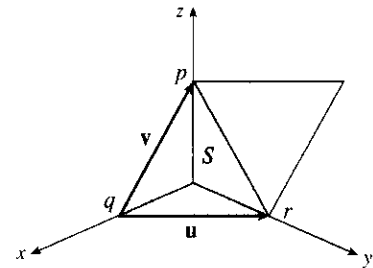
Another method: We introduce a coordinate system, as shown.

Recall that the area of the parallelogram spanned by two vectors is equal to the length of their cross product, so since

$\mathbf{u} \times \mathbf{v} = \langle -q, r, 0 \rangle \times \langle -q, 0, p \rangle = \langle pr, pq, qr \rangle$, we have

$|\mathbf{u} \times \mathbf{v}| = \sqrt{(pr)^2 + (pq)^2 + (qr)^2}$, and therefore

$$\begin{aligned}D^2 &= \left(\frac{1}{2}|\mathbf{u} \times \mathbf{v}|\right)^2 = \frac{1}{4}[(pr)^2 + (pq)^2 + (qr)^2] \\ &= \left(\frac{1}{2}pr\right)^2 + \left(\frac{1}{2}pq\right)^2 + \left(\frac{1}{2}qr\right)^2 = A^2 + B^2 + C^2.\end{aligned}$$



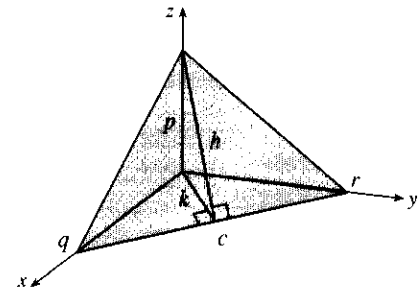
A third method: We draw a line from S perpendicular to QR , as shown. Now $D = \frac{1}{2}ch$, so $D^2 = \frac{1}{4}c^2h^2$. Substituting

$h^2 = p^2 + k^2$, we get $D^2 = \frac{1}{4}c^2(p^2 + k^2) = \frac{1}{4}c^2p^2 + \frac{1}{4}c^2k^2$.

But $C = \frac{1}{2}ck$, so $D^2 = \frac{1}{4}c^2p^2 + C^2$. Now substituting

$c^2 = q^2 + r^2$ gives

$$D^2 = \frac{1}{4}p^2q^2 + \frac{1}{4}q^2r^2 + C^2 = A^2 + B^2 + C^2.$$



13.5 Equations of Lines and Planes

ET 12.5

- (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.
- (b) False; for example, the x - and y -axes are both perpendicular to the z -axis, yet the x - and y -axes are not parallel.
- (c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.
- (d) False; for example, the xy - and yz -planes are not parallel, yet they are both perpendicular to the xz -plane.
- (e) False; the x - and y -axes are not parallel, yet they are both parallel to the plane $z = 1$.
- (f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

- (g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x -axis.
- (h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.
- (i) True; see Figure 9 and the accompanying discussion.
- (j) False; they can be skew, as in Example 3.
- (k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ , $0^\circ \leq \theta < 90^\circ$, and the line will intersect the plane at an angle $90^\circ - \theta$.

2. For this line, we have $\mathbf{r}_0 = \mathbf{i} - 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, so a vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (\mathbf{i} - 3\mathbf{k}) + t(2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = (1 + 2t)\mathbf{i} - 4t\mathbf{j} + (-3 + 5t)\mathbf{k} \text{ and parametric equations are } x = 1 + 2t, y = -4t, z = -3 + 5t.$$

3. For this line, we have $\mathbf{r}_0 = -2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 8\mathbf{k}$, so a vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (-2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}) + t(3\mathbf{i} + \mathbf{j} - 8\mathbf{k}) = (-2 + 3t)\mathbf{i} + (4 + t)\mathbf{j} + (10 - 8t)\mathbf{k} \text{ and parametric equations are } x = -2 + 3t, y = 4 + t, z = 10 - 8t.$$

4. This line has the same direction as the given line, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Here $\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, so a vector equation is $\mathbf{r} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2t\mathbf{i} - t\mathbf{j} + 3t\mathbf{k}$ and parametric equations are $x = 2t, y = -t, z = 3t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1, 3, 1 \rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = (1 + t)\mathbf{i} + 3t\mathbf{j} + (6 + t)\mathbf{k}$, and parametric equations are $x = 1 + t, y = 3t, z = 6 + t$.

6. The vector $\mathbf{v} = \langle 1 - 0, 2 - 0, 3 - 0 \rangle = \langle 1, 2, 3 \rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 0 + 1 \cdot t = t, y = 0 + 2 \cdot t = 2t, z = 0 + 3 \cdot t = 3t$, while symmetric equations are $x = \frac{y}{2} = \frac{z}{3}$.

7. The vector $\mathbf{v} = \langle -4 - 1, 3 - 3, 0 - 2 \rangle = \langle -5, 0, -2 \rangle$ is parallel to the line. Letting $P_0 = (1, 3, 2)$, parametric equations are $x = 1 - 5t, y = 3 + 0t = 3, z = 2 - 2t$, while symmetric equations are $\frac{x-1}{-5} = \frac{z-2}{-2}, y = 3$.

Notice here that the direction number $b = 0$, so rather than writing $\frac{y-3}{0}$ in the symmetric equation we must write the equation $y = 3$ separately.

8. $\mathbf{v} = \langle 2 - 6, 4 - 1, 5 - (-3) \rangle = \langle -4, 3, 8 \rangle$, and letting $P_0 = (6, 1, -3)$, parametric equations are $x = 6 - 4t, y = 1 + 3t, z = -3 + 8t$, while symmetric equations are $\frac{x-6}{-4} = \frac{y-1}{3} = \frac{z+3}{8}$.

9. $\mathbf{v} = \langle 2 - 0, 1 - \frac{1}{2}, -3 - 1 \rangle = \langle 2, \frac{1}{2}, -4 \rangle$, and letting $P_0 = (2, 1, -3)$, parametric equations are $x = 2 + 2t, y = 1 + \frac{1}{2}t, z = -3 - 4t$, while symmetric equations are $\frac{x-2}{2} = \frac{y-1}{1/2} = \frac{z+3}{-4}$ or $\frac{x-2}{2} = 2y-2 = \frac{z+3}{-4}$.

10. $\mathbf{v} = (\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ is the direction of the line perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

With $P_0 = (2, 1, 0)$, parametric equations are $x = 2 + t, y = 1 - t, z = t$ and symmetric equations are

$$x - 2 = \frac{y - 1}{-1} = z \text{ or } x - 2 = 1 - y = z.$$

11. The line has direction $\mathbf{v} = \langle 1, 2, 1 \rangle$. Letting $P_0 = (1, -1, 1)$, parametric equations are $x = 1 + t$, $y = -1 + 2t$, $z = 1 + t$ and symmetric equations are $x - 1 = \frac{y + 1}{2} = z - 1$.
12. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$ is the direction of this line. Taking the point $(0, 1, 0)$ as P_0 , parametric equations are $x = t$, $y = 1$, $z = -t$, and symmetric equations are $x = -z$, $y = 1$.
13. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2 - (-4), 0 - (-6), -3 - 1 \rangle = \langle 2, 6, -4 \rangle$ and $\mathbf{v}_2 = \langle 5 - 10, 3 - 18, 14 - 4 \rangle = \langle -5, -15, 10 \rangle$, and since $\mathbf{v}_2 = -\frac{5}{2}\mathbf{v}_1$, the direction vectors and thus the lines are parallel.
14. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2, 4, 4 \rangle$ and $\mathbf{v}_2 = \langle 8, -1, 4 \rangle$. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = -16 - 4 + 16 \neq 0$, the vectors and thus the lines are not perpendicular.
15. (a) A direction vector of the line with parametric equations $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$ is $\mathbf{v} = \langle 2, 3, -7 \rangle$ and the desired parallel line must also have $\mathbf{v}$ as a direction vector. Here $P_0 = (0, 2, -1)$, so symmetric equations for the line are $\frac{x}{2} = \frac{y - 2}{3} = \frac{z + 1}{-7}$.
- (b) The line intersects the xy -plane when $z = 0$, so we need $\frac{x}{2} = \frac{y - 2}{3} = \frac{1}{-7}$ or $x = -\frac{2}{7}$, $y = \frac{11}{7}$. Thus the point of intersection with the xy -plane is $(-\frac{2}{7}, \frac{11}{7}, 0)$. Similarly for the yz -plane, we need $x = 0 \Leftrightarrow 0 = \frac{y - 2}{3} = \frac{z + 1}{-7} \Leftrightarrow y = 2, z = -1$. Thus the line intersects the yz -plane at $(0, 2, -1)$. For the xz -plane, we need $y = 0 \Leftrightarrow \frac{x}{2} = -\frac{2}{3} = \frac{z + 1}{-7} \Leftrightarrow x = -\frac{4}{3}, z = \frac{11}{3}$. So the line intersects the xz -plane at $(-\frac{4}{3}, 0, \frac{11}{3})$.
16. (a) A vector normal to the plane $2x - y + z = 1$ is $\mathbf{n} = \langle 2, -1, 1 \rangle$, and since the line is to be perpendicular to the plane, $\mathbf{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x = 5 + 2t$, $y = 1 - t$, $z = t$.
- (b) On the xy -plane, $z = 0$. So $z = t = 0$ in the parametric equations of the line, and therefore $x = 5$ and $y = 1$, giving the point of intersection $(5, 1, 0)$. For the yz -plane, $x = 0$ which implies $t = -\frac{5}{2}$, so $y = \frac{7}{2}$ and $z = -\frac{5}{2}$ and the point is $(0, \frac{7}{2}, -\frac{5}{2})$. For the xz -plane, $y = 0$ which implies $t = 1$, so $x = 7$ and $z = 1$ and the point of intersection is $(7, 0, 1)$.
17. From Equation 4, the line segment from $\mathbf{r}_0 = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ to $\mathbf{r}_1 = 4\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ is $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 = (1 - t)(2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) + t(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) = (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) + t(2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k})$, $0 \leq t \leq 1$.
18. From Equation 4, the line segment from $\mathbf{r}_0 = 10\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ to $\mathbf{r}_1 = 5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ is $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 = (1 - t)(10\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + t(5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$
 $= (10\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + t(-5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$, $0 \leq t \leq 1$
 The corresponding parametric equations are $x = 10 - 5t$, $y = 3 + 3t$, $z = 1 - 4t$, $0 \leq t \leq 1$.
19. Since the direction vectors are $\mathbf{v}_1 = \langle -6, 9, -3 \rangle$ and $\mathbf{v}_2 = \langle 2, -3, 1 \rangle$, we have $\mathbf{v}_1 = -3\mathbf{v}_2$ so the lines are parallel.
20. The lines aren't parallel since the direction vectors $\langle 2, 3, -1 \rangle$ and $\langle 1, 1, 3 \rangle$ aren't parallel. For the lines to intersect we must be able to find one value of t and one value of s that produce the same point from the respective parametric

equations. Thus we need to satisfy the following three equations: $1 + 2t = -1 + s$, $3t = 4 + s$, $2 - t = 1 + 3s$. Solving the first two equations we get $t = 6$, $s = 14$ and checking, we see that these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

21. Since the direction vectors $\langle 1, 2, 3 \rangle$ and $\langle -4, -3, 2 \rangle$ are not scalar multiples of each other, the lines are not parallel, so we check to see if the lines intersect. The parametric equations of the lines are $L_1: x = t, y = 1 + 2t, z = 2 + 3t$ and $L_2: x = 3 - 4s, y = 2 - 3s, z = 1 + 2s$. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $t = 3 - 4s$, $1 + 2t = 2 - 3s$, $2 + 3t = 1 + 2s$. Solving the first two equations we get $t = -1$, $s = 1$ and checking, we see that these values don't satisfy the third equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.
22. Since the direction vectors $\langle 2, 2, -1 \rangle$ and $\langle 1, -1, 3 \rangle$ aren't parallel, the lines aren't parallel. Here the parametric equations are $L_1: x = 1 + 2t, y = 3 + 2t, z = 2 - t$ and $L_2: x = 2 + s, y = 6 - s, z = -2 + 3s$. Thus, for the lines to intersect, the three equations $1 + 2t = 2 + s$, $3 + 2t = 6 - s$, and $2 - t = -2 + 3s$ must be satisfied simultaneously. Solving the first two equations gives $t = 1$, $s = 1$ and, checking, we see that these values do satisfy the third equation, so the lines intersect when $t = 1$ and $s = 1$, that is, at the point $(3, 5, 1)$.
23. Since the plane is perpendicular to the vector $\langle -2, 1, 5 \rangle$, we can take $\langle -2, 1, 5 \rangle$ as a normal vector to the plane. $(6, 3, 2)$ is a point on the plane, so setting $a = -2$, $b = 1$, $c = 5$ and $x_0 = 6$, $y_0 = 3$, $z_0 = 2$ in Equation 7 gives $-2(x - 6) + 1(y - 3) + 5(z - 2) = 0$ or $-2x + y + 5z = 1$ to be an equation of the plane.
24. $\mathbf{j} + 2\mathbf{k} = \langle 0, 1, 2 \rangle$ is a normal vector to the plane and $(4, 0, -3)$ is a point on the plane, so setting $a = 0$, $b = 1$, $c = 2$, $x_0 = 4$, $y_0 = 0$, $z_0 = -3$ in Equation 7 gives $0(x - 4) + 1(y - 0) + 2(z - (-3)) = 0$ or $y + 2z = -6$ to be an equation of the plane.
25. $\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle 1, 1, -1 \rangle$ is a normal vector to the plane and $(1, -1, 1)$ is a point on the plane, so setting $a = 1$, $b = 1$, $c = -1$, $x_0 = 1$, $y_0 = -1$, $z_0 = 1$ in Equation 7 gives $1(x - 1) + 1(y - (-1)) - 1(z - 1) = 0$ or $x + y - z = -1$ to be an equation of the plane.
26. Since the line is perpendicular to the plane, its direction vector $\langle 1, 2, -3 \rangle$ is a normal vector to the plane. An equation of the plane, then, is $1[x - (-2)] + 2(y - 8) - 3(z - 10) = 0$ or $x + 2y - 3z = -16$.
27. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 2, -1, 3 \rangle$, and an equation of the plane is $2(x - 0) - 1(y - 0) + 3(z - 0) = 0$ or $2x - y + 3z = 0$.
28. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 1, 1, 1 \rangle$, and an equation of the plane is $1[x - (-1)] + 1(y - 6) + 1[z - (-5)] = 0$ or $x + y + z = 0$.
29. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 3, 0, -7 \rangle$, and an equation of the plane is $3(x - 4) + 0[y - (-2)] - 7(z - 3) = 0$ or $3x - 7z = -9$.
30. First, a normal vector for the plane $2x + 4y + 8z = 17$ is $\mathbf{n} = \langle 2, 4, 8 \rangle$. A direction vector for the line is $\mathbf{v} = \langle 2, 1, -1 \rangle$, and since $\mathbf{n} \cdot \mathbf{v} = 0$ we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t = 0$, we know the point $(3, 0, 8)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n} = \langle 2, 4, 8 \rangle$, so an equation of the plane is $2(x - 3) + 4(y - 0) + 8(z - 8) = 0$ or $x + 2y + 4z = 35$.
31. Here the vectors $\mathbf{a} = \langle 1 - 0, 0 - 1, 1 - 1 \rangle = \langle 1, -1, 0 \rangle$ and $\mathbf{b} = \langle 1 - 0, 1 - 1, 0 - 1 \rangle = \langle 1, 0, -1 \rangle$ lie in the plane, so $\mathbf{a} \times \mathbf{b}$ is a normal vector to the plane. Thus, we can take $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1 - 0, 0 + 1, 0 + 1 \rangle = \langle 1, 1, 1 \rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 2$.

32. Here the vectors $\mathbf{a} = \langle 2, -4, 6 \rangle$ and $\mathbf{b} = \langle 5, 1, 3 \rangle$ lie in the plane, so
 $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -12 - 6, 30 - 6, 2 + 20 \rangle = \langle -18, 24, 22 \rangle$ is a normal vector to the plane and an equation of the plane is $-18(x - 0) + 24(y - 0) + 22(z - 0) = 0$ or $-18x + 24y + 22z = 0$.
33. Here the vectors $\mathbf{a} = \langle 8 - 3, 2 - (-1), 4 - 2 \rangle = \langle 5, 3, 2 \rangle$ and
 $\mathbf{b} = \langle -1 - 3, -2 - (-1), -3 - 2 \rangle = \langle -4, -1, -5 \rangle$ lie in the plane, so a normal vector to the plane is
 $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -15 + 2, -8 + 25, -5 + 12 \rangle = \langle -13, 17, 7 \rangle$ and an equation of the plane is
 $-13(x - 3) + 17[y - (-1)] + 7(z - 2) = 0$ or $-13x + 17y + 7z = -42$.
34. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 3, 1, -1 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, 3)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(0, 1, 2)$ is on the line, so $\mathbf{b} = \langle 1 - 0, 2 - 1, 3 - 2 \rangle = \langle 1, 1, 1 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1 + 1, -1 - 3, 3 - 1 \rangle = \langle 2, -4, 2 \rangle$. Thus, an equation of the plane is $2(x - 1) - 4(y - 2) + 2(z - 3) = 0$ or $2x - 4y + 2z = 0$. (Equivalently, we can write $x - 2y + z = 0$.)
35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -2, 5, 4 \rangle$ is one vector in the plane. We can verify that the given point $(6, 0, -2)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, 3, 7)$ is on the line, so $\mathbf{b} = \langle 6 - 4, 0 - 3, -2 - 7 \rangle = \langle 2, -3, -9 \rangle$ and
 $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -45 + 12, 8 - 18, 6 - 10 \rangle = \langle -33, -10, -4 \rangle$. Thus, an equation of the plane is
 $-33(x - 6) - 10(y - 0) - 4(z - (-2)) = 0$ or $33x + 10y + 4z = 190$.
36. Since the line $x = 2y = 3z$, or $x = \frac{y}{1/2} = \frac{z}{1/3}$, lies in the plane, its direction vector $\mathbf{a} = \langle 1, \frac{1}{2}, \frac{1}{3} \rangle$ is parallel to the plane. The point $(0, 0, 0)$ is on the line (put $t = 0$), and we can verify that the given point $(1, -1, 1)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 1, -1, 1 \rangle$, is therefore parallel to the plane, but not parallel to $\langle 1, 2, 3 \rangle$. Then $\mathbf{a} \times \mathbf{b} = \langle \frac{1}{2} + \frac{1}{3}, \frac{1}{3} - 1, -1 - \frac{1}{2} \rangle = \langle \frac{5}{6}, -\frac{2}{3}, -\frac{3}{2} \rangle$ is a normal vector to the plane, and an equation of the plane is $\frac{5}{6}(x - 0) - \frac{2}{3}(y - 0) - \frac{3}{2}(z - 0) = 0$ or $5x - 4y - 9z = 0$.
37. A direction vector for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, -1 \rangle \times \langle 2, -1, 3 \rangle = \langle 2, -5, -3 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. Setting $x = 0$, the equations of the planes reduce to $y - z = 2$ and $-y + 3z = 1$ with simultaneous solution $y = \frac{7}{2}$ and $z = \frac{3}{2}$. So a point on the line is $(0, \frac{7}{2}, \frac{3}{2})$ and another vector parallel to the plane is $\langle -1, -\frac{3}{2}, -\frac{1}{2} \rangle$. Then a normal vector to the plane is
 $\mathbf{n} = \langle 2, -5, -3 \rangle \times \langle -1, -\frac{3}{2}, -\frac{1}{2} \rangle = \langle -2, 4, -8 \rangle$ and an equation of the plane is
 $-2(x + 1) + 4(y - 2) - 8(z - 1) = 0$ or $x - 2y + 4z = -1$.
38. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, a normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = \langle 3, 3, 3 \rangle$, or we can use $\langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \Leftrightarrow x + y + z = 4$.
39. Substitute the parametric equations of the line into the equation of the plane: $(3 - t) - (2 + t) + 2(5t) = 9 \Rightarrow 8t = 8 \Rightarrow t = 1$. Therefore, the point of intersection of the line and the plane is given by $x = 3 - 1 = 2$, $y = 2 + 1 = 3$, and $z = 5(1) = 5$, that is, the point $(2, 3, 5)$.

40. Substitute the parametric equations of the line into the equation of the plane: $(1 + 2t) + 2(4t) - (2 - 3t) + 1 = 0$
 $\Rightarrow 13t = 0 \Rightarrow t = 0$. Therefore, the point of intersection of the line and the plane is given by
 $x = 1 + 2(0) = 1$, $y = 4(0) = 0$, and $z = 2 - 3(0) = 2$, that is, the point $(1, 0, 2)$.
41. Parametric equations for the line are $x = t$, $y = 1 + t$, $z = \frac{1}{2}t$ and substituting into the equation of the plane gives
 $4(t) - (1 + t) + 3(\frac{1}{2}t) = 8 \Rightarrow \frac{9}{2}t = 9 \Rightarrow t = 2$. Thus $x = 2$, $y = 1 + 2 = 3$, $z = \frac{1}{2}(2) = 1$ and the point
of intersection is $(2, 3, 1)$.
42. A direction vector for the line through $(1, 0, 1)$ and $(4, -2, 2)$ is $\mathbf{v} = \langle 3, -2, 1 \rangle$ and, taking $P_0 = (1, 0, 1)$,
parametric equations for the line are $x = 1 + 3t$, $y = -2t$, $z = 1 + t$. Substitution of the parametric equations into
the equation of the plane gives $1 + 3t - 2t + 1 + t = 6 \Rightarrow t = 2$. Then $x = 1 + 3(2) = 7$, $y = -2(2) = -4$,
and $z = 1 + 2 = 3$ so the point of intersection is $(7, -4, 3)$.
43. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so that they do in fact have a line of
intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$ is the direction of this line. Therefore, direction
numbers of the intersecting line are $1, 0, -1$.
44. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the
two planes are $\langle 1, 1, 1 \rangle$ and $\langle 1, 2, 3 \rangle$. The cosine of the angle θ between these two planes is

$$\cos \theta = \frac{\langle 1, 1, 1 \rangle \cdot \langle 1, 2, 3 \rangle}{|\langle 1, 1, 1 \rangle| |\langle 1, 2, 3 \rangle|} = \frac{1 + 2 + 3}{\sqrt{1+1+1} \sqrt{1+4+9}} = \frac{6}{\sqrt{42}} = \sqrt{\frac{6}{7}}.$$
45. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 4, -3 \rangle$ and $\mathbf{n}_2 = \langle -3, 6, 7 \rangle$, so the normals (and thus the planes) aren't
parallel. But $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 + 24 - 21 = 0$, so the normals (and thus the planes) are perpendicular.
46. Normal vectors for the planes are $\mathbf{n}_1 = \langle -1, 4, -2 \rangle$ and $\mathbf{n}_2 = \langle 3, -12, 6 \rangle$. Since $\mathbf{n}_2 = -3\mathbf{n}_1$, the normals
(and thus the planes) are parallel.
47. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, -1, 1 \rangle$. The normals are not parallel, so neither are
the planes. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 - 1 + 1 = 1 \neq 0$, so the planes aren't perpendicular. The angle between them
is given by $\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{1}{\sqrt{3} \sqrt{3}} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}(\frac{1}{3}) \approx 70.5^\circ$.
48. The normals are $\mathbf{n}_1 = \langle 2, -3, 4 \rangle$ and $\mathbf{n}_2 = \langle 1, 6, 4 \rangle$ so the planes aren't parallel. Since $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 18 + 16 = 0$,
the normals (and thus the planes) are perpendicular.
49. The normals are $\mathbf{n}_1 = \langle 1, -4, 2 \rangle$ and $\mathbf{n}_2 = \langle 2, -8, 4 \rangle$. Since $\mathbf{n}_2 = 2\mathbf{n}_1$, the normals (and thus the planes) are
parallel.
50. The normal vectors are $\mathbf{n}_1 = \langle 1, 2, 2 \rangle$ and $\mathbf{n}_2 = \langle 2, -1, 2 \rangle$. The normals are not parallel, so neither are the planes.
Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 2 + 4 = 4 \neq 0$, so the planes aren't perpendicular. The angle between them is given by

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{4}{\sqrt{9} \sqrt{9}} = \frac{4}{9} \Rightarrow \theta = \cos^{-1}(\frac{4}{9}) \approx 63.6^\circ.$$
51. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z = 0$. (This will only
work if the line of intersection crosses the xy -plane; otherwise, try setting x or y equal to 0.) Then the equations
of the planes reduce to $x + y = 2$ and $3x - 4y = 6$. Solving these two equations gives $x = 2$, $y = 0$.
So a point on the line of intersection is $(2, 0, 0)$. The direction of the line is
 $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 5 - 4, -3 - 5, -4 - 3 \rangle = \langle 1, -8, -7 \rangle$, and symmetric equations for the line are

$$x - 2 = \frac{y}{-8} = \frac{z}{-7}.$$

(b) The angle between the planes satisfies $\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{3 - 4 - 5}{\sqrt{3} \sqrt{50}} = -\frac{\sqrt{6}}{5}$. Therefore

$$\theta = \cos^{-1}\left(-\frac{\sqrt{6}}{5}\right) \approx 119^\circ \text{ (or } 61^\circ\text{)}.$$

52. (a) $x - 2y + z = 1 \Rightarrow \mathbf{n}_1 = \langle 1, -2, 1 \rangle$ and $2x + y + z = 1 \Rightarrow \mathbf{n}_2 = \langle 2, 1, 1 \rangle$. The vector that gives the direction of the line of intersection of these two planes is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -2 - 1, 2 - 1, 1 + 4 \rangle = \langle -3, 1, 5 \rangle$.

Setting $x = y = 0$, we see that both planes contain $(0, 0, 1)$ so that this point must lie on their line of

intersection. Then symmetric equations for this line are $\frac{x}{-3} = y = \frac{z - 1}{5}$.

(b) $\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{2 - 2 + 1}{\sqrt{1 + 4 + 1} \sqrt{4 + 1 + 1}} = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{6}\right) \approx 80^\circ.$

53. Setting $x = 0$, the equations of the two planes become $z = y$ and $5y + z = -1$, which intersect at $y = -\frac{1}{6}$ and $z = -\frac{1}{6}$. Thus we can choose $(x_0, y_0, z_0) = (0, -\frac{1}{6}, -\frac{1}{6})$. The vector giving the direction of this intersecting line, $\mathbf{v}$, is perpendicular to the normal vectors of both planes. So $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -5, -1 \rangle \times \langle 1, 1, -1 \rangle = \langle 6, 1, 7 \rangle$. Therefore, by Equations 2, parametric equations for this line are $x = 6t, y = -\frac{1}{6} + t, z = -\frac{1}{6} + 7t$.

54. Setting $y = 0$, the equations of the two planes become $2x + 5z = -3$ and $x + z = -2$, which intersect at $x = -\frac{7}{3}$ and $z = \frac{1}{3}$. Thus we can choose $(x_0, y_0, z_0) = (-\frac{7}{3}, 0, \frac{1}{3})$. The vector giving the direction of this intersecting line, $\mathbf{v}$, is perpendicular to the normal vectors of both planes. So $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, 0, 5 \rangle \times \langle 1, -3, 1 \rangle = \langle 15, 5 - 2, -6 \rangle = 3 \langle 5, 1, -2 \rangle$. Therefore, by Equations 2, parametric equations of the line of intersection of the two planes are $x = -\frac{7}{3} + 5t, y = t, z = \frac{1}{3} - 2t$.

55. The plane contains all perpendicular bisectors of the line segment joining $(1, 1, 0)$ and $(0, 1, 1)$. All of these bisectors pass through the midpoint of this segment $(\frac{1}{2}, \frac{1+1}{2}, \frac{1}{2}) = (\frac{1}{2}, 1, \frac{1}{2})$. The direction of this line segment $\langle 1 - 0, 1 - 1, 0 - 1 \rangle = \langle 1, 0, -1 \rangle$ is perpendicular to the plane so that we can choose this to be $\mathbf{n}$. Therefore the equation of the plane is $1(x - \frac{1}{2}) + 0(y - 1) - 1(z - \frac{1}{2}) = 0 \Leftrightarrow x = z$.

56. The plane will contain all perpendicular bisectors of the line segment joining the two points. Thus, a point in the plane is $P_0 = (-1, -1, 2)$, the midpoint of the line segment joining the two given points, and a normal to the plane is $\mathbf{n} = \langle 6, -6, 2 \rangle$, the vector connecting the two points. So an equation of the plane is $6(x + 1) - 6(y + 1) + 2(z - 2) = 0$ or $3x - 3y + z = 2$.

57. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0 \rangle$ and $\mathbf{b} = \langle -a, 0, c \rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab \rangle = \langle bc, ac, ab \rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. This is a good equation to remember!

58. (a) For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1 + t = 2 - s$, $1 - t = s$ and $2t = 2$. From the third we get $t = 1$, and putting this in the second gives $s = 0$. These values of s and t do satisfy the first equation, so the lines intersect at the point $P_0 = (1 + 1, 1 - 1, 2(1)) = (2, 0, 2)$.

- (b) The direction vectors of the lines are $\langle 1, -1, 2 \rangle$ and $\langle -1, 1, 0 \rangle$, so a normal vector for the plane is $\langle -1, 1, 0 \rangle \times \langle 1, -1, 2 \rangle = \langle 2, 2, 0 \rangle$ and it contains the point $(2, 0, 2)$. Then the equation of the plane is $2(x - 2) + 2(y - 0) + 0(z - 2) = 0 \Leftrightarrow x + y = 2$.

59. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1, 1, 1 \rangle$, and a direction vector for the given line, $\langle 1, -1, 2 \rangle$. So a direction vector for the required line is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \langle 3, -1, -2 \rangle$. Thus L is given by $\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t\langle 3, -1, -2 \rangle$, or in parametric form, $x = 3t$, $y = 1 - t$, $z = 2 - 2t$.
60. Let L be the given line. Then $(1, 1, 0)$ is the point on L corresponding to $t = 0$. L is in the direction of $\mathbf{a} = \langle 1, -1, 2 \rangle$ and $\mathbf{b} = \langle -1, 0, 2 \rangle$ is the vector joining $(1, 1, 0)$ and $(0, 1, 2)$. Then $\mathbf{b} - \text{proj}_{\mathbf{a}} \mathbf{b} = \langle -1, 0, 2 \rangle - \frac{\langle 1, -1, 2 \rangle \cdot \langle -1, 0, 2 \rangle}{1^2 + (-1)^2 + 2^2} \langle 1, -1, 2 \rangle = \langle -1, 0, 2 \rangle - \frac{1}{2} \langle 1, -1, 2 \rangle = \langle -\frac{3}{2}, \frac{1}{2}, 1 \rangle$ is a direction vector for the required line. Thus $2\langle -\frac{3}{2}, \frac{1}{2}, 1 \rangle = \langle -3, 1, 2 \rangle$ is also a direction vector, and the line has parametric equations $x = -3t$, $y = 1 + t$, $z = 2 + 2t$. (Notice that this is the same line as in Exercise 59.)
61. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1 = \langle 4, -2, 6 \rangle$, $\mathbf{n}_2 = \langle 4, -2, -2 \rangle$, $\mathbf{n}_3 = \langle -6, 3, -9 \rangle$, $\mathbf{n}_4 = \langle 2, -1, -1 \rangle$. Now $\mathbf{n}_1 = -\frac{2}{3}\mathbf{n}_3$, so $\mathbf{n}_1$ and $\mathbf{n}_3$ are parallel, and hence P_1 and P_3 are parallel; similarly P_2 and P_4 are parallel because $\mathbf{n}_2 = 2\mathbf{n}_4$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. $(0, 0, \frac{1}{2})$ lies on P_1 , but not on P_3 , so they are not the same plane, but both P_2 and P_4 contain the point $(0, 0, -3)$, so these two planes are identical.
62. Let L_i have direction vector $\mathbf{v}_i$. Then $\mathbf{v}_1 = \langle 1, 1, -5 \rangle$, $\mathbf{v}_2 = \langle 1, 1, -1 \rangle$, $\mathbf{v}_3 = \langle 1, 1, -1 \rangle$, $\mathbf{v}_4 = \langle 2, 2, -10 \rangle$. $\mathbf{v}_2$ and $\mathbf{v}_3$ are equal so they're parallel. $\mathbf{v}_4 = 2\mathbf{v}_1$, so L_4 and L_1 are parallel. L_3 contains the point $(1, 4, 1)$, but this point does not lie on L_2 , so they're not equal. $(2, 1, -3)$ lies on L_4 , and on L_1 , with $t = 1$. So L_1 and L_4 are identical.
63. Let $Q = (2, 2, 0)$ and $R = (3, -1, 5)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (1, 2, 3)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1, -3, 5 \rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle -1, 0, 3 \rangle$. The distance is
$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -3, 5 \rangle \times \langle -1, 0, 3 \rangle|}{|\langle 1, -3, 5 \rangle|} = \frac{|\langle -9, -8, -3 \rangle|}{|\langle 1, -3, 5 \rangle|} = \frac{\sqrt{9^2 + 8^2 + 3^2}}{\sqrt{1^2 + 3^2 + 5^2}} = \frac{\sqrt{154}}{\sqrt{35}} = \sqrt{\frac{22}{5}}.$$
64. Let $Q = (5, 0, 1)$ and $R = (4, 3, 3)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (1, 0, -1)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle -1, 3, 2 \rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle -4, 0, -2 \rangle$. The distance is
$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle -1, 3, 2 \rangle \times \langle -4, 0, -2 \rangle|}{|\langle -1, 3, 2 \rangle|} = \frac{|\langle -6, -10, 12 \rangle|}{|\langle -1, 3, 2 \rangle|} = \frac{2\sqrt{3^2 + 5^2 + 6^2}}{\sqrt{1^2 + 3^2 + 2^2}} = \frac{2\sqrt{70}}{\sqrt{14}} = 2\sqrt{5}.$$
65. By Equation 9, the distance is $D = \frac{1}{\sqrt{1+4+4}} [(1)(2) + (-2)(8) + (-2)(5) - 1] = \frac{25}{3}.$
66. By Equation 9, the distance is $D = \frac{1}{\sqrt{16+36+1}} [4(3) + (-6)(-2) + 1(7) - 5] = \frac{26}{\sqrt{53}}.$
67. Put $y = z = 0$ in the equation of the first plane to get the point $(-1, 0, 0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(-1, 0, 0)$ to the second plane. By Equation 9,
$$D = \frac{|3(-1) + 6(0) - 3(0) - 4|}{\sqrt{3^2 + 6^2 + (-3)^2}} = \frac{7}{3\sqrt{6}} \text{ or } \frac{7\sqrt{6}}{18}.$$
68. Put $y = z = 0$ in the equation of the first plane to get the point $(\frac{4}{3}, 0, 0)$ on the plane. Because the planes are parallel the distance D between them is the distance from $(\frac{4}{3}, 0, 0)$ to the second plane. By Equation 9,
$$D = \frac{|1(\frac{4}{3}) + 2(0) - 3(0) - 1|}{\sqrt{1^2 + 2^2 + (-3)^2}} = \frac{1}{3\sqrt{14}}.$$

69. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax + by + cz + d_1 = 0$. Then

$ax_0 + by_0 + cz_0 + d_1 = 0$ and the distance between P_0 and the plane given by $ax + by + cz + d_2 = 0$ is, from

$$\text{Equation 9, } D = \frac{|ax_0 + by_0 + cz_0 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

70. The planes must have parallel normal vectors, so if $ax + by + cz + d = 0$ is such a plane, then for some $t \neq 0$, $\langle a, b, c \rangle = t\langle 1, 2, -2 \rangle = \langle t, 2t, -2t \rangle$. So this plane is given by the equation $x + 2y - 2z + e = 0$, where $e = d/t$.

$$\text{By Exercise 69, the distance between the planes is } 2 = \frac{|1 - e|}{\sqrt{1^2 + 2^2 + (-2)^2}} \Leftrightarrow 6 = |1 - e| \Leftrightarrow e = 7 \text{ or } -5.$$

So the desired planes have equations $x + 2y - 2z = 7$ and $x + 2y - 2z = -5$.

71. $L_1: x = y = z \Rightarrow x = y$ (1). $L_2: x + 1 = y/2 = z/3 \Rightarrow x + 1 = y/2$ (2). The solution of (1) and (2) is $x = y = -2$. However, when $x = -2$, $x = z \Rightarrow z = -2$, but $x + 1 = z/3 \Rightarrow z = -3$, a contradiction.

Hence the lines do not intersect. For L_1 , $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$, and for L_2 , $\mathbf{v}_2 = \langle 1, 2, 3 \rangle$, so the lines are not parallel.

Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1, 1, 1 \rangle$ and $\langle 1, 2, 3 \rangle$, the direction vectors of the two lines. So set $\mathbf{n} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle = \langle 3 - 2, -3 + 1, 2 - 1 \rangle = \langle 1, -2, 1 \rangle$. From above, we know that

$(-2, -2, -2)$ and $(-2, -2, -3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 8,

$$1(-2) - 2(-2) + 1(-2) + d_1 = 0 \Rightarrow d_1 = 0 \text{ and } 1(-2) - 2(-2) + 1(-3) + d_2 = 0 \Rightarrow d_2 = 1.$$

$$\text{By Exercise 69, the distance between these two skew lines is } D = \frac{|0 - 1|}{\sqrt{1 + 4 + 1}} = \frac{1}{\sqrt{6}}.$$

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is

$\mathbf{n} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle = \langle 1, -2, 1 \rangle$. Pick any point on each of the lines, say $(-2, -2, -2)$ and $(-2, -2, -3)$,

and form the vector $\mathbf{b} = \langle 0, 0, 1 \rangle$ connecting the two points. The distance between the two skew lines is the

$$\text{absolute value of the scalar projection of } \mathbf{b} \text{ along } \mathbf{n}, \text{ that is, } D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|1 \cdot 0 - 2 \cdot 0 + 1 \cdot 1|}{\sqrt{1 + 4 + 1}} = \frac{1}{\sqrt{6}}.$$

72. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the

planes must be perpendicular to both $\mathbf{v}_1 = \langle 1, 6, 2 \rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6 \rangle$, the direction vectors of the two lines respectively. Thus set $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 36 - 30, 4 - 6, 15 - 12 \rangle = \langle 6, -2, 3 \rangle$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 8, $6 - 2 + 0 + d_1 = 0 \Rightarrow d_1 = -4$ and

$6 - 10 - 6 + d_2 = 0 \Rightarrow d_2 = 10$. Then by Exercise 69, the distance between the two skew lines is given by

$$D = \frac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \frac{14}{7} = 2.$$

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are

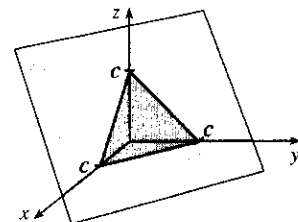
$\mathbf{v}_1 = \langle 1, 6, 2 \rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6 \rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 6, -2, 3 \rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 1, 0)$ and $(1, 5, -2)$, and form the vector $\mathbf{b} = \langle 0, 4, -2 \rangle$ connecting the two points.

Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is,

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{36 + 4 + 9}} |0 - 8 - 6| = \frac{14}{7} = 2.$$

73. If $a \neq 0$, then $ax + by + cz + d = 0 \Rightarrow a(x + d/a) + b(y - 0) + c(z - 0) = 0$ which by (7) is the scalar equation of the plane through the point $(-d/a, 0, 0)$ with normal vector $\langle a, b, c \rangle$. Similarly, if $b \neq 0$ (or if $c \neq 0$) the equation of the plane can be rewritten as $a(x - 0) + b(y + d/b) + c(z - 0) = 0$ [or as $a(x - 0) + b(y - 0) + c(z + d/c) = 0$] which by (7) is the scalar equation of a plane through the point $(0, -d/b, 0)$ [or the point $(0, 0, -d/c)$] with normal vector $\langle a, b, c \rangle$.

74. (a) The planes $x + y + z = c$ have normal vector $\langle 1, 1, 1 \rangle$, so they are all parallel. Their x -, y -, and z -intercepts are all c . When $c > 0$ their intersection with the first octant is an equilateral triangle and when $c < 0$ their intersection with the octant diagonally opposite the first is an equilateral triangle.



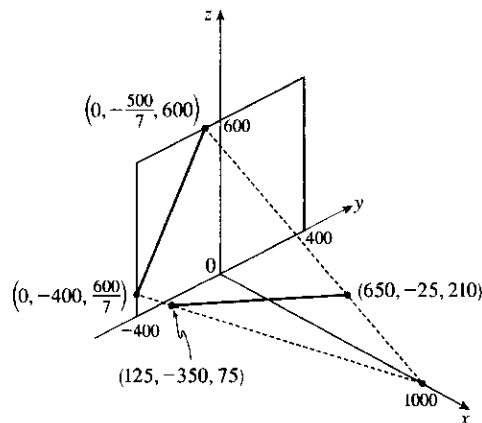
- (b) The planes $x + y + cz = 1$ have x -intercept 1, y -intercept 1, and z -intercept $1/c$. The plane with $c = 0$ is parallel to the z -axis. As c gets larger, the planes get closer to the xy -plane.
- (c) The planes $y \cos \theta + z \sin \theta = 1$ have normal vectors $\langle 0, \cos \theta, \sin \theta \rangle$, which are perpendicular to the x -axis, and so the planes are parallel to the x -axis. We look at their intersection with the yz -plane. These are lines that are perpendicular to $\langle \cos \theta, \sin \theta \rangle$ and pass through $(\cos \theta, \sin \theta)$, since $\cos^2 \theta + \sin^2 \theta = 1$. So these are the tangent lines to the unit circle. Thus the family consists of all planes tangent to the circular cylinder with radius 1 and axis the x -axis.

LABORATORY PROJECT Putting 3D in Perspective

1. If we view the screen from the camera's location, the vertical clipping plane on the left passes through the points $(1000, 0, 0)$, $(0, -400, 0)$, and $(0, -400, 600)$. A vector from the first point to the second is $\mathbf{v}_1 = \langle -1000, -400, 0 \rangle$ and a vector from the first point to the third is $\mathbf{v}_2 = \langle -1000, -400, 600 \rangle$. A normal vector for the clipping plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -240,000 \mathbf{i} + 600,000 \mathbf{j}$ or $-2 \mathbf{i} + 5 \mathbf{j}$, and an equation for the plane is $-2(x - 1000) + 5(y - 0) + 0(z - 0) = 0 \Rightarrow 2x - 5y = 2000$. By symmetry, the vertical clipping plane on the right is given by $2x + 5y = 2000$. The lower clipping plane is $z = 0$. The upper clipping plane passes through the points $(1000, 0, 0)$, $(0, -400, 600)$, and $(0, 400, 600)$. Vectors from the first point to the second and third points are $\mathbf{v}_1 = \langle -1000, -400, 600 \rangle$ and $\mathbf{v}_2 = \langle -1000, 400, 600 \rangle$, and a normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -480,000 \mathbf{i} - 800,000 \mathbf{k}$ or $3 \mathbf{i} + 5 \mathbf{k}$. An equation for the plane is $3(x - 1000) + 0(y - 0) + 5(z - 0) = 0 \Rightarrow 3x + 5z = 3000$.

A direction vector for the line L is $\mathbf{v} = \langle 630, 390, 162 \rangle$ and taking $P_0 = (230, -285, 102)$, parametric equations are $x = 230 + 630t$, $y = -285 + 390t$, $z = 102 + 162t$. L intersects the left clipping plane when $2(230 + 630t) - 5(-285 + 390t) = 2000 \Rightarrow t = -\frac{1}{6}$. The corresponding point is $(125, -350, 75)$. L intersects the right clipping plane when $2(230 + 630t) + 5(-285 + 390t) = 2000 \Rightarrow t = \frac{593}{642}$. The corresponding point is approximately $(811.9, 75.2, 251.6)$, but this point is not contained within the viewing volume. L intersects the upper clipping plane when $3(230 + 630t) + 5(102 + 162t) = 3000 \Rightarrow t = \frac{2}{3}$, corresponding to the point $(650, -25, 210)$, and L intersects the lower clipping plane when $z = 0 \Rightarrow 102 + 162t = 0 \Rightarrow t = -\frac{17}{27}$. The corresponding point is approximately $(-166.7, -530.6, 0)$, which is not contained within the viewing volume. Thus L should be clipped at the points $(125, -350, 75)$ and $(650, -25, 210)$.

2. A sight line from the camera at $(1000, 0, 0)$ to the left endpoint $(125, -350, 75)$ of the clipped line has direction $\mathbf{v} = \langle -875, -350, 75 \rangle$. Parametric equations are $x = 1000 - 875t$, $y = -350t$, $z = 75t$. This line intersects the screen when $x = 0 \Rightarrow 1000 - 875t = 0 \Rightarrow t = \frac{8}{7}$, corresponding to the point $(0, -400, \frac{600}{7})$. Similarly, a sight line from the camera to the right endpoint $(650, -25, 210)$ of the clipped line has direction $\langle -350, -25, 210 \rangle$ and parametric equations are $x = 1000 - 350t$, $y = -25t$, $z = 210t$. $x = 0 \Rightarrow 1000 - 350t = 0 \Rightarrow t = \frac{20}{7}$, corresponding to the point $(0, -\frac{500}{7}, 600)$. Thus the projection of the clipped line is the line segment between the points $(0, -400, \frac{600}{7})$ and $(0, -\frac{500}{7}, 600)$.
3. From Equation 13.5.4 [ET 12.5.4], equations for the four sides of the screen are $\mathbf{r}_1(t) = (1-t)\langle 0, -400, 0 \rangle + t\langle 0, -400, 600 \rangle$, $\mathbf{r}_2(t) = (1-t)\langle 0, -400, 600 \rangle + t\langle 0, 400, 600 \rangle$, $\mathbf{r}_3(t) = (1-t)\langle 0, 400, 0 \rangle + t\langle 0, 400, 600 \rangle$, and $\mathbf{r}_4(t) = (1-t)\langle 0, -400, 0 \rangle + t\langle 0, 400, 0 \rangle$. The clipped line segment connects the points $(125, -350, 75)$ and $(650, -25, 210)$, so an equation for the segment is $\mathbf{r}_5(t) = (1-t)\langle 125, -350, 75 \rangle + t\langle 650, -25, 210 \rangle$. The projection of the clipped segment connects the points $(0, -400, \frac{600}{7})$ and $(0, -\frac{500}{7}, 600)$, so an equation is $\mathbf{r}_6(t) = (1-t)\langle 0, -400, \frac{600}{7} \rangle + t\langle 0, -\frac{500}{7}, 600 \rangle$. The sight line on the left connects the points $(1000, 0, 0)$ and $(0, -400, \frac{600}{7})$, so an equation is $\mathbf{r}_7(t) = (1-t)\langle 1000, 0, 0 \rangle + t\langle 0, -400, \frac{600}{7} \rangle$. The other sight line connects $(1000, 0, 0)$ to $(0, -\frac{500}{7}, 600)$, so an equation is $\mathbf{r}_8(t) = (1-t)\langle 1000, 0, 0 \rangle + t\langle 0, -\frac{500}{7}, 600 \rangle$.

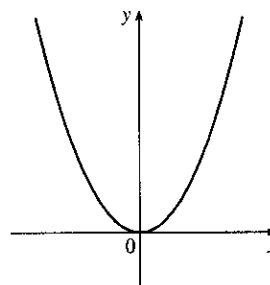


4. The vector from $(621, -147, 206)$ to $(563, 31, 242)$, $\mathbf{v}_1 = \langle -58, 178, 36 \rangle$, lies in the plane of the rectangle, as does the vector from $(621, -147, 206)$ to $(657, -111, 86)$, $\mathbf{v}_2 = \langle 36, 36, -120 \rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle -1888, -142, -708 \rangle$ or $\langle 8, 2, 3 \rangle$, and an equation of the plane is $8x + 2y + 3z = 5292$. The line L intersects this plane when $8(230 + 630t) + 2(-285 + 390t) + 3(102 + 162t) = 5292 \Rightarrow t = \frac{1858}{3153} \approx 0.589$. The corresponding point is approximately $(601.25, -55.18, 197.46)$. Starting at this point, a portion of the line is hidden behind the rectangle. The line becomes visible again at the left edge of the rectangle, specifically the edge between the points $(621, -147, 206)$ and $(657, -111, 86)$. (This is most easily determined by graphing the rectangle and the line.) A plane through these two points and the camera's location, $(1000, 0, 0)$, will clip the line at the point it becomes visible. Two vectors in this plane are $\mathbf{v}_1 = \langle -379, -147, 206 \rangle$ and $\mathbf{v}_2 = \langle -343, -111, 86 \rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 10224, -38064, -8352 \rangle$ and an equation of the plane is $213x - 793y - 174z = 213,000$. L intersects this plane when $213(230 + 630t) - 793(-285 + 390t) - 174(102 + 162t) = 213,000 \Rightarrow t = \frac{44,247}{203,268} \approx 0.2177$. The corresponding point is approximately $(367.14, -200.11, 137.26)$. Thus the portion of L that should be removed is the segment between the points $(601.25, -55.18, 197.46)$ and $(367.14, -200.11, 137.26)$.

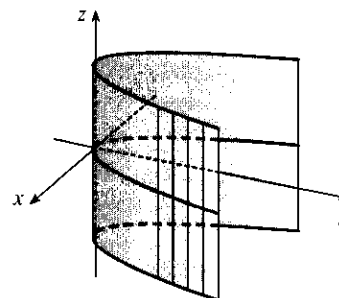
13.6 Cylinders and Quadric Surfaces

ET 12.6

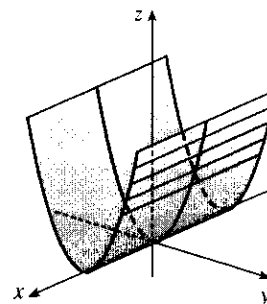
1. (a) In $\mathbb{R}^2$, the equation $y = x^2$ represents a parabola.



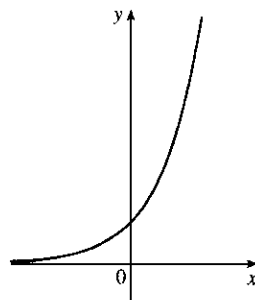
- (b) In $\mathbb{R}^3$, the equation $y = x^2$ doesn't involve z , so any horizontal plane with equation $z = k$ intersects the graph in a curve with equation $y = x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z -axis.



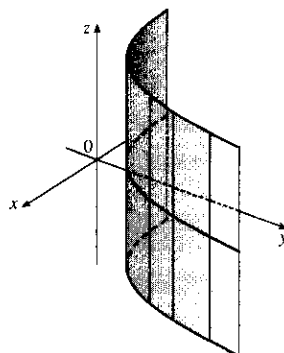
- (c) In $\mathbb{R}^3$, the equation $z = y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z = y^2$ in the direction of the x -axis. Thus, the rulings of the cylinder are parallel to the x -axis.



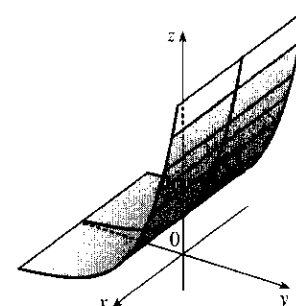
2. (a)



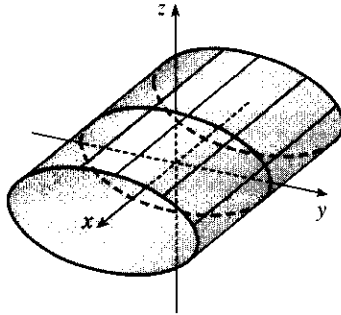
- (b) Since the equation $y = e^x$ doesn't involve z , horizontal traces are copies of the curve $y = e^x$. The rulings are parallel to the z -axis.



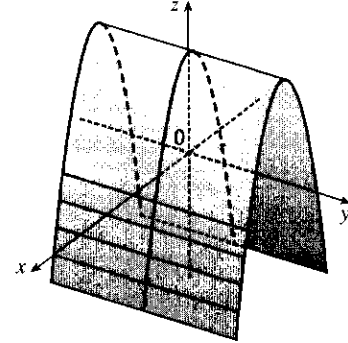
- (c) The equation $z = e^y$ doesn't involve x , so vertical traces in $x = k$ (parallel to the yz -plane) are copies of the curve $z = e^y$. The rulings are parallel to the x -axis.



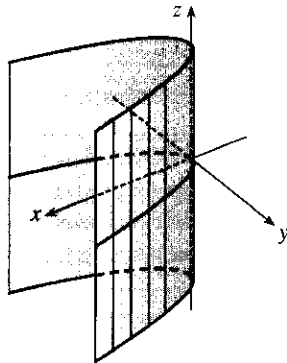
3. Since x is missing from the equation, the vertical traces $y^2 + 4z^2 = 4$, $x = k$, are copies of the same ellipse in the plane $x = k$. Thus, the surface $y^2 + 4z^2 = 4$ is an elliptic cylinder with rulings parallel to the x -axis.



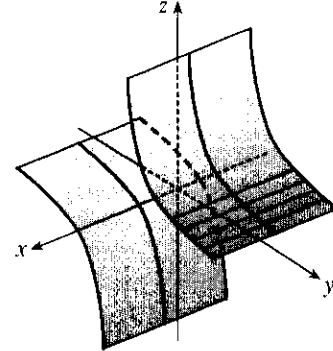
4. Since y is missing from the equation, each vertical trace $z = 4 - x^2$, $y = k$, is a copy of the same parabola in the plane $y = k$. Thus, the surface $z = 4 - x^2$ is a parabolic cylinder with rulings parallel to the y -axis.



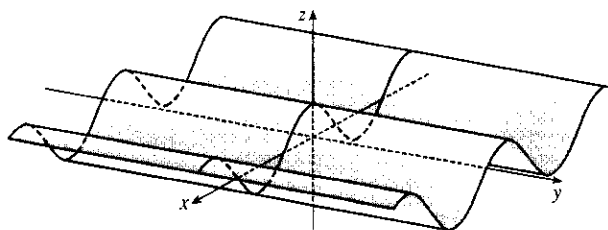
5. Since z is missing, each horizontal trace $x = y^2$, $z = k$, is a copy of the same parabola in the plane $z = k$. Thus, the surface $x - y^2 = 0$ is a parabolic cylinder with rulings parallel to the z -axis.



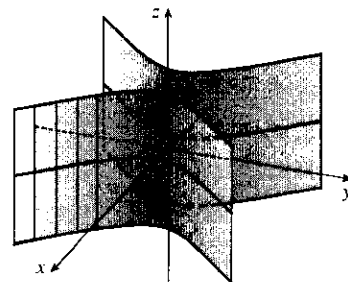
6. Since x is missing, each vertical trace $yz = 4$, $x = k$ is a copy of the same hyperbola in the plane $x = k$. Thus, the surface $yz = 4$ is a hyperbolic cylinder with rulings parallel to the x -axis.



7. Since y is missing, each vertical trace $z = \cos x$, $y = k$ is a copy of a cosine curve in the plane $y = k$. Thus, the surface $z = \cos x$ is a cylindrical surface with rulings parallel to the y -axis.

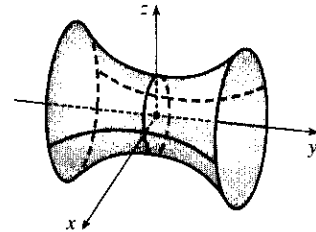


8. Since z is missing, each horizontal trace $x^2 - y^2 = 1$, $z = k$ is a copy of the same hyperbola in the plane $z = k$. Thus, the surface $x^2 - y^2 = 1$ is a hyperbolic cylinder with rulings parallel to the z -axis.

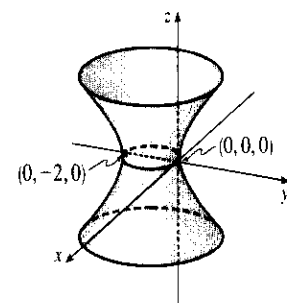


9. (a) The traces of $x^2 + y^2 - z^2 = 1$ in $x = k$ are $y^2 - z^2 = 1 - k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1 < k < 1$ than for $k < -1$ or $k > 1$.) The traces in $y = k$ are $x^2 - z^2 = 1 - k^2$, a similar family of hyperbolas. The traces in $z = k$ are $x^2 + y^2 = 1 + k^2$, a family of circles. For $k = 0$, the trace in the xy -plane, the circle is of radius 1. As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.

- (b) The shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y -axis. Traces in $y = k$ are circles, while traces in $x = k$ and $z = k$ are hyperbolas.

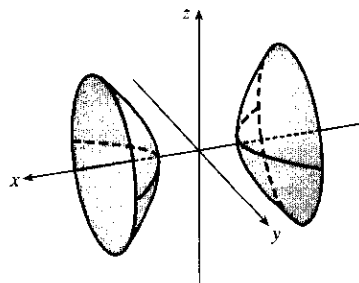


- (c) Completing the square in y gives $x^2 + (y + 1)^2 - z^2 = 1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y -direction.

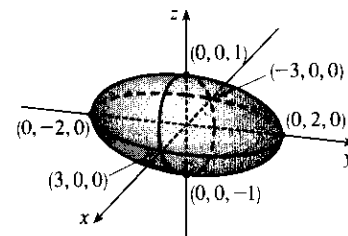


10. (a) The traces of $-x^2 - y^2 + z^2 = 1$ in $x = k$ are $-y^2 + z^2 = 1 + k^2$, a family of hyperbolas, as are the traces in $y = k$, $-x^2 + z^2 = 1 + k^2$. The traces in $z = k$ are $x^2 + y^2 = k^2 - 1$, a family of circles for $|k| > 1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k| < 1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 1.

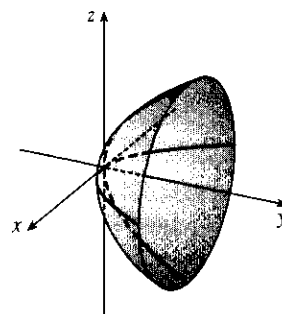
- (b) The graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x -axis. Traces in $x = k$, $|k| > 1$, are circles, while traces in $y = k$ and $z = k$ are hyperbolas.



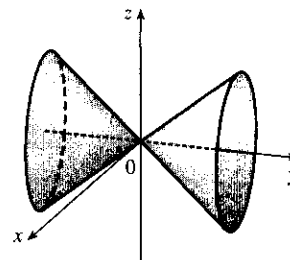
11. Traces: $x = k$, $9y^2 + 36z^2 = 36 - 4k^2$, an ellipse for $|k| < 3$;
 $y = k$, $4x^2 + 36z^2 = 36 - 9k^2$, an ellipse for $|k| < 2$; $z = k$,
 $4x^2 + 9y^2 = 36(1 - k^2)$, an ellipse for $|k| < 1$. Thus the surface is an ellipsoid with center at the origin and axes along the x -, y - and z -axes.



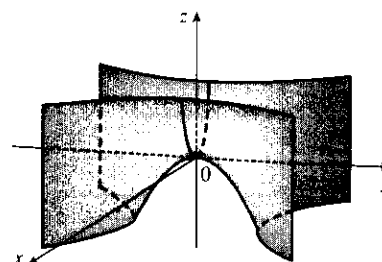
12. Traces: $x = k$, $4y = k^2 + z^2$, a parabola; $y = k$, $4k = x^2 + z^2$, a circle for $k > 0$; $z = k$, $4y = x^2 + k^2$ a parabola. Thus the surface is a circular paraboloid with axis the y -axis and vertex at $(0, 0, 0)$.



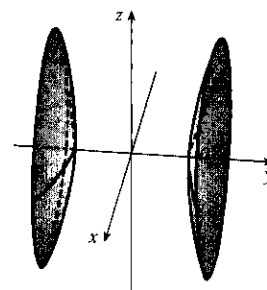
13. Traces: $x = k$, $y^2 = k^2 + z^2$ or $y^2 - z^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$; $y = k$, $x^2 + z^2 = k^2$, a circle for $k \neq 0$; $z = k$, $y^2 = x^2 + k^2$ or $y^2 - x^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$. Thus the surface is a cone (right circular) with axis the y -axis and vertex the origin.



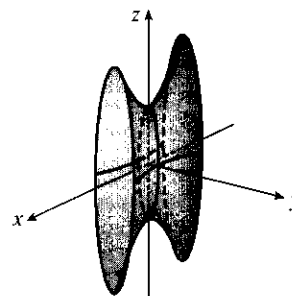
14. Traces: $x = k$, $z - k^2 = -y^2$, a parabola; $y = k$, $z + k^2 = x^2$, a parabola; $z = k$, $x^2 - y^2 = k$, a hyperbola. Thus the surface is a hyperbolic paraboloid with saddle point $(0, 0, 0)$ (and since $c > 0$, the saddle is upside down).



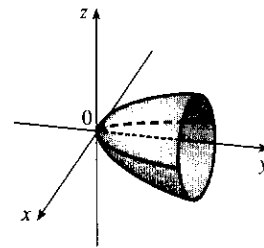
15. Traces: $x = k$, $4y^2 - z^2 = 4 + k^2$, a hyperbola; $y = k$, $x^2 + z^2 = 4k^2 - 4$, a circle for $|k| > 1$; $z = k$, $4y^2 - x^2 = 4 + k^2$, a hyperbola. Thus the surface is a hyperboloid of two sheets with axis the y -axis.



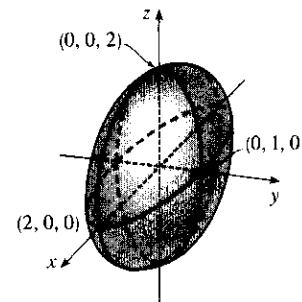
16. Traces: $x = k$, $25y^2 + z^2 = 100 + 4k^2$, an ellipse; $y = k$, $25k^2 + z^2 = 100 + 4x^2$ or $z^2 - 4x^2 = 100 - 25k^2$, a hyperbola for $|k| < 2$; $z = k$, $25y^2 + k^2 = 100 + 4x^2$ or $25y^2 - 4x^2 = 100 - k^2$, a hyperbola for $|k| < 10$. Thus the surface is a hyperboloid of one sheet with axis the x -axis.



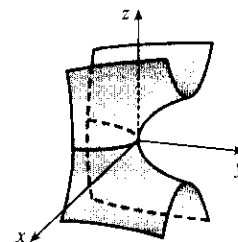
17. Traces: $x = k, k^2 + 4z^2 - y = 0$ or $y - k^2 = 4z^2$, a parabola;
 $y = k, x^2 + 4z^2 = k$, an ellipse for $k > 0$; $z = k, x^2 + 4k^2 - y = 0$
 or $y - 4k^2 = x^2$, a parabola. Thus the surface is an elliptic paraboloid
 with axis the y -axis and vertex the origin.



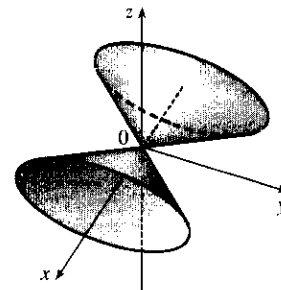
18. Traces: $x = k, |k| \leq 2 \Rightarrow y^2 + \frac{z^2}{4} = 1 - \frac{k^2}{4}$, ellipses;
 $y = k, |k| \leq 1 \Rightarrow x^2 + z^2 = 4(1 - k^2)$, circles; $z = k, |k| \leq 2$
 $\Rightarrow \frac{x^2}{4} + y^2 = 1 - \frac{k^2}{4}$, ellipses. $x^2 + 4y^2 + z^2 = 4 \Leftrightarrow$
 $\frac{x^2}{2^2} + \frac{y^2}{1^2} + \frac{z^2}{2^2} = 1$, which is the equation of an ellipsoid.



19. $y = z^2 - x^2$. The traces in $x = k$ are the parabolas $y = z^2 - k^2$;
 the traces in $y = k$ are $k = z^2 - x^2$, which are hyperbolas (note the
 hyperbolas are oriented differently for $k > 0$ than for $k < 0$); and the
 traces in $z = k$ are the parabolas $y = k^2 - x^2$. Thus, $\frac{y}{1} = \frac{z^2}{1^2} - \frac{x^2}{1^2}$
 is a hyperbolic paraboloid.



20. Traces: $x = k \Rightarrow y^2 + 4z^2 = 16k^2$, ellipses; $y = k \Rightarrow$
 $16x^2 - 4z^2 = k^2$, hyperbolas if $k \neq 0$ and two intersecting lines if
 $k = 0$; $z = k \Rightarrow 16x^2 - y^2 = 4k^2$, hyperbolas if $k \neq 0$ and two
 intersecting lines if $k = 0$.
 $16x^2 = y^2 + 4z^2 \Leftrightarrow x^2 = \frac{y^2}{4^2} + \frac{z^2}{2^2}$ is an elliptic cone with axis
 the x -axis and vertex the origin.



21. This is the equation of an ellipsoid: $x^2 + 4y^2 + 9z^2 = x^2 + \frac{y^2}{(1/2)^2} + \frac{z^2}{(1/3)^2} = 1$, with x -intercepts ± 1 ,
 y -intercepts $\pm \frac{1}{2}$ and z -intercepts $\pm \frac{1}{3}$. So the major axis is the x -axis and the only possible graph is VII.
22. This is the equation of an ellipsoid: $9x^2 + 4y^2 + z^2 = \frac{x^2}{(1/3)^2} + \frac{y^2}{(1/2)^2} + z^2 = 1$, with x -intercepts $\pm \frac{1}{3}$,
 y -intercepts $\pm \frac{1}{2}$ and z -intercepts ± 1 . So the major axis is the z -axis and the only possible graph is IV.
23. This is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the
 axis of the hyperboloid is the y -axis, hence the correct graph is II.
24. This is a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz -plane at all, so the
 axis of the hyperboloid is the y -axis and the graph is III.

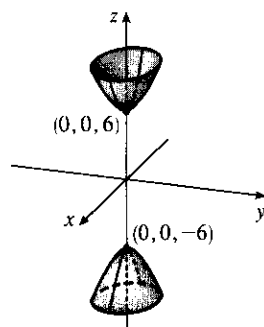
25. There are no real values of x and z that satisfy this equation for $y < 0$, so this surface does not extend to the left of the xz -plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y -axis. Its graph is VI.

26. This is the equation of a cone with axis the y -axis, so the graph is I.

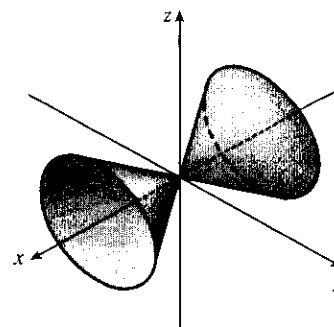
27. This surface is a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz -plane is an ellipse. So the graph is VIII.

28. This is the equation of a hyperbolic paraboloid. The trace in the xy -plane is the parabola $y = x^2$. So the correct graph is V.

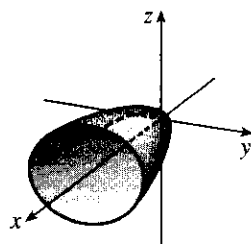
29. $z^2 = 4x^2 + 9y^2 + 36$ or $-4x^2 - 9y^2 + z^2 = 36$
or $-\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{36} = 1$ represents a hyperboloid
of two sheets with axis the z -axis.



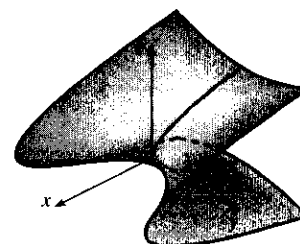
30. $x^2 = 2y^2 + 3z^2$ or $x^2 = \frac{y^2}{1/2} + \frac{z^2}{1/3}$ or
 $\frac{x^2}{6} = \frac{y^2}{3} + \frac{z^2}{2}$ represents an elliptic cone with
vertex $(0, 0, 0)$ and axis the x -axis.



31. $x = 2y^2 + 3z^2$ or $x = \frac{y^2}{1/2} + \frac{z^2}{1/3}$ or
 $\frac{x}{6} = \frac{y^2}{3} + \frac{z^2}{2}$ represents an elliptic paraboloid
with vertex $(0, 0, 0)$ and axis the x -axis.



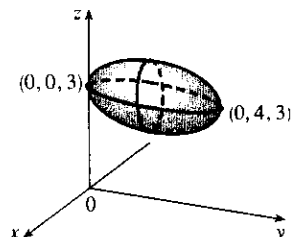
32. $4x - y^2 + 4z^2 = 0$ or $4x = y^2 - 4z^2$ or
 $x = \frac{y^2}{4} - z^2$ represents a hyperbolic paraboloid
with center $(0, 0, 0)$.



33. Completing squares in
- y
- and
- z
- gives

$$4x^2 + (y - 2)^2 + 4(z - 3)^2 = 4 \text{ or}$$

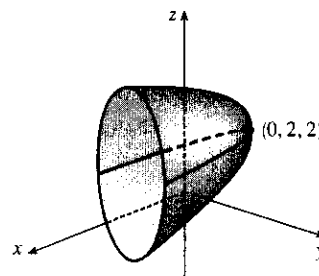
$x^2 + \frac{(y - 2)^2}{4} + (z - 3)^2 = 1$, an ellipsoid with center $(0, 2, 3)$.



34. Completing squares in
- y
- and
- z
- gives

$$4(y - 2)^2 + (z - 2)^2 - x = 0 \text{ or}$$

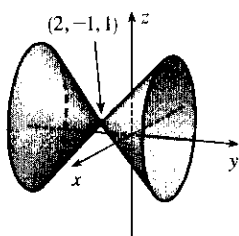
$\frac{x}{4} = (y - 2)^2 + \frac{(z - 2)^2}{4}$, an elliptic paraboloid with vertex $(0, 2, 2)$ and axis the horizontal line $y = 2, z = 2$.



35. Completing squares in all three variables gives

$$(x - 2)^2 - (y + 1)^2 + (z - 1)^2 = 0 \text{ or}$$

$(y + 1)^2 = (x - 2)^2 + (z - 1)^2$, a circular cone with center $(2, -1, 1)$ and axis the horizontal line $x = 2, z = 1$.

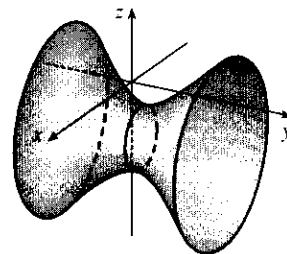


36. Completing squares in all three variables gives

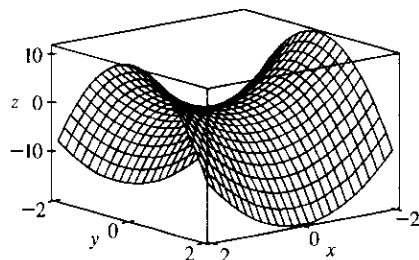
$$(x - 1)^2 - (y - 1)^2 + (z + 2)^2 = 2 \text{ or}$$

$$\frac{(x - 1)^2}{2} - \frac{(y - 1)^2}{2} + \frac{(z + 2)^2}{2} = 1, \text{ a}$$

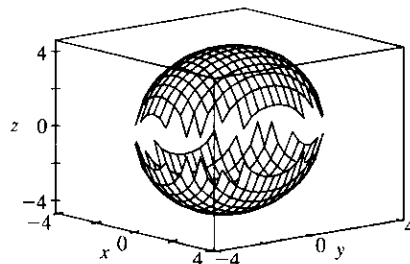
hyperboloid of one sheet with center $(1, 1, -2)$ and axis the horizontal line $x = 1, z = -2$.



37.

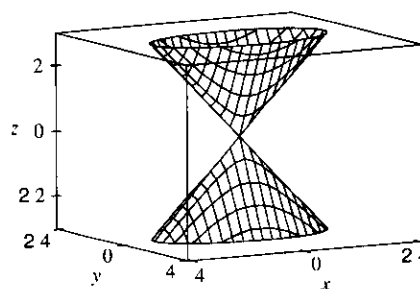
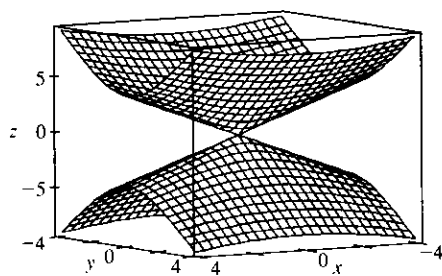


38.



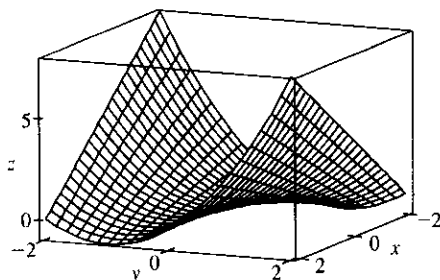
In Section 17.6 [ET 16.6], we will be able to graph ellipsoids without gaps; see Exercise 17.6.53 [ET 16.6.53].

39.

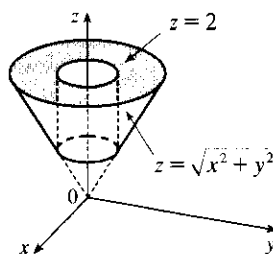


To restrict the z -range as in the second graph, we can use the option `view = -2..2` in Maple's `plot3d` command, or `PlotRange -> {-2, 2}` in Mathematica's `Plot3D` command.

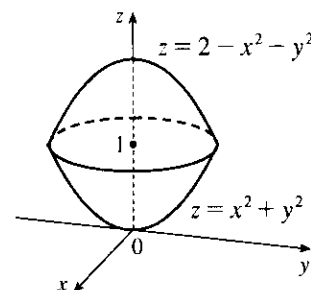
40.



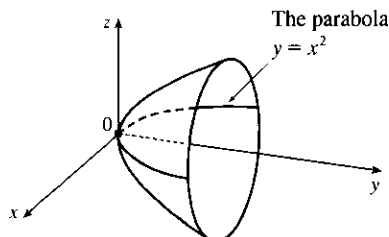
41.



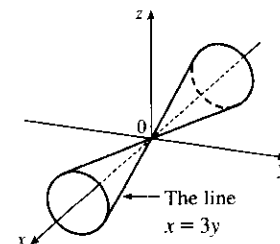
42.



43. The surface is a paraboloid of revolution (circular paraboloid) with vertex at the origin, axis the y -axis and opens to the right. Thus the trace in the yz -plane is also a parabola: $y = z^2, x = 0$. The equation is $y = x^2 + z^2$.



44. The surface is a right circular cone with vertex at $(0, 0, 0)$ and axis the x -axis. For $x = k \neq 0$, the trace is a circle with center $(k, 0, 0)$ and radius $r = y = \frac{x}{3} = \frac{k}{3}$. Thus the equation is $\frac{1}{3}x^2 = y^2 + z^2$.



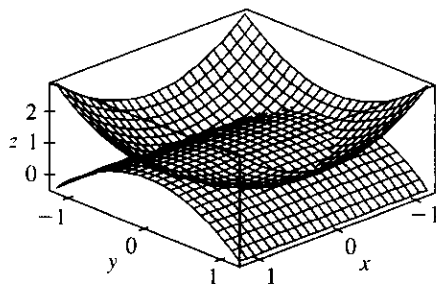
45. Let $P = (x, y, z)$ be an arbitrary point equidistant from $(-1, 0, 0)$ and the plane $x = 1$. Then the distance from P to $(-1, 0, 0)$ is $\sqrt{(x+1)^2 + y^2 + z^2}$ and the distance from P to the plane $x = 1$ is $|x-1|/\sqrt{1^2} = |x-1|$ (by Equation 13.5.9 [ET 12.5.9]). So $|x-1| = \sqrt{(x+1)^2 + y^2 + z^2} \Leftrightarrow (x-1)^2 = (x+1)^2 + y^2 + z^2 \Leftrightarrow x^2 - 2x + 1 = x^2 + 2x + 1 + y^2 + z^2 \Leftrightarrow -4x = y^2 + z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x -axis, which opens in the negative direction.

46. Let $P = (x, y, z)$ be an arbitrary point whose distance from the x -axis is twice its distance from the yz -plane. The distance from P to the x -axis is $\sqrt{(x-x)^2 + y^2 + z^2} = \sqrt{y^2 + z^2}$ and the distance from P to the yz -plane ($x = 0$) is $|x|/1 = |x|$. Thus $\sqrt{y^2 + z^2} = 2|x| \Leftrightarrow y^2 + z^2 = 4x^2 \Leftrightarrow x^2 = (y^2/2^2) + (z^2/2^2)$. So the surface is a right circular cone with vertex the origin and axis the x -axis.

47. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. $L_1: x = a + t, y = b + t, z = c + 2(b-a)t$.
 $L_2: x = a + t, y = b - t, z = c - 2(b+a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \Rightarrow$
 $c + 2(b-a)t = (b+t)^2 - (a+t)^2 = b^2 - a^2 + 2(b-a)t \Rightarrow c = b^2 - a^2$. As this is true for all values of t , L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \Rightarrow$
 $c - 2(b+a)t = (b-t)^2 - (a+t)^2 = b^2 - a^2 - 2(b+a)t \Rightarrow c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

48. Any point on the curve of intersection must satisfy both $2x^2 + 4y^2 - 2z^2 + 6x = 2$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$. Subtracting, we get $6x + 5y = 2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

49.



The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy -plane is the set of points $(x, y, 0)$ which satisfy $x^2 + y^2 = 1 - y^2 \Leftrightarrow x^2 + 2y^2 = 1 \Leftrightarrow x^2 + \frac{y^2}{(1/\sqrt{2})^2} = 1$. This is an equation of an ellipse.

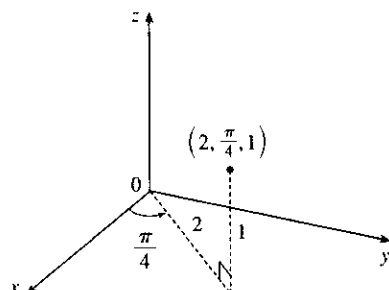
13.7 Cylindrical and Spherical Coordinates

ET 12.7

1. See Figure 1 and the accompanying discussion; see the paragraph accompanying Figure 3.

2. See Figure 5 and the accompanying discussion.

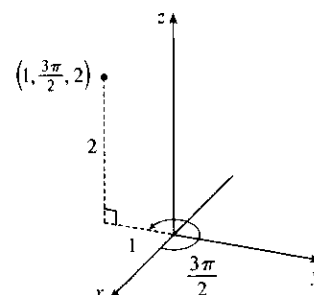
3.



$$x = 2 \cos \frac{\pi}{4} = \sqrt{2}, y = 2 \sin \frac{\pi}{4} = \sqrt{2},$$

$z = 1$, so the point is $(\sqrt{2}, \sqrt{2}, 1)$ in rectangular coordinates.

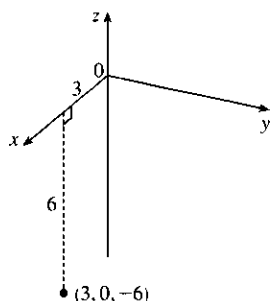
4.



$$x = 1 \cos \frac{3\pi}{2} = 0, y = 1 \sin \frac{3\pi}{2} = -1,$$

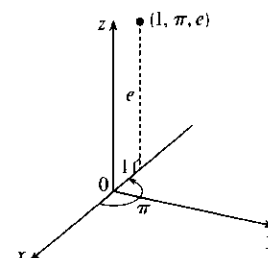
$z = 2$, so the point is $(0, -1, 2)$ in rectangular coordinates.

5.



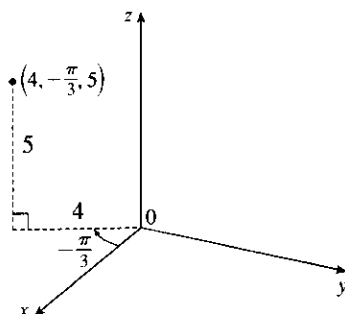
$x = 3 \cos 0 = 3, y = 3 \sin 0 = 0$, and $z = -6$, so the point is $(3, 0, -6)$ in rectangular coordinates.

6.



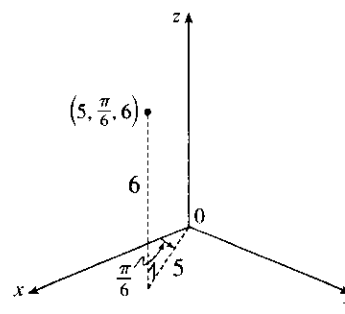
$x = 1 \cos \pi = -1, y = 1 \sin \pi = 0$, and $z = e$, so the point is $(-1, 0, e)$ in rectangular coordinates.

7.



$x = 4 \cos(-\frac{\pi}{3}) = 2$,
 $y = 4 \sin(-\frac{\pi}{3}) = -2\sqrt{3}$, and $z = 5$, so the point is $(2, -2\sqrt{3}, 5)$ in rectangular coordinates.

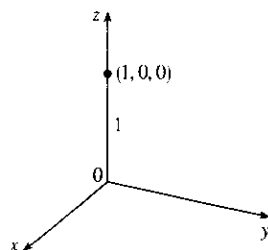
8.



$x = 5 \cos(\frac{\pi}{6}) = \frac{5\sqrt{3}}{2}, y = 5 \sin(\frac{\pi}{6}) = \frac{5}{2}$,
 and $z = 6$, so the point is $(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 6)$ in rectangular coordinates.

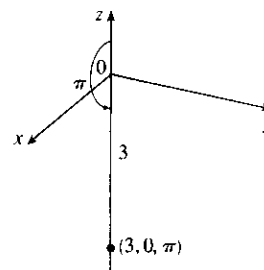
9. $r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$ so $r = \sqrt{2}$; $\tan \theta = \frac{y}{x} = \frac{-1}{1} = -1$ and the point $(1, -1)$ is in the fourth quadrant of the xy -plane, so $\theta = \frac{7\pi}{4} + 2n\pi$; $z = 4$. Thus, one set of cylindrical coordinates is $(\sqrt{2}, \frac{7\pi}{4}, 4)$.
10. $r^2 = x^2 + y^2 = 3^2 + 3^2 = 18$ so $r = \sqrt{18} = 3\sqrt{2}$; $\tan \theta = \frac{y}{x} = \frac{3}{3} = 1$ and the point $(3, 3)$ is in the first quadrant of the xy -plane, so $\theta = \frac{\pi}{4} + 2n\pi$; $z = -2$. Thus, one set of cylindrical coordinates is $(3\sqrt{2}, \frac{\pi}{4}, -2)$.
11. $r^2 = (-1)^2 + (-\sqrt{3})^2 = 4$ so $r = 2$; $\tan \theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$ and the point $(-1, -\sqrt{3})$ is in the third quadrant of the xy -plane, so $\theta = \frac{4\pi}{3} + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $(2, \frac{4\pi}{3}, 2)$.
12. $r^2 = 3^2 + 4^2 = 25$ so $r = 5$; $\tan \theta = \frac{4}{3}$ and the point $(3, 4)$ is in the first quadrant of the xy -plane, so $\theta = \tan^{-1}(\frac{4}{3}) + 2n\pi \approx 0.93 + 2n\pi$; $z = 5$. Thus, one set of cylindrical coordinates is $(5, \tan^{-1}(\frac{4}{3}), 5) \approx (5, 0.93, 5)$.

13.



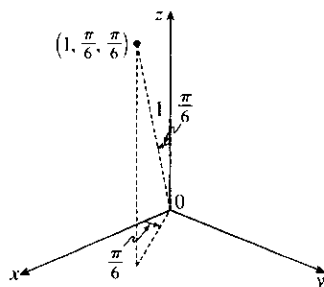
$x = \rho \sin \phi \cos \theta = (1) \sin 0 \cos 0 = 0$,
 $y = \rho \sin \phi \sin \theta = (1) \sin 0 \sin 0 = 0$, and
 $z = \rho \cos \phi = (1) \cos 0 = 1$ so the point is $(0, 0, 1)$ in rectangular coordinates.

14.



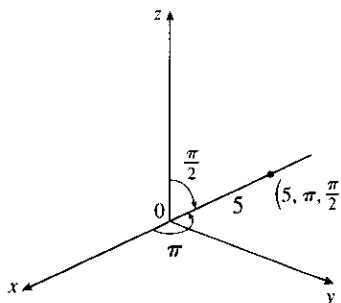
$x = 3 \sin \pi \cos 0 = 0$, $y = 3 \sin \pi \sin 0 = 0$,
 $z = 3 \cos \pi = -3$ and in rectangular coordinates the point is $(0, 0, -3)$.

15.



$x = \sin \frac{\pi}{6} \cos \frac{\pi}{6} = \frac{\sqrt{3}}{4}$, $y = \sin \frac{\pi}{6} \sin \frac{\pi}{6} = \frac{1}{4}$, and
 $z = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so the point is $(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2})$ in rectangular coordinates.

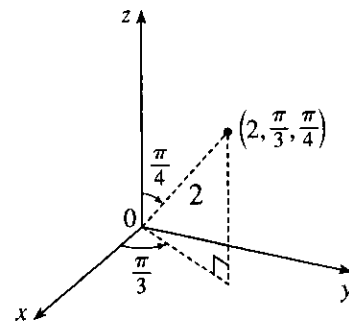
16.



$x = 5 \sin \frac{\pi}{2} \cos \pi = -5$, $y = 5 \sin \frac{\pi}{2} \sin \pi = 0$,
 $z = 5 \cos \frac{\pi}{2} = 0$ so the point is $(-5, 0, 0)$ in rectangular coordinates.

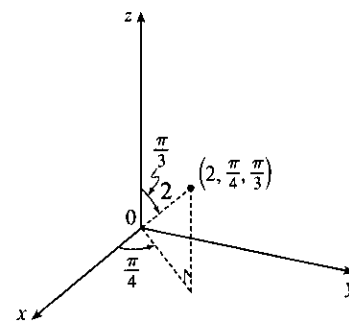
$$17. x = 2 \sin \frac{\pi}{4} \cos \frac{\pi}{3} = \frac{\sqrt{2}}{2}, y = 2 \sin \frac{\pi}{4} \sin \frac{\pi}{3} = \frac{\sqrt{6}}{2},$$

$z = 2 \cos \frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in rectangular coordinates.



$$18. x = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = \frac{\sqrt{6}}{2}, y = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{\sqrt{6}}{2}, z = 2 \cos \frac{\pi}{3} = 1$$

so the point is $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$ in rectangular coordinates.



$$19. \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 3 + 12} = 4, \cos \phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}, \text{ and}$$

$$\cos \theta = \frac{x}{\rho \sin \phi} = \frac{1}{4 \sin(\pi/6)} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ (since } y > 0 \text{). Thus spherical coordinates are } \left(4, \frac{\pi}{3}, \frac{\pi}{6}\right).$$

$$20. \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 3 + 1} = 2, \cos \phi = \frac{z}{\rho} = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}, \text{ and } \cos \theta = \frac{x}{\rho \sin \phi} = \frac{0}{2 \sin(\pi/3)} = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} \text{ (since } y > 0 \text{). Thus spherical coordinates are } \left(2, \frac{\pi}{2}, \frac{\pi}{3}\right).$$

$$21. \rho = \sqrt{0 + 1 + 1} = \sqrt{2}, \cos \phi = \frac{-1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4}, \text{ and } \cos \theta = \frac{0}{\sqrt{2} \sin(3\pi/4)} = 0 \Rightarrow \theta = \frac{3\pi}{2}$$

$$\text{(since } y < 0 \text{). Thus spherical coordinates are } \left(\sqrt{2}, \frac{3\pi}{2}, \frac{3\pi}{4}\right).$$

$$22. \rho = \sqrt{1 + 1 + 6} = 2\sqrt{2}, \cos \phi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}, \text{ and } \cos \theta = \frac{-1}{2\sqrt{2} \sin(\pi/6)} = -\frac{1}{\sqrt{2}} \Rightarrow$$

$$\theta = \frac{3\pi}{4} \text{ (since } y > 0 \text{). Thus spherical coordinates are } \left(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right).$$

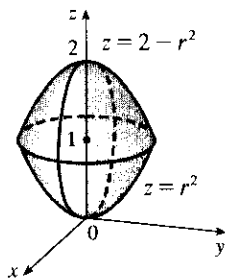
$$23. \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} = \sqrt{1 + 3} = 2; \theta = \frac{\pi}{6}; \cos \phi = \frac{z}{\rho} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}, \text{ thus in spherical coordinates the point is } \left(2, \frac{\pi}{6}, \frac{\pi}{6}\right).$$

$$24. \rho = \sqrt{r^2 + z^2} = \sqrt{6 + 2} = 2\sqrt{2}; \theta = \frac{\pi}{4}; \cos \phi = \frac{z}{\rho} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}, \text{ thus in spherical coordinates the point is } \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right).$$

25. $\rho = \sqrt{r^2 + z^2} = \sqrt{3 + 1} = 2$; $\theta = \frac{\pi}{2}$; $\cos \phi = \frac{z}{\rho} = \frac{-1}{2} \Rightarrow \phi = \frac{2\pi}{3}$, so in spherical coordinates the point is $\left(2, \frac{\pi}{2}, \frac{2\pi}{3}\right)$.
26. $\rho = \sqrt{16 + 9} = 5$; $\theta = \frac{\pi}{8}$; $\cos \phi = \frac{3}{5} \Rightarrow \phi = \cos^{-1}\left(\frac{3}{5}\right)$, so in spherical coordinates the point is $\left(5, \frac{\pi}{8}, \cos^{-1}\left(\frac{3}{5}\right)\right) \approx (5, \frac{\pi}{8}, 0.927)$.
27. $z = \rho \cos \phi = 2 \cos 0 = 2$, $\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2 \Rightarrow r = \sqrt{\rho^2 - z^2} = \sqrt{2^2 - 2^2} = 0$, (or $r = 2 \sin 0 = 0$), $\theta = 0$ and the point is $(0, 0, 2)$.
28. $z = 2\sqrt{2} \cos \frac{\pi}{2} = 0$, $r = 2\sqrt{2} \sin \frac{\pi}{2} = 2\sqrt{2}$, $\theta = \frac{3\pi}{2}$ and the point is $(2\sqrt{2}, \frac{3\pi}{2}, 0)$.
29. $z = 8 \cos \frac{\pi}{2} = 0$, $r = 8 \sin \frac{\pi}{2} = 8$, $\theta = \frac{\pi}{6}$ and the point is $(8, \frac{\pi}{6}, 0)$.
30. $z = 4 \cos \frac{\pi}{3} = 2$, $r = 4 \sin \frac{\pi}{3} = 2\sqrt{3}$, $\theta = \frac{\pi}{4}$ and the point is $(2\sqrt{3}, \frac{\pi}{4}, 2)$.
31. Since $r = 3$, $x^2 + y^2 = 9$ and the surface is a circular cylinder with radius 3 and axis the z -axis.
32. Since $\rho = 3$, $x^2 + y^2 + z^2 = 9$ and the surface is a sphere with center the origin and radius 3.
33. Since $\phi = 0$, $x = 0$ and $y = 0$ while $z = \rho \geq 0$. Thus the "surface" is the positive z -axis including the origin.
34. Since $\phi = \frac{\pi}{2}$, $z = 0$ but there are no restrictions on x and y ($x = \rho \cos \theta$, $y = \rho \sin \theta$). Thus the surface is the xy -plane.
35. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z -axis.
36. Whether spherical or cylindrical coordinates, since $\theta = \frac{\pi}{3}$ the surface is a half-plane including the z -axis and intersecting the xy -plane in the half-line $y = \sqrt{3}x$, $x > 0$.
37. $z = r^2 = x^2 + y^2$, so the surface is a circular paraboloid with vertex at the origin and axis the positive z -axis.
38. Since $r = 4 \sin \theta$ and $y = r \sin \theta$, $y = 4 \sin^2 \theta$. Also $r^2 = x^2 + y^2$ so $x^2 + y^2 = 16 \sin^2 \theta$. Thus $x^2 + y^2 - 4y = 16 \sin^2 \theta - 16 \sin^2 \theta = 0$ or $x^2 + (y - 2)^2 = 4$, a circular cylinder of radius 2 and with axis parallel to the z -axis.
39. $2 = \rho \cos \phi = z$ is a plane through the point $(0, 0, 2)$ and parallel to the xy -plane.
40. Since $\rho \sin \phi = 2$ and $x = \rho \sin \phi \cos \theta$, $x = 2 \cos \theta$. Also $y = \rho \sin \phi \sin \theta$ so $y = 2 \sin \theta$. Then $x^2 + y^2 = 4 \cos^2 \theta + 4 \sin^2 \theta = 4$, a circular cylinder of radius 2 about the z -axis.
41. $r = 2 \cos \theta \Rightarrow r^2 = x^2 + y^2 = 2r \cos \theta = 2x \Leftrightarrow (x - 1)^2 + y^2 = 1$, which is the equation of a circular cylinder with radius 1, whose axis is the vertical line $x = 1$, $y = 0$, $z = z$.

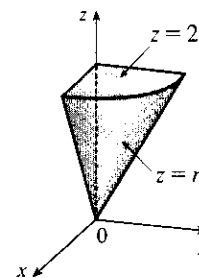
42. $\rho = 2 \cos \phi \Rightarrow \rho^2 = 2\rho \cos \phi = 2z \Leftrightarrow x^2 + y^2 + z^2 = 2z \Leftrightarrow x^2 + y^2 + (z - 1)^2 = 1$. Therefore, the surface is a sphere of radius 1 centered at $(0, 0, 1)$.
43. Since $r^2 + z^2 = 25$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + z^2 = 25$, a sphere with radius 5 and center at the origin.
44. Since $r^2 - 2z^2 = 4$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 - 2z^2 = 4$ or $\frac{1}{4}x^2 + \frac{1}{4}y^2 - \frac{1}{2}z^2 = 1$, a hyperboloid of one sheet with axis the z -axis.
45. Since $x^2 = \rho^2 \sin^2 \phi \cos^2 \theta$ and $z^2 = \rho^2 \cos^2 \phi$, the equation of the surface in rectangular coordinates is $x^2 + z^2 = 4$. Thus the surface is a circular cylinder of radius 2 about the y -axis.
46. Since $\rho^2(\sin^2 \phi - 4 \cos^2 \phi) = 1$, $\rho^2(\sin^2 \phi - 4 \cos^2 \phi) + \rho^2 \cos^2 \phi - \rho^2 \cos^2 \phi = 1$ or $\rho^2(\sin^2 \phi + \cos^2 \phi - 5 \cos^2 \phi) = 1$ or $\rho^2(1 - 5 \cos^2 \phi) = 1$. But $\rho^2 = x^2 + y^2 + z^2$ and $z^2 = \rho^2 \cos^2 \phi$, so we can rewrite the equation of the surface as $x^2 + y^2 + z^2 - 5z^2 = 1$ or $x^2 + y^2 - 4z^2 = 1$. Thus the surface is a hyperboloid of one sheet with axis the z -axis.
47. Since $r^2 - r = 0$, $r = 0$ or $r = 1$. But $x^2 + y^2 = r^2$. Thus the surface consists of the right circular cylinder of radius 1 and axis the z -axis along with the surface given by $x^2 + y^2 = 0$, that is, the z -axis.
48. Since $\rho^2 - 6\rho + 8 = 0$, either $\rho = 2$ or $\rho = 4$. Thus the surface consists of two concentric spheres (centered at the origin), one with radius 2 and the other with radius 4.
49. (a) $x^2 + y^2 = r^2$, so the equation becomes $z = r^2$.
(b) $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$, so the equation becomes $\rho \cos \phi = (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2$ or $\rho \cos \phi = \rho^2 \sin^2 \phi$ or $\rho \sin^2 \phi = \cos \phi$.
50. (a) $x^2 + y^2 = r^2$, so the equation becomes $r^2 + z^2 = 2$.
(b) $x^2 + y^2 + z^2 = \rho^2$, so the equation becomes $\rho^2 = 2$ or $\rho = \sqrt{2}$.
51. (a) $x = r \cos \theta$, so the equation becomes $r \cos \theta = 3$ or $r = 3 \sec \theta$ (since $\cos \theta \neq 0$ here).
(b) $x = \rho \sin \phi \cos \theta$, so the equation becomes $\rho \sin \phi \cos \theta = 3$.
52. (a) $x^2 + y^2 = r^2$, so the equation becomes $r^2 + z^2 + 2z = 0$ or $r^2 + (z + 1)^2 = 1$.
(b) $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, so the equation becomes $\rho^2 + 2\rho \cos \phi = 0$ or $\rho = -2 \cos \phi$.
53. (a) $r^2(\cos^2 \theta - \sin^2 \theta) - 2z^2 = 4$ or $2z^2 = r^2 \cos 2\theta - 4$.
(b) $\rho^2(\sin^2 \phi \cos^2 \theta - \sin^2 \phi \sin^2 \theta - 2 \cos^2 \phi) = 4$ or $\rho^2(\sin^2 \phi \cos 2\theta - 2 \cos^2 \phi) = 4$.
54. (a) $r^2 \sin^2 \theta + z^2 = 1$
(b) $\rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi = 1$ or $\rho^2(\sin^2 \phi \sin^2 \theta + \cos^2 \phi) = 1$.
55. (a) $r^2 = 2r \sin \theta$ or $r = 2 \sin \theta$.
(b) $\rho^2 \sin^2 \phi(\cos^2 \theta + \sin^2 \theta) = 2\rho \sin \phi \sin \theta$ or $\rho \sin^2 \phi = 2 \sin \phi \sin \theta$ or $\rho \sin \phi = 2 \sin \theta$.
56. (a) $z = r^2(\cos^2 \theta - \sin^2 \theta)$ or $z = r^2 \cos 2\theta$.
(b) $\rho \cos \phi = \rho^2 \sin^2 \phi(\cos^2 \theta - \sin^2 \theta)$ or $\cos \phi = \rho \sin^2 \phi \cos 2\theta$.

57.



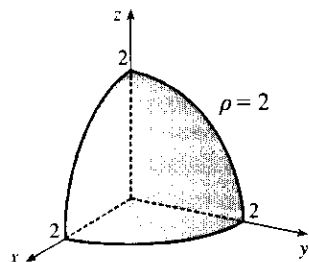
$z = r^2 = x^2 + y^2$ is a circular paraboloid with vertex $(0, 0, 0)$, opening upward. $z = 2 - r^2$
 $\Rightarrow z - 2 = -(x^2 + y^2)$ is a circular paraboloid with vertex $(0, 0, 2)$ opening downward. Thus $r^2 \leq z \leq 2 - r^2$ is the solid region enclosed by these two surfaces.

58.



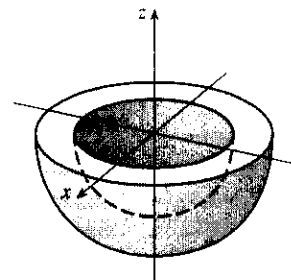
$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward. Thus $r \leq z \leq 2$ is the region above this cone and beneath the horizontal plane $z = 2$. $0 \leq \theta \leq \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

59.



$\rho = 2$ represents a sphere of radius 2, centered at the origin, so $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy -plane, and $0 \leq \theta \leq \frac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.

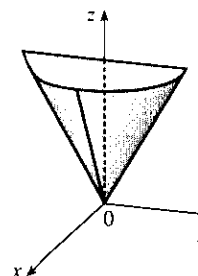
60.



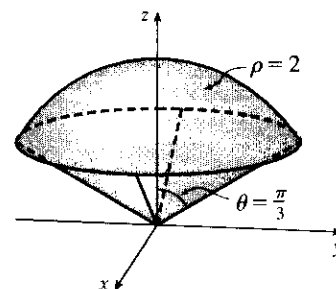
$2 \leq \rho \leq 3$ represents the solid region between and including the spheres of radii 2 and 3, centered at the origin. $\frac{\pi}{2} \leq \phi \leq \pi$ restricts the solid to that portion on or below the xy -plane.

61. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ restricts the solid to the 4 octants in which x is

positive. $\rho = \sec \phi \Rightarrow \rho \cos \phi = z = 1$, which is the equation of a horizontal plane. $0 \leq \phi \leq \frac{\pi}{6}$ describes a cone, opening upward. So the solid lies above the cone $\phi = \frac{\pi}{6}$ and below the plane $z = 1$.



62. $\rho = 2 \Leftrightarrow x^2 + y^2 + z^2 = 4$, which is a sphere of radius 2, centered at the origin. Hence $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{3}$ restricts the solid to that section of this ball that lies above the cone $\phi = \frac{\pi}{3}$.

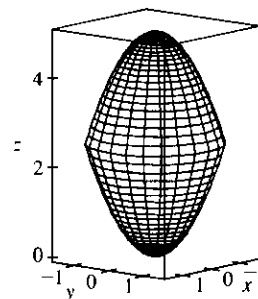


63. We can position the cylindrical shell vertically so that its axis coincides with the z -axis and its base lies in the xy -plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \leq r \leq 7, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 20$.
64. (a) The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \leq \rho \leq 15, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$.
- (b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy -plane which is described by $14.5 \leq \rho \leq 15, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$.

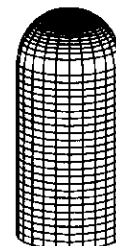
65. $z \geq \sqrt{x^2 + y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \geq x^2 + y^2 \Rightarrow 2z^2 \geq x^2 + y^2 + z^2 = \rho^2 \Rightarrow z^2 = \rho^2 \cos^2 \phi \geq \frac{1}{2}\rho^2 \Rightarrow \cos^2 \phi \geq \frac{1}{2}$. The cone opens upward so that the inequality is $\cos \phi \geq \frac{1}{\sqrt{2}}$, or equivalently $0 \leq \phi \leq \frac{\pi}{4}$. In spherical coordinates the sphere $x^2 + y^2 + z^2 = 1$ is $\rho \cos \phi = 1 \Rightarrow \rho = \sec \phi, 0 \leq \rho \leq \sec \phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \leq \rho \leq \sec \phi, 0 \leq \phi \leq \frac{\pi}{4}$.

66. In cylindrical coordinates, the equations are $z = r^2$ and $z = 5 - r^2$.

The curve of intersection is $r^2 = 5 - r^2$ or $r = \sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0 \leq r \leq \sqrt{5/2}$. In Maple, we can use either the `coords=cylindrical` option in a regular `plot` command, or the `plots[cylinderplot]` command. In Mathematica, we can use `ParametricPlot3d`.



67. In cylindrical coordinates, the equation of the cylinder is $r = 3$, $0 \leq z \leq 10$. The hemisphere is the upper part of the sphere radius 3, center $(0, 0, 10)$, equation $r^2 + (z - 10)^2 = 3^2, z \geq 10$. In Maple, we can use either the `coords=cylindrical` option in a regular `plot` command, or the `plots[cylinderplot]` command. In Mathematica, we can use `ParametricPlot3d`.



68. We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho = 3960$ mi	$\rho = 3960$ mi
$\theta = 360^\circ - 73.60^\circ = 286.40^\circ$	$\theta = 360^\circ - 118.25^\circ = 241.75^\circ$
$\phi = 90^\circ - 45.50^\circ = 44.50^\circ$	$\phi = 90^\circ - 34.06^\circ = 55.94^\circ$

Now we change the above to Cartesian coordinates using $x = \rho \cos \theta \sin \phi$, $y = \rho \sin \theta \sin \phi$ and $z = \rho \cos \phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the Earth). In particular:

Montréal: $\langle 783.67, -2662.67, 2824.47 \rangle$

Los Angeles: $\langle -1552.80, -2889.91, 2217.84 \rangle$

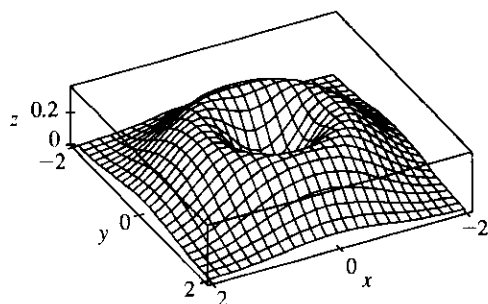
To find the angle α between these two vectors we use the dot product:

$$\langle 783.67, -2662.67, 2824.47 \rangle \cdot \langle -1552.80, -2889.91, 2217.84 \rangle = (3960)^2 \cos \alpha \Rightarrow \cos \alpha \approx 0.8126 \Rightarrow$$

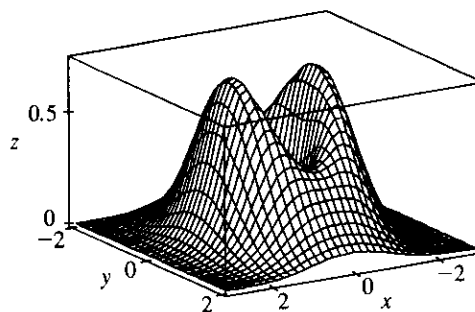
$\alpha \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\theta \approx 3960(0.6223) \approx 2464$ mi.

LABORATORY PROJECT Families of Surfaces

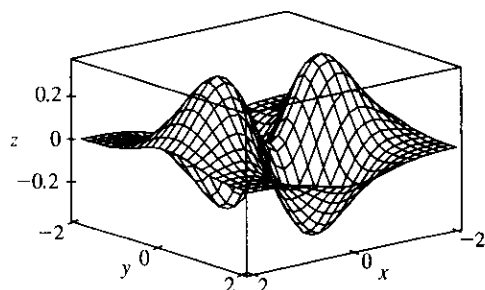
1. $z = (ax^2 + by^2)e^{-x^2-y^2}$. There are only three basic shapes which can be obtained (the fourth and fifth graphs are the reflections of the first and second ones in the xy -plane). Interchanging a and b rotates the graph by 90° about the z -axis.



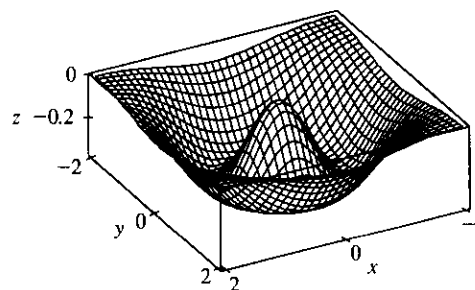
$$a = 1, b = 1$$



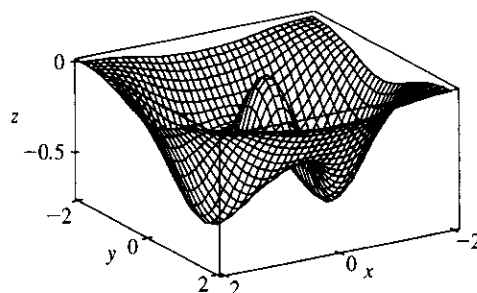
$$a = 2, b = 1$$



$$a = 1, b = -1$$



$$a = -1, b = -1$$

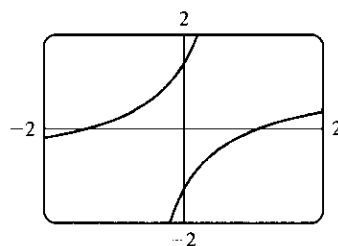


$$a = -2, b = -1$$

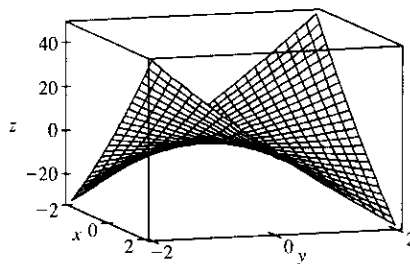
If a and b are both positive ($a \neq b$), we see that the graph has two maximum points whose height increases as a and b increase. If a and b have opposite signs, the graph has two maximum points and two minimum points, and if a and b are both negative, the graph has one maximum point and two minimum points.

2. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See graph below.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

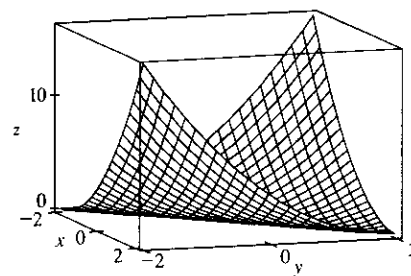
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0, z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5, z = 2$



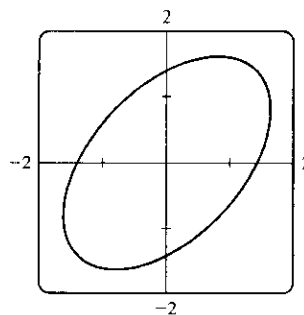
$c = -10$



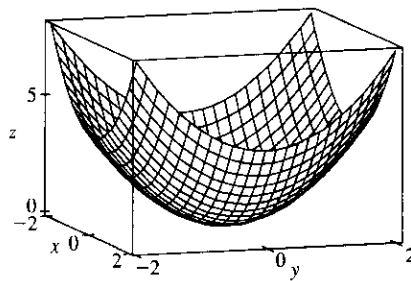
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y . If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

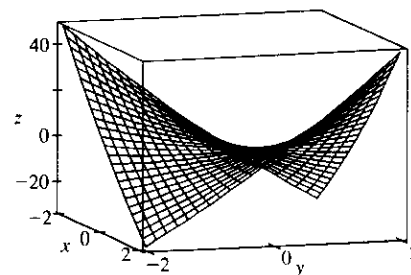
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1, z = 2$



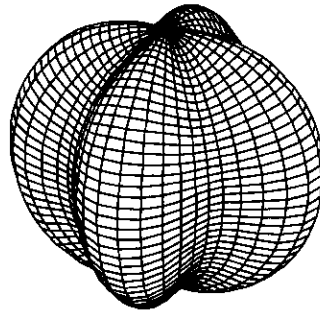
$c = 0$



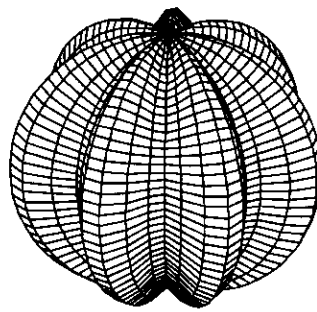
$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

3. $\rho = 1 + 0.2 \sin m\theta \sin n\phi$. If we start with $m = 1, n = 1$ the equation is $\rho = 1 + 0.2 \sin \theta \sin \phi$, whose graph appears spherical or nearly spherical in shape. First we investigate varying just m . Values of $m > 1$ produce vertical ridges in the sphere, the number of ridges corresponding to the value of m . We graph two examples.

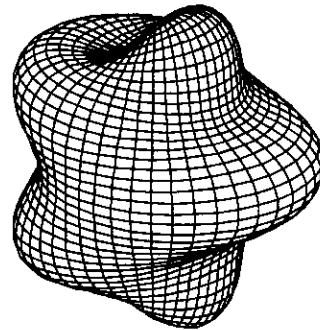


$m = 4, n = 1$

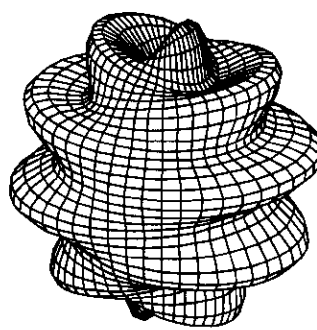


$m = 7, n = 1$

If we leave m fixed at 1 and vary n , we see horizontal ridges that span half the sphere arranged in a staggered fashion. Again, the number of “bumps” coincides with the value of n .

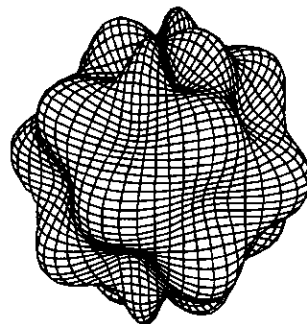


$m = 1, n = 5$

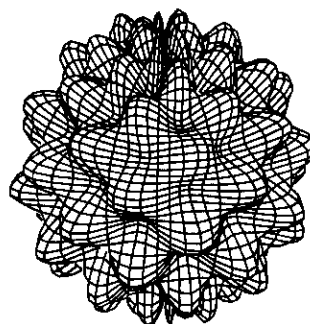


$m = 1, n = 10$

If we allow both m and n to vary, we get combinations of the vertical and horizontal bumps.



$m = 4, n = 5$



$m = 7, n = 10$

The graph on the left shows $m = 4, n = 5$. Looking at the top of the bumpy sphere, we can see the 4 vertical ridges which become perturbed horizontally as they progress down the sphere. We can also see the 5 horizontal rows of bumps. (Consequently, there are 20 bumps on the surface.) The graph on the right shows $m = 7, n = 10$ which should have 70 bumps.

13 Review

ET 12

CONCEPT CHECK

1. A scalar is a real number, while a vector is a quantity that has both a real-valued magnitude and a direction.
2. To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law, as illustrated in Figures 3 and 4 in Section 13.2 [ET 12.2]. Algebraically, we add the corresponding components of the vectors.
3. For $c > 0$, $c\mathbf{a}$ is a vector with the same direction as $\mathbf{a}$ and length c times the length of $\mathbf{a}$. If $c < 0$, $c\mathbf{a}$ points in the opposite direction as $\mathbf{a}$ and has length $|c|$ times the length of $\mathbf{a}$. (See Figures 7 and 15 in Section 13.2 [ET 12.2].) Algebraically, to find $c\mathbf{a}$ we multiply each component of $\mathbf{a}$ by c .
4. See (1) in Section 13.2 [ET 12.2].
5. See Theorem 13.3.3 [ET 12.3.3] and Definition 13.3.1 [ET 12.3.1].
6. The dot product can be used to find the angle between two vectors and the scalar projection of one vector onto another. In particular, the dot product can determine if two vectors are orthogonal. Also, the dot product can be used to determine the work done moving an object given the force and displacement vectors.
7. See the boxed equations on page 847 [ET 811] as well as Figures 4 and 5 and the accompanying discussion on pages 846–47 [ET 810–11].
8. See Theorem 13.4.6 [ET 12.4.6] and the preceding discussion; use either (1) or (4) in Section 13.4 [ET 12.4].
9. The cross product can be used to create a vector orthogonal to two given vectors as well as to determine if two vectors are parallel. The cross product can also be used to find the area of a parallelogram determined by two vectors. In addition, the cross product can be used to determine torque if the force and position vectors are known.
10. (a) The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is the length of the cross product: $|\mathbf{a} \times \mathbf{b}|$.
(b) The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.
11. If an equation of the plane is known, it can be written as $ax + by + cz + d = 0$. A normal vector, which is perpendicular to the plane, is $\langle a, b, c \rangle$ (or any scalar multiple of $\langle a, b, c \rangle$). If an equation is not known, we can use points on the plane to find two non-parallel vectors which lie in the plane. The cross product of these vectors is a vector perpendicular to the plane.
12. The angle between two intersecting planes is defined as the acute angle between their normal vectors. We can find this angle using Corollary 13.3.6 [ET 12.3.6].
13. See (1), (2), and (3) in Section 13.5 [ET 12.5].
14. See (5), (6), and (7) in Section 13.5 [ET 12.5].
15. (a) Two (nonzero) vectors are parallel if and only if one is a scalar multiple of the other. In addition, two nonzero vectors are parallel if and only if their cross product is $\mathbf{0}$.
(b) Two vectors are perpendicular if and only if their dot product is 0.
(c) Two planes are parallel if and only if their normal vectors are parallel.
16. (a) Determine the vectors $\overrightarrow{PQ} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{PR} = \langle b_1, b_2, b_3 \rangle$. If there is a scalar t such that $\langle a_1, a_2, a_3 \rangle = t \langle b_1, b_2, b_3 \rangle$, then the vectors are parallel and the points must all lie on the same line. Alternatively, if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P , Q , and R are collinear. Thirdly, an algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

- (b) Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P , Q and R , and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. (Or use the reasoning in Example 5 in Section 13.4 [ET 12.4].)

Alternatively, find an equation for the plane determined by three of the points and check whether or not the fourth point satisfies this equation.

17. (a) See Exercise 13.4.39 [ET 12.4.39].
 (b) See Example 8 in Section 13.5 [ET 12.5].
 (c) See Example 10 in Section 13.5 [ET 12.5].
18. The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz -plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz -plane) and $z = k$ (parallel to the xy -plane) are found similarly.
19. See Table 1 in Section 13.6 [ET 12.6].
20. (a) See (1) and the discussion accompanying Figure 3 in Section 13.7 [ET 12.7].
 (b) See (3) and Figures 6–8, and the accompanying discussion, in Section 13.7 [ET 12.7].

TRUE-FALSE QUIZ

- True, by Theorem 13.3.2 [ET 12.3.2] #2.
- False. Theorem 13.4.8 [ET 12.4.8] #1 says that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 13.4.6 [ET 12.4.6],
 $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta = |\mathbf{v}| |\mathbf{u}| \sin \theta = |\mathbf{v} \times \mathbf{u}|$.
 (Or, by Theorem 13.4.8 [ET 12.4.8], $|\mathbf{u} \times \mathbf{v}| = |-\mathbf{v} \times \mathbf{u}| = |-1| |\mathbf{v} \times \mathbf{u}| = |\mathbf{v} \times \mathbf{u}|$.)
- This is true by Theorem 13.3.2 [ET 12.3.2] #4.
- Theorem 13.4.8 [ET 12.4.8] #2 tells us that this is true.
- This is true by Theorem 13.4.8 [ET 12.4.8] #4.
- This is true by Theorem 13.4.8 [ET 12.4.8] #5.
- In general, this assertion is false; a counterexample is $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. (See the paragraph preceding Theorem 13.4.8 [ET 12.4.8].)
- This is true because $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 13.4.5 [ET 12.4.5]), and the dot product of two orthogonal vectors is 0.
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{v}$ (by Theorem 13.4.8 [ET 12.4.8] #4)
 $= \mathbf{u} \times \mathbf{v} + \mathbf{0}$ (by Example 13.4.2 [ET 12.4.2])
 $= \mathbf{u} \times \mathbf{v}$, so this is true.
- If $|\mathbf{u}| = 1$, $|\mathbf{v}| = 1$ and θ is the angle between these two vectors (so $0 \leq \theta \leq \pi$), then by Theorem 13.4.6 [ET 12.4.6], $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta = \sin \theta$, which is equal to 1 if and only if $\theta = \frac{\pi}{2}$ (that is, if and only if the two vectors are orthogonal). Therefore, the assertion that the cross product of two unit vectors is a unit vector is false.
- This is false, because according to Equation 13.5.8 [ET 12.5.8], $ax + by + cz + d = 0$ is the general equation of a plane.

$$(f) \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 9 + 15 - 6 = 18$$

(g) $\mathbf{c} \times \mathbf{c} = \mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \times (9\mathbf{i} + 15\mathbf{j} + 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix} = (3 + 30)\mathbf{i} - (3 + 18)\mathbf{j} + (15 - 9)\mathbf{k} \\ &= 33\mathbf{i} - 21\mathbf{j} + 6\mathbf{k}. \end{aligned}$$

(i) The scalar projection is $\text{comp}_{\mathbf{a}} \mathbf{b} = |\mathbf{b}| \cos \theta = \mathbf{a} \cdot \mathbf{b} / |\mathbf{a}| = -\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\text{proj}_{\mathbf{a}} \mathbf{b} = -\frac{1}{\sqrt{6}}(\mathbf{a} / |\mathbf{a}|) = -\frac{1}{6}(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

$$(k) \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-1}{\sqrt{6} \sqrt{14}} = \frac{-1}{2\sqrt{21}} \text{ and } \theta = \cos^{-1}\left(\frac{-1}{2\sqrt{21}}\right) \approx 96^\circ.$$

5. For the two vectors to be orthogonal, we need $\langle 3, 2, x \rangle \cdot \langle 2x, 4, x \rangle = 0 \Leftrightarrow$

$$(3)(2x) + (2)(4) + (x)(x) = 0 \Leftrightarrow x^2 + 6x + 8 = 0 \Leftrightarrow (x + 2)(x + 4) = 0 \Leftrightarrow x = -2 \text{ or } x = -4.$$

6. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$(\mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = [3 - (-4)]\mathbf{i} - (0 - 2)\mathbf{j} + (0 - 1)\mathbf{k} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}. \text{ Then two unit vectors}$$

orthogonal to both given vectors are $\pm \frac{7\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{7^2 + 2^2 + (-1)^2}} = \pm \frac{1}{3\sqrt{6}}(7\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, that is,

$$\frac{7}{3\sqrt{6}}\mathbf{i} + \frac{2}{3\sqrt{6}}\mathbf{j} - \frac{1}{3\sqrt{6}}\mathbf{k} \text{ and } -\frac{7}{3\sqrt{6}}\mathbf{i} - \frac{2}{3\sqrt{6}}\mathbf{j} + \frac{1}{3\sqrt{6}}\mathbf{k}.$$

7. (a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$

$$(b) \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \mathbf{u} \cdot [-(\mathbf{v} \times \mathbf{w})] = -\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -2$$

$$(c) \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} = -(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = -2$$

$$(d) (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{0} = 0$$

8. $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}] \mathbf{c} - [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c}] \mathbf{a}$

(see Exercise 13.4.42 [ET 12.4.42])

$$= (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}] \mathbf{c} = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$= [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0, 0, 0)$ to $(1, 1, 1)$ and $(1, 0, 0)$ to $(0, 1, 1)$ are $\langle 1, 1, 1 \rangle$ and $\langle -1, 1, 1 \rangle$. Let θ be the angle between these two vectors. $\langle 1, 1, 1 \rangle \cdot \langle -1, 1, 1 \rangle = -1 + 1 + 1 = 1 = |\langle 1, 1, 1 \rangle| |\langle -1, 1, 1 \rangle| \cos \theta = 3 \cos \theta$
 $\Rightarrow \cos \theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ.$

10. $\overrightarrow{AB} = \langle 1, 3, -1 \rangle$, $\overrightarrow{AC} = \langle -2, 1, 3 \rangle$ and $\overrightarrow{AD} = \langle -1, 3, 1 \rangle$. By Equation 13.4.10 [ET 12.4.10],

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} - 3 \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8 - 3 + 5 = -6.$$

The volume is $|\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| = 6$ cubic units.

11. $\overrightarrow{AB} = \langle 1, 0, -1 \rangle$, $\overrightarrow{AC} = \langle 0, 4, 3 \rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0 + 4, -(3 + 0), 4 - 0 \rangle = \langle 4, -3, 4 \rangle$.

(b) $\frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{16 + 9 + 16} = \frac{\sqrt{41}}{2}$.

12. $\mathbf{D} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$, $W = \mathbf{F} \cdot \mathbf{D} = 12 + 15 + 60 = 87$ joules

13. Let F_1 be the magnitude of the force directed 20° away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives $F_1 \cos 20^\circ + F_2 \cos 30^\circ = 255$ (1), and $F_1 \sin 20^\circ - F_2 \sin 30^\circ = 0 \Rightarrow$

$$F_1 = F_2 \frac{\sin 30^\circ}{\sin 20^\circ} \quad (2). \text{ Substituting (2) into (1) gives } F_2(\sin 30^\circ \cot 20^\circ + \cos 30^\circ) = 255 \Rightarrow F_2 \approx 114 \text{ N.}$$

Substituting this into (2) gives $F_1 \approx 166 \text{ N}$.

14. $|\tau| = |\mathbf{r}| |\mathbf{F}| \sin \theta = (0.40)(50) \sin(90^\circ - 30^\circ) \approx 17.3$ joules

15. The line has direction $\mathbf{v} = \langle -3, 2, 3 \rangle$. Letting $P_0 = (4, -1, 2)$, parametric equations are $x = 4 - 3t$, $y = -1 + 2t$, $z = 2 + 3t$.

16. A direction vector for the line is $\mathbf{v} = \langle 3, 2, 1 \rangle$, so parametric equations for the line are $x = 1 + 3t$, $y = 2t$, $z = -1 + t$.

17. A direction vector for the line is a normal vector for the plane, $\mathbf{n} = \langle 2, -1, 5 \rangle$, and parametric equations for the line are $x = -2 + 2t$, $y = 2 - t$, $z = 4 + 5t$.

18. Since the two planes are parallel, they will have the same normal vectors. Then we can take $\mathbf{n} = \langle 1, 4, -3 \rangle$ and an equation of the plane is $1(x - 2) + 4(y - 1) - 3(z - 0) = 0$ or $x + 4y - 3z = 6$.

19. Here the vectors $\mathbf{a} = \langle 4 - 3, 0 - (-1), 2 - 1 \rangle = \langle 1, 1, 1 \rangle$ and $\mathbf{b} = \langle 6 - 3, 3 - (-1), 1 - 1 \rangle = \langle 3, 4, 0 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -4, 3, 1 \rangle$ is a normal vector to the plane and an equation of the plane is $-4(x - 3) + 3(y - (-1)) + 1(z - 1) = 0$ or $-4x + 3y + z = -14$.

20. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 2, -1, 3 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, -2)$ does not lie on this line. The point $(0, 3, 1)$ is on the line (obtained by putting $t = 0$) and hence in the plane, so the vector $\mathbf{b} = \langle 0 - 1, 3 - 2, 1 - (-2) \rangle = \langle -1, 1, 3 \rangle$ lies in the plane, and a normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -6, -9, 1 \rangle$. Thus an equation of the plane is $-6(x - 1) - 9(y - 2) + (z + 2) = 0$ or $6x + 93y - z = 26$.

21. Substitution of the parametric equations into the equation of the plane gives

$$2x - y + z = 2(2 - t) - (1 + 3t) + 4t = 2 \Rightarrow -t + 3 = 2 \Rightarrow t = 1. \text{ When } t = 1, \text{ the parametric equations give } x = 2 - 1 = 1, y = 1 + 3 = 4 \text{ and } z = 4. \text{ Therefore, the point of intersection is } (1, 4, 4).$$

22. Use the formula proven in Exercise 13.4.39 [ET 12.4.39]. In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1, -1, 2 \rangle$. A point on the line is $(1, 2, -1)$ (setting $t = 0$), and therefore

$$\mathbf{b} = \langle 1 - 0, 2 - 0, -1 - 0 \rangle = \langle 1, 2, -1 \rangle. \text{ Hence}$$

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -1, 2 \rangle \times \langle 1, 2, -1 \rangle|}{\sqrt{1 + 1 + 4}} = \frac{|\langle -3, 3, 3 \rangle|}{\sqrt{6}} = \frac{\sqrt{27}}{\sqrt{6}} = \frac{3}{\sqrt{2}}.$$

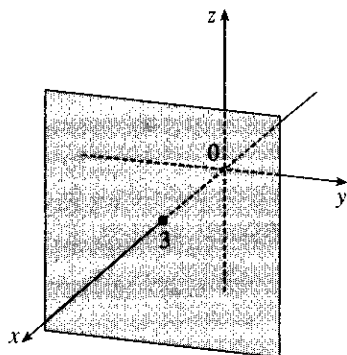
23. Since the direction vectors $\langle 2, 3, 4 \rangle$ and $\langle 6, -1, 2 \rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

24. (a) The normal vectors are $\langle 1, 1, -1 \rangle$ and $\langle 2, -3, 4 \rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

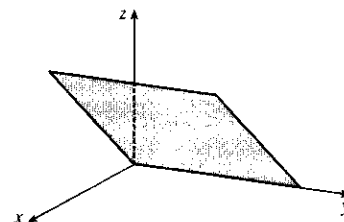
(b) $\cos \theta = \frac{\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle}{\sqrt{3} \sqrt{29}} = -\frac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122^\circ$ (or we can say $\approx 58^\circ$).

25. By Exercise 13.5.69 [ET 12.5.69], $D = \frac{|2 - 24|}{\sqrt{26}} = \frac{22}{\sqrt{26}}$.

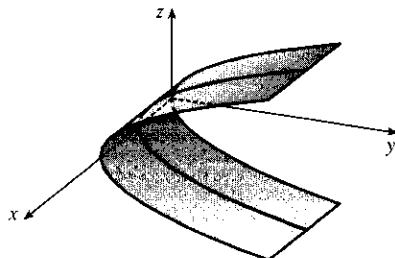
26. The equation $x = 3$ represents a plane parallel to the yz -plane and 3 units in front of it.



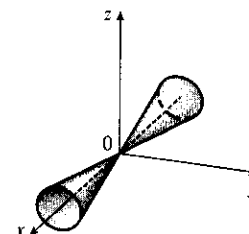
27. The equation $x = z$ represents a plane perpendicular to the xz -plane and intersecting the xz -plane in the line $x = z, y = 0$.



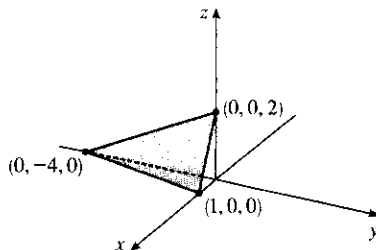
28. A parabolic cylinder whose trace in the xz -plane is the x -axis and which opens to the right.



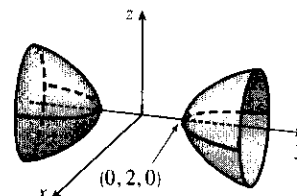
29. A (right elliptical) cone with vertex at the origin and axis the x -axis.



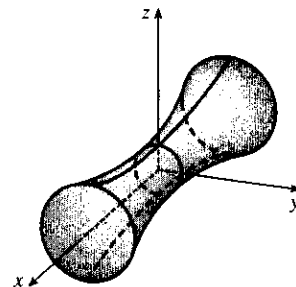
30. $4x - y + 2z = 4$ is a plane with intercepts $(1, 0, 0)$, $(0, -4, 0)$, and $(0, 0, 2)$.



31. An equivalent equation is $-x^2 + \frac{y^2}{4} - z^2 = 1$, a hyperboloid of two sheets with axis the y -axis. For $|y| > 2$, traces parallel to the xz -plane are circles.



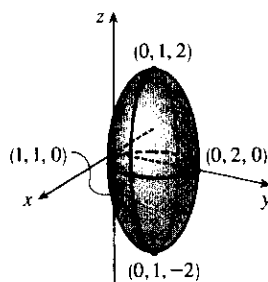
32. An equivalent equation is $-x^2 + y^2 + z^2 = 1$, a hyperboloid of one sheet with axis the x -axis.



33. Completing the square in y gives

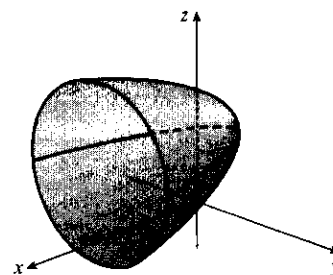
$$4x^2 + 4(y-1)^2 + z^2 = 4 \text{ or}$$

$$x^2 + (y-1)^2 + \frac{z^2}{4} = 1, \text{ an ellipsoid centered at } (0, 1, 0).$$



34. Completing the square in y and z gives

$$x = (y-1)^2 + (z-2)^2, \text{ a circular paraboloid with vertex } (0, 1, 2) \text{ and axis the horizontal line } y=1, z=2.$$



35. $4x^2 + y^2 = 16 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1$. The equation of the ellipsoid is $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z=0$ must be the original ellipse. The traces of the ellipsoid in the yz -plane must be circles since the surface is obtained by rotation about the x -axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \Leftrightarrow 4x^2 + y^2 + z^2 = 16.$$

36. The distance from a point $P(x, y, z)$ to the plane $y=1$ is $|y-1|$, so the given condition becomes

$$|y-1| = 2\sqrt{(x-0)^2 + (y+1)^2 + (z-0)^2} \Rightarrow |y-1| = 2\sqrt{x^2 + (y+1)^2 + z^2} \Rightarrow$$

$$(y-1)^2 = 4x^2 + 4(y+1)^2 + 4z^2 \Leftrightarrow -3 = 4x^2 + (3y^2 + 10y) + 4z^2 \Leftrightarrow$$

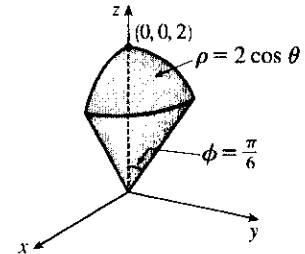
$$\frac{16}{3} = 4x^2 + 3\left(y + \frac{5}{3}\right)^2 + 4z^2 \Rightarrow \frac{3}{4}x^2 + \frac{9}{16}\left(y + \frac{5}{3}\right)^2 + \frac{3}{4}z^2 = 1. \text{ This is the equation of an ellipsoid whose center is } \left(0, -\frac{5}{3}, 0\right).$$

37. $x = r \cos \theta = 2\sqrt{3} \cos \frac{\pi}{3} = 2\sqrt{3} \cdot \frac{1}{2} = \sqrt{3}$, $y = r \sin \theta = 2\sqrt{3} \sin \frac{\pi}{3} = 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 3$, $z = 2$, so in rectangular coordinates the point is $(\sqrt{3}, 3, 2)$. $\rho = \sqrt{r^2 + z^2} = \sqrt{12 + 4} = 4$, $\theta = \frac{\pi}{3}$, and $\cos \phi = \frac{z}{\rho} = \frac{1}{2}$, so $\phi = \frac{\pi}{3}$ and spherical coordinates are $\left(4, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

38. $r = \sqrt{4 + 4} = 2\sqrt{2}$, $z = -1$, $\cos \theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$ so $\theta = \frac{\pi}{4}$ and in cylindrical coordinates the point is $(2\sqrt{2}, \frac{\pi}{4}, -1)$. $\rho = \sqrt{4 + 4 + 1} = 3$, $\cos \phi = -\frac{1}{3}$, so the spherical coordinates are $\left(3, \frac{\pi}{4}, \cos^{-1}\left(-\frac{1}{3}\right)\right)$.

39. $x = \rho \sin \phi \cos \theta = 8 \sin \frac{\pi}{6} \cos \frac{\pi}{4} = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$, $y = \rho \sin \phi \sin \theta = 8 \sin \frac{\pi}{6} \sin \frac{\pi}{4} = 2\sqrt{2}$, and $z = \rho \cos \phi = 8 \cos \frac{\pi}{6} = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$. Thus rectangular coordinates for the point are $(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$. $r^2 = x^2 + y^2 = 8 + 8 = 16 \Rightarrow r = 4$, $\theta = \frac{\pi}{4}$, and $z = 4\sqrt{3}$, so cylindrical coordinates are $(4, \frac{\pi}{4}, 4\sqrt{3})$.

40. $\phi = \frac{\pi}{4}$. This is one frustum of a circular cone with vertex the origin and axis the positive z -axis.
41. $\theta = \frac{\pi}{4}$. In spherical coordinates, this is a half-plane including the z -axis and intersecting the xy -plane in the half-line $x = y, x > 0$.
42. Since $r = \cos \theta$ and $x = r \cos \theta, x = \cos^2 \theta$. Also $r^2 = x^2 + y^2$ so $x^2 + y^2 = 2 \cos^2 \theta$. Thus $x^2 + y^2 - 2x = 0$ or $(x - 1)^2 + y^2 = 1$. Thus the surface is a circular cylinder with axis the line $x = 1, y = 0, z = z$.
43. Since $\rho = 3 \sec \phi, \rho \cos \phi = 3$ or $z = 3$. Thus the surface is a plane parallel to the xy -plane and through the point $(0, 0, 3)$.
44. $x^2 + y^2 = 4$. In cylindrical coordinates: $r^2 = 4$. In spherical coordinates: $\rho^2 - z^2 = 4$ or $\rho^2 - \rho^2 \cos^2 \phi = 4$ or $\rho^2 \sin^2 \phi = 4$ or $\rho \sin \phi = 2$.
45. $x^2 + y^2 + z^2 = 4$. In cylindrical coordinates, this becomes $r^2 + z^2 = 4$. In spherical coordinates, it becomes $\rho^2 = 4$ or $\rho = 2$.
46. In cylindrical coordinates: $r^2 + z^2 = 2r \cos \theta$ or $z^2 = r(2 \cos \theta - r)$.
In spherical coordinates: $\rho^2 = 2\rho \sin \phi \cos \theta$ or $\rho = 2 \sin \phi \cos \theta$.
47. The resulting surface is a circular paraboloid with equation $z = 4x^2 + 4y^2$. Changing to cylindrical coordinates we have $z = 4(x^2 + y^2) = 4r^2$.
48. $\rho = 2 \cos \phi \Rightarrow \rho^2 = 2\rho \cos \phi \Rightarrow x^2 + y^2 + z^2 = 2z \Rightarrow x^2 + y^2 + (z - 1)^2 = 1$. This is the equation of a sphere with radius 1, centered at $(0, 0, 1)$. Therefore, $0 \leq \rho \leq 2 \cos \phi$ is the solid ball whose boundary is this sphere. $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{6}$ restrict the solid to the section of this ball that lies above the cone $\phi = \frac{\pi}{6}$ and is in the first octant.

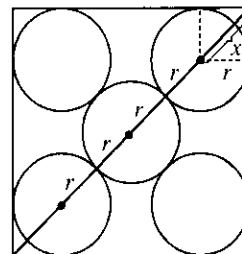


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□ PROBLEMS PLUS

1. Since three-dimensional situations are often difficult to visualize and work with, let us first try to find an analogous problem in two dimensions. The analogue of a cube is a square and the analogue of a sphere is a circle. Thus a similar problem in two dimensions is the following: if five circles with the same radius r are contained in a square of side 1 m so that the circles touch each other and four of the circles touch two sides of the square, find r .



The diagonal of the square is $\sqrt{2}$. The diagonal is also $4r + 2x$. But x is the diagonal of a smaller square of side r .

$$\text{Therefore } x = \sqrt{2}r \Rightarrow \sqrt{2} = 4r + 2x = 4r + 2\sqrt{2}r = (4 + 2\sqrt{2})r \Rightarrow r = \frac{\sqrt{2}}{4 + 2\sqrt{2}}.$$

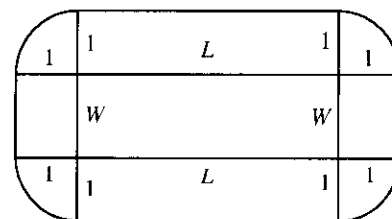
Let us use these ideas to solve the original three-dimensional problem. The diagonal of the cube is

$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The diagonal of the cube is also $4r + 2x$ where x is the diagonal of a smaller cube with edge r . Therefore $x = \sqrt{r^2 + r^2 + r^2} = \sqrt{3}r \Rightarrow \sqrt{3} = 4r + 2x = 4r + 2\sqrt{3}r = (4 + 2\sqrt{3})r$. Thus

$$r = \frac{\sqrt{3}}{4 + 2\sqrt{3}} = \frac{2\sqrt{3} - 3}{2}. \text{ The radius of each ball is } \left(\sqrt{3} - \frac{3}{2}\right) \text{ m.}$$

2. Try an analogous problem in two dimensions. Consider a rectangle with length L and width W and find the area of S in terms of L and W . Since S contains B , it has area

$$\begin{aligned} A(S) = & LW + \text{the area of two } L \times 1 \text{ rectangles} \\ & + \text{the area of two } 1 \times W \text{ rectangles} \\ & + \text{the area of four quarter-circles of radius 1} \end{aligned}$$



as seen in the diagram. So $A(S) = LW + 2L + 2W + \pi \cdot 1^2$.

Now in three dimensions, the volume of S is

$$\begin{aligned} & LWH + 2(L \times W \times 1) + 2(1 \times W \times H) + 2(L \times 1 \times H) \\ & + \text{the volume of 4 quarter-cylinders with radius 1 and height } W \\ & + \text{the volume of 4 quarter-cylinders with radius 1 and height } L \\ & + \text{the volume of 4 quarter-cylinders with radius 1 and height } H \\ & + \text{the volume of 8 eighths of a sphere of radius 1} \end{aligned}$$

So

$$\begin{aligned} V(S) = & LWH + 2LW + 2WH + 2LH + \pi \cdot 1^2 \cdot W + \pi \cdot 1^2 \cdot L + \pi \cdot 1^2 \cdot H + \frac{4}{3}\pi \cdot 1^3 \\ = & LWH + 2(LW + WH + LH) + \pi(L + W + H) + \frac{4}{3}\pi. \end{aligned}$$

3. (a) We find the line of intersection L as in Example 13.5.7(b) [ET 12.5.7(b)]. Observe that the point $(-1, c, c)$ lies on both planes. Now since L lies in both planes, it is perpendicular to both of the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and

thus parallel to their cross product $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & 1 & 1 \\ 1 & -c & c \end{vmatrix} = \langle 2c, -c^2 + 1, -c^2 - 1 \rangle$. So symmetric equations

of L can be written as $\frac{x+1}{-2c} = \frac{y-c}{c^2-1} = \frac{z-c}{c^2+1}$, provided that $c \neq 0, \pm 1$.

If $c = 0$, then the two planes are given by $y + z = 0$ and $x = -1$, so symmetric equations of L are $x = -1$, $y = -z$. If $c = -1$, then the two planes are given by $-x + y + z = -1$ and $x + y + z = -1$, and they intersect in the line $x = 0$, $y = -z - 1$. If $c = 1$, then the two planes are given by $x + y + z = 1$ and $x - y + z = 1$, and they intersect in the line $y = 0$, $x = 1 - z$.

- (b) If we set $z = t$ in the symmetric equations and solve for x and y separately, we get $x + 1 = \frac{(t-c)(-2c)}{c^2+1}$,

$$y - c = \frac{(t-c)(c^2-1)}{c^2+1} \Rightarrow x = \frac{-2ct + (c^2-1)}{c^2+1}, y = \frac{(c^2-1)t + 2c}{c^2+1}.$$

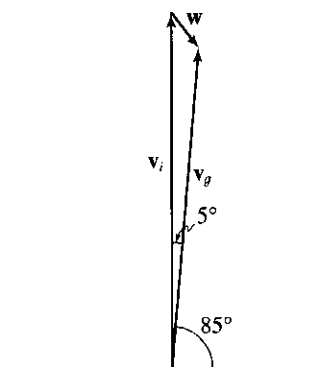
Eliminating c from these equations, we have $x^2 + y^2 = t^2 + 1$. So the curve traced out by L in the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2 + 1}$.

- (c) The area of a horizontal cross-section of the solid is $A(z) = \pi(z^2 + 1)$, so

$$V = \int_0^1 A(z) dz = \pi \left[\frac{1}{3} z^3 + z \right]_0^1 = \frac{4\pi}{3}.$$

4. (a) We consider velocity vectors for the plane and the wind. Let $\mathbf{v}_i$ be the initial, intended velocity for the plane and $\mathbf{v}_g$ the actual velocity relative to the ground. If $\mathbf{w}$ is the velocity of the wind, $\mathbf{v}_g$ is the resultant, that is, the vector sum $\mathbf{v}_i + \mathbf{w}$ as shown in the figure. We know $\mathbf{v}_i = 180\mathbf{j}$, and since the plane actually flew 80 km in $\frac{1}{2}$ hour, $|\mathbf{v}_g| = 160$. Thus $\mathbf{v}_g = (160 \cos 85^\circ)\mathbf{i} + (160 \sin 85^\circ)\mathbf{j} \approx 13.9\mathbf{i} + 159.4\mathbf{j}$. Finally, $\mathbf{v}_i + \mathbf{w} = \mathbf{v}_g$, so $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_i \approx 13.9\mathbf{i} - 20.6\mathbf{j}$. Thus, the wind velocity is about $13.9\mathbf{i} - 20.6\mathbf{j}$, and the wind speed is

$$|\mathbf{w}| \approx \sqrt{(13.9)^2 + (-20.6)^2} \approx 24.9 \text{ km/h}.$$

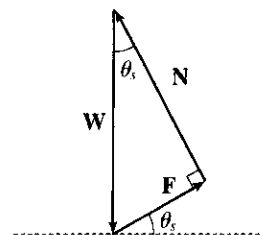


- (b) Let $\mathbf{v}$ be the velocity the pilot should take. With the effect of wind, the actual velocity (with respect to the ground) will be $\mathbf{v} + \mathbf{w}$, which we want to be $\mathbf{v}_i$. Thus

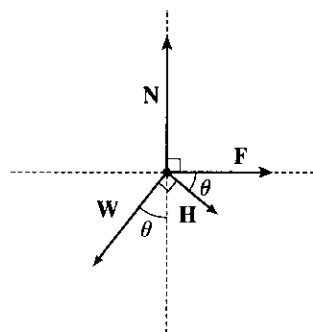
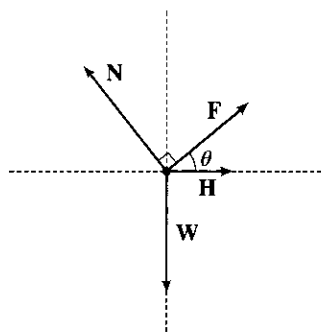
$$\mathbf{v} = \mathbf{v}_i - \mathbf{w} \approx 180\mathbf{j} - (13.9\mathbf{i} - 20.6\mathbf{j}) \approx -13.9\mathbf{i} + 200.6\mathbf{j}.$$

$$\tan \theta \approx \frac{200.6}{-13.9} \Rightarrow \theta \approx 94.0^\circ, \text{ or } 4.0^\circ \text{ west of north.}$$

5. (a) When $\theta = \theta_s$, the block is not moving, so the sum of the forces on the block must be 0, thus $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This relationship is illustrated geometrically in the figure. Since the vectors form a right triangle, we have $\tan(\theta_s) = \frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{\mu_s n}{n} = \mu_s$.



- (b) We place the block at the origin and sketch the force vectors acting on the block, including the additional horizontal force $\mathbf{H}$, with initial points at the origin. We then rotate this system so that $\mathbf{F}$ lies along the positive x -axis and the inclined plane is parallel to the x -axis.



$|\mathbf{F}|$ is maximal, so $|\mathbf{F}| = \mu_s n$ for $\theta > \theta_s$. Then the vectors, in terms of components parallel and perpendicular to the inclined plane, are

$$\mathbf{N} = n \mathbf{j} \quad \mathbf{F} = (\mu_s n) \mathbf{i}$$

$$\mathbf{W} = (-mg \sin \theta) \mathbf{i} + (-mg \cos \theta) \mathbf{j}$$

$$\mathbf{H} = (h_{\min} \cos \theta) \mathbf{i} + (-h_{\min} \sin \theta) \mathbf{j}$$

Equating components, we have

$$\mu_s n - mg \sin \theta + h_{\min} \cos \theta = 0 \Rightarrow h_{\min} \cos \theta + \mu_s n = mg \sin \theta \quad (1)$$

$$n - mg \cos \theta - h_{\min} \sin \theta = 0 \Rightarrow h_{\min} \sin \theta + mg \cos \theta = n \quad (2)$$

- (c) Since (2) is solved for n , we substitute into (1):

$$\begin{aligned} h_{\min} \cos \theta + \mu_s (h_{\min} \sin \theta + mg \cos \theta) &= mg \sin \theta \Rightarrow \\ h_{\min} \cos \theta + h_{\min} \mu_s \sin \theta &= mg \sin \theta - mg \mu_s \cos \theta \Rightarrow \end{aligned}$$

$$h_{\min} = mg \left(\frac{\sin \theta - \mu_s \cos \theta}{\cos \theta + \mu_s \sin \theta} \right) = mg \left(\frac{\tan \theta - \mu_s}{1 + \mu_s \tan \theta} \right)$$

From part (a) we know $\mu_s = \tan \theta_s$, so this becomes $h_{\min} = mg \left(\frac{\tan \theta - \tan \theta_s}{1 + \tan \theta_s \tan \theta} \right)$ and using a trigonometric identity, this is $mg \tan(\theta - \theta_s)$ as desired.

Note for $\theta = \theta_s$, $h_{\min} = mg \tan 0 = 0$, which makes sense since the block is at rest for θ_s , thus no additional force $\mathbf{H}$ is necessary to prevent it from moving. As θ increases, the factor $\tan(\theta - \theta_s)$, and hence the value of $h_{\min}$, increases slowly for small values of $\theta - \theta_s$ but much more rapidly as $\theta - \theta_s$ becomes significant. This seems reasonable, as the steeper the inclined plane, the less the horizontal components of the various forces affect the movement of the block, so we would need a much larger magnitude of horizontal force to keep the block motionless. If we allow $\theta \rightarrow 90^\circ$, corresponding to the inclined plane being placed vertically, the value of $h_{\min}$ is quite large; this is to be expected, as it takes a great amount of horizontal force to keep an object from moving vertically. In fact, without friction (so $\theta_s = 0$), we would have $\theta \rightarrow 90^\circ \Rightarrow h_{\min} \rightarrow \infty$, and it would be impossible to keep the block from slipping.

- (d) Since $h_{\max}$ is the largest value of h that keeps the block from slipping, the force of friction is keeping the block from moving *up* the inclined plane; thus, $\mathbf{F}$ is directed *down* the plane. Our system of forces is similar to that in part (b), then, except that we have $\mathbf{F} = -(\mu_s n) \mathbf{i}$. (Note that $|\mathbf{F}|$ is again maximal.) Following our procedure in parts (b) and (c), we equate components:

$$-\mu_s n - mg \sin \theta + h_{\max} \cos \theta = 0 \quad \Rightarrow \quad h_{\max} \cos \theta - \mu_s n = mg \sin \theta$$

$$n - mg \cos \theta - h_{\max} \sin \theta = 0 \quad \Rightarrow \quad h_{\max} \sin \theta + mg \cos \theta = n$$

Then substituting,

$$h_{\max} \cos \theta - \mu_s (h_{\max} \sin \theta + mg \cos \theta) = mg \sin \theta \quad \Rightarrow$$

$$h_{\max} \cos \theta - h_{\max} \mu_s \sin \theta = mg \sin \theta + mg \mu_s \cos \theta \quad \Rightarrow$$

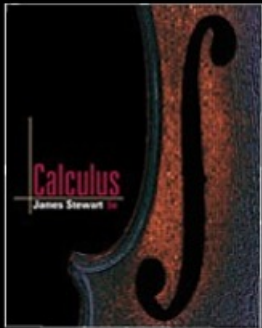
$$h_{\max} = mg \left(\frac{\sin \theta + \mu_s \cos \theta}{\cos \theta - \mu_s \sin \theta} \right) = mg \left(\frac{\tan \theta + \mu_s}{1 - \mu_s \tan \theta} \right)$$

$$= mg \left(\frac{\tan \theta + \tan \theta_s}{1 - \tan \theta_s \tan \theta} \right) = mg \tan(\theta + \theta_s)$$

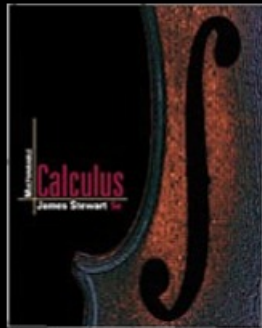
We would expect $h_{\max}$ to increase as θ increases, with similar behavior as we established for $h_{\min}$, but with $h_{\max}$ values always larger than $h_{\min}$. We can see that this is the case if we graph $h_{\max}$ as a function of θ , as the curve is the graph of $h_{\min}$ translated $2\theta_s$ to the left, so the equation does seem reasonable. Notice that the equation predicts $h_{\max} \rightarrow \infty$ as $\theta \rightarrow (90^\circ - \theta_s)$. In fact, as $h_{\max}$ increases, the normal force increases as well. When $(90^\circ - \theta_s) \leq \theta \leq 90^\circ$, the horizontal force is completely counteracted by the sum of the normal and frictional forces, so no part of the horizontal force contributes to moving the block up the plane no matter how large its magnitude.

Chapter 14

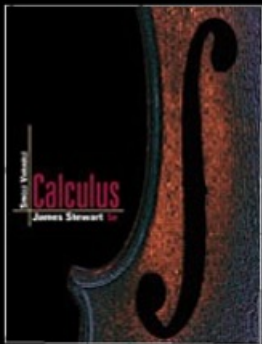
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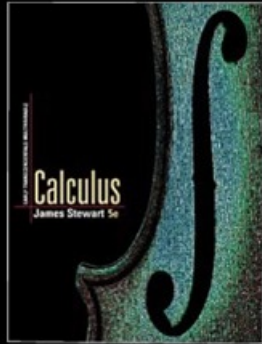
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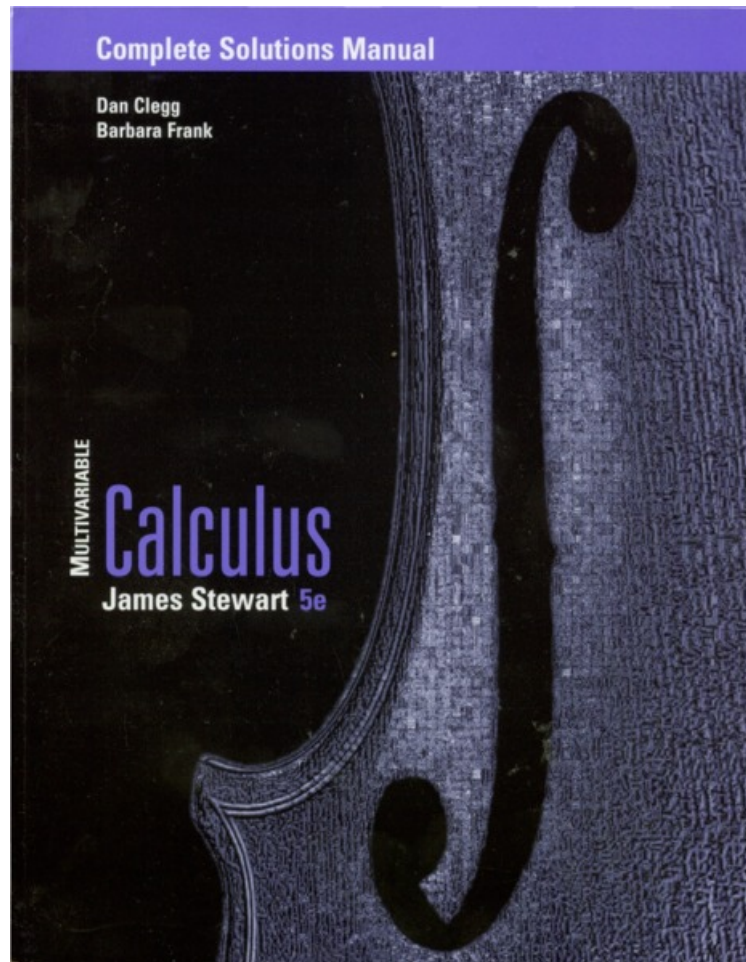
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14 □ VECTOR FUNCTIONS

□ ET 13

14.1 Vector Functions and Space Curves

ET 13.1

1. The component functions t^2 , $\sqrt{t-1}$, and $\sqrt{5-t}$ are all defined when $t-1 \geq 0 \Rightarrow t \geq 1$ and $5-t \geq 0 \Rightarrow t \leq 5$, so the domain of $\mathbf{r}(t)$ is $[1, 5]$.

2. The component functions $\frac{t-2}{t+2}$, $\sin t$, and $\ln(9-t^2)$ are all defined when $t \neq -2$ and $9-t^2 > 0 \Rightarrow -3 < t < 3$, so the domain of $\mathbf{r}(t)$ is $(-3, -2) \cup (-2, 3)$.

3. $\lim_{t \rightarrow 0^+} \cos t = \cos 0 = 1$, $\lim_{t \rightarrow 0^+} \sin t = \sin 0 = 0$, $\lim_{t \rightarrow 0^+} t \ln t = \lim_{t \rightarrow 0^+} \frac{\ln t}{1/t} = \lim_{t \rightarrow 0^+} \frac{1/t}{-1/t^2} = \lim_{t \rightarrow 0^+} -t = 0$
[by l'Hospital's Rule]. Thus $\lim_{t \rightarrow 0^+} \langle \cos t, \sin t, t \ln t \rangle = \left\langle \lim_{t \rightarrow 0^+} \cos t, \lim_{t \rightarrow 0^+} \sin t, \lim_{t \rightarrow 0^+} t \ln t \right\rangle = \langle 1, 0, 0 \rangle$.

4. $\lim_{t \rightarrow 0} \frac{e^t - 1}{t} = \lim_{t \rightarrow 0} \frac{e^t}{1} = 1$ [using l'Hospital's Rule],

$$\lim_{t \rightarrow 0} \frac{\sqrt{1+t} - 1}{t} = \lim_{t \rightarrow 0} \frac{\sqrt{1+t} - 1}{t} \cdot \frac{\sqrt{1+t} + 1}{\sqrt{1+t} + 1} = \lim_{t \rightarrow 0} \frac{1}{\sqrt{1+t} + 1} = \frac{1}{2}, \lim_{t \rightarrow 0} \frac{3}{1+t} = 3.$$

Thus the given limit equals $\langle 1, \frac{1}{2}, 3 \rangle$.

5. $\lim_{t \rightarrow 1} \sqrt{t+3} = 2$, $\lim_{t \rightarrow 1} \frac{t-1}{t^2-1} = \lim_{t \rightarrow 1} \frac{1}{t+1} = \frac{1}{2}$, $\lim_{t \rightarrow 1} \left(\frac{\tan t}{t} \right) = \tan 1$.

Thus the given limit equals $\langle 2, \frac{1}{2}, \tan 1 \rangle$.

6. $\lim_{t \rightarrow \infty} \arctan t = \frac{\pi}{2}$, $\lim_{t \rightarrow \infty} e^{-2t} = 0$, $\lim_{t \rightarrow \infty} \frac{\ln t}{t} = \lim_{t \rightarrow \infty} \frac{1/t}{1} = 0$ [by l'Hospital's Rule].

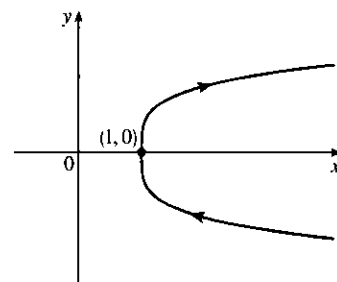
Thus $\lim_{t \rightarrow \infty} \left\langle \arctan t, e^{-2t}, \frac{\ln t}{t} \right\rangle = \left\langle \frac{\pi}{2}, 0, 0 \right\rangle$.

7. The corresponding parametric equations for this curve are

$x = t^4 + 1$, $y = t$. We can make a table of values, or we can

eliminate the parameter: $t = y \Rightarrow x = y^4 + 1$, with $y \in \mathbb{R}$.

By comparing different values of t , we find the direction in which t increases as indicated in the graph.



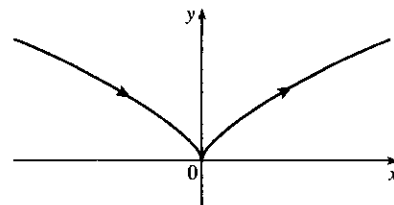
8. The corresponding parametric equations for this curve are

$x = t^3$, $y = t^2$. We can make a table of values, or we can

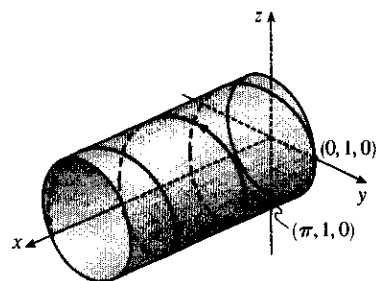
eliminate the parameter: $x = t^3 \Rightarrow t = \sqrt[3]{x} \Rightarrow$

$y = t^2 = (\sqrt[3]{x})^2 = x^{2/3}$, with $t \in \mathbb{R} \Rightarrow x \in \mathbb{R}$. By

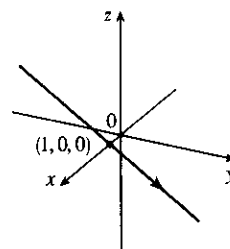
comparing different values of t , we find the direction in which t increases as indicated in the graph.



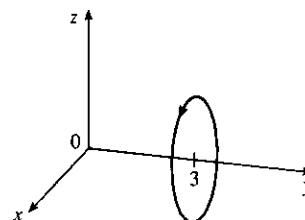
9. The corresponding parametric equations are $x = t$, $y = \cos 2t$, $z = \sin 2t$. Note that $y^2 + z^2 = \cos^2 2t + \sin^2 2t = 1$, so the curve lies on the circular cylinder $y^2 + z^2 = 1$. Since $x = t$, the curve is a helix.



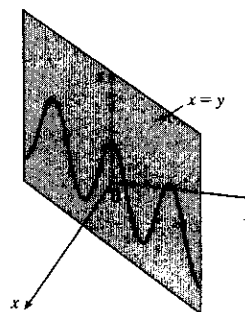
10. The corresponding parametric equations are $x = 1 + t$, $y = 3t$, $z = -t$, which are parametric equations of a line through the point $(1, 0, 0)$ and with direction vector $\langle 1, 3, -1 \rangle$.



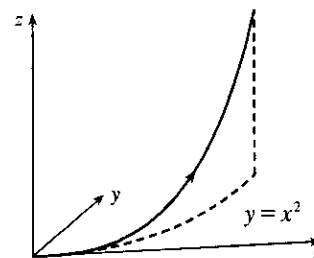
11. The parametric equations give $x^2 + z^2 = \sin^2 t + \cos^2 t = 1$, $y = 3$, which is a circle of radius 1, center $(0, 3, 0)$ in the plane $y = 3$.



12. The parametric equations are $x = t$, $y = t$, $z = \cos t$. Thus $x = y$, so the curve must lie in the plane $x = y$. Combine this with $z = \cos t$ to determine that the curve traces out the cosine curve in the vertical plane $x = y$.

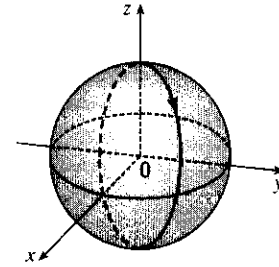


13. The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the xy -plane is $y = x^2$, $y > 0$, a half parabola. On the xz -plane $z = x^3$, $z > 0$, a half cubic, and the yz -plane, $y^3 = z^2$.



14. The parametric equations give

$x^2 + y^2 + z^2 = 2 \sin^2 t + 2 \cos^2 t = 2$, so the curve lies on the sphere with radius $\sqrt{2}$ and center $(0, 0, 0)$. Furthermore $x = y = \sin t$, so the curve is the intersection of this sphere with the plane $x = y$, that is, the curve is the circle of radius $\sqrt{2}$, center $(0, 0, 0)$ in the plane $x = y$.



15. Taking $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$ and $\mathbf{r}_1 = \langle 1, 2, 3 \rangle$, we have from Equation 13.5.4 [ET 12.5.4]

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 = (1-t)\langle 0, 0, 0 \rangle + t\langle 1, 2, 3 \rangle, \quad 0 \leq t \leq 1 \text{ or } \mathbf{r}(t) = \langle t, 2t, 3t \rangle, \quad 0 \leq t \leq 1.$$

Parametric equations are $x = t, y = 2t, z = 3t, 0 \leq t \leq 1$.

16. Taking $\mathbf{r}_0 = \langle 1, 0, 1 \rangle$ and $\mathbf{r}_1 = \langle 2, 3, 1 \rangle$, we have from Equation 13.5.4 [ET 12.5.4]

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 = (1-t)\langle 1, 0, 1 \rangle + t\langle 2, 3, 1 \rangle, \quad 0 \leq t \leq 1 \text{ or } \mathbf{r}(t) = \langle 1+t, 3t, 1 \rangle, \quad 0 \leq t \leq 1.$$

Parametric equations are $x = 1+t, y = 3t, z = 1, 0 \leq t \leq 1$.

17. Taking $\mathbf{r}_0 = \langle 1, -1, 2 \rangle$ and $\mathbf{r}_1 = \langle 4, 1, 7 \rangle$, we have $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 = (1-t)\langle 1, -1, 2 \rangle + t\langle 4, 1, 7 \rangle$,

$$0 \leq t \leq 1 \text{ or } \mathbf{r}(t) = \langle 1+3t, -1+2t, 2+5t \rangle, \quad 0 \leq t \leq 1. \text{ Parametric equations are } x = 1+3t, y = -1+2t, z = 2+5t, 0 \leq t \leq 1.$$

18. Taking $\mathbf{r}_0 = \langle -2, 4, 0 \rangle$ and $\mathbf{r}_1 = \langle 6, -1, 2 \rangle$, we have $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 = (1-t)\langle -2, 4, 0 \rangle + t\langle 6, -1, 2 \rangle$,

$$0 \leq t \leq 1 \text{ or } \mathbf{r}(t) = \langle -2+8t, 4-5t, 2t \rangle, \quad 0 \leq t \leq 1. \text{ Parametric equations are } x = -2+8t, y = 4-5t, z = 2t, 0 \leq t \leq 1.$$

19. $x = \cos 4t, y = t, z = \sin 4t$. At any point (x, y, z) on the curve, $x^2 + z^2 = \cos^2 4t + \sin^2 4t = 1$. So the curve lies on a circular cylinder with axis the y -axis. Since $y = t$, this is a helix. So the graph is VI.

20. $x = t, y = t^2, z = e^{-t}$. At any point on the curve, $y = x^2$. So the curve lies on the parabolic cylinder $y = x^2$.

Note that y and z are positive for all t , and the point $(0, 0, 1)$ is on the curve (when $t = 0$). As $t \rightarrow \infty$, $(x, y, z) \rightarrow (\infty, \infty, 0)$, while as $t \rightarrow -\infty$, $(x, y, z) \rightarrow (-\infty, \infty, \infty)$, so the graph must be II.

21. $x = t, y = 1/(1+t^2), z = t^2$. Note that y and z are positive for all t . The curve passes through $(0, 1, 0)$ when $t = 0$. As $t \rightarrow \infty$, $(x, y, z) \rightarrow (\infty, 0, \infty)$, and as $t \rightarrow -\infty$, $(x, y, z) \rightarrow (-\infty, 0, \infty)$. So the graph is IV.

22. $x = e^{-t} \cos 10t, y = e^{-t} \sin 10t, z = e^{-t}$.

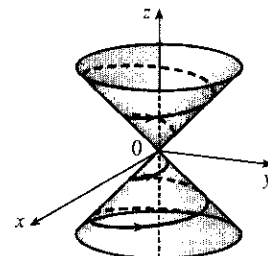
$x^2 + y^2 = e^{-2t} \cos^2 10t + e^{-2t} \sin^2 10t = e^{-2t}(\cos^2 10t + \sin^2 10t) = e^{-2t} = z^2$, so the curve lies on the cone $x^2 + y^2 = z^2$. Also, z is always positive; the graph must be I.

23. $x = \cos t, y = \sin t, z = \sin 5t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z -axis. Each of x, y and z is periodic, and at $t = 0$ and $t = 2\pi$ the curve passes through the same point, so the curve repeats itself and the graph is V.

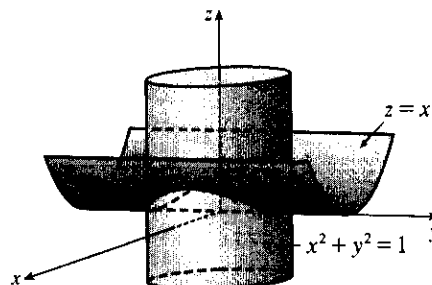
24. $x = \cos t, y = \sin t, z = \ln t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z -axis. As $t \rightarrow 0$, $z \rightarrow -\infty$, so the graph is III.

25. If $x = t \cos t, y = t \sin t$, and $z = t$, then

$x^2 + y^2 = t^2 \cos^2 t + t^2 \sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.



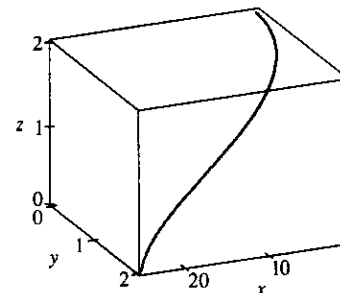
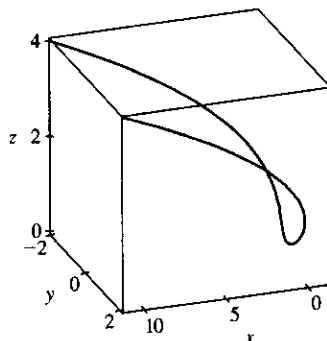
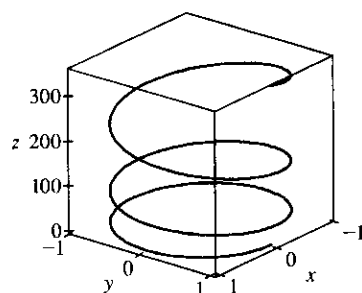
26. Here $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$.



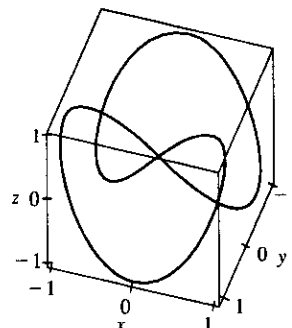
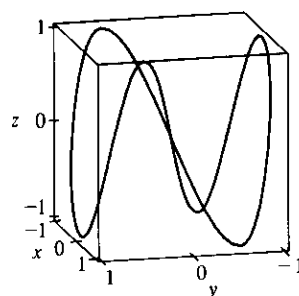
27. $\mathbf{r}(t) = \langle \sin t, \cos t, t^2 \rangle$

28. $\mathbf{r}(t) = \langle t^4 - t^2 + 1, t, t^2 \rangle$

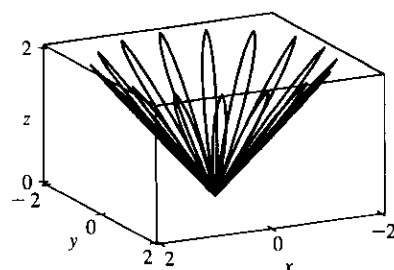
29. $\mathbf{r}(t) = \langle t^2, \sqrt{t-1}, \sqrt{5-t} \rangle$



30. We have the computer plot the parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$, $0 \leq t \leq 2\pi$. The shape of the curve is not clear from just one viewpoint, so we include a second plot drawn from a different angle.



31.



$$x = (1 + \cos 16t) \cos t, y = (1 + \cos 16t) \sin t, z = 1 + \cos 16t.$$

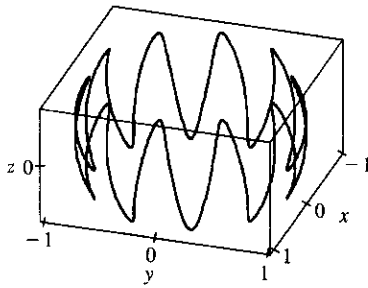
At any point on the graph,

$$x^2 + y^2 = (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t$$

$$= (1 + \cos 16t)^2 = z^2, \text{ so the graph lies on the cone}$$

$x^2 + y^2 = z^2$. From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

32.



$x = \sqrt{1 - 0.25 \cos^2 10t} \cos t$, $y = \sqrt{1 - 0.25 \cos^2 10t} \sin t$,
 $z = 0.5 \cos 10t$. At any point on the graph,
 $x^2 + y^2 + z^2 = (1 - 0.25 \cos^2 10t) \cos^2 t$
 $+ (1 - 0.25 \cos^2 10t) \sin^2 t + 0.25 \cos^2 t$
 $= 1 - 0.25 \cos^2 10t + 0.25 \cos^2 10t = 1$,
 so the graph lies on the sphere $x^2 + y^2 + z^2 = 1$, and since
 $z = 0.5 \cos 10t$ the graph resembles a trigonometric curve with ten
 peaks projected onto the sphere. The graph is generated by $t \in [0, 2\pi]$.

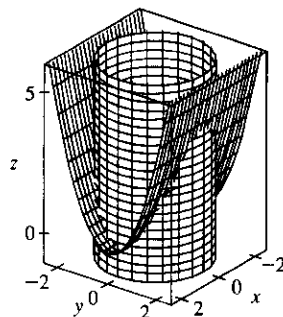
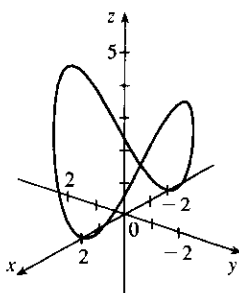
33. If $t = -1$, then $x = 1$, $y = 4$, $z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then
 $x = 9$, $y = -8$, $z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the
 curve, we require $y = 1 - 3t = 7 \Rightarrow t = -2$. But then $z = 1 + (-2)^3 = -7 \neq -6$, so $(4, 7, -6)$ is not on the
 curve.

34. The projection of the curve C of intersection onto the xy -plane is the circle $x^2 + y^2 = 4$, $z = 0$.
 Then we can write $x = 2 \cos t$, $y = 2 \sin t$, $0 \leq t \leq 2\pi$. Since C also lies on the surface $z = xy$, we have
 $z = xy = (2 \cos t)(2 \sin t) = 4 \cos t \sin t$, or $2 \sin(2t)$. Then parametric equations for C are
 $x = 2 \cos t$, $y = 2 \sin t$, $z = 2 \sin(2t)$, $0 \leq t \leq 2\pi$, and the corresponding vector function is
 $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 2 \sin(2t) \mathbf{k}$, $0 \leq t \leq 2\pi$.

35. Both equations are solved for z , so we can substitute to eliminate z : $\sqrt{x^2 + y^2} = 1 + y \Rightarrow$
 $x^2 + y^2 = 1 + 2y + y^2 \Rightarrow x^2 = 1 + 2y \Rightarrow y = \frac{1}{2}(x^2 - 1)$. We can form parametric equations for the curve
 C of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2 - 1)$ and $z = 1 + y = 1 + \frac{1}{2}(t^2 - 1) = \frac{1}{2}(t^2 + 1)$.
 Thus a vector function representing C is $\mathbf{r}(t) = t \mathbf{i} + \frac{1}{2}(t^2 - 1) \mathbf{j} + \frac{1}{2}(t^2 + 1) \mathbf{k}$.

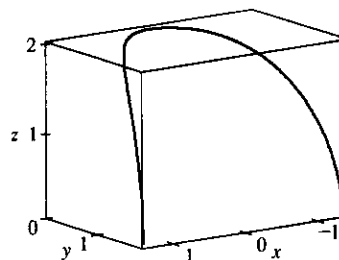
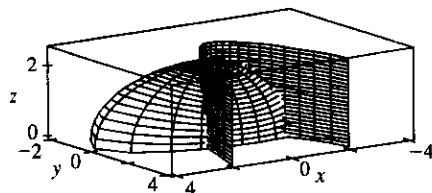
36. The projection of the curve C of intersection onto the xy -plane is the parabola $y = x^2$, $z = 0$. Then we can choose
 the parameter $x = t \Rightarrow y = t^2$. Since C also lies on the surface $z = 4x^2 + y^2$, we have
 $z = 4x^2 + y^2 = 4t^2 + (t^2)^2$. Then parametric equations for C are $x = t$, $y = t^2$, $z = 4t^2 + t^4$, and the
 corresponding vector function is $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + (4t^2 + t^4) \mathbf{k}$.

37.



The projection of the curve C of intersection onto
 the xy -plane is the circle $x^2 + y^2 = 4$, $z = 0$.
 Then we can write $x = 2 \cos t$, $y = 2 \sin t$,
 $0 \leq t \leq 2\pi$. Since C also lies on the surface
 $z = x^2$, we have $z = x^2 = (2 \cos t)^2 = 4 \cos^2 t$.
 Then parametric equations for C are $x = 2 \cos t$,
 $y = 2 \sin t$, $z = 4 \cos^2 t$, $0 \leq t \leq 2\pi$.

38.



$x = t \Rightarrow y = t^2 \Rightarrow 4z^2 = 16 - x^2 - 4y^2 = 16 - t^2 - 4t^4 \Rightarrow z = \sqrt{4 - (\frac{1}{2}t)^2 - t^4}$. Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x = t, y = t^2, z = \sqrt{4 - \frac{1}{4}t^2 - t^4}$.

39. For the particles to collide, we require $\mathbf{r}_1(t) = \mathbf{r}_2(t) \Leftrightarrow \langle t^2, 7t - 12, t^2 \rangle = \langle 4t - 3, t^2, 5t - 6 \rangle$. Equating components gives $t^2 = 4t - 3, 7t - 12 = t^2$, and $t^2 = 5t - 6$. From the first equation, $t^2 - 4t + 3 = 0 \Leftrightarrow (t - 3)(t - 1) = 0$ so $t = 1$ or $t = 3$. $t = 1$ does not satisfy the other two equations, but $t = 3$ does. The particles collide when $t = 3$, at the point $(9, 9, 9)$.

40. The particles collide provided $\mathbf{r}_1(t) = \mathbf{r}_2(t) \Leftrightarrow \langle t, t^2, t^3 \rangle = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$. Equating components gives $t = 1 + 2t, t^2 = 1 + 6t$, and $t^3 = 1 + 14t$. The first equation gives $t = -1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for t and a value for s where $\mathbf{r}_1(t) = \mathbf{r}_2(s) \Leftrightarrow \langle t, t^2, t^3 \rangle = \langle 1 + 2s, 1 + 6s, 1 + 14s \rangle$. Equating components, $t = 1 + 2s, t^2 = 1 + 6s$, and $t^3 = 1 + 14s$. Substituting the first equation into the second gives $(1 + 2s)^2 = 1 + 6s \Rightarrow 4s^2 - 2s = 0 \Rightarrow 2s(2s - 1) = 0 \Rightarrow s = 0$ or $s = \frac{1}{2}$. From the first equation, $s = 0 \Rightarrow t = 1$ and $s = \frac{1}{2} \Rightarrow t = 2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1, 1, 1)$ when $s = 0$ and $t = 1$, and at $(2, 4, 8)$ when $s = \frac{1}{2}$ and $t = 2$.

41. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real-valued functions.

(a) $\lim_{t \rightarrow a} \mathbf{u}(t) + \lim_{t \rightarrow a} \mathbf{v}(t) = \left\langle \lim_{t \rightarrow a} u_1(t), \lim_{t \rightarrow a} u_2(t), \lim_{t \rightarrow a} u_3(t) \right\rangle + \left\langle \lim_{t \rightarrow a} v_1(t), \lim_{t \rightarrow a} v_2(t), \lim_{t \rightarrow a} v_3(t) \right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \rightarrow a$. Then adding the two vectors and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned} \lim_{t \rightarrow a} \mathbf{u}(t) + \lim_{t \rightarrow a} \mathbf{v}(t) &= \left\langle \lim_{t \rightarrow a} u_1(t) + \lim_{t \rightarrow a} v_1(t), \lim_{t \rightarrow a} u_2(t) + \lim_{t \rightarrow a} v_2(t), \lim_{t \rightarrow a} u_3(t) + \lim_{t \rightarrow a} v_3(t) \right\rangle \\ &= \left\langle \lim_{t \rightarrow a} [u_1(t) + v_1(t)], \lim_{t \rightarrow a} [u_2(t) + v_2(t)], \lim_{t \rightarrow a} [u_3(t) + v_3(t)] \right\rangle \\ &= \lim_{t \rightarrow a} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t) \rangle \quad [\text{using (1) backward}] \\ &= \lim_{t \rightarrow a} [\mathbf{u}(t) + \mathbf{v}(t)] \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \lim_{t \rightarrow a} c\mathbf{u}(t) &= \lim_{t \rightarrow a} \langle cu_1(t), cu_2(t), cu_3(t) \rangle = \left\langle \lim_{t \rightarrow a} cu_1(t), \lim_{t \rightarrow a} cu_2(t), \lim_{t \rightarrow a} cu_3(t) \right\rangle \\
 &= \left\langle c \lim_{t \rightarrow a} u_1(t), c \lim_{t \rightarrow a} u_2(t), c \lim_{t \rightarrow a} u_3(t) \right\rangle = c \left\langle \lim_{t \rightarrow a} u_1(t), \lim_{t \rightarrow a} u_2(t), \lim_{t \rightarrow a} u_3(t) \right\rangle \\
 &= c \lim_{t \rightarrow a} \langle u_1(t), u_2(t), u_3(t) \rangle = c \lim_{t \rightarrow a} \mathbf{u}(t)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \lim_{t \rightarrow a} \mathbf{u}(t) \cdot \lim_{t \rightarrow a} \mathbf{v}(t) &= \left\langle \lim_{t \rightarrow a} u_1(t), \lim_{t \rightarrow a} u_2(t), \lim_{t \rightarrow a} u_3(t) \right\rangle \cdot \left\langle \lim_{t \rightarrow a} v_1(t), \lim_{t \rightarrow a} v_2(t), \lim_{t \rightarrow a} v_3(t) \right\rangle \\
 &= \left[\lim_{t \rightarrow a} u_1(t) \right] \left[\lim_{t \rightarrow a} v_1(t) \right] + \left[\lim_{t \rightarrow a} u_2(t) \right] \left[\lim_{t \rightarrow a} v_2(t) \right] + \left[\lim_{t \rightarrow a} u_3(t) \right] \left[\lim_{t \rightarrow a} v_3(t) \right] \\
 &= \lim_{t \rightarrow a} u_1(t)v_1(t) + \lim_{t \rightarrow a} u_2(t)v_2(t) + \lim_{t \rightarrow a} u_3(t)v_3(t) \\
 &= \lim_{t \rightarrow a} [u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)] = \lim_{t \rightarrow a} [\mathbf{u}(t) \cdot \mathbf{v}(t)]
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \lim_{t \rightarrow a} \mathbf{u}(t) \times \lim_{t \rightarrow a} \mathbf{v}(t) &= \left\langle \lim_{t \rightarrow a} u_1(t), \lim_{t \rightarrow a} u_2(t), \lim_{t \rightarrow a} u_3(t) \right\rangle \times \left\langle \lim_{t \rightarrow a} v_1(t), \lim_{t \rightarrow a} v_2(t), \lim_{t \rightarrow a} v_3(t) \right\rangle \\
 &= \left\langle \left[\lim_{t \rightarrow a} u_2(t) \right] \left[\lim_{t \rightarrow a} v_3(t) \right] - \left[\lim_{t \rightarrow a} u_3(t) \right] \left[\lim_{t \rightarrow a} v_2(t) \right], \right. \\
 &\quad \left[\lim_{t \rightarrow a} u_3(t) \right] \left[\lim_{t \rightarrow a} v_1(t) \right] - \left[\lim_{t \rightarrow a} u_1(t) \right] \left[\lim_{t \rightarrow a} v_3(t) \right], \\
 &\quad \left. \left[\lim_{t \rightarrow a} u_1(t) \right] \left[\lim_{t \rightarrow a} v_2(t) \right] - \left[\lim_{t \rightarrow a} u_2(t) \right] \left[\lim_{t \rightarrow a} v_1(t) \right] \right\rangle \\
 &= \left\langle \lim_{t \rightarrow a} [u_2(t)v_3(t) - u_3(t)v_2(t)], \lim_{t \rightarrow a} [u_3(t)v_1(t) - u_1(t)v_3(t)], \right. \\
 &\quad \left. \lim_{t \rightarrow a} [u_1(t)v_2(t) - u_2(t)v_1(t)] \right\rangle \\
 &= \lim_{t \rightarrow a} \langle u_2(t)v_3(t) - u_3(t)v_2(t), u_3(t)v_1(t) - u_1(t)v_3(t), \\
 &\quad u_1(t)v_2(t) - u_2(t)v_1(t) \rangle \\
 &= \lim_{t \rightarrow a} [\mathbf{u}(t) \times \mathbf{v}(t)]
 \end{aligned}$$

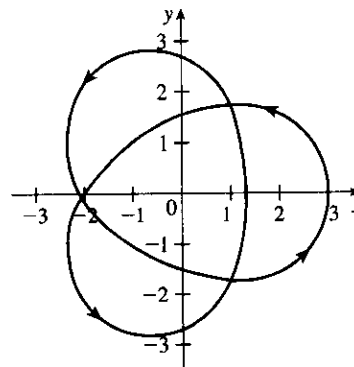
42. The projection of the curve onto the xy -plane is given by the parametric equations $x = (2 + \cos 1.5t) \cos t$, $y = (2 + \cos 1.5t) \sin t$. If we convert to polar coordinates, we have

$$\begin{aligned}
 r^2 &= x^2 + y^2 = [(2 + \cos 1.5t) \cos t]^2 + [(2 + \cos 1.5t) \sin t]^2 \\
 &= (2 + \cos 1.5t)^2 (\cos^2 t + \sin^2 t) \\
 &= (2 + \cos 1.5t)^2
 \end{aligned}$$

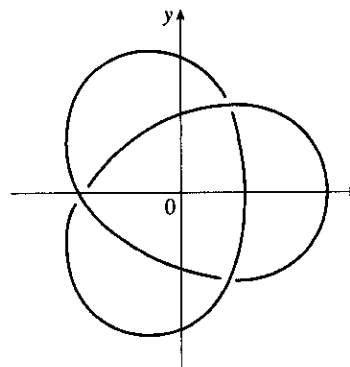
$$\Rightarrow r = 2 + \cos 1.5t. \text{ Also, } \tan \theta = \frac{y}{x} = \frac{(2 + \cos 1.5t) \sin t}{(2 + \cos 1.5t) \cos t} = \tan t \Rightarrow \theta = t.$$

Thus the polar equation of the curve is $r = 2 + \cos 1.5\theta$. At $\theta = 0$, we have $r = 3$, and r decreases to 1 as θ increases to $\frac{2\pi}{3}$. For $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$, r increases to 3; r decreases to 1 again at $\theta = 2\pi$, increases to 3 at $\theta = \frac{8\pi}{3}$,

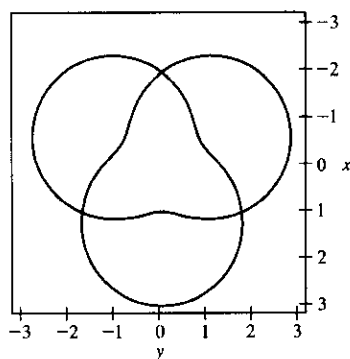
decreases to 1 at $\theta = \frac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta = 4\pi$. We sketch an approximate graph as shown in the figure.



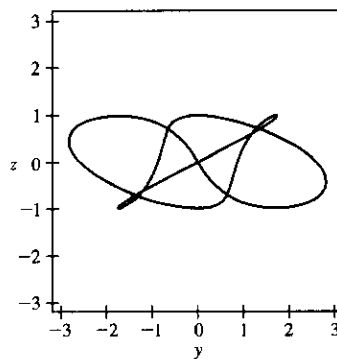
We can determine how the curve passes over itself by investigating the maximum and minimum values of z for $t = \theta \in [0, 4\pi]$. Since $z = \sin 1.5t$, z is maximized where $\sin 1.5t = 1 \Rightarrow 1.5t = \frac{\pi}{2}, \frac{5\pi}{2}, \text{ or } \frac{9\pi}{2} \Rightarrow t = \frac{\pi}{3}, \frac{5\pi}{3}, \text{ or } 3\pi$. z is minimized where $\sin 1.5t = -1 \Rightarrow 1.5t = \frac{3\pi}{2}, \frac{7\pi}{2}, \text{ or } \frac{11\pi}{2} \Rightarrow t = \pi, \frac{7\pi}{3}, \text{ or } \frac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t = 0 \Rightarrow r = 2$, and on the graph of the projection, these six points appear to be at the three self-intersections we see. Comparing the maximum and minimum values of z at these intersections, we can determine where the curve passes over itself, as indicated in the figure.



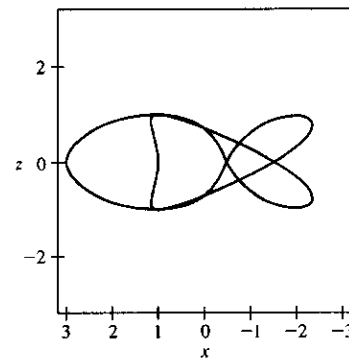
We show a computer-drawn graph of the curve from above, as well as views from the front and from the right side.



Top view

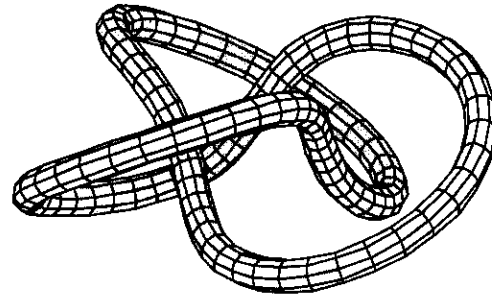
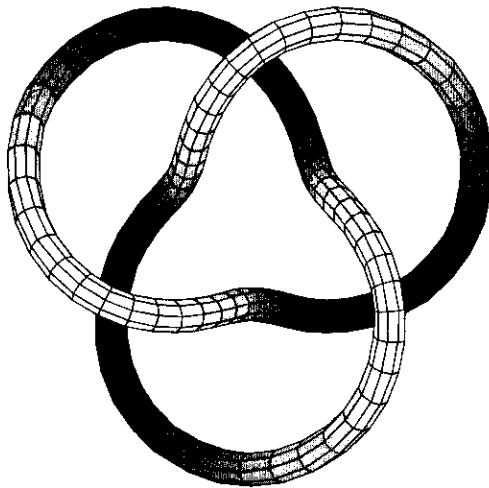
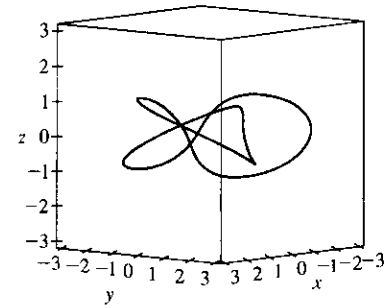
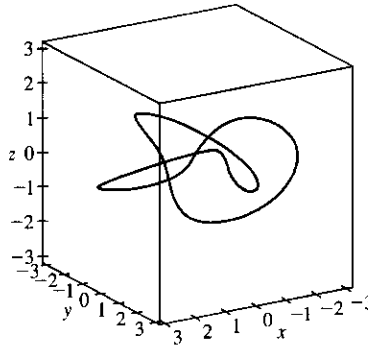
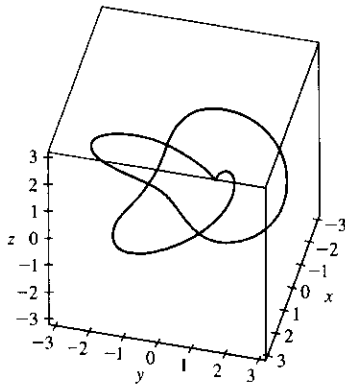


Front view



Side view

The top view graph shows a more accurate representation of the projection of the trefoil knot on the xy -plane (the axes are rotated 90°). Notice the indentations the graph exhibits at the points corresponding to $r = 1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.



43. Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. If $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{b}$, then $\lim_{t \rightarrow a} \mathbf{r}(t)$ exists, so by (1),

$$\mathbf{b} = \lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle. \text{ By the definition of equal vectors we have } \lim_{t \rightarrow a} f(t) = b_1,$$

$\lim_{t \rightarrow a} g(t) = b_2$ and $\lim_{t \rightarrow a} h(t) = b_3$. But these are limits of real-valued functions, so by the definition of limits, for

every $\epsilon > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ so $|f(t) - b_1| < \epsilon/3$ whenever $0 < |t - a| < \delta_1$,

$|g(t) - b_2| < \epsilon/3$ whenever $0 < |t - a| < \delta_2$, and $|h(t) - b_3| < \epsilon/3$ whenever $0 < |t - a| < \delta_3$. Letting

$\delta = \text{minimum of } \{\delta_1, \delta_2, \delta_3\}$, we have $|f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$

whenever $0 < |t - a| < \delta$. But $|\mathbf{r}(t) - \mathbf{b}| = |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle|$

$$= \sqrt{(f(t) - b_1)^2 + (g(t) - b_2)^2 + (h(t) - b_3)^2} \leq \sqrt{[f(t) - b_1]^2} + \sqrt{[g(t) - b_2]^2} + \sqrt{[h(t) - b_3]^2}$$

$$= |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3|. \text{ Thus for every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that}$$

$|\mathbf{r}(t) - \mathbf{b}| \leq |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. Conversely, if for every $\epsilon > 0$,

there exists $\delta > 0$ such that $|\mathbf{r}(t) - \mathbf{b}| < \epsilon$ whenever $0 < |t - a| < \delta$, then

$$|\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle| < \epsilon \Leftrightarrow \sqrt{[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2} < \epsilon \Leftrightarrow$$

$[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$. But each term on the left side of this

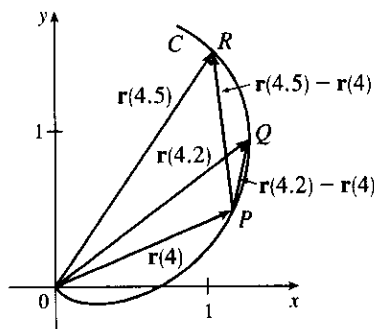
inequality is positive so $[f(t) - b_1]^2 < \epsilon^2$, $[g(t) - b_2]^2 < \epsilon^2$ and $[h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$, or

taking the square root of both sides in each of the above we have $|f(t) - b_1| < \epsilon$, $|g(t) - b_2| < \epsilon$ and $|h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. And by definition of limits of real-valued functions we have $\lim_{t \rightarrow a} f(t) = b_1$, $\lim_{t \rightarrow a} g(t) = b_2$ and $\lim_{t \rightarrow a} h(t) = b_3$. But by (1), $\lim_{t \rightarrow a} \mathbf{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$, so $\lim_{t \rightarrow a} \mathbf{r}(t) = \langle b_1, b_2, b_3 \rangle = \mathbf{b}$.

14.2 Derivatives and Integrals of Vector Functions

ET 13.2

1. (a)



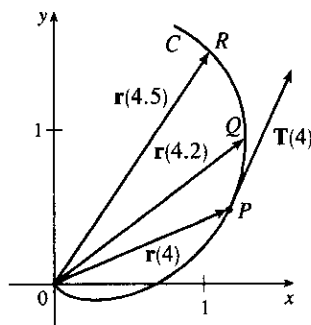
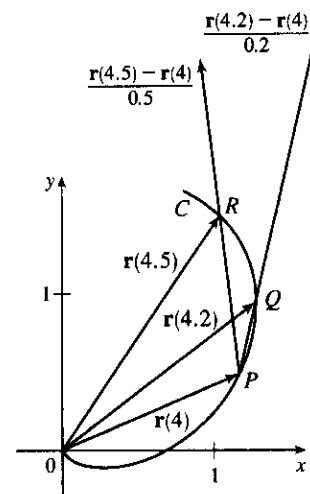
(b) $\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector

$\mathbf{r}(4.5) - \mathbf{r}(4)$. $\frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.

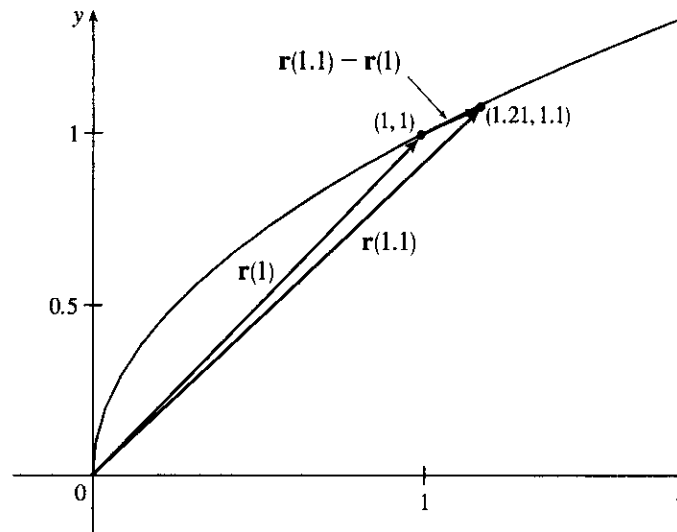
(c) By Definition 1, $\mathbf{r}'(4) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(4+h) - \mathbf{r}(4)}{h}$.

$$\mathbf{T}(4) = \frac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}.$$

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.

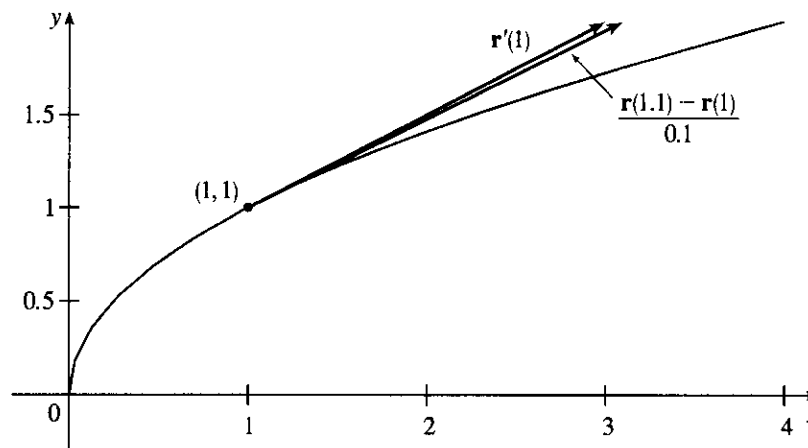


2. (a) The curve can be represented by the parametric equations $x = t^2$, $y = t$, $0 \leq t \leq 2$. Eliminating the parameter, we have $x = y^2$, $0 \leq y \leq 2$, a portion of which we graph here, along with the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.



- (b) Since $\mathbf{r}(t) = \langle t^2, t \rangle$, we differentiate components, giving $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, so $\mathbf{r}'(1) = \langle 2, 1 \rangle$.

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} = \frac{\langle 1.21, 1.1 \rangle - \langle 1, 1 \rangle}{0.1} = 10 \langle 0.21, 0.1 \rangle = \langle 2.1, 1 \rangle.$$

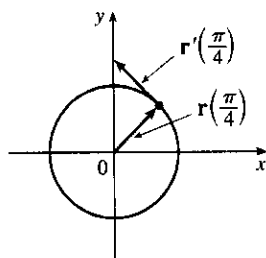


As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'(1)$ is defined to be

$$\lim_{h \rightarrow 0} \frac{\mathbf{r}(1+h) - \mathbf{r}(1)}{h}, \text{ and we recognize } \frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} \text{ as the expression after the limit sign with } h = 0.1.$$

Since h is close to 0, we would expect $\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ to be a vector close to $\mathbf{r}'(1)$.

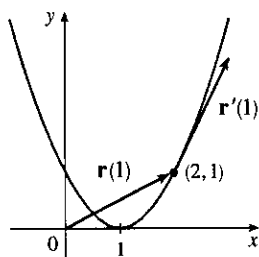
3. (a), (c)



(b) $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$

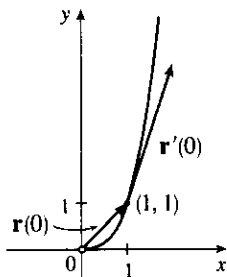
5. Since $(x-1)^2 = t^2 = y$, the curve is a parabola.

(a), (c)



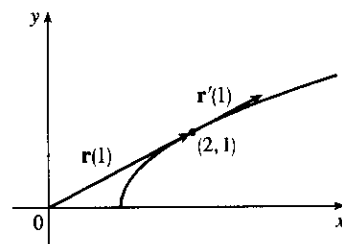
(b) $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

7. (a), (c)



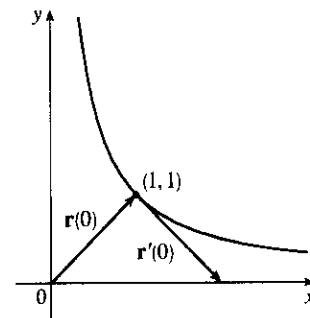
(b) $\mathbf{r}'(t) = e^t \mathbf{i} + 3e^{3t} \mathbf{j}$

4. (a), (c)



(b) $\mathbf{r}'(t) = \left\langle 1, \frac{1}{2\sqrt{t}} \right\rangle$

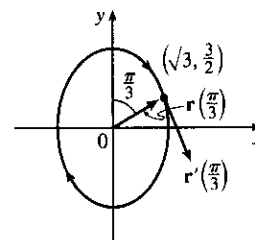
6. (a), (c)



(b) $\mathbf{r}'(t) = e^t \mathbf{i} - e^{-t} \mathbf{j}$

8. $x = 2 \sin t$, $y = 3 \cos t$, so $(x/2)^2 + (y/3)^2 = \sin^2 t + \cos^2 t = 1$ and the curve is an ellipse.

(a), (c)



(b) $\mathbf{r}'(t) = 2 \cos t \mathbf{i} - 3 \sin t \mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \frac{d}{dt} [t^2], \frac{d}{dt} [1-t], \frac{d}{dt} [\sqrt{t}] \right\rangle = \left\langle 2t, -1, \frac{1}{2\sqrt{t}} \right\rangle$

10. $\mathbf{r}(t) = \langle \cos 3t, t, \sin 3t \rangle \Rightarrow \mathbf{r}'(t) = \langle -3 \sin 3t, 1, 3 \cos 3t \rangle$

11. $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + e^{4t} \mathbf{k} \Rightarrow \mathbf{r}'(t) = 0\mathbf{i} + 0\mathbf{j} + 4e^{4t} \mathbf{k} = 4e^{4t} \mathbf{k}$

12. $\mathbf{r}(t) = \sin^{-1} t \mathbf{i} + \sqrt{1-t^2} \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}'(t) = \frac{1}{\sqrt{1-t^2}} \mathbf{i} - \frac{t}{\sqrt{1-t^2}} \mathbf{j}$

13. $\mathbf{r}(t) = e^{t^2} \mathbf{i} - \mathbf{j} + \ln(1+3t) \mathbf{k} \Rightarrow \mathbf{r}'(t) = 2te^{t^2} \mathbf{i} + \frac{3}{1+3t} \mathbf{k}$

$$\begin{aligned}
 14. \mathbf{r}'(t) &= [at(-3\sin 3t) + a\cos 3t]\mathbf{i} + b \cdot 3\sin^2 t \cos t \mathbf{j} + c \cdot 3\cos^2 t(-\sin t)\mathbf{k} \\
 &= (a\cos 3t - 3at\sin 3t)\mathbf{i} + 3b\sin^2 t \cos t \mathbf{j} - 3c\cos^2 t \sin t \mathbf{k}
 \end{aligned}$$

$$15. \mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\mathbf{c} = \mathbf{b} + 2t\mathbf{c} \text{ by Formulas 1 and 3 of Theorem 3.}$$

$$16. \text{ To find } \mathbf{r}'(t), \text{ we first expand } \mathbf{r}(t) = t\mathbf{a} \times (\mathbf{b} + t\mathbf{c}) = t(\mathbf{a} \times \mathbf{b}) + t^2(\mathbf{a} \times \mathbf{c}), \text{ so } \mathbf{r}'(t) = \mathbf{a} \times \mathbf{b} + 2t(\mathbf{a} \times \mathbf{c}).$$

$$17. \mathbf{r}'(t) = \langle 30t^4, 12t^2, 2 \rangle \Rightarrow \mathbf{r}'(1) = \langle 30, 12, 2 \rangle. \text{ So } |\mathbf{r}'(1)| = \sqrt{30^2 + 12^2 + 2^2} = \sqrt{1048} = 2\sqrt{262} \text{ and}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{2\sqrt{262}} \langle 30, 12, 2 \rangle = \left\langle \frac{15}{\sqrt{262}}, \frac{6}{\sqrt{262}}, \frac{1}{\sqrt{262}} \right\rangle.$$

$$18. \mathbf{r}'(t) = \frac{2}{\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}'(1) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}. \text{ Thus}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{2^2 + 2^2 + 1^2}} (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}.$$

$$19. \mathbf{r}'(t) = -\sin t\mathbf{i} + 3\mathbf{j} + 4\cos 2t\mathbf{k} \Rightarrow \mathbf{r}'(0) = 3\mathbf{j} + 4\mathbf{k}. \text{ Thus}$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{0^2 + 3^2 + 4^2}} (3\mathbf{j} + 4\mathbf{k}) = \frac{1}{5}(3\mathbf{j} + 4\mathbf{k}) = \frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}.$$

$$20. \mathbf{r}'(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} + \sec^2 t\mathbf{k} \Rightarrow \mathbf{r}'\left(\frac{\pi}{4}\right) = \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k} \text{ and } |\mathbf{r}'\left(\frac{\pi}{4}\right)| = \sqrt{2 + 2 + 4} = 2\sqrt{2}.$$

$$\text{Thus } \mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{|\mathbf{r}'\left(\frac{\pi}{4}\right)|} = \frac{1}{2\sqrt{2}} (\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}) = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

$$21. \mathbf{r}(t) = \langle t, t^2, t^3 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle. \text{ Then } \mathbf{r}'(1) = \langle 1, 2, 3 \rangle \text{ and } |\mathbf{r}'(1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \text{ so}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle. \mathbf{r}''(t) = \langle 0, 2, 6t \rangle, \text{ so}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} \mathbf{k}$$

$$= (12t^2 - 6t^2)\mathbf{i} - (6t - 0)\mathbf{j} + (2 - 0)\mathbf{k} = \langle 6t^2, -6t, 2 \rangle.$$

$$22. \mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle \Rightarrow \mathbf{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t} \rangle \Rightarrow$$

$$\mathbf{r}'(0) = \langle 2e^0, -2e^0, (0+1)e^0 \rangle = \langle 2, -2, 1 \rangle \text{ and } |\mathbf{r}'(0)| = \sqrt{2^2 + (-2)^2 + 1^2} = 3. \text{ Then}$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{3} \langle 2, -2, 1 \rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle.$$

$$\mathbf{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t} \rangle \Rightarrow \mathbf{r}''(0) = \langle 4e^0, 4e^0, (0+4)e^0 \rangle = \langle 4, 4, 4 \rangle.$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t} \rangle \cdot \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t} \rangle$$

$$= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t}) + ((2t+1)e^{2t})((4t+4)e^{2t})$$

$$= 8e^{4t} - 8e^{-4t} + (8t^2 + 12t + 4)e^{4t} = (8t^2 + 12t + 12)e^{4t} - 8e^{-4t}$$

$$23. \text{ The vector equation for the curve is } \mathbf{r}(t) = \langle t^5, t^4, t^3 \rangle, \text{ so } \mathbf{r}'(t) = \langle 5t^4, 4t^3, 3t^2 \rangle. \text{ The point } (1, 1, 1) \text{ corresponds to } t = 1, \text{ so the tangent vector there is } \mathbf{r}'(1) = \langle 5, 4, 3 \rangle. \text{ Thus, the tangent line goes through the point } (1, 1, 1) \text{ and is parallel to the vector } \langle 5, 4, 3 \rangle. \text{ Parametric equations are } x = 1 + 5t, y = 1 + 4t, z = 1 + 3t.$$

$$24. \text{ The vector equation for the curve is } \mathbf{r}(t) = \langle t^2 - 1, t^2 + 1, t + 1 \rangle, \text{ so } \mathbf{r}'(t) = \langle 2t, 2t, 1 \rangle. \text{ The point } (-1, 1, 1) \text{ corresponds to } t = 0, \text{ so the tangent vector there is } \mathbf{r}'(0) = \langle 0, 0, 1 \rangle. \text{ Thus, the tangent line is parallel to the vector } \langle 0, 0, 1 \rangle \text{ and parametric equations are } x = -1 + 0 \cdot t = -1, y = 1 + 0 \cdot t = 1, z = 1 + 1 \cdot t = 1 + t.$$

25. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t, e^{-t} \rangle$, so

$$\begin{aligned}\mathbf{r}'(t) &= \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t} \cos t + (\sin t)(-e^{-t}), (-e^{-t}) \rangle \\ &= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t} \rangle.\end{aligned}$$

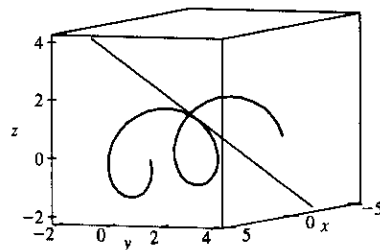
The point $(1, 0, 1)$ corresponds to $t = 0$, so the tangent vector there is

$$\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0 \rangle = \langle -1, 1, -1 \rangle. \text{ Thus, the tangent line is parallel to the vector } \langle -1, 1, -1 \rangle \text{ and parametric equations are } x = 1 + (-1)t = 1 - t, y = 0 + 1 \cdot t = t, z = 1 + (-1)t = 1 - t.$$

26. $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2 \rangle$, $\mathbf{r}'(t) = \langle 1/t, 1/\sqrt{t}, 2t \rangle$. At $(0, 2, 1)$, $t = 1$ and $\mathbf{r}'(1) = \langle 1, 1, 2 \rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 2 + t$, $z = 1 + 2t$.

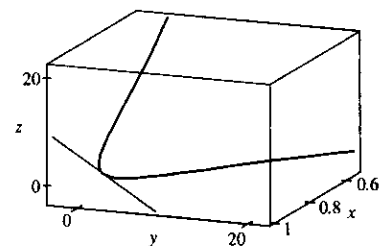
27. $\mathbf{r}(t) = \langle t, \sqrt{2} \cos t, \sqrt{2} \sin t \rangle \Rightarrow$

$$\begin{aligned}\mathbf{r}'(t) &= \langle 1, -\sqrt{2} \sin t, \sqrt{2} \cos t \rangle. \text{ At } \left(\frac{\pi}{4}, 1, 1\right), t = \frac{\pi}{4} \text{ and} \\ \mathbf{r}'\left(\frac{\pi}{4}\right) &= \langle 1, -1, 1 \rangle. \text{ Thus, parametric equations of the tangent line are } x = \frac{\pi}{4} + t, y = 1 - t, z = 1 + t.\end{aligned}$$



28. $\mathbf{r}(t) = \langle \cos t, 3e^{2t}, 3e^{-2t} \rangle$, $\mathbf{r}'(t) = \langle -\sin t, 6e^{2t}, -6e^{-2t} \rangle$.

At $(1, 3, 3)$, $t = 0$ and $\mathbf{r}'(0) = \langle 0, 6, -6 \rangle$. Thus, parametric equations of the tangent line are $x = 1$, $y = 3 + 6t$, $z = 3 - 6t$.



29. (a) $\mathbf{r}(t) = \langle t^3, t^4, t^5 \rangle \Rightarrow \mathbf{r}'(t) = \langle 3t^2, 4t^3, 5t^4 \rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0 \rangle = \mathbf{0}$, the curve is not smooth.

(b) $\mathbf{r}(t) = \langle t^3 + t, t^4, t^5 \rangle \Rightarrow \mathbf{r}'(t) = \langle 3t^2 + 1, 4t^3, 5t^4 \rangle$. $\mathbf{r}'(t)$ is continuous since its component functions are continuous. Also, $\mathbf{r}'(t) \neq \mathbf{0}$, as the y - and z -components are 0 only for $t = 0$, but $\mathbf{r}'(0) = \langle 1, 0, 0 \rangle \neq \mathbf{0}$. Thus, the curve is smooth.

(c) $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle \Rightarrow \mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle$. Since $\mathbf{r}'(0) = \langle -3\cos^2 0 \sin 0, 3\sin^2 0 \cos 0 \rangle = \langle 0, 0 \rangle = \mathbf{0}$, the curve is not smooth.

30. (a) The tangent line at $t = 0$ is the line through the point with

$$\text{position vector } \mathbf{r}(0) = \langle \sin 0, 2 \sin 0, \cos 0 \rangle = \langle 0, 0, 1 \rangle,$$

and in the direction of the tangent vector,

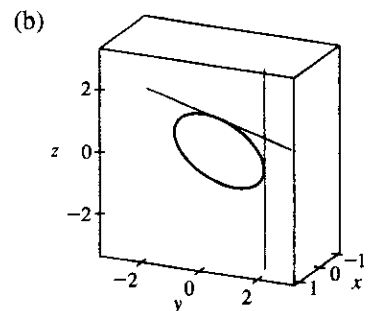
$$\mathbf{r}'(0) = \langle \pi \cos 0, 2\pi \cos 0, -\pi \sin 0 \rangle = \langle \pi, 2\pi, 0 \rangle.$$

So an equation of the line is

$$\langle x, y, z \rangle = \mathbf{r}(0) + u \mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1 \rangle = \langle \pi u, 2\pi u, 1 \rangle.$$

$$\mathbf{r}\left(\frac{1}{2}\right) = \left\langle \sin \frac{\pi}{2}, 2 \sin \frac{\pi}{2}, \cos \frac{\pi}{2} \right\rangle = \langle 1, 2, 0 \rangle, \mathbf{r}'\left(\frac{1}{2}\right) = \left\langle \pi \cos \frac{\pi}{2}, 2\pi \cos \frac{\pi}{2}, -\pi \sin \frac{\pi}{2} \right\rangle = \langle 0, 0, -\pi \rangle.$$

So the equation of the second line is $\langle x, y, z \rangle = \langle 1, 2, 0 \rangle + v \langle 0, 0, -\pi \rangle = \langle 1, 2, -\pi v \rangle$. The lines intersect where $\langle \pi u, 2\pi u, 1 \rangle = \langle 1, 2, -\pi v \rangle$, so the point of intersection is $(1, 2, 1)$.



31. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}'_1(t) = \langle 1, 2t, 3t^2 \rangle$ and $t = 0$ at $(0, 0, 0)$, $\mathbf{r}'_1(0) = \langle 1, 0, 0 \rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0, 0, 0)$. Similarly, $\mathbf{r}'_2(t) = \langle \cos t, 2 \cos 2t, 1 \rangle$ and since $\mathbf{r}_2(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}'_2(0) = \langle 1, 2, 1 \rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0, 0, 0)$. If θ is the angle between these two tangent vectors, then $\cos \theta = \frac{1}{\sqrt{1+6}} \langle 1, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$.

32. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1, 0, 4)$. To find the angle θ of intersection, we proceed as in Exercise 31. The tangent vectors to the respective curves at $(1, 0, 4)$ are $\mathbf{r}'_1(1) = \langle 1, -1, 2 \rangle$ and $\mathbf{r}'_2(2) = \langle -1, 1, 4 \rangle$. So $\cos \theta = \frac{1}{\sqrt{6}\sqrt{18}} (-1 - 1 + 8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

Note: In Exercise 31, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

33. $\int_0^1 (16t^3 \mathbf{i} - 9t^2 \mathbf{j} + 25t^4 \mathbf{k}) dt = \left(\int_0^1 16t^3 dt\right) \mathbf{i} - \left(\int_0^1 9t^2 dt\right) \mathbf{j} + \left(\int_0^1 25t^4 dt\right) \mathbf{k}$
 $= [4t^4]_0^1 \mathbf{i} - [3t^3]_0^1 \mathbf{j} + [5t^5]_0^1 \mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$
34. $\int_0^1 \left(\frac{4}{1+t^2} \mathbf{j} + \frac{2t}{1+t^2} \mathbf{k}\right) dt = [4 \tan^{-1} t \mathbf{j} + \ln(1+t^2) \mathbf{k}]_0^1$
 $= [4 \tan^{-1} 1 \mathbf{j} + \ln 2 \mathbf{k}] - [4 \tan^{-1} 0 \mathbf{j} + \ln 1 \mathbf{k}] = 4\left(\frac{\pi}{4}\right) \mathbf{j} + \ln 2 \mathbf{k} - 0\mathbf{j} - 0\mathbf{k} = \pi \mathbf{j} + \ln 2 \mathbf{k}$
35. $\int_0^{\pi/2} (3 \sin^2 t \cos t \mathbf{i} + 3 \sin t \cos^2 t \mathbf{j} + 2 \sin t \cos t \mathbf{k}) dt$
 $= \left(\int_0^{\pi/2} 3 \sin^2 t \cos t dt\right) \mathbf{i} + \left(\int_0^{\pi/2} 3 \sin t \cos^2 t dt\right) \mathbf{j} + \left(\int_0^{\pi/2} 2 \sin t \cos t dt\right) \mathbf{k}$
 $= [\sin^3 t]_0^{\pi/2} \mathbf{i} + [-\cos^3 t]_0^{\pi/2} \mathbf{j} + [\sin^2 t]_0^{\pi/2} \mathbf{k}$
 $= (1 - 0) \mathbf{i} + (0 + 1) \mathbf{j} + (1 - 0) \mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
36. $\int_1^4 (\sqrt{t} \mathbf{i} + te^{-t} \mathbf{j} + t^{-2} \mathbf{k}) dt = \left[\frac{2}{3} t^{3/2} \mathbf{i} - t^{-1} \mathbf{k}\right]_1^4 + \left([-te^{-t}]_1^4 + \int_1^4 e^{-t} dt\right) \mathbf{j}$
 $= \left(\frac{16}{3} - \frac{2}{3}\right) \mathbf{i} - \left(\frac{1}{4} - 1\right) \mathbf{k} + (-4e^{-4} + e^{-1} - e^{-4} + e^{-1}) \mathbf{j} = \frac{14}{3} \mathbf{i} + e^{-1}(2 - 5e^{-3}) \mathbf{j} + \frac{3}{4} \mathbf{k}$
37. $\int (e^t \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k}) dt = \left(\int e^t dt\right) \mathbf{i} + \left(\int 2t dt\right) \mathbf{j} + \left(\int \ln t dt\right) \mathbf{k}$
 $= e^t \mathbf{i} + t^2 \mathbf{j} + (t \ln t - t) \mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a vector constant of integration.
38. $\int (\cos \pi t \mathbf{i} + \sin \pi t \mathbf{j} + t \mathbf{k}) dt = \left(\int \cos \pi t dt\right) \mathbf{i} + \left(\int \sin \pi t dt\right) \mathbf{j} + \left(\int t dt\right) \mathbf{k}$
 $= \frac{1}{\pi} \sin \pi t \mathbf{i} - \frac{1}{\pi} \cos \pi t \mathbf{j} + \frac{1}{2} t^2 \mathbf{k} + \mathbf{C}$
39. $\mathbf{r}'(t) = t^2 \mathbf{i} + 4t^3 \mathbf{j} - t^2 \mathbf{k} \Rightarrow \mathbf{r}(t) = \frac{1}{3} t^3 \mathbf{i} + t^4 \mathbf{j} - \frac{1}{3} t^3 \mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector.
 But $\mathbf{j} = \mathbf{r}(0) = (0) \mathbf{i} + (0) \mathbf{j} - (0) \mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = \mathbf{j}$ and
 $\mathbf{r}(t) = \frac{1}{3} t^3 \mathbf{i} + t^4 \mathbf{j} - \frac{1}{3} t^3 \mathbf{k} + \mathbf{j} = \frac{1}{3} t^3 \mathbf{i} + (t^4 + 1) \mathbf{j} - \frac{1}{3} t^3 \mathbf{k}$.
40. $\mathbf{r}'(t) = \sin t \mathbf{i} - \cos t \mathbf{j} + 2t \mathbf{k} \Rightarrow \mathbf{r}(t) = (-\cos t) \mathbf{i} - (\sin t) \mathbf{j} + t^2 \mathbf{k} + \mathbf{C}$.
 But $\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \mathbf{r}(0) = -\mathbf{i} + (0) \mathbf{j} + (0) \mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and
 $\mathbf{r}(t) = (2 - \cos t) \mathbf{i} + (1 - \sin t) \mathbf{j} + (2 + t^2) \mathbf{k}$.

For Exercises 41–44, let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

$$\begin{aligned}
 41. \quad \frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] &= \frac{d}{dt} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t) \rangle \\
 &= \left\langle \frac{d}{dt} [u_1(t) + v_1(t)], \frac{d}{dt} [u_2(t) + v_2(t)], \frac{d}{dt} [u_3(t) + v_3(t)] \right\rangle \\
 &= \langle u_1'(t) + v_1'(t), u_2'(t) + v_2'(t), u_3'(t) + v_3'(t) \rangle \\
 &= \langle u_1'(t), u_2'(t), u_3'(t) \rangle + \langle v_1'(t), v_2'(t), v_3'(t) \rangle = \mathbf{u}'(t) + \mathbf{v}'(t).
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \frac{d}{dt} [f(t) \mathbf{u}(t)] &= \frac{d}{dt} \langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t) \rangle \\
 &= \left\langle \frac{d}{dt} [f(t)u_1(t)], \frac{d}{dt} [f(t)u_2(t)], \frac{d}{dt} [f(t)u_3(t)] \right\rangle \\
 &= \langle f'(t)u_1(t) + f(t)u_1'(t), f'(t)u_2(t) + f(t)u_2'(t), f'(t)u_3(t) + f(t)u_3'(t) \rangle \\
 &= f'(t) \langle u_1(t), u_2(t), u_3(t) \rangle + f(t) \langle u_1'(t), u_2'(t), u_3'(t) \rangle \\
 &= f'(t) \mathbf{u}(t) + f(t) \mathbf{u}'(t)
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] &= \frac{d}{dt} \langle u_2(t)v_3(t) - u_3(t)v_2(t), u_3(t)v_1(t) - u_1(t)v_3(t), u_1(t)v_2(t) - u_2(t)v_1(t) \rangle \\
 &= \langle u_2'(t)v_3(t) + u_2(t)v_3'(t) - u_3'(t)v_2(t) - u_3(t)v_2'(t), \\
 &\quad u_3'(t)v_1(t) + u_3(t)v_1'(t) - u_1'(t)v_3(t) - u_1(t)v_3'(t), \\
 &\quad u_1'(t)v_2(t) + u_1(t)v_2'(t) - u_2'(t)v_1(t) - u_2(t)v_1'(t) \rangle \\
 &= \langle u_2'(t)v_3(t) - u_3'(t)v_2(t), u_3'(t)v_1(t) - u_1'(t)v_3(t), u_1'(t)v_2(t) - u_2'(t)v_1(t) \rangle \\
 &\quad + \langle u_2(t)v_3'(t) - u_3(t)v_2'(t), u_3(t)v_1'(t) - u_1(t)v_3'(t), u_1(t)v_2'(t) - u_2(t)v_1'(t) \rangle \\
 &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)
 \end{aligned}$$

Alternate solution: Let $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$. Then

$$\begin{aligned}
 \mathbf{r}(t+h) - \mathbf{r}(t) &= [\mathbf{u}(t+h) \times \mathbf{v}(t+h)] - [\mathbf{u}(t) \times \mathbf{v}(t)] \\
 &= [\mathbf{u}(t+h) \times \mathbf{v}(t+h)] - [\mathbf{u}(t) \times \mathbf{v}(t)] + [\mathbf{u}(t+h) \times \mathbf{v}(t)] - [\mathbf{u}(t+h) \times \mathbf{v}(t)] \\
 &= \mathbf{u}(t+h) \times [\mathbf{v}(t+h) - \mathbf{v}(t)] + [\mathbf{u}(t+h) - \mathbf{u}(t)] \times \mathbf{v}(t)
 \end{aligned}$$

(Be careful of the order of the cross product.)

Dividing through by h and taking the limit as $h \rightarrow 0$ we have

$$\begin{aligned}
 \mathbf{r}'(t) &= \lim_{h \rightarrow 0} \frac{\mathbf{u}(t+h) \times [\mathbf{v}(t+h) - \mathbf{v}(t)]}{h} + \lim_{h \rightarrow 0} \frac{[\mathbf{u}(t+h) - \mathbf{u}(t)] \times \mathbf{v}(t)}{h} \\
 &= \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)
 \end{aligned}$$

by Exercise 14.1.41(a) [ET 13.1.41(a)] and Definition 1.

$$\begin{aligned} 44. \frac{d}{dt} [\mathbf{u}(f(t))] &= \frac{d}{dt} \langle u_1(f(t)), u_2(f(t)), u_3(f(t)) \rangle = \left\langle \frac{d}{dt} [u_1(f(t))], \frac{d}{dt} [u_2(f(t))], \frac{d}{dt} [u_3(f(t))] \right\rangle \\ &= \langle f'(t)u'_1(f(t)), f'(t)u'_2(f(t)), f'(t)u'_3(f(t)) \rangle = f'(t) \mathbf{u}'(t) \end{aligned}$$

$$\begin{aligned} 45. \frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \quad [\text{by Formula 4 of Theorem 3}] \\ &= (-4t\mathbf{j} + 9t^2\mathbf{k}) \cdot (t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}) + (\mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}) \cdot (\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}) \\ &= -4t \cos t + 9t^2 \sin t + 1 + 2t^2 \sin t + 3t^3 \cos t \\ &= 1 - 4t \cos t + 11t^2 \sin t + 3t^3 \cos t \end{aligned}$$

$$\begin{aligned} 46. \frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \quad [\text{by Formula 5 of Theorem 3}] \\ &= (-4t\mathbf{j} + 9t^2\mathbf{k}) \times (t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}) + (\mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}) \times (\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}) \\ &= (-4t \sin t - 9t^2 \cos t)\mathbf{i} + (9t^3 - 0)\mathbf{j} + (0 + 4t^2)\mathbf{k} \\ &\quad + (-2t^2 \cos t + 3t^3 \sin t)\mathbf{i} + (3t^3 - \cos t)\mathbf{j} + (-\sin t + 2t^2)\mathbf{k} \\ &= [(\sin t)(3t^3 - 4t) - 11t^2 \cos t]\mathbf{i} + (12t^3 - \cos t)\mathbf{j} + (6t^2 - \sin t)\mathbf{k} \end{aligned}$$

$$\begin{aligned} 47. \frac{d}{dt} [\mathbf{r}(t) \times \mathbf{r}'(t)] &= \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t) \text{ by Formula 5 of Theorem 3. But } \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{0} \\ &\text{(see Example 13.4.2 [ET 12.4.2]). Thus, } \frac{d}{dt} [\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t). \end{aligned}$$

$$\begin{aligned} 48. \frac{d}{dt} (\mathbf{u}(t) \cdot [\mathbf{v}(t) \times \mathbf{w}(t)]) &= \mathbf{u}'(t) \cdot [\mathbf{v}(t) \times \mathbf{w}(t)] + \mathbf{u}(t) \cdot \frac{d}{dt} [\mathbf{v}(t) \times \mathbf{w}(t)] \\ &= \mathbf{u}'(t) \cdot [\mathbf{v}(t) \times \mathbf{w}(t)] + \mathbf{u}(t) \cdot [\mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t)] \\ &= \mathbf{u}'(t) \cdot [\mathbf{v}(t) \times \mathbf{w}(t)] + \mathbf{u}(t) \cdot [\mathbf{v}'(t) \times \mathbf{w}(t)] + \mathbf{u}(t) \cdot [\mathbf{v}(t) \times \mathbf{w}'(t)] \\ &= \mathbf{u}'(t) \cdot [\mathbf{v}(t) \times \mathbf{w}(t)] - \mathbf{v}'(t) \cdot [\mathbf{u}(t) \times \mathbf{w}(t)] + \mathbf{w}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] \end{aligned}$$

$$49. \frac{d}{dt} |\mathbf{r}(t)| = \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{r}(t)]^{1/2} = \frac{1}{2} [\mathbf{r}(t) \cdot \mathbf{r}(t)]^{-1/2} [2\mathbf{r}(t) \cdot \mathbf{r}'(t)] = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t)$$

$$50. \text{ Since } \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0, \text{ we have } 0 = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{r}(t)] = \frac{d}{dt} |\mathbf{r}(t)|^2. \text{ Thus } |\mathbf{r}(t)|^2, \text{ and so } |\mathbf{r}(t)|, \text{ is a constant, and hence the curve lies on a sphere with center the origin.}$$

$$51. \text{ Since } \mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)],$$

$$\begin{aligned} \mathbf{u}'(t) &= \mathbf{r}'(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)] + \mathbf{r}(t) \cdot \frac{d}{dt} [\mathbf{r}'(t) \times \mathbf{r}''(t)] \\ &= 0 + \mathbf{r}(t) \cdot [\mathbf{r}''(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'''(t)] \quad [\text{since } \mathbf{r}'(t) \perp \mathbf{r}'(t) \times \mathbf{r}''(t)] \\ &= \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)] \quad [\text{since } \mathbf{r}''(t) \times \mathbf{r}''(t) = \mathbf{0}] \end{aligned}$$

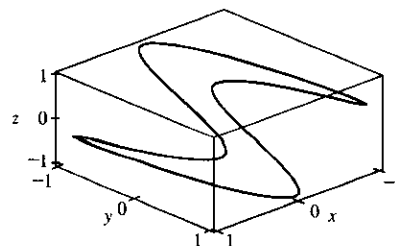
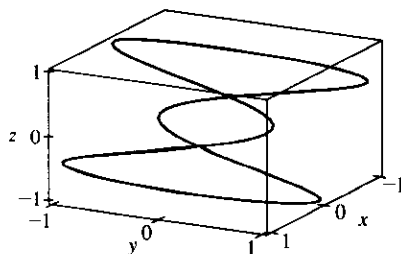
14.3 Arc Length and Curvature

ET 13.3

1. $\mathbf{r}'(t) = \langle 2 \cos t, 5, -2 \sin t \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{(2 \cos t)^2 + 5^2 + (-2 \sin t)^2} = \sqrt{29}$. Then using Formula 3, we have $L = \int_{-10}^{10} |\mathbf{r}'(t)| dt = \int_{-10}^{10} \sqrt{29} dt = \sqrt{29} t \Big|_{-10}^{10} = 20\sqrt{29}$.
2. $\mathbf{r}'(t) = \langle 2t, \cos t + t \sin t - \cos t, -\sin t + t \cos t + \sin t \rangle = \langle 2t, t \sin t, t \cos t \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{(2t)^2 + (t \sin t)^2 + (t \cos t)^2} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5} |t| = \sqrt{5} t$ for $0 \leq t \leq \pi$. Then using Formula 3, we have $L = \int_0^\pi |\mathbf{r}'(t)| dt = \int_0^\pi \sqrt{5} t dt = \sqrt{5} \frac{t^2}{2} \Big|_0^\pi = \frac{\sqrt{5}}{2} \pi^2$.
3. $\mathbf{r}'(t) = \sqrt{2} \mathbf{i} + e^t \mathbf{j} - e^{-t} \mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ (since $e^t + e^{-t} > 0$). Then $L = \int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 (e^t + e^{-t}) dt = [e^t - e^{-t}]_0^1 = e - e^{-1}$.
4. $\mathbf{r}'(t) = \langle 2t, 2, 1/t \rangle, |\mathbf{r}'(t)| = \sqrt{4t^2 + 4 + (1/t)^2} = \frac{1 + 2t^2}{|t|} = \frac{1 + 2t^2}{t}$ for $1 \leq t \leq e$.

$$L = \int_1^e \frac{1 + 2t^2}{t} dt = \int_1^e \left(\frac{1}{t} + 2t \right) dt = [\ln t + t^2]_1^e = e^2$$
5. $\mathbf{r}'(t) = 2t \mathbf{j} + 3t^2 \mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ (since $t \geq 0$). Then $L = \int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 t\sqrt{4 + 9t^2} dt = \frac{1}{18} \cdot \frac{2}{3} (4 + 9t^2)^{3/2} \Big|_0^1 = \frac{1}{27} (13^{3/2} - 4^{3/2}) = \frac{1}{27} (13^{3/2} - 8)$.
6. $\mathbf{r}'(t) = 12 \mathbf{i} + 12\sqrt{t} \mathbf{j} + 6t \mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{144 + 144t + 36t^2} = \sqrt{36(t+2)^2} = 6|t+2| = 6(t+2)$ for $0 \leq t \leq 1$. Then $L = \int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 6(t+2) dt = [3t^2 + 12t]_0^1 = 15$.
7. The point $(2, 4, 8)$ corresponds to $t = 2$, so by Equation 2, $L = \int_0^2 \sqrt{(1)^2 + (2t)^2 + (3t^2)^2} dt$.
 If $f(t) = \sqrt{1 + 4t^2 + 9t^4}$, then Simpson's Rule gives

$$L \approx \frac{2-0}{10 \cdot 3} [f(0) + 4f(0.2) + 2f(0.4) + \cdots + 4f(1.8) + f(2)] \approx 9.5706.$$
8. Here are two views of the curve with parametric equations $x = \cos t, y = \sin 3t, z = \sin t$:



The complete curve is given by the parameter interval $[0, 2\pi]$, so

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (3 \cos 3t)^2 + (\cos t)^2} dt = \int_0^{2\pi} \sqrt{1 + 9 \cos^2 3t} dt \approx 13.9744.$$

9. $\mathbf{r}'(t) = 2 \mathbf{i} - 3 \mathbf{j} + 4 \mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4 + 9 + 16} = \sqrt{29}$. Then $s = s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \sqrt{29} du = \sqrt{29} t$. Therefore, $t = \frac{1}{\sqrt{29}} s$, and substituting for t in the original equation, we have $\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}} s \mathbf{i} + \left(1 - \frac{3}{\sqrt{29}} s\right) \mathbf{j} + \left(5 + \frac{4}{\sqrt{29}} s\right) \mathbf{k}$.

$$10. \mathbf{r}'(t) = 2e^{2t}(\cos 2t - \sin 2t)\mathbf{i} + 2e^{2t}(\cos 2t + \sin 2t)\mathbf{k},$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 2e^{2t} \sqrt{(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2} = 2e^{2t} \sqrt{2\cos^2 2t + 2\sin^2 2t} = 2\sqrt{2}e^{2t}.$$

$$s = s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t 2\sqrt{2}e^{2u} du = \sqrt{2}e^{2u} \Big|_0^t = \sqrt{2}(e^{2t} - 1) \Rightarrow$$

$$\frac{s}{\sqrt{2}} + 1 = e^{2t} \Rightarrow t = \frac{1}{2} \ln\left(\frac{s}{\sqrt{2}} + 1\right). \text{ Substituting, we have}$$

$$\begin{aligned} \mathbf{r}(t(s)) &= e^{2\left(\frac{1}{2} \ln\left(\frac{s}{\sqrt{2}} + 1\right)\right)} \cos 2\left(\frac{1}{2} \ln\left(\frac{s}{\sqrt{2}} + 1\right)\right) \mathbf{i} + 2\mathbf{j} + e^{2\left(\frac{1}{2} \ln\left(\frac{s}{\sqrt{2}} + 1\right)\right)} \sin 2\left(\frac{1}{2} \ln\left(\frac{s}{\sqrt{2}} + 1\right)\right) \mathbf{k} \\ &= \left(\frac{s}{\sqrt{2}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right) \mathbf{i} + 2\mathbf{j} + \left(\frac{s}{\sqrt{2}} + 1\right) \sin\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right) \mathbf{k}. \end{aligned}$$

$$11. |\mathbf{r}'(t)| = \sqrt{(3\cos t)^2 + 16 + (-3\sin t)^2} = \sqrt{9 + 16} = 5 \text{ and } s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t 5 du = 5t \Rightarrow$$

$$t(s) = \frac{1}{5}s. \text{ Therefore, } \mathbf{r}(t(s)) = 3\sin\left(\frac{1}{5}s\right)\mathbf{i} + \frac{4}{5}s\mathbf{j} + 3\cos\left(\frac{1}{5}s\right)\mathbf{k}.$$

$$12. \mathbf{r}'(t) = \frac{-4t}{(t^2+1)^2}\mathbf{i} + \frac{-2t^2+2}{(t^2+1)^2}\mathbf{j}.$$

$$\begin{aligned} \frac{ds}{dt} = |\mathbf{r}'(t)| &= \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4 + 8t^2 + 4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}} \\ &= \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1} \end{aligned}$$

Since the initial point $(1, 0)$ corresponds to $t = 0$, the arc length function

$$s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \frac{2}{u^2+1} du = 2 \arctan t. \text{ Then } \arctan t = \frac{1}{2}s \Rightarrow t = \tan \frac{1}{2}s. \text{ Substituting, we have}$$

$$\begin{aligned} \mathbf{r}(t(s)) &= \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right) + 1} - 1 \right] \mathbf{i} + \frac{2 \tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right) + 1} \mathbf{j} = \frac{1 - \tan^2\left(\frac{1}{2}s\right)}{1 + \tan^2\left(\frac{1}{2}s\right)} \mathbf{i} + \frac{2 \tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)} \mathbf{j} \\ &= \frac{1 - \tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)} \mathbf{i} + 2 \tan\left(\frac{1}{2}s\right) \cos^2\left(\frac{1}{2}s\right) \mathbf{j} \\ &= [\cos^2\left(\frac{1}{2}s\right) - \sin^2\left(\frac{1}{2}s\right)] \mathbf{i} + 2 \sin\left(\frac{1}{2}s\right) \cos\left(\frac{1}{2}s\right) \mathbf{j} = \cos s \mathbf{i} + \sin s \mathbf{j} \end{aligned}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1, 0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ (k an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

$$13. (a) \mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}. \text{ Then}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{29}} \langle 2\cos t, 5, -2\sin t \rangle \text{ or } \left\langle \frac{2}{\sqrt{29}} \cos t, \frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}} \sin t \right\rangle.$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{29}} \langle -2\sin t, 0, -2\cos t \rangle \Rightarrow |\mathbf{T}'(t)| = \frac{1}{\sqrt{29}} \sqrt{4\sin^2 t + 0 + 4\cos^2 t} = \frac{2}{\sqrt{29}}. \text{ Thus}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1/\sqrt{29}}{2/\sqrt{29}} \langle -2\sin t, 0, -2\cos t \rangle = \langle -\sin t, 0, -\cos t \rangle.$$

$$(b) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{2/\sqrt{29}}{\sqrt{29}} = \frac{2}{29}.$$

$$14. (a) \mathbf{r}'(t) = \langle 2t, t \sin t, t \cos t \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{4t^2 + t^2 \sin^2 t + t^2 \cos^2 t} = \sqrt{5t^2} = \sqrt{5} t$$

$$(\text{since } t > 0). \text{ Then } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{5} t} \langle 2t, t \sin t, t \cos t \rangle = \frac{1}{\sqrt{5}} \langle 2, \sin t, \cos t \rangle.$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}} \langle 0, \cos t, -\sin t \rangle \Rightarrow |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}} \sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}. \text{ Thus}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1/\sqrt{5}}{1/\sqrt{5}} \langle 0, \cos t, -\sin t \rangle = \langle 0, \cos t, -\sin t \rangle.$$

$$(b) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1/\sqrt{5}}{\sqrt{5} t} = \frac{1}{5t}.$$

$$15. (a) \mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}. \text{ Then}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{e^t + e^{-t}} \langle \sqrt{2}, e^t, -e^{-t} \rangle = \frac{1}{e^{2t} + 1} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle \quad \left(\text{after multiplying by } \frac{e^t}{e^t} \right) \text{ and}$$

$$\begin{aligned} \mathbf{T}'(t) &= \frac{1}{e^{2t} + 1} \langle \sqrt{2}e^t, 2e^{2t}, 0 \rangle - \frac{2e^{2t}}{(e^{2t} + 1)^2} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle \\ &= \frac{1}{(e^{2t} + 1)^2} \left[(e^{2t} + 1) \langle \sqrt{2}e^t, 2e^{2t}, 0 \rangle - 2e^{2t} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle \right] \\ &= \frac{1}{(e^{2t} + 1)^2} \langle \sqrt{2}e^t (1 - e^{2t}), 2e^{2t}, 2e^{2t} \rangle \end{aligned}$$

Then

$$\begin{aligned} |\mathbf{T}'(t)| &= \frac{1}{(e^{2t} + 1)^2} \sqrt{2e^{2t}(1 - 2e^{2t} + e^{4t}) + 4e^{4t} + 4e^{4t}} = \frac{1}{(e^{2t} + 1)^2} \sqrt{2e^{2t}(1 + 2e^{2t} + e^{4t})} \\ &= \frac{1}{(e^{2t} + 1)^2} \sqrt{2e^{2t}(1 + e^{2t})^2} = \frac{\sqrt{2}e^t(1 + e^{2t})}{(e^{2t} + 1)^2} = \frac{\sqrt{2}e^t}{e^{2t} + 1} \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{e^{2t} + 1}{\sqrt{2}e^t} \frac{1}{(e^{2t} + 1)^2} \langle \sqrt{2}e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t} \rangle \\ &= \frac{1}{\sqrt{2}e^t(e^{2t} + 1)} \langle \sqrt{2}e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t} \rangle \\ &= \frac{1}{e^{2t} + 1} \langle 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t \rangle \end{aligned}$$

$$(b) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}e^t}{e^{2t} + 1} \cdot \frac{1}{e^t + e^{-t}} = \frac{\sqrt{2}e^t}{e^{3t} + 2e^t + e^{-t}} = \frac{\sqrt{2}e^{2t}}{e^{4t} + 2e^{2t} + 1} = \frac{\sqrt{2}e^{2t}}{(e^{2t} + 1)^2}.$$

$$16. (a) \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4t^2 + 4 + (1/t)^2}} \langle 2t, 2, 1/t \rangle = \frac{|t|}{2t^2 + 1} \langle 2t, 2, 1/t \rangle. \text{ But since the}$$

$$\mathbf{k}\text{-component is } \ln t, t \text{ is positive, } |t| = t \text{ and } \mathbf{T}(t) = \frac{1}{2t^2 + 1} \langle 2t^2, 2t, 1 \rangle. \text{ Then}$$

$$\mathbf{T}'(t) = \frac{1}{2t^2 + 1} \langle 4t, 2, 0 \rangle - (2t^2 + 1)^{-2} (4t) \langle 2t^2, 2t, 1 \rangle = \frac{1}{(2t^2 + 1)^2} \langle 4t, 2 - 4t^2, -4t \rangle, \text{ so}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle 4t, 2 - 4t^2, -4t \rangle}{\sqrt{(4t)^2 + (2 - 4t^2)^2 + (-4t)^2}} = \frac{1}{2t^2 + 1} \langle 2t, 1 - 2t^2, -2t \rangle.$$

$$(b) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{2}{2t^2 + 1} \left(\frac{t}{2t^2 + 1} \right) = \frac{2t}{(2t^2 + 1)^2}$$

$$17. \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{k}, \mathbf{r}''(t) = 2\mathbf{i}, |\mathbf{r}'(t)| = \sqrt{(2t)^2 + 0^2 + 1^2} = \sqrt{4t^2 + 1}, \mathbf{r}'(t) \times \mathbf{r}''(t) = 2\mathbf{j}, |\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2.$$

$$\text{Then } \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2}{(\sqrt{4t^2 + 1})^3} = \frac{2}{(4t^2 + 1)^{3/2}}.$$

$$18. \mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 2t\mathbf{k}, \mathbf{r}''(t) = 2\mathbf{k}, |\mathbf{r}'(t)| = \sqrt{1^2 + 1^2 + (2t)^2} = \sqrt{4t^2 + 2},$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\mathbf{i} - 2\mathbf{j}, |\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{2^2 + 2^2 + 0^2} = \sqrt{8} = 2\sqrt{2}. \text{ Then}$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{2}}{(\sqrt{4t^2 + 2})^3} = \frac{2\sqrt{2}}{(\sqrt{2}\sqrt{2t^2 + 1})^3} = \frac{1}{(2t^2 + 1)^{3/2}}.$$

$$19. \mathbf{r}'(t) = 3\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}, \mathbf{r}''(t) = -4\sin t\mathbf{j} - 4\cos t\mathbf{k},$$

$$|\mathbf{r}'(t)| = \sqrt{9 + 16\cos^2 t + 16\sin^2 t} = \sqrt{9 + 16} = 5, \mathbf{r}'(t) \times \mathbf{r}''(t) = -16\mathbf{i} + 12\cos t\mathbf{j} - 12\sin t\mathbf{k},$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{256 + 144\cos^2 t + 144\sin^2 t} = \sqrt{400} = 20. \text{ Then } \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{20}{5^3} = \frac{4}{25}.$$

$$20. \mathbf{r}'(t) = \langle e^t \cos t - e^t \sin t, e^t \cos t + e^t \sin t, 1 \rangle. \text{ The point } (1, 0, 0) \text{ corresponds to } t = 0, \text{ and}$$

$$\mathbf{r}'(0) = \langle 1, 1, 1 \rangle \Rightarrow |\mathbf{r}'(0)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

$$\mathbf{r}''(t) = \langle e^t \cos t - e^t \sin t - e^t \cos t - e^t \sin t, e^t \cos t - e^t \sin t + e^t \cos t + e^t \sin t, 0 \rangle$$

$$= \langle -2e^t \sin t, 2e^t \cos t, 0 \rangle \Rightarrow \mathbf{r}''(0) = \langle 0, 2, 0 \rangle.$$

$$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -2, 0, 2 \rangle. \quad |\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

$$\text{Then } \kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{2\sqrt{2}}{(\sqrt{3})^3} = \frac{2\sqrt{2}}{3\sqrt{3}} \text{ or } \frac{2\sqrt{6}}{9}.$$

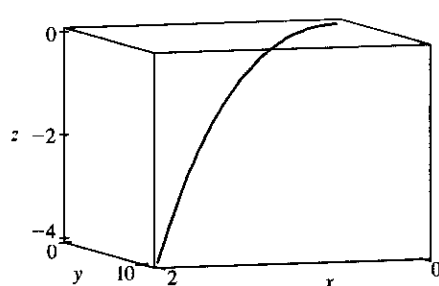
$$21. \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle. \text{ The point } (1, 1, 1) \text{ corresponds to } t = 1, \text{ and}$$

$$\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \Rightarrow |\mathbf{r}'(1)| = \sqrt{1 + 4 + 9} = \sqrt{14}. \quad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle \Rightarrow \mathbf{r}''(1) = \langle 0, 2, 6 \rangle.$$

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 6, -6, 2 \rangle, \text{ so } |\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{36 + 36 + 4} = \sqrt{76}. \text{ Then}$$

$$\kappa(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{\sqrt{76}}{\sqrt{14}^3} = \frac{1}{7} \sqrt{\frac{19}{14}}.$$

22.



$$\mathbf{r}(t) = \langle t, 4t^{3/2}, -t^2 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, 6t^{1/2}, -2t \rangle,$$

$$\mathbf{r}''(t) = \langle 0, 3t^{-1/2}, -2 \rangle, |\mathbf{r}'(t)|^3 = (1 + 36t + 4t^2)^{3/2},$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -12t^{1/2} + 6t^{1/2}, 2, 3t^{-1/2} \rangle \Rightarrow$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t + 4 + 9t^{-1}} = \left[\frac{36t^2 + 4t + 9}{t} \right]^{1/2}$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \left(\frac{36t^2 + 4t + 9}{t} \right)^{1/2} \frac{1}{(1 + 36t + 4t^2)^{3/2}} = \frac{\sqrt{36t^2 + 4t + 9}}{t^{1/2}(1 + 36t + 4t^2)^{3/2}}.$$

$$\text{The point } (1, 4, -1) \text{ corresponds to } t = 1, \text{ so the curvature at this point is } \kappa(1) = \frac{\sqrt{36 + 4 + 9}}{(1 + 36 + 4)^{3/2}} = \frac{7}{41\sqrt{41}}.$$

$$23. f(x) = x^3, f'(x) = 3x^2, f''(x) = 6x, \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{6|x|}{(1 + 9x^4)^{3/2}}$$

24. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|-\cos x|}{[1 + (-\sin x)^2]^{3/2}} = \frac{|\cos x|}{(1 + \sin^2 x)^{3/2}}$$

25. $f(x) = 4x^{5/2}$, $f'(x) = 10x^{3/2}$, $f''(x) = 15x^{1/2}$,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|15x^{1/2}|}{[1 + (10x^{3/2})^2]^{3/2}} = \frac{15\sqrt{x}}{(1 + 100x^3)^{3/2}}$$

26. $y' = \frac{1}{x}$, $y'' = -\frac{1}{x^2}$,

$$\kappa(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{\left|-\frac{1}{x^2}\right|}{\left[1 + \left(\frac{1}{x}\right)^2\right]^{3/2}} = \frac{1}{x^2} \frac{(x^2)^{3/2}}{(x^2 + 1)^{3/2}} = \frac{|x|}{(x^2 + 1)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}$$

(since $x > 0$). To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2 + 1)^{3/2} - x\left(\frac{3}{2}\right)(x^2 + 1)^{1/2}(2x)}{[(x^2 + 1)^{3/2}]^2} = \frac{(x^2 + 1)^{1/2}[(x^2 + 1) - 3x^2]}{(x^2 + 1)^3} = \frac{1 - 2x^2}{(x^2 + 1)^{5/2}};$$

$\kappa'(x) = 0 \Rightarrow 1 - 2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln \frac{1}{\sqrt{2}}\right)$. Since $\lim_{x \rightarrow \infty} \frac{x}{(x^2 + 1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \rightarrow \infty$.

27. Since $y' = y'' = e^x$, the curvature is $\kappa(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}} = e^x(1 + e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = e^x(1 + e^{2x})^{-3/2} + e^x\left(-\frac{3}{2}\right)(1 + e^{2x})^{-5/2}(2e^{2x}) = e^x \frac{1 + e^{2x} - 3e^{2x}}{(1 + e^{2x})^{5/2}} = e^x \frac{1 - 2e^{2x}}{(1 + e^{2x})^{5/2}}.$$

$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2} \ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2} \ln 2$ and $1 - 2e^{2x} < 0$ for $x > -\frac{1}{2} \ln 2$, the maximum curvature is attained at the point

$$\left(-\frac{1}{2} \ln 2, e^{(-\frac{1}{2} \ln 2)/2}\right) = \left(-\frac{1}{2} \ln 2, \frac{1}{\sqrt{2}}\right). \text{ Since } \lim_{x \rightarrow \infty} e^x(1 + e^{2x})^{-3/2} = 0, \kappa(x) \text{ approaches 0 as } x \rightarrow \infty.$$

28. We can take the parabola as having its vertex at the origin and opening upward, so the equation is

$$f(x) = ax^2, a > 0. \text{ Then by Equation 11, } \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|2a|}{[1 + (2ax)^2]^{3/2}} = \frac{2a}{(1 + 4a^2x^2)^{3/2}},$$

thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

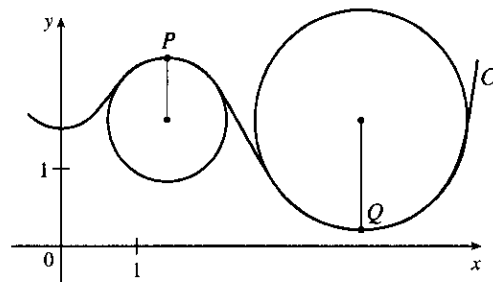
29. (a) C appears to be changing direction more quickly at P than Q , so we would expect the curvature to be greater at P .

(b) First we sketch approximate osculating circles at P and Q . Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units,

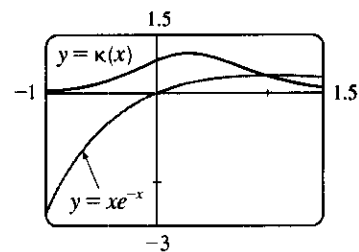
$$\text{thus } \rho = \frac{1}{\kappa} \Rightarrow \kappa = \frac{1}{\rho} \approx \frac{1}{0.8} \approx 1.3. \text{ Similarly, we}$$

estimate the radius of the osculating circle at Q to be

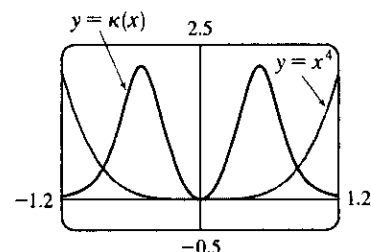
$$1.4 \text{ units, so } \kappa = \frac{1}{\rho} \approx \frac{1}{1.4} \approx 0.7.$$



30. $y = xe^{-x} \Rightarrow y' = e^{-x}(1-x), y'' = e^{-x}(x-2)$, and
- $$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{e^{-x}|x-2|}{[1 + e^{-2x}(1-x)^2]^{3/2}}.$$
- The graph of the curvature here is what we would expect. The graph of xe^{-x} is bending most sharply slightly to the right of the origin. As $x \rightarrow \infty$, the graph of xe^{-x} is asymptotic to the x -axis, and so the curvature approaches zero.

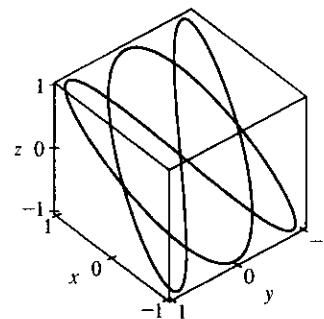


31. $y = x^4 \Rightarrow y' = 4x^3, y'' = 12x^2$, and
- $$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{12x^2}{(1 + 16x^6)^{3/2}}.$$
- The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^4$ is very flat around the origin, and so here the curvature is zero.



32. Notice that the curve a is highest for the same x -values at which curve b is turning more sharply, and a is 0 or near 0 where b is nearly straight. So, a must be the graph of $y = \kappa(x)$, and b is the graph of $y = f(x)$.
33. Notice that the curve b has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve a isn't near 0 there. Thus, a must be the graph of $y = f(x)$ rather than the graph of curvature, and b is the graph of $y = \kappa(x)$.

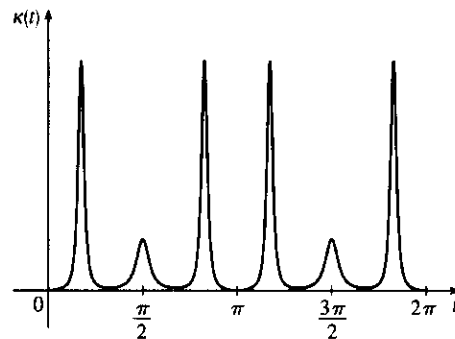
34. (a) The complete curve is given by $0 \leq t \leq 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



- (b) Using a CAS, we find (after simplifying)

$$\kappa(t) = \frac{3\sqrt{2}\sqrt{(5\sin t + \sin 5t)^2}}{(9\cos 6t + 2\cos 4t + 11)^{3/2}}. \quad (\text{To compute}$$

cross products in Maple, use the `Linalg` package and the `crossprod(a, b)` command; in Mathematica, use `Cross[a, b]`.) The graph shows 6 local (or absolute) maximum points for $0 \leq t \leq 2\pi$, as observed in part (a).



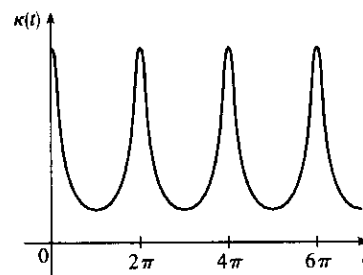
35. Using a CAS, we find (after simplifying)

$$\kappa(t) = \frac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}. \text{ (To compute cross}$$

products in Maple, use the Linalg package and the

crossprod(a, b) command; in Mathematica, use

Cross[a, b].) Curvature is largest at integer multiples of 2π .



36. Here $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$, $\mathbf{r}''(t) = \langle f''(t), g''(t) \rangle$,

$$|\mathbf{r}'(t)|^3 = \left[\sqrt{(f'(t))^2 + (g'(t))^2} \right]^3 = [(f'(t))^2 + (g'(t))^2]^{3/2} = (\dot{x}^2 + \dot{y}^2)^{3/2}, \text{ and}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, f'(t)g''(t) - f''(t)g'(t) \rangle| = [(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2]^{1/2} = |\dot{x}\ddot{y} - \ddot{x}\dot{y}|.$$

$$\text{Thus } \kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

37. $x = e^t \cos t \Rightarrow \dot{x} = e^t(\cos t - \sin t) \Rightarrow \ddot{x} = e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) = -2e^t \sin t$,
 $y = e^t \sin t \Rightarrow \dot{y} = e^t(\cos t + \sin t) \Rightarrow \ddot{y} = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t \cos t$. Then

$$\begin{aligned} \kappa(t) &= \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|e^t(\cos t - \sin t)(2e^t \cos t) - e^t(\cos t + \sin t)(-2e^t \sin t)|}{([e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2)^{3/2}} \\ &= \frac{|2e^{2t}(\cos^2 t - \sin t \cos t + \sin t \cos t + \sin^2 t)|}{[e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t)]^{3/2}} \\ &= \frac{|2e^{2t}(1)|}{[e^{2t}(1 + 1)]^{3/2}} = \frac{2e^{2t}}{e^{3t}(2)^{3/2}} = \frac{1}{\sqrt{2}e^t} \end{aligned}$$

38. $x = 1 + t^3 \Rightarrow \dot{x} = 3t^2 \Rightarrow \ddot{x} = 6t$, $y = t + t^2 \Rightarrow \dot{y} = 1 + 2t \Rightarrow \ddot{y} = 2$. Then

$$\begin{aligned} \kappa(t) &= \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(3t^2)(2) - (1 + 2t)(6t)|}{[(3t^2)^2 + (1 + 2t)^2]^{3/2}} = \frac{|-6t^2 - 6t|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}} \\ &= \frac{6|t^2 + t|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}} \end{aligned}$$

39. $(1, \frac{2}{3}, 1)$ corresponds to $t = 1$. $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \frac{\langle 2t, 2t^2, 1 \rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$.

$$\mathbf{T}'(t) = -4t(2t^2 + 1)^{-2} \langle 2t, 2t^2, 1 \rangle + (2t^2 + 1)^{-1} \langle 2, 4t, 0 \rangle \quad [\text{by Theorem 14.2.3 [ET 13.2.3] \#3}]$$

$$= (2t^2 + 1)^{-2} \langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t \rangle = 2(2t^2 + 1)^{-2} \langle 1 - 2t^2, 2t, -2t \rangle$$

$$\begin{aligned} \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2 + 1)^{-2} \langle 1 - 2t^2, 2t, -2t \rangle}{2(2t^2 + 1)^{-2} \sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\langle 1 - 2t^2, 2t, -2t \rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} \\ &= \frac{\langle 1 - 2t^2, 2t, -2t \rangle}{1 + 2t^2} \end{aligned}$$

$$\mathbf{N}(1) = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle \text{ and } \mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \langle -\frac{4}{9} - \frac{2}{9}, -(-\frac{4}{9} + \frac{1}{9}), \frac{4}{9} + \frac{2}{9} \rangle = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle.$$

40. $(1, 0, 1)$ corresponds to $t = 0$. $\mathbf{r}(t) = e^t \langle 1, \sin t, \cos t \rangle$, so

$$\mathbf{r}'(t) = e^t \langle 1, \sin t, \cos t \rangle + e^t \langle 0, \cos t, -\sin t \rangle = e^t \langle 1, \sin t + \cos t, \cos t - \sin t \rangle \text{ and}$$

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{e^t \langle 1, \sin t + \cos t, \cos t - \sin t \rangle}{e^t \sqrt{1 + \sin^2 t + 2 \sin t \cos t + \cos^2 t + \cos^2 t - 2 \sin t \cos t + \sin^2 t}} \\ &= \frac{\langle 1, \sin t + \cos t, \cos t - \sin t \rangle}{\sqrt{3}}, \end{aligned}$$

$$\mathbf{T}(0) = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle. \quad \mathbf{T}'(t) = \frac{1}{\sqrt{3}} \langle 0, \cos t - \sin t, -\sin t - \cos t \rangle, \text{ so}$$

$$\begin{aligned} \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{1}{\sqrt{3}} \langle 0, \cos t - \sin t, -\sin t - \cos t \rangle}{\frac{1}{\sqrt{3}} \sqrt{0^2 + \cos^2 t - 2 \cos t \sin t + \sin^2 t + \sin^2 t + 2 \sin t \cos t + \cos^2 t}} \\ &= \frac{1}{\sqrt{2}} \langle 0, \cos t - \sin t, -\sin t - \cos t \rangle. \end{aligned}$$

$$\mathbf{N}(0) = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \text{ and } \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle.$$

41. $(0, \pi, -2)$ corresponds to $t = \pi$. $\mathbf{r}(t) = \langle 2 \sin 3t, t, 2 \cos 3t \rangle \Rightarrow$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 6 \cos 3t, 1, -6 \sin 3t \rangle}{\sqrt{36 \cos^2 3t + 1 + 36 \sin^2 3t}} = \frac{1}{\sqrt{37}} \langle 6 \cos 3t, 1, -6 \sin 3t \rangle.$$

$\mathbf{T}(\pi) = \frac{1}{\sqrt{37}} \langle -6, 1, 0 \rangle$ is a normal vector for the normal plane, and so $\langle -6, 1, 0 \rangle$ is also normal. Thus an equation for the plane is $-6(x - 0) + 1(y - \pi) + 0(z + 2) = 0$ or $y - 6x = \pi$.

$$\mathbf{T}'(t) = \frac{1}{\sqrt{37}} \langle -18 \sin 3t, 0, -18 \cos 3t \rangle \Rightarrow |\mathbf{T}'(t)| = \frac{\sqrt{18^2 \sin^2 3t + 18^2 \cos^2 3t}}{\sqrt{37}} = \frac{18}{\sqrt{37}} \Rightarrow$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin 3t, 0, -\cos 3t \rangle. \text{ So } \mathbf{N}(\pi) = \langle 0, 0, 1 \rangle \text{ and}$$

$\mathbf{B}(\pi) = \frac{1}{\sqrt{37}} \langle -6, 1, 0 \rangle \times \langle 0, 0, 1 \rangle = \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$. Since $\mathbf{B}(\pi)$ is a normal to the osculating plane, so is $\langle 1, 6, 0 \rangle$ and an equation for the plane is $1(x - 0) + 6(y - \pi) + 0(z + 2) = 0$ or $x + 6y = 6\pi$.

42. $t = 1$ at $(1, 1, 1)$. $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle$ is normal to the normal plane, so an equation for this plane is $1(x - 1) + 2(y - 1) + 3(z - 1) = 0$, or $x + 2y + 3z = 6$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \langle 1, 2t, 3t^2 \rangle. \text{ Using the product rule on each term of } \mathbf{T}(t) \text{ gives}$$

$$\begin{aligned} \mathbf{T}'(t) &= \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \left\langle -\frac{1}{2}(8t + 36t^3), 2(1 + 4t^2 + 9t^4) - \frac{1}{2}(8t + 36t^3)2t, \right. \\ &\quad \left. 6t(1 + 4t^2 + 9t^4) - \frac{1}{2}(8t + 36t^3)3t^2 \right\rangle \\ &= \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \langle -4t - 18t^3, 2 - 18t^4, 6t + 12t^3 \rangle = \frac{-2}{(14)^{3/2}} \langle 11, 8, -9 \rangle \text{ when } t = 1. \end{aligned}$$

$\mathbf{N}(1) \parallel \mathbf{T}'(1) \parallel \langle 11, 8, -9 \rangle$ and $\mathbf{T}(1) \parallel \mathbf{r}'(1) = \langle 1, 2, 3 \rangle \Rightarrow$ a normal vector to the osculating plane is $\langle 11, 8, -9 \rangle \times \langle 1, 2, 3 \rangle = \langle 42, -42, 14 \rangle$ or equivalently $\langle 3, -3, 1 \rangle$. An equation for the plane is $3(x - 1) - 3(y - 1) + (z - 1) = 0$ or $3x - 3y + z = 1$.

43. The ellipse is given by the parametric equations $x = 2 \cos t$, $y = 3 \sin t$, so using the result from Exercise 36,

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(-2 \sin t)(-3 \sin t) - (3 \cos t)(-2 \cos t)|}{(4 \sin^2 t + 9 \cos^2 t)^{3/2}} = \frac{6}{(4 \sin^2 t + 9 \cos^2 t)^{3/2}}$$

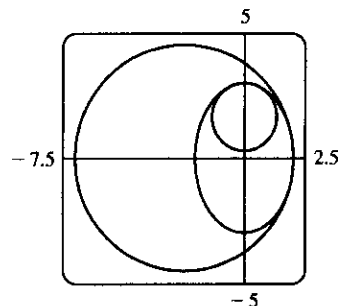
At $(2, 0)$, $t = 0$. Now $\kappa(0) = \frac{6}{27} = \frac{2}{9}$, so the radius of the

osculating circle is $1/\kappa(0) = \frac{9}{2}$ and its center is $(-\frac{5}{2}, 0)$. Its

equation is therefore $(x + \frac{5}{2})^2 + y^2 = \frac{81}{4}$. At $(0, 3)$, $t = \frac{\pi}{2}$, and

$\kappa(\frac{\pi}{2}) = \frac{6}{8} = \frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its

center is $(0, \frac{5}{3})$. Hence its equation is $x^2 + (y - \frac{5}{3})^2 = \frac{16}{9}$.



44. $y = \frac{1}{2}x^2 \Rightarrow y' = x$ and $y'' = 1$, so Formula 11 gives $\kappa(x) = \frac{1}{(1+x^2)^{3/2}}$. So the curvature at $(0, 0)$ is

$\kappa(0) = 1$ and the osculating circle has radius 1 and center $(0, 1)$, and hence equation $x^2 + (y - 1)^2 = 1$.

The curvature at $(1, \frac{1}{2})$ is $\kappa(1) = \frac{1}{(1+1^2)^{3/2}} = \frac{1}{2\sqrt{2}}$. The tangent line to the parabola at $(1, \frac{1}{2})$ has slope 1,

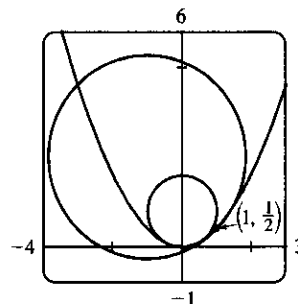
so the normal line has slope -1 . Thus the center of the osculating

circle lies in the direction of the unit vector $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. The

circle has radius $2\sqrt{2}$, so its center has position vector

$\langle 1, \frac{1}{2} \rangle + 2\sqrt{2} \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \langle -1, \frac{5}{2} \rangle$. So the equation of the

circle is $(x + 1)^2 + (y - \frac{5}{2})^2 = 8$.



45. The tangent vector is normal to the normal plane, and the vector $\langle 6, 6, -8 \rangle$ is normal to the given plane. But

$\mathbf{T}(t) \parallel \mathbf{r}'(t)$ and $\langle 6, 6, -8 \rangle \parallel \langle 3, 3, -4 \rangle$, so we need to find t such that $\mathbf{r}'(t) \parallel \langle 3, 3, -4 \rangle$. $\mathbf{r}(t) = \langle t^3, 3t, t^4 \rangle \Rightarrow$

$\mathbf{r}'(t) = \langle 3t^2, 3, 4t^3 \rangle \parallel \langle 3, 3, -4 \rangle$ when $t = -1$. So the planes are parallel at the point $\mathbf{r}(-1) = (-1, -3, 1)$.

46. To find the osculating plane, we first calculate the tangent and normal vectors.

In Maple, we set $x := t^3$; $y := 3t$; and $z := t^4$; and then calculate the components of the tangent vector

$\mathbf{T}(t)$ using the `diff` command. We find that $\mathbf{T}(t) = \frac{\langle 3t^2, 3, 4t^3 \rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. Differentiating the components of $\mathbf{T}(t)$,

we find that $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle -6t(8t^6 - 9), 3(48t^5 + 18t^3), 36t^2(t^4 + 3) \rangle}{\sqrt{144t(8t^6 - 9)^2 + 9(96t^5 + 36t^3)^2 + 5,184t^{12} + 31,104t^8 + 46,656t^4}}$.

In Maple, we can calculate $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ using the `linalg` package. First we define $\mathbf{T}$

and $\mathbf{N}$ using $\mathbf{T} := \text{array}([f, g, h])$; and $\mathbf{N} := \text{array}([F, G, H])$; where f, g, h, F, G , and H are the

components of $\mathbf{T}$ and $\mathbf{N}$. Then we use the command $\mathbf{B} := \text{crossprod}(\mathbf{T}, \mathbf{N})$. After normalization and

simplification, we find that $\mathbf{B}(t) = b \langle 6t, -2t^3, -3 \rangle$, where

$$b = \frac{t \sqrt{16t^6 + 9t^4 + 9}}{\sqrt{16t^2(8t^6 - 9)^2 + (96t^5 + 36t^3)^2 + 576t^{12} + 3456t^8 + 5184t^4}}$$

In Mathematica, we use the command `Dt` to differentiate the components of $\mathbf{r}(t)$ and subsequently $\mathbf{T}(t)$, and then load the vector analysis package with the command `<<Calculus`VectorAnalysis``. After setting

$\mathbf{T} = \{f, g, h\}$ and $\mathbf{N} = \{F, G, H\}$, we use `CrossProduct[T, N]` to find $\mathbf{B}$ (before normalization).

Now $\mathbf{B}(t)$ is parallel to $\langle 6t, -2t^3, -3 \rangle$, so if $\mathbf{B}(t)$ is parallel to $\langle 1, 1, 1 \rangle$ for some t , then $6t = 1 \Rightarrow t = \frac{1}{6}$,

but $-2(\frac{1}{6})^3 \neq 1$. So there is no such osculating plane.

$$47. \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right| = \frac{|d\mathbf{T}/dt|}{|ds/dt|} \text{ and } \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \text{ so } \kappa\mathbf{N} = \frac{\left| \frac{d\mathbf{T}}{dt} \right| \frac{d\mathbf{T}}{dt}}{\left| \frac{d\mathbf{T}}{dt} \right| \frac{ds}{dt}} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{d\mathbf{T}}{ds} \text{ by the Chain Rule.}$$

48. For a plane curve, $\mathbf{T} = |\mathbf{T}| \cos \phi \mathbf{i} + |\mathbf{T}| \sin \phi \mathbf{j} = \cos \phi \mathbf{i} + \sin \phi \mathbf{j}$. Then

$$\frac{d\mathbf{T}}{ds} = \left(\frac{d\mathbf{T}}{d\phi} \right) \left(\frac{d\phi}{ds} \right) = (-\sin \phi \mathbf{i} + \cos \phi \mathbf{j}) \left(\frac{d\phi}{ds} \right) \text{ and } \left| \frac{d\mathbf{T}}{ds} \right| = |-\sin \phi \mathbf{i} + \cos \phi \mathbf{j}| \left| \frac{d\phi}{ds} \right| = \left| \frac{d\phi}{ds} \right|. \text{ Hence for a plane curve, the curvature is } \kappa = |d\phi/ds|.$$

$$49. (a) |\mathbf{B}| = 1 \Rightarrow \mathbf{B} \cdot \mathbf{B} = 1 \Rightarrow \frac{d}{ds}(\mathbf{B} \cdot \mathbf{B}) = 0 \Rightarrow 2 \frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0 \Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{B}$$

$$(b) \mathbf{B} = \mathbf{T} \times \mathbf{N} \Rightarrow$$

$$\begin{aligned} \frac{d\mathbf{B}}{ds} &= \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \frac{d}{dt}(\mathbf{T} \times \mathbf{N}) \frac{1}{ds/dt} = \frac{d}{dt}(\mathbf{T} \times \mathbf{N}) \frac{1}{|\mathbf{r}'(t)|} \\ &= [(\mathbf{T}' \times \mathbf{N}) + (\mathbf{T} \times \mathbf{N}')] \frac{1}{|\mathbf{r}'(t)|} = \left[\left(\mathbf{T}' \times \frac{\mathbf{T}'}{|\mathbf{T}'|} \right) + (\mathbf{T} \times \mathbf{N}') \right] \frac{1}{|\mathbf{r}'(t)|} = \frac{\mathbf{T} \times \mathbf{N}'}{|\mathbf{r}'(t)|} \\ &\Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{T} \end{aligned}$$

(c) $\mathbf{B} = \mathbf{T} \times \mathbf{N} \Rightarrow \mathbf{T} \perp \mathbf{N}, \mathbf{B} \perp \mathbf{T}$ and $\mathbf{B} \perp \mathbf{N}$. So $\mathbf{B}, \mathbf{T}$ and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Therefore, $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$, where $\tau(s)$ is a scalar.

(d) Since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, $\mathbf{T} \perp \mathbf{N}$ and both $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is a unit vector mutually perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. For a plane curve, $\mathbf{T}$ and $\mathbf{N}$ always lie in the plane of the curve, so that $\mathbf{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\mathbf{B}/ds = 0$, but $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ and $\mathbf{N} \neq \mathbf{0}$, so $\tau(s) = 0$.

$$50. \mathbf{N} = \mathbf{B} \times \mathbf{T} \Rightarrow$$

$$\begin{aligned} \frac{d\mathbf{N}}{ds} &= \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} && [\text{by Theorem 14.2.3 [ET 13.2.3] \#5}] \\ &= -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa\mathbf{N} && [\text{by Formulas 3 and 1}] \\ &= -\tau(\mathbf{N} \times \mathbf{T}) + \kappa(\mathbf{B} \times \mathbf{N}) && [\text{by Theorem 13.4.8 [ET 12.4.8] \#2}] \end{aligned}$$

$$\text{But } \mathbf{B} \times \mathbf{N} = \mathbf{B} \times (\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \cdot \mathbf{T})\mathbf{B} - (\mathbf{B} \cdot \mathbf{B})\mathbf{T} \quad [\text{by Theorem 13.4.8 [ET 12.4.8] \#6}] = -\mathbf{T} \Rightarrow$$

$$d\mathbf{N}/ds = \tau(\mathbf{T} \times \mathbf{N}) - \kappa\mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

51. (a) $\mathbf{r}' = s' \mathbf{T} \Rightarrow \mathbf{r}'' = s'' \mathbf{T} + s' \mathbf{T}' = s'' \mathbf{T} + s' \frac{d\mathbf{T}}{ds} s' = s'' \mathbf{T} + \kappa(s')^2 \mathbf{N}$ by the first Serret-Frenet formula.

(b) Using part (a), we have

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= (s' \mathbf{T}) \times [s'' \mathbf{T} + \kappa(s')^2 \mathbf{N}] \\ &= [(s' \mathbf{T}) \times (s'' \mathbf{T})] + [(s' \mathbf{T}) \times (\kappa(s')^2 \mathbf{N})] \quad [\text{by Theorem 13.4.8 [ET 12.4.8] \#3}] \\ &= (s' s'')(\mathbf{T} \times \mathbf{T}) + \kappa(s')^3 (\mathbf{T} \times \mathbf{N}) = \mathbf{0} + \kappa(s')^3 \mathbf{B} = \kappa(s')^3 \mathbf{B}\end{aligned}$$

(c) Using part (a), we have

$$\begin{aligned}\mathbf{r}''' &= [s'' \mathbf{T} + \kappa(s')^2 \mathbf{N}]' = s''' \mathbf{T} + s'' \mathbf{T}' + \kappa'(s')^2 \mathbf{N} + 2\kappa s' s'' \mathbf{N} + \kappa(s')^2 \mathbf{N}' \\ &= s''' \mathbf{T} + s'' \frac{d\mathbf{T}}{ds} s' + \kappa'(s')^2 \mathbf{N} + 2\kappa s' s'' \mathbf{N} + \kappa(s')^2 \frac{d\mathbf{N}}{ds} s' \\ &= s''' \mathbf{T} + s'' s' \kappa \mathbf{N} + \kappa'(s')^2 \mathbf{N} + 2\kappa s' s'' \mathbf{N} + \kappa(s')^3 (-\kappa \mathbf{T} + \tau \mathbf{B}) \quad [\text{by the second formula}] \\ &= [s''' - \kappa^2(s')^3] \mathbf{T} + [3\kappa s' s'' + \kappa'(s')^2] \mathbf{N} + \kappa \tau (s')^3 \mathbf{B}\end{aligned}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B} \cdot \mathbf{T} = 0$, $\mathbf{B} \cdot \mathbf{N} = 0$, and $\mathbf{B} \cdot \mathbf{B} = 1$, we get

$$\begin{aligned}\frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} &= \frac{\kappa(s')^3 \mathbf{B} \cdot \{[s''' - \kappa^2(s')^3] \mathbf{T} + [3\kappa s' s'' + \kappa'(s')^2] \mathbf{N} + \kappa \tau (s')^3 \mathbf{B}\}}{[\kappa(s')^3 \mathbf{B}]^2} \\ &= \frac{\kappa(s')^3 \kappa \tau (s')^3}{[\kappa(s')^3]^2} = \tau\end{aligned}$$

52. First we find the quantities required to compute κ :

$$\mathbf{r}'(t) = \langle -a \sin t, a \cos t, b \rangle \Rightarrow \mathbf{r}''(t) = \langle -a \cos t, -a \sin t, 0 \rangle \Rightarrow \mathbf{r}'''(t) = \langle a \sin t, -a \cos t, 0 \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(-a \sin t)^2 + (a \cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = ab \sin t \mathbf{i} - ab \cos t \mathbf{j} + a^2 \mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(ab \sin t)^2 + (-ab \cos t)^2 + (a^2)^2} = \sqrt{a^2 b^2 + a^4}$$

$$(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = (ab \sin t)(a \sin t) + (-ab \cos t)(-a \cos t) + (a^2)(0) = a^2 b$$

Then by Theorem 10,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{a^2 b^2 + a^4}}{(\sqrt{a^2 + b^2})^3} = \frac{a \sqrt{a^2 + b^2}}{(\sqrt{a^2 + b^2})^3} = \frac{a}{a^2 + b^2}$$

which is a constant.

From Exercise 51(d), the torsion τ is given by

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{a^2 b}{(\sqrt{a^2 b^2 + a^4})^2} = \frac{b}{a^2 + b^2}$$

which is also a constant.

$$53. \mathbf{r} = \langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle \Rightarrow \mathbf{r}' = \langle 1, t, t^2 \rangle, \mathbf{r}'' = \langle 0, 1, 2t \rangle, \mathbf{r}''' = \langle 0, 0, 2 \rangle \Rightarrow \mathbf{r}' \times \mathbf{r}'' = \langle t^2, -2t, 1 \rangle \Rightarrow$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle t^2, -2t, 1 \rangle \cdot \langle 0, 0, 2 \rangle}{t^4 + 4t^2 + 1} = \frac{2}{t^4 + 4t^2 + 1}$$

$$54. \mathbf{r} = \langle \sinh t, \cosh t, t \rangle \Rightarrow \mathbf{r}' = \langle \cosh t, \sinh t, 1 \rangle, \mathbf{r}'' = \langle \sinh t, \cosh t, 0 \rangle, \mathbf{r}''' = \langle \cosh t, \sinh t, 0 \rangle \Rightarrow$$

$$\mathbf{r}' \times \mathbf{r}'' = \langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t \rangle = \langle -\cosh t, \sinh t, 1 \rangle \Rightarrow$$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle -\cosh t, \sinh t, 1 \rangle|}{|\langle \cosh t, \sinh t, 1 \rangle|^3} = \frac{\sqrt{\cosh^2 t + \sinh^2 t + 1}}{(\cosh^2 t + \sinh^2 t + 1)^{3/2}} = \frac{1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2 \cosh^2 t},$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle -\cosh t, \sinh t, 1 \rangle \cdot \langle \cosh t, \sinh t, 0 \rangle}{\cosh^2 t + \sinh^2 t + 1} = \frac{-\cosh^2 t + \sinh^2 t}{2 \cosh^2 t} = \frac{-1}{2 \cosh^2 t}$$

So at the point $(0, 1, 0)$, $t = 0$, and $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{2}$.

55. For one helix, the vector equation is $\mathbf{r}(t) = \langle 10 \cos t, 10 \sin t, 34t/(2\pi) \rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t . Using the arc length formula, letting t go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$L = \int_0^{2.9 \times 10^8 \times 2\pi} |\mathbf{r}'(t)| dt = \int_0^{2.9 \times 10^8 \times 2\pi} \sqrt{(-10 \sin t)^2 + (10 \cos t)^2 + \left(\frac{34}{2\pi}\right)^2} dt$$

$$= \sqrt{100 + \left(\frac{34}{2\pi}\right)^2} \int_0^{2.9 \times 10^8 \times 2\pi} dt = 2.9 \times 10^8 \times 2\pi \sqrt{100 + \left(\frac{34}{2\pi}\right)^2}$$

$$\approx 2.07 \times 10^{10} \text{ \AA} \text{ — more than two meters!}$$

56. (a) For the function $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ to be continuous, we must have $P(0) = 0$ and $P(1) = 1$.

For F' to be continuous, we must have $P'(0) = P'(1) = 0$. The curvature of the curve $y = F(x)$ at the point $(x, F(x))$ is $\kappa(x) = \frac{|F''(x)|}{(1 + [F'(x)]^2)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0) = P''(1) = 0$.

Write $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Then $P'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ and $P''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$. Our six conditions are:

$$P(0) = 0 \Rightarrow f = 0 \quad (1)$$

$$P(1) = 1 \Rightarrow a + b + c + d + e + f = 1 \quad (2)$$

$$P'(0) = 0 \Rightarrow e = 0 \quad (3)$$

$$P'(1) = 0 \Rightarrow 5a + 4b + 3c + 2d + e = 0 \quad (4)$$

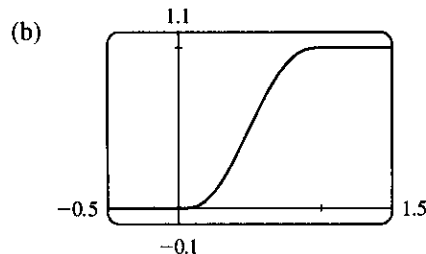
$$P''(0) = 0 \Rightarrow d = 0 \quad (5)$$

$$P''(1) = 0 \Rightarrow 20a + 12b + 6c + 2d = 0 \quad (6)$$

From (1), (3), and (5), we have $d = e = f = 0$. Thus (2), (4) and (6) become (7) $a + b + c = 1$,

(8) $5a + 4b + 3c = 0$, and (9) $10a + 6b + 3c = 0$. Subtracting (8) from (9) gives (10) $5a + 2b = 0$.

Multiplying (7) by 3 and subtracting from (8) gives (11) $2a + b = -3$. Multiplying (11) by 2 and subtracting from (10) gives $a = 6$. By (10), $b = -15$. By (7), $c = 10$. Thus, $P(x) = 6x^5 - 15x^4 + 10x^3$.



14.4 Motion in Space: Velocity and Acceleration

ET 13.4

1. (a) If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is the position vector of the particle at time t , then the average velocity over the time interval $[0, 1]$ is

$$\mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \frac{(4.5\mathbf{i} + 6.0\mathbf{j} + 3.0\mathbf{k}) - (2.7\mathbf{i} + 9.8\mathbf{j} + 3.7\mathbf{k})}{1} = 1.8\mathbf{i} - 3.8\mathbf{j} - 0.7\mathbf{k}.$$

Similarly, over the other intervals we have

$$\begin{aligned} [0.5, 1]: \quad \mathbf{v}_{\text{ave}} &= \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\mathbf{i} + 6.0\mathbf{j} + 3.0\mathbf{k}) - (3.5\mathbf{i} + 7.2\mathbf{j} + 3.3\mathbf{k})}{0.5} \\ &= 2.0\mathbf{i} - 2.4\mathbf{j} - 0.6\mathbf{k} \end{aligned}$$

$$\begin{aligned} [1, 2]: \quad \mathbf{v}_{\text{ave}} &= \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\mathbf{i} + 7.8\mathbf{j} + 2.7\mathbf{k}) - (4.5\mathbf{i} + 6.0\mathbf{j} + 3.0\mathbf{k})}{1} \\ &= 2.8\mathbf{i} + 1.8\mathbf{j} - 0.3\mathbf{k} \end{aligned}$$

$$\begin{aligned} [1, 1.5]: \quad \mathbf{v}_{\text{ave}} &= \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\mathbf{i} + 6.4\mathbf{j} + 2.8\mathbf{k}) - (4.5\mathbf{i} + 6.0\mathbf{j} + 3.0\mathbf{k})}{0.5} \\ &= 2.8\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k} \end{aligned}$$

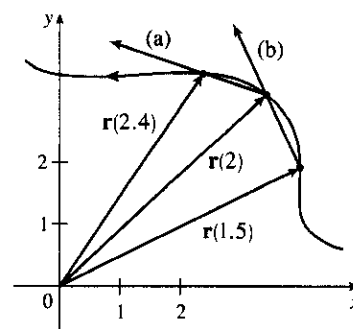
- (b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals $[0.5, 1]$ and $[1, 1.5]$: $\mathbf{v}(1) \approx \frac{1}{2}[(2\mathbf{i} - 2.4\mathbf{j} - 0.6\mathbf{k}) + (2.8\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k})] = 2.4\mathbf{i} - 0.8\mathbf{j} - 0.5\mathbf{k}$. Then the speed is $|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$.

2. (a) The average velocity over $2 \leq t \leq 2.4$ is

$$\frac{\mathbf{r}(2.4) - \mathbf{r}(2)}{2.4 - 2} = 2.5 [\mathbf{r}(2.4) - \mathbf{r}(2)], \text{ so we sketch a vector}$$

in the same direction but 2.5 times the length of

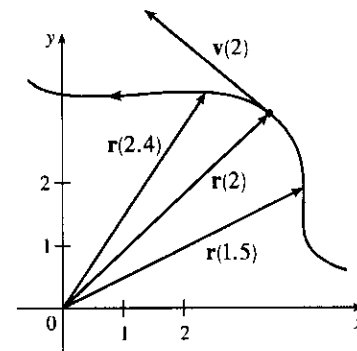
$$[\mathbf{r}(2.4) - \mathbf{r}(2)].$$



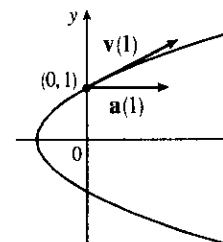
(b) The average velocity over $1.5 \leq t \leq 2$ is $\frac{\mathbf{r}(2) - \mathbf{r}(1.5)}{2 - 1.5} = 2[\mathbf{r}(2) - \mathbf{r}(1.5)]$, so we sketch a vector in the same direction but twice the length of $[\mathbf{r}(2) - \mathbf{r}(1.5)]$.

(c) Using Equation 2 we have $\mathbf{v}(2) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(2+h) - \mathbf{r}(2)}{h}$.

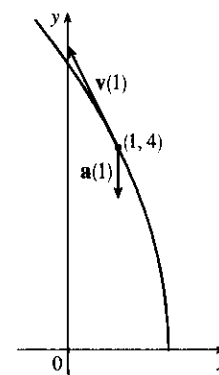
(d) $\mathbf{v}(2)$ is tangent to the curve at $\mathbf{r}(2)$ and points in the direction of increasing t . Its length is the speed of the particle at $t = 2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2 \leq t \leq 2.4$ and $1.5 \leq t \leq 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7. Thus, we estimate the speed at $t = 2$ to be $|\mathbf{v}(2)| \approx \frac{1}{2}(2.8 + 2.7) = 2.75$ and we draw the velocity vector $\mathbf{v}(2)$ with this length.



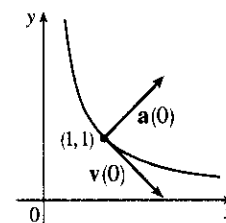
$$\begin{aligned} 3. \mathbf{r}(t) &= \langle t^2 - 1, t \rangle \Rightarrow & \text{At } t = 1: \\ \mathbf{v}(t) &= \mathbf{r}'(t) = \langle 2t, 1 \rangle, & \mathbf{v}(1) = \langle 2, 1 \rangle \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \langle 2, 0 \rangle, & \mathbf{a}(1) = \langle 2, 0 \rangle \\ |\mathbf{v}(t)| &= \sqrt{4t^2 + 1} \end{aligned}$$



$$\begin{aligned} 4. \mathbf{r}(t) &= \langle 2 - t, 4\sqrt{t} \rangle \Rightarrow & \text{At } t = 1: \\ \mathbf{v}(t) &= \mathbf{r}'(t) = \langle -1, 2/\sqrt{t} \rangle, & \mathbf{v}(1) = \langle -1, 2 \rangle \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \langle 0, -1/t^{3/2} \rangle, & \mathbf{a}(1) = \langle 0, -1 \rangle \\ |\mathbf{v}(t)| &= \sqrt{1 + 4/t} \end{aligned}$$



$$\begin{aligned} 5. \mathbf{r}(t) &= e^t \mathbf{i} + e^{-t} \mathbf{j} \Rightarrow & \text{At } t = 0: \\ \mathbf{v}(t) &= e^t \mathbf{i} - e^{-t} \mathbf{j}, & \mathbf{v}(0) = \mathbf{i} - \mathbf{j}, \\ \mathbf{a}(t) &= e^t \mathbf{i} + e^{-t} \mathbf{j} & \mathbf{a}(0) = \mathbf{i} + \mathbf{j} \\ |\mathbf{v}(t)| &= \sqrt{e^{2t} + e^{-2t}} = e^{-t} \sqrt{e^{4t} + 1} \end{aligned}$$



Since $x = e^t$, $t = \ln x$ and $y = e^{-t} = e^{-\ln x} = 1/x$, and $x > 0$, $y > 0$.

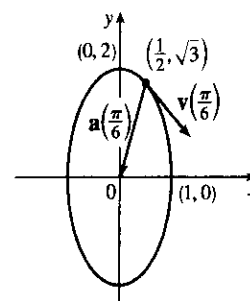
$$6. \mathbf{r}(t) = \sin t \mathbf{i} + 2 \cos t \mathbf{j} \Rightarrow$$

$$\mathbf{v}(t) = \cos t \mathbf{i} - 2 \sin t \mathbf{j}, \mathbf{v}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \mathbf{i} - \mathbf{j}$$

$$\mathbf{a}(t) = -\sin t \mathbf{i} - 2 \cos t \mathbf{j}, \mathbf{a}\left(\frac{\pi}{6}\right) = -\frac{1}{2} \mathbf{i} - \sqrt{3} \mathbf{j}$$

$$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 4 \sin^2 t} = \sqrt{1 + 3 \sin^2 t}$$

And $x^2 + y^2/4 = \sin^2 t + \cos^2 t = 1$, an ellipse.



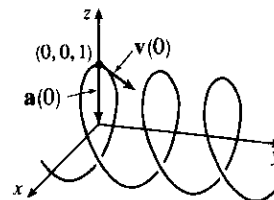
$$7. \mathbf{r}(t) = \sin t \mathbf{i} + t \mathbf{j} + \cos t \mathbf{k} \Rightarrow$$

$$\mathbf{v}(t) = \cos t \mathbf{i} + \mathbf{j} - \sin t \mathbf{k}, \mathbf{v}(0) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(t) = -\sin t \mathbf{i} - \cos t \mathbf{k}, \mathbf{a}(0) = -\mathbf{k}$$

$$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 1 + \sin^2 t} = \sqrt{2}$$

Since $x^2 + z^2 = 1$, $y = t$, the path of the particle is a helix about the y -axis.



$$8. \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k} \Rightarrow$$

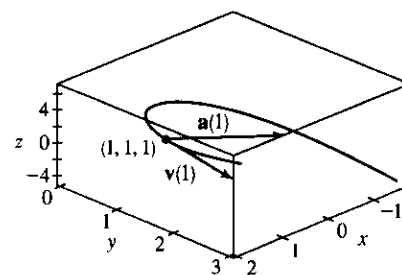
$$\mathbf{v}(t) = \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}, \mathbf{v}(1) = \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k}$$

$$\mathbf{a}(t) = 2 \mathbf{j} + 6t \mathbf{k}, \mathbf{a}(1) = 2 \mathbf{j} + 6 \mathbf{k}$$

$$|\mathbf{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

The path is a "twisted cubic"

(see Example 14.1.7 [ET 13.1.7]).



$$9. \mathbf{r}(t) = \langle t^2 + 1, t^3, t^2 - 1 \rangle \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 3t^2, 2t \rangle, \mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 2 \rangle,$$

$$|\mathbf{v}(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 8t^2} = |t| \sqrt{9t^2 + 8}.$$

$$10. \mathbf{r}(t) = \langle 2 \cos t, 3t, 2 \sin t \rangle \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2 \sin t, 3, 2 \cos t \rangle, \mathbf{a}(t) = \mathbf{v}'(t) = \langle -2 \cos t, 0, -2 \sin t \rangle,$$

$$|\mathbf{v}(t)| = \sqrt{4 \sin^2 t + 9 + 4 \cos^2 t} = \sqrt{13}.$$

$$11. \mathbf{r}(t) = \sqrt{2} t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2} \mathbf{i} + e^t \mathbf{j} - e^{-t} \mathbf{k}, \mathbf{a}(t) = \mathbf{v}'(t) = e^t \mathbf{j} + e^{-t} \mathbf{k},$$

$$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$$

$$12. \mathbf{r}(t) = t^2 \mathbf{i} + \ln t \mathbf{j} + t \mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = 2t \mathbf{i} + t^{-1} \mathbf{j} + \mathbf{k}, \mathbf{a}(t) = \mathbf{v}'(t) = 2 \mathbf{i} - t^{-2} \mathbf{j},$$

$$|\mathbf{v}(t)| = \sqrt{4t^2 + t^{-2} + 1}.$$

$$13. \mathbf{r}(t) = e^t \langle \cos t, \sin t, t \rangle \Rightarrow$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = e^t \langle \cos t, \sin t, t \rangle + e^t \langle -\sin t, \cos t, 1 \rangle = e^t \langle \cos t - \sin t, \sin t + \cos t, t + 1 \rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = e^t \langle \cos t - \sin t - \sin t - \cos t, \sin t + \cos t + \cos t - \sin t, t + 1 + 1 \rangle$$

$$= e^t \langle -2 \sin t, 2 \cos t, t + 2 \rangle$$

$$|\mathbf{v}(t)| = e^t \sqrt{\cos^2 t + \sin^2 t - 2 \cos t \sin t + \sin^2 t + \cos^2 t + 2 \sin t \cos t + t^2 + 2t + 1}$$

$$= e^t \sqrt{t^2 + 2t + 3}$$

$$14. \mathbf{r}(t) = t \sin t \mathbf{i} + t \cos t \mathbf{j} + t^2 \mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = (\sin t + t \cos t) \mathbf{i} + (\cos t - t \sin t) \mathbf{j} + 2t \mathbf{k},$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = (2 \cos t - t \sin t) \mathbf{i} + (-2 \sin t - t \cos t) \mathbf{j} + 2 \mathbf{k},$$

$$|\mathbf{v}(t)| = \sqrt{(\sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t) + (\cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t) + 4t^2} = \sqrt{5t^2 + 1}.$$

15. $\mathbf{a}(t) = \mathbf{k} \Rightarrow \mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \mathbf{k} dt = t\mathbf{k} + \mathbf{c}_1$ and $\mathbf{i} - \mathbf{j} = \mathbf{v}(0) = 0\mathbf{k} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + t\mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int (\mathbf{i} - \mathbf{j} + t\mathbf{k}) dt = t\mathbf{i} - t\mathbf{j} + \frac{1}{2}t^2\mathbf{k} + \mathbf{c}_2$. But $\mathbf{0} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$.

16. $\mathbf{a}(t) = -10\mathbf{k} \Rightarrow \mathbf{v}(t) = \int (-10\mathbf{k}) dt = -10t\mathbf{k} + \mathbf{c}_1$, and $\mathbf{i} + \mathbf{j} - \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - (10t + 1)\mathbf{k}$. $\mathbf{r}(t) = \int [\mathbf{i} + \mathbf{j} - (10t + 1)\mathbf{k}] dt = t\mathbf{i} + t\mathbf{j} - (5t^2 + t)\mathbf{k} + \mathbf{c}_2$. But $2\mathbf{i} + 3\mathbf{j} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{r}(t) = (t + 2)\mathbf{i} + (t + 3)\mathbf{j} - (5t^2 + t)\mathbf{k}$.

17. (a) $\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow$

$$\mathbf{v}(t) = \int (\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}) dt = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} + \mathbf{c}_1, \text{ and}$$

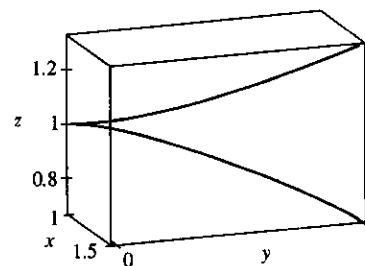
$$\mathbf{0} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1, \text{ so } \mathbf{c}_1 = \mathbf{0} \text{ and } \mathbf{v}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}.$$

$$\mathbf{r}(t) = \int (t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}) dt = \frac{1}{2}t^2\mathbf{i} + t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k} + \mathbf{c}_2.$$

$$\text{But } \mathbf{i} + \mathbf{k} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2, \text{ so } \mathbf{c}_2 = \mathbf{i} + \mathbf{k} \text{ and}$$

$$\mathbf{r}(t) = \left(1 + \frac{1}{2}t^2\right)\mathbf{i} + t^2\mathbf{j} + \left(1 + \frac{1}{3}t^3\right)\mathbf{k}.$$

(b)



18. (a) $\mathbf{a}(t) = t\mathbf{i} + t^2\mathbf{j} + \cos 2t\mathbf{k} \Rightarrow$

$$\mathbf{v}(t) = \int (t\mathbf{i} + t^2\mathbf{j} + \cos 2t\mathbf{k}) dt$$

$$= \frac{t^2}{2}\mathbf{i} + \frac{t^3}{3}\mathbf{j} + \frac{\sin 2t}{2}\mathbf{k} + \mathbf{c}_1$$

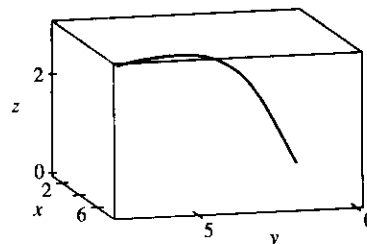
$$\text{and } \mathbf{i} + \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1, \text{ so } \mathbf{c}_1 = \mathbf{i} + \mathbf{k} \text{ and}$$

$$\mathbf{v}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + \frac{1}{3}t^3\mathbf{j} + \left(1 + \frac{1}{2}\sin 2t\right)\mathbf{k}.$$

$$\mathbf{r}(t) = \int \left[\left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + \frac{1}{3}t^3\mathbf{j} + \left(1 + \frac{1}{2}\sin 2t\right)\mathbf{k}\right] dt = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \frac{1}{12}t^4\mathbf{j} + \left(t - \frac{1}{4}\cos 2t\right)\mathbf{k} + \mathbf{c}_2$$

$$\text{But } \mathbf{j} = \mathbf{r}(0) = -\frac{1}{4}\mathbf{k} + \mathbf{c}_2, \text{ so } \mathbf{c}_2 = \mathbf{j} + \frac{1}{4}\mathbf{k} \text{ and } \mathbf{r}(t) = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \left(1 + \frac{1}{12}t^4\right)\mathbf{j} + \left(\frac{1}{4} + t - \frac{1}{4}\cos 2t\right)\mathbf{k}.$$

(b)



19. $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle \Rightarrow \mathbf{v}(t) = \langle 2t, 5, 2t - 16 \rangle,$

$$|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281} \text{ and}$$

$$\frac{d}{dt} |\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64). \text{ This is zero if and only if the numerator is zero, that is,}$$

$16t - 64 = 0$ or $t = 4$. Since $\frac{d}{dt} |\mathbf{v}(t)| < 0$ for $t < 4$ and $\frac{d}{dt} |\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained at $t = 4$ units of time.

20. Since $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = 6t\mathbf{i} + 2\mathbf{j} + 6t\mathbf{k}$. By Newton's Second Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = 6mt\mathbf{i} + 2m\mathbf{j} + 6mt\mathbf{k} \text{ is the required force.}$$

21. $|\mathbf{F}(t)| = 20$ N in the direction of the positive z -axis, so $\mathbf{F}(t) = 20\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$.

Since $20\mathbf{k} = \mathbf{F}(t) = 4\mathbf{a}(t)$, $\mathbf{a}(t) = 5\mathbf{k}$. Then $\mathbf{v}(t) = 5t\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\mathbf{k}$ and the

speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + \frac{5}{2}t^2\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$ and

$$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + \frac{5}{2}t^2\mathbf{k}.$$

22. The argument here is the same as that in Example 14.2.5 [ET 13.2.5] with $\mathbf{r}(t)$ replaced by $\mathbf{v}(t)$ and $\mathbf{r}'(t)$ replaced by $\mathbf{a}(t)$.

23. $|\mathbf{v}(0)| = 500$ m/s and since the angle of elevation is 30° , the direction of the velocity is $\frac{1}{2}(\sqrt{3}\mathbf{i} + \mathbf{j})$. Thus $\mathbf{v}(0) = 250(\sqrt{3}\mathbf{i} + \mathbf{j})$ and if we set up the axes so the projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the only force is that due to gravity, so $\mathbf{F}(t) = -mg\mathbf{j}$ where $g \approx 9.8$ m/s². Thus $\mathbf{a}(t) = -g\mathbf{j}$ and $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{c}_1$. But $250(\sqrt{3}\mathbf{i} + \mathbf{j}) = \mathbf{v}(0) = \mathbf{c}_1$, so $\mathbf{v}(t) = 250\sqrt{3}\mathbf{i} + (250 - gt)\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}t\mathbf{i} + (250t - \frac{1}{2}gt^2)\mathbf{j} + \mathbf{c}_2$ where $\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2$. Thus $\mathbf{r}(t) = 250\sqrt{3}t\mathbf{i} + (250t - \frac{1}{2}gt^2)\mathbf{j}$.

(a) Setting $250t - \frac{1}{2}gt^2 = 0$ gives $t = 0$ or $t = \frac{500}{g} \approx 51.0$ s. So the range is $250\sqrt{3} \cdot \frac{500}{g} \approx 22$ km.

(b) $0 = \frac{d}{dt}(250t - \frac{1}{2}gt^2) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s.

Thus, the maximum height is $(250)(250/g) - g(250/g)^2 \frac{1}{2} = (250)^2/(2g) \approx 3.2$ km.

(c) From part (a), impact occurs at $t = 500/g \approx 51.0$. Thus, the velocity at impact is

$$\mathbf{v}(500/g) = 250\sqrt{3}\mathbf{i} + [250 - g(500/g)]\mathbf{j} = 250\sqrt{3}\mathbf{i} - 250\mathbf{j} \text{ and the speed is}$$

$$|\mathbf{v}(500/g)| = 250\sqrt{3+1} = 500 \text{ m/s.}$$

24. As in Exercise 23, $\mathbf{v}(t) = 250\sqrt{3}\mathbf{i} + (250 - gt)\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}t\mathbf{i} + (250t - \frac{1}{2}gt^2)\mathbf{j} + \mathbf{c}_2$. But $\mathbf{r}(0) = 200\mathbf{j}$, so $\mathbf{c}_2 = 200\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}t\mathbf{i} + (200 + 250t - \frac{1}{2}gt^2)\mathbf{j}$.

(a) $200 + 250t - \frac{1}{2}gt^2 = 0$ implies that $gt^2 - 500t - 400 = 0$ or $t = \frac{500 \pm \sqrt{500^2 + 1600g}}{2g}$. Taking the

positive t -value gives $t = \frac{500 + \sqrt{250,000 + 1600g}}{2g} \approx 51.8$ s. Thus the range is

$$(250\sqrt{3}) \frac{500 + \sqrt{250,000 + 1600g}}{2g} \approx 22.4 \text{ km.}$$

(b) $0 = \frac{d}{dt}(200 + 250t - \frac{1}{2}gt^2) = 250 - gt$ implies that the maximum height is attained

when $t = 250/g \approx 25.5$ s and thus the maximum height is

$$\left[200 + (250) \left(\frac{250}{g} \right) - \frac{g}{2} \left(\frac{250}{g} \right)^2 \right] = 200 + \frac{(250)^2}{2g} \approx 3.4 \text{ km.}$$

Alternate solution: Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 200 m higher, the maximum height reached is 200 m higher than that found in Exercise 23, that is, $3.2 \text{ km} + 200 \text{ m} = 3.4 \text{ km}$.

(c) From part (a), impact occurs at $t = \frac{500 + \sqrt{250,000 + 1600g}}{2g}$. Thus the velocity at impact is

$$250\sqrt{3}\mathbf{i} + \left[250 - g \frac{500 + \sqrt{250,000 + 1600g}}{2g} \right] \mathbf{j}, \text{ so } |\mathbf{v}| \approx \sqrt{(250)^2(3) + (250 - 51.8g)^2} \approx 504 \text{ m/s.}$$

25. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45^\circ)t\mathbf{i} + [(v_0 \sin 45^\circ)t - \frac{1}{2}gt^2]\mathbf{j} = \frac{1}{2}[v_0\sqrt{2}t\mathbf{i} + (v_0\sqrt{2}t - gt^2)\mathbf{j}]$. Then the

ball lands at $t = \frac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = \frac{1}{2}v_0\sqrt{2} \frac{v_0\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity

is $v_0 = \sqrt{90g} \approx 30$ m/s.

26. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 30^\circ)t \mathbf{i} + [(v_0 \sin 30^\circ)t - \frac{1}{2}gt^2] \mathbf{j} = \frac{1}{2}[v_0\sqrt{3}t \mathbf{i} + (v_0t - gt^2)\mathbf{j}]$ and then $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{1}{2}[v_0\sqrt{3} \mathbf{i} + (v_0 - 2gt)\mathbf{j}]$. The shell reaches its maximum height when the vertical component of velocity is zero, so $\frac{1}{2}(v_0 - 2gt) = 0 \Rightarrow t = \frac{v_0}{2g}$. The vertical height of the shell at that time is 500 m, so

$$\frac{1}{2}\left[v_0\left(\frac{v_0}{2g}\right) - g\left(\frac{v_0}{2g}\right)^2\right] = 500 \Rightarrow \frac{v_0^2}{8g} = 500 \Rightarrow v_0 = \sqrt{4000g} = \sqrt{4000(9.8)} \approx 198 \text{ m/s}.$$

27. Let α be the angle of elevation. Then $v_0 = 150$ m/s and from Example 5, the horizontal distance traveled by the projectile is $d = \frac{v_0^2 \sin 2\alpha}{g}$. Thus $\frac{150^2 \sin 2\alpha}{g} = 800 \Rightarrow \sin 2\alpha = \frac{800g}{150^2} \approx 0.3484 \Rightarrow 2\alpha \approx 20.4^\circ$ or $180 - 20.4 = 159.6^\circ$. Two angles of elevation then are $\alpha \approx 10.2^\circ$ and $\alpha \approx 79.8^\circ$.

28. Here $v_0 = 115$ ft/s, the angle of elevation is $\alpha = 50^\circ$, and if we place the origin at home plate, then $\mathbf{r}(0) = 3\mathbf{j}$. As in Example 5, we have $\mathbf{r}(t) = -\frac{1}{2}gt^2 \mathbf{j} + t\mathbf{v}_0 + \mathbf{D}$ where $\mathbf{D} = \mathbf{r}(0) = 3\mathbf{j}$ and $\mathbf{v}_0 = v_0 \cos \alpha \mathbf{i} + v_0 \sin \alpha \mathbf{j}$, so $\mathbf{r}(t) = (v_0 \cos \alpha)t \mathbf{i} + [(v_0 \sin \alpha)t - \frac{1}{2}gt^2 + 3]\mathbf{j}$. Thus, parametric equations for the trajectory of the ball are $x = (v_0 \cos \alpha)t$, $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 + 3$. The ball reaches the fence when $x = 400 \Rightarrow (v_0 \cos \alpha)t = 400 \Rightarrow t = \frac{400}{v_0 \cos \alpha} = \frac{400}{115 \cos 50^\circ} \approx 5.41$ s. At this time, the height of the ball is $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 + 3 \approx (115 \sin 50^\circ)(5.41) - \frac{1}{2}(32)(5.41)^2 + 3 \approx 11.2$ ft. Since the fence is 10 ft high, the ball clears the fence.

29. (a) After t seconds, the boat will be $5t$ meters west of point A. The velocity of the water at that location is

$$\frac{3}{400}(5t)(40 - 5t)\mathbf{j}.$$

The velocity of the boat in still water is $5\mathbf{i}$, so the resultant velocity of the boat is

$$\mathbf{v}(t) = 5\mathbf{i} + \frac{3}{400}(5t)(40 - 5t)\mathbf{j} = 5\mathbf{i} + \left(\frac{3}{4}t - \frac{3}{16}t^2\right)\mathbf{j}.$$

Integrating, we obtain $\mathbf{r}(t) = 5t\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j} + \mathbf{C}$.

If we place the origin at A (and consider $\mathbf{j}$ to coincide with the northern direction) then $\mathbf{r}(0) = \mathbf{0} \Rightarrow \mathbf{C} = \mathbf{0}$

and we have $\mathbf{r}(t) = 5t\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j}$. The boat reaches the east bank after 8 s, and it is located at

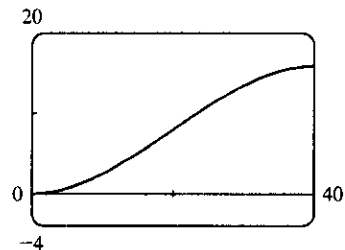
$$\mathbf{r}(8) = 5(8)\mathbf{i} + \left(\frac{3}{4}(8)^2 - \frac{1}{16}(8)^3\right)\mathbf{j} = 40\mathbf{i} + 16\mathbf{j}.$$

Thus the boat is 16 m downstream.

- (b) Let α be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos \alpha)\mathbf{i} + 5(\sin \alpha)\mathbf{j}$. At t seconds, the boat is $5(\cos \alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos \alpha)t][40 - 5(\cos \alpha)t]\mathbf{j}$. The resultant velocity of the boat is given by

$$\begin{aligned} \mathbf{v}(t) &= 5(\cos \alpha)\mathbf{i} + \left[5\sin \alpha + \frac{3}{400}(5t \cos \alpha)(40 - 5t \cos \alpha)\right]\mathbf{j} \\ &= (5 \cos \alpha)\mathbf{i} + \left(5 \sin \alpha + \frac{3}{2}t \cos \alpha - \frac{3}{16}t^2 \cos^2 \alpha\right)\mathbf{j} \end{aligned}$$

Integrating, $\mathbf{r}(t) = (5t \cos \alpha)\mathbf{i} + \left(5t \sin \alpha + \frac{3}{4}t^2 \cos \alpha - \frac{1}{16}t^3 \cos^2 \alpha\right)\mathbf{j}$ (where we have again placed



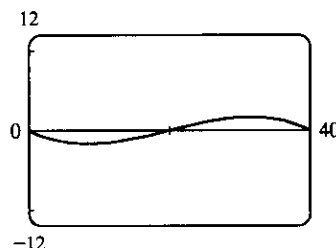
the origin at A). The boat will reach the east bank when $5t \cos \alpha = 40 \Rightarrow t = \frac{40}{5 \cos \alpha} = \frac{8}{\cos \alpha}$.

In order to land at point $B(40, 0)$ we need $5t \sin \alpha + \frac{3}{4}t^2 \cos \alpha - \frac{1}{16}t^3 \cos^2 \alpha = 0 \Rightarrow$

$$5\left(\frac{8}{\cos \alpha}\right) \sin \alpha + \frac{3}{4}\left(\frac{8}{\cos \alpha}\right)^2 \cos \alpha - \frac{1}{16}\left(\frac{8}{\cos \alpha}\right)^3 \cos^2 \alpha = 0 \Rightarrow \frac{1}{\cos \alpha} (40 \sin \alpha + 48 - 32) = 0 \Rightarrow$$

$40 \sin \alpha + 16 = 0 \Rightarrow \sin \alpha = -\frac{2}{5}$. Thus $\alpha = \sin^{-1}\left(-\frac{2}{5}\right) \approx -23.6^\circ$, so the boat should head 23.6° south of east (upstream).

The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point B .



30. As in Exercise 29(b), let α be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos \alpha) \mathbf{i} + 5(\sin \alpha) \mathbf{j}$. At t seconds, the boat is $5(\cos \alpha)t$ meters from the west bank, at which point the velocity of the water is $3 \sin(\pi x/40) \mathbf{j} = 3 \sin[\pi \cdot 5(\cos \alpha)t/40] \mathbf{j} = 3 \sin(\frac{\pi}{8} t \cos \alpha) \mathbf{j}$. The resultant velocity of the boat then is given by $\mathbf{v}(t) = 5(\cos \alpha) \mathbf{i} + [5 \sin \alpha + 3 \sin(\frac{\pi}{8} t \cos \alpha)] \mathbf{j}$. Integrating,

$$\mathbf{r}(t) = (5t \cos \alpha) \mathbf{i} + \left[5t \sin \alpha - \frac{24}{\pi \cos \alpha} \cos\left(\frac{\pi}{8} t \cos \alpha\right) \right] \mathbf{j} + \mathbf{C}.$$

If we place the origin at A then $\mathbf{r}(0) = \mathbf{0} \Rightarrow -\frac{24}{\pi \cos \alpha} \mathbf{j} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = \frac{24}{\pi \cos \alpha} \mathbf{j}$ and

$$\mathbf{r}(t) = (5t \cos \alpha) \mathbf{i} + \left[5t \sin \alpha - \frac{24}{\pi \cos \alpha} \cos\left(\frac{\pi}{8} t \cos \alpha\right) + \frac{24}{\pi \cos \alpha} \right] \mathbf{j}.$$

The boat will reach the east bank when $5t \cos \alpha = 40 \Rightarrow t = \frac{8}{\cos \alpha}$. In order to land

at point $B(40, 0)$ we need $5t \sin \alpha - \frac{24}{\pi \cos \alpha} \cos\left(\frac{\pi}{8} t \cos \alpha\right) + \frac{24}{\pi \cos \alpha} = 0$

$$\Rightarrow 5\left(\frac{8}{\cos \alpha}\right) \sin \alpha - \frac{24}{\pi \cos \alpha} \cos\left[\frac{\pi}{8}\left(\frac{8}{\cos \alpha}\right) \cos \alpha\right] + \frac{24}{\pi \cos \alpha} = 0 \Rightarrow$$

$$\frac{1}{\cos \alpha} \left(40 \sin \alpha - \frac{24}{\pi} \cos \pi + \frac{24}{\pi} \right) = 0 \Rightarrow 40 \sin \alpha + \frac{48}{\pi} = 0 \Rightarrow \sin \alpha = -\frac{6}{5\pi}. \text{ Thus}$$

$$\alpha = \sin^{-1}\left(-\frac{6}{5\pi}\right) \approx -22.5^\circ, \text{ so the boat should head } 22.5^\circ \text{ south of east.}$$

31. $\mathbf{r}(t) = (3t - t^3) \mathbf{i} + 3t^2 \mathbf{j} \Rightarrow \mathbf{r}'(t) = (3 - 3t^2) \mathbf{i} + 6t \mathbf{j}$,

$$|\mathbf{r}'(t)| = \sqrt{(3 - 3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3 + 3t^2)^2} = 3 + 3t^2,$$

$\mathbf{r}''(t) = -6t \mathbf{i} + 6 \mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = (18 + 18t^2) \mathbf{k}$. Then Equation 9 gives

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(3 - 3t^2)(-6t) + (6t)(6)}{3 + 3t^2} = \frac{18t + 18t^3}{3 + 3t^2} = \frac{18t(1 + t^2)}{3(1 + t^2)} = 6t \quad \left[\text{or by Equation 8,} \right]$$

$$a_T = v' = \frac{d}{dt} [3 + 3t^2] = 6t \quad \text{and Equation 10 gives } a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{18 + 18t^2}{3 + 3t^2} = \frac{18(1 + t^2)}{3(1 + t^2)} = 6.$$

$$\begin{aligned}
 32. \mathbf{r}(t) &= (1+t)\mathbf{i} + (t^2-2t)\mathbf{j} \Rightarrow \mathbf{r}'(t) = \mathbf{i} + (2t-2)\mathbf{j}, |\mathbf{r}'(t)| = \sqrt{1^2 + (2t-2)^2} = \sqrt{4t^2 - 8t + 5}, \\
 \mathbf{r}''(t) &= 2\mathbf{j}, \mathbf{r}'(t) \times \mathbf{r}''(t) = 2\mathbf{k}. \text{ Then Equation 9 gives } a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{2(2t-2)}{\sqrt{4t^2 - 8t + 5}} \text{ and Equation 10} \\
 \text{gives } a_N &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{2}{\sqrt{4t^2 - 8t + 5}}.
 \end{aligned}$$

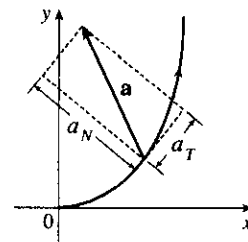
$$\begin{aligned}
 33. \mathbf{r}(t) &= \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \Rightarrow \mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}, |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}, \\
 \mathbf{r}''(t) &= -\cos t \mathbf{i} - \sin t \mathbf{j}, \mathbf{r}'(t) \times \mathbf{r}''(t) = \sin t \mathbf{i} - \cos t \mathbf{j} + \mathbf{k}. \\
 \text{Then } a_T &= \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{\sin t \cos t - \sin t \cos t}{\sqrt{2}} = 0 \text{ and} \\
 a_N &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.
 \end{aligned}$$

$$\begin{aligned}
 34. \mathbf{r}(t) &= t \mathbf{i} + t^2 \mathbf{j} + 3t \mathbf{k} \Rightarrow \mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + 3 \mathbf{k}, |\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + 3^2} = \sqrt{4t^2 + 10}, \mathbf{r}''(t) = 2 \mathbf{j}, \\
 \mathbf{r}'(t) \times \mathbf{r}''(t) &= -6 \mathbf{i} + 2 \mathbf{k}. \text{ Then } a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{4t}{\sqrt{4t^2 + 10}} \text{ and } a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{2\sqrt{10}}{\sqrt{4t^2 + 10}}.
 \end{aligned}$$

$$\begin{aligned}
 35. \mathbf{r}(t) &= e^t \mathbf{i} + \sqrt{2}t \mathbf{j} + e^{-t} \mathbf{k} \Rightarrow \mathbf{r}'(t) = e^t \mathbf{i} + \sqrt{2} \mathbf{j} - e^{-t} \mathbf{k}, \\
 |\mathbf{r}'(t)| &= \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}, \mathbf{r}''(t) = e^t \mathbf{i} + e^{-t} \mathbf{k}. \\
 \text{Then } a_T &= \frac{e^{2t} - e^{-2t}}{e^t + e^{-t}} = \frac{(e^t + e^{-t})(e^t - e^{-t})}{e^t + e^{-t}} = e^t - e^{-t} = 2 \sinh t \text{ and} \\
 a_N &= \frac{|\sqrt{2}e^{-t} \mathbf{i} - 2 \mathbf{j} - \sqrt{2}e^t \mathbf{k}|}{e^t + e^{-t}} = \frac{\sqrt{2(e^{-2t} + 2 + e^{2t})}}{e^t + e^{-t}} = \sqrt{2} \frac{e^t + e^{-t}}{e^t + e^{-t}} = \sqrt{2}.
 \end{aligned}$$

$$\begin{aligned}
 36. \mathbf{r}(t) &= t \mathbf{i} + \cos^2 t \mathbf{j} + \sin^2 t \mathbf{k} \Rightarrow \mathbf{r}'(t) = \mathbf{i} - 2 \cos t \sin t \mathbf{j} + 2 \sin t \cos t \mathbf{k} = \mathbf{i} - \sin 2t \mathbf{j} + \sin 2t \mathbf{k}, \\
 |\mathbf{r}'(t)| &= \sqrt{1 + 2 \sin^2 2t}, \mathbf{r}''(t) = 2(\sin^2 t - \cos^2 t) \mathbf{j} + 2(\cos^2 t - \sin^2 t) \mathbf{k} = -2 \cos 2t \mathbf{j} + 2 \cos 2t \mathbf{k}. \text{ So} \\
 a_T &= \frac{2 \sin 2t \cos 2t + 2 \sin 2t \cos 2t}{\sqrt{1 + 2 \sin^2 2t}} = \frac{4 \sin 2t \cos 2t}{\sqrt{1 + 2 \sin^2 2t}} \text{ and } a_N = \frac{|-2 \cos 2t \mathbf{j} - 2 \cos 2t \mathbf{k}|}{\sqrt{1 + 2 \sin^2 2t}} = \frac{2\sqrt{2} |\cos 2t|}{\sqrt{1 + 2 \sin^2 2t}}.
 \end{aligned}$$

37. The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide). Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0. Thus $a_T \approx 4.5 \text{ cm/s}^2$ and $a_N \approx 9.0 \text{ cm/s}^2$.



$$38. \mathbf{L}(t) = m \mathbf{r}(t) \times \mathbf{v}(t) \Rightarrow$$

$$\begin{aligned}
 \mathbf{L}'(t) &= m[\mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)] \quad [\text{by Theorem 14.2.3 [ET 13.2.3] \#5}] \\
 &= m[\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{a}(t)] = m[\mathbf{0} + \mathbf{r}(t) \times \mathbf{a}(t)] = \boldsymbol{\tau}(t)
 \end{aligned}$$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{0}$ for all t , and so $\mathbf{L}(t)$ is constant.

39. If the engines are turned off at time t , then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need

$$\text{a } t \text{ such that for some scalar } s > 0, \mathbf{r}(t) + s\mathbf{v}(t) = \langle 6, 4, 9 \rangle. \quad \mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{8t}{(t^2+1)^2}\mathbf{k} \Rightarrow$$

$$\mathbf{r}(t) + s\mathbf{v}(t) = \left\langle 3+t+s, 2+\ln t + \frac{s}{t}, 7 - \frac{4}{t^2+1} + \frac{8st}{(t^2+1)^2} \right\rangle \Rightarrow 3+t+s=6 \Rightarrow s=3-t,$$

$$\text{so } 7 - \frac{4}{t^2+1} + \frac{8(3-t)t}{(t^2+1)^2} = 9 \Leftrightarrow \frac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \Leftrightarrow t^4 + 8t^2 - 12t + 3 = 0. \text{ It is easily seen that}$$

$$t=1 \text{ is a root of this polynomial. Also } 2 + \ln 1 + \frac{3-1}{1} = 4, \text{ so } t=1 \text{ is the desired solution.}$$

40. (a) $m \frac{d\mathbf{v}}{dt} = \frac{dm}{dt} \mathbf{v}_e \Leftrightarrow \frac{d\mathbf{v}}{dt} = \frac{1}{m} \frac{dm}{dt} \mathbf{v}_e$. Integrating both sides of this equation with respect to t gives

$$\int_0^t \frac{d\mathbf{v}}{du} du = \mathbf{v}_e \int_0^t \frac{1}{m} \frac{dm}{du} du \Rightarrow \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}_e \int_{m(0)}^{m(t)} \frac{dm}{m} \quad [\text{Substitution Rule}] \Rightarrow$$

$$\mathbf{v}(t) - \mathbf{v}(0) = \ln\left(\frac{m(t)}{m(0)}\right) \mathbf{v}_e \Rightarrow \mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\frac{m(0)}{m(t)}\right) \mathbf{v}_e.$$

$$(b) |\mathbf{v}(t)| = 2|\mathbf{v}_e|, \text{ and } |\mathbf{v}(0)| = 0. \text{ Therefore, by part (a), } 2|\mathbf{v}_e| = \left| -\ln\left(\frac{m(0)}{m(t)}\right) \mathbf{v}_e \right| \Rightarrow$$

$$2|\mathbf{v}_e| = \ln\left(\frac{m(0)}{m(t)}\right) |\mathbf{v}_e|. \quad \left[\text{Note: } m(0) > m(t) \text{ so that } \ln\left(\frac{m(0)}{m(t)}\right) > 0 \right] \Rightarrow m(t) = e^{-2} m(0).$$

$$\text{Thus } \frac{m(0) - e^{-2}m(0)}{m(0)} = 1 - e^{-2} \text{ is the fraction of the initial mass that is burned as fuel.}$$

APPLIED PROJECT Kepler's Laws

1. With $\mathbf{r} = (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j}$ and $\mathbf{h} = \alpha \mathbf{k}$ where $\alpha > 0$,

$$(a) \mathbf{h} = \mathbf{r} \times \mathbf{r}' = [(r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j}] \times \left[\left(r' \cos \theta - r \sin \theta \frac{d\theta}{dt} \right) \mathbf{i} + \left(r' \sin \theta + r \cos \theta \frac{d\theta}{dt} \right) \mathbf{j} \right]$$

$$= \left[rr' \cos \theta \sin \theta + r^2 \cos^2 \theta \frac{d\theta}{dt} - rr' \cos \theta \sin \theta + r^2 \sin^2 \theta \frac{d\theta}{dt} \right] \mathbf{k} = r^2 \frac{d\theta}{dt} \mathbf{k}$$

$$(b) \text{ Since } \mathbf{h} = \alpha \mathbf{k}, \alpha > 0, \alpha = |\mathbf{h}|. \text{ But by part (a), } \alpha = |\mathbf{h}| = r^2 (d\theta/dt).$$

$$(c) A(t) = \frac{1}{2} \int_{\theta_0}^{\theta} |\mathbf{r}|^2 d\theta = \frac{1}{2} \int_{t_0}^t r^2 (d\theta/dt) dt \text{ in polar coordinates. Thus, by the Fundamental Theorem}$$

$$\text{of Calculus, } \frac{dA}{dt} = \frac{r^2}{2} \frac{d\theta}{dt}.$$

$$(d) \frac{dA}{dt} = \frac{r^2}{2} \frac{d\theta}{dt} = \frac{h}{2} = \text{constant since } \mathbf{h} \text{ is a constant vector and } h = |\mathbf{h}|.$$

2. (a) Since $dA/dt = \frac{1}{2}h$, a constant, $A(t) = \frac{1}{2}ht + c_1$. But $A(0) = 0$, so $A(t) = \frac{1}{2}ht$. But

$$A(T) = \text{area of the ellipse} = \pi ab \text{ and } A(T) = \frac{1}{2}hT, \text{ so } T = 2\pi ab/h.$$

$$(b) h^2/(GM) = ed \text{ where } e \text{ is the eccentricity of the ellipse. But } a = ed/(1 - e^2) \text{ or } ed = a(1 - e^2) \text{ and } 1 - e^2 = b^2/a^2. \text{ Hence } h^2/(GM) = ed = b^2/a.$$

$$(c) T^2 = \frac{4\pi a^2 b^2}{h^2} = 4\pi^2 a^2 b^2 \frac{a}{GM b^2} = \frac{4\pi^2}{GM} a^3.$$

3. From Problem 2, $T^2 = \frac{4\pi^2}{GM} a^3$. $T \approx 365.25 \text{ days} \times 24 \cdot 60^2 \frac{\text{seconds}}{\text{day}} \approx 3.1558 \times 10^7 \text{ seconds}$. Therefore
- $$a^3 = \frac{GMT^2}{4\pi^2} \approx \frac{(6.67 \times 10^{-11})(1.99 \times 10^{30})(3.1558 \times 10^7)^2}{4\pi^2} \approx 3.348 \times 10^{33} \text{ m}^3 \Rightarrow a \approx 1.496 \times 10^{11} \text{ m}.$$
- Thus, the length of the major axis of Earth's orbit (that is, $2a$) is approximately $2.99 \times 10^{11} \text{ m} = 2.99 \times 10^8 \text{ km}$.

4. We can adapt the equation $T^2 = \frac{4\pi^2}{GM} a^3$ from Problem 2(c) with Earth at the center of the system, so T is the period of the satellite's orbit about Earth, M is the mass of Earth, and a is the length of the semimajor axis of the satellite's orbit (measured from Earth's center). Since we want the satellite to remain fixed above a particular point on Earth's equator, T must coincide with the period of Earth's own rotation, so $T = 24 \text{ h} = 86,400 \text{ s}$.

The mass of Earth is $M = 5.98 \times 10^{24} \text{ kg}$, so

$$a = \left(\frac{T^2 GM}{4\pi^2} \right)^{1/3} \approx \left[\frac{(86,400)^2 (6.67 \times 10^{-11}) (5.98 \times 10^{24})}{4\pi^2} \right]^{1/3} \approx 4.23 \times 10^7 \text{ m}.$$

If we assume a circular orbit, the radius of the orbit is a , and since the radius of Earth is $6.37 \times 10^6 \text{ m}$, the required altitude above Earth's surface for the satellite is $4.23 \times 10^7 - 6.37 \times 10^6 \approx 3.59 \times 10^7 \text{ m}$, or 35,900 km.

14 Review

ET 13

CONCEPT CHECK

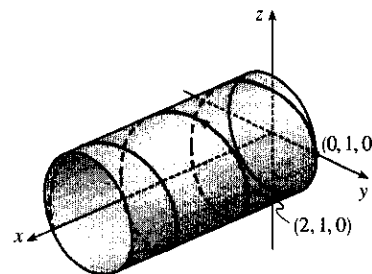
- A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component of the vector function.
- The tip of the moving vector $\mathbf{r}(t)$ of a continuous vector function traces out a space curve.
- (a) A curve represented by the vector function $\mathbf{r}(t)$ is smooth if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on its parametric domain (except possibly at the endpoints).
(b) The tangent vector to a smooth curve at a point P with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at P is the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.
- (a)–(f) See Theorem 14.2.3 [ET 13.2.3].
- Use Formula 14.3.2 [ET 13.3.2], or equivalently 14.3.3 [ET 13.3.3].
- (a) The curvature of a curve is $\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$ where $\mathbf{T}$ is the unit tangent vector.
(b) $\kappa(t) = \left| \frac{\mathbf{T}'(t)}{\mathbf{r}'(t)} \right|$ (c) $\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ (d) $\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$
- (a) The unit normal vector: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. The binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.
(b) See the discussion preceding Example 7 in Section 14.3 [ET 13.3].
- (a) If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.
(b) $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ where $a_T = v'$ and $a_N = \kappa v^2$.
- See the statement of Kepler's Laws on page 912 [ET 876].

TRUE-FALSE QUIZ

1. True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\mathbf{i} + 2u\mathbf{j} + 3u\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
2. True. $\mathbf{r}'(t) = \langle 1, 3t^2, 5t^4 \rangle$ is continuous for all t (since its component functions are each continuous) and since $x'(t) = 1$, we have $\mathbf{r}'(t) \neq \mathbf{0}$ for all t .
3. False. $\mathbf{r}'(t) = \langle -\sin t, 2t, 4t^3 \rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0 \rangle = \mathbf{0}$, the curve is not smooth.
4. True. See Theorem 14.2.2 [ET 13.2.2].
5. False. By Formula 5 of Theorem 14.2.3 [ET 13.2.3], $\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.
6. False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \Rightarrow \frac{d}{dt} |\mathbf{r}(t)| = 0$, but $|\mathbf{r}'(t)| = | \langle -\sin t, \cos t \rangle | = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.
7. False. κ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s , not with respect to t .
8. False. The binormal vector, by the definition given in Section 14.3 [ET 13.3], is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -[\mathbf{N}(t) \times \mathbf{T}(t)]$.
9. True. See the discussion preceding Example 7 in Section 14.3 [ET 13.3].
10. False. For example, $\mathbf{r}_1(t) = \langle t, t \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t \rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}'_1(t) = \langle 1, 1 \rangle$ for all t , while $\mathbf{r}'_2(t) = \langle 2, 2 \rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

EXERCISES

1. (a) The corresponding parametric equations for the curve are $x = t$, $y = \cos \pi t$, $z = \sin \pi t$. Since $y^2 + z^2 = 1$, the curve is contained in a circular cylinder with axis the x -axis. Since $x = t$, the curve is a helix.

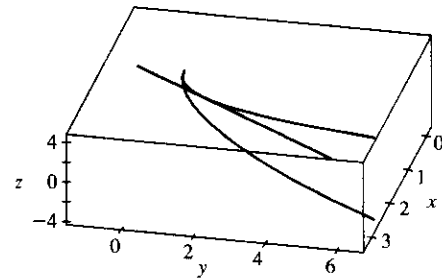


- (b) $\mathbf{r}(t) = t\mathbf{i} + \cos \pi t\mathbf{j} + \sin \pi t\mathbf{k} \Rightarrow \mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\mathbf{j} + \pi \cos \pi t\mathbf{k} \Rightarrow \mathbf{r}''(t) = -\pi^2 \cos \pi t\mathbf{j} - \pi^2 \sin \pi t\mathbf{k}$
2. (a) The expressions $\sqrt{2-t}$, $(e^t - 1)/t$, and $\ln(t+1)$ are all defined when $2-t \geq 0 \Rightarrow t \leq 2$, $t \neq 0$, and $t+1 > 0 \Rightarrow t > -1$. Thus the domain of $\mathbf{r}$ is $(-1, 0) \cup (0, 2]$.
 - (b) $\lim_{t \rightarrow 0} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow 0} \sqrt{2-t}, \lim_{t \rightarrow 0} \frac{e^t - 1}{t}, \lim_{t \rightarrow 0} \ln(t+1) \right\rangle = \left\langle \sqrt{2-0}, \lim_{t \rightarrow 0} \frac{e^t}{1}, \ln(0+1) \right\rangle = \langle \sqrt{2}, 1, 0 \rangle$
(using l'Hospital's Rule in the y -component).
 - (c) $\mathbf{r}'(t) = \left\langle \frac{d}{dt} \sqrt{2-t}, \frac{d}{dt} \frac{e^t - 1}{t}, \frac{d}{dt} \ln(t+1) \right\rangle = \left\langle -\frac{1}{2\sqrt{2-t}}, \frac{te^t - e^t + 1}{t^2}, \frac{1}{t+1} \right\rangle$

3. The projection of the curve C of intersection onto the xy -plane is the circle $x^2 + y^2 = 16, z = 0$. So we can write $x = 4 \cos t, y = 4 \sin t, 0 \leq t \leq 2\pi$. From the equation of the plane, we have $z = 5 - x = 5 - 4 \cos t$, so parametric equations for C are $x = 4 \cos t, y = 4 \sin t, z = 5 - 4 \cos t, 0 \leq t \leq 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + (5 - 4 \cos t) \mathbf{k}, 0 \leq t \leq 2\pi$.

4. The curve is given by $\mathbf{r}(t) = \langle t^2, t^4, t^3 \rangle$, so $\mathbf{r}'(t) = \langle 2t, 4t^3, 3t^2 \rangle$.

The point $(1, 1, 1)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 2, 4, 3 \rangle$. Then the tangent line has direction vector $\langle 2, 4, 3 \rangle$ and includes the point $(1, 1, 1)$, so parametric equations are $x = 1 + 2t, y = 1 + 4t, z = 1 + 3t$.



$$\begin{aligned} 5. \int_0^1 (t^2 \mathbf{i} + t \cos \pi t \mathbf{j} + \sin \pi t \mathbf{k}) dt &= \left(\int_0^1 t^2 dt \right) \mathbf{i} + \left(\int_0^1 t \cos \pi t dt \right) \mathbf{j} + \left(\int_0^1 \sin \pi t dt \right) \mathbf{k} \\ &= \left[\frac{1}{3} t^3 \right]_0^1 \mathbf{i} + \left(\left[\frac{t}{\pi} \sin \pi t \right]_0^1 - \int_0^1 \frac{1}{\pi} \sin \pi t dt \right) \mathbf{j} + \left[-\frac{1}{\pi} \cos \pi t \right]_0^1 \mathbf{k} \\ &= \frac{1}{3} \mathbf{i} + \left[\frac{1}{\pi^2} \cos \pi t \right]_0^1 \mathbf{j} + \frac{2}{\pi} \mathbf{k} = \frac{1}{3} \mathbf{i} - \frac{2}{\pi^2} \mathbf{j} + \frac{2}{\pi} \mathbf{k} \end{aligned}$$

where we integrated by parts in the y -component.

6. (a) C intersects the xz -plane where $y = 0 \Rightarrow 2t - 1 = 0 \Rightarrow t = \frac{1}{2}$, so the point is

$$\left(2 - \left(\frac{1}{2} \right)^3, 0, \ln \frac{1}{2} \right) = \left(\frac{15}{8}, 0, -\ln 2 \right).$$

- (b) The curve is given by $\mathbf{r}(t) = \langle 2 - t^3, 2t - 1, \ln t \rangle$, so $\mathbf{r}'(t) = \langle -3t^2, 2, 1/t \rangle$. The point $(1, 1, 0)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$. Then the tangent line has direction vector $\langle -3, 2, 1 \rangle$ and includes the point $(1, 1, 0)$, so parametric equations are $x = 1 - 3t, y = 1 + 2t, z = t$.

- (c) The normal plane has normal vector $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$ and equation $-3(x - 1) + 2(y - 1) + z = 0$ or $3x - 2y - z = 1$.

7. $t = 1$ at $(1, 4, 2)$ and $t = 4$ at $(2, 1, 17)$, so

$$\begin{aligned} L &= \int_1^4 \sqrt{\frac{1}{4t} + \frac{16}{t^4} + 4t^2} dt \\ &\approx \frac{4-1}{3 \cdot 4} \left[\sqrt{\frac{1}{4} + 16 + 4} + 4 \cdot \sqrt{\frac{1}{4 \cdot \frac{7}{4}} + \frac{16}{(\frac{7}{4})^4} + 4 \left(\frac{7}{4} \right)^2} + 2 \cdot \sqrt{\frac{1}{4 \cdot \frac{10}{4}} + \frac{16}{(\frac{10}{4})^4} + 4 \left(\frac{10}{4} \right)^2} \right. \\ &\quad \left. + 4 \cdot \sqrt{\frac{1}{4 \cdot \frac{13}{4}} + \frac{16}{(\frac{13}{4})^4} + 4 \left(\frac{13}{4} \right)^2} + \sqrt{\frac{1}{4 \cdot 4} + \frac{16}{4^4} + 4 \cdot 4^2} \right] \\ &\approx 15.9241 \end{aligned}$$

8. $\mathbf{r}'(t) = \langle 3t^{1/2}, -2 \sin 2t, 2 \cos 2t \rangle$, $|\mathbf{r}'(t)| = \sqrt{9t + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{9t + 4}$. Thus

$$L = \int_0^1 \sqrt{9t + 4} dt = \int_4^{13} \frac{1}{9} u^{1/2} du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_4^{13} = \frac{2}{27} (13^{3/2} - 8).$$

9. The angle of intersection of the two curves, θ , is the angle between their respective tangents at the point of intersection. For both curves the point $(1, 0, 0)$ occurs when $t = 0$. $\mathbf{r}'_1(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}'_1(0) = \mathbf{j} + \mathbf{k}$ and $\mathbf{r}'_2(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow \mathbf{r}'_2(0) = \mathbf{i}$. $\mathbf{r}'_1(0) \cdot \mathbf{r}'_2(0) = (\mathbf{j} + \mathbf{k}) \cdot \mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = \frac{\pi}{2}$.

10. The parametric value corresponding to the point $(1, 0, 1)$ is $t = 0$.

$$\mathbf{r}'(t) = e^t \mathbf{i} + e^t(\cos t + \sin t)\mathbf{j} + e^t(\cos t - \sin t)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}'(t)| = e^t \sqrt{1 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{3} e^t$$

$$\text{and } s(t) = \int_0^t e^u \sqrt{3} du = \sqrt{3}(e^t - 1) \Rightarrow t = \ln\left(1 + \frac{1}{\sqrt{3}}s\right). \text{ Therefore,}$$

$$\mathbf{r}(t(s)) = \left(1 + \frac{1}{\sqrt{3}}s\right) \mathbf{i} + \left(1 + \frac{1}{\sqrt{3}}s\right) \sin \ln\left(1 + \frac{1}{\sqrt{3}}s\right) \mathbf{j} + \left(1 + \frac{1}{\sqrt{3}}s\right) \cos \ln\left(1 + \frac{1}{\sqrt{3}}s\right) \mathbf{k}.$$

$$11. (a) \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle t^2, t, 1 \rangle}{|\langle t^2, t, 1 \rangle|} = \frac{\langle t^2, t, 1 \rangle}{\sqrt{t^4 + t^2 + 1}}$$

$$(b) \mathbf{T}'(t) = -\frac{1}{2}(t^4 + t^2 + 1)^{-3/2}(4t^3 + 2t)\langle t^2, t, 1 \rangle + (t^4 + t^2 + 1)^{-1/2}\langle 2t, 1, 0 \rangle$$

$$= \frac{-2t^3 - t}{(t^4 + t^2 + 1)^{3/2}} \langle t^2, t, 1 \rangle + \frac{1}{(t^4 + t^2 + 1)^{1/2}} \langle 2t, 1, 0 \rangle$$

$$= \frac{\langle -2t^5 - t^3, -2t^4 - t^2, -2t^3 - t \rangle + \langle 2t^5 + 2t^3 + 2t, t^4 + t^2 + 1, 0 \rangle}{(t^4 + t^2 + 1)^{3/2}}$$

$$= \frac{\langle 2t, -t^4 + 1, -2t^3 - t \rangle}{(t^4 + t^2 + 1)^{3/2}}$$

$$|\mathbf{T}'(t)| = \frac{\sqrt{4t^2 + t^8 - 2t^4 + 1 + 4t^6 + 4t^4 + t^2}}{(t^4 + t^2 + 1)^{3/2}} = \frac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4 + t^2 + 1)^{3/2}}, \text{ and}$$

$$\mathbf{N}(t) = \frac{\langle 2t, 1 - t^4, -2t^3 - t \rangle}{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}.$$

$$(c) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4 + t^2 + 1)^2}$$

12. Using Exercise 14.3.36 [ET 13.3.36], we have $\mathbf{r}'(t) = \langle -3 \sin t, 4 \cos t \rangle$, $\mathbf{r}''(t) = \langle -3 \cos t, -4 \sin t \rangle$,

$$|\mathbf{r}'(t)|^3 = \left(\sqrt{9 \sin^2 t + 16 \cos^2 t} \right)^3 \text{ and then}$$

$$\kappa(t) = \frac{|(-3 \sin t)(-4 \sin t) - (4 \cos t)(-3 \cos t)|}{(9 \sin^2 t + 16 \cos^2 t)^{3/2}} = \frac{12}{(9 \sin^2 t + 16 \cos^2 t)^{3/2}}.$$

$$\text{At } (3, 0), t = 0 \text{ and } \kappa(0) = 12/(16)^{3/2} = \frac{12}{64} = \frac{3}{16}. \text{ At } (0, 4), t = \frac{\pi}{2} \text{ and } \kappa\left(\frac{\pi}{2}\right) = 12/9^{3/2} = \frac{12}{27} = \frac{4}{9}.$$

$$13. y' = 4x^3, y'' = 12x^2 \text{ and } \kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|12x^2|}{(1 + 16x^6)^{3/2}}, \text{ so } \kappa(1) = \frac{12}{17^{3/2}}.$$

to gravity, so as in Example 14.4.5 [ET 13.4.5] we have $\mathbf{a} = -g\mathbf{j}$ where here $g \approx 32 \text{ ft/s}^2$. Since $\mathbf{v}'(t) = \mathbf{a}(t)$, we integrate, giving $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{C}$ where $\mathbf{C} = \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \Rightarrow \mathbf{v}(t) = \frac{43}{\sqrt{2}}\mathbf{i} + \left(\frac{43}{\sqrt{2}} - gt\right)\mathbf{j}$. Since

$\mathbf{r}'(t) = \mathbf{v}(t)$ we integrate again, so $\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{D} = \mathbf{r}(0) = 7\mathbf{j} \Rightarrow$

$$\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7\right)\mathbf{j}.$$

(a) At 2 seconds, the shot is at $\mathbf{r}(2) = \frac{43}{\sqrt{2}}(2)\mathbf{i} + \left(\frac{43}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 7\right)\mathbf{j} \approx 60.8\mathbf{i} + 3.8\mathbf{j}$, so the shot is about 3.8 ft above the ground, at a horizontal distance of 60.8 ft from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{43}{\sqrt{2}} - gt = 0 \Rightarrow$

$$t = \frac{43}{\sqrt{2}g} \approx 0.95 \text{ s. Then } \mathbf{r}(0.95) \approx 28.9\mathbf{i} + 21.4\mathbf{j}, \text{ so the maximum height is approximately 21.4 ft.}$$

(c) The shot hits the ground when the vertical component of $\mathbf{r}(t)$ is 0, so $\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7 = 0 \Rightarrow$

$$-16t^2 + \frac{43}{\sqrt{2}}t + 7 = 0 \Rightarrow t \approx 2.11 \text{ s. } \mathbf{r}(2.11) \approx 64.2\mathbf{i} - 0.08\mathbf{j}, \text{ thus the shot lands approximately 64.2 ft from the athlete.}$$

$$20. \mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}, \mathbf{r}''(t) = 2\mathbf{k}, |\mathbf{r}'(t)| = \sqrt{1 + 4 + 4t^2} = \sqrt{4t^2 + 5}.$$

$$\text{Then } a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{4t}{\sqrt{4t^2 + 5}} \text{ and } a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{|4\mathbf{i} - 2\mathbf{j}|}{\sqrt{4t^2 + 5}} = \frac{2\sqrt{5}}{\sqrt{4t^2 + 5}}.$$

21. (a) Instead of proceeding directly, we use Formula 3 of Theorem 14.2.3 [ET 13.2.3]:

$$\mathbf{r}(t) = t\mathbf{R}(t) \Rightarrow \mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\mathbf{R}'(t) = \cos \omega t\mathbf{i} + \sin \omega t\mathbf{j} + t\mathbf{v}_d.$$

(b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\mathbf{R}'(t)$, we have

$$\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\mathbf{R}''(t) = 2\mathbf{R}'(t) + t\mathbf{R}''(t) = 2\mathbf{v}_d + t\mathbf{a}_d.$$

(c) Here we have $\mathbf{r}(t) = e^{-t} \cos \omega t\mathbf{i} + e^{-t} \sin \omega t\mathbf{j} = e^{-t} \mathbf{R}(t)$. So, as in parts (a) and (b),

$$\mathbf{v} = \mathbf{r}'(t) = e^{-t} \mathbf{R}'(t) - e^{-t} \mathbf{R}(t) = e^{-t} [\mathbf{R}'(t) - \mathbf{R}(t)] \Rightarrow$$

$$\begin{aligned} \mathbf{a} = \mathbf{v}' &= e^{-t} [\mathbf{R}''(t) - \mathbf{R}'(t)] - e^{-t} [\mathbf{R}'(t) - \mathbf{R}(t)] = e^{-t} [\mathbf{R}''(t) - 2\mathbf{R}'(t) + \mathbf{R}(t)] \\ &= e^{-t} \mathbf{a}_d - 2e^{-t} \mathbf{v}_d + e^{-t} \mathbf{R} \end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving $\mathbf{a}_d$) is $-2e^{-t} \mathbf{v}_d + e^{-t} \mathbf{R}$.

$$22. (a) F(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ \sqrt{2}-x & \text{if } x \geq \frac{1}{\sqrt{2}} \end{cases} \Rightarrow F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x/\sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -1 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases} \Rightarrow$$

$$F''(x) = \begin{cases} 0 & \text{if } x < 0 \\ -1/(1-x^2)^{3/2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

$$\text{since } \frac{d}{dx}[-x(1-x^2)^{-1/2}] = -(1-x^2)^{-1/2} - x^2(1-x^2)^{-3/2} = -(1-x^2)^{-3/2}.$$

Now $\lim_{x \rightarrow 0^+} \sqrt{1-x^2} = 1 = F(0)$ and $\lim_{x \rightarrow (1/\sqrt{2})^-} \sqrt{1-x^2} = \frac{1}{\sqrt{2}} = F\left(\frac{1}{\sqrt{2}}\right)$, so F is continuous. Also, since

$\lim_{x \rightarrow 0^+} F'(x) = 0 = \lim_{x \rightarrow 0^-} F'(x)$ and $\lim_{x \rightarrow (1/\sqrt{2})^-} F'(x) = -1 = \lim_{x \rightarrow (1/\sqrt{2})^+} F'(x)$, F' is continuous. But

$\lim_{x \rightarrow 0^+} F''(x) = -1 \neq 0 = \lim_{x \rightarrow 0^-} F''(x)$, so F'' is not continuous at $x = 0$. (The same is true at $x = \frac{1}{\sqrt{2}}$.)

So F does not have continuous curvature.

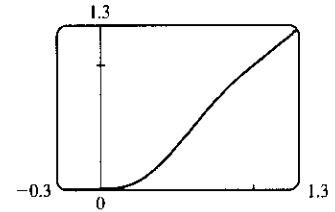
- (b) Set $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. The continuity conditions on P are $P(0) = 0$, $P(1) = 1$, $P'(0) = 0$ and $P'(1) = 1$. Also the curvature must be continuous. For $x \leq 0$ and $x \geq 1$, $\kappa(x) = 0$; elsewhere

$$\kappa(x) = \frac{|P''(x)|}{(1 + [P'(x)]^2)^{3/2}}, \text{ so we need } P''(0) = 0 \text{ and } P''(1) = 0.$$

The conditions $P(0) = P'(0) = P''(0) = 0$ imply that $d = e = f = 0$.

The other conditions imply that $a + b + c = 1$, $5a + 4b + 3c = 1$, and $10a + 6b + 3c = 0$. From these, we find that $a = 3$, $b = -8$, and $c = 6$.

Therefore $P(x) = 3x^5 - 8x^4 + 6x^3$. Since there was no solution with $a = 0$, this could not have been done with a polynomial of degree 4.



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□ PROBLEMS PLUS

1. (a) $\mathbf{r}(t) = R \cos \omega t \mathbf{i} + R \sin \omega t \mathbf{j} \Rightarrow \mathbf{v} = \mathbf{r}'(t) = -\omega R \sin \omega t \mathbf{i} + \omega R \cos \omega t \mathbf{j}$, so $\mathbf{r} = R(\cos \omega t \mathbf{i} + \sin \omega t \mathbf{j})$ and $\mathbf{v} = \omega R(-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j})$. $\mathbf{v} \cdot \mathbf{r} = \omega R^2(-\cos \omega t \sin \omega t + \sin \omega t \cos \omega t) = 0$, so $\mathbf{v} \perp \mathbf{r}$. Since $\mathbf{r}$ points along a radius of the circle, and $\mathbf{v} \perp \mathbf{r}$, $\mathbf{v}$ is tangent to the circle. Because it is a velocity vector, $\mathbf{v}$ points in the direction of motion.
- (b) In (a), we wrote $\mathbf{v}$ in the form $\omega R \mathbf{u}$, where $\mathbf{u}$ is the unit vector $-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}$. Clearly $|\mathbf{v}| = \omega R |\mathbf{u}| = \omega R$. At speed ωR , the particle completes one revolution, a distance $2\pi R$, in time $T = \frac{2\pi R}{\omega R} = \frac{2\pi}{\omega}$.
- (c) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\omega^2 R \cos \omega t \mathbf{i} - \omega^2 R \sin \omega t \mathbf{j} = -\omega^2 R(\cos \omega t \mathbf{i} + \sin \omega t \mathbf{j})$, so $\mathbf{a} = -\omega^2 \mathbf{r}$. This shows that $\mathbf{a}$ is proportional to $\mathbf{r}$ and points in the opposite direction (toward the origin). Also, $|\mathbf{a}| = \omega^2 |\mathbf{r}| = \omega^2 R$.
- (d) By Newton's Second Law (see Section 14.4 [ET 13.4]), $\mathbf{F} = m\mathbf{a}$, so

$$|\mathbf{F}| = m|\mathbf{a}| = mR\omega^2 = \frac{m(\omega R)^2}{R} = \frac{m|\mathbf{v}|^2}{R}.$$

2. (a) Dividing the equation $|\mathbf{F}| \sin \theta = \frac{mv_R^2}{R}$ by the equation $|\mathbf{F}| \cos \theta = mg$, we obtain $\tan \theta = \frac{v_R^2}{Rg}$,

$$\text{so } v_R^2 = Rg \tan \theta.$$

- (b) $R = 400$ ft and $\theta = 12^\circ$, so $v_R = \sqrt{Rg \tan \theta} \approx \sqrt{400 \cdot 32 \cdot \tan 12^\circ} \approx 52.16$ ft/s ≈ 36 mi/h.

- (c) We want to choose a new radius R_1 for which the new rated speed is $\frac{3}{2}$ of the old one:

$$\sqrt{R_1 g \tan 12^\circ} = \frac{3}{2} \sqrt{Rg \tan 12^\circ}. \text{ Squaring, we get } R_1 g \tan 12^\circ = \frac{9}{4} Rg \tan 12^\circ, \text{ so } R_1 = \frac{9}{4} R = \frac{9}{4}(400) = 900 \text{ ft.}$$

3. (a) The projectile reaches maximum height when $0 = \frac{dy}{dt} = \frac{d}{dt}[(v_0 \sin \alpha)t - \frac{1}{2}gt^2] = v_0 \sin \alpha - gt$; that is, when

$$t = \frac{v_0 \sin \alpha}{g} \text{ and } y = (v_0 \sin \alpha) \left(\frac{v_0 \sin \alpha}{g} \right) - \frac{1}{2}g \left(\frac{v_0 \sin \alpha}{g} \right)^2 = \frac{v_0^2 \sin^2 \alpha}{2g}. \text{ This is the maximum height attained when the projectile is fired with an angle of elevation } \alpha. \text{ This maximum height is largest when } \alpha = \frac{\pi}{2}.$$

$$\text{In that case, } \sin \alpha = 1 \text{ and the maximum height is } \frac{v_0^2}{2g}.$$

- (b) Let $R = v_0^2/g$. We are asked to consider the parabola $x^2 + 2Ry - R^2 = 0$ which can be rewritten as

$$y = -\frac{1}{2R}x^2 + \frac{R}{2}. \text{ The points on or inside this parabola are those for which } -R \leq x \leq R \text{ and}$$

$$0 \leq y \leq -\frac{1}{2R}x^2 + \frac{R}{2}. \text{ When the projectile is fired at angle of elevation } \alpha, \text{ the points } (x, y) \text{ along its path satisfy the relations } x = (v_0 \cos \alpha)t \text{ and } y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2, \text{ where } 0 \leq t \leq (2v_0 \sin \alpha)/g \text{ (as in}$$

$$\text{Example 14.4.5 [ET 13.4.5]). Thus } |x| \leq \left| v_0 \cos \alpha \left(\frac{2v_0 \sin \alpha}{g} \right) \right| = \left| \frac{v_0^2}{g} \sin 2\alpha \right| \leq \left| \frac{v_0^2}{g} \right| = |R|. \text{ This shows that } -R \leq x \leq R.$$

$$\text{For } t \text{ in the specified range, we also have } y = t(v_0 \sin \alpha - \frac{1}{2}gt) = \frac{1}{2}gt \left(\frac{2v_0 \sin \alpha}{g} - t \right) \geq 0 \text{ and}$$

$$y = (v_0 \sin \alpha) \frac{x}{v_0 \cos \alpha} - \frac{g}{2} \left(\frac{x}{v_0 \cos \alpha} \right)^2 = (\tan \alpha)x - \frac{g}{2v_0^2 \cos^2 \alpha} x^2 = -\frac{1}{2R \cos^2 \alpha} x^2 + (\tan \alpha)x.$$

Thus

$$\begin{aligned} y - \left(\frac{-1}{2R}x^2 + \frac{R}{2} \right) &= \frac{-1}{2R \cos^2 \alpha} x^2 + \frac{1}{2R} x^2 + (\tan \alpha) x - \frac{R}{2} \\ &= \frac{x^2}{2R} \left(1 - \frac{1}{\cos^2 \alpha} \right) + (\tan \alpha) x - \frac{R}{2} = \frac{x^2(1 - \sec^2 \alpha) + 2R(\tan \alpha)x - R^2}{2R} \\ &= \frac{-(\tan^2 \alpha)x^2 + 2R(\tan \alpha)x - R^2}{2R} = \frac{-[(\tan \alpha)x - R]^2}{2R} \leq 0 \end{aligned}$$

We have shown that every target that can be hit by the projectile lies on or inside the parabola

$$y = -\frac{1}{2R}x^2 + \frac{R}{2}. \text{ Now let } (a, b) \text{ be any point on or inside the parabola } y = -\frac{1}{2R}x^2 + \frac{R}{2}. \text{ Then}$$

$$-R \leq a \leq R \text{ and } 0 \leq b \leq -\frac{1}{2R}a^2 + \frac{R}{2}. \text{ We seek an angle } \alpha \text{ such that } (a, b) \text{ lies in the path of the projectile;}$$

$$\text{that is, we wish to find an angle } \alpha \text{ such that } b = -\frac{1}{2R \cos^2 \alpha} a^2 + (\tan \alpha) a \text{ or}$$

$$\text{equivalently } b = \frac{-1}{2R} (\tan^2 \alpha + 1) a^2 + (\tan \alpha) a. \text{ Rearranging this equation we get}$$

$$\frac{a^2}{2R} \tan^2 \alpha - a \tan \alpha + \left(\frac{a^2}{2R} + b \right) = 0 \text{ or } a^2 (\tan \alpha)^2 - 2aR(\tan \alpha) + (a^2 + 2bR) = 0 \quad (*). \text{ This quadratic}$$

equation for $\tan \alpha$ has real solutions exactly when the discriminant is nonnegative. Now $B^2 - 4AC \geq 0 \Leftrightarrow$

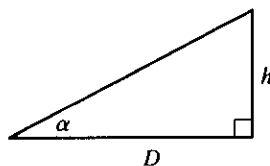
$$(-2aR)^2 - 4a^2(a^2 + 2bR) \geq 0 \Leftrightarrow 4a^2(R^2 - a^2 - 2bR) \geq 0 \Leftrightarrow -a^2 - 2bR + R^2 \geq 0 \Leftrightarrow$$

$$b \leq \frac{1}{2R}(R^2 - a^2) \Leftrightarrow b \leq -\frac{1}{2R}a^2 + \frac{R}{2}. \text{ This condition is satisfied since } (a, b) \text{ is on or inside the parabola}$$

$$y = -\frac{1}{2R}x^2 + \frac{R}{2}. \text{ It follows that } (a, b) \text{ lies in the path of the projectile when } \tan \alpha \text{ satisfies } (*), \text{ that is, when}$$

$$\tan \alpha = \frac{2aR \pm \sqrt{4a^2(R^2 - a^2 - 2bR)}}{2a^2} = \frac{R \pm \sqrt{R^2 - 2bR - a^2}}{a}.$$

(c)



If the gun is pointed at a target with height h at a distance D downrange, then $\tan \alpha = h/D$. When the projectile reaches a distance D downrange (remember we are assuming that it doesn't hit the ground first), we have

$$D = x = (v_0 \cos \alpha)t, \text{ so } t = \frac{D}{v_0 \cos \alpha} \text{ and}$$

$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = D \tan \alpha - \frac{gD^2}{2v_0^2 \cos^2 \alpha}. \text{ Meanwhile, the target, whose } x\text{-coordinate is also } D, \text{ has}$$

$$\text{fallen from height } h \text{ to height } h - \frac{1}{2}gt^2 = D \tan \alpha - \frac{gD^2}{2v_0^2 \cos^2 \alpha}. \text{ Thus the projectile hits the target.}$$

4. (a) As in Problem 3, $\mathbf{r}(t) = (v_0 \cos \alpha)t \mathbf{i} + [(v_0 \sin \alpha)t - \frac{1}{2}gt^2] \mathbf{j}$, so $x = (v_0 \cos \alpha)t$ and

$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2. \text{ The difference here is that the projectile travels until it reaches a point where } x > 0 \text{ and } y = -(\tan \theta)x. \text{ (Here } 0 \leq \theta \leq \frac{\pi}{2}.) \text{ From the parametric equations, we obtain } t = \frac{x}{v_0 \cos \alpha} \text{ and}$$

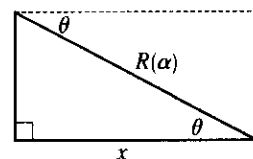
$$y = \frac{(v_0 \sin \alpha)x}{v_0 \cos \alpha} - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = (\tan \alpha)x - \frac{gx^2}{2v_0^2 \cos^2 \alpha}. \text{ Thus the projectile hits the inclined plane at the}$$

$$\text{point where } (\tan \alpha)x - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = -(\tan \theta)x. \text{ Since } \frac{gx^2}{2v_0^2 \cos^2 \alpha} = (\tan \alpha + \tan \theta)x \text{ and } x > 0,$$

$$\text{we must have } \frac{gx}{2v_0^2 \cos^2 \alpha} = \tan \alpha + \tan \theta. \text{ It follows that } x = \frac{2v_0^2 \cos^2 \alpha}{g}(\tan \alpha + \tan \theta) \text{ and}$$

$t = \frac{x}{v_0 \cos \alpha} = \frac{2v_0 \cos \alpha}{g} (\tan \alpha + \tan \theta)$. This means that the parametric equations are defined for t in the interval $\left[0, \frac{2v_0 \cos \alpha}{g} (\tan \alpha + \tan \theta)\right]$.

- (b) The downhill range (that is, the distance to the projectile's landing point as measured along the inclined plane) is $R(\alpha) = x \sec \theta$, where x is the coordinate of the landing point calculated in part (a). Thus



$$\begin{aligned} R(\alpha) &= \frac{2v_0^2 \cos^2 \alpha}{g} (\tan \alpha + \tan \theta) \sec \theta = \frac{2v_0^2}{g} \left(\frac{\sin \alpha \cos \alpha}{\cos \theta} + \frac{\cos^2 \alpha \sin \theta}{\cos^2 \theta} \right) \\ &= \frac{2v_0^2 \cos \alpha}{g \cos^2 \theta} (\sin \alpha \cos \theta + \cos \alpha \sin \theta) = \frac{2v_0^2 \cos \alpha \sin(\alpha + \theta)}{g \cos^2 \theta} \end{aligned}$$

$R(\alpha)$ is maximized when

$$\begin{aligned} 0 &= R'(\alpha) = \frac{2v_0^2}{g \cos^2 \theta} [-\sin \alpha \sin(\alpha + \theta) + \cos \alpha \cos(\alpha + \theta)] \\ &= \frac{2v_0^2}{g \cos^2 \theta} \cos[(\alpha + \theta) + \alpha] = \frac{2v_0^2 \cos(2\alpha + \theta)}{g \cos^2 \theta} \end{aligned}$$

This condition implies that $\cos(2\alpha + \theta) = 0 \Rightarrow 2\alpha + \theta = \frac{\pi}{2} \Rightarrow \alpha = \frac{1}{2}(\frac{\pi}{2} - \theta)$.

- (c) The solution is similar to the solutions to parts (a) and (b). This time the projectile travels until it reaches a point where $x > 0$ and $y = (\tan \theta)x$. Since $\tan \theta = -\tan(-\theta)$, we obtain the solution from the previous one by replacing θ with $-\theta$. The desired angle is $\alpha = \frac{1}{2}(\frac{\pi}{2} + \theta)$.

- (d) As observed in part (c), firing the projectile up an inclined plane with angle of inclination θ involves the same equations as in parts (a) and (b) but with θ replaced by $-\theta$. So if R is the distance up an inclined plane, we know

$$\text{from part (b) that } R = \frac{2v_0^2 \cos \alpha \sin(\alpha - \theta)}{g \cos^2(-\theta)} \Rightarrow v_0^2 = \frac{Rg \cos^2 \theta}{2 \cos \alpha \sin(\alpha - \theta)}. v_0^2 \text{ is minimized}$$

(and hence v_0 is minimized) with respect to α when

$$\begin{aligned} 0 &= \frac{d}{d\alpha} (v_0^2) = \frac{Rg \cos^2 \theta}{2} \cdot \frac{-(\cos \alpha \cos(\alpha - \theta) - \sin \alpha \sin(\alpha - \theta))}{[\cos \alpha \sin(\alpha - \theta)]^2} \\ &= \frac{-Rg \cos^2 \theta}{2} \cdot \frac{\cos[\alpha + (\alpha - \theta)]}{[\cos \alpha \sin(\alpha - \theta)]^2} = \frac{-Rg \cos^2 \theta}{2} \cdot \frac{\cos(2\alpha - \theta)}{[\cos \alpha \sin(\alpha - \theta)]^2} \end{aligned}$$

Since $\theta < \alpha < \frac{\pi}{2}$, this implies $\cos(2\alpha - \theta) = 0 \Leftrightarrow 2\alpha - \theta = \frac{\pi}{2} \Rightarrow \alpha = \frac{1}{2}(\frac{\pi}{2} + \theta)$. Thus the initial speed, and hence the energy required, is minimized for $\alpha = \frac{1}{2}(\frac{\pi}{2} + \theta)$.

$$5. (a) m \frac{d^2 \mathbf{R}}{dt^2} = -mg \mathbf{j} - k \frac{d\mathbf{R}}{dt} \Rightarrow \frac{d}{dt} \left(m \frac{d\mathbf{R}}{dt} + k \mathbf{R} + mgt \mathbf{j} \right) = \mathbf{0} \Rightarrow m \frac{d\mathbf{R}}{dt} + k \mathbf{R} + mgt \mathbf{j} = \mathbf{c}$$

($\mathbf{c}$ is a constant vector in the xy -plane). At $t = 0$, this says that $m \mathbf{v}(0) + k \mathbf{R}(0) = \mathbf{c}$. Since $\mathbf{v}(0) = \mathbf{v}_0$ and

$\mathbf{R}(0) = \mathbf{0}$, we have $\mathbf{c} = m \mathbf{v}_0$. Therefore $\frac{d\mathbf{R}}{dt} + \frac{k}{m} \mathbf{R} + gt \mathbf{j} = \mathbf{v}_0$, or $\frac{d\mathbf{R}}{dt} + \frac{k}{m} \mathbf{R} = \mathbf{v}_0 - gt \mathbf{j}$.

- (b) Multiplying by $e^{(k/m)t}$ gives $e^{(k/m)t} \frac{d\mathbf{R}}{dt} + \frac{k}{m} e^{(k/m)t} \mathbf{R} = e^{(k/m)t} \mathbf{v}_0 - gte^{(k/m)t} \mathbf{j}$ or

$$\frac{d}{dt} (e^{(k/m)t} \mathbf{R}) = e^{(k/m)t} \mathbf{v}_0 - gte^{(k/m)t} \mathbf{j}. \text{ Integrating gives}$$

$$e^{(k/m)t} \mathbf{R} = \frac{m}{k} e^{(k/m)t} \mathbf{v}_0 - \left[\frac{mg}{k} t e^{(k/m)t} - \frac{m^2 g}{k^2} e^{(k/m)t} \right] \mathbf{j} + \mathbf{b} \text{ for some constant vector } \mathbf{b}.$$

Setting $t = 0$ yields the relation $\mathbf{R}(0) = \frac{m}{k} \mathbf{v}_0 + \frac{m^2 g}{k^2} \mathbf{j} + \mathbf{b}$, so $\mathbf{b} = -\frac{m}{k} \mathbf{v}_0 - \frac{m^2 g}{k^2} \mathbf{j}$. Thus

$$e^{(k/m)t} \mathbf{R} = \frac{m}{k} [e^{(k/m)t} - 1] \mathbf{v}_0 - \left[\frac{mg}{k} t e^{(k/m)t} - \frac{m^2 g}{k^2} (e^{(k/m)t} - 1) \right] \mathbf{j} \text{ and}$$

$$\mathbf{R}(t) = \frac{m}{k} [1 - e^{-kt/m}] \mathbf{v}_0 + \frac{mg}{k} \left[\frac{m}{k} (1 - e^{-kt/m}) - t \right] \mathbf{j}.$$

6. By the Fundamental Theorem of Calculus, $\mathbf{r}'(t) = \langle \sin(\pi t^2/2), \cos(\pi t^2/2) \rangle$, $|\mathbf{r}'(t)| = 1$ and so $\mathbf{T}(t) = \mathbf{r}'(t)$.

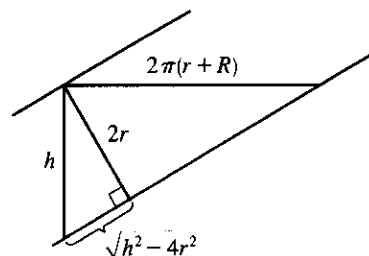
Thus $\mathbf{T}'(t) = \pi t \langle \sin(\pi t^2/2), \cos(\pi t^2/2) \rangle$ and the curvature is $\kappa = |\mathbf{T}'(t)| = \sqrt{(\pi t)^2(1)} = \pi |t|$.

7. (a) $\mathbf{a} = -g\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{v}_0 - gt\mathbf{j} = 2\mathbf{i} - gt\mathbf{j} \Rightarrow \mathbf{s} = \mathbf{s}_0 + 2t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} = 3.5\mathbf{j} + 2t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} \Rightarrow \mathbf{s} = 2t\mathbf{i} + (3.5 - \frac{1}{2}gt^2)\mathbf{j}$. Therefore $y = 0$ when $t = \sqrt{7/g}$ seconds. At that instant, the ball is $2\sqrt{7/g} \approx 0.94$ ft to the right of the table top. Its coordinates (relative to an origin on the floor directly under the table's edge) are $(0.94, 0)$. At impact, the velocity is $\mathbf{v} = 2\mathbf{i} - \sqrt{7g}\mathbf{j}$, so the speed is $|\mathbf{v}| = \sqrt{4+7g} \approx 15$ ft/s.
- (b) The slope of the curve when $t = \sqrt{7/g}$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-gt}{2} = \frac{-g\sqrt{7/g}}{2} = \frac{-\sqrt{7g}}{2}$. Thus $\cot \theta = \frac{\sqrt{7g}}{2}$ and $\theta \approx 7.6^\circ$.
- (c) From (a), $|\mathbf{v}| = \sqrt{4+7g}$. So the ball rebounds with speed $0.8\sqrt{4+7g} \approx 12.08$ ft/s at angle of inclination $90^\circ - \theta \approx 82.3886^\circ$. By Example 14.4.5 [ET 13.4.5], the horizontal distance traveled between bounces is $d = \frac{v_0^2 \sin 2\alpha}{g}$, where $v_0 \approx 12.08$ ft/s and $\alpha \approx 82.3886^\circ$. Therefore, $d \approx 1.197$ ft. So the ball strikes the floor at about $2\sqrt{7/g} + 1.197 \approx 2.13$ ft to the right of the table's edge.

8. As the cable is wrapped around the spool, think of the top or bottom of the cable forming a helix of radius $R + r$. Let h be the vertical distance between coils. Then, from similar triangles,

$$\frac{2r}{\sqrt{h^2 - 4r^2}} = \frac{2\pi(r+R)}{h} \Rightarrow h^2 r^2 = \pi^2 (r+R)^2 (h^2 - 4r^2)$$

$$\Rightarrow h = \frac{2\pi r(r+R)}{\sqrt{\pi^2 (r+R)^2 - r^2}}.$$



If we parametrize the helix by $x(t) = (R+r)\cos t$, $y(t) = (R+r)\sin t$, then we must have $z(t) = [h/(2\pi)]t$.

The length of one complete cycle is

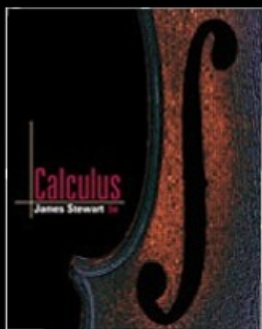
$$\begin{aligned} \ell &= \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt = \int_0^{2\pi} \sqrt{(R+r)^2 + \left(\frac{h}{2\pi}\right)^2} dt = 2\pi \sqrt{(R+r)^2 + \left(\frac{h}{2\pi}\right)^2} \\ &= 2\pi \sqrt{(R+r)^2 + \frac{r^2(R+r)^2}{\pi^2(R+r)^2 - r^2}} = 2\pi(R+r) \sqrt{1 + \frac{r^2}{\pi^2(R+r)^2 - r^2}} = \frac{2\pi^2(R+r)^2}{\sqrt{\pi^2(R+r)^2 - r^2}} \end{aligned}$$

The number of complete cycles is $\lceil L/\ell \rceil$, and so the shortest length along the spool is

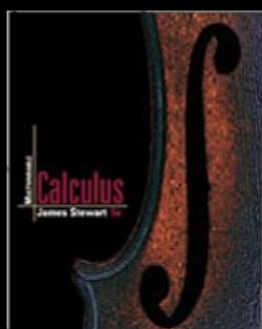
$$h \left\lceil \frac{L}{\ell} \right\rceil = \frac{2\pi r(R+r)}{\sqrt{\pi^2(R+r)^2 - r^2}} \left\lceil \frac{L \sqrt{\pi^2(R+r)^2 - r^2}}{2\pi^2(R+r)^2} \right\rceil$$

Chapter 15

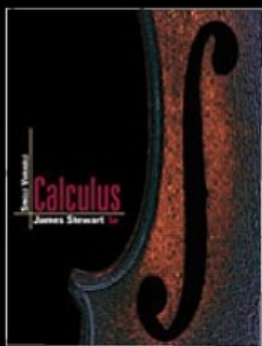
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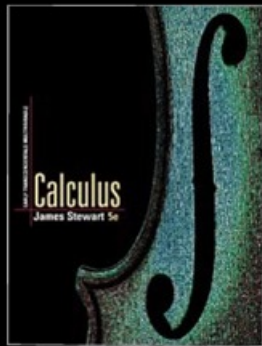
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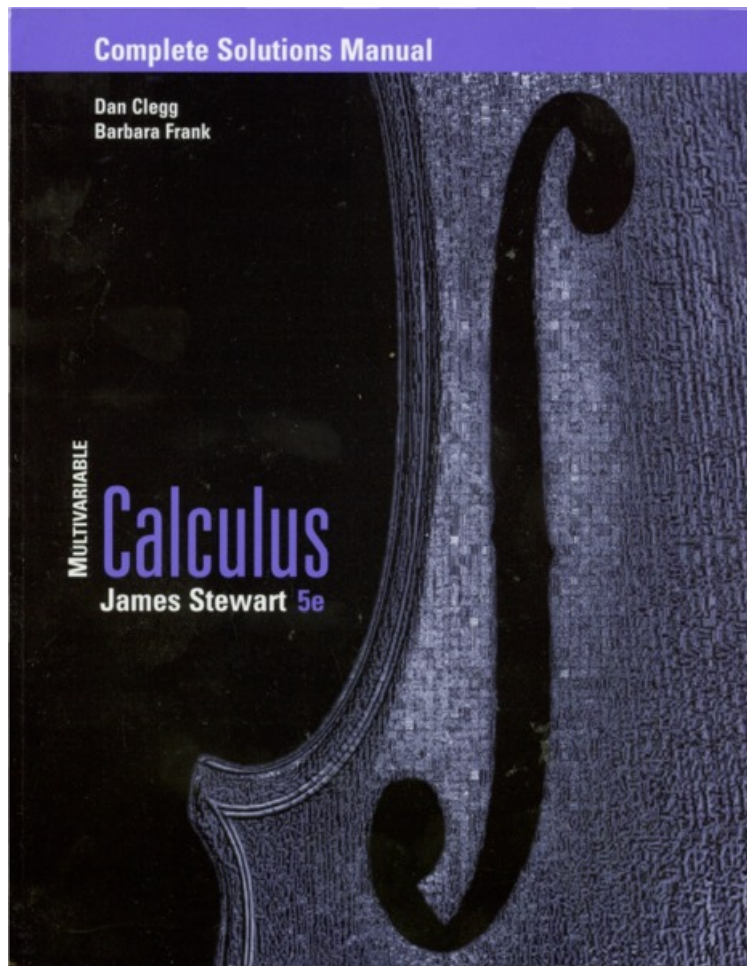
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15 ☐ PARTIAL DERIVATIVES

☐ ET 14

15.1 Functions of Several Variables

ET 14.1

1. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is -15°C and the wind speed is 40 km/h, then the air would feel equivalent to approximately -27°C without wind.
(b) The question is asking: when the temperature is -20°C , what wind speed gives a wind-chill index of -30°C ?
From Table 1, the speed is 20 km/h.
(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of -49°C ?
From Table 1, the temperature is -35°C .
(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is -5°C . From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.
(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h. From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.
2. (a) From the table, $f(95, 70) = 124$, which means that when the actual temperature is 95°F and the relative humidity is 70%, the perceived air temperature is approximately 124°F .
(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.
(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.
(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is 80°F . Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is 100°F . Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately 1°F per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of 5°F per 10% relative humidity) and at an increasing rate (approximately 12°F per 10% relative humidity for larger values of h).

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$\begin{aligned} P(2L, 2K) &= 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^1)1.01L^{0.75}K^{0.25} \\ &= 2P(L, K) \end{aligned}$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^\alpha K^{1-\alpha}$:

$$P(2L, 2K) = b(2L)^\alpha(2K)^{1-\alpha} = b(2^\alpha)(2^{1-\alpha})L^\alpha K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^\alpha K^{1-\alpha} = 2P(L, K).$$

4. We compare the values for the wind-chill index given by Table 1 with those given by the model function:

Modeled Wind-Chill Index Values $W(T, v)$

		Wind Speed (km/h)										
Actual temperature (°C)	$T \backslash v$	5	10	15	20	25	30	40	50	60	70	80
	5	4.08	2.66	1.74	1.07	0.52	0.05	-0.71	-1.33	-1.85	-2.30	-2.70
	0	-1.59	-3.31	-4.42	-5.24	-5.91	-6.47	-7.40	-8.14	-8.77	-9.32	-9.80
	-5	-7.26	-9.29	-10.58	-11.55	-12.34	-13.00	-14.08	-14.96	-15.70	-16.34	-16.91
	-10	-12.93	-15.26	-16.75	-17.86	-18.76	-19.52	-20.77	-21.77	-22.62	-23.36	-24.01
	-15	-18.61	-21.23	-22.91	-24.17	-25.19	-26.04	-27.45	-28.59	-29.54	-30.38	-31.11
	-20	-24.28	-27.21	-29.08	-30.48	-31.61	-32.57	-34.13	-35.40	-36.47	-37.40	-38.22
	-25	-29.95	-33.18	-35.24	-36.79	-38.04	-39.09	-40.82	-42.22	-43.39	-44.42	-45.32
	-30	-35.62	-39.15	-41.41	-43.10	-44.46	-45.62	-47.50	-49.03	-50.32	-51.44	-52.43
	-35	-41.30	-45.13	-47.57	-49.41	-50.89	-52.14	-54.19	-55.84	-57.24	-58.46	-59.53
	-40	-46.97	-51.10	-53.74	-55.72	-57.31	-58.66	-60.87	-62.66	-64.17	-65.48	-66.64

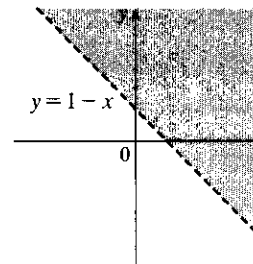
The values given by the function appear to be fairly close (within 0.5) to the values in Table 1.

5. (a) According to the table, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.
- (b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.
- (c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

6. (a) $f(1, 1) = \ln(1 + 1 - 1) = \ln 1 = 0$

(b) $f(e, 1) = \ln(e + 1 - 1) = \ln e = 1$

(c) $\ln(x + y - 1)$ is defined only when $x + y - 1 > 0$, that is,
 $y > 1 - x$. So the domain of f is $\{(x, y) \mid y > 1 - x\}$.



(d) Since $\ln(x + y - 1)$ can be any real number, the range is $\mathbb{R}$.

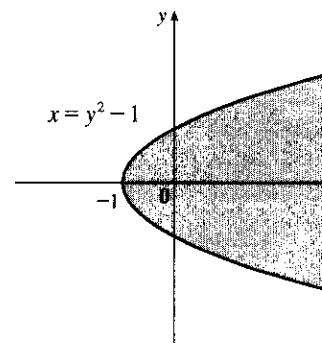
7. (a) $f(2, 0) = 2^2 e^{3(2)(0)} = 4(1) = 4$

(b) Since both x^2 and the exponential function are defined everywhere, $x^2 e^{3xy}$ is defined for all choices of values for x and y . Thus, the domain of f is $\mathbb{R}^2$.

(c) Because the range of $g(x, y) = 3xy$ is $\mathbb{R}$, and the range of e^x is $(0, \infty)$, the range of $e^{g(x, y)} = e^{3xy}$ is $(0, \infty)$. The range of x^2 is $[0, \infty)$, so the range of the product $x^2 e^{3xy}$ is $[0, \infty)$.

8. $\sqrt{1 + x - y^2}$ is defined only when $1 + x - y^2 \geq 0 \Rightarrow$
 $x \geq y^2 - 1$, so the domain of f is $\{(x, y) \mid x \geq y^2 - 1\}$, all
those points on or to the right of the parabola $x = y^2 - 1$.

The range of f is $[0, \infty)$.



9. (a) $f(2, -1, 6) = e^{\sqrt{6 - 2^2 - (-1)^2}} = e^{\sqrt{1}} = e$.

(b) $e^{\sqrt{z - x^2 - y^2}}$ is defined when $z - x^2 - y^2 \geq 0 \Rightarrow z \geq x^2 + y^2$. Thus the domain of f is
 $\{(x, y, z) \mid z \geq x^2 + y^2\}$.

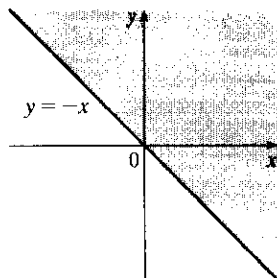
(c) Since $\sqrt{z - x^2 - y^2} \geq 0$, we have $e^{\sqrt{z - x^2 - y^2}} \geq 1$. Thus the range of f is $[1, \infty)$.

10. (a) $g(2, -2, 4) = \ln(25 - 2^2 - (-2)^2 - 4^2) = \ln 1 = 0$.

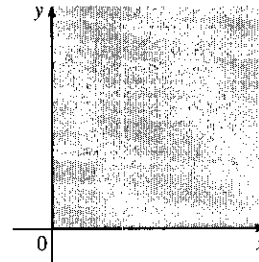
(b) For the logarithmic function to be defined, we need $25 - x^2 - y^2 - z^2 > 0$. Thus the domain of g is
 $\{(x, y, z) \mid x^2 + y^2 + z^2 < 25\}$, the interior of the sphere $x^2 + y^2 + z^2 = 25$.

(c) Since $0 < 25 - x^2 - y^2 - z^2 \leq 25$ for (x, y, z) in the domain of g , $\ln(25 - x^2 - y^2 - z^2) \leq \ln 25$. Thus the range of g is $(-\infty, \ln 25]$.

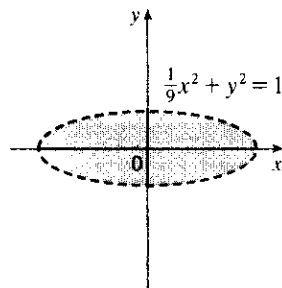
11. $\sqrt{x+y}$ is defined only when $x+y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x, y) \mid y \geq -x\}$.



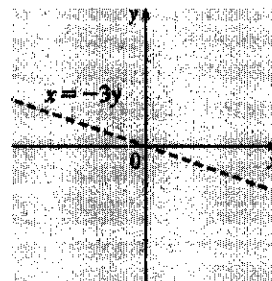
12. We need $x \geq 0$ and $y \geq 0$, so $D = \{(x, y) \mid x \geq 0 \text{ and } y \geq 0\}$, the first quadrant.



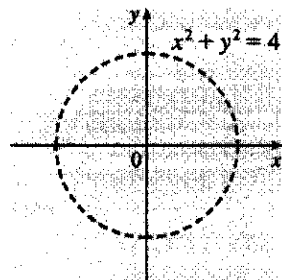
13. $\ln(9 - x^2 - 9y^2)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9}x^2 + y^2 < 1$. So the domain of f is $\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\}$, the interior of an ellipse.



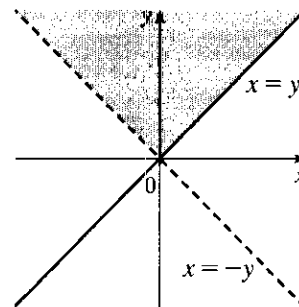
14. $\frac{x-3y}{x+3y}$ is defined only when $x+3y \neq 0$, or $x \neq -3y$. So the domain of f is $\{(x, y) \mid x \neq -3y\}$.



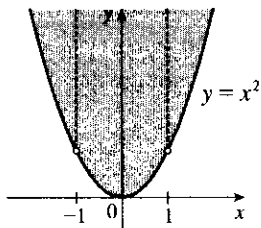
15. $\frac{3x+5y}{x^2+y^2-4}$ is defined only when $x^2+y^2-4 \neq 0$, or $x^2+y^2 \neq 4$. So the domain of f is $\{(x, y) \mid x^2+y^2 \neq 4\}$.



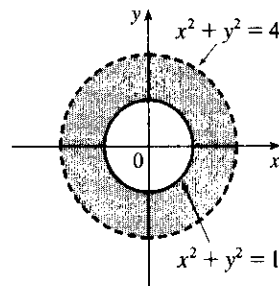
16. We need $y-x \geq 0$ or $y \geq x$ and $y+x > 0$ or $x > -y$. Thus $D = \{(x, y) \mid -y < x \leq y, y > 0\}$.



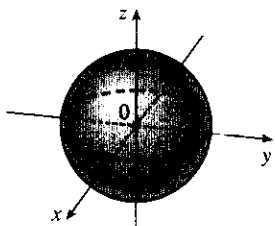
17. $\sqrt{y - x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0 \Rightarrow x = \pm 1$. Thus the domain of f is $\{(x, y) \mid y \geq x^2, x \neq \pm 1\}$.



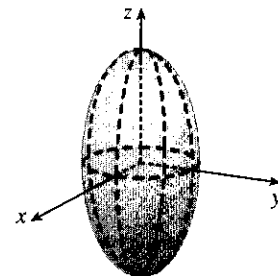
18. f is defined only when $x^2 + y^2 - 1 \geq 0 \Rightarrow x^2 + y^2 \geq 1$ and $4 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 4$. Thus $D = \{(x, y) \mid 1 \leq x^2 + y^2 < 4\}$.



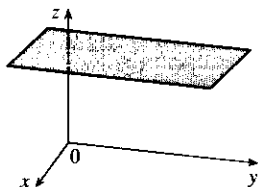
19. We need $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, so $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ (the points inside or on the sphere of radius 1, center the origin).



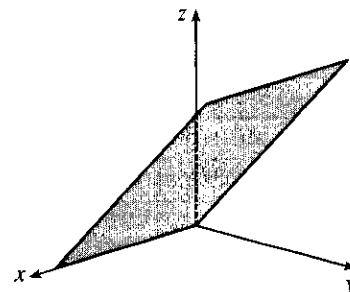
20. f is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} < 1$. Thus, $D = \{(x, y, z) \mid \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} < 1\}$, that is, the points inside the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} = 1$.



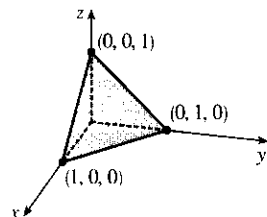
21. $z = 3$, a horizontal plane through the point $(0, 0, 3)$.



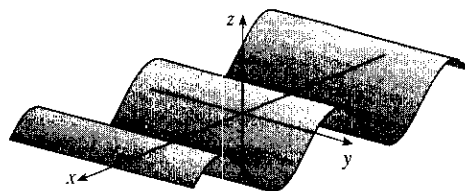
22. $z = y$, a plane which intersects the yz -plane in the line $z = y, x = 0$. The portion of this plane that lies in the first octant is shown.



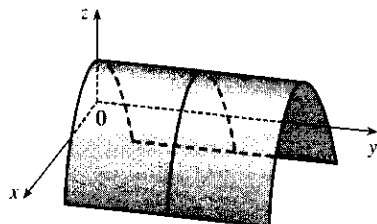
23. $z = 1 - x - y$ or $x + y + z = 1$, a plane with intercepts 1, 1, and 1.



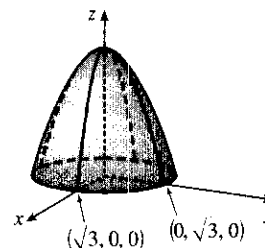
24. $z = \cos x$, a "wave."



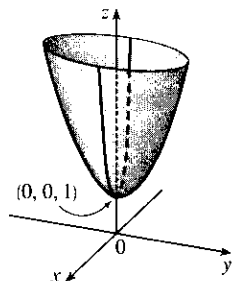
25. $z = 1 - x^2$, a parabolic cylinder.



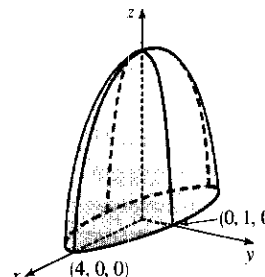
26. $z = 3 - x^2 - y^2$, a circular paraboloid with vertex at $(0, 0, 3)$.



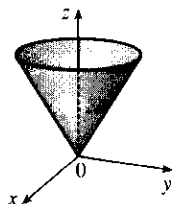
27. $z = 4x^2 + y^2 + 1$, an elliptic paraboloid with vertex at $(0, 0, 1)$.



28. $z = \sqrt{16 - x^2 - 16y^2}$ so $z \geq 0$ and $z^2 + x^2 + 16y^2 = 16$, the top half of an ellipsoid.



29. $z = \sqrt{x^2 + y^2}$ so $x^2 + y^2 = z^2$ and $z \geq 0$, the top half of a right circular cone.



30. All six graphs have different traces in the planes $x = 0$ and $y = 0$, so we investigate these for each function.

(a) $f(x, y) = |x| + |y|$. The trace in $x = 0$ is $z = |y|$, and in $y = 0$ is $z = |x|$, so it must be graph VI.

(b) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and in $y = 0$ is $z = 0$, so it must be graph V.

(c) $f(x, y) = \frac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \frac{1}{1 + y^2}$, and in $y = 0$ is $z = \frac{1}{1 + x^2}$. In addition, we can see that f is close to 0 for large values of x and y , so this is graph I.

(d) $f(x, y) = (x^2 - y^2)^2$. The trace in $x = 0$ is $z = y^4$, and in $y = 0$ is $z = x^4$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x^2 - y^2)^2 \Rightarrow y = \pm x$, so it must be graph IV.

(e) $f(x, y) = (x - y)^2$. The trace in $x = 0$ is $z = y^2$, and in $y = 0$ is $z = x^2$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x - y)^2 \Rightarrow y = x$, so it must be graph II.

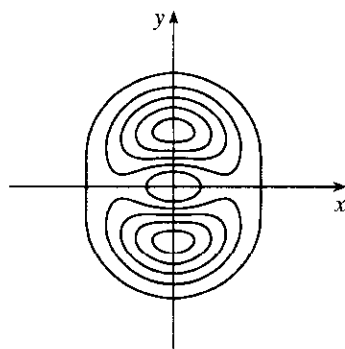
(f) $f(x, y) = \sin(|x| + |y|)$. The trace in $x = 0$ is $z = \sin|y|$, and in $y = 0$ is $z = \sin|x|$. In addition, notice that the oscillating nature of the graph is characteristic of trigonometric functions. So this is graph III.

31. The point $(-3, 3)$ lies between the level curves with z -values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z -values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

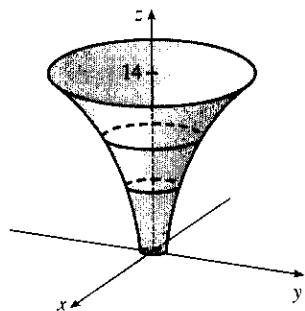
32. If we start at the origin and move along the x -axis, for example, the z -values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z -values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

33. Near A , the level curves are very close together, indicating that the terrain is quite steep. At B , the level curves are much farther apart, so we would expect the terrain to be much less steep than near A , perhaps almost flat.

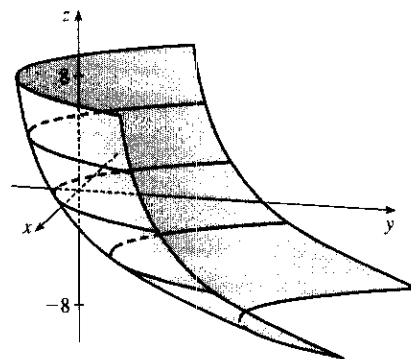
34.



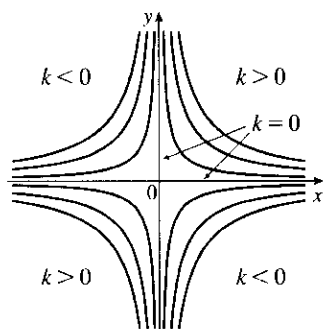
35.



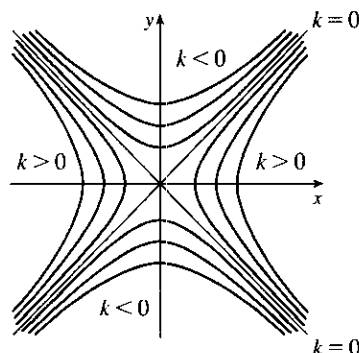
36.



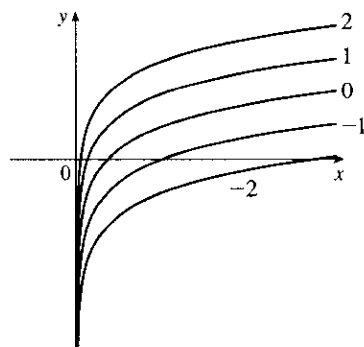
37. The level curves are $xy = k$. For $k = 0$ the curves are the coordinate axis; if $k > 0$, they are hyperbolas in the first and third quadrants; if $k < 0$, they are hyperbolas in the second and fourth quadrants.



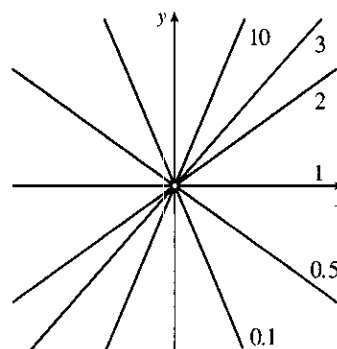
38. The level curves are $k = x^2 - y^2$. When $k = 0$, these are the lines $y = \pm x$. When $k > 0$, the curves are hyperbolas with axis the x -axis and when $k < 0$, they are hyperbolas with axis the y -axis.



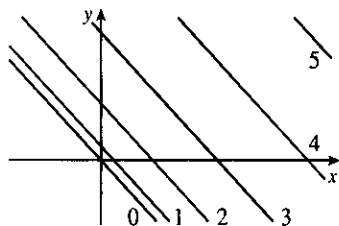
39. The level curves are $y - \ln x = k$ or $y = \ln x + k$.



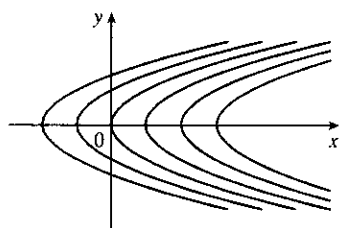
40. The level curves are $e^{y/x} = k$ or equivalently $y = x \ln k$ ($x \neq 0$), a family of lines with slope $\ln k$ ($k > 0$) without the origin.



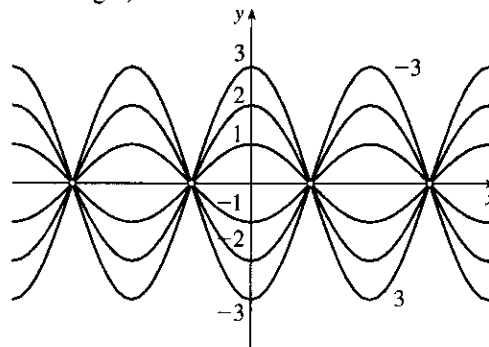
41. $k = \sqrt{x+y}$ or for $x+y \geq 0$, $k^2 = x+y$,
or $y = -x + k^2$.
Note: $k \geq 0$ since $k = \sqrt{x+y}$.



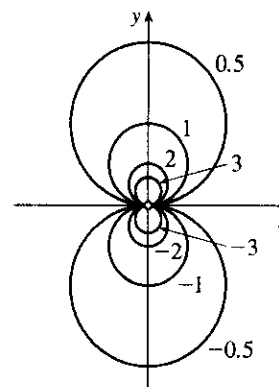
43. $k = x - y^2$, or $x - k = y^2$, a family
of parabolas with vertex $(k, 0)$.



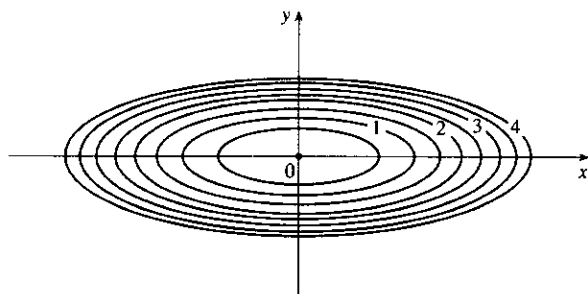
42. $k = y \sec x$ or $y = k \cos x$, $x \neq \frac{\pi}{2} + n\pi$
(n an integer).



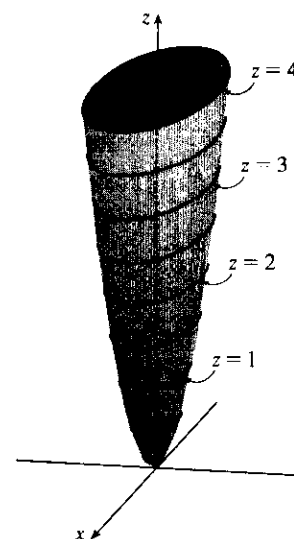
44. For $k \neq 0$ and $(x, y) \neq (0, 0)$, $k = \frac{y}{x^2 + y^2} \Leftrightarrow$
 $x^2 + y^2 - \frac{y}{k} = 0 \Leftrightarrow x^2 + (y - \frac{1}{2k})^2 = \frac{1}{4k^2}$, a family
of circles with center $(0, \frac{1}{2k})$ and radius $\frac{1}{2k}$ (without the
origin). If $k = 0$, the level curve is the x -axis.



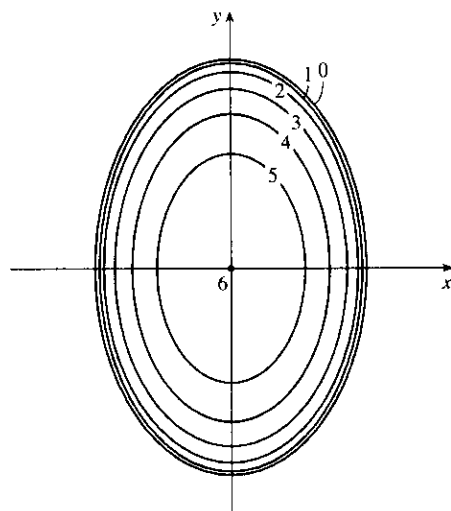
45. The contour map consists of the level curves $k = x^2 + 9y^2$, a family
of ellipses with major axis the x -axis. (Or, if $k = 0$, the origin.)
The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic
paraboloid.



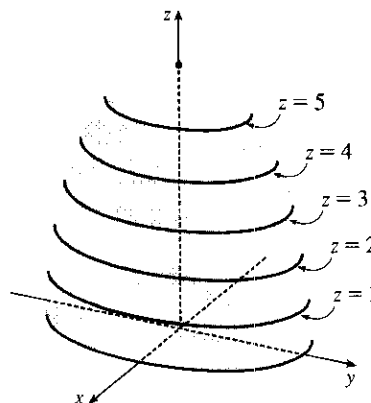
If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map
to the plane $z = k$, we have horizontal traces that indicate the shape
of the graph of f .



46.



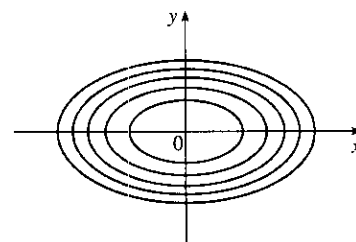
The contour map consists of the level curves $k = \sqrt{36 - 9x^2 - 4y^2}$
 $\Rightarrow 9x^2 + 4y^2 = 36 - k^2, k \geq 0$, a family of ellipses with major axis the y -axis. (Or, if $k = 6$, the origin.)



The graph of $f(x, y)$ is the surface $z = \sqrt{36 - 9x^2 - 4y^2}$, or equivalently the upper half of the ellipsoid $9x^2 + 4y^2 + z^2 = 36$. If we visualize lifting each ellipse $k = \sqrt{36 - 9x^2 - 4y^2}$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f .

47. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or

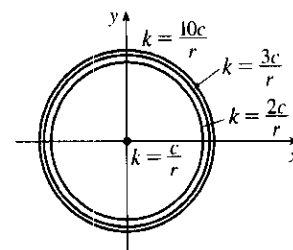
$x^2 + 2y^2 = (100 - k)/k$ ($0 < k \leq 100$), a family of ellipses.



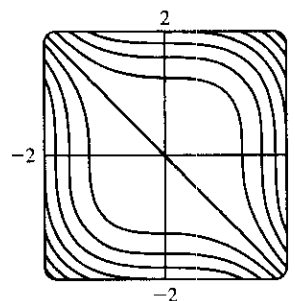
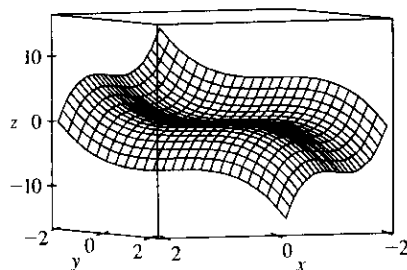
48. The equipotential curves are $k = \frac{c}{\sqrt{r^2 - x^2 - y^2}}$ or

$x^2 + y^2 = r^2 - \left(\frac{c}{k}\right)^2$, a family of circles ($k \geq c/r$).

Note: As $k \rightarrow \infty$, the radius of the circle approaches r .

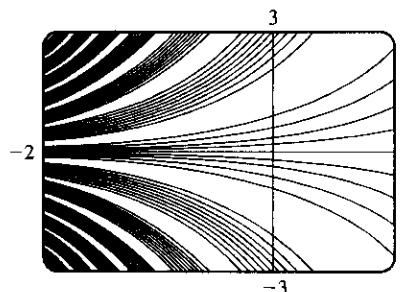
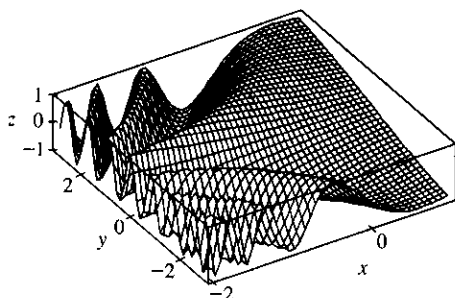


49. $f(x, y) = x^3 + y^3$



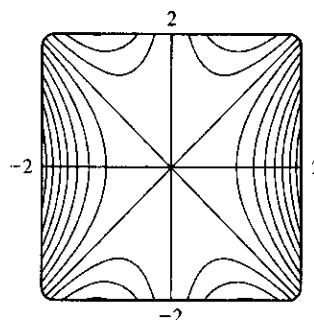
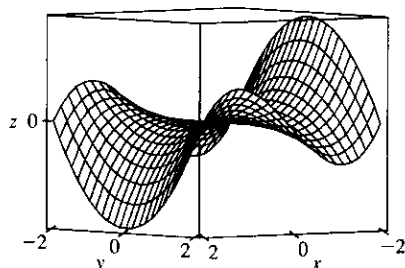
Note that the function is 0 along the line $y = -x$.

50. $f(x, y) = \sin(ye^{-x})$



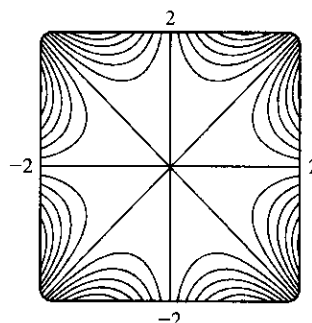
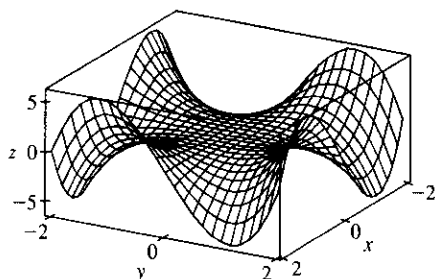
Cross-sections parallel to the yz -plane (such as the left-front trace in the first graph above) are sine-like curves. The periods of these curves decrease as x decreases.

51. $f(x, y) = xy^2 - x^3$



The traces parallel to the yz -plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz -plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

52. $f(x, y) = xy^3 - yx^3$



The traces parallel to either the yz -plane or the xz -plane are cubic curves.

53. (a) B *Reasons:* This function is constant on any circle centered at the origin, a description which matches only B and III.
 (b) III
54. (a) C *Reasons:* This function is the same if x is interchanged with y , so its graph is symmetric about the plane $x = y$. Also, $z(0, 0) = 0$ and the values of z approach 0 as we use points farther from the origin. These conditions are satisfied only by C and II.
 (b) II

55. (a) F *Reasons:* z increases without bound as we use points closer to the origin, a condition satisfied only
 (b) V by F and V.

56. (a) A *Reasons:* Along the lines $y = \pm \frac{1}{\sqrt{3}}x$ and $x = 0$, this function is 0.
 (b) VI

57. (a) D *Reasons:* This function is periodic in both x and y , with period 2π in each variable.
 (b) IV

58. (a) E *Reasons:* This function is periodic along the x -axis, and increases as $|y|$ increases.
 (b) I

59. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

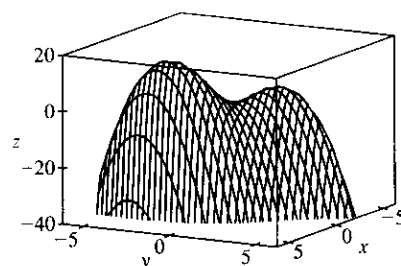
60. $k = x^2 + 3y^2 + 5z^2$ is a family of ellipsoids for $k > 0$ and the origin for $k = 0$.

61. $k = x^2 - y^2 + z^2$ are the equations of the level surfaces. For $k = 0$, the surface is a right circular cone with vertex the origin and axis the y -axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the y -axis. For $k < 0$, we have a family of hyperboloids of two sheets with axis the y -axis.

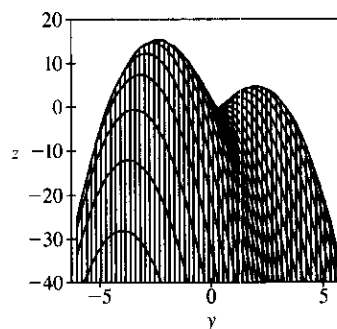
62. $k = x^2 - y^2$ is a family of hyperbolic cylinders. The cross section of this family in the xy -plane has the same graph as the level curves in Exercise 38.

63. (a) The graph of g is the graph of f shifted upward 2 units.
 (b) The graph of g is the graph of f stretched vertically by a factor of 2.
 (c) The graph of g is the graph of f reflected about the xy -plane.
 (d) The graph of $g(x, y) = -f(x, y) + 2$ is the graph of f reflected about the xy -plane and then shifted upward 2 units.
64. (a) The graph of g is the graph of f shifted 2 units in the positive x -direction.
 (b) The graph of g is the graph of f shifted 2 units in the negative y -direction.
 (c) The graph of g is the graph of f shifted 3 units in the negative x -direction and 4 units in the positive y -direction.

65. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



Three-dimensional view

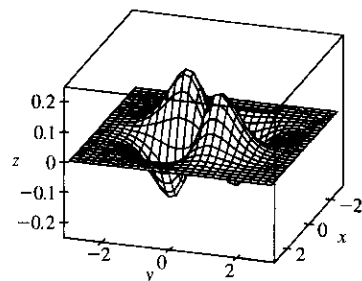


Front view

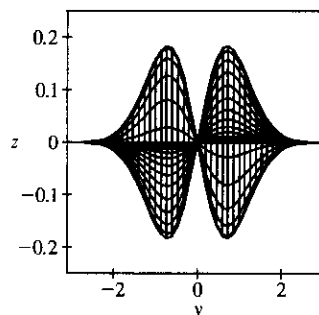
It does appear that the function has a maximum value, at the higher of the two “hilltops.” From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as

the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

66. $f(x, y) = xye^{-x^2-y^2}$

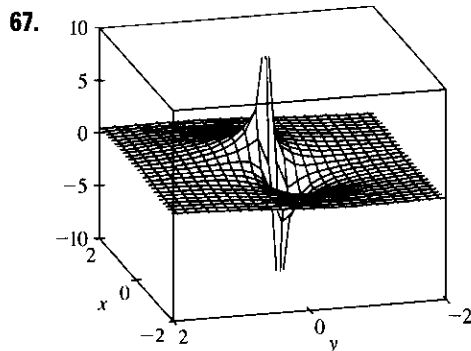


Three-dimensional view

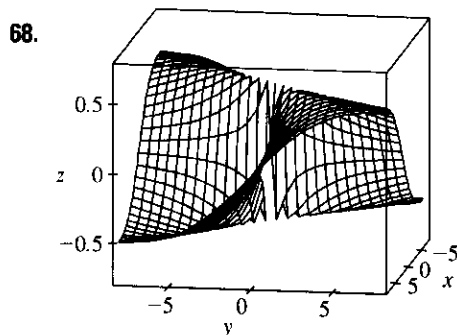


Front view

The function does have a maximum value, which it appears to achieve at two different points (the two “hilltops”). From the front view graph, we can estimate the maximum value to be approximately 0.18. These same two points can also be considered local maximum points. The two “valley bottoms” visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f .

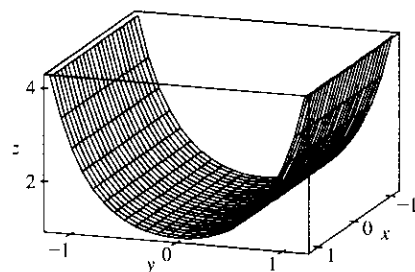


$f(x, y) = \frac{x+y}{x^2+y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x, y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x, y) \rightarrow \infty$, while in others $f(x, y) \rightarrow -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x, y)$ approaches 0 along the line $y = -x$.



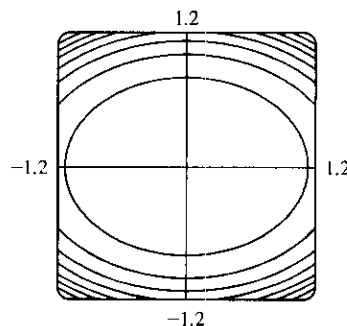
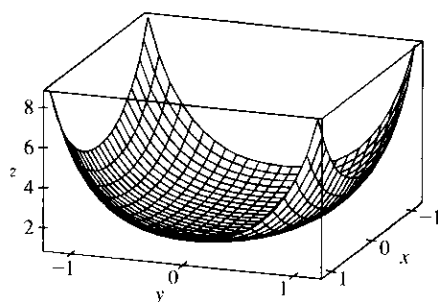
$f(x, y) = \frac{xy}{x^2+y^2}$. The graph exhibits different limiting values as x and y become large or as (x, y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y = x$, regardless of the distance from the origin. Along the line $y = -x$, the value is always $-\frac{1}{2}$. Along the axes, $f(x, y) = 0$ for all values of (x, y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

69. $f(x, y) = e^{cx^2 + y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y -axis remains fixed while the sides of the surface in the x -direction “curl” upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



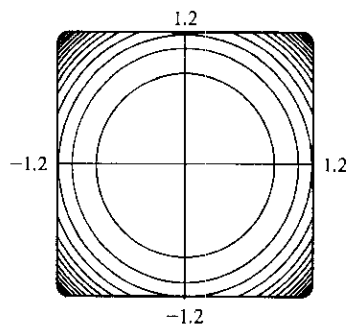
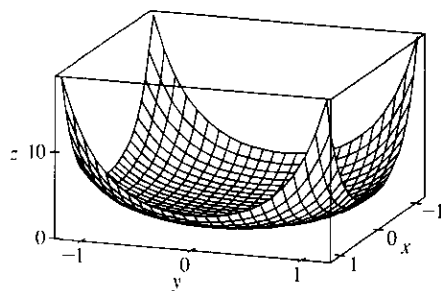
$c = 0$

For $0 < c < 1$, the ellipses have major axis the x -axis and the eccentricity increases as $c \rightarrow 0$.



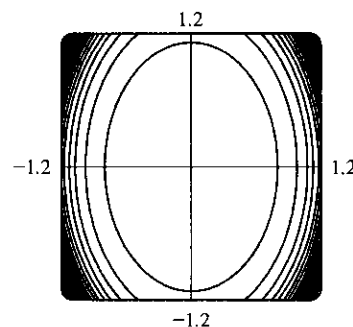
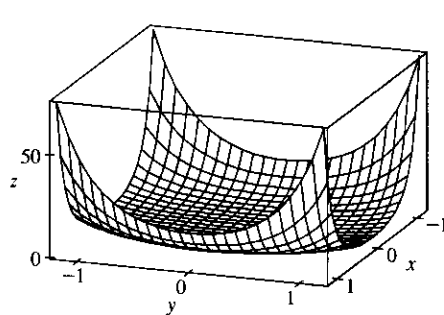
$c = 0.5$ (level curves in increments of 1)

For $c = 1$ the level curves are circles centered at the origin.



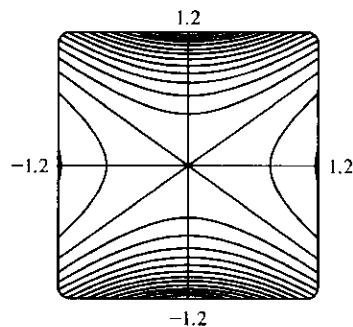
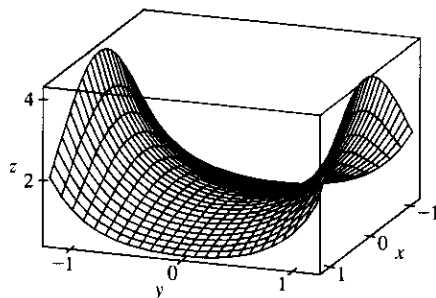
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y -axis, and the eccentricity increases as c increases.

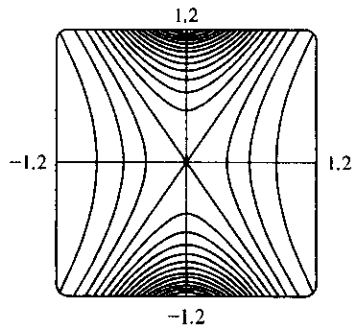
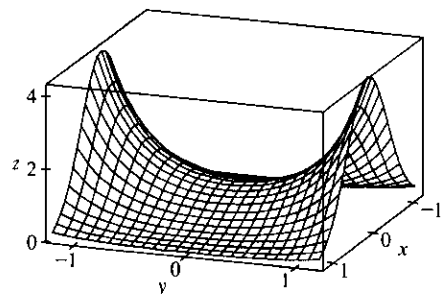


$c = 2$ (level curves in increments of 4)

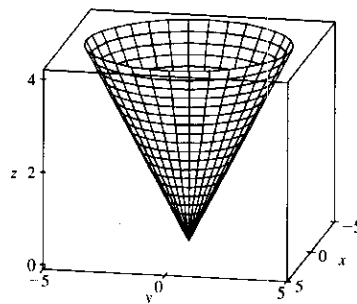
For values of $c < 0$, the sides of the surface in the x -direction curl downward and approach the xy -plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x -direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.



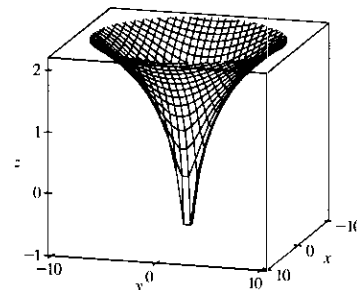
$c = -0.5$ (level curves in increments of 0.25)



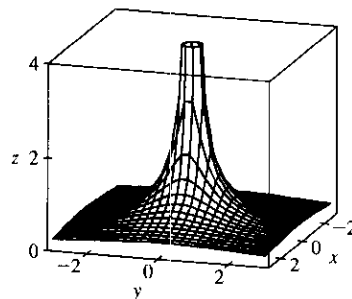
$c = -2$ (level curves in increments of 0.25)



$$z = r, r \geq 0$$



$$f(x, y) = \ln \sqrt{x^2 + y^2}$$



$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

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71. (a) $P = bL^\alpha K^{1-\alpha} \Rightarrow \frac{P}{K} = bL^\alpha K^{-\alpha} \Rightarrow \frac{P}{K} = b\left(\frac{L}{K}\right)^\alpha \Rightarrow \ln \frac{P}{K} = \ln\left(b\left(\frac{L}{K}\right)^\alpha\right) \Rightarrow$
 $\ln \frac{P}{K} = \ln b + \alpha \ln\left(\frac{L}{K}\right)$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1899	0	0
1900	-0.02	-0.06
1901	-0.04	-0.02
1902	-0.04	0
1903	-0.07	-0.05
1904	-0.13	-0.12
1905	-0.18	-0.04
1906	-0.20	-0.07
1907	-0.23	-0.15
1908	-0.41	-0.38
1909	-0.33	-0.24
1910	-0.35	-0.27

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1911	-0.38	-0.34
1912	-0.38	-0.24
1913	-0.41	-0.25
1914	-0.47	-0.37
1915	-0.53	-0.34
1916	-0.49	-0.28
1917	-0.53	-0.39
1918	-0.60	-0.50
1919	-0.68	-0.57
1920	-0.74	-0.57
1921	-1.05	-0.85
1922	-0.98	-0.59

After entering the (x, y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \Rightarrow b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^\alpha K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

15.2 Limits and Continuity

ET 14.2

- In general, we can't say anything about $f(3, 1)$! $\lim_{(x,y) \rightarrow (3,1)} f(x, y) = 6$ means that the values of $f(x, y)$ approach 6 as (x, y) approaches, but is not equal to, $(3, 1)$. If f is continuous, we know that $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$, so $\lim_{(x,y) \rightarrow (3,1)} f(x, y) = f(3, 1) = 6$.
- The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.
 - Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.
 - The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x, y) = \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of (x, y) points near the origin.

$x \backslash y$	-0.2	-0.1	-0.05	0	0.05	0.1	0.2
-0.2	-2.551	-2.525	-2.513	-2.500	-2.488	-2.475	-2.451
-0.1	-2.525	-2.513	-2.506	-2.500	-2.494	-2.488	-2.475
-0.05	-2.513	-2.506	-2.503	-2.500	-2.497	-2.494	-2.488
0	-2.500	-2.500	-2.500		-2.500	-2.500	-2.500
0.05	-2.488	-2.494	-2.497	-2.500	-2.503	-2.506	-2.513
0.1	-2.475	-2.488	-2.494	-2.500	-2.506	-2.513	-2.525
0.2	-2.451	-2.475	-2.488	-2.500	-2.513	-2.525	-2.551

As the table shows, the values of $f(x, y)$ seem to approach -2.5 as (x, y) approaches the origin from a variety of different directions. This suggests that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = -2.5$.

Since f is a rational function, it is continuous on its domain. f is defined at $(0, 0)$, so we can use direct substitution to establish that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \frac{0^2 0^3 + 0^3 0^2 - 5}{2 - 0 \cdot 0} = -\frac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x, y) = \frac{2xy}{x^2 + 2y^2}$ for a set of (x, y) points near the origin.

$x \backslash y$	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
-0.3	0.667	0.706	0.545	0.000	-0.545	-0.706	-0.667
-0.2	0.545	0.667	0.667	0.000	-0.667	-0.667	-0.545
-0.1	0.316	0.444	0.667	0.000	-0.667	-0.444	-0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	-0.316	-0.444	-0.667	0.000	0.667	0.444	0.316
0.2	-0.545	-0.667	-0.667	0.000	0.667	0.667	0.545
0.3	-0.667	-0.706	-0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x, y)$ are not approaching a single value as (x, y) approaches the origin. For verification, if we first approach $(0, 0)$ along the x -axis, we have $f(x, 0) = 0$, so $f(x, y) \rightarrow 0$. But if we approach $(0, 0)$ along the line $y = x$, $f(x, x) = \frac{2x^2}{x^2 + 2x^2} = \frac{2}{3}$ ($x \neq 0$), so $f(x, y) \rightarrow \frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x, y) = x^5 + 4x^3y - 5xy^2$ is a polynomial, and hence continuous, so $\lim_{(x,y) \rightarrow (5,-2)} f(x, y) = f(5, -2) = 5^5 + 4(5)^3(-2) - 5(5)(-2)^2 = 2025$.
6. $x - 2y$ is a polynomial and therefore continuous. Since $\cos t$ is a continuous function, the composition $\cos(x - 2y)$ is also continuous. xy is also a polynomial, and hence continuous, so the product $f(x, y) = xy \cos(x - 2y)$ is a continuous function. Then $\lim_{(x,y) \rightarrow (6,3)} f(x, y) = f(6, 3) = (6)(3) \cos(6 - 2 \cdot 3) = 18$.

7. $f(x, y) = x^2/(x^2 + y^2)$. First approach $(0, 0)$ along the x -axis. Then $f(x, 0) = x^2/x^2 = 1$ for $x \neq 0$, so $f(x, y) \rightarrow 1$. Now approach $(0, 0)$ along the y -axis. Then for $y \neq 0$, $f(0, y) = 0$, so $f(x, y) \rightarrow 0$. Since f has two different limits along two different lines, the limit does not exist.
8. $f(x, y) = (x^2 + \sin^2 y)/(2x^2 + y^2)$. First approach $(0, 0)$ along the x -axis. Then $f(x, 0) = x^2/2x^2 = \frac{1}{2}$ for $x \neq 0$, so $f(x, y) \rightarrow \frac{1}{2}$. Next approach $(0, 0)$ along the y -axis. For $y \neq 0$, $f(0, y) = \frac{\sin^2 y}{y^2} = \left(\frac{\sin y}{y}\right)^2$ and $\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$, so $f(x, y) \rightarrow 1$. Since f has two different limits along two different lines, the limit does not exist.
9. $f(x, y) = (xy \cos y)/(3x^2 + y^2)$. On the x -axis, $f(x, 0) = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along the x -axis. Approaching $(0, 0)$ along the line $y = x$, $f(x, x) = (x^2 \cos x)/4x^2 = \frac{1}{4} \cos x$ for $x \neq 0$, so $f(x, y) \rightarrow \frac{1}{4}$ along this line. Thus the limit does not exist.
10. $f(x, y) = 6x^3y/(2x^4 + y^4)$. On the x -axis, $f(x, 0) = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along the x -axis. Approaching $(0, 0)$ along the line $y = x$ gives $f(x, x) = 6x^4/(3x^4) = 2$ for $x \neq 0$, so along this line $f(x, y) \rightarrow 2$ as $(x, y) \rightarrow (0, 0)$. Thus the limit does not exist.
11. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$. We can see that the limit along any line through $(0, 0)$ is 0, as well as along other paths through $(0, 0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. $0 \leq \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq |x|$ since $|y| \leq \sqrt{x^2 + y^2}$, and $|x| \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. So $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$.
12. $f(x, y) = (x^4 - y^4)/(x^2 + y^2) = (x^2 + y^2)(x^2 - y^2)/(x^2 + y^2) = x^2 - y^2$ for $(x, y) \neq (0, 0)$. Thus the limit as $(x, y) \rightarrow (0, 0)$ is 0.
13. Let $f(x, y) = \frac{2x^2y}{x^4 + y^2}$. Then $f(x, 0) = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along the x -axis. But $f(x, x^2) = \frac{2x^4}{2x^4} = 1$ for $x \neq 0$, so $f(x, y) \rightarrow 1$ as $(x, y) \rightarrow (0, 0)$ along the parabola $y = x^2$. Thus the limit doesn't exist.
14. We can use the Squeeze Theorem to show that $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = 0$:
 $0 \leq \frac{x^2 \sin^2 y}{x^2 + 2y^2} \leq \sin^2 y$ since $\frac{x^2}{x^2 + 2y^2} \leq 1$, and $\sin^2 y \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$, so $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = 0$.
15.
$$\begin{aligned} \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} &= \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \cdot \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1} \\ &= \lim_{(x, y) \rightarrow (0, 0)} \frac{(x^2 + y^2)(\sqrt{x^2 + y^2 + 1} + 1)}{x^2 + y^2} \\ &= \lim_{(x, y) \rightarrow (0, 0)} (\sqrt{x^2 + y^2 + 1} + 1) = 2 \end{aligned}$$
16. $f(x, y) = xy^4/(x^2 + y^8)$. On the x -axis, $f(x, 0) = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along the x -axis. Approaching $(0, 0)$ along the curve $x = y^4$ gives $f(y^4, y) = y^8/2y^8 = \frac{1}{2}$ for $y \neq 0$, so along this path $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$. Thus the limit does not exist.

17. e^{-xy} and $\sin(\pi z/2)$ are each compositions of continuous functions, and hence continuous, so their product

$f(x, y, z) = e^{-xy} \sin(\pi z/2)$ is a continuous function. Then

$$\lim_{(x,y,z) \rightarrow (3,0,1)} f(x, y, z) = f(3, 0, 1) = e^{-(3)(0)} \sin(\pi \cdot 1/2) = 1.$$

18. $f(x, y, z) = \frac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2}$. Then $f(x, 0, 0) = \frac{x^2 + 0 + 0}{x^2 + 0 + 0} = 1$ for $x \neq 0$, so $f(x, y, z) \rightarrow 1$ as

$(x, y, z) \rightarrow (0, 0, 0)$ along the x -axis. But $f(0, y, 0) = \frac{0 + 2y^2 + 0}{0 + y^2 + 0} = 2$ for $y \neq 0$, so $f(x, y, z) \rightarrow 2$ as

$(x, y, z) \rightarrow (0, 0, 0)$ along the y -axis. Thus, the limit doesn't exist.

19. $f(x, y, z) = \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$. Then $f(x, 0, 0) = 0/x^2 = 0$ for $x \neq 0$, so as $(x, y, z) \rightarrow (0, 0, 0)$ along the x -axis,

$f(x, y, z) \rightarrow 0$. But $f(x, x, 0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x, y, z) \rightarrow (0, 0, 0)$ along the line $y = x, z = 0$,

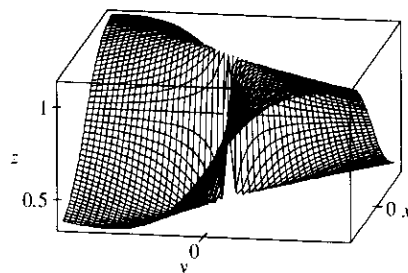
$f(x, y, z) \rightarrow \frac{1}{2}$. Thus the limit doesn't exist.

20. $f(x, y, z) = \frac{xy + yz + zx}{x^2 + y^2 + z^2}$. Then $f(x, 0, 0) = 0$ for $x \neq 0$, so as $(x, y, z) \rightarrow (0, 0, 0)$ along the x -axis,

$f(x, y, z) \rightarrow 0$. But $f(x, x, 0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x, y, z) \rightarrow (0, 0, 0)$ along the line $y = x, z = 0$,

$f(x, y, z) \rightarrow \frac{1}{2}$. Thus the limit doesn't exist.

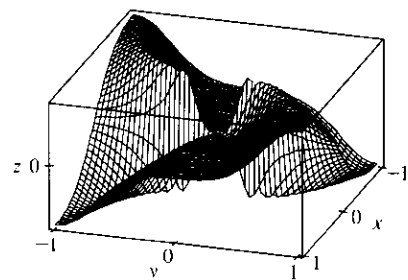
21.



From the ridges on the graph, we see that as $(x, y) \rightarrow (0, 0)$ along the lines under the two ridges, $f(x, y)$ approaches different values.

So the limit does not exist.

22.



From the graph, it appears that as we approach the origin along the lines $x = 0$ or $y = 0$, the function is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3, y) = y^6/(2y^6) = \frac{1}{2}$ for $y \neq 0$, so

$f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along the curve $x = y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

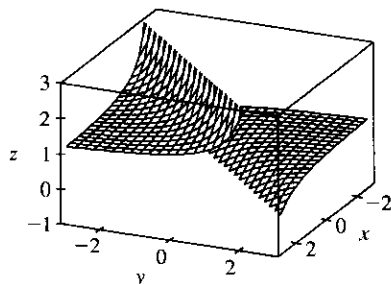
23. $h(x, y) = g(f(x, y)) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain

$D = \{(x, y) \mid 2x + 3y - 6 \geq 0\} = \{(x, y) \mid y \geq -\frac{2}{3}x + 2\}$, which consists of all points on or above the line $y = -\frac{2}{3}x + 2$.

24. $h(x, y) = g(f(x, y)) = (\sqrt{x^2 - y} - 1) / (\sqrt{x^2 - y} + 1)$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain

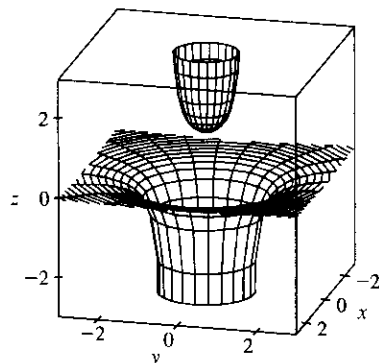
$D = \{(x, y) \mid x^2 - y \geq 0\} = \{(x, y) \mid y \leq x^2\}$ which consists of all points below or on the parabola $y = x^2$.

25.



From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x, y) = e^{1/(x-y)}$ as a composition of functions, $g(x, y) = 1/(x - y)$ is a rational function and therefore continuous except where $x - y = 0 \Rightarrow y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x, y)) = e^{1/(x-y)} = f(x, y)$ is continuous except along the line $y = x$, as we suspected.

26.



We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous. [Note: For a more accurate graph, try converting to cylindrical coordinates first.] Since $f(x, y) = \frac{1}{1 - x^2 - y^2}$ is a rational function, it is continuous except where $1 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 1$, confirming our observation that f is discontinuous on the circle $x^2 + y^2 = 1$.

27. The functions $\sin(xy)$ and $e^x - y^2$ are continuous everywhere, so $F(x, y) = \frac{\sin(xy)}{e^x - y^2}$ is continuous except where

$$e^x - y^2 = 0 \Rightarrow y^2 = e^x \Rightarrow y = \pm\sqrt{e^x} = \pm e^{\frac{1}{2}x}. \text{ Thus } F \text{ is continuous on its domain } \{(x, y) \mid y \neq \pm e^{x/2}\}.$$

28. $F(x, y) = \frac{x - y}{1 + x^2 + y^2}$ is a rational function and thus is continuous on its domain $\mathbb{R}^2$ (since the denominator is never zero).

29. $F(x, y) = \arctan(x + \sqrt{y}) = g(f(x, y))$ where $f(x, y) = x + \sqrt{y}$, continuous on its domain $\{(x, y) \mid y \geq 0\}$, and $g(t) = \arctan t$ is continuous everywhere. Thus F is continuous on its domain $\{(x, y) \mid y \geq 0\}$.

30. e^{x^2y} is continuous on $\mathbb{R}^2$ and $\sqrt{x + y^2}$ is continuous on its domain $\{(x, y) \mid x + y^2 \geq 0\} = \{(x, y) \mid x \geq -y^2\}$, so $F(x, y) = e^{x^2y} + \sqrt{x + y^2}$ is continuous on the set $\{(x, y) \mid x \geq -y^2\}$.

31. $G(x, y) = \ln(x^2 + y^2 - 4) = g(f(x, y))$ where $f(x, y) = x^2 + y^2 - 4$, continuous on $\mathbb{R}^2$, and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Thus G is continuous on its domain $\{(x, y) \mid x^2 + y^2 - 4 > 0\} = \{(x, y) \mid x^2 + y^2 > 4\}$, the exterior of the circle $x^2 + y^2 = 4$.

32. $G(x, y) = g(f(x, y))$ where $f(x, y) = x^2 + y^2$, continuous on $\mathbb{R}^2$, and $g(t) = \sin^{-1} t$, continuous on its domain $\{t \mid -1 \leq t \leq 1\}$. Thus G is continuous on its domain $D = \{(x, y) \mid -1 \leq x^2 + y^2 \leq 1\} = \{(x, y) \mid x^2 + y^2 \leq 1\}$, inside and on the circle $x^2 + y^2 = 1$.

33. $\sqrt{y}$ is continuous on its domain $\{y \mid y \geq 0\}$ and $x^2 - y^2 + z^2$ is continuous everywhere, so

$$f(x, y, z) = \frac{\sqrt{y}}{x^2 - y^2 + z^2} \text{ is continuous for } y \geq 0 \text{ and } x^2 - y^2 + z^2 \neq 0 \Rightarrow y^2 \neq x^2 + z^2, \text{ that is, } \{(x, y, z) \mid y \geq 0, y \neq \sqrt{x^2 + z^2}\}.$$

34. $f(x, y, z) = \sqrt{x+y+z} = h(g(x, y, z))$ where $g(x, y, z) = x + y + z$, continuous everywhere, and $h(t) = \sqrt{t}$ is continuous on its domain $\{t \mid t \geq 0\}$. Thus f is continuous on its domain $\{(x, y, z) \mid x + y + z \geq 0\}$, so f is continuous on and above the plane $z = -x - y$.

35. $f(x, y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$ The first piece of f is a rational function defined everywhere except

at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2 \leq 2x^2 + y^2$, we have $|x^2 y^3 / (2x^2 + y^2)| \leq |y^3|$. We know that $|y^3| \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. So, by the Squeeze Theorem,

$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^3}{2x^2 + y^2} = 0$. But $f(0, 0) = 1$, so f is discontinuous at $(0, 0)$. Therefore, f is continuous on the set $\{(x, y) \mid (x, y) \neq (0, 0)\}$.

36. $f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ The first piece of f is a rational function defined everywhere

except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along the x -axis. But $f(x, x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x, y) \rightarrow \frac{1}{3}$ as $(x, y) \rightarrow (0, 0)$ along the line $y = x$. Thus $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ doesn't exist, so f is not continuous at $(0, 0)$ and the largest set on which f is continuous is $\{(x, y) \mid (x, y) \neq (0, 0)\}$.

37. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \rightarrow 0^+} \frac{(r \cos \theta)^3 + (r \sin \theta)^3}{r^2} = \lim_{r \rightarrow 0^+} (r \cos^3 \theta + r \sin^3 \theta) = 0$

38. $\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2) = \lim_{r \rightarrow 0^+} r^2 \ln r^2 = \lim_{r \rightarrow 0^+} \frac{\ln r^2}{1/r^2}$
 $= \lim_{r \rightarrow 0^+} \frac{(1/r^2)(2r)}{-2/r^3} \quad [\text{using l'Hospital's Rule}] = \lim_{r \rightarrow 0^+} (-r^2) = 0$

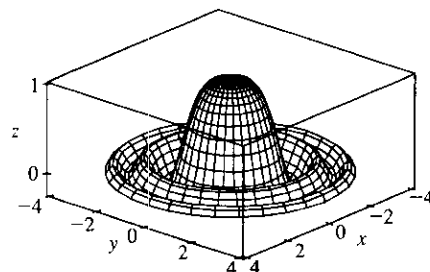
39. $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} = \lim_{\rho \rightarrow 0^+} \frac{(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi)}{\rho^2}$
 $= \lim_{\rho \rightarrow 0^+} (\rho \sin^2 \phi \cos \phi \sin \theta \cos \theta) = 0$

40. $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0^+} \frac{\sin(r^2)}{r^2}$, which is an indeterminate

form of type $0/0$. Using l'Hospital's Rule, we get

$$\lim_{r \rightarrow 0^+} \frac{\sin(r^2)}{r^2} \stackrel{H}{=} \lim_{r \rightarrow 0^+} \frac{2r \cos(r^2)}{2r} = \lim_{r \rightarrow 0^+} \cos(r^2) = 1.$$

Or: Use the fact that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.



41. Since $|\mathbf{x} - \mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}|\cos \theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $||\mathbf{x}| - |\mathbf{a}|| \leq |\mathbf{x} - \mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then whenever $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $||\mathbf{x}| - |\mathbf{a}|| \leq |\mathbf{x} - \mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x} \rightarrow \mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

42. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$ or $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$. But $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| = |\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})| \leq |\mathbf{c}| |\mathbf{x} - \mathbf{a}|$ by Exercise 13.3.57 [ET 12.3.57] (the Cauchy-Schwartz Inequality). Let $\epsilon > 0$ be given and set $\delta = \epsilon/|\mathbf{c}|$. Then whenever $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| \leq |\mathbf{c}| |\mathbf{x} - \mathbf{a}| < |\mathbf{c}| \delta = |\mathbf{c}| (\epsilon/|\mathbf{c}|) = \epsilon$. So f is continuous on $\mathbb{R}^n$.

15.3 Partial Derivatives

ET 14.3

- (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x , which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y , which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t , which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158°W , latitude 21°N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.
- By Definition 4, $f_T(92, 60) = \lim_{h \rightarrow 0} \frac{f(92+h, 60) - f(92, 60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(92, 60) \approx \frac{f(94, 60) - f(92, 60)}{2} = \frac{111 - 105}{2} = 3$, $f_T(92, 60) \approx \frac{f(90, 60) - f(92, 60)}{-2} = \frac{100 - 105}{-2} = 2.5$. Averaging these values, we estimate $f_T(92, 60)$ to be approximately 2.75. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 2.75°F for every degree that the actual temperature rises.

Similarly, $f_H(92, 60) = \lim_{h \rightarrow 0} \frac{f(92, 60+h) - f(92, 60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H(92, 60) \approx \frac{f(92, 65) - f(92, 60)}{5} = \frac{108 - 105}{5} = 0.6$, $f_H(92, 60) \approx \frac{f(92, 55) - f(92, 60)}{-5} = \frac{103 - 105}{-5} = 0.4$. Averaging these values, we estimate $f_H(92, 60)$ to be approximately 0.5. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 0.5°F for every percent that the relative humidity increases.

3. (a) By Definition 4, $f_T(-15, 30) = \lim_{h \rightarrow 0} \frac{f(-15 + h, 30) - f(-15, 30)}{h}$, which we can approximate by

considering $h = 5$ and $h = -5$ and using the values given in the table:

$$f_T(-15, 30) \approx \frac{f(-10, 30) - f(-15, 30)}{5} = \frac{-20 - (-26)}{5} = \frac{6}{5} = 1.2,$$

$$f_T(-15, 30) \approx \frac{f(-20, 30) - f(-15, 30)}{-5} = \frac{-33 - (-26)}{-5} = \frac{-7}{-5} = 1.4. \text{ Averaging these values, we}$$

estimate $f_T(-15, 30)$ to be approximately 1.3. Thus, when the actual temperature is -15°C and the wind speed is 30 km/h, the apparent temperature rises by about 1.3°C for every degree that the actual temperature rises.

Similarly, $f_v(-15, 30) = \lim_{h \rightarrow 0} \frac{f(-15, 30 + h) - f(-15, 30)}{h}$ which we can approximate by considering

$$h = 10 \text{ and } h = -10: f_v(-15, 30) \approx \frac{f(-15, 40) - f(-15, 30)}{10} = \frac{-27 - (-26)}{10} = \frac{-1}{10} = -0.1,$$

$$f_v(-15, 30) \approx \frac{f(-15, 20) - f(-15, 30)}{-10} = \frac{-24 - (-26)}{-10} = \frac{2}{-10} = -0.2. \text{ Averaging these values, we}$$

estimate $f_v(-15, 30)$ to be approximately -0.15 . Thus, when the actual temperature is -15°C and the wind speed is 30 km/h, the apparent temperature decreases by about 0.15°C for every km/h that the wind speed increases.

- (b) For a fixed wind speed v , the values of the wind-chill index W increase as temperature T increases (look at a column of the table), so $\frac{\partial W}{\partial T}$ is positive. For a fixed temperature T , the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\frac{\partial W}{\partial v}$ is negative (or perhaps 0).
- (c) For fixed values of T , the function values $f(T, v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim_{v \rightarrow \infty} (\partial W / \partial v) = 0$.

4. (a) $\partial h / \partial v$ represents the rate of change of h when we fix t and consider h as a function of v , which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h / \partial t$ represents the rate of change of h when we fix v and consider h as a function of t , which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

- (b) By Definition 4, $f_v(40, 15) = \lim_{h \rightarrow 0} \frac{f(40 + h, 15) - f(40, 15)}{h}$ which we can approximate by considering

$h = 10$ and $h = -10$ and using the values given in the table:

$$f_v(40, 15) \approx \frac{f(50, 15) - f(40, 15)}{10} = \frac{36 - 25}{10} = 1.1,$$

$$f_v(40, 15) \approx \frac{f(30, 15) - f(40, 15)}{-10} = \frac{16 - 25}{-10} = 0.9. \text{ Averaging these values, we have } f_v(40, 15) \approx 1.0.$$

Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly,

$f_t(40, 15) = \lim_{h \rightarrow 0} \frac{f(40, 15 + h) - f(40, 15)}{h}$ which we can approximate by considering

$$h = 5 \text{ and } h = -5: f_t(40, 15) \approx \frac{f(40, 20) - f(40, 15)}{5} = \frac{28 - 25}{5} = 0.6,$$

$$f_t(40, 15) \approx \frac{f(40, 10) - f(40, 15)}{-5} = \frac{21 - 25}{-5} = 0.8. \text{ Averaging these values, we have } f_t(40, 15) \approx 0.7.$$

Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.

- (c) For fixed values of v , the function values $f(v, t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that

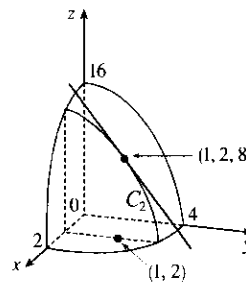
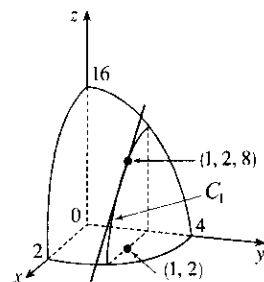
$$\lim_{t \rightarrow \infty} (\partial h / \partial t) = 0.$$

5. (a) If we start at $(1, 2)$ and move in the positive x -direction, the graph of f increases. Thus $f_x(1, 2)$ is positive.
 (b) If we start at $(1, 2)$ and move in the positive y -direction, the graph of f decreases. Thus $f_y(1, 2)$ is negative.
6. (a) The graph of f decreases if we start at $(-1, 2)$ and move in the positive x -direction, so $f_x(-1, 2)$ is negative.
 (b) The graph of f decreases if we start at $(-1, 2)$ and move in the positive y -direction, so $f_y(-1, 2)$ is negative.
 (c) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x -direction. f_x is negative at $(-1, 2)$ and if we move in the positive x -direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1, 2)$ is positive.
 (d) f_{yy} is the rate of change of f_y in the y -direction. f_y is negative at $(-1, 2)$ and if we move in the positive y -direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1, 2)$ is negative.
7. First of all, if we start at the point $(3, -3)$ and move in the positive y -direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3, -1.5)$, while a is 0 at this point. So a is definitely the graph of f_y , and one of b and c is the graph of f . To see which is which, we start at the point $(-3, -1.5)$ and move in the positive x -direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x -derivative of c . So c is the graph of f , b is the graph of f_x , and a is the graph of f_y .
8. $f_x(2, 1)$ is the rate of change of f at $(2, 1)$ in the x -direction. If we start at $(2, 1)$, where $f(2, 1) = 10$, and move in the positive x -direction, we reach the next contour line (where $f(x, y) = 12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2, 1)$ from the left (moving in the positive x -direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2, 1)$ would be the average of these two, so $f_x(2, 1) \approx 2.8$. Similarly, $f_y(2, 1)$ is the rate of change of f at $(2, 1)$ in the y -direction. If we approach $(2, 1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2 . If we start at $(2, 1)$ and move in the positive y -direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2, 1) \approx -2.1$.

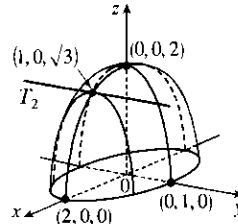
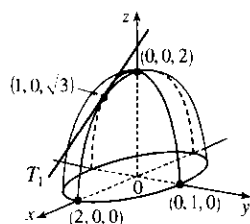
9. $f(x, y) = 16 - 4x^2 - y^2 \Rightarrow f_x(x, y) = -8x$ and $f_y(x, y) = -2y \Rightarrow f_x(1, 2) = -8$ and $f_y(1, 2) = -4$.

The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2, y = 2$ (the curve C_1 in the first figure).

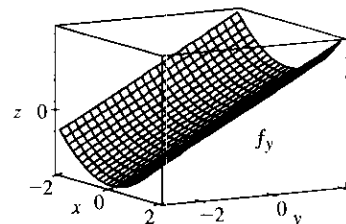
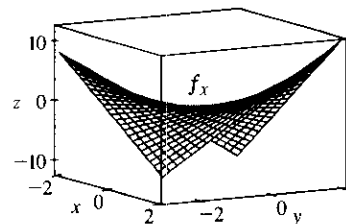
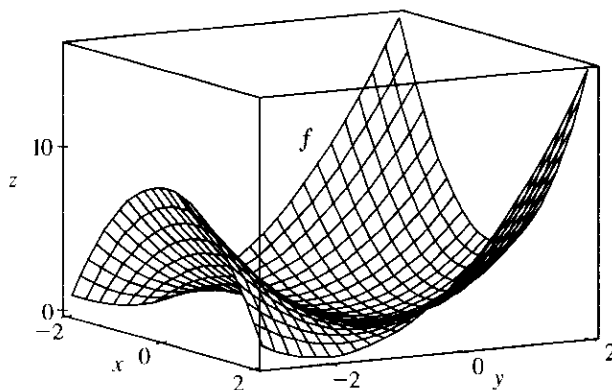
The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2, x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.



10. $f(x, y) = (4 - x^2 - 4y^2)^{1/2} \Rightarrow f_x(x, y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x, y) = -4y(4 - x^2 - 4y^2)^{-1/2} \Rightarrow f_x(1, 0) = -\frac{1}{\sqrt{3}}, f_y(1, 0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4, z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $(1, 0, \sqrt{3})$ is $f_x(1, 0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3, z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $(1, 0, \sqrt{3})$ is $f_y(1, 0) = 0$.



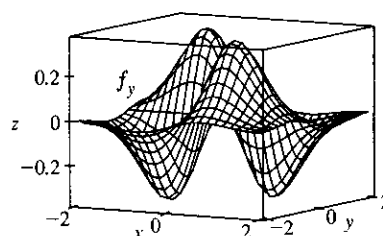
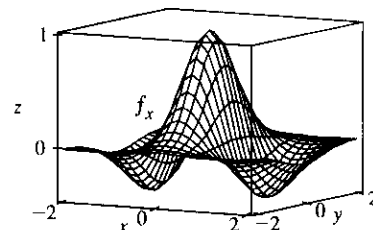
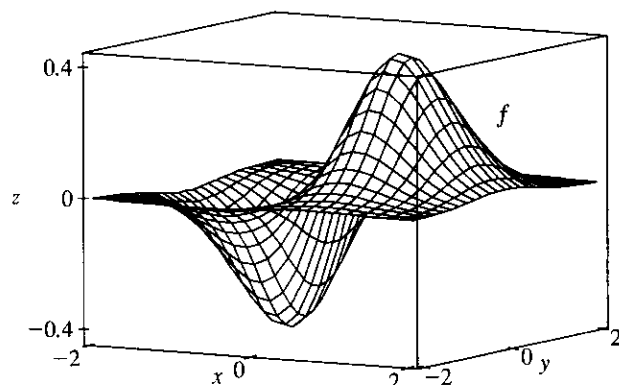
11. $f(x, y) = x^2 + y^2 + x^2y \Rightarrow f_x = 2x + 2xy, f_y = 2y + x^2$



Note that the traces of f in planes parallel to the xz -plane are parabolas which open downward for $y < -1$ and

upward for $y > -1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of f in planes parallel to the yz -plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

$$12. f(x, y) = xe^{-x^2-y^2} \Rightarrow f_x = x(-2xe^{-x^2-y^2}) + e^{-x^2-y^2} = e^{-x^2-y^2}(1-2x^2), f_y = -2xye^{-x^2-y^2}$$



Note that traces of f in planes parallel to the xz -plane have two extreme values, while traces of f_x in these planes have two zeros. Traces of f in planes parallel to the yz -plane have only one extreme value (a minimum if $x < 0$, a maximum if $x > 0$), and traces of f_y in these planes have only one zero (going from negative to positive if $x < 0$ and from positive to negative if $x > 0$).

$$13. f(x, y) = 3x - 2y^4 \Rightarrow f_x(x, y) = 3 - 0 = 3, f_y(x, y) = 0 - 8y^3 = -8y^3$$

$$14. f(x, y) = x^5 + 3x^3y^2 + 3xy^4 \Rightarrow f_x(x, y) = 5x^4 + 3 \cdot 3x^2 \cdot y^2 + 3 \cdot 1 \cdot y^4 = 5x^4 + 9x^2y^2 + 3y^4, \\ f_y(x, y) = 0 + 3x^3 \cdot 2y + 3x \cdot 4y^3 = 6x^3y + 12xy^3.$$

$$15. z = xe^{3y} \Rightarrow \frac{\partial z}{\partial x} = e^{3y}, \frac{\partial z}{\partial y} = 3xe^{3y}$$

$$16. z = y \ln x \Rightarrow \frac{\partial z}{\partial x} = \frac{y}{x}, \frac{\partial z}{\partial y} = \ln x$$

$$17. f(x, y) = \frac{x-y}{x+y} \Rightarrow f_x(x, y) = \frac{(1)(x+y) - (x-y)(1)}{(x+y)^2} = \frac{2y}{(x+y)^2}, \\ f_y(x, y) = \frac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} = -\frac{2x}{(x+y)^2}$$

$$18. f(x, y) = x^y \Rightarrow f_x(x, y) = yx^{y-1}, f_y(x, y) = x^y \ln x$$

$$19. w = \sin \alpha \cos \beta \Rightarrow \frac{\partial w}{\partial \alpha} = \cos \alpha \cos \beta, \frac{\partial w}{\partial \beta} = -\sin \alpha \sin \beta$$

$$20. f(s, t) = \frac{st^2}{s^2 + t^2} \Rightarrow$$

$$f_s(s, t) = \frac{t^2(s^2 + t^2) - st^2(2s)}{(s^2 + t^2)^2} = \frac{t^4 - s^2t^2}{(s^2 + t^2)^2}, f_t(s, t) = \frac{2st(s^2 + t^2) - st^2(2t)}{(s^2 + t^2)^2} = \frac{2s^3t}{(s^2 + t^2)^2}$$

$$21. f(r, s) = r \ln(r^2 + s^2) \Rightarrow f_r(r, s) = r \cdot \frac{2r}{r^2 + s^2} + \ln(r^2 + s^2) \cdot 1 = \frac{2r^2}{r^2 + s^2} + \ln(r^2 + s^2),$$

$$f_s(r, s) = r \cdot \frac{2s}{r^2 + s^2} + 0 = \frac{2rs}{r^2 + s^2}$$

$$22. f(x, t) = \arctan(x\sqrt{t}) \Rightarrow f_x(x, t) = \frac{1}{1 + (x\sqrt{t})^2} \cdot \sqrt{t} = \frac{\sqrt{t}}{1 + x^2 t},$$

$$f_t(x, t) = \frac{1}{1 + (x\sqrt{t})^2} \cdot x \left(\frac{1}{2} t^{-1/2} \right) = \frac{x}{2\sqrt{t}(1 + x^2 t)}$$

$$23. u = te^{w/t} \Rightarrow \frac{\partial u}{\partial t} = t \cdot e^{w/t}(-wt^{-2}) + e^{w/t} \cdot 1 = e^{w/t} - \frac{w}{t}e^{w/t} = e^{w/t} \left(1 - \frac{w}{t} \right), \frac{\partial u}{\partial w} = te^{w/t} \cdot \frac{1}{t} = e^{w/t}$$

$$24. f(x, y) = \int_y^x \cos(t^2) dt \Rightarrow f_x(x, y) = \frac{\partial}{\partial x} \int_y^x \cos(t^2) dt = \cos(x^2) \text{ by the Fundamental Theorem of Calculus, Part 1; } f_y(x, y) = \frac{\partial}{\partial y} \int_y^x \cos(t^2) dt = -\frac{\partial}{\partial y} \cos(y^2) = -\cos(y^2).$$

$$25. f(x, y, z) = xy^2z^3 + 3yz \Rightarrow f_x(x, y, z) = y^2z^3, f_y(x, y, z) = 2xyz^3 + 3z, f_z(x, y, z) = 3xy^2z^2 + 3y$$

$$26. f(x, y, z) = x^2e^{yz} \Rightarrow f_x(x, y, z) = 2xe^{yz}, f_y(x, y, z) = x^2e^{yz}(z) = x^2ze^{yz},$$

$$f_z(x, y, z) = x^2e^{yz}(y) = x^2ye^{yz}$$

$$27. w = \ln(x + 2y + 3z) \Rightarrow \frac{\partial w}{\partial x} = \frac{1}{x + 2y + 3z}, \frac{\partial w}{\partial y} = \frac{2}{x + 2y + 3z}, \frac{\partial w}{\partial z} = \frac{3}{x + 2y + 3z}$$

$$28. w = \sqrt{r^2 + s^2 + t^2} \Rightarrow \frac{\partial w}{\partial r} = \frac{1}{2}(r^2 + s^2 + t^2)^{-1/2}(2r) = \frac{r}{\sqrt{r^2 + s^2 + t^2}}, \frac{\partial w}{\partial s} = \frac{s}{\sqrt{r^2 + s^2 + t^2}},$$

$$\frac{\partial w}{\partial t} = \frac{t}{\sqrt{r^2 + s^2 + t^2}}$$

$$29. u = xe^{-t} \sin \theta \Rightarrow \frac{\partial u}{\partial x} = e^{-t} \sin \theta, \frac{\partial u}{\partial t} = -xe^{-t} \sin \theta, \frac{\partial u}{\partial \theta} = xe^{-t} \cos \theta$$

$$30. u = x^{y/z} \Rightarrow u_x = \frac{y}{z}x^{(y/z)-1}, u_y = x^{y/z} \ln x \cdot \frac{1}{z} = \frac{x^{y/z}}{z} \ln x, u_z = x^{y/z} \ln x \cdot \frac{-y}{z^2} = -\frac{yx^{y/z}}{z^2} \ln x$$

$$31. f(x, y, z, t) = xyz^2 \tan(yt) \Rightarrow f_x(x, y, z, t) = yz^2 \tan(yt),$$

$$f_y(x, y, z, t) = xyz^2 \cdot \sec^2(yt) \cdot t + xz^2 \tan(yt) = xyz^2 t \sec^2(yt) + xz^2 \tan(yt),$$

$$f_z(x, y, z, t) = 2xyz \tan(yt), f_t(x, y, z, t) = xyz^2 \sec^2(yt) \cdot y = xy^2 z^2 \sec^2(yt).$$

$$32. f(x, y, z, t) = \frac{xy^2}{t + 2z} \Rightarrow$$

$$f_x(x, y, z, t) = \frac{y^2}{t + 2z}, f_y(x, y, z, t) = \frac{2xy}{t + 2z},$$

$$f_z(x, y, z, t) = xy^2(-1)(t + 2z)^{-2}(2) = -\frac{2xy^2}{(t + 2z)^2}, f_t(x, y, z, t) = xy^2(-1)(t + 2z)^{-2}(1) = -\frac{xy^2}{(t + 2z)^2}.$$

$$33. u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \text{ For each } i = 1, \dots, n,$$

$$u_{x_i} = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2}(2x_i) = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}.$$

34. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$. For each $i = 1, \dots, n$, $u_{x_i} = i \cos(x_1 + 2x_2 + \cdots + nx_n)$.

35. $f(x, y) = \sqrt{x^2 + y^2} \Rightarrow f_x(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$, so $f_x(3, 4) = \frac{3}{\sqrt{3^2 + 4^2}} = \frac{3}{5}$.

36. $f(x, y) = \sin(2x + 3y) \Rightarrow f_y(x, y) = \cos(2x + 3y) \cdot 3 = 3 \cos(2x + 3y)$, so
 $f_y(-6, 4) = 3 \cos[2(-6) + 3(4)] = 3 \cos 0 = 3$.

37. $f(x, y, z) = \frac{x}{y+z} = x(y+z)^{-1} \Rightarrow f_z(x, y, z) = x(-1)(y+z)^{-2} = -\frac{x}{(y+z)^2}$, so
 $f_z(3, 2, 1) = -\frac{3}{(2+1)^2} = -\frac{1}{3}$.

38. $f(u, v, w) = w \tan(uv) \Rightarrow f_v(u, v, w) = w \sec^2(uv) \cdot u = uw \sec^2(uv)$, so
 $f_v(2, 0, 3) = (2)(3) \sec^2(2 \cdot 0) = 6$.

39. $f(x, y) = x^2 - xy + 2y^2 \Rightarrow$

$$\begin{aligned} f_x(x, y) &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - (x+h)y + 2y^2 - (x^2 - xy + 2y^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x - y + h)}{h} = \lim_{h \rightarrow 0} (2x - y + h) = 2x - y \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{x^2 - x(y+h) + 2(y+h)^2 - (x^2 - xy + 2y^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(4y - x + 2h)}{h} = \lim_{h \rightarrow 0} (4y - x + 2h) = 4y - x \end{aligned}$$

40. $f(x, y) = \sqrt{3x - y} \Rightarrow$

$$\begin{aligned} f_x(x, y) &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{3(x+h) - y} - \sqrt{3x - y}}{h} \cdot \frac{\sqrt{3(x+h) - y} + \sqrt{3x - y}}{\sqrt{3(x+h) - y} + \sqrt{3x - y}} \\ &= \lim_{h \rightarrow 0} \frac{3}{\sqrt{3(x+h) - y} + \sqrt{3x - y}} = \frac{3}{2\sqrt{3x - y}} \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{3x - (y+h)} - \sqrt{3x - y}}{h} \cdot \frac{\sqrt{3x - (y+h)} + \sqrt{3x - y}}{\sqrt{3x - (y+h)} + \sqrt{3x - y}} \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{3x - (y+h)} + \sqrt{3x - y}} = \frac{-1}{2\sqrt{3x - y}} \end{aligned}$$

41. $x^2 + y^2 + z^2 = 3xyz \Rightarrow \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = \frac{\partial}{\partial x}(3xyz) \Rightarrow 2x + 0 + 2z \frac{\partial z}{\partial x} = 3y \left(x \frac{\partial z}{\partial x} + z \cdot 1 \right)$
 $\Leftrightarrow 2z \frac{\partial z}{\partial x} - 3xy \frac{\partial z}{\partial x} = 3yz - 2x \Leftrightarrow (2z - 3xy) \frac{\partial z}{\partial x} = 3yz - 2x$, so $\frac{\partial z}{\partial x} = \frac{3yz - 2x}{2z - 3xy}$.

$$\begin{aligned}\frac{\partial}{\partial y}(x^2 + y^2 + z^2) &= \frac{\partial}{\partial y}(3xyz) \Rightarrow 0 + 2y + 2z \frac{\partial z}{\partial y} = 3x \left(y \frac{\partial z}{\partial y} + z \cdot 1 \right) \Leftrightarrow \\ 2z \frac{\partial z}{\partial y} - 3xy \frac{\partial z}{\partial y} &= 3xz - 2y \Leftrightarrow (2z - 3xy) \frac{\partial z}{\partial y} = 3xz - 2y, \text{ so } \frac{\partial z}{\partial y} = \frac{3xz - 2y}{2z - 3xy}.\end{aligned}$$

$$\begin{aligned}42. \ yz = \ln(x + z) &\Rightarrow \frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}(\ln(x + z)) \Rightarrow y \frac{\partial z}{\partial x} = \frac{1}{x + z} \left(1 + \frac{\partial z}{\partial x} \right) \Leftrightarrow \\ \left(y - \frac{1}{x + z} \right) \frac{\partial z}{\partial x} &= \frac{1}{x + z}, \text{ so } \frac{\partial z}{\partial x} = \frac{1/(x + z)}{y - 1/(x + z)} = \frac{1}{y(x + z) - 1}.\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}(yz) &= \frac{\partial}{\partial y}(\ln(x + z)) \Rightarrow y \frac{\partial z}{\partial y} + z \cdot 1 = \frac{1}{x + z} \left(0 + \frac{\partial z}{\partial y} \right) \Leftrightarrow \left(y - \frac{1}{x + z} \right) \frac{\partial z}{\partial y} = -z, \text{ so} \\ \frac{\partial z}{\partial y} &= \frac{-z}{y - 1/(x + z)} = \frac{z(x + z)}{1 - y(x + z)}.\end{aligned}$$

$$\begin{aligned}43. \ x - z = \arctan(yz) &\Rightarrow \frac{\partial}{\partial x}(x - z) = \frac{\partial}{\partial x}(\arctan(yz)) \Rightarrow 1 - \frac{\partial z}{\partial x} = \frac{1}{1 + (yz)^2} \cdot y \frac{\partial z}{\partial x} \Leftrightarrow \\ 1 &= \left(\frac{y}{1 + y^2 z^2} + 1 \right) \frac{\partial z}{\partial x} \Leftrightarrow 1 = \left(\frac{y + 1 + y^2 z^2}{1 + y^2 z^2} \right) \frac{\partial z}{\partial x}, \text{ so } \frac{\partial z}{\partial x} = \frac{1 + y^2 z^2}{1 + y + y^2 z^2}.\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}(x - z) &= \frac{\partial}{\partial y}(\arctan(yz)) \Rightarrow 0 - \frac{\partial z}{\partial y} = \frac{1}{1 + (yz)^2} \cdot \left(y \frac{\partial z}{\partial y} + z \cdot 1 \right) \Leftrightarrow \\ -\frac{z}{1 + y^2 z^2} &= \left(\frac{y}{1 + y^2 z^2} + 1 \right) \frac{\partial z}{\partial y} \Leftrightarrow -\frac{z}{1 + y^2 z^2} = \left(\frac{y + 1 + y^2 z^2}{1 + y^2 z^2} \right) \frac{\partial z}{\partial y} \Leftrightarrow \frac{\partial z}{\partial y} = -\frac{z}{1 + y + y^2 z^2}.\end{aligned}$$

$$\begin{aligned}44. \ \sin(xyz) = x + 2y + 3z &\Rightarrow \frac{\partial}{\partial x}(\sin(xyz)) = \frac{\partial}{\partial x}(x + 2y + 3z) \Rightarrow \\ \cos(xyz) \cdot y \left(x \frac{\partial z}{\partial x} + z \right) &= 1 + 3 \frac{\partial z}{\partial x} \Leftrightarrow (xy \cos(xyz) - 3) \frac{\partial z}{\partial x} = 1 - yz \cos(xyz), \text{ so} \\ \frac{\partial z}{\partial x} &= \frac{1 - yz \cos(xyz)}{xy \cos(xyz) - 3}.\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}(\sin(xyz)) &= \frac{\partial}{\partial y}(x + 2y + 3z) \Rightarrow \cos(xyz) \cdot x \left(y \frac{\partial z}{\partial y} + z \right) = 2 + 3 \frac{\partial z}{\partial y} \Leftrightarrow \\ (xy \cos(xyz) - 3) \frac{\partial z}{\partial y} &= 2 - xz \cos(xyz), \text{ so } \frac{\partial z}{\partial y} = \frac{2 - xz \cos(xyz)}{xy \cos(xyz) - 3}.\end{aligned}$$

$$45. \text{ (a) } z = f(x) + g(y) \Rightarrow \frac{\partial z}{\partial x} = f'(x), \frac{\partial z}{\partial y} = g'(y)$$

$$\begin{aligned}\text{(b) } z = f(x + y). \text{ Let } u = x + y. \text{ Then } \frac{\partial z}{\partial x} &= \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du}(1) = f'(u) = f'(x + y), \\ \frac{\partial z}{\partial y} &= \frac{df}{du} \frac{\partial u}{\partial y} = \frac{df}{du}(1) = f'(u) = f'(x + y).\end{aligned}$$

46. (a) $z = f(x)g(y) \Rightarrow \frac{\partial z}{\partial x} = f'(x)g(y), \frac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\frac{\partial u}{\partial x} = y$ and $\frac{\partial u}{\partial y} = x$. Hence $\frac{\partial z}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du} \cdot y = yf'(u) = yf'(xy)$
and $\frac{\partial z}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y} = \frac{df}{du} \cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\frac{x}{y}\right)$. Let $u = \frac{x}{y}$. Then $\frac{\partial u}{\partial x} = \frac{1}{y}$ and $\frac{\partial u}{\partial y} = -\frac{x}{y^2}$. Hence $\frac{\partial z}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = f'(u) \frac{1}{y} = \frac{f'(x/y)}{y}$ and
 $\frac{\partial z}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y} = f'(u) \left(-\frac{x}{y^2}\right) = -\frac{xf'(x/y)}{y^2}$.

47. $f(x, y) = x^4 - 3x^2y^3 \Rightarrow f_x(x, y) = 4x^3 - 6xy^3, f_y(x, y) = -9x^2y^2$. Then $f_{xx}(x, y) = 12x^2 - 6y^3$,
 $f_{xy}(x, y) = -18xy^2, f_{yx}(x, y) = -18xy^2$, and $f_{yy}(x, y) = -18x^2y$.

48. $f(x, y) = \ln(3x + 5y) \Rightarrow f_x(x, y) = \frac{3}{3x + 5y}, f_y(x, y) = \frac{5}{3x + 5y}$. Then
 $f_{xx}(x, y) = 3(-1)(3x + 5y)^{-2}(3) = -\frac{9}{(3x + 5y)^2}, f_{xy}(x, y) = -\frac{15}{(3x + 5y)^2}, f_{yx}(x, y) = -\frac{15}{(3x + 5y)^2}$,
and $f_{yy}(x, y) = -\frac{25}{(3x + 5y)^2}$.

49. $z = \frac{x}{x+y} = x(x+y)^{-1} \Rightarrow z_x = \frac{1(x+y) - 1(x)}{(x+y)^2} = \frac{y}{(x+y)^2}, z_y = x(-1)(x+y)^{-2} = -\frac{x}{(x+y)^2}$.
Then $z_{xx} = y(-2)(x+y)^{-3} = -\frac{2y}{(x+y)^3}, z_{xy} = \frac{1(x+y)^2 - y(2)(x+y)}{[(x+y)^2]^2} = \frac{x+y-2y}{(x+y)^3} = \frac{x-y}{(x+y)^3}$,
 $z_{yx} = -\frac{1(x+y)^2 - x(2)(x+y)}{[(x+y)^2]^2} = -\frac{-x^2 + xy + y^2}{(x+y)^2} = \frac{(x+y)(x-y)}{(x+y)^2} = \frac{x-y}{(x+y)^3}$, and
 $z_{yy} = -x(-2)(x+y)^{-3} = \frac{2x}{(x+y)^3}$.

50. $z = y \tan 2x \Rightarrow z_x = y \sec^2(2x) \cdot 2 = 2y \sec^2(2x), z_y = \tan 2x$. Then
 $z_{xx} = 2y(2) \sec(2x) \cdot \sec(2x) \tan(2x) \cdot 2 = 8y \sec^2(2x) \tan(2x), z_{xy} = 2 \sec^2(2x)$,
 $z_{yx} = \sec^2(2x) \cdot 2 = 2 \sec^2(2x)$, and $z_{yy} = 0$.

51. $u = e^{-s} \sin t \Rightarrow u_s = -e^{-s} \sin t, u_t = e^{-s} \cos t$. Then $u_{ss} = e^{-s} \sin t, u_{st} = -e^{-s} \cos t$,
 $u_{ts} = -e^{-s} \cos t$, and $u_{tt} = -e^{-s} \sin t$.

52. $v = \sqrt{x+y^2} \Rightarrow v_x = \frac{1}{2}(x+y^2)^{-1/2} = \frac{1}{2\sqrt{x+y^2}},$
 $v_y = \frac{1}{2}(x+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x+y^2}}$. Then $v_{xx} = \frac{1}{2}\left(-\frac{1}{2}\right)(x+y^2)^{-3/2} = -\frac{1}{4(x+y^2)^{3/2}},$
 $v_{xy} = \frac{1}{2}\left(-\frac{1}{2}\right)(x+y^2)^{-3/2}(2y) = -\frac{y}{2(x+y^2)^{3/2}}, v_{yx} = y\left(-\frac{1}{2}\right)(x+y^2)^{-3/2} = -\frac{y}{2(x+y^2)^{3/2}},$
and $v_{yy} = \frac{1\sqrt{x+y^2} - y\left(\frac{1}{2}\right)(x+y^2)^{-1/2}(2y)}{\left(\sqrt{x+y^2}\right)^2} = \frac{(x+y^2) - y^2}{(x+y^2)^{3/2}} = \frac{x}{(x+y^2)^{3/2}}.$

$$53. u = x \sin(x + 2y) \Rightarrow u_x = x \cdot \cos(x + 2y)(1) + \sin(x + 2y) \cdot 1 = x \cos(x + 2y) + \sin(x + 2y),$$

$$u_{xy} = x(-\sin(x + 2y)(2)) + \cos(x + 2y)(2) = 2 \cos(x + 2y) - 2x \sin(x + 2y) \text{ and}$$

$$u_y = x \cos(x + 2y)(2) = 2x \cos(x + 2y),$$

$$u_{yx} = 2x \cdot (-\sin(x + 2y)(1)) + \cos(x + 2y) \cdot 2 = 2 \cos(x + 2y) - 2x \sin(x + 2y). \text{ Thus } u_{xy} = u_{yx}.$$

$$54. u = x^4 y^2 - 2xy^5 \Rightarrow u_x = 4x^3 y^2 - 2y^5, u_{xy} = 8x^3 y - 10y^4 \text{ and } u_y = 2x^4 y - 10xy^4, u_{yx} = 8x^3 y - 10y^4.$$

$$\text{Thus } u_{xy} = u_{yx}.$$

$$55. u = \ln \sqrt{x^2 + y^2} = \ln(x^2 + y^2)^{1/2} = \frac{1}{2} \ln(x^2 + y^2) \Rightarrow u_x = \frac{1}{2} \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2},$$

$$u_{xy} = x(-1)(x^2 + y^2)^{-2}(2y) = -\frac{2xy}{(x^2 + y^2)^2} \text{ and } u_y = \frac{1}{2} \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2},$$

$$u_{yx} = y(-1)(x^2 + y^2)^{-2}(2x) = -\frac{2xy}{(x^2 + y^2)^2}. \text{ Thus } u_{xy} = u_{yx}.$$

$$56. u = xye^y \Rightarrow u_x = ye^y, u_{xy} = ye^y + e^y = (y + 1)e^y \text{ and } u_y = x(ye^y + e^y) = x(y + 1)e^y,$$

$$u_{yx} = (y + 1)e^y. \text{ Thus } u_{xy} = u_{yx}.$$

$$57. f(x, y) = 3xy^4 + x^3 y^2 \Rightarrow f_x = 3y^4 + 3x^2 y^2, f_{xx} = 6xy^2, f_{xy} = 12xy \text{ and } f_y = 12xy^3 + 2x^3 y,$$

$$f_{yy} = 36xy^2 + 2x^3, f_{yyy} = 72xy.$$

$$58. f(x, t) = x^2 e^{-ct} \Rightarrow f_t = x^2(-ce^{-ct}), f_{tt} = x^2(c^2 e^{-ct}), f_{ttt} = x^2(-c^3 e^{-ct}) = -c^3 x^2 e^{-ct} \text{ and}$$

$$f_{tx} = 2x(-ce^{-ct}), f_{txx} = 2(-ce^{-ct}) = -2ce^{-ct}.$$

$$59. f(x, y, z) = \cos(4x + 3y + 2z) \Rightarrow$$

$$f_x = -\sin(4x + 3y + 2z)(4) = -4 \sin(4x + 3y + 2z),$$

$$f_{xy} = -4 \cos(4x + 3y + 2z)(3) = -12 \cos(4x + 3y + 2z),$$

$$f_{xyz} = -12(-\sin(4x + 3y + 2z))(2) = 24 \sin(4x + 3y + 2z) \text{ and}$$

$$f_y = -\sin(4x + 3y + 2z)(3) = -3 \sin(4x + 3y + 2z),$$

$$f_{yz} = -3 \cos(4x + 3y + 2z)(2) = -6 \cos(4x + 3y + 2z),$$

$$f_{yzz} = -6(-\sin(4x + 3y + 2z))(2) = 12 \sin(4x + 3y + 2z).$$

$$60. f(r, s, t) = r \ln(rs^2 t^3) \Rightarrow$$

$$f_r = r \cdot \frac{1}{rs^2 t^3} (s^2 t^3) + \ln(rs^2 t^3) \cdot 1 = \frac{rs^2 t^3}{rs^2 t^3} + \ln(rs^2 t^3) = 1 + \ln(rs^2 t^3),$$

$$f_{rs} = \frac{1}{rs^2 t^3} (2rst^3) = \frac{2}{s} = 2s^{-1}, f_{rss} = -2s^{-2} = -\frac{2}{s^2} \text{ and } f_{rst} = 0.$$

$$61. u = e^{r\theta} \sin \theta \Rightarrow \frac{\partial u}{\partial \theta} = e^{r\theta} \cos \theta + \sin \theta \cdot e^{r\theta} (r) = e^{r\theta} (\cos \theta + r \sin \theta),$$

$$\frac{\partial^2 u}{\partial r \partial \theta} = e^{r\theta} (\sin \theta) + (\cos \theta + r \sin \theta) e^{r\theta} (\theta) = e^{r\theta} (\sin \theta + \theta \cos \theta + r\theta \sin \theta),$$

$$\frac{\partial^3 u}{\partial r^2 \partial \theta} = e^{r\theta} (\theta \sin \theta) + (\sin \theta + \theta \cos \theta + r\theta \sin \theta) \cdot e^{r\theta} (\theta) = \theta e^{r\theta} (2 \sin \theta + \theta \cos \theta + r\theta \sin \theta).$$

$$62. z = u \sqrt{v - w} = u(v - w)^{1/2} \Rightarrow \frac{\partial z}{\partial w} = u \left[\frac{1}{2} (v - w)^{-1/2} (-1) \right] = -\frac{1}{2} u (v - w)^{-1/2},$$

$$\frac{\partial^2 z}{\partial v \partial w} = -\frac{1}{2} u \left(-\frac{1}{2} (v - w)^{-3/2} (1) \right) = \frac{1}{4} u (v - w)^{-3/2}, \frac{\partial^3 z}{\partial u \partial v \partial w} = \frac{1}{4} (v - w)^{-3/2}.$$

$$63. w = \frac{x}{y+2z} = x(y+2z)^{-1} \Rightarrow \frac{\partial w}{\partial x} = (y+2z)^{-1}, \frac{\partial^2 w}{\partial y \partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2},$$

$$\frac{\partial^3 w}{\partial z \partial y \partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \frac{4}{(y+2z)^3} \text{ and}$$

$$\frac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2}, \frac{\partial^2 w}{\partial x \partial y} = -(y+2z)^{-2}, \frac{\partial^3 w}{\partial x^2 \partial y} = 0.$$

$$64. u = x^a y^b z^c. \text{ If } a = 0, \text{ or if } b = 0 \text{ or } 1, \text{ or if } c = 0, 1, \text{ or } 2, \text{ then } \frac{\partial^6 u}{\partial x \partial y^2 \partial z^3} = 0. \text{ Otherwise } \frac{\partial u}{\partial z} = cx^a y^b z^{c-1},$$

$$\frac{\partial^2 u}{\partial z^2} = c(c-1)x^a y^b z^{c-2}, \frac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)x^a y^b z^{c-3}, \frac{\partial^4 u}{\partial y \partial z^3} = bc(c-1)(c-2)x^a y^{b-1} z^{c-3},$$

$$\frac{\partial^5 u}{\partial y^2 \partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}, \text{ and } \frac{\partial^6 u}{\partial x \partial y^2 \partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}.$$

$$65. \text{ By Definition 4, } f_x(3, 2) = \lim_{h \rightarrow 0} \frac{f(3+h, 2) - f(3, 2)}{h} \text{ which we can approximate by considering } h = 0.5$$

$$\text{and } h = -0.5: f_x(3, 2) \approx \frac{f(3.5, 2) - f(3, 2)}{0.5} = \frac{22.4 - 17.5}{0.5} = 9.8,$$

$$f_x(3, 2) \approx \frac{f(2.5, 2) - f(3, 2)}{-0.5} = \frac{10.2 - 17.5}{-0.5} = 14.6. \text{ Averaging these values, we estimate } f_x(3, 2) \text{ to be}$$

$$\text{approximately } 12.2. \text{ Similarly, } f_x(3, 2.2) = \lim_{h \rightarrow 0} \frac{f(3+h, 2.2) - f(3, 2.2)}{h} \text{ which we can approximate by}$$

$$\text{considering } h = 0.5 \text{ and } h = -0.5: f_x(3, 2.2) \approx \frac{f(3.5, 2.2) - f(3, 2.2)}{0.5} = \frac{26.1 - 15.9}{0.5} = 20.4,$$

$$f_x(3, 2.2) \approx \frac{f(2.5, 2.2) - f(3, 2.2)}{-0.5} = \frac{9.3 - 15.9}{-0.5} = 13.2. \text{ Averaging these values, we have } f_x(3, 2.2) \approx 16.8.$$

To estimate $f_{xy}(3, 2)$, we first need an estimate for $f_x(3, 1.8)$:

$$f_x(3, 1.8) \approx \frac{f(3.5, 1.8) - f(3, 1.8)}{0.5} = \frac{20.0 - 18.1}{0.5} = 3.8,$$

$$f_x(3, 1.8) \approx \frac{f(2.5, 1.8) - f(3, 1.8)}{-0.5} = \frac{12.5 - 18.1}{-0.5} = 11.2. \text{ Averaging these values, we get } f_x(3, 1.8) \approx 7.5.$$

Now $f_{xy}(x, y) = \frac{\partial}{\partial y} [f_x(x, y)]$ and $f_x(x, y)$ is itself a function of 2 variables, so Definition 4 says that

$$f_{xy}(x, y) = \frac{\partial}{\partial y} [f_x(x, y)] = \lim_{h \rightarrow 0} \frac{f_x(x, y+h) - f_x(x, y)}{h} \Rightarrow f_{xy}(3, 2) = \lim_{h \rightarrow 0} \frac{f_x(3, 2+h) - f_x(3, 2)}{h}.$$

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$$f_{xy}(3, 2) \approx \frac{f_x(3, 2.2) - f_x(3, 2)}{0.2} = \frac{16.8 - 12.2}{0.2} = 23,$$

$$f_{xy}(3, 2) \approx \frac{f_x(3, 1.8) - f_x(3, 2)}{-0.2} = \frac{7.5 - 12.2}{-0.2} = 23.5. \text{ Averaging these values, we estimate } f_{xy}(3, 2) \text{ to be}$$

approximately 23.25.

66. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x -direction, so f_x is negative at P .

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y -direction, so f_y is positive at P .

(c) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x -direction) than at points to the left of P , demonstrating that f decreases less quickly with respect to x to the right of P . So as we move through P in the positive x -direction the (negative) value of f_x increases, hence $\frac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at P .

(d) $f_{xy} = \frac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x -direction) at points above P than at those below P , demonstrating that f decreases more quickly with respect to x for y -values above P . So as we move through P in the positive y -direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \frac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y -direction) at points above P than at those below P , demonstrating that f increases more quickly with respect to y above P . So as we move through P in the positive y -direction the (positive) value of f_y increases, hence $\frac{\partial}{\partial y}(f_y) = f_{yy}$ is positive at P .

67. $u = e^{-\alpha^2 k^2 t} \sin kx \Rightarrow u_x = k e^{-\alpha^2 k^2 t} \cos kx, u_{xx} = -k^2 e^{-\alpha^2 k^2 t} \sin kx$, and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx$.
Thus $\alpha^2 u_{xx} = u_t$.

68. (a) $u = x^2 + y^2 \Rightarrow u_x = 2x, u_{xx} = 2; u_y = 2y, u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2, u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2, u_{xx} = 6x; u_y = 6xy, u_{yy} = 6x$.

(d) $u = \ln \sqrt{x^2 + y^2}$ is a solution: $u_x = \frac{1}{\sqrt{x^2 + y^2}} \left(\frac{1}{2} \right) (x^2 + y^2)^{-1/2} (2x) = \frac{x}{x^2 + y^2}$,

$$u_{xx} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \text{ By symmetry, } u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \text{ so } u_{xx} + u_{yy} = 0.$$

(e) $u = \sin x \cosh y + \cos x \sinh y$ is a solution:

$$u_x = \cos x \cosh y - \sin x \sinh y, u_{xx} = -\sin x \cosh y - \cos x \sinh y, \text{ and } u_y = \sin x \sinh y + \cos x \cosh y, \\ u_{yy} = \sin x \cosh y + \cos x \sinh y.$$

(f) $u = e^{-x} \cos y - e^{-y} \cos x$ is a solution: $u_x = -e^{-x} \cos y + e^{-y} \sin x, u_{xx} = e^{-x} \cos y + e^{-y} \cos x$, and $u_y = -e^{-x} \sin y + e^{-y} \cos x, u_{yy} = -e^{-x} \cos y - e^{-y} \cos x$.

69. $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \Rightarrow u_x = \left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}$ and

$$u_{xx} = -(x^2 + y^2 + z^2)^{-3/2} - x\left(-\frac{3}{2}\right)(x^2 + y^2 + z^2)^{-5/2}(2x) = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$ and $u_{zz} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$.

Thus $u_{xx} + u_{yy} + u_{zz} = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} = 0$.

70. (a) $u = \sin(kx) \sin(akt) \Rightarrow u_t = ak \sin(kx) \cos(akt), u_{tt} = -a^2 k^2 \sin(kx) \sin(akt), \\ u_x = k \cos(kx) \sin(akt), u_{xx} = -k^2 \sin(kx) \sin(akt). \text{ Thus } u_{tt} = a^2 u_{xx}.$

$$\begin{aligned}
 \text{(b) } u &= \frac{t}{a^2 t^2 - x^2} \Rightarrow u_t = \frac{(a^2 t^2 - x^2) - t(2a^2 t)}{(a^2 t^2 - x^2)^2} = -\frac{a^2 t^2 + x^2}{(a^2 t^2 - x^2)^2}, \\
 u_{tt} &= \frac{-2a^2 t(a^2 t^2 - x^2)^2 + (a^2 t^2 - x^2)(2)(a^2 t^2 - x^2)(2a^2 t)}{(a^2 t^2 - x^2)^4} = \frac{2a^4 t^3 + 6a^2 t x^2}{a^2 t^2 - x^2}, \\
 u_x &= t(-1)(a^2 t^2 - x^2)^{-2}(2x) = \frac{2tx}{(a^2 t^2 - x^2)^2}, \\
 u_{xx} &= \frac{2t(a^2 t^2 - x^2)^2 - 2tx(2)(a^2 t^2 - x^2)(-2x)}{(a^2 t^2 - x^2)^4} = \frac{2a^2 t^3 - 2tx^2 + 8tx^2}{(a^2 t^2 - x^2)^3} = \frac{2a^2 t^3 + 6tx^2}{(a^2 t^2 - x^2)^4}.
 \end{aligned}$$

$$\text{Thus } u_{tt} = a^2 u_{xx}.$$

$$\begin{aligned}
 \text{(c) } u &= (x - at)^6 + (x + at)^6 \Rightarrow u_t = -6a(x - at)^5 + 6a(x + at)^5, \\
 u_{tt} &= 30a^2(x - at)^4 + 30a^2(x + at)^4, u_x = 6(x - at)^5 + 6(x + at)^5, u_{xx} = 30(x - at)^4 + 30(x + at)^4.
 \end{aligned}$$

$$\text{Thus } u_{tt} = a^2 u_{xx}.$$

$$\begin{aligned}
 \text{(d) } u &= \sin(x - at) + \ln(x + at) \Rightarrow u_t = -a \cos(x - at) + \frac{a}{x + at}, u_{tt} = -a^2 \sin(x - at) - \frac{a^2}{(x + at)^2}, \\
 u_x &= \cos(x - at) + \frac{1}{x + at}, u_{xx} = -\sin(x - at) - \frac{1}{(x + at)^2}. \text{ Thus } u_{tt} = a^2 u_{xx}.
 \end{aligned}$$

71. Let $v = x + at$, $w = x - at$. Then $u_t = \frac{\partial[f(v) + g(w)]}{\partial t} = \frac{df(v)}{dv} \frac{\partial v}{\partial t} + \frac{dg(w)}{dw} \frac{\partial w}{\partial t} = af'(v) - ag'(w)$ and $u_{tt} = \frac{\partial[af'(v) - ag'(w)]}{\partial t} = a[af''(v) + ag''(w)] = a^2[f''(v) + g''(w)]$. Similarly, by using the Chain Rule we have $u_x = f'(v) + g'(w)$ and $u_{xx} = f''(v) + g''(w)$. Thus $u_{tt} = a^2 u_{xx}$.

72. For each i , $i = 1, \dots, n$, $\partial u / \partial x_i = a_i e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}$ and $\partial^2 u / \partial x_i^2 = a_i^2 e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}$. Then $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = (a_1^2 + a_2^2 + \dots + a_n^2) e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} = e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} = u$ since $a_1^2 + a_2^2 + \dots + a_n^2 = 1$.

73. $z_x = e^y + ye^x$, $z_{xx} = ye^x$, $\partial^3 z / \partial x^3 = ye^x$. By symmetry $z_y = xe^y + e^x$, $z_{yy} = xe^y$ and $\partial^3 z / \partial y^3 = xe^y$. Then $\partial^3 z / \partial x \partial y^2 = e^y$ and $\partial^3 z / \partial x^2 \partial y = e^x$. Thus $z = xe^y + ye^x$ satisfies the given partial differential equation.

74. $P = bL^\alpha K^\beta$, so $\frac{\partial P}{\partial L} = \alpha bL^{\alpha-1} K^\beta$ and $\frac{\partial P}{\partial K} = \beta bL^\alpha K^{\beta-1}$. Then

$$\begin{aligned}
 L \frac{\partial P}{\partial L} + K \frac{\partial P}{\partial K} &= L(\alpha bL^{\alpha-1} K^\beta) + K(\beta bL^\alpha K^{\beta-1}) = \alpha bL^{1+\alpha-1} K^\beta + \beta bL^\alpha K^{1+\beta-1} \\
 &= (\alpha + \beta) bL^\alpha K^\beta = (\alpha + \beta) P
 \end{aligned}$$

75. If we fix $K = K_0$, $P(L, K_0)$ is a function of a single variable L , and $\frac{dP}{dL} = \alpha \frac{P}{L}$ is a separable differential

$$\text{equation. Then } \frac{dP}{P} = \alpha \frac{dL}{L} \Rightarrow \int \frac{dP}{P} = \int \alpha \frac{dL}{L} \Rightarrow \ln |P| = \alpha \ln |L| + C(K_0), \text{ where } C(K_0)$$

can depend on K_0 . Then $|P| = e^{\alpha \ln |L| + C(K_0)}$, and since $P > 0$ and $L > 0$, we have

$$P = e^{\alpha \ln L} e^{C(K_0)} = e^{C(K_0)} e^{\ln L^\alpha} = C_1(K_0) L^\alpha \text{ where } C_1(K_0) = e^{C(K_0)}.$$

76. (a) $\partial T/\partial x = -60(2x)/(1+x^2+y^2)^2$, so at $(2, 1)$, $T_x = -240/(1+4+1)^2 = -\frac{20}{3}$.

(b) $\partial T/\partial y = -60(2y)/(1+x^2+y^2)^2$, so at $(2, 1)$, $T_y = -120/36 = -\frac{10}{3}$. Thus from the point $(2, 1)$ the temperature is decreasing at a rate of $\frac{20}{3}^\circ\text{C}/\text{m}$ in the x -direction and is decreasing at a rate of $\frac{10}{3}^\circ\text{C}/\text{m}$ in the y -direction.

77. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$$\frac{\partial R^{-1}}{\partial R} \frac{\partial R}{\partial R_1} = \frac{\partial [(1/R_1) + (1/R_2) + (1/R_3)]}{\partial R_1} \text{ or } -R^{-2} \frac{\partial R}{\partial R_1} = -R_1^{-2}. \text{ Thus } \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.$$

78. $P = \frac{mRT}{V}$ so $\frac{\partial P}{\partial V} = \frac{-mRT}{V^2}$; $V = \frac{mRT}{P}$, so $\frac{\partial V}{\partial T} = \frac{mR}{P}$; $T = \frac{PV}{mR}$, so $\frac{\partial T}{\partial P} = \frac{V}{mR}$.

Thus $\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = \frac{-mRT}{V^2} \cdot \frac{mR}{P} \cdot \frac{V}{mR} = \frac{-mRT}{PV} = -1$, since $PV = mRT$.

79. By Exercise 78, $PV = mRT \Rightarrow P = \frac{mRT}{V}$, so $\frac{\partial P}{\partial T} = \frac{mR}{V}$. Also, $PV = mRT \Rightarrow V = \frac{mRT}{P}$

and $\frac{\partial V}{\partial T} = \frac{mR}{P}$. Since $T = \frac{PV}{mR}$, we have $T \frac{\partial P}{\partial T} \frac{\partial V}{\partial T} = \frac{PV}{mR} \cdot \frac{mR}{V} \cdot \frac{mR}{P} = mR$.

80. $\frac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15^\circ\text{C}$ and $v = 30 \text{ km/h}$, $\frac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$,

so we would expect the apparent temperature to drop by approximately 1.3°C if the actual temperature decreases by

1°C . $\frac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15^\circ\text{C}$ and $v = 30 \text{ km/h}$,

$\frac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent

temperature to drop by approximately 0.16°C if the wind speed increases by 1 km/h .

81. $\frac{\partial K}{\partial m} = \frac{1}{2}V^2$, $\frac{\partial K}{\partial V} = mV$, $\frac{\partial^2 K}{\partial V^2} = m$. Thus $\frac{\partial K}{\partial m} \cdot \frac{\partial^2 K}{\partial V^2} = \frac{1}{2}V^2m = K$.

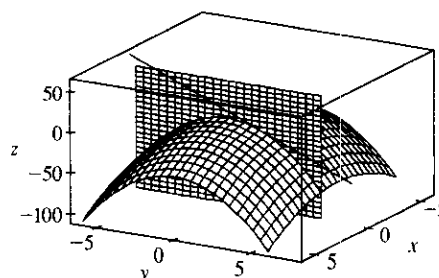
82. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc \cos A$. Thus $\frac{\partial(a^2)}{\partial a} = \frac{\partial(b^2 + c^2 - 2ab \cos A)}{\partial a}$ or

$2a = -2bc(-\sin A) \frac{\partial A}{\partial a}$, implying that $\frac{\partial A}{\partial a} = \frac{a}{bc \sin A}$. Taking the partial derivative of both sides with respect to

b gives $0 = 2b - 2c(\cos A) - 2bc(-\sin A) \frac{\partial A}{\partial b}$. Thus $\frac{\partial A}{\partial b} = \frac{c \cos A - b}{bc \sin A}$. By symmetry $\frac{\partial A}{\partial c} = \frac{b \cos A - c}{bc \sin A}$.

83. $f_x(x, y) = x + 4y \Rightarrow f_{xy}(x, y) = 4$ and $f_y(x, y) = 3x - y \Rightarrow f_{yx}(x, y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x, y) \neq f_{yx}(x, y)$, Clairaut's Theorem implies that such a function $f(x, y)$ does not exist.

84. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1, 2, -4)$ the slope is -8 . Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



85. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1, 2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z(\partial z/\partial x) = 0 \Rightarrow \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1, 2) = -2$. Thus the tangent line is given by $z - 2 = -2(x - 1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

86. $T(x, t) = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$

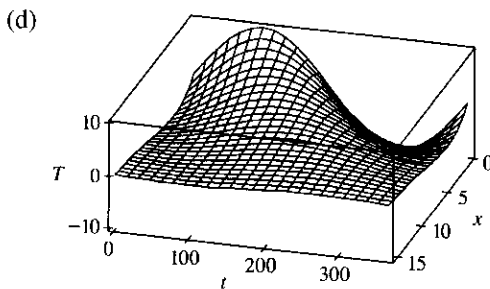
$$\begin{aligned} \text{(a)} \quad \partial T/\partial x &= T_1 e^{-\lambda x} [\cos(\omega t - \lambda x)(-\lambda)] + T_1(-\lambda e^{-\lambda x}) \sin(\omega t - \lambda x) \\ &= -\lambda T_1 e^{-\lambda x} [\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)] \end{aligned}$$

This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t .

- (b) $\partial T/\partial t = T_1 e^{-\lambda x} [\cos(\omega t - \lambda x)(\omega)] = \omega T_1 e^{-\lambda x} \cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x .

$$\begin{aligned} \text{(c)} \quad T_{xx} &= \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) \\ &= -\lambda T_1 (e^{-\lambda x} [\cos(\omega t - \lambda x)(-\lambda) - \sin(\omega t - \lambda x)(-\lambda)] \\ &\quad + e^{-\lambda x}(-\lambda) [\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)]) \\ &= 2\lambda^2 T_1 e^{-\lambda x} \cos(\omega t - \lambda x) \end{aligned}$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x} \cos(\omega t - \lambda x) = \frac{\omega}{2\lambda^2} T_{xx}$. So with $k = \frac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

- (e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 150$, and at a depth of 10 feet, at $t \approx 220$.

87. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

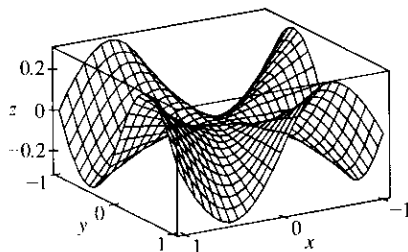
88. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n n th order partial derivatives.
- (b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all n th order partial derivatives with p partials with respect to x and $n - p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an n th order partial derivative can range from 0 to n , a function of two variables has $n + 1$ distinct partial derivatives of order n if these partial derivatives are all continuous.
- (c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n n th order partial derivatives of a function of three variables.

89. Let $g(x) = f(x, 0) = x(x^2)^{-3/2}e^0 = x|x|^{-3}$. But we are using the point $(1, 0)$, so near $(1, 0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1, 0) = g'(1) = -2$.

90. $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{(h^3 + 0)^{1/3} - 0}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$

Or: Let $g(x) = f(x, 0) = \sqrt[3]{x^3 + 0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0, 0) = g'(0) = 1$.

91. (a)



(b) For $(x, y) \neq (0, 0)$, $f_x(x, y) = \frac{(3x^2y - y^3)(x^2 + y^2) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$, and by

symmetry $f_y(x, y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}.$

(c) $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{(0/h^2) - 0}{h} = 0$ and $f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = 0.$

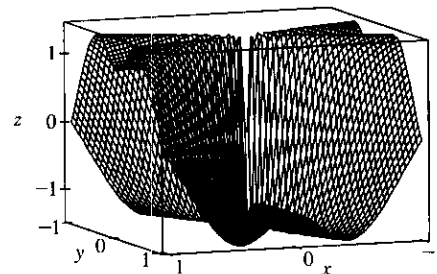
(d) By (3), $f_{xy}(0, 0) = \frac{\partial f_x}{\partial y} = \lim_{h \rightarrow 0} \frac{f_x(0, h) - f_x(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$f_{yx}(0, 0) = \frac{\partial f_y}{\partial x} = \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^5/h^4}{h} = 1.$

(e) For $(x, y) \neq (0, 0)$, we use a CAS to compute

$$f_{xy}(x, y) = \frac{x^6 + 9x^4y^2 - 4x^2y^4 + 4y^6}{(x^2 + y^2)^3}.$$

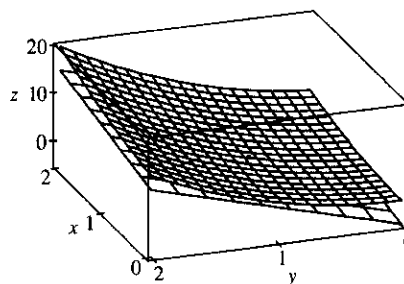
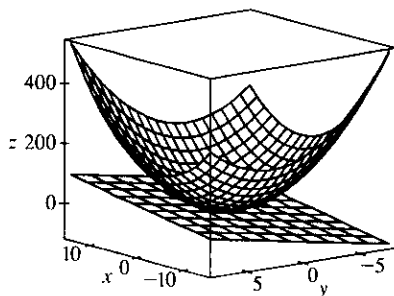
Now as $(x, y) \rightarrow (0, 0)$ along the x -axis, $f_{xy}(x, y) \rightarrow 1$ while as $(x, y) \rightarrow (0, 0)$ along the y -axis, $f_{xy}(x, y) \rightarrow 4$. Thus f_{xy} isn't continuous at $(0, 0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



15.4 Tangent Planes and Linear Approximations

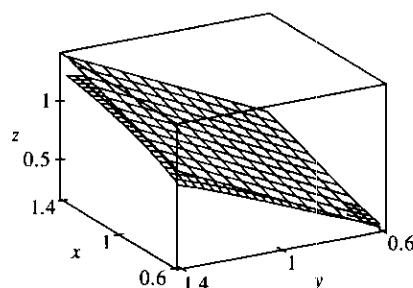
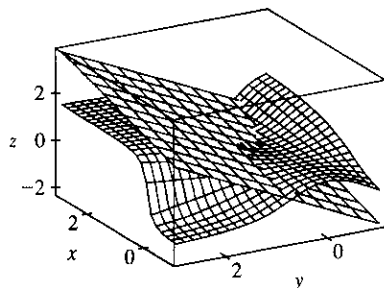
ET 14.4

1. $z = f(x, y) = 4x^2 - y^2 + 2y \Rightarrow f_x(x, y) = 8x, f_y(x, y) = -2y + 2$, so $f_x(-1, 2) = -8, f_y(-1, 2) = -2$.
By Equation 2, an equation of the tangent plane is $z - 4 = f_x(-1, 2)[x - (-1)] + f_y(-1, 2)(y - 2) \Rightarrow$
 $z - 4 = -8(x + 1) - 2(y - 2)$ or $z = -8x - 2y$.
2. $z = f(x, y) = 9x^2 + y^2 + 6x - 3y + 5 \Rightarrow f_x(x, y) = 18x + 6, f_y(x, y) = 2y - 3$, so $f_x(1, 2) = 24$ and
 $f_y(1, 2) = 1$. By Equation 2, an equation of the tangent plane is $z - 18 = f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) \Rightarrow$
 $z - 18 = 24(x - 1) + 1(y - 2)$ or $z = 24x + y - 8$.
3. $z = f(x, y) = \sqrt{4 - x^2 - 2y^2} \Rightarrow f_x(x, y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{4 - x^2 - 2y^2}},$
 $f_y(x, y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-4y) = -\frac{2y}{\sqrt{4 - x^2 - 2y^2}},$ so $f_x(1, -1) = -1$ and $f_y(1, -1) = 2$. Thus, an
equation of the tangent plane is $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)(y - (-1)) \Rightarrow$
 $z - 1 = -1(x - 1) + 2(y + 1)$ or $x - 2y + z = 4$.
4. $z = f(x, y) = y \ln x \Rightarrow f_x(x, y) = y/x, f_y(x, y) = \ln x$, so $f_x(1, 4) = 4, f_y(1, 4) = 0$, and an equation of
the tangent plane is $z - 0 = f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4) \Rightarrow z = 4(x - 1) + 0(y - 4)$ or $z = 4x - 4$.
5. $z = f(x, y) = y \cos(x - y) \Rightarrow f_x = y(-\sin(x - y)(1)) = -y \sin(x - y),$
 $f_y = y(-\sin(x - y)(-1)) + \cos(x - y) = y \sin(x - y) + \cos(x - y)$, so $f_x(2, 2) = -2 \sin(0) = 0,$
 $f_y(2, 2) = 2 \sin(0) + \cos(0) = 1$ and an equation of the tangent plane is $z - 2 = 0(x - 2) + 1(y - 2)$ or $z = y$.
6. $z = f(x, y) = e^{x^2 - y^2} \Rightarrow f_x(x, y) = 2xe^{x^2 - y^2}, f_y(x, y) = -2ye^{x^2 - y^2}$, so $f_x(1, -1) = 2, f_y(1, -1) = 2$.
By Equation 2, an equation of the tangent plane is $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)(y - (-1)) \Rightarrow$
 $z - 1 = 2(x - 1) + 2(y + 1)$ or $z = 2x + 2y + 1$.
7. $z = f(x, y) = x^2 + xy + 3y^2$, so $f_x(x, y) = 2x + y \Rightarrow f_x(1, 1) = 3, f_y(x, y) = x + 6y \Rightarrow f_y(1, 1) = 7$
and an equation of the tangent plane is $z - 5 = 3(x - 1) + 7(y - 1)$ or $z = 3x + 7y - 5$. After zooming in, the
surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we
zoom in farther, the surface and the tangent plane will appear to coincide.



8. $z = f(x, y) = \arctan(xy^2) \Rightarrow f_x = \frac{1}{1 + (xy^2)^2}(y^2) = \frac{y^2}{1 + x^2y^4}, f_y = \frac{1}{1 + (xy^2)^2}(2xy) = \frac{2xy}{1 + x^2y^4},$
 $f_x(1, 1) = \frac{1}{1+1} = \frac{1}{2}, f_y(1, 1) = \frac{2}{1+1} = 1$, so an equation of the tangent plane is $z - \frac{\pi}{4} = \frac{1}{2}(x - 1) + 1(y - 1)$ or

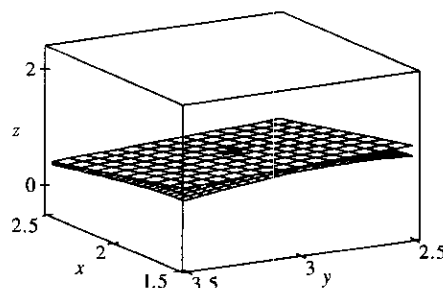
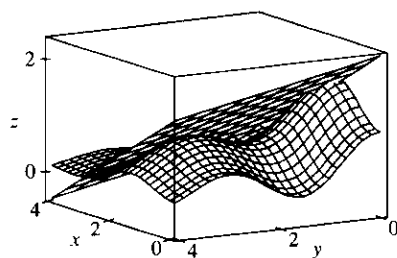
$z = \frac{1}{2}x + y - \frac{3}{2} + \frac{\pi}{4}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



9. $f(x, y) = e^{-(x^2+y^2)/15}(\sin^2 x + \cos^2 y)$. A CAS gives

$$f_x = -\frac{2}{15}e^{-(x^2+y^2)/15}(x \sin^2 x + x \cos^2 y - 15 \sin x \cos x) \text{ and}$$

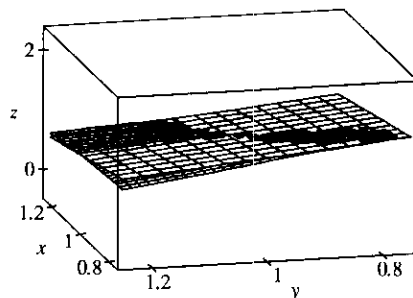
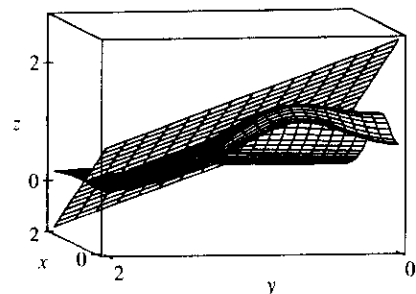
$f_y = -\frac{2}{15}e^{-(x^2+y^2)/15}(y \sin^2 x + y \cos^2 y + 15 \sin y \cos y)$. We use the CAS to evaluate these at $(2, 3)$, and then substitute the results into Equation 2 in order to plot the tangent plane. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



10. $f(x, y) = \frac{\sqrt{1+4x^2+4y^2}}{1+x^4+y^4}$. A CAS gives $f_x = \frac{4x(1-3x^4+y^4-x^2-4x^2y^2)}{\sqrt{1+4x^2+4y^2}(1+x^4+y^4)^2}$ and

$$f_y = \frac{4y(1-3y^4+x^4-y^2-4x^2y^2)}{\sqrt{1+4x^2+4y^2}(1+x^4+y^4)^2}.$$
 We use the CAS to evaluate these at $(1, 1)$, and then substitute the results

into Equation 2 to get an equation of the tangent plane: $z = \frac{25-8x-8y}{9}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



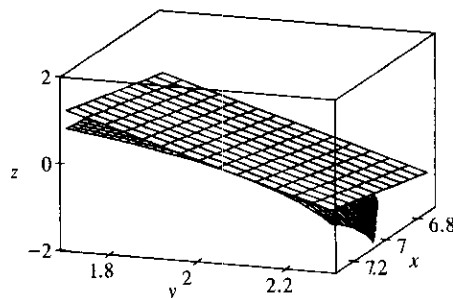
11. $f(x, y) = x\sqrt{y}$. The partial derivatives are $f_x(x, y) = \sqrt{y}$ and $f_y(x, y) = \frac{x}{2\sqrt{y}}$, so $f_x(1, 4) = 2$ and $f_y(1, 4) = \frac{1}{4}$. Both f_x and f_y are continuous functions for $y > 0$, so by Theorem 8, f is differentiable at $(1, 4)$. By Equation 3, the linearization of f at $(1, 4)$ is given by
- $$L(x, y) = f(1, 4) + f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4) = 2 + 2(x - 1) + \frac{1}{4}(y - 4) = 2x + \frac{1}{4}y - 1.$$
12. $f(x, y) = \frac{x}{y}$. The partial derivatives are $f_x(x, y) = \frac{1}{y}$ and $f_y(x, y) = -\frac{x}{y^2}$, so $f_x(6, 3) = \frac{1}{3}$ and $f_y(6, 3) = -\frac{2}{3}$. Both f_x and f_y are continuous functions for $y \neq 0$, so f is differentiable at $(6, 3)$ by Theorem 8. The linearization of f at $(6, 3)$ is given by
- $$L(x, y) = f(6, 3) + f_x(6, 3)(x - 6) + f_y(6, 3)(y - 3) = \frac{2}{3} + \frac{1}{3}(x - 6) - \frac{2}{3}(y - 3) = \frac{1}{3}x - \frac{2}{3}y + 2.$$
13. $f(x, y) = e^x \cos xy$. The partial derivatives are $f_x(x, y) = e^x(\cos xy - y \sin xy)$ and $f_y(x, y) = -xe^x \sin xy$, so $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0, 0)$ by Theorem 8. The linearization of f at $(0, 0)$ is given by
- $$L(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 1 + 1(x - 0) + 0(y - 0) = x + 1.$$
14. $f(x, y) = \sqrt{x + e^{4y}} = (x + e^{4y})^{1/2}$. The partial derivatives are $f_x(x, y) = \frac{1}{2}(x + e^{4y})^{-1/2}$ and $f_y(x, y) = \frac{1}{2}(x + e^{4y})^{-1/2}(4e^{4y}) = 2e^{4y}(x + e^{4y})^{-1/2}$, so $f_x(3, 0) = \frac{1}{2}(3 + e^0)^{-1/2} = \frac{1}{4}$ and $f_y(3, 0) = 2e^0(3 + e^0)^{-1/2} = 1$. Both f_x and f_y are continuous functions near $(3, 0)$, so f is differentiable at $(3, 0)$ by Theorem 8. The linearization of f at $(3, 0)$ is
- $$L(x, y) = f(3, 0) + f_x(3, 0)(x - 3) + f_y(3, 0)(y - 0) = 2 + \frac{1}{4}(x - 3) + 1(y - 0) = \frac{1}{4}x + y + \frac{5}{4}.$$
15. $f(x, y) = \tan^{-1}(x + 2y)$. The partial derivatives are $f_x(x, y) = \frac{1}{1 + (x + 2y)^2}$ and $f_y(x, y) = \frac{2}{1 + (x + 2y)^2}$, so $f_x(1, 0) = \frac{1}{2}$ and $f_y(1, 0) = 1$. Both f_x and f_y are continuous functions, so f is differentiable at $(1, 0)$, and the linearization of f at $(1, 0)$ is
- $$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = \frac{\pi}{4} + \frac{1}{2}(x - 1) + 1(y) = \frac{1}{2}x + y + \frac{\pi}{4} - \frac{1}{2}.$$
16. $f(x, y) = \sin(2x + 3y)$. The partial derivatives are $f_x(x, y) = 2\cos(2x + 3y)$ and $f_y(x, y) = 3\cos(2x + 3y)$, so $f_x(-3, 2) = 2$ and $f_y(-3, 2) = 3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3, 2)$, and the linearization of f at $(-3, 2)$ is
- $$L(x, y) = f(-3, 2) + f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) = 0 + 2(x + 3) + 3(y - 2) = 2x + 3y.$$
17. $f(x, y) = \sqrt{20 - x^2 - 7y^2} \Rightarrow f_x(x, y) = -\frac{x}{\sqrt{20 - x^2 - 7y^2}}$ and $f_y(x, y) = -\frac{7y}{\sqrt{20 - x^2 - 7y^2}}$, so $f_x(2, 1) = -\frac{2}{3}$ and $f_y(2, 1) = -\frac{7}{3}$. Then the linear approximation of f at $(2, 1)$ is given by
- $$\begin{aligned} f(x, y) &\approx f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1) \\ &= -\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3} \end{aligned}$$
- Thus $f(1.95, 1.08) \approx -\frac{2}{3}(1.95) - \frac{7}{3}(1.08) + \frac{20}{3} = 2.84\bar{6}$.

18. $f(x, y) = \ln(x - 3y) \Rightarrow f_x(x, y) = \frac{1}{x - 3y}$ and $f_y(x, y) = -\frac{3}{x - 3y}$, so $f_x(7, 2) = 1$ and $f_y(7, 2) = -3$.

Then the linear approximation of f at $(7, 2)$ is given by

$$\begin{aligned} f(x, y) &\approx f(7, 2) + f_x(7, 2)(x - 7) + f_y(7, 2)(y - 2) \\ &= 0 + 1(x - 7) - 3(y - 2) = x - 3y - 1 \end{aligned}$$

Thus $f(6.9, 2.06) \approx 6.9 - 3(2.06) - 1 = -0.28$. The graph shows that our approximated value is slightly greater than the actual value.



19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \Rightarrow f_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$, $f_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$, and $f_z(x, y, z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$, so $f_x(3, 2, 6) = \frac{3}{7}$, $f_y(3, 2, 6) = \frac{2}{7}$, and $f_z(3, 2, 6) = \frac{6}{7}$. Then the linear approximation of f at $(3, 2, 6)$ is given by

$$\begin{aligned} f(x, y, z) &\approx f(3, 2, 6) + f_x(3, 2, 6)(x - 3) + f_y(3, 2, 6)(y - 2) + f_z(3, 2, 6)(z - 6) \\ &= 7 + \frac{3}{7}(x - 3) + \frac{2}{7}(y - 2) + \frac{6}{7}(z - 6) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

20. From the table, $f(40, 20) = 28$. To estimate $f_v(40, 20)$ and $f_t(40, 20)$ we follow the procedure used in

Exercise 15.3.4 [ET 14.3.4]. Since $f_v(40, 20) = \lim_{h \rightarrow 0} \frac{f(40 + h, 20) - f(40, 20)}{h}$, we approximate this quantity

with $h = \pm 10$ and use the values given in the table: $f_v(40, 20) \approx \frac{f(50, 20) - f(40, 20)}{10} = \frac{40 - 28}{10} = 1.2$,

$f_v(40, 20) \approx \frac{f(30, 20) - f(40, 20)}{-10} = \frac{17 - 28}{-10} = 1.1$. Averaging these values gives $f_v(40, 20) \approx 1.15$.

Similarly, $f_t(40, 20) = \lim_{h \rightarrow 0} \frac{f(40, 20 + h) - f(40, 20)}{h}$, so we use $h = 10$ and $h = -5$:

$f_t(40, 20) \approx \frac{f(40, 30) - f(40, 20)}{10} = \frac{31 - 28}{10} = 0.3$, $f_t(40, 20) \approx \frac{f(40, 15) - f(40, 20)}{-5} = \frac{25 - 28}{-5} = 0.6$.

Averaging these values gives $f_t(40, 15) \approx 0.45$. The linear approximation, then, is

$$\begin{aligned} f(v, t) &\approx f(40, 20) + f_v(40, 20)(v - 40) + f_t(40, 20)(t - 20) \\ &\approx 28 + 1.15(v - 40) + 0.45(t - 20) \end{aligned}$$

When $v = 43$ and $t = 24$, we estimate $f(43, 24) \approx 28 + 1.15(43 - 40) + 0.45(24 - 20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.

21. From the table, $f(94, 80) = 127$. To estimate $f_T(94, 80)$ and $f_H(94, 80)$ we follow the procedure used in

Section 15.3 [ET 14.3]. Since $f_T(94, 80) = \lim_{h \rightarrow 0} \frac{f(94 + h, 80) - f(94, 80)}{h}$, we approximate this quantity

with $h = \pm 2$ and use the values given in the table: $f_T(94, 80) \approx \frac{f(96, 80) - f(94, 80)}{2} = \frac{135 - 127}{2} = 4$,

$$f_T(94, 80) \approx \frac{f(92, 80) - f(94, 80)}{-2} = \frac{119 - 127}{-2} = 4.$$

Averaging these values gives $f_T(94, 80) \approx 4$. Similarly, $f_H(94, 80) = \lim_{h \rightarrow 0} \frac{f(94, 80 + h) - f(94, 80)}{h}$,

$$\text{so we use } h = \pm 5: f_H(94, 80) \approx \frac{f(94, 85) - f(94, 80)}{5} = \frac{132 - 127}{5} = 1,$$

$f_H(94, 80) \approx \frac{f(94, 75) - f(94, 80)}{-5} = \frac{122 - 127}{-5} = 1$. Averaging these values gives $f_H(94, 80) \approx 1$. The linear approximation, then, is

$$\begin{aligned} f(T, H) &\approx f(94, 80) + f_T(94, 80)(T - 94) + f_H(94, 80)(H - 80) \\ &\approx 127 + 4(T - 94) + 1(H - 80) \end{aligned}$$

Thus when $T = 95$ and $H = 78$, $f(95, 78) \approx 127 + 4(95 - 94) + 1(78 - 80) = 129$, so we estimate the heat index to be approximately 129°F .

- 22.** From the table, $f(-15, 50) = -29$. To estimate $f_T(-15, 50)$ and $f_v(-15, 50)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(-15, 50) = \lim_{h \rightarrow 0} \frac{f(-15 + h, 50) - f(-15, 50)}{h}$, we approximate this quantity with $h = \pm 5$ and use the values given in the table:

$$f_T(-15, 50) \approx \frac{f(-10, 50) - f(-15, 50)}{5} = \frac{-22 - (-29)}{5} = 1.4,$$

$$f_T(-15, 50) \approx \frac{f(-20, 50) - f(-15, 50)}{-5} = \frac{-35 - (-29)}{-5} = 1.2.$$

Averaging these values gives $f_T(-15, 50) \approx 1.3$. Similarly $f_v(-15, 50) = \lim_{h \rightarrow 0} \frac{f(-15, 50 + h) - f(-15, 50)}{h}$ so

$$\text{we use } h = \pm 10: f_v(-15, 50) \approx \frac{f(-15, 60) - f(-15, 50)}{10} = \frac{-30 - (-29)}{10} = -0.1,$$

$$f_v(-15, 50) \approx \frac{f(-15, 40) - f(-15, 50)}{-10} = \frac{-27 - (-29)}{-10} = -0.2.$$

Averaging these values gives $f_v(-15, 50) \approx -0.15$. The linear approximation to the wind-chill index function, then, is

$$\begin{aligned} f(T, v) &\approx f(-15, 50) + f_T(-15, 50)(T - (-15)) + f_v(-15, 50)(v - 50) \\ &\approx -29 + (1.3)(T + 15) - (0.15)(v - 50) \end{aligned}$$

Thus when $T = -17^\circ\text{C}$ and $v = 55 \text{ km/h}$, $f(-17, 55) \approx -29 + (1.3)(-17 + 15) - (0.15)(55 - 50) = -32.35$, so we estimate the wind-chill index to be approximately -32.35°C .

- 23.** $z = x^3 \ln(y^2) \Rightarrow$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 3x^2 \ln(y^2) dx + x^3 \cdot \frac{1}{y^2}(2y) dy = 3x^2 \ln(y^2) dx + \frac{2x^3}{y} dy.$$

- 24.** $v = y \cos xy \Rightarrow$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = y(-\sin xy)y dx + [y(-\sin xy)x + \cos xy] dy = -y^2 \sin xy dx + (\cos xy - xy \sin xy) dy$$

- 25.** $u = e^t \sin \theta \Rightarrow du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial \theta} d\theta = e^t \sin \theta dt + e^t \cos \theta d\theta$

26. $u = \frac{r}{s+2t} \Rightarrow$

$$\begin{aligned} du &= \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt = \frac{1}{s+2t} dr + r(-1)(s+2t)^{-2} ds + r(-1)(s+2t)^{-2}(2) dt \\ &= \frac{1}{s+2t} dr - \frac{r}{(s+2t)^2} ds - \frac{2r}{(s+2t)^2} dt \end{aligned}$$

27. $w = \ln \sqrt{x^2 + y^2 + z^2} \Rightarrow$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz \\ &= \left(\frac{1}{2} \right) \frac{2x(x^2 + y^2 + z^2)^{-1/2} dx + 2y(x^2 + y^2 + z^2)^{-1/2} dy + 2z(x^2 + y^2 + z^2)^{-1/2} dz}{(x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2} \end{aligned}$$

28. $w = xye^{xz} \Rightarrow$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz = (xyz e^{xz} + ye^{xz}) dx + xe^{xz} dy + x^2 ye^{xz} dz \\ &= (xz + 1)ye^{xz} dx + xe^{xz} dy + x^2 ye^{xz} dz. \end{aligned}$$

29. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$, $dz = z_x(1, 2) dx + z_y(1, 2) dy = (10)(0.05) + (4)(0.1) = 0.9$ while $\Delta z = f(1.05, 2.1) - f(1, 2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

30. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$, $dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189$.

31. $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = y dx + x dy$ and $|\Delta x| \leq 0.1$, $|\Delta y| \leq 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4 \text{ cm}^2$.

32. Let S be surface area. Then $S = 2(xy + xz + yz)$ and $dS = 2(y + z) dx + 2(x + z) dy + 2(x + y) dz$. The maximum error occurs with $\Delta x = \Delta y = \Delta z = 0.2$. Using $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$ we find the maximum error in calculated surface area to be about $dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152 \text{ cm}^2$.

33. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi r h dr + \pi r^2 dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08 \text{ cm}^3$. Thus the amount of tin is about 16 cm^3 .

34. Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi r h dr + \pi r^2 dh$ is an estimate of the amount of metal. With $dr = 0.05$ and $dh = 0.2$ we get $dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.2) = 2.80\pi \approx 8.8 \text{ cm}^3$.

35. The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y dx + x dy$, so with $dx = dy = \frac{3 \pm 3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150 \text{ ft}^2$. Thus there are approximately 150 ft^2 of paint in the stripe.

36. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31 \frac{T}{V}$, so

$$dP = \left(\frac{8.31}{V} \right) dT - \frac{8.31 \cdot T}{V^2} dV = 8.31 \left[-\frac{5}{12} - \frac{310}{144} \cdot \frac{3}{10} \right] \approx -8.83.$$

Thus the pressure will drop by about 8.83 kPa.

37. First we find $\frac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1 :

$$\frac{\partial}{\partial R_1} \left[\frac{1}{R} \right] = \frac{\partial [(1/R_1) + (1/R_2) + (1/R_3)]}{\partial R_1} \Rightarrow -R^{-2} \frac{\partial R}{\partial R_1} = -R_1^{-2} \Rightarrow \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}. \text{ Then by}$$

$$\text{symmetry, } \frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2}, \frac{\partial R}{\partial R_3} = \frac{R^2}{R_3^2}. \text{ When } R_1 = 25, R_2 = 40 \text{ and } R_3 = 50, \frac{1}{R} = \frac{17}{200} \Leftrightarrow R = \frac{200}{17} \text{ ohms.}$$

Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005 R_i$. So

$$\begin{aligned} \Delta R \approx dR &= \frac{\partial R}{\partial R_1} \Delta R_1 + \frac{\partial R}{\partial R_2} \Delta R_2 + \frac{\partial R}{\partial R_3} \Delta R_3 = (0.005) R^2 \left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right] \\ &= (0.005) R = \frac{1}{17} \approx 0.059 \text{ ohms} \end{aligned}$$

38. Let x, y, z and w be the four numbers with $p(x, y, z, w) = xyzw$. Since the largest error due to rounding for each number is 0.05, the maximum error in the calculated product is approximated by

$dp = (yzw)(0.05) + (xzw)(0.05) + (xyw)(0.05) + (xyz)(0.05)$. Furthermore, each of the numbers is positive but less than 50, so the product of any three is between 0 and $(50)^3$. Thus $dp \leq 4(50)^3(0.05) = 25,000$.

39. $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)^2 + (b + \Delta y)^2 - (a^2 + b^2)$

$$= a^2 + 2a \Delta x + (\Delta x)^2 + b^2 + 2b \Delta y + (\Delta y)^2 - a^2 - b^2 = 2a \Delta x + (\Delta x)^2 + 2b \Delta y + (\Delta y)^2$$

But $f_x(a, b) = 2a$ and $f_y(a, b) = 2b$ and so $\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \Delta x \Delta x + \Delta y \Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = \Delta y$. Hence f is differentiable.

40. $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)(b + \Delta y) - 5(b + \Delta y)^2 - (ab - 5b^2)$

$$= ab + a \Delta y + b \Delta x + \Delta x \Delta y - 5b^2 - 10b \Delta y - 5(\Delta y)^2 - ab + 5b^2$$

$$= (a - 10b) \Delta y + b \Delta x + \Delta x \Delta y - 5 \Delta y \Delta y,$$

but $f_x(a, b) = b$ and $f_y(a, b) = a - 10b$ and so $\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \Delta x \Delta y - 5 \Delta y \Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta y$ and $\varepsilon_2 = -5 \Delta y$. Hence f is differentiable.

41. To show that f is continuous at (a, b) we need to show that $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$ or equivalently

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b). \text{ Since } f \text{ is differentiable at } (a, b),$$

$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where ε_1 and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$. Thus $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \rightarrow (0, 0)$ gives $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Thus f is

continuous at (a, b) .

42. (a) $\lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$ and $\lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$. Thus

$f_x(0, 0) = f_y(0, 0) = 0$. To show that f isn't differentiable at $(0, 0)$ we need only show that f is not continuous at $(0, 0)$ and apply Exercise 41. As $(x, y) \rightarrow (0, 0)$ along the x -axis $f(x, y) = 0/x^2 = 0$ for $x \neq 0$ so

$f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along the x -axis. But as $(x, y) \rightarrow (0, 0)$ along the line $y = x$,

$f(x, x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along this line. Thus $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$

doesn't exist, so f is discontinuous at $(0, 0)$ and thus not differentiable there.

(b) For $(x, y) \neq (0, 0)$, $f_x(x, y) = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$. If we approach $(0, 0)$ along the y -axis,

then $f_x(x, y) = f_x(0, y) = \frac{y^3}{y^4} = \frac{1}{y}$, so $f_x(x, y) \rightarrow \pm\infty$ as $(x, y) \rightarrow (0, 0)$. Thus $\lim_{(x,y) \rightarrow (0,0)} f_x(x, y)$ does

not exist and $f_x(x, y)$ is not continuous at $(0, 0)$. Similarly, $f_y(x, y) = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$

for $(x, y) \neq (0, 0)$, and if we approach $(0, 0)$ along the x -axis, then $f_y(x, y) = f_y(x, 0) = \frac{x^3}{x^4} = \frac{1}{x}$. Thus

$\lim_{(x,y) \rightarrow (0,0)} f_y(x, y)$ does not exist and $f_y(x, y)$ is not continuous at $(0, 0)$.

15.5 The Chain Rule

ET 14.5

1. $z = x^2y + xy^2, x = 2 + t^4, y = 1 - t^3 \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (2xy + y^2)(4t^3) + (x^2 + 2xy)(-3t^2) = 4(2xy + y^2)^3 - 3(x^2 + 2xy)t^2$$

2. $z = \sqrt{x^2 + y^2}, x = e^{2t}, y = e^{-2t} \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) \cdot e^{2t}(2) + \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) \cdot e^{-2t}(-2) = \frac{2xe^{2t} - 2ye^{2t}}{\sqrt{x^2 + y^2}}$$

3. $z = \sin x \cos y, x = \pi t, y = \sqrt{t} \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \cos x \cos y \cdot \pi + \sin x (-\sin y) \cdot \frac{1}{2}t^{-1/2} = \pi \cos x \cos y - \frac{1}{2\sqrt{t}} \sin x \sin y$$

4. $z = x \ln(x + 2y), x = \sin t, y = \cos t \Rightarrow$

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \left[x \cdot \frac{1}{x + 2y} + 1 \cdot \ln(x + 2y) \right] \cos t + x \cdot \frac{1}{x + 2y} (2) \cdot (-\sin t) \\ &= \left[\frac{x}{x + 2y} + \ln(x + 2y) \right] \cos t - \frac{2x}{x + 2y} (\sin t) \end{aligned}$$

5. $w = xe^{y/z}, x = t^2, y = 1 - t, z = 1 + 2t \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = e^{y/z} \cdot 2t + xe^{y/z} \left(\frac{1}{z} \right) \cdot (-1) + xe^{y/z} \left(-\frac{y}{z^2} \right) \cdot 2 = e^{y/z} \left(2t - \frac{x}{z} - \frac{2xy}{z^2} \right)$$

6. $w = xy + yz^2, x = e^t, y = e^t \sin t, z = e^t \cos t \Rightarrow$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = y \cdot e^t + (x + z^2) \cdot (e^t \cos t + e^t \sin t) + 2yz \cdot (-e^t \sin t + e^t \cos t) \\ &= e^t [y + (x + z^2)(\cos t + \sin t) + 2yz(\cos t - \sin t)] \end{aligned}$$

7. $z = x^2 + xy + y^2, x = s + t, y = st \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(t) = 2x + y + xt + 2yt$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (2x + y)(1) + (x + 2y)(s) = 2x + y + xs + 2ys$$

8. $z = x/y, x = se^t, y = 1 + se^{-t} \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{1}{y}(e^t) + \left(-\frac{x}{y^2}\right)(e^{-t}) = \frac{1}{y}e^t - \frac{x}{y^2}e^{-t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{1}{y}(se^t) + \left(-\frac{x}{y^2}\right)(-se^{-t}) = \frac{s}{y}e^t + \frac{xs}{y^2}e^{-t}$$

9. $z = \arctan(2x + y), x = s^2t, y = s \ln t \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{2}{1 + (2x + y)^2} \cdot 2st + \frac{1}{1 + (2x + y)^2} \cdot \ln t = \frac{4st + \ln t}{1 + (2x + y)^2}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{2}{1 + (2x + y)^2} \cdot s^2 + \frac{1}{1 + (2x + y)^2} \cdot \frac{s}{t} = \frac{2s^2 + s/t}{1 + (2x + y)^2}$$

10. $z = e^{xy} \tan y, x = s + 2t, y = s/t \Rightarrow$

$$\frac{\partial z}{\partial s} = ye^{xy} \tan y \cdot 1 + (e^{xy} \sec^2 y + xe^{xy} \tan y) \cdot \frac{1}{t} = ye^{xy} \tan y + \frac{e^{xy}}{t} (\sec^2 y + x \tan y)$$

$$\frac{\partial z}{\partial t} = ye^{xy} \tan y \cdot 2 + (e^{xy} \sec^2 y + xe^{xy} \tan y) \left(\frac{-s}{t^2}\right) = 2ye^{xy} \tan y - \frac{se^{xy}}{t^2} (\sec^2 y + x \tan y)$$

11. $z = e^r \cos \theta, r = st, \theta = \sqrt{s^2 + t^2} \Rightarrow$

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial s} = e^r \cos \theta \cdot t + e^r (-\sin \theta) \cdot \frac{1}{2}(s^2 + t^2)^{-1/2}(2s) \\ &= te^r \cos \theta - e^r \sin \theta \cdot \frac{s}{\sqrt{s^2 + t^2}} = e^r \left(t \cos \theta - \frac{s}{\sqrt{s^2 + t^2}} \sin \theta \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial t} = e^r \cos \theta \cdot s + e^r (-\sin \theta) \cdot \frac{1}{2}(s^2 + t^2)^{-1/2}(2t) \\ &= se^r \cos \theta - e^r \sin \theta \cdot \frac{t}{\sqrt{s^2 + t^2}} = e^r \left(s \cos \theta - \frac{t}{\sqrt{s^2 + t^2}} \sin \theta \right) \end{aligned}$$

12. $z = \sin \alpha \tan \beta, \alpha = 3s + t, \beta = s - t \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial s} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial s} = \cos \alpha \tan \beta \cdot 3 + \sin \alpha \sec^2 \beta \cdot 1 = 3 \cos \alpha \tan \beta + \sin \alpha \sec^2 \beta$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial t} = \cos \alpha \tan \beta \cdot 1 + \sin \alpha \sec^2 \beta \cdot (-1) = \cos \alpha \tan \beta - \sin \alpha \sec^2 \beta$$

13. When $t = 3, x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = f_x(2, 7)g'(3) + f_y(2, 7)h'(3) = (6)(5) + (-8)(-4) = 62.$$

14. By the Chain Rule (3), $\frac{\partial W}{\partial s} = \frac{\partial W}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial W}{\partial v} \frac{\partial v}{\partial s}$. Then

$$\begin{aligned} W_s(1, 0) &= F_u(u(1, 0), v(1, 0)) u_s(1, 0) + F_v(u(1, 0), v(1, 0)) v_s(1, 0) \\ &= F_u(2, 3) u_s(1, 0) + F_v(2, 3) v_s(1, 0) = (-1)(-2) + (10)(5) = 52 \end{aligned}$$

Similarly, $\frac{\partial W}{\partial t} = \frac{\partial W}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial W}{\partial v} \frac{\partial v}{\partial t} \Rightarrow$

$$\begin{aligned} W_t(1, 0) &= F_u(u(1, 0), v(1, 0)) u_t(1, 0) + F_v(u(1, 0), v(1, 0)) v_t(1, 0) \\ &= F_u(2, 3) u_t(1, 0) + F_v(2, 3) v_t(1, 0) = (-1)(6) + (10)(4) = 34 \end{aligned}$$

15. $g(u, v) = f(x(u, v), y(u, v))$ where $x = e^u + \sin v$, $y = e^u + \cos v \Rightarrow \frac{\partial x}{\partial u} = e^u$, $\frac{\partial x}{\partial v} = \cos v$, $\frac{\partial y}{\partial u} = e^u$, $\frac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}$. Then

$$\begin{aligned} g_u(0, 0) &= f_x(x(0, 0), y(0, 0)) x_u(0, 0) + f_y(x(0, 0), y(0, 0)) y_u(0, 0) \\ &= f_x(1, 2)(e^0) + f_y(1, 2)(e^0) = 2(1) + 5(1) = 7 \end{aligned}$$

Similarly $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}$. Then

$$\begin{aligned} g_v(0, 0) &= f_x(x(0, 0), y(0, 0)) x_v(0, 0) + f_y(x(0, 0), y(0, 0)) y_v(0, 0) \\ &= f_x(1, 2)(\cos 0) + f_y(1, 2)(-\sin 0) = 2(1) + 5(0) = 2 \end{aligned}$$

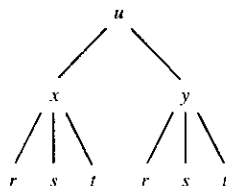
16. $g(r, s) = f(x(r, s), y(r, s))$ where $x = 2r - s$, $y = s^2 - 4r \Rightarrow \frac{\partial x}{\partial r} = 2$, $\frac{\partial x}{\partial s} = -1$, $\frac{\partial y}{\partial r} = -4$, $\frac{\partial y}{\partial s} = 2s$. By the Chain Rule (3) $\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$. Then

$$\begin{aligned} g_r(1, 2) &= f_x(x(1, 2), y(1, 2)) x_r(1, 2) + f_y(x(1, 2), y(1, 2)) y_r(1, 2) \\ &= f_x(0, 0)(2) + f_y(0, 0)(-4) = 4(2) + 8(-4) = -24 \end{aligned}$$

Similarly $\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$. Then

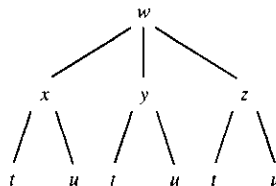
$$\begin{aligned} g_s(1, 2) &= f_x(x(1, 2), y(1, 2)) x_s(1, 2) + f_y(x(1, 2), y(1, 2)) y_s(1, 2) \\ &= f_x(0, 0)(-1) + f_y(0, 0)(4) = 4(-1) + 8(4) = 28 \end{aligned}$$

17.



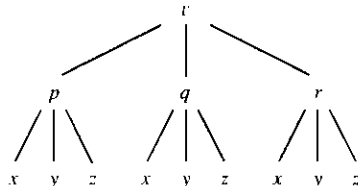
$$\begin{aligned} u &= f(x, y), x = x(r, s, t), y = y(r, s, t) \Rightarrow \\ \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}, \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}, \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \end{aligned}$$

18.



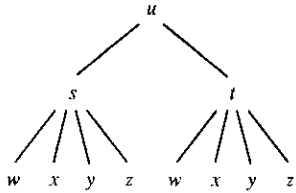
$$\begin{aligned} w &= f(x, y, z), x = x(t, u), y = y(t, u), z = z(t, u) \Rightarrow \\ \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}, \\ \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \end{aligned}$$

19.



$$\begin{aligned} v &= f(p, q, r), p = p(x, y, z), q = q(x, y, z), r = r(x, y, z) \Rightarrow \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial v}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial v}{\partial r} \frac{\partial r}{\partial x}, \frac{\partial v}{\partial y} = \frac{\partial v}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial v}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial v}{\partial r} \frac{\partial r}{\partial y}, \\ \frac{\partial v}{\partial z} &= \frac{\partial v}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial v}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial v}{\partial r} \frac{\partial r}{\partial z} \end{aligned}$$

20.



$$u = f(s, t), s = s(w, x, y, z), t = t(w, x, y, z) \Rightarrow$$

$$\frac{\partial u}{\partial w} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial w} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial w}, \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y}, \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial z}$$

$$21. z = x^2 + xy^3, x = uv^2 + w^3, y = u + ve^w \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w),$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w). \text{ When } u = 2, v = 1, \text{ and } w = 0, \text{ we have } x = 2,$$

$$y = 3, \text{ so } \frac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85, \frac{\partial z}{\partial v} = (31)(4) + (54)(1) = 178, \frac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54.$$

$$22. u = (r^2 + s^2)^{1/2}, r = y + x \cos t, s = x + y \sin t \Rightarrow$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{1}{2}(r^2 + s^2)^{-1/2}(2r)(\cos t) + \frac{1}{2}(r^2 + s^2)^{-1/2}(2s)(1) = (r \cos t + s)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} = \frac{1}{2}(r^2 + s^2)^{-1/2}(2r)(1) + \frac{1}{2}(r^2 + s^2)^{-1/2}(2s)(\sin t) = (r + s \sin t)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial t} = \frac{1}{2}(r^2 + s^2)^{-1/2}(2r)(-x \sin t) + \frac{1}{2}(r^2 + s^2)^{-1/2}(2s)(y \cos t) = \frac{-rx \sin t + sy \cos t}{\sqrt{r^2 + s^2}}.$$

$$\text{When } x = 1, y = 2, \text{ and } t = 0 \text{ we have } r = 3 \text{ and } s = 1, \text{ so } \frac{\partial u}{\partial x} = \frac{4}{\sqrt{10}}, \frac{\partial u}{\partial y} = \frac{3}{\sqrt{10}}, \text{ and } \frac{\partial u}{\partial t} = \frac{2}{\sqrt{10}}.$$

$$23. R = \ln(u^2 + v^2 + w^2), u = x + 2y, v = 2x - y, w = 2xy \Rightarrow$$

$$\begin{aligned} \frac{\partial R}{\partial x} &= \frac{\partial R}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial R}{\partial w} \frac{\partial w}{\partial x} = \frac{2u}{u^2 + v^2 + w^2}(1) + \frac{2v}{u^2 + v^2 + w^2}(2) + \frac{2w}{u^2 + v^2 + w^2}(2y) \\ &= \frac{2u + 4v + 4wy}{u^2 + v^2 + w^2}, \end{aligned}$$

$$\begin{aligned} \frac{\partial R}{\partial y} &= \frac{\partial R}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial R}{\partial w} \frac{\partial w}{\partial y} = \frac{2u}{u^2 + v^2 + w^2}(2) + \frac{2v}{u^2 + v^2 + w^2}(-1) + \frac{2w}{u^2 + v^2 + w^2}(2x) \\ &= \frac{4u - 2v + 4wx}{u^2 + v^2 + w^2}. \end{aligned}$$

$$\text{When } x = y = 1 \text{ we have } u = 3, v = 1, \text{ and } w = 2, \text{ so } \frac{\partial R}{\partial x} = \frac{9}{7} \text{ and } \frac{\partial R}{\partial y} = \frac{9}{7}.$$

$$24. M = xe^{y-z^2}, x = 2uv, y = u - v, z = u + v \Rightarrow$$

$$\begin{aligned} \frac{\partial M}{\partial u} &= \frac{\partial M}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial M}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial M}{\partial z} \frac{\partial z}{\partial u} = e^{y-z^2}(2v) + xe^{y-z^2}(1) + x(-2z)e^{y-z^2}(1) \\ &= e^{y-z^2}(2v + x - 2xz), \end{aligned}$$

$$\begin{aligned} \frac{\partial M}{\partial v} &= \frac{\partial M}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial M}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial M}{\partial z} \frac{\partial z}{\partial v} = e^{y-z^2}(2u) + xe^{y-z^2}(-1) + x(-2z)e^{y-z^2}(1) \\ &= e^{y-z^2}(2u - x - 2xz). \end{aligned}$$

$$\text{When } u = 3, v = -1 \text{ we have } x = -6, y = 4, \text{ and } z = 2, \text{ so } \frac{\partial M}{\partial u} = 16 \text{ and } \frac{\partial M}{\partial v} = 36.$$

25. $u = x^2 + yz$, $x = pr \cos \theta$, $y = pr \sin \theta$, $z = p + r \Rightarrow$

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial p} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial p} = (2x)(r \cos \theta) + (z)(r \sin \theta) + (y)(1) = 2xr \cos \theta + zr \sin \theta + y,$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r} = (2x)(p \cos \theta) + (z)(p \sin \theta) + (y)(1) = 2xp \cos \theta + zp \sin \theta + y,$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \theta} = (2x)(-pr \sin \theta) + (z)(pr \cos \theta) + (y)(0) = -2xpr \sin \theta + zpr \cos \theta.$$

When $p = 2$, $r = 3$, and $\theta = 0$ we have $x = 6$, $y = 0$, and $z = 5$, so $\frac{\partial u}{\partial p} = 36$, $\frac{\partial u}{\partial r} = 24$, and $\frac{\partial u}{\partial \theta} = 30$.

26. $Y = w \tan^{-1}(uv)$, $u = r + s$, $v = s + t$, $w = t + r \Rightarrow$

$$\begin{aligned} \frac{\partial Y}{\partial r} &= \frac{\partial Y}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial Y}{\partial v} \frac{\partial v}{\partial r} + \frac{\partial Y}{\partial w} \frac{\partial w}{\partial r} = \frac{w}{1+(uv)^2} (v)(1) + \frac{w}{1+(uv)^2} (u)(0) + \tan^{-1}(uv)(1) \\ &= \frac{vw}{1+u^2v^2} + \tan^{-1}(uv) \end{aligned}$$

$$\begin{aligned} \frac{\partial Y}{\partial s} &= \frac{\partial Y}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial Y}{\partial v} \frac{\partial v}{\partial s} + \frac{\partial Y}{\partial w} \frac{\partial w}{\partial s} = \frac{wv}{1+u^2v^2} (1) + \frac{wu}{1+u^2v^2} (1) + \tan^{-1}(uv)(0) \\ &= \frac{w(v+u)}{1+u^2v^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial Y}{\partial t} &= \frac{\partial Y}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v} \frac{\partial v}{\partial t} + \frac{\partial Y}{\partial w} \frac{\partial w}{\partial t} = \frac{wv}{1+u^2v^2} (0) + \frac{wu}{1+u^2v^2} (1) + \tan^{-1}(uv)(1) \\ &= \frac{wu}{1+u^2v^2} + \tan^{-1}(uv) \end{aligned}$$

When $r = 1$, $s = 0$, and $t = 1$, we have $u = 1$, $v = 1$, and $w = 2$, so $\frac{\partial Y}{\partial r} = 1 + \frac{\pi}{4}$, $\frac{\partial Y}{\partial s} = 2$, and $\frac{\partial Y}{\partial t} = 1 + \frac{\pi}{4}$.

27. $\sqrt{xy} = 1 + x^2y$, so let $F(x, y) = (xy)^{1/2} - 1 - x^2y = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{1}{2}(xy)^{-1/2}(y) - 2xy}{\frac{1}{2}(xy)^{-1/2}(x) - x^2} = -\frac{y - 4xy\sqrt{xy}}{x - 2x^2\sqrt{xy}} = \frac{4(xy)^{3/2} - y}{x - 2x^2\sqrt{xy}}.$$

28. $y^5 + x^2y^3 = 1 + ye^{x^2}$, so let $F(x, y) = y^5 + x^2y^3 - 1 - ye^{x^2} = 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy^3 - 2xye^{x^2}}{5y^4 + 3x^2y^2 - e^{x^2}} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}.$$

29. $\cos(x - y) = xe^y$, so let $F(x, y) = \cos(x - y) - xe^y = 0$.

$$\text{Then } \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(x - y) - e^y}{-\sin(x - y)(-1) - xe^y} = \frac{\sin(x - y) + e^y}{\sin(x - y) - xe^y}.$$

30. $\sin x + \cos y = \sin x \cos y$, so let $F(x, y) = \sin x + \cos y - \sin x \cos y = 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\cos x - \cos x \cos y}{-\sin y + \sin x \sin y} = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}.$$

31. $x^2 + y^2 + z^2 = 3xyz$, so let $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy} = \frac{3yz - 2x}{2z - 3xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - 3xz}{2z - 3xy} = \frac{3xz - 2y}{2z - 3xy}.$$

32. $xyz = \cos(x + y + z)$. Let $F(x, y, z) = xyz - \cos(x + y + z) = 0$, so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}.$$

33. $x - z = \arctan(yz)$, so let $F(x, y, z) = x - z - \arctan(yz) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-1 - \frac{1}{1+(yz)^2}(y)} = \frac{1+y^2z^2}{1+y+y^2z^2} \text{ and}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-\frac{1}{1+(yz)^2}(z)}{-1 - \frac{1}{1+(yz)^2}(y)} = -\frac{\frac{z}{1+y^2z^2}}{\frac{1+y^2z^2+y}{1+y^2z^2}} = -\frac{z}{1+y+y^2z^2}.$$

34. $yz = \ln(x+z)$, so let $F(x, y, z) = yz - \ln(x+z) = 0$. Then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-\frac{1}{x+z}(1)}{y - \frac{1}{x+z}(1)} = \frac{1}{y(x+z)-1},$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z}{y - \frac{1}{x+z}} = -\frac{z(x+z)}{y(x+z)-1}.$$

35. Since x and y are each functions of t , $T(x, y)$ is a function of t , so by the Chain Rule, $\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt}.$

$$\text{After 3 seconds, } x = \sqrt{1+t} = \sqrt{1+3} = 2, y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3, \frac{dx}{dt} = \frac{1}{2\sqrt{1+t}} = \frac{1}{2\sqrt{1+3}} = \frac{1}{4},$$

and $\frac{dy}{dt} = \frac{1}{3}$. Then $\frac{dT}{dt} = T_x(2, 3) \frac{dx}{dt} + T_y(2, 3) \frac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2°C/s .

36. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of 0.15°C/year , we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year , we know $dR/dt = -0.1$. Then, by the Chain Rule,

$$\frac{dW}{dt} = \frac{\partial W}{\partial T} \frac{dT}{dt} + \frac{\partial W}{\partial R} \frac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1. \text{ Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.}$$

37. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\frac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and

$$\frac{\partial C}{\partial D} = 0.016. \text{ According to the graph, the diver is experiencing a temperature of approximately } 12.5^\circ\text{C at}$$

$t = 20$ minutes, so $\frac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the

graphs given, we estimate $\frac{dD}{dt} \approx \frac{1}{2}$ and $\frac{dT}{dt} \approx -\frac{1}{10}$. Then, by the Chain Rule,

$$\frac{dC}{dt} = \frac{\partial C}{\partial T} \frac{dT}{dt} + \frac{\partial C}{\partial D} \frac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33. \text{ Thus the speed of sound experienced by the diver is decreasing at a rate of approximately } 0.33 \text{ m/s per minute.}$$

38. $V = \pi r^2 h/3$, so $\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = \frac{2\pi r h}{3} 1.8 + \frac{\pi r^2}{3} (-2.5) = 20,160\pi - 12,000\pi = 8160\pi \text{ in}^3/\text{s}.$

39. (a) $V = \ell wh$, so by the Chain Rule,

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial \ell} \frac{d\ell}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = wh \frac{d\ell}{dt} + \ell h \frac{dw}{dt} + \ell w \frac{dh}{dt} \\ &= 2 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot (-3) = 6 \text{ m}^3/\text{s} \end{aligned}$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell} \frac{d\ell}{dt} + \frac{\partial S}{\partial w} \frac{dw}{dt} + \frac{\partial S}{\partial h} \frac{dh}{dt} = 2(w + h) \frac{d\ell}{dt} + 2(\ell + h) \frac{dw}{dt} + 2(\ell + w) \frac{dh}{dt} \\ &= 2(2 + 2)2 + 2(1 + 2)2 + 2(1 + 2)(-3) = 10 \text{ m}^2/\text{s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \Rightarrow 2L \frac{dL}{dt} = 2\ell \frac{d\ell}{dt} + 2w \frac{dw}{dt} + 2h \frac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \Rightarrow$
 $dL/dt = 0 \text{ m/s}.$

40. $I = \frac{V}{R} \Rightarrow \frac{dI}{dt} = \frac{\partial I}{\partial V} \frac{dV}{dt} + \frac{\partial I}{\partial R} \frac{dR}{dt} = \frac{1}{R} \frac{dV}{dt} - \frac{V}{R^2} \frac{dR}{dt} = \frac{1}{R} \frac{dV}{dt} - \frac{I}{R} \frac{dR}{dt}$
 $= \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03) = -0.000031 \text{ A/s}$

41. $\frac{dP}{dt} = 0.05$, $\frac{dT}{dt} = 0.15$, $V = 8.31 \frac{T}{P}$ and $\frac{dV}{dt} = \frac{8.31}{P} \frac{dT}{dt} - 8.31 \frac{T}{P^2} \frac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,
 $\frac{dV}{dt} = 8.31 \left[\frac{0.15}{20} - \frac{(0.05)(320)}{400} \right] \approx -0.27 \text{ L/s}.$

42. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt = -90$, $dy/dt = -80$ and $z^2 = x^2 + y^2$. When $x = 0.3$ and $y = 0.4$, $z = \sqrt{0.25} = 0.5$ and $2z (dz/dt) = 2x (dx/dt) + 2y (dy/dt)$ or $dz/dt = 0.6(-90) + 0.8(-80) = -118 \text{ km/h}.$

43. (a) By the Chain Rule, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} r \cos \theta$.

(b) $\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta$,
 $\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2 \theta - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} r^2 \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2 \theta$. Thus
 $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right] (\cos^2 \theta + \sin^2 \theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$

44. By the Chain Rule, $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t$, $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} (-e^s \sin t) + \frac{\partial u}{\partial y} e^s \cos t$. Then

$$\begin{aligned}\left(\frac{\partial u}{\partial s}\right)^2 &= \left(\frac{\partial u}{\partial x}\right)^2 e^{2s} \cos^2 t + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} e^{2s} \cos t \sin t + \left(\frac{\partial u}{\partial y}\right)^2 e^{2s} \sin^2 t \text{ and} \\ \left(\frac{\partial u}{\partial t}\right)^2 &= \left(\frac{\partial u}{\partial x}\right)^2 e^{2s} \sin^2 t - 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} e^{2s} \cos t \sin t + \left(\frac{\partial u}{\partial y}\right)^2 e^{2s} \cos^2 t. \text{ Thus} \\ \left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 \right] e^{-2s} &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.\end{aligned}$$

45. Let $u = x - y$. Then $\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x} = \frac{dz}{du}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du} (-1)$. Thus $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

46. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$. Thus $\frac{\partial z}{\partial s} \frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$.

47. Let $u = x + at$, $v = x - at$. Then $z = f(u) + g(v)$, so $\partial z / \partial u = f'(u)$ and $\partial z / \partial v = g'(v)$.

Thus $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$$\frac{\partial^2 z}{\partial t^2} = a \frac{\partial}{\partial t} [f'(u) - g'(v)] = a \left(\frac{df'(u)}{du} \frac{\partial u}{\partial t} - \frac{dg'(v)}{dv} \frac{\partial v}{\partial t} \right) = a^2 f''(u) + a^2 g''(v).$$

Similarly $\frac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\frac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$.

48. By the Chain Rule, $\frac{\partial u}{\partial s} = e^s \cos t \frac{\partial u}{\partial x} + e^s \sin t \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t} = -e^s \sin t \frac{\partial u}{\partial x} + e^s \cos t \frac{\partial u}{\partial y}$. Then

$$\frac{\partial^2 u}{\partial s^2} = e^s \cos t \frac{\partial u}{\partial x} + e^s \cos t \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial x} \right) + e^s \sin t \frac{\partial u}{\partial y} + e^s \sin t \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial y} \right).$$

But $\frac{\partial}{\partial s} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y \partial x} \frac{\partial y}{\partial s} = e^s \cos t \frac{\partial^2 u}{\partial x^2} + e^s \sin t \frac{\partial^2 u}{\partial x \partial y}$ and

$$\frac{\partial}{\partial s} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial s} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial s} = e^s \sin t \frac{\partial^2 u}{\partial y^2} + e^s \cos t \frac{\partial^2 u}{\partial x \partial y}.$$
 Also, by continuity of the partials,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}. \text{ Thus}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial s^2} &= e^s \cos t \frac{\partial u}{\partial x} + e^s \cos t \left(e^s \cos t \frac{\partial^2 u}{\partial x^2} + e^s \sin t \frac{\partial^2 u}{\partial x \partial y} \right) + e^s \sin t \frac{\partial u}{\partial y} \\ &\quad + e^s \sin t \left(e^s \sin t \frac{\partial^2 u}{\partial y^2} + e^s \cos t \frac{\partial^2 u}{\partial x \partial y} \right) \\ &= e^s \cos t \frac{\partial u}{\partial x} + e^s \sin t \frac{\partial u}{\partial y} + e^{2s} \cos^2 t \frac{\partial^2 u}{\partial x^2} + 2e^{2s} \cos t \sin t \frac{\partial^2 u}{\partial x \partial y} + e^{2s} \sin^2 t \frac{\partial^2 u}{\partial y^2} \end{aligned}$$

Similarly

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) - e^s \sin t \frac{\partial u}{\partial y} + e^s \cos t \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} \right) \\ &= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \left(-e^s \sin t \frac{\partial^2 u}{\partial x^2} + e^s \cos t \frac{\partial^2 u}{\partial x \partial y} \right) \\ &\quad - e^s \sin t \frac{\partial u}{\partial y} + e^s \cos t \left(e^s \cos t \frac{\partial^2 u}{\partial y^2} - e^s \sin t \frac{\partial^2 u}{\partial x \partial y} \right) \\ &= -e^s \cos t \frac{\partial u}{\partial x} - e^s \sin t \frac{\partial u}{\partial y} + e^{2s} \sin^2 t \frac{\partial^2 u}{\partial x^2} - 2e^{2s} \cos t \sin t \frac{\partial^2 u}{\partial x \partial y} + e^{2s} \cos^2 t \frac{\partial^2 u}{\partial y^2} \end{aligned}$$

Thus $e^{-2s} \left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right) = (\cos^2 t + \sin^2 t) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, as desired.

49. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} 2s + \frac{\partial z}{\partial y} 2r$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial s} &= \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x} 2s \right) + \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial y} 2r \right) \\ &= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} 2s + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) \frac{\partial y}{\partial r} 2s + \frac{\partial z}{\partial x} \frac{\partial}{\partial r} 2s + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} 2r + \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \frac{\partial x}{\partial r} 2r + \frac{\partial z}{\partial y} 2 \\ &= 4rs \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y \partial x} 4s^2 + 0 + 4rs \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} 4r^2 + 2 \frac{\partial z}{\partial y}\end{aligned}$$

By the continuity of the partials, $\frac{\partial^2 z}{\partial r \partial s} = 4rs \frac{\partial^2 z}{\partial x^2} + 4rs \frac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2) \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial z}{\partial y}$.

50. By the Chain Rule,

(a) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$

(b) $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$

(c)
$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial \theta} &= \frac{\partial^2 z}{\partial \theta \partial r} = \frac{\partial}{\partial \theta} \left(\frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \frac{\partial}{\partial \theta} \left(\frac{\partial z}{\partial x} \right) + \cos \theta \frac{\partial z}{\partial y} + \sin \theta \frac{\partial}{\partial \theta} \left(\frac{\partial z}{\partial y} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \left(\frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) + \cos \theta \frac{\partial z}{\partial y} + \sin \theta \left(\frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial \theta} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \left(-r \sin \theta \frac{\partial^2 z}{\partial x^2} + r \cos \theta \frac{\partial^2 z}{\partial y \partial x} \right) + \cos \theta \frac{\partial z}{\partial y} \\ &\quad + \sin \theta \left(r \cos \theta \frac{\partial^2 z}{\partial y^2} - r \sin \theta \frac{\partial^2 z}{\partial x \partial y} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} - r \cos \theta \sin \theta \frac{\partial^2 z}{\partial x^2} + r \cos^2 \theta \frac{\partial^2 z}{\partial y \partial x} + \cos \theta \frac{\partial z}{\partial y} \\ &\quad + r \cos \theta \sin \theta \frac{\partial^2 z}{\partial y^2} - r \sin^2 \theta \frac{\partial^2 z}{\partial y \partial x} \\ &= \cos \theta \frac{\partial z}{\partial y} - \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \sin \theta \left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} \right) + r(\cos^2 \theta - \sin^2 \theta) \frac{\partial^2 z}{\partial y \partial x}\end{aligned}$$

51. $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$ and $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos \theta \left(\frac{\partial^2 z}{\partial x^2} \cos \theta + \frac{\partial^2 z}{\partial y \partial x} \sin \theta \right) + \sin \theta \left(\frac{\partial^2 z}{\partial y^2} \sin \theta + \frac{\partial^2 z}{\partial x \partial y} \cos \theta \right) \\ &= \cos^2 \theta \frac{\partial^2 z}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= -r \cos \theta \frac{\partial z}{\partial x} + (-r \sin \theta) \left(\frac{\partial^2 z}{\partial x^2} (-r \sin \theta) + \frac{\partial^2 z}{\partial y \partial x} r \cos \theta \right) \\ &\quad - r \sin \theta \frac{\partial z}{\partial y} + r \cos \theta \left(\frac{\partial^2 z}{\partial y^2} r \cos \theta + \frac{\partial^2 z}{\partial x \partial y} (-r \sin \theta) \right) \\ &= -r \cos \theta \frac{\partial z}{\partial x} - r \sin \theta \frac{\partial z}{\partial y} + r^2 \sin^2 \theta \frac{\partial^2 z}{\partial x^2} - 2r^2 \cos \theta \sin \theta \frac{\partial^2 z}{\partial x \partial y} + r^2 \cos^2 \theta \frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 z}{\partial x^2} + (\sin^2 \theta + \cos^2 \theta) \frac{\partial^2 z}{\partial y^2} - \frac{1}{r} \cos \theta \frac{\partial z}{\partial x} \\ &\quad - \frac{1}{r} \sin \theta \frac{\partial z}{\partial y} + \frac{1}{r} \left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right) \\ &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}\end{aligned}$$

52. (a) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2} \frac{\partial z}{\partial x} + \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y} \\ &= \frac{\partial^2 z}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial t^2} \frac{\partial x}{\partial x} + \frac{\partial^2 z}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial y}{\partial t} \frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y} \\ &= \frac{\partial^2 z}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 x}{\partial t^2} \frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \frac{\partial^2 z}{\partial s \partial t} &= \frac{\partial}{\partial s} \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \left(\frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial s} \right) \frac{\partial x}{\partial t} + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial s \partial t} + \left(\frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial s} \right) \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial s \partial t} \\ &= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial s} \frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y} \left(\frac{\partial y}{\partial s} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} \frac{\partial x}{\partial s} \right) + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial s \partial t} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial s} \frac{\partial y}{\partial t}\end{aligned}$$

53. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$$\begin{aligned}f(tx, ty) &= (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3 x^2 y + 2t^3 x y^2 + 5t^3 y^3 \\ &= t^3(x^2 y + 2x y^2 + 5y^3) = t^3 f(x, y)\end{aligned}$$

Thus, f is homogeneous of degree 3.

- (b) Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(tx, ty) = \frac{\partial}{\partial t} [t^n f(x, y)] \Leftrightarrow$$

$$\frac{\partial}{\partial(tx)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} = x \frac{\partial}{\partial(tx)} f(tx, ty) + y \frac{\partial}{\partial(ty)} f(tx, ty) = nt^{n-1} f(x, y).$$

$$\text{Setting } t = 1: x \frac{\partial}{\partial x} f(x, y) + y \frac{\partial}{\partial y} f(x, y) = nf(x, y).$$

54. Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial(tx)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} = x \frac{\partial}{\partial(tx)} f(tx, ty) + y \frac{\partial}{\partial(ty)} f(tx, ty) = nt^{n-1} f(x, y) \text{ and}$$

differentiating again with respect to t gives

$$x \left[\frac{\partial^2}{\partial(tx)^2} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial^2}{\partial(ty) \partial(tx)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} \right] \\ + y \left[\frac{\partial^2}{\partial(tx) \partial(ty)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial^2}{\partial(ty)^2} f(tx, ty) \cdot \frac{\partial(ty)}{\partial t} \right] = n(n-1)t^{n-1} f(x, y).$$

Setting $t = 1$ and using the fact that $f_{yx} = f_{xy}$, we have $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f(x, y)$.

55. Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x} f(tx, ty) = \frac{\partial}{\partial x} [t^n f(x, y)] \Leftrightarrow \\ \frac{\partial}{\partial(tx)} f(tx, ty) \cdot \frac{\partial(tx)}{\partial x} + \frac{\partial}{\partial(ty)} f(tx, ty) \cdot \frac{\partial(ty)}{\partial x} = t^n \frac{\partial}{\partial x} f(x, y) \Leftrightarrow t f_x(tx, ty) = t^n f_x(x, y).$$

Thus $f_x(tx, ty) = t^{n-1} f_x(x, y)$.

56. $F(x, y, z) = 0$ is assumed to define z as a function of x and y , that is, $z = f(x, y)$. So by (7), $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ since

$F_z \neq 0$. Similarly, it is assumed that $F(x, y, z) = 0$ defines x as a function of y and z , that is $x = h(y, z)$. Then

$F(h(y, z), y, z) = 0$ and by the Chain Rule, $F_x \frac{\partial x}{\partial y} + F_y \frac{\partial y}{\partial y} + F_z \frac{\partial z}{\partial y} = 0$. But $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial y}{\partial y} = 1$, so

$$F_x \frac{\partial x}{\partial y} + F_y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{F_y}{F_x}. \text{ A similar calculation shows that } \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}. \text{ Thus}$$

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = \left(-\frac{F_x}{F_z} \right) \left(-\frac{F_y}{F_x} \right) \left(-\frac{F_z}{F_y} \right) = -1.$$

15.6 Directional Derivatives and the Gradient Vector

ET 14.6

1. First we draw a line passing through Raleigh and the eye of the hurricane. We can approximate the directional derivative at Raleigh in the direction of the eye of the hurricane by the average rate of change of pressure between the points where this line intersects the contour lines closest to Raleigh. In the direction of the eye of the hurricane, the pressure changes from 996 millibars to 992 millibars. We estimate the distance between these two points to be approximately 40 miles, so the rate of change of pressure in the direction given is approximately

$$\frac{992 - 996}{40} = -0.1 \text{ millibar/mi.}$$

2. First we draw a line passing through Muskegon and Ludington. We approximate the directional derivative at Muskegon in the direction of Ludington by the average rate of change of snowfall between the points where the line

intersects the contour lines closest to Muskegon. In the direction of Ludington, the snowfall changes from 60 to 70 inches. We estimate the distance between these two points to be approximately 28 miles, so the rate of change of annual snowfall in the direction given is approximately $\frac{70-60}{28} \approx 0.36$ in./mi. [If we talk of snowfall (rather than annual snowfall), the units are (in/year)/mi.]

$$3. D_{\mathbf{u}} f(-20, 30) = \nabla f(-20, 30) \cdot \mathbf{u} = f_T(-20, 30) \left(\frac{1}{\sqrt{2}} \right) + f_v(-20, 30) \left(\frac{1}{\sqrt{2}} \right).$$

$f_T(-20, 30) = \lim_{h \rightarrow 0} \frac{f(-20+h, 30) - f(-20, 30)}{h}$, so we can approximate $f_T(-20, 30)$ by considering $h = \pm 5$

and using the values given in the table: $f_T(-20, 30) \approx \frac{f(-15, 30) - f(-20, 30)}{5} = \frac{-26 - (-33)}{5} = 1.4$,

$f_T(-20, 30) \approx \frac{f(-25, 30) - f(-20, 30)}{-5} = \frac{-39 - (-33)}{-5} = 1.2$. Averaging these values gives

$f_T(-20, 30) \approx 1.3$. Similarly, $f_v(-20, 30) = \lim_{h \rightarrow 0} \frac{f(-20, 30+h) - f(-20, 30)}{h}$, so we can approximate $f_v(-20, 30)$ with $h = \pm 10$:

$f_v(-20, 30) \approx \frac{f(-20, 40) - f(-20, 30)}{10} = \frac{-34 - (-33)}{10} = -0.1$,

$f_v(-20, 30) \approx \frac{f(-20, 20) - f(-20, 30)}{-10} = \frac{-30 - (-33)}{-10} = -0.3$. Averaging these values gives

$f_v(-20, 30) \approx -0.2$. Then $D_{\mathbf{u}} f(-20, 30) \approx 1.3 \left(\frac{1}{\sqrt{2}} \right) + (-0.2) \left(\frac{1}{\sqrt{2}} \right) \approx 0.778$.

$$4. f(x, y) = x^2 y^3 - y^4 \Rightarrow f_x(x, y) = 2xy^3 \text{ and } f_y(x, y) = 3x^2 y^2 - 4y^3. \text{ If } \mathbf{u} \text{ is a unit vector in the direction of } \theta = \frac{\pi}{4}, \text{ then from Equation 6, } D_{\mathbf{u}} f(2, 1) = f_x(2, 1) \cos\left(\frac{\pi}{4}\right) + f_y(2, 1) \sin\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} + 8 \cdot \frac{\sqrt{2}}{2} = 6\sqrt{2}.$$

$$5. f(x, y) = \sqrt{5x-4y} \Rightarrow f_x(x, y) = \frac{1}{2}(5x-4y)^{-1/2}(5) = \frac{5}{2\sqrt{5x-4y}} \text{ and}$$

$f_y(x, y) = \frac{1}{2}(5x-4y)^{-1/2}(-4) = -\frac{2}{\sqrt{5x-4y}}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\frac{\pi}{6}$, then from

Equation 6, $D_{\mathbf{u}} f(4, 1) = f_x(4, 1) \cos\left(-\frac{\pi}{6}\right) + f_y(4, 1) \sin\left(-\frac{\pi}{6}\right) = \frac{5}{8} \cdot \frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right) = \frac{5\sqrt{3}}{16} + \frac{1}{4}$.

$$6. f(x, y) = x \sin(xy) \Rightarrow f_x(x, y) = x \cos(xy) \cdot y + \sin(xy) = xy \cos(xy) + \sin(xy) \text{ and } f_y(x, y) = x \cos(xy) \cdot x = x^2 \cos(xy). \text{ If } \mathbf{u} \text{ is a unit vector in the direction of } \theta = \frac{\pi}{3}, \text{ then from Equation 6,}$$

$D_{\mathbf{u}} f(2, 0) = f_x(2, 0) \cos \frac{\pi}{3} + f_y(2, 0) \sin \frac{\pi}{3} = 0 + 4 \left(\frac{\sqrt{3}}{2} \right) = 2\sqrt{3}$.

$$7. f(x, y) = 5xy^2 - 4x^3y$$

$$(a) \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \langle 5y^2 - 12x^2y, 10xy - 4x^3 \rangle$$

$$(b) \nabla f(1, 2) = \langle 5(2)^2 - 12(1)^2(2), 10(1)(2) - 4(1)^3 \rangle = \langle -4, 16 \rangle$$

$$(c) \text{ By Equation 9, } D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \langle -4, 16 \rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle = (-4) \left(\frac{5}{13} \right) + (16) \left(\frac{12}{13} \right) = \frac{172}{13}.$$

$$8. f(x, y) = y \ln x$$

$$(a) \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \langle y/x, \ln x \rangle \quad (b) \nabla f(1, -3) = \left\langle \frac{-3}{1}, \ln 1 \right\rangle = \langle -3, 0 \rangle$$

$$(c) \text{ By Equation 9, } D_{\mathbf{u}} f(1, -3) = \nabla f(1, -3) \cdot \mathbf{u} = \langle -3, 0 \rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = \frac{12}{5}.$$

9. $f(x, y, z) = xe^{2yz}$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \langle e^{2yz}, 2xz e^{2yz}, 2xy e^{2yz} \rangle$

(b) $\nabla f(3, 0, 2) = \langle 1, 12, 0 \rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(3, 0, 2) = \nabla f(3, 0, 2) \cdot \mathbf{u} = \langle 1, 12, 0 \rangle \cdot \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}$.

10. $f(x, y, z) = \sqrt{x + yz} = (x + yz)^{1/2}$

(a) $\nabla f(x, y, z) = \left\langle \frac{1}{2}(x + yz)^{-1/2}(1), \frac{1}{2}(x + yz)^{-1/2}(z), \frac{1}{2}(x + yz)^{-1/2}(y) \right\rangle$
 $= \langle 1/(2\sqrt{x + yz}), z/(2\sqrt{x + yz}), y/(2\sqrt{x + yz}) \rangle$

(b) $\nabla f(1, 3, 1) = \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \rangle$

(c) $D_{\mathbf{u}} f(1, 3, 1) = \nabla f(1, 3, 1) \cdot \mathbf{u} = \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \rangle \cdot \langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \rangle = \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}$.

11. $f(x, y) = 1 + 2x\sqrt{y} \Rightarrow \nabla f(x, y) = \langle 2\sqrt{y}, 2x \cdot \frac{1}{2}y^{-1/2} \rangle = \langle 2\sqrt{y}, x/\sqrt{y} \rangle$, $\nabla f(3, 4) = \langle 4, \frac{3}{2} \rangle$,

and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2 + (-3)^2}} \langle 4, -3 \rangle = \langle \frac{4}{5}, -\frac{3}{5} \rangle$, so

$D_{\mathbf{u}} f(3, 4) = \nabla f(3, 4) \cdot \mathbf{u} = \langle 4, \frac{3}{2} \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle = \frac{23}{10}$.

12. $f(x, y) = \ln(x^2 + y^2) \Rightarrow \nabla f(x, y) = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle$, $\nabla f(2, 1) = \langle \frac{4}{5}, \frac{2}{5} \rangle$, and a unit

vector in the direction of $\mathbf{v} = \langle -1, 2 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{1 + 4}} \langle -1, 2 \rangle = \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$, so

$D_{\mathbf{u}} f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = \langle \frac{4}{5}, \frac{2}{5} \rangle \cdot \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = -\frac{4}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} = 0$.

13. $g(s, t) = s^2 e^t \Rightarrow \nabla g(s, t) = 2se^t \mathbf{i} + s^2 e^t \mathbf{j}$, $\nabla g(2, 0) = 4\mathbf{i} + 4\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is

$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$, so $D_{\mathbf{u}} g(2, 0) = \nabla g(2, 0) \cdot \mathbf{u} = (4\mathbf{i} + 4\mathbf{j}) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{8}{\sqrt{2}} = 4\sqrt{2}$.

14. $g(r, \theta) = e^{-r} \sin \theta \Rightarrow \nabla g(r, \theta) = (-e^{-r} \sin \theta) \mathbf{i} + (e^{-r} \cos \theta) \mathbf{j}$, $\nabla g(0, \frac{\pi}{3}) = -\frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{j}$, and

a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$, so

$D_{\mathbf{u}} g(0, \frac{\pi}{3}) = \nabla g(0, \frac{\pi}{3}) \cdot \mathbf{u} = \left(-\frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{j}\right) \cdot \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j}) = -\frac{3\sqrt{3}}{2\sqrt{13}} - \frac{1}{\sqrt{13}} = -\frac{3\sqrt{3} + 2}{2\sqrt{13}}$.

15. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \Rightarrow \nabla f(x, y, z) = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$,

$\nabla f(1, 2, -2) = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{9} \langle -6, 6, -3 \rangle = \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle$, so

$D_{\mathbf{u}} f(1, 2, -2) = \nabla f(1, 2, -2) \cdot \mathbf{u} = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle \cdot \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle = \frac{4}{9}$.

16. $f(x, y, z) = \frac{x}{y + z} \Rightarrow \nabla f(x, y, z) = \left\langle \frac{1}{y + z}, -\frac{x}{(y + z)^2}, -\frac{x}{(y + z)^2} \right\rangle$,

$\nabla f(4, 1, 1) = \langle \frac{1}{2}, -1, -1 \rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle$, so

$D_{\mathbf{u}} f(4, 1, 1) = \nabla f(4, 1, 1) \cdot \mathbf{u} = \langle \frac{1}{2}, -1, -1 \rangle \cdot \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle = -\frac{9}{2\sqrt{14}}$.

$$17. g(x, y, z) = (x + 2y + 3z)^{3/2} \Rightarrow$$

$$\begin{aligned} \nabla g(x, y, z) &= \left\langle \frac{3}{2}(x + 2y + 3z)^{1/2}(1), \frac{3}{2}(x + 2y + 3z)^{1/2}(2), \frac{3}{2}(x + 2y + 3z)^{1/2}(3) \right\rangle \\ &= \left\langle \frac{3}{2}\sqrt{x + 2y + 3z}, 3\sqrt{x + 2y + 3z}, \frac{9}{2}\sqrt{x + 2y + 3z} \right\rangle, \nabla g(1, 1, 2) = \left\langle \frac{9}{2}, 9, \frac{27}{2} \right\rangle, \end{aligned}$$

and a unit vector in the direction of $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$ is $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$, so

$$D_{\mathbf{u}}g(1, 1, 2) = \left\langle \frac{9}{2}, 9, \frac{27}{2} \right\rangle \cdot \left\langle 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle = \frac{18}{\sqrt{5}} - \frac{27}{2\sqrt{5}} = \frac{9}{2\sqrt{5}}.$$

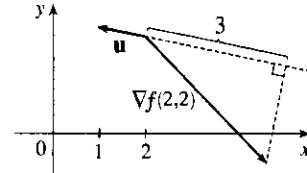
$$18. D_{\mathbf{u}}f(2, 2) = \nabla f(2, 2) \cdot \mathbf{u}, \text{ the scalar projection of } \nabla f(2, 2)$$

onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2, 2)$

to the line containing $\mathbf{u}$. We can use the point $(2, 2)$ to

determine the scale of the axes, and we estimate the length of

the projection to be approximately 3.0 units. Since the angle between $\nabla f(2, 2)$ and $\mathbf{u}$ is greater than 90° , the scalar projection is negative. Thus $D_{\mathbf{u}}f(2, 2) \approx -3$.



$$19. f(x, y) = \sqrt{xy} \Rightarrow \nabla f(x, y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle, \text{ so}$$

$\nabla f(2, 8) = \left\langle 1, \frac{1}{4} \right\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 5 - 2, 4 - 8 \rangle = \langle 3, -4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$, so

$$D_{\mathbf{u}}f(2, 8) = \nabla f(2, 8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}.$$

$$20. f(x, y, z) = x^2 + y^2 + z^2 \Rightarrow \nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle, \text{ so } \nabla f(2, 1, 3) = \langle 4, 2, 6 \rangle. \text{ The}$$

unit vector in the direction of $\overrightarrow{PO} = \langle -2, -1, -3 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{14}}\langle -2, -1, -3 \rangle$, so

$$D_{\mathbf{u}}f(2, 1, 3) = \nabla f(2, 1, 3) \cdot \mathbf{u} = \langle 4, 2, 6 \rangle \cdot \frac{1}{\sqrt{14}}\langle -2, -1, -3 \rangle = -\frac{28}{\sqrt{14}} = -2\sqrt{14}.$$

$$21. f(x, y) = y^2/x = y^2x^{-1} \Rightarrow \nabla f(x, y) = \langle -y^2x^{-2}, 2yx^{-1} \rangle = \langle -y^2/x^2, 2y/x \rangle.$$

$\nabla f(2, 4) = \langle -4, 4 \rangle$, or equivalently $\langle -1, 1 \rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2, 4)| = \sqrt{16 + 16} = 4\sqrt{2}$.

$$22. f(p, q) = qe^{-p} + pe^{-q} \Rightarrow \nabla f(p, q) = \langle -qe^{-p} + e^{-q}, e^{-p} - pe^{-q} \rangle.$$

$\nabla f(0, 0) = \langle 1, 1 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(0, 0)| = \sqrt{2}$.

$$23. f(x, y) = \sin(xy) \Rightarrow \nabla f(x, y) = \langle y \cos(xy), x \cos(xy) \rangle, \nabla f(1, 0) = \langle 0, 1 \rangle. \text{ Thus the maximum rate of change is } |\nabla f(1, 0)| = 1 \text{ in the direction } \langle 0, 1 \rangle.$$

$$24. f(x, y, z) = x^2y^3z^4 \Rightarrow \nabla f(x, y, z) = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle, \nabla f(1, 1, 1) = \langle 2, 3, 4 \rangle. \text{ Thus the maximum rate of change is } |\nabla f(1, 1, 1)| = \sqrt{29} \text{ in the direction } \langle 2, 3, 4 \rangle.$$

$$25. f(x, y, z) = \ln(xy^2z^3) \Rightarrow \nabla f(x, y, z) = \left\langle \frac{y^2z^3}{xy^2z^3}, \frac{2xyz^3}{xy^2z^3}, \frac{3xy^2z^2}{xy^2z^3} \right\rangle = \left\langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right\rangle.$$

$\nabla f(1, -2, -3) = \langle 1, -1, -1 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(1, -2, -3)| = \sqrt{3}$.

26. $f(x, y, z) = \tan(x + 2y + 3z) \Rightarrow$

$$\nabla f(x, y, z) = \langle \sec^2(x + 2y + 3z)(1), \sec^2(x + 2y + 3z)(2), \sec^2(x + 2y + 3z)(3) \rangle.$$

$\nabla f(-5, 1, 1) = \langle \sec^2(0), 2\sec^2(0), 3\sec^2(0) \rangle = \langle 1, 2, 3 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(-5, 1, 1)| = \sqrt{14}$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}} f = |\nabla f| \cos \theta$. Since the minimum value of $\cos \theta$ is -1 occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}} f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq 0$).

(b) $f(x, y) = x^4 y - x^2 y^3 \Rightarrow \nabla f(x, y) = \langle 4x^3 y - 2xy^3, x^4 - 3x^2 y^2 \rangle$, so f decreases fastest at the point $(2, -3)$ in the direction $-\nabla f(2, -3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$.

28. $f(x, y) = x^2 + \sin xy \Rightarrow f_x(x, y) = 2x + y \cos xy, f_y(x, y) = x \cos xy$ and

$f_x(1, 0) = 2(1) + (0) \cos 0 = 2, f_y(1, 0) = (1) \cos 0 = 1$. If $\mathbf{u}$ is a unit vector which makes an angle θ with the positive x -axis, then $D_{\mathbf{u}} f(1, 0) = f_x(1, 0) \cos \theta + f_y(1, 0) \sin \theta = 2 \cos \theta + \sin \theta$. We want $D_{\mathbf{u}} f(1, 0) = 1$, so

$$2 \cos \theta + \sin \theta = 1 \Rightarrow \sin \theta = 1 - 2 \cos \theta \Rightarrow \sin^2 \theta = (1 - 2 \cos \theta)^2 \Rightarrow$$

$$1 - \cos^2 \theta = 1 - 4 \cos \theta + 4 \cos^2 \theta \Rightarrow 5 \cos^2 \theta - 4 \cos \theta = 0 \Rightarrow \cos \theta(5 \cos \theta - 4) = 0 \Rightarrow$$

$$\cos \theta = 0 \text{ or } \cos \theta = \frac{4}{5} \Rightarrow \theta = \frac{\pi}{2} \text{ or } \theta = 2\pi - \cos^{-1}\left(\frac{4}{5}\right) \approx 5.64.$$

29. The direction of fastest change is $\nabla f(x, y) = (2x - 2)\mathbf{i} + (2y - 4)\mathbf{j}$, so we need to find all points (x, y) where $\nabla f(x, y)$ is parallel to $\mathbf{i} + \mathbf{j} \Leftrightarrow (2x - 2)\mathbf{i} + (2y - 4)\mathbf{j} = k(\mathbf{i} + \mathbf{j}) \Leftrightarrow k = 2x - 2$ and $k = 2y - 4$. Then $2x - 2 = 2y - 4 \Rightarrow y = x + 1$, so the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x + 1$.

30. The fisherman is traveling in the direction $\langle -80, -60 \rangle$. A unit vector in this direction is

$\mathbf{u} = \frac{1}{100} \langle -80, -60 \rangle = \langle -\frac{4}{5}, -\frac{3}{5} \rangle$, and if the depth of the lake is given by $f(x, y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x, y) = \langle 0.04x, -0.003y^2 \rangle$. $D_{\mathbf{u}} f(80, 60) = \nabla f(80, 60) \cdot \mathbf{u} = \langle 3.2, -10.8 \rangle \cdot \langle -\frac{4}{5}, -\frac{3}{5} \rangle = 3.92$. Since $D_{\mathbf{u}} f(80, 60)$ is positive, the depth of the lake is increasing near $(80, 60)$ in the direction toward the buoy.

31. $T = \frac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $120 = T(1, 2, 2) = \frac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \frac{\langle 1, -1, 1 \rangle}{\sqrt{3}},$

$$\begin{aligned} D_{\mathbf{u}} T(1, 2, 2) &= \nabla T(1, 2, 2) \cdot \mathbf{u} = \left[-360(x^2 + y^2 + z^2)^{-3/2} \langle x, y, z \rangle \right]_{(1, 2, 2)} \cdot \mathbf{u} \\ &= -\frac{40}{3} \langle 1, 2, 2 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, -1, 1 \rangle = -\frac{40}{3\sqrt{3}} \end{aligned}$$

- (b) From (a), $\nabla T = -360(x^2 + y^2 + z^2)^{-3/2} \langle x, y, z \rangle$, and since $\langle x, y, z \rangle$ is the position vector of the point (x, y, z) , the vector $-\langle x, y, z \rangle$, and thus ∇T , always points toward the origin.

32. $\nabla T = -400e^{-x^2 - 3y^2 - 9z^2} \langle x, 3y, 9z \rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle, \nabla T(2, -1, 2) = -400e^{-43} \langle 2, -3, 18 \rangle$ and

$$D_{\mathbf{u}} T(2, -1, 2) = \left(-\frac{400e^{-43}}{\sqrt{6}} \right) (26) = -\frac{5200\sqrt{6}}{3e^{43}} \text{ } ^\circ\text{C/m}.$$

(b) $\nabla T(2, -1, 2) = 400e^{-43} \langle -2, 3, -18 \rangle$ or equivalently $\langle -2, 3, -18 \rangle$.

(c) $|\nabla T| = 400e^{-x^2 - 3y^2 - 9z^2} \sqrt{x^2 + 9y^2 + 8z^2} \text{ } ^\circ\text{C/m}$ is the maximum rate of increase. At $(2, -1, 2)$ the maximum rate of increase is $400e^{-43} \sqrt{337} \text{ } ^\circ\text{C/m}$.

33. $\nabla V(x, y, z) = \langle 10x - 3y + yz, xz - 3x, xy \rangle$, $\nabla V(3, 4, 5) = \langle 38, 6, 12 \rangle$

(a) $D_{\mathbf{u}} V(3, 4, 5) = \langle 38, 6, 12 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3, 4, 5) = \langle 38, 6, 12 \rangle$ or equivalently $\langle 19, 3, 6 \rangle$.

(c) $|\nabla V(3, 4, 5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

34. $z = f(x, y) = 1000 - 0.01x^2 - 0.02y^2 \Rightarrow \nabla f(x, y) = \langle -0.02x, -0.04y \rangle$ and $\nabla f(50, 80) = \langle -1, -3.2 \rangle$

(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and

$D_{\mathbf{u}} f(50, 80) = \nabla f(50, 80) \cdot \langle 0, -1 \rangle = \langle -1, -3.2 \rangle \cdot \langle 0, -1 \rangle = 0 + 3.2 = 3.2$. Thus, if you walk due south from $(50, 80, 847)$ you will ascend at a rate of 3.2 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$ and

$D_{\mathbf{u}} f(50, 80) = \nabla f(50, 80) \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = \langle -1, -3.2 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = -\frac{2.2}{\sqrt{2}} \approx -1.56$. Thus, if you walk northwest from $(50, 80, 847)$ you will descend at a rate of approximately 1.56 vertical meters per horizontal meter.

(c) $\nabla f(50, 80) = \langle -1, -3.2 \rangle$ is the direction of largest slope with a rate of ascent

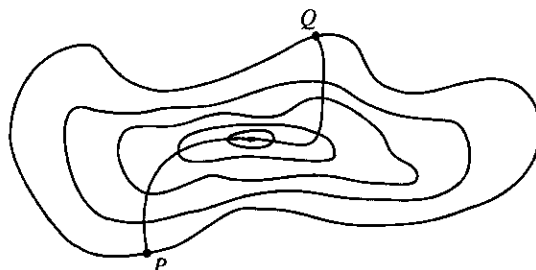
$|\nabla f(50, 80)| = \sqrt{11.24} \approx 3.35$. The angle above the horizontal in which the path begins is given by $\tan \theta \approx 3.35 \Rightarrow \theta \approx \tan^{-1}(3.35) \approx 73.4^\circ$.

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$.

Thus $D_{\overrightarrow{AB}} f(1, 3) = f_x(1, 3) = 3$ and $D_{\overrightarrow{AC}} f(1, 3) = f_y(1, 3) = 26$. Therefore

$\nabla f(1, 3) = \langle f_x(1, 3), f_y(1, 3) \rangle = \langle 3, 26 \rangle$, and by definition, $D_{\overrightarrow{AD}} f(1, 3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\langle \frac{5}{13}, \frac{12}{13} \rangle$. Therefore, $D_{\overrightarrow{AD}} f(1, 3) = \langle 3, 26 \rangle \cdot \langle \frac{5}{13}, \frac{12}{13} \rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

36. The curve of steepest ascent is perpendicular to all of the contour lines.



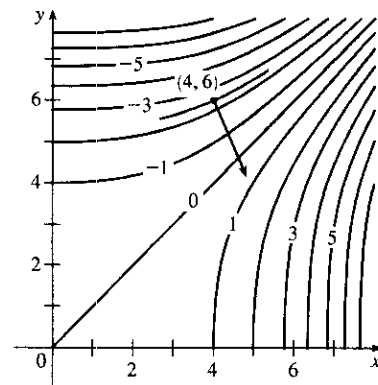
37. (a) $\nabla(au + bv) = \left\langle \frac{\partial(au + bv)}{\partial x}, \frac{\partial(au + bv)}{\partial y} \right\rangle = \left\langle a \frac{\partial u}{\partial x} + b \frac{\partial v}{\partial x}, a \frac{\partial u}{\partial y} + b \frac{\partial v}{\partial y} \right\rangle$
 $= a \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle + b \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = a \nabla u + b \nabla v$

(b) $\nabla(uv) = \left\langle v \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x}, v \frac{\partial u}{\partial y} + u \frac{\partial v}{\partial y} \right\rangle = v \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle + u \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = v \nabla u + u \nabla v$

$$(c) \nabla \left(\frac{u}{v} \right) = \left\langle \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2}, \frac{v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}}{v^2} \right\rangle = \frac{v \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle - u \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle}{v^2} = \frac{v \nabla u - u \nabla v}{v^2}$$

$$(d) \nabla u^n = \left\langle \frac{\partial(u^n)}{\partial x}, \frac{\partial(u^n)}{\partial y} \right\rangle = \left\langle nu^{n-1} \frac{\partial u}{\partial x}, nu^{n-1} \frac{\partial u}{\partial y} \right\rangle = nu^{n-1} \nabla u.$$

38. If we place the initial point of the gradient vector $\nabla f(4, 6)$ at $(4, 6)$, the vector is perpendicular to the level curve of f that includes $(4, 6)$, so we sketch a portion of the level curve through $(4, 6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4, 6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4, 6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\frac{-2 - (-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



39. Let $F(x, y, z) = x^2 + 2y^2 + 3z^2$. Then $x^2 + 2y^2 + 3z^2 = 21$ is a level surface of F . $F_x(x, y, z) = 2x \Rightarrow F_x(4, -1, 1) = 8$, $F_y(x, y, z) = 4y \Rightarrow F_y(4, -1, 1) = -4$, and $F_z(x, y, z) = 6z \Rightarrow F_z(4, -1, 1) = 6$.

(a) Equation 19 gives an equation of the tangent plane at $(4, -1, 1)$ as $8(x - 4) - 4[y - (-1)] + 6(z - 1) = 0$ or $4x - 2y + 3z = 21$.

(b) By Equation 20, the normal line has symmetric equations

$$\frac{x - 4}{8} = \frac{y + 1}{-4} = \frac{z - 1}{6} \text{ or } \frac{x - 4}{4} = \frac{y + 1}{-2} = \frac{z - 1}{3}.$$

40. Let $F(x, y, z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 - 2$ is the level surface $F(x, y, z) = 2$.

$$F_x(x, y, z) = -1 \Rightarrow F_x(-1, 1, 0) = -1, \quad F_y(x, y, z) = 2y \Rightarrow F_y(-1, 1, 0) = 2,$$

$$\text{and } F_z(x, y, z) = 2z \Rightarrow F_z(-1, 1, 0) = 0.$$

(a) An equation of the tangent plane is $-1(x + 1) + 2(y - 1) + 0(z - 0) = 0$ or $-x + 2y = 3$.

(b) The normal line has symmetric equations $\frac{x + 1}{-1} = \frac{y - 1}{2}, z = 0$.

41. Let $F(x, y, z) = x^2 - 2y^2 + z^2 + yz$. Then $x^2 - 2y^2 + z^2 + yz = 2$ is a level surface of F

$$\text{and } \nabla F(x, y, z) = \langle 2x, -4y + z, 2z + y \rangle.$$

(a) $\nabla F(2, 1, -1) = \langle 4, -5, -1 \rangle$ is a normal vector for the tangent plane at $(2, 1, -1)$, so an equation of the tangent plane is $4(x - 2) - 5(y - 1) - 1(z + 1) = 0$ or $4x - 5y - z = 4$.

(b) The normal line has direction $\langle 4, -5, -1 \rangle$, so parametric equations are $x = 2 + 4t, y = 1 - 5t, z = -1 - t$, and symmetric equations are $\frac{x - 2}{4} = \frac{y - 1}{-5} = \frac{z + 1}{-1}$.

42. Let $F(x, y, z) = x - z - 4 \arctan(yz)$. Then $x - z = 4 \arctan(yz)$ is the level surface $F(x, y, z) = 0$,

$$\text{and } \nabla F(x, y, z) = \left\langle 1, -\frac{4z}{1+y^2z^2}, -1 - \frac{4y}{1+y^2z^2} \right\rangle.$$

- (a) $\nabla F(1 + \pi, 1, 1) = \langle 1, -2, -3 \rangle$ and an equation of the tangent plane is

$$1(x - (1 + \pi)) - 2(y - 1) - 3(z - 1) = 0 \text{ or } x - 2y - 3z = -4 + \pi.$$

- (b) The normal line has direction $\langle 1, -2, -3 \rangle$, so parametric equations are $x = 1 + \pi + t, y = 1 - 2t, z = 1 - 3t$,

$$\text{and symmetric equations are } x - 1 - \pi = \frac{y - 1}{-2} = \frac{z - 1}{-3}.$$

43. $F(x, y, z) = -z + xe^y \cos z \Rightarrow \nabla F(x, y, z) = \langle e^y \cos z, xe^y \cos z, -1 - xe^y \sin z \rangle$,

$$\nabla F(1, 0, 0) = \langle 1, 1, -1 \rangle$$

- (a) $1(x - 1) + 1(y - 0) - 1(z - 0) = 0$ or $x + y - z = 1$

$$(b) \ x - 1 = y = -z$$

44. $F(x, y, z) = yz - \ln(x + z) \Rightarrow \nabla F(x, y, z) = \left\langle -\frac{1}{x + z}, z, y - \frac{1}{x + z} \right\rangle, \nabla F(0, 0, 1) = \langle -1, 1, -1 \rangle$.

- (a) $(-1)(x - 0) + (1)(y - 0) - 1(z - 1) = 0$ or $x - y + z = 1$.

- (b) Parametric equations are $x = -t, y = t, z = 1 - t$ and symmetric equations are $\frac{x}{-1} = \frac{y}{1} = \frac{z - 1}{-1}$
or $-x = y = 1 - z$.

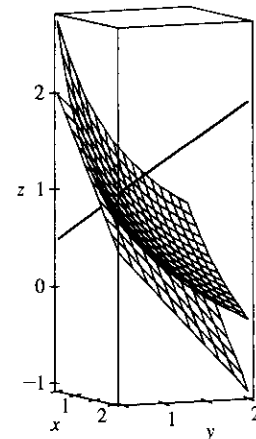
45. $F(x, y, z) = xy + yz + zx$,

$$\nabla F(x, y, z) = \langle y + z, x + z, y + x \rangle,$$

$$\nabla F(1, 1, 1) = \langle 2, 2, 2 \rangle, \text{ so an equation of the tangent plane}$$

$$\text{is } 2x + 2y + 2z = 6 \text{ or } x + y + z = 3, \text{ and the normal line is}$$

$$\text{given by } x - 1 = y - 1 = z - 1 \text{ or } x = y = z.$$



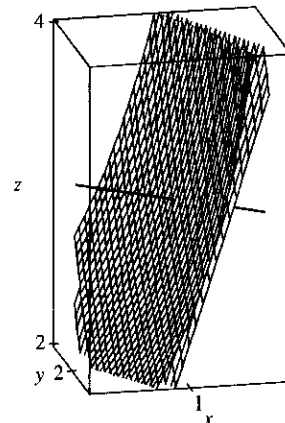
46. $F(x, y, z) = xyz, \nabla F(x, y, z) = \langle yz, xz, yx \rangle$,

$$\nabla F(1, 2, 3) = \langle 6, 3, 2 \rangle, \text{ so an equation of the tangent plane}$$

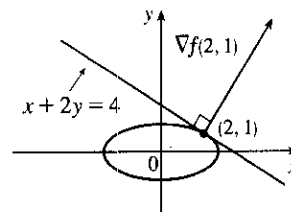
$$\text{is } 6x + 3y + 2z = 18, \text{ and the normal line is given by}$$

$$\frac{x - 1}{6} = \frac{y - 2}{3} = \frac{z - 3}{2} \text{ or } x = 1 + 6t, y = 2 + 3t,$$

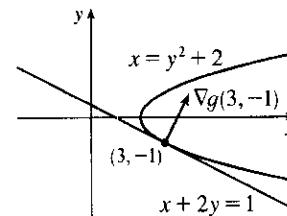
$$z = 3 + 2t.$$



47. $\nabla f(x, y) = \langle 2x, 8y \rangle$, $\nabla f(2, 1) = \langle 4, 8 \rangle$. The tangent line has equation $\nabla f(2, 1) \cdot \langle x - 2, y - 1 \rangle = 0 \Rightarrow 4(x - 2) + 8(y - 1) = 0$, which simplifies to $x + 2y = 4$.



48. $\nabla g(x, y) = \langle 1, -2y \rangle$, $\nabla g(3, -1) = \langle 1, 2 \rangle$. The tangent line has equation $\nabla g(3, -1) \cdot \langle x - 3, y + 1 \rangle = 0 \Rightarrow 1(x - 3) + 2(y + 1) = 0$, which simplifies to $x + 2y = 1$.



49. $\nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0, y_0, z_0) is $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence $\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z = 1$ is an equation of the tangent plane.

50. $\nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, -\frac{2z_0}{c^2} \right\rangle$, so an equation of the tangent plane at (x_0, y_0, z_0) is $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y - \frac{2z_0}{c^2}z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 2$ or $\frac{x_0}{a^2}x + \frac{y_0}{b^2}y - \frac{z_0}{c^2}z = 1$.

51. $\nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, -\frac{1}{c} \right\rangle$, so an equation of the tangent plane is $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y - \frac{1}{c}z = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} - \frac{z_0}{c}$ or $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y = \frac{z}{c} + 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right) - \frac{z_0}{c}$. But $\frac{z_0}{c} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$, so the equation can be written as $\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y = \frac{z + z_0}{c}$.

52. Since $\nabla f(x_0, y_0, z_0) = \langle 2x_0, 4y_0, 6z_0 \rangle$ and $\langle 3, -1, 3 \rangle$ are both normal vectors to the surface at (x_0, y_0, z_0) , we need $\langle 2x_0, 4y_0, 6z_0 \rangle = c\langle 3, -1, 3 \rangle$ or $\langle x_0, 2y_0, 3z_0 \rangle = k\langle 3, -1, 3 \rangle$. Thus $x_0 = 3k$, $y_0 = -\frac{1}{2}k$ and $z_0 = k$. But $x_0^2 + 2y_0^2 + 3z_0^2 = 1$ or $(9 + \frac{1}{2} + 3)k^2 = 1$, so $k = \pm \frac{\sqrt{2}}{5}$ and there are two such points: $\left(\pm \frac{3\sqrt{2}}{5}, \mp \frac{1}{5\sqrt{2}}, \pm \frac{\sqrt{2}}{5}\right)$.

53. $\nabla f(x_0, y_0, z_0) = \langle 2x_0, -2y_0, 4z_0 \rangle$ and the given line has direction numbers 2, 4, 6, so $\langle 2x_0, -2y_0, 4z_0 \rangle = k\langle 2, 4, 6 \rangle$ or $x_0 = k$, $y_0 = -2k$ and $z_0 = \frac{3}{2}k$. But $x_0^2 - y_0^2 + 2z_0^2 = 1$ or $(1 - 4 + \frac{9}{2})k^2 = 1$, so $k = \pm \sqrt{\frac{2}{3}} = \pm \frac{\sqrt{6}}{3}$ and there are two such points: $\left(\pm \frac{\sqrt{6}}{3}, \mp \frac{2\sqrt{6}}{3}, \pm \frac{\sqrt{6}}{2}\right)$.

54. First note that the point $(1, 1, 2)$ is on both surfaces. For the ellipsoid, an equation of the tangent plane at $(1, 1, 2)$ is $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$, and for the sphere, an equation of the tangent plane at $(1, 1, 2)$ is $(2 - 8)x + (2 - 6)y + (4 - 8)z = -18$ or $-6x - 4y - 4z = -18$ or $3x + 2y + 2z = 9$. Since these tangent planes are the same, the surfaces are tangent to each other at the point $(1, 1, 2)$.

55. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. Then an equation of the tangent plane to the cone at this point is $2x_0x + 2y_0y - 2z_0z = 2(x_0^2 + y_0^2 - z_0^2)$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

56. Let (x_0, y_0, z_0) be a point on the sphere. Then the normal line is given by $\frac{x - x_0}{2x_0} = \frac{y - y_0}{2y_0} = \frac{z - z_0}{2z_0}$. For the center $(0, 0, 0)$ to be on the line, we need $-\frac{x_0}{2x_0} = -\frac{y_0}{2y_0} = -\frac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

57. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is

$\frac{x}{2\sqrt{x_0}} + \frac{y}{2\sqrt{y_0}} + \frac{z}{2\sqrt{z_0}} = \frac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is $\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{c}$. The x -, y -, and z -intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x -intercept is found by setting $y = z = 0$ and solving the resulting equation for x , and the y - and z -intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = c$, a constant.

58. Here the equation of the tangent plane to the point (x_0, y_0, z_0) is $y_0z_0x + x_0z_0y + x_0y_0z = 3x_0y_0z_0$ or

$\frac{x}{3x_0} + \frac{y}{3y_0} + \frac{z}{3z_0} = 1$. Then the x -, y -, and z -intercepts are $3x_0$, $3y_0$ and $3z_0$ respectively, and their product is $27x_0y_0z_0 = 27c^3$, a constant.

59. If $f(x, y, z) = z - x^2 - y^2$ and $g(x, y, z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1, 1, 2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line. We have:

$$\nabla f(x, y, z) = \langle -2x, -2y, 1 \rangle \Rightarrow \nabla f(-1, 1, 2) = \langle 2, -2, 1 \rangle, \text{ and } \nabla g(x, y, z) = \langle 8x, 2y, 2z \rangle \Rightarrow$$

$$\nabla g(-1, 1, 2) = \langle -8, 2, 4 \rangle. \text{ Hence } \mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\mathbf{i} - 16\mathbf{j} - 12\mathbf{k}. \text{ Parametric equations}$$

are: $x = -1 - 10t, y = 1 - 16t, z = 2 - 12t$.

60. (a) Let $f(x, y, z) = y + z$ and $g(x, y, z) = x^2 + y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1, 2, 1)$ and the vector $\mathbf{v} = \nabla f \times \nabla g$ is parallel to the tangent line. We have

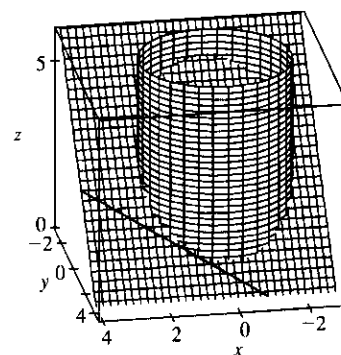
$$\nabla f(x, y, z) = \langle 0, 1, 1 \rangle \Rightarrow \nabla f(1, 2, 1) = \langle 0, 1, 1 \rangle, \text{ and}$$

$$\nabla g(x, y, z) = \langle 2x, 2y, 0 \rangle \Rightarrow \nabla g(1, 2, 1) = \langle 2, 4, 0 \rangle. \text{ Hence}$$

$$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}. \text{ So parametric}$$

equations of the desired tangent line are $x = 1 - 4t, y = 2 + 2t, z = 1 - 2t$.

(b)



61. (a) The direction of the normal line of F is given by ∇F , and that of G by ∇G . Assuming that

$$\nabla F \neq 0 \neq \nabla G, \text{ the two normal lines are perpendicular at } P \text{ if } \nabla F \cdot \nabla G = 0 \text{ at } P \Leftrightarrow$$

$$\langle \partial F / \partial x, \partial F / \partial y, \partial F / \partial z \rangle \cdot \langle \partial G / \partial x, \partial G / \partial y, \partial G / \partial z \rangle = 0 \text{ at } P \Leftrightarrow F_x G_x + F_y G_y + F_z G_z = 0 \text{ at } P.$$

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so

$\nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point $\langle x, y, z \rangle$ lies on the graph of $F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

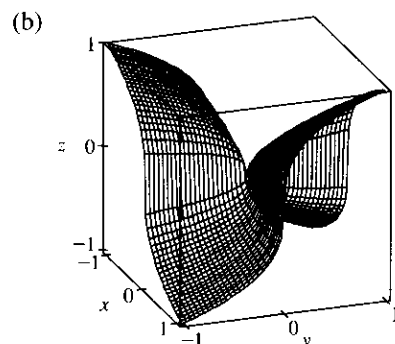
62. (a) The function $f(x, y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 15.2.8 [ET 14.2.8].)

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{(h \cdot 0)^{1/3} - 0}{h} = 0,$$

$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{(0 \cdot h)^{1/3} - 0}{h} = 0$. Therefore, $f_x(0, 0)$ and $f_y(0, 0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ where $a \neq 0$ and $b \neq 0$. Thus

$$D_{\mathbf{u}} f(0, 0) = \lim_{h \rightarrow 0} \frac{f(0 + ha, 0 + hb) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{(ha)(hb)}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{ab}}{h^{1/3}}$$

and this limit does not exist, so $D_{\mathbf{u}} f(0, 0)$ does not exist.



Notice that if we start at the origin and proceed in the direction of the x - or y -axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

63. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$. Then we know that at the given point, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y , and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y \rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{dD_{\mathbf{u}} f - bD_{\mathbf{v}} f}{ad - bc}, \frac{aD_{\mathbf{v}} f - cD_{\mathbf{u}} f}{ad - bc} \right\rangle.$$

64. Since $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 15.4.7 [ET 14.4.7] we have

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \text{ where } \varepsilon_1, \varepsilon_2 \rightarrow 0 \text{ as } (\Delta x, \Delta y) \rightarrow (0, 0).$$

$$\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0), \langle \Delta x, \Delta y \rangle = \mathbf{x} - \mathbf{x}_0 \text{ so } (\Delta x, \Delta y) \rightarrow (0, 0) \text{ is equivalent to } \mathbf{x} \rightarrow \mathbf{x}_0 \text{ and}$$

$$\langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle = \nabla f(\mathbf{x}_0). \text{ Substituting into (15.4.7 [ET 14.4.7]) gives}$$

$$f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \varepsilon_1, \varepsilon_2 \rangle \cdot \langle \Delta x, \Delta y \rangle \text{ or}$$

$$\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0), \text{ and so}$$

$$\frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \frac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}. \text{ But } \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \text{ is a unit vector so}$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{\langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0 \text{ since } \varepsilon_1, \varepsilon_2 \rightarrow 0 \text{ as } \mathbf{x} \rightarrow \mathbf{x}_0. \text{ Hence}$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0.$$

15.7 Maximum and Minimum Values

ET 14.7

- (a) First we compute $D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$ by the Second Derivatives Test.

(b) $D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1, 1) < 0$, f has a saddle point at $(1, 1)$ by the Second Derivatives Test.
- (a) $D = g_{xx}(0, 2)g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0, 2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0, 2)g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0, 2) < 0$, g has a local maximum at $(0, 2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0, 2)g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0, 2)$.
- In the figure, a point at approximately $(1, 1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1, 1)$. The level curves near $(0, 0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

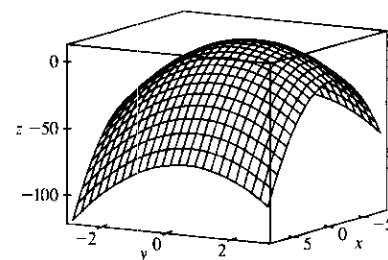
To verify our predictions, we have $f(x, y) = 4 + x^3 + y^3 - 3xy \Rightarrow f_x(x, y) = 3x^2 - 3y$,
 $f_y(x, y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$,
 $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \Rightarrow$
 $3x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$. Then we have two critical points, $(0, 0)$ and $(1, 1)$. The second partial derivatives are $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -3$, and $f_{yy}(x, y) = 6y$, so
 $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then
 $D(0, 0) = 36(0)(0) - 9 = -9$, and $D(1, 1) = 36(1)(1) - 9 = 27$. Since $D(0, 0) < 0$, f has a saddle point at $(0, 0)$ by the Second Derivatives Test. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$.
- In the figure, points at approximately $(-1, 1)$ and $(-1, -1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1, \pm 1)$. Similarly, the point $(1, 0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1, 0)$, $(1, 1)$, and $(1, -1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x, y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x, y) = 3 - 3x^2$,
 $f_y(x, y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \Rightarrow x = \pm 1$ and

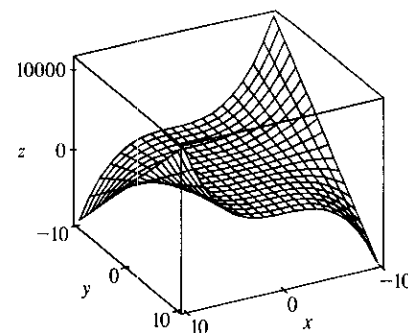
$-4y + 4y^3 = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y = 0, \pm 1$. So our critical points are $(\pm 1, 0), (\pm 1, \pm 1)$. The second partial derivatives are $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = 0$, and $f_{yy}(x, y) = 12y^2 - 4$, so $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1, 0)$	24	-6	$D > 0, f_{xx} < 0 \Rightarrow f$ has a local maximum at $(1, 0)$
$(1, 1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, 1)$
$(1, -1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, -1)$
$(-1, 0)$	-24		$D < 0 \Rightarrow f$ has a saddle point at $(-1, 0)$
$(-1, 1)$	48	6	$D > 0, f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, 1)$
$(-1, -1)$	48	6	$D > 0, f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, -1)$

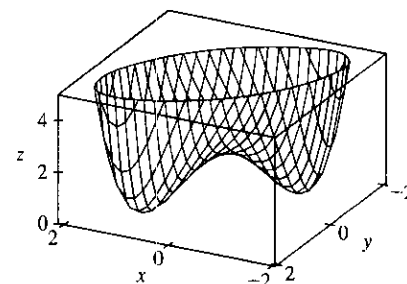
5. $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2 \Rightarrow f_x = -2 - 2x$,
 $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and
 $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is
 $(-1, \frac{1}{2})$. $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and
since $D(-1, \frac{1}{2}) = 16 > 0$ and $f_{xx}(-1, \frac{1}{2}) = -2 < 0$,
 $f(-1, \frac{1}{2}) = 11$ is a local maximum by the Second Derivatives Test.



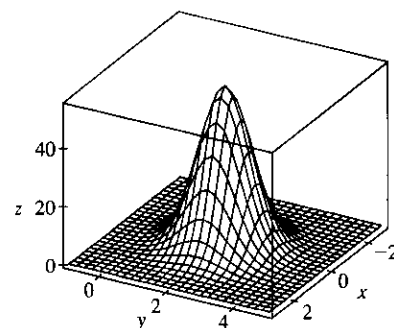
6. $f(x, y) = x^3y + 12x^2 - 8y \Rightarrow f_x = 3x^2y + 24x$,
 $f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$. Then $f_y = 0$
implies $x = 2$, and substitution into $f_x = 0$ gives $12y + 48 = 0 \Rightarrow$
 $y = -4$. Thus, the only critical point is $(2, -4)$.
 $D(2, -4) = (-24)(0) - 12^2 = -144 < 0$, so $(2, -4)$ is a saddle
point.



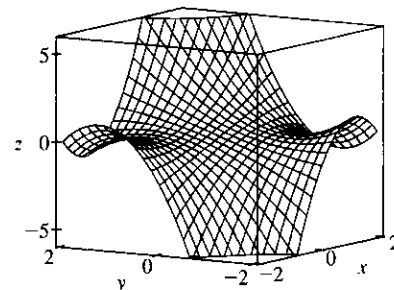
7. $f(x, y) = x^4 + y^4 - 4xy + 2 \Rightarrow f_x = 4x^3 - 4y$,
 $f_y = 4y^3 - 4x$, $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Then $f_x = 0$
implies $y = x^3$, and substitution into $f_y = 0 \Rightarrow x = y^3$ gives
 $x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0 \Rightarrow x = 0$ or $x = \pm 1$.
Thus the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$. Now
 $D(0, 0) = 0 \cdot 0 - (-4)^2 = -16 < 0$, so $(0, 0)$ is a saddle point.
 $D(1, 1) = (12)(12) - (-4)^2 > 0$ and $f_{xx}(1, 1) = 12 > 0$, so
 $f(1, 1) = 0$ is a local minimum. $D(-1, -1) = (12)(12) - (-4)^2 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so
 $f(-1, -1) = 0$ is also a local minimum.



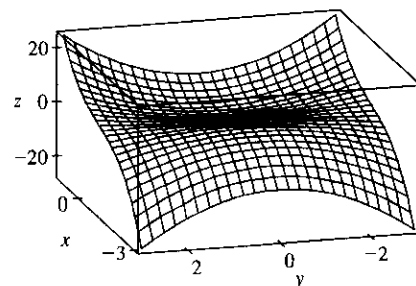
8. $f(x, y) = e^{4y-x^2-y^2} \Rightarrow f_x = -2xe^{4y-x^2-y^2}$,
 $f_y = (4-2y)e^{4y-x^2-y^2}$, $f_{xx} = (4x^2-2)e^{4y-x^2-y^2}$,
 $f_{xy} = -2x(4-2y)e^{4y-x^2-y^2}$,
 $f_{yy} = (4y^2-16y+14)e^{4y-x^2-y^2}$. Then $f_x = 0$ and $f_y = 0$
 implies $x = 0$ and $y = 2$, so the only critical point is $(0, 2)$.
 $D(0, 2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$ and
 $f_{xx}(0, 2) = -2e^4 < 0$, so $f(0, 2) = e^4$ is a local maximum.



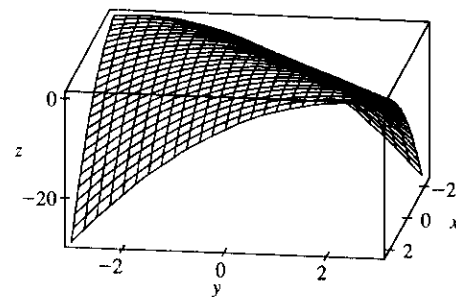
9. $f(x, y) = (1+xy)(x+y) = x+y+x^2y+xy^2 \Rightarrow$
 $f_x = 1+2xy+y^2$, $f_y = 1+x^2+2xy$, $f_{xx} = 2y$, $f_{xy} = 2x+2y$,
 $f_{yy} = 2x$. Then $f_x = 0$ implies $1+2xy+y^2 = 0$ and $f_y = 0$
 implies $1+x^2+2xy = 0$. Subtracting the second equation from the
 first gives $y^2 - x^2 = 0 \Rightarrow y = \pm x$, but if $y = x$ then
 $1+2xy+y^2 = 0 \Rightarrow 1+3x^2 = 0$ which has no real solution. If
 $y = -x$ then $1+2xy+y^2 = 0 \Rightarrow 1-x^2 = 0 \Rightarrow x = \pm 1$,
 so critical points are $(1, -1)$ and $(-1, 1)$. $D(1, -1) = (-2)(2) - 0 < 0$ and $D(-1, 1) = (2)(-2) - 0 < 0$,
 so $(-1, 1)$ and $(1, -1)$ are saddle points.



10. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow f_x = 6x^2 + y^2 + 10x$,
 $f_y = 2xy + 2y$, $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$. Then
 $f_y = 0$ implies $y = 0$ or $x = -1$. Substituting into $f_x = 0$ gives the
 critical points $(0, 0)$, $(-\frac{5}{3}, 0)$, $(-1, \pm 2)$. Now $D(0, 0) = 20 > 0$
 and $f_{xx}(0, 0) = 10 > 0$, so $f(0, 0) = 0$ is a local minimum. Also
 $f_{xx}(-\frac{5}{3}, 0) < 0$, $D(-\frac{5}{3}, 0) > 0$, and $D(-1, \pm 2) < 0$. Hence
 $f(-\frac{5}{3}, 0) = \frac{125}{27}$ is a local maximum while $(-1, \pm 2)$ are saddle
 points.



11. $f(x, y) = 1 + 2xy - x^2 - y^2 \Rightarrow f_x = 2y - 2x$,
 $f_y = 2x - 2y$, $f_{xx} = f_{yy} = -2$, $f_{xy} = 2$. Then $f_x = 0$ and
 $f_y = 0$ implies $x = y$ so the critical points are all points of the form
 (x_0, x_0) . But $D(x_0, x_0) = 4 - 4 = 0$ so the Second Derivatives
 Test gives no information. However
 $1 + 2xy - x^2 - y^2 = 1 - (x - y)^2$ and $1 - (x - y)^2 \leq 1$ for all
 (x, y) , with equality if and only if $x = y$. Thus $f(x_0, x_0) = 1$ are
 local maxima.



12. $f(x, y) = xy(1 - x - y) \Rightarrow f_x = y - 2xy - y^2,$

$$f_y = x - x^2 - 2xy, f_{xx} = -2y, f_{yy} = -2x, f_{xy} = 1 - 2x - 2y.$$

Then $f_x = 0$ implies $y = 0$ or $y = 1 - 2x$. Substituting $y = 0$ into

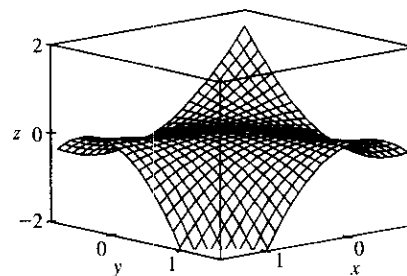
$f_y = 0$ gives $x = 0$ or $x = 1$ and substituting $y = 1 - 2x$ into

$f_y = 0$ gives $3x^2 - x = 0$ so $x = 0$ or $\frac{1}{3}$. Thus the critical points are

$(0, 0)$, $(1, 0)$, $(0, 1)$ and $(\frac{1}{3}, \frac{1}{3})$.

$D(0, 0) = D(1, 0) = D(0, 1) = -1$ while $D(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$ and $f_{xx}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3} < 0$. Thus $(0, 0)$, $(1, 0)$

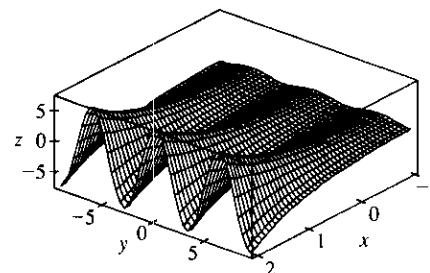
and $(0, 1)$ are saddle points, and $f(\frac{1}{3}, \frac{1}{3}) = \frac{1}{27}$ is a local maximum.



13. $f(x, y) = e^x \cos y \Rightarrow f_x = e^x \cos y, f_y = -e^x \sin y.$

Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer.

But $\sin(\frac{\pi}{2} + n\pi) \neq 0$, so there are no critical points.



14. $f(x, y) = x^2 + y^2 + \frac{1}{x^2 y^2} \Rightarrow f_x = 2x - 2x^{-3}y^{-2},$

$$f_y = 2y - 2x^{-2}y^{-3}, f_{xx} = 2 + 6x^{-4}y^{-2}, f_{yy} = 2 + 6x^{-2}y^{-4},$$

$$f_{xy} = 4x^{-3}y^{-3}. \text{ Then } f_x = 0 \text{ implies } 2x^4 y^2 - 2 = 0 \text{ or } x^4 y^2 = 1$$

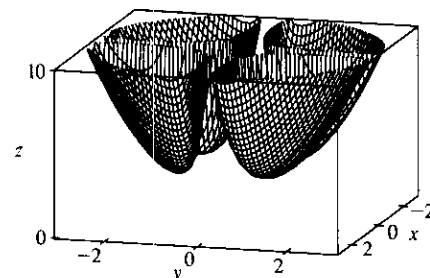
or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$

implies $2x^2 y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies

$$2x^{-6} - 2 = 0 \text{ or } x^6 = 1. \text{ Thus } x = \pm 1 \text{ and if } x = 1, y = \pm 1;$$

if $x = -1, y = \pm 1$. So the critical points are $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$. Now

$D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(\pm 1, \pm 1) = f(\pm 1, \mp 1) = 3$ are local minima.



15. $f(x, y) = x \sin y \Rightarrow f_x = \sin y, f_y = x \cos y, f_{xx} = 0,$

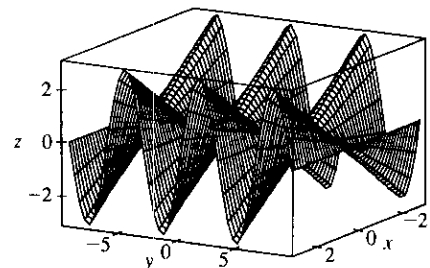
$$f_{yy} = -x \sin y \text{ and } f_{xy} = \cos y. \text{ Then } f_x = 0 \text{ if and only if}$$

$y = n\pi$, n an integer, and substituting into $f_y = 0$ requires $x = 0$

for each of these y -values. Thus the critical points are $(0, n\pi)$, n

an integer. But $D(0, n\pi) = -\cos^2(n\pi) < 0$ so each critical point

is a saddle point.



$$16. f(x, y) = (2x - x^2)(2y - y^2) \Rightarrow f_x = (2 - 2x)(2y - y^2),$$

$$f_y = (2x - x^2)(2 - 2y), f_{xx} = -2(2y - y^2), f_{yy} = -2(2x - x^2)$$

$$\text{and } f_{xy} = (2 - 2x)(2 - 2y). \text{ Then } f_x = 0 \text{ implies } x = 1 \text{ or } y = 0$$

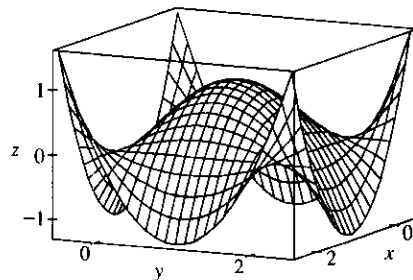
$$\text{or } y = 2 \text{ and when } x = 1, f_y = 0 \text{ implies } y = 1, \text{ when } y = 0,$$

$$f_y = 0 \text{ implies } x = 0 \text{ or } x = 2 \text{ and when } y = 2, f_y = 0 \text{ implies}$$

$$x = 0 \text{ or } x = 2. \text{ Thus the critical points are } (1, 1), (0, 0), (2, 0),$$

$$(0, 2) \text{ and } (2, 2). \text{ Now } D(0, 0) = D(2, 0) = D(0, 2) = D(2, 2) = -16 \text{ so these critical points are saddle points,}$$

$$\text{and } D(1, 1) = 4 \text{ with } f_{xx}(1, 1) = -2, \text{ so } f(1, 1) = 1 \text{ is a local maximum.}$$



$$17. f(x, y) = (x^2 + y^2)e^{y^2 - x^2} \Rightarrow f_x = (x^2 + y^2)e^{y^2 - x^2}(-2x) + 2xe^{y^2 - x^2} = 2xe^{y^2 - x^2}(1 - x^2 - y^2),$$

$$f_y = (x^2 + y^2)e^{y^2 - x^2}(2y) + 2ye^{y^2 - x^2} = 2ye^{y^2 - x^2}(1 + x^2 + y^2),$$

$$f_{xx} = 2xe^{y^2 - x^2}(-2x) + (1 - x^2 - y^2)(2x(-2xe^{y^2 - x^2}) + 2e^{y^2 - x^2})$$

$$= 2e^{y^2 - x^2}((1 - x^2 - y^2)(1 - 2x^2) - 2x^2),$$

$$f_{xy} = 2xe^{y^2 - x^2}(-2y) + 2x(2y)e^{y^2 - x^2}(1 - x^2 - y^2) = -4xye^{y^2 - x^2}(x^2 + y^2),$$

$$f_{yy} = 2ye^{y^2 - x^2}(2y) + (1 + x^2 + y^2)(2y(2ye^{y^2 - x^2}) + 2e^{y^2 - x^2})$$

$$= 2e^{y^2 - x^2}((1 + x^2 + y^2)(1 + 2y^2) + 2y^2).$$

$$f_y = 0 \text{ implies } y = 0, \text{ and substituting into } f_x = 0 \text{ gives}$$

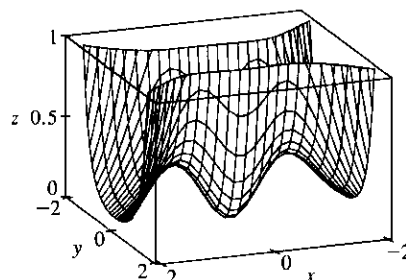
$$2xe^{-x^2}(1 - x^2) = 0 \Rightarrow x = 0 \text{ or } x = \pm 1.$$

$$\text{Thus the critical points are } (0, 0) \text{ and } (\pm 1, 0).$$

$$D(0, 0) = (2)(2) - 0 > 0 \text{ and } f_{xx}(0, 0) = 2 > 0, \text{ so } f(0, 0) = 0$$

$$\text{is a local minimum. } D(\pm 1, 0) = (-4e^{-1})(4e^{-1}) - 0 < 0 \text{ so}$$

$$(\pm 1, 0) \text{ are saddle points.}$$



$$18. f(x, y) = x^2ye^{-x^2 - y^2} \Rightarrow$$

$$f_x = x^2ye^{-x^2 - y^2}(-2x) + 2xye^{-x^2 - y^2} = 2xy(1 - x^2)e^{-x^2 - y^2},$$

$$f_y = x^2ye^{-x^2 - y^2}(-2y) + x^2e^{-x^2 - y^2} = x^2(1 - 2y^2)e^{-x^2 - y^2},$$

$$f_{xx} = 2y(2x^4 - 5x^2 + 1)e^{-x^2 - y^2},$$

$$f_{xy} = 2x(1 - x^2)(1 - 2y^2)e^{-x^2 - y^2},$$

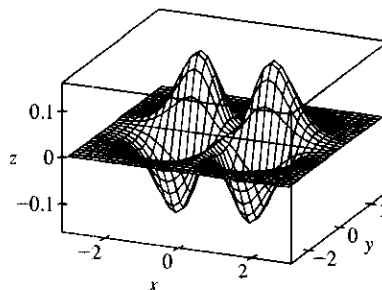
$$f_{yy} = 2x^2y(2y^2 - 3)e^{-x^2 - y^2}.$$

$$f_x = 0 \text{ implies } x = 0, y = 0, \text{ or } x = \pm 1. \text{ If } x = 0 \text{ then } f_y = 0 \text{ for any } y\text{-value, so all points of the form } (0, y) \text{ are}$$

$$\text{critical points. If } y = 0 \text{ then } f_y = 0 \Rightarrow x^2e^{-x^2} = 0 \Rightarrow x = 0, \text{ so } (0, 0) \text{ (already included above) is a}$$

$$\text{critical point. If } x = \pm 1 \text{ then } (1 - 2y^2)e^{-1 - y^2} = 0 \Rightarrow y = \pm \frac{1}{\sqrt{2}}, \text{ so } \left(1, \pm \frac{1}{\sqrt{2}}\right) \text{ and } \left(-1, \pm \frac{1}{\sqrt{2}}\right) \text{ are critical}$$

$$\text{points. } D(0, y) = 0, \text{ so the Second Derivatives Test gives no information. However, if } y > 0 \text{ then}$$

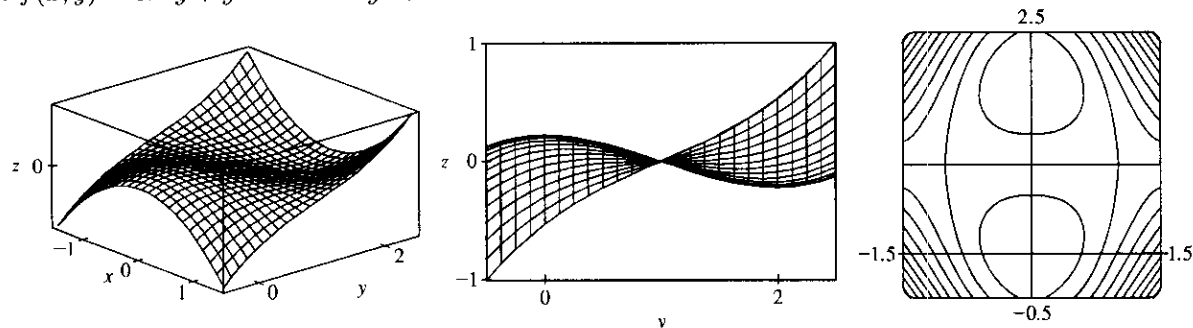


$x^2ye^{-x^2-y^2} \geq 0$ with equality only when $x = 0$, so we have local minimum values $f(0, y) = 0$, $y > 0$. Similarly, if $y < 0$ then $x^2ye^{-x^2-y^2} \leq 0$ with equality when $x = 0$ so $f(0, y) = 0$, $y < 0$ are local maximum values, and $(0, 0)$ is a saddle point.

$$D\left(\pm 1, \frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0, f_{xx}\left(\pm 1, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}e^{-3/2} < 0 \text{ and}$$

$D\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0, f_{xx}\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}e^{-3/2} > 0$, so $f\left(\pm 1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while $f\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points.

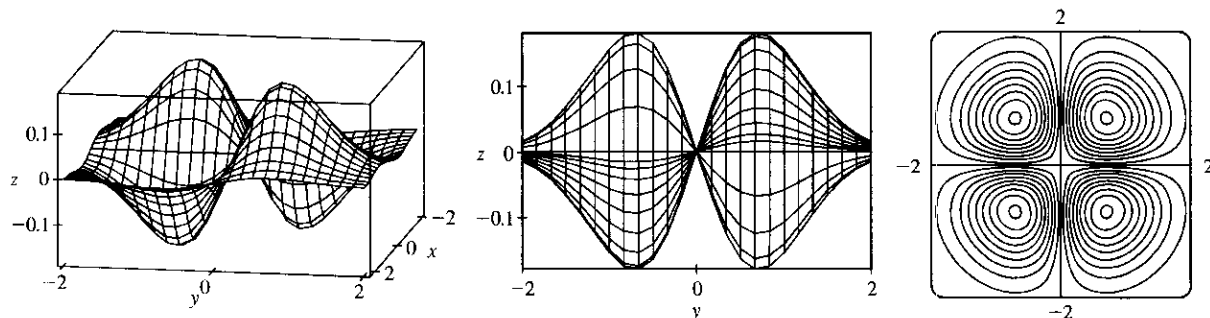
19. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$



From the graphs, it appears that f has a local maximum $f(0, 0) \approx 2$ and a local minimum $f(0, 2) \approx -2$. There appear to be saddle points near $(\pm 1, 1)$.

$f_x = 6xy - 6x$, $f_y = 3x^2 + 3y^2 - 6y$. Then $f_x = 0$ implies $x = 0$ or $y = 1$ and when $x = 0$, $f_y = 0$ implies $y = 0$ or $y = 2$; when $y = 1$, $f_y = 0$ implies $x^2 = 1$ or $x = \pm 1$. Thus the critical points are $(0, 0)$, $(0, 2)$, $(\pm 1, 1)$. Now $f_{xx} = 6y - 6$, $f_{yy} = 6y - 6$ and $f_{xy} = 6x$, so $D(0, 0) = D(0, 2) = 36 > 0$ while $D(\pm 1, 1) = -36 < 0$ and $f_{xx}(0, 0) = -6$, $f_{xx}(0, 2) = 6$. Hence $(\pm 1, 1)$ are saddle points while $f(0, 0) = 2$ is a local maximum and $f(0, 2) = -2$ is a local minimum.

20. $f(x, y) = xye^{-x^2-y^2}$



There appear to be local maxima of about $f(\pm 0.7, \pm 0.7) \approx 0.18$ and local minima of about $f(\pm 0.7, \mp 0.7) \approx -0.18$. Also, there seems to be a saddle point at the origin.

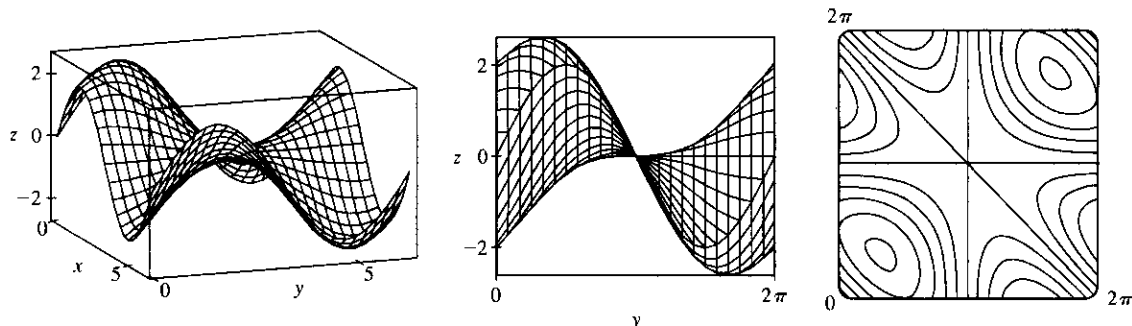
$$f_x = ye^{-x^2-y^2}(1 - 2x^2), f_y = xe^{-x^2-y^2}(1 - 2y^2), f_{xx} = 2xye^{-x^2-y^2}(2x^2 - 3),$$

$$f_{yy} = 2xye^{-x^2-y^2}(2y^2 - 3), f_{xy} = (1 - 2x^2)e^{-x^2-y^2}(1 - 2y^2). \text{ Then } f_x = 0 \text{ implies } y = 0 \text{ or } x = \pm \frac{1}{\sqrt{2}}.$$

Substituting these values into $f_y = 0$ gives the critical points $(0, 0)$, $\left(\frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$. Then

$D(x, y) = e^{2(-x^2-y^2)}[4x^2y^2(2x^2-3)(2y^2-3) - (1-2x^2)^2(1-2y^2)^2]$, so $D(0, 0) = -1$, while $D\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$ and $D\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$. But $f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$, $f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) > 0$, $f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > 0$ and $f_{xx}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$. Hence $(0, 0)$ is a saddle point; $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$ are local minima and $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$ are local maxima.

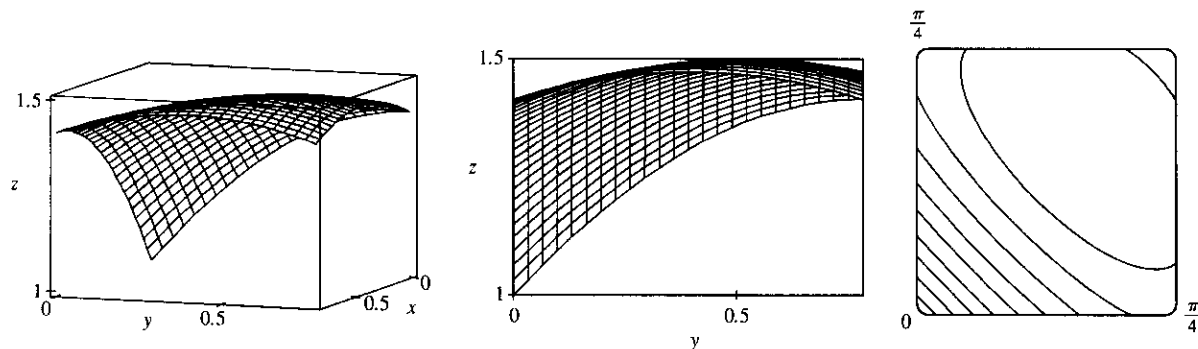
21. $f(x, y) = \sin x + \sin y + \sin(x + y)$, $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$



From the graphs it appears that f has a local maximum at about $(1, 1)$ with value approximately 2.6, a local minimum at about $(5, 5)$ with value approximately -2.6 , and a saddle point at about $(3, 3)$.

$f_x = \cos x + \cos(x + y)$, $f_y = \cos y + \cos(x + y)$, $f_{xx} = -\sin x - \sin(x + y)$, $f_{yy} = -\sin y - \sin(x + y)$, $f_{xy} = -\sin(x + y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi, \frac{\pi}{3}$, or $\frac{5\pi}{3}$, yielding the critical points (π, π) , $(\frac{\pi}{3}, \frac{\pi}{3})$ and $(\frac{5\pi}{3}, \frac{5\pi}{3})$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π, π) . Now $D(x, y) = \sin x \sin y + \sin x \sin(x + y) + \sin y \sin(x + y)$. So $D(\pi, \pi) = 0$ and the Second Derivatives Test doesn't apply. $D(\frac{\pi}{3}, \frac{\pi}{3}) = \frac{9}{4} > 0$ and $f_{xx}(\frac{\pi}{3}, \frac{\pi}{3}) < 0$ so $f(\frac{\pi}{3}, \frac{\pi}{3}) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D(\frac{5\pi}{3}, \frac{5\pi}{3}) = \frac{9}{4} > 0$ and $f_{xx}(\frac{5\pi}{3}, \frac{5\pi}{3}) > 0$, so $f(\frac{5\pi}{3}, \frac{5\pi}{3}) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

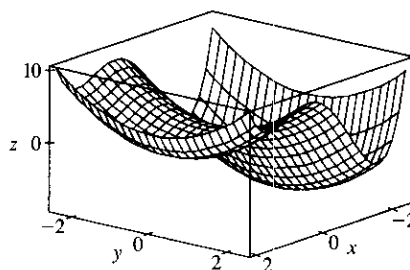
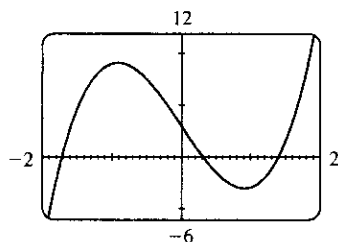
22. $f(x, y) = \sin x + \sin y + \cos(x + y)$, $0 \leq x \leq \frac{\pi}{4}$, $0 \leq y \leq \frac{\pi}{4}$



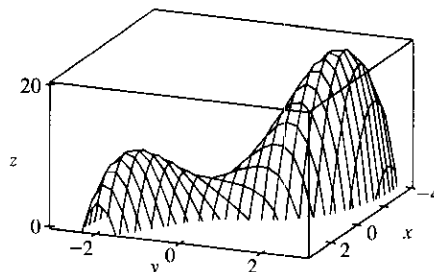
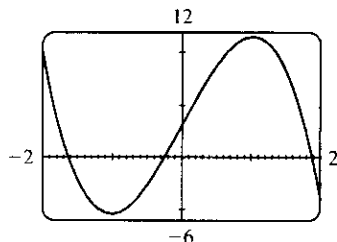
From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$. $f_x = \cos x - \sin(x + y)$, $f_y = \cos y - \sin(x + y)$, $f_{xx} = -\sin x - \cos(x + y)$, $f_{yy} = -\sin y - \cos(x + y)$,

$f_{xy} = -\cos(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \neq 0$ for $0 \leq x \leq \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $(\frac{\pi}{6}, \frac{\pi}{6})$. Here $f_{xx}(\frac{\pi}{6}, \frac{\pi}{6}) = -1 < 0$ and $D(\frac{\pi}{6}, \frac{\pi}{6}) = (-1)^2 - \frac{1}{4} > 0$. Thus $f(\frac{\pi}{6}, \frac{\pi}{6}) = \frac{3}{2}$ is a local maximum.

23. $f(x, y) = x^4 - 5x^2 + y^2 + 3x + 2 \Rightarrow f_x(x, y) = 4x^3 - 10x + 3$ and $f_y(x, y) = 2y$. $f_y = 0 \Rightarrow y = 0$, and the graph of f_x shows that the roots of $f_x = 0$ are approximately $x = -1.714, 0.312$ and 1.402 . (Alternatively, we could have used a calculator or a CAS to find these roots.) So to three decimal places, the critical points are $(-1.714, 0)$, $(1.402, 0)$, and $(0.312, 0)$. Now since $f_{xx} = 12x^2 - 10$, $f_{xy} = 0$, $f_{yy} = 2$, and $D = 24x^2 - 20$, we have $D(-1.714, 0) > 0$, $f_{xx}(-1.714, 0) > 0$, $D(1.402, 0) > 0$, $f_{xx}(1.402, 0) > 0$, and $D(0.312, 0) < 0$. Therefore $f(-1.714, 0) \approx -9.200$ and $f(1.402, 0) \approx 0.242$ are local minima, and $(0.312, 0)$ is a saddle point. The lowest point on the graph is approximately $(-1.714, 0, -9.200)$.

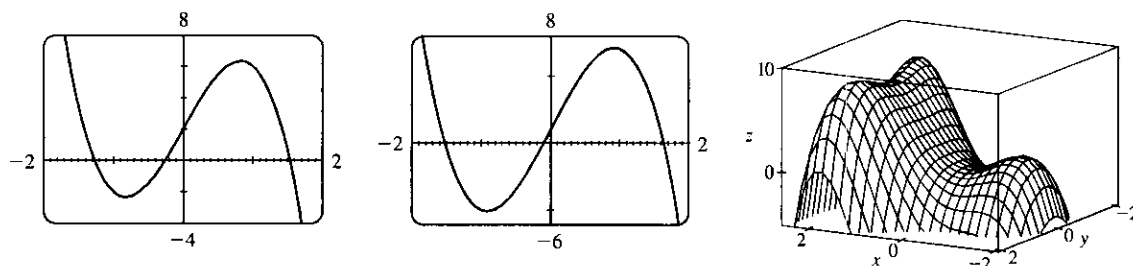


24. $f(x, y) = 5 - 10xy - 4x^2 + 3y - y^4 \Rightarrow f_x(x, y) = -10y - 8x$, $f_y(x, y) = -10x + 3 - 4y^3$. Now $f_x = 0 \Rightarrow x = -\frac{5}{4}y$, so using a graph, we find solutions to $0 = f_y(-\frac{5}{4}y, y) = -10(-\frac{5}{4}y) + 3 - 4y^3 = -4y^3 + \frac{25}{2}y + 3$. (Alternatively, we could have found the roots of $f_x = f_y = 0$ directly, using a calculator or a CAS.) To three decimal places, the solutions are $y \approx 1.877, -0.245$ and -1.633 , so f has critical points at approximately $(-2.347, 1.877)$, $(0.306, -0.245)$, and $(2.041, -1.633)$. Now since $f_{xx} = -8$, $f_{xy} = -10$, $f_{yy} = -12y^2$, and $D = 96y^2 - 100$, we have $D(-2.347, 1.877) > 0$, $D(0.306, -0.245) < 0$, and $D(2.041, -1.633) > 0$. Therefore, since $f_{xx} < 0$ everywhere, $f(-2.347, 1.877) \approx 20.238$ and $f(2.041, -1.633) \approx 9.657$ are local maxima, and $(0.306, -0.245)$ is a saddle point. The highest point on the graph is approximately $(-2.347, 1.877, 20.238)$.

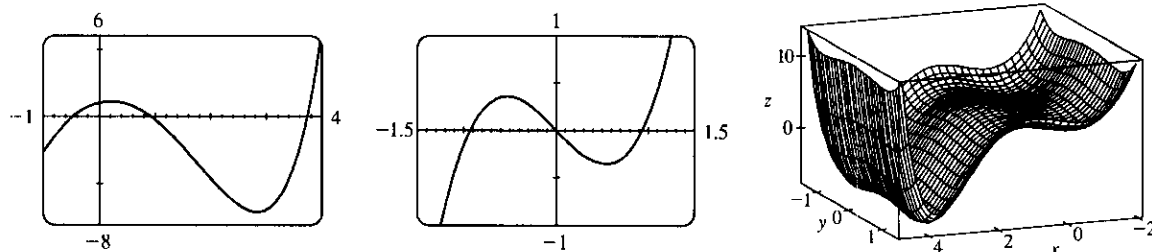


25. $f(x, y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4 \Rightarrow f_x(x, y) = 2 + 8x + 2y^2 - 4x^3$, $f_y(x, y) = -2y + 4xy - 4y^3$. Now $f_y = 0 \Leftrightarrow 2y(2y^2 - 2x + 1) = 0 \Leftrightarrow y = 0$ or $y^2 = x - \frac{1}{2}$.

The first of these implies that $f_x = -4x^3 + 8x + 2$, and the second implies that $f_x = 2 + 8x + 2(x - \frac{1}{2}) - 4x^3 = -4x^3 + 10x + 1$. From the graphs, we see that the first possibility for f_x has roots at approximately -1.267 , -0.259 , and 1.526 , and the second has a root at approximately 1.629 (the negative roots do not give critical points, since $y^2 = x - \frac{1}{2}$ must be positive). So to three decimal places, f has critical points at $(-1.267, 0)$, $(-0.259, 0)$, $(1.526, 0)$, and $(1.629, \pm 1.063)$. Now since $f_{xx} = 8 - 12x^2$, $f_{xy} = 4y$, $f_{yy} = 4x - 12y^2$, and $D = (8 - 12x^2)(4x - 12y^2) - 16y^2$, we have $D(-1.267, 0) > 0$, $f_{xx}(-1.267, 0) > 0$, $D(-0.259, 0) < 0$, $D(1.526, 0) < 0$, $D(1.629, \pm 1.063) > 0$, and $f_{xx}(1.629, \pm 1.063) < 0$. Therefore, to three decimal places, $f(-1.267, 0) \approx 1.310$ and $f(1.629, \pm 1.063) \approx 8.105$ are local maxima, and $(-0.259, 0)$ and $(1.526, 0)$ are saddle points. The highest points on the graph are approximately $(1.629, \pm 1.063, 8.105)$.

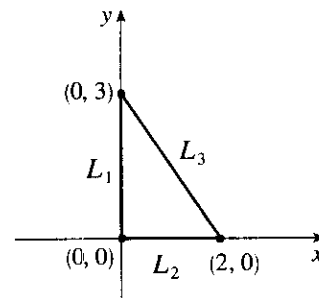


26. $f(x, y) = e^x + y^4 - x^3 + 4 \cos y \Rightarrow f_x(x, y) = e^x - 3x^2$ and $f_y(x, y) = 4y^3 - 4 \sin y$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -0.459, 0.910$, or 3.733 , and $f_y = 0$ when $y \approx 0$ or ± 0.929 . (Alternatively, we could have used a calculator or a CAS to find the roots of $f_x = 0$ and $f_y = 0$.) So, to three decimal places, f has critical points at $(-0.459, 0)$, $(-0.459, \pm 0.929)$, $(0.910, 0)$, $(0.910, \pm 0.929)$, $(3.733, 0)$, and $(3.733, \pm 0.929)$. Now $f_{xx} = e^x - 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4 \cos y$, and $D = (e^x - 6x)(12y^2 - 4 \cos y)$. Therefore $D(-0.459, 0) < 0$, $D(-0.459, \pm 0.929) > 0$, $f_{xx}(-0.459, \pm 0.929) > 0$, $D(0.910, 0) > 0$, $f_{xx}(0.910, 0) < 0$, $D(0.910, \pm 0.929) < 0$, $D(3.733, 0) < 0$, $D(3.733, \pm 0.929) > 0$, and $f_{xx}(3.733, \pm 0.929) > 0$. So $f(-0.459, \pm 0.929) \approx 3.868$ and $f(3.733, \pm 0.929) \approx -7.077$ are local minima, $f(0.910, 0) \approx 5.731$ is a local maximum, and $(-0.459, 0)$, $(0.910, \pm 0.929)$, and $(3.733, 0)$ are saddle points. The lowest points on the graph are approximately $(3.733, \pm 0.929, -7.077)$.

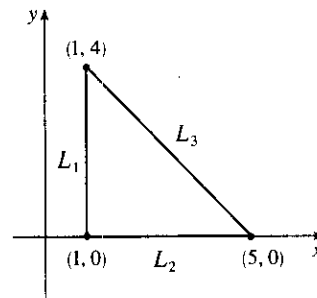


27. Since f is a polynomial it is continuous on D , so an absolute maximum and minimum exist. Here $f_x = 4$, $f_y = -5$ so there are no critical points inside D . Thus the absolute extrema must both occur on the boundary. Along L_1 , $x = 0$ and $f(0, y) = 1 - 5y$ for $0 \leq y \leq 3$, a decreasing function in y , so the maximum value is $f(0, 0) = 1$ and

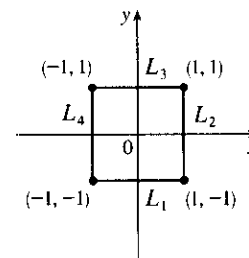
the minimum value is $f(0, 3) = -14$. Along L_2 , $y = 0$ and $f(x, 0) = 1 + 4x$ for $0 \leq x \leq 2$, an increasing function in x , so the minimum value is $f(0, 0) = 1$ and the maximum value is $f(2, 0) = 9$. Along L_3 , $y = -\frac{3}{2}x + 3$ and $f(x, -\frac{3}{2}x + 3) = \frac{23}{2}x - 14$ for $0 \leq x \leq 2$, an increasing function in x , so the minimum value is $f(0, 3) = -14$ and the maximum value is $f(2, 0) = 9$. Thus the absolute maximum of f on D is $f(2, 0) = 9$ and the absolute minimum is $f(0, 3) = -14$.



28. Since f is a polynomial it is continuous on D , so an absolute maximum and minimum exist. $f_x = y - 1$, $f_y = x - 2$, and setting $f_x = f_y = 0$ gives $(2, 1)$ as the only critical point, where $f(2, 1) = 1$. Along L_1 : $x = 1$ and $f(1, y) = 2 - y$ for $0 \leq y \leq 4$, a decreasing function in y , so the maximum value is $f(1, 0) = 2$ and the minimum value is $f(1, 4) = -2$. Along L_2 : $y = 0$ and $f(x, 0) = 3 - x$ for $1 \leq x \leq 5$, a decreasing function in x , so the maximum value is $f(1, 0) = 2$ and the minimum value is $f(5, 0) = -2$. Along L_3 , $y = 5 - x$ and $f(x, 5 - x) = -x^2 + 6x - 7 = -(x - 3)^2 + 2$ for $1 \leq x \leq 5$, which has a maximum at $x = 3$ where $f(3, 2) = 2$ and a minimum at both $x = 1$ and $x = 5$, where $f(1, 4) = f(5, 0) = -2$. Thus the absolute maximum of f on D is $f(1, 0) = f(3, 2) = 2$ and the absolute minimum is $f(1, 4) = f(5, 0) = -2$.

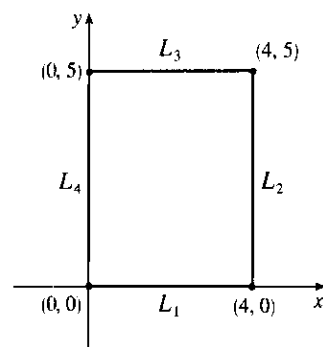


29. $f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D , with $f(0, 0) = 4$. On L_1 : $y = -1$, $f(x, -1) = 5$, a constant. On L_2 : $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $(1, -\frac{1}{2})$, $f(1, -\frac{1}{2}) = \frac{17}{4}$. On L_3 : $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$. On L_4 : $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $(-1, -\frac{1}{2})$, $f(-1, -\frac{1}{2}) = \frac{17}{4}$. Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.

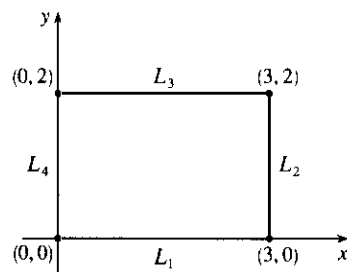


30. $f_x(x, y) = 4 - 2x$ and $f_y(x, y) = 6 - 2y$, so the only critical point is $(2, 3)$ (which is in D) where $f(2, 3) = 13$. Along L_1 : $y = 0$, so $f(x, 0) = 4x - x^2 = -(x - 2)^2 + 4$, $0 \leq x \leq 4$, which has a maximum value when $x = 2$ where $f(2, 0) = 4$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0, 0) = f(4, 0) = 0$. Along L_2 : $x = 4$, so $f(4, y) = 6y - y^2 = -(y - 3)^2 + 9$, $0 \leq y \leq 5$, which has a maximum value when $y = 3$ where $f(4, 3) = 9$ and a minimum value when $y = 0$ where $f(4, 0) = 0$. Along L_3 : $y = 5$, so $f(x, 5) = -x^2 + 4x + 5 = -(x - 2)^2 + 9$, $0 \leq x \leq 4$, which has a maximum value when $x = 2$ where $f(2, 5) = 9$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0, 5) = f(4, 5) = 5$.

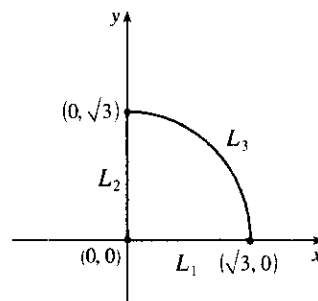
Along L_4 : $x = 0$, so $f(0, y) = 6y - y^2 = -(y - 3)^2 + 9$,
 $0 \leq y \leq 5$, which has a maximum value when $y = 3$ where
 $f(0, 3) = 9$ and a minimum value when $y = 0$ where $f(0, 0) = 0$.
 Thus the absolute maximum is $f(2, 3) = 13$ and the absolute
 minimum is attained at both $(0, 0)$ and $(4, 0)$, where
 $f(0, 0) = f(4, 0) = 0$.



31. $f(x, y) = x^4 + y^4 - 4xy + 2$ is a polynomial and hence continuous on D , so it has an absolute maximum and minimum on D . In Exercise 7, we found the critical points of f ; only $(1, 1)$ with $f(1, 1) = 0$ is inside D . On L_1 : $y = 0$, $f(x, 0) = x^4 + 2$, $0 \leq x \leq 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3, 0) = 83$, and its minimum at $x = 0$, $f(0, 0) = 2$. On L_2 : $x = 3$, $f(3, y) = y^4 - 12y + 83$, $0 \leq y \leq 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$, $f(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3, 0) = 83$. On L_3 : $y = 2$, $f(x, 2) = x^4 - 8x + 18$, $0 \leq x \leq 3$, a polynomial in x which attains its minimum at $x = \sqrt[3]{2}$, $f(\sqrt[3]{2}, 2) = 18 - 6\sqrt[3]{2} \approx 10.4$, and its maximum at $x = 3$, $f(3, 2) = 75$. On L_4 : $x = 0$, $f(0, y) = y^4 + 2$, $0 \leq y \leq 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0, 2) = 18$, and its minimum at $y = 0$, $f(0, 0) = 2$. Thus the absolute maximum of f on D is $f(3, 0) = 83$ and the absolute minimum is $f(1, 1) = 0$.



32. $f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \Leftrightarrow y = 0$, there are no critical points in the interior of D . Along L_1 , $y = 0$ and $f(x, 0) = 0$. Along L_2 , $x = 0$ and $f(0, y) = 0$. Along L_3 , $y = \sqrt{3 - x^2}$, so let $g(x) = f(x, \sqrt{3 - x^2}) = 3x - x^3$ for $0 \leq x \leq \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \Leftrightarrow x = 1$. The maximum value is $f(1, \sqrt{2}) = 2$ and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f(0, \sqrt{3}) = f(\sqrt{3}, 0) = 0$. Thus the absolute maximum of f on D is $f(1, \sqrt{2}) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2 .

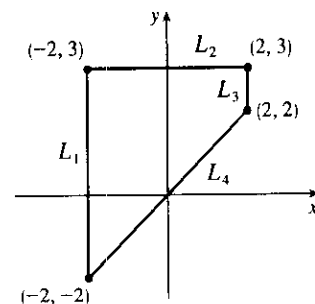


33. $f_x(x, y) = 6x^2$ and $f_y(x, y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0, 0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x, y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$, $-1 \leq x \leq 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \Rightarrow x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f(\frac{1}{2}, \pm \frac{\sqrt{3}}{2}) = g(\frac{1}{2}) = \frac{13}{16}$, and $(-2, -3)$ is not in D . Checking the endpoints, we get $f(-1, 0) = g(-1) = -2$ and $f(1, 0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1, 0) = 2$ and $f(-1, 0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos \theta$, $y = \sin \theta$, so

$$f(\cos \theta, \sin \theta) = 2 \cos^3 \theta + \sin^4 \theta, 0 \leq \theta \leq 2\pi.$$

34. $f_x(x, y) = 3x^2 - 3$ and $f_y(x, y) = -3y^2 + 12$ and the critical points are $(1, 2)$, $(1, -2)$, $(-1, 2)$, and $(-1, -2)$. But only $(1, 2)$ and $(-1, 2)$ are in D and $f(1, 2) = 14$, $f(-1, 2) = 18$. Along L_1 : $x = -2$ and $f(-2, y) = -2 - y^3 + 12y$, $-2 \leq y \leq 3$, which has a maximum at $y = 2$ where $f(-2, 2) = 14$ and a minimum at $y = -2$ where $f(-2, -2) = -18$. Along L_2 : $x = 2$ and $f(2, y) = 2 - y^3 + 12y$, $2 \leq y \leq 3$, which has a maximum at $y = 2$ where $f(2, 2) = 18$ and a minimum at $y = 3$ where $f(2, 3) = 11$. Along L_3 : $y = 3$ and $f(x, 3) = x^3 - 3x + 9$, $-2 \leq x \leq 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1, 3) = f(2, 3) = 11$ and a minimum at $x = 1$ and $x = -2$ where $f(1, 3) = f(-2, 3) = 7$. Along L_4 : $y = x$ and $f(x, x) = 9x$, $-2 \leq x \leq 2$, which has a maximum at $x = 2$ where $f(2, 2) = 18$ and a minimum at $x = -2$ where $f(-2, -2) = -18$. So the absolute maximum value of f on D is $f(2, 2) = 18$ and the minimum is $f(-2, -2) = -18$.



35. $f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 \Rightarrow f_x(x, y) = -2(x^2 - 1)(2x) - 2(x^2y - x - 1)(2xy - 1)$ and $f_y(x, y) = -2(x^2y - x - 1)x^2$. Setting $f_y(x, y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$. There are no critical points for $x = 0$, since $f_x(0, y) = -2$, so we set $x^2y - x - 1 = 0 \Leftrightarrow y = \frac{x+1}{x^2}$ ($x \neq 0$), so

$$f_x\left(x, \frac{x+1}{x^2}\right) = -2(x^2 - 1)(2x) - 2\left(x^2 \frac{x+1}{x^2} - x - 1\right)\left(2x \frac{x+1}{x^2} - 1\right) = -4x(x^2 - 1). \text{ Therefore}$$

$f_x(x, y) = f_y(x, y) = 0$ at the points $(1, 2)$ and $(-1, 0)$. To classify these critical points, we calculate

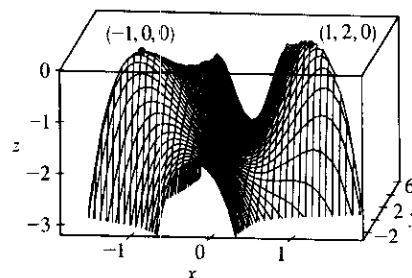
$$f_{xx}(x, y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2, f_{yy}(x, y) = -2x^4, \text{ and } f_{xy}(x, y) = -8x^3y + 6x^2 + 4x.$$

In order to use the Second Derivatives Test we calculate

$$\begin{aligned} D(-1, 0) &= f_{xx}(-1, 0) f_{yy}(-1, 0) - [f_{xy}(-1, 0)]^2 \\ &= 16 > 0, \end{aligned}$$

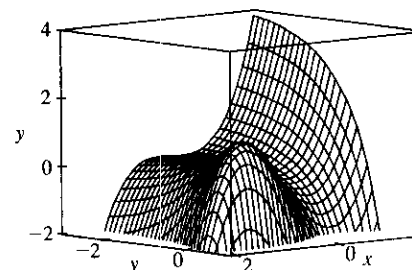
$$f_{xx}(-1, 0) = -10 < 0, D(1, 2) = 16 > 0, \text{ and}$$

$f_{xx}(1, 2) = -26 < 0$, so both $(-1, 0)$ and $(1, 2)$ give local maxima.



36. $f(x, y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that (1) $f_x = 3e^y - 3x^2 = 0$ and (2) $f_y = 3xe^y - 3e^{3y} = 0$. From (1) we obtain $e^y = x^2$, and then (2) gives $3x^3 - 3x^6 = 0 \Rightarrow x = 1$ or 0 , but only $x = 1$ is valid, since $x = 0$ makes (1) impossible. So substituting $x = 1$ into (1) gives $y = 0$, and the only critical point is $(1, 0)$.

The Second Derivatives Test shows that this gives a local maximum, since



$D(1, 0) = [-6x(3xe^y - 9e^{3y}) - (3e^y)^2]_{(1,0)} = 27 > 0$ and $f_{xx}(1, 0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1, 0) = 1$ is not an absolute maximum because, for instance, $f(-3, 0) = 17$. This can also be seen from the graph.

- 37.** Let d be the distance from $(2, 1, -1)$ to any point (x, y, z) on the plane $x + y - z = 1$, so

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2} \text{ where } z = x + y - 1, \text{ and we minimize}$$

$$d^2 = f(x, y) = (x-2)^2 + (y-1)^2 + (x+y)^2. \text{ Then } f_x(x, y) = 2(x-2) + 2(x+y) = 4x + 2y - 4,$$

$f_y(x, y) = 2(y-1) + 2(x+y) = 2x + 4y - 2$. Solving $4x + 2y - 4 = 0$ and $2x + 4y - 2 = 0$ simultaneously gives $x = 1, y = 0$. An absolute minimum exists (since there is a minimum distance from the point to the plane)

and it must occur at a critical point, so the shortest distance occurs for $x = 1, y = 0$ for which

$$d = \sqrt{(1-2)^2 + (0-1)^2 + (1+0)^2} = \sqrt{3}.$$

- 38.** Here the distance d from a point on the plane to the point $(1, 2, 3)$ is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$,

where $z = 4 - x + y$. We can minimize $d^2 = f(x, y) = (x-1)^2 + (y-2)^2 + (1-x+y)^2$, so

$$f_x(x, y) = 2(x-1) + 2(1-x+y)(-1) = 4x - 2y - 4 \text{ and}$$

$$f_y(x, y) = 2(y-2) + 2(1-x+y) = 4y - 2x - 2. \text{ Solving } 4x - 2y - 4 = 0 \text{ and } 4y - 2x - 2 = 0$$

simultaneously gives $x = \frac{5}{3}$ and $y = \frac{4}{3}$, so the only critical point is $(\frac{5}{3}, \frac{4}{3})$. This point must correspond to the minimum distance, so the point on the plane closest to $(1, 2, 3)$ is $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$.

- 39.** Minimize $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1$. Then $f_x = 2x + y, f_y = 2y + x$ so the critical point is $(0, 0)$ and $D(0, 0) = 4 - 1 > 0$ with $f_{xx}(0, 0) = 2$ so this is a minimum. Thus $z^2 = 1$ or $z = \pm 1$ and the points on the surface are $(0, 0, \pm 1)$.

- 40.** Since $z = 1/(x^2y^2)$ on the surface, we minimize $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + x^{-4}y^{-4} = f(x, y)$.

$$f_x = 2x - \frac{4}{x^5y^4}, f_y = 2y - \frac{4}{x^4y^5}, \text{ so the critical points occur when } 2x = \frac{4}{x^5y^4} \text{ and } 2y = \frac{4}{x^4y^5} \text{ or}$$

$x^6y^4 = 2 = x^4y^6$, so $x^2 = y^2 \Rightarrow x = \pm y$ and $x^{10} = 2 \Rightarrow x = \pm 2^{1/10}, y = \pm 2^{1/10}$. The four critical points are $(\pm 2^{1/10}, \pm 2^{1/10})$. The absolute minimum must occur at these points (there is no maximum since the surface is infinite in extent). Thus the points on the surface closest to the origin are $(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5})$.

- 41.** $x + y + z = 100$, so maximize $f(x, y) = xy(100 - x - y)$. $f_x = 100y - 2xy - y^2, f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y, f_{yy} = -2x, f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y = 0$ or $y = 100 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 100$ and substituting $y = 100 - 2x$ into $f_y = 0$ gives $3x^2 - 100x = 0$ so $x = 0$ or $\frac{100}{3}$. Thus the critical points are $(0, 0), (100, 0), (0, 100)$ and $(\frac{100}{3}, \frac{100}{3})$.

$$D(0, 0) = D(100, 0) = D(0, 100) = -10,000 \text{ while } D(\frac{100}{3}, \frac{100}{3}) = \frac{10,000}{3} \text{ and } f_{xx}(\frac{100}{3}, \frac{100}{3}) = -\frac{200}{3} < 0.$$

Thus $(0, 0), (100, 0)$ and $(0, 100)$ are saddle points whereas $f(\frac{100}{3}, \frac{100}{3})$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$.

42. Maximize
- $f(x, y) = x^a y^b (100 - x - y)^c$
- .

$$f_x = ax^{a-1}y^b(100 - x - y)^c - cx^a y^b(100 - x - y)^{c-1} = x^{a-1}y^b(100 - x - y)^{c-1}[a(100 - x - y) - cx]$$

and $f_y = x^a y^{b-1}(100 - x - y)^{c-1}[b(100 - x - y) - cy]$. Since x, y and z are all positive, the only critical point

occurs when $x = a \frac{100 - y}{a + c}$ and $y = \frac{100b}{a + b + c}$. Thus the point is $\left(\frac{100a}{a + b + c}, \frac{100b}{a + b + c}\right)$ and the numbers are

$$x = \frac{100a}{a + b + c}, y = \frac{100b}{a + b + c}, z = \frac{100c}{a + b + c}.$$

43. Maximize
- $f(x, y) = xy(36 - 9x^2 - 36y^2)^{1/2}/2$
- with
- (x, y, z)
- in first octant. Then

$$f_x = \frac{y(36 - 9x^2 - 36y^2)^{1/2}}{2} + \frac{-9x^2 y(36 - 9x^2 - 36y^2)^{-1/2}}{2} = \frac{(36y - 18x^2 y - 36y^3)}{2(36 - 9x^2 - 36y^2)^{1/2}} \text{ and}$$

$$f_y = \frac{36x - 9x^3 - 72xy^2}{2(36 - 9x^2 - 36y^2)^{1/2}}. \text{ Setting } f_x = 0 \text{ gives } y = 0 \text{ or } y^2 = \frac{2 - x^2}{2} \text{ but } y > 0, \text{ so only the latter solution}$$

applies. Substituting this y into $f_y = 0$ gives $x^2 = \frac{4}{3}$ or $x = \frac{2}{\sqrt{3}}, y = \frac{1}{\sqrt{3}}$ and then $z^2 = (36 - 12 - 12)/4 = 3$.

The fact that this gives a maximum volume follows from the geometry. This maximum volume is

$$V = (2x)(2y)(2z) = 8\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)(\sqrt{3}) = \frac{16}{\sqrt{3}}.$$

44. Here maximize
- $f(x, y) = xy \frac{(a^2 b^2 c^2 - b^2 c^2 x^2 - a^2 c^2 y^2)^{1/2}}{a^2 b^2}$
- . Then

$$f_x = yc^2 \frac{a^2 b^2 - 2b^2 x^2 - a^2 y^2}{a^2 b^2 (a^2 b^2 c^2 - b^2 c^2 x^2 - a^2 c^2 y^2)^{1/2}} \text{ and } f_y = xc^2 \frac{a^2 b^2 - 2a^2 y^2 - b^2 x^2}{a^2 b^2 (a^2 b^2 c^2 - b^2 c^2 x^2 - a^2 c^2 y^2)^{1/2}}. \text{ Then } f_x = 0$$

(with $x, y > 0$) implies $y^2 = \frac{a^2 b^2 - 2b^2 x^2}{a^2}$ and substituting into $f_y = 0$ implies $3b^2 x^2 = a^2 b^2$ or $x = \frac{1}{\sqrt{3}} a$,

$y = \frac{1}{\sqrt{3}} b$ and then $z = \frac{1}{\sqrt{3}} c$. Thus the maximum volume of such a rectangle is $V = (2x)(2y)(2z) = \frac{8}{3\sqrt{3}} abc$.

45. Maximize
- $f(x, y) = \frac{xy}{3} (6 - x - 2y)$
- , then the maximum volume is
- $V = xyz$
- .

$$f_x = \frac{1}{3}(6y - 2xy - y^2) = \frac{1}{3}y(6 - 2x - 2y) \text{ and } f_y = \frac{1}{3}x(6 - x - 4y). \text{ Setting } f_x = 0 \text{ and } f_y = 0 \text{ gives the}$$

critical point $(2, 1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is

$$V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}.$$

46. Surface area
- $= 2(xy + xz + yz) = 64 \text{ cm}^2$
- , so
- $xy + xz + yz = 32$
- or
- $z = \frac{32 - xy}{x + y}$
- . Maximize the volume

$$f(x, y) = xy \frac{32 - xy}{x + y}. \text{ Then } f_x = \frac{32y^2 - 2xy^3 - x^2 y^2}{(x + y)^2} = y^2 \frac{32 - 2xy - x^2}{(x + y)^2} \text{ and}$$

$$f_y = x^2 \frac{32 - 2xy - y^2}{(x + y)^2}. \text{ Setting } f_x = 0 \text{ implies } y = \frac{32 - x^2}{2x} \text{ and substituting into } f_y = 0 \text{ gives}$$

$$32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0 \text{ or } 3x^4 + 64x^2 - (32)^2 = 0. \text{ Thus } x^2 = \frac{64}{6} \text{ or } x = \frac{8}{\sqrt{6}},$$

$$y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}} \text{ and } z = \frac{8}{\sqrt{6}}. \text{ Thus the box is a cube with edge length } \frac{8}{\sqrt{6}} \text{ cm.}$$

47. Let the dimensions be x , y , and z ; then $4x + 4y + 4z = c$ and the volume is

$$V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2, \quad x > 0, y > 0. \text{ Then } V_x = \frac{1}{4}cy - 2xy - y^2 \text{ and}$$

$V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

48. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$.

$$\text{Then } C_x = 5y - 2Vx^{-2}, C_y = 5x - 2Vy^{-2}, f_x = 0 \text{ implies } y = 2V/(5x^2), f_y = 0 \text{ implies } x = \sqrt[3]{\frac{2}{5}}V = y.$$

Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}}V$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

49. Let the dimensions be x , y and z , then minimize $xy + 2(xz + yz)$ if $xyz = 32,000 \text{ m}^3$. Then

$$f(x, y) = xy + [64,000(x + y)/xy] = xy + 64,000(x^{-1} + y^{-1}), f_x = y - 64,000x^{-2}, f_y = x - 64,000y^{-2}.$$

And $f_x = 0$ implies $y = 64,000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64,000$ or $x = 40$ and then $y = 40$.

Now $D(x, y) = [(2)(64,000)]^2 x^{-3} y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum.

Thus the dimensions of the box are $x = y = 40 \text{ cm}$, $z = 20 \text{ cm}$.

50. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$.

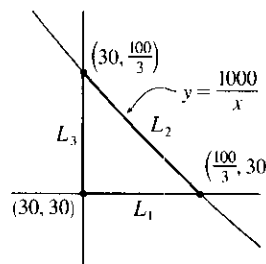
The volume is 4000 m^3 , so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function

$$h(x, y) = 6xy + 80,000/x + 64,000/y.$$

$$(a) \text{ Since } z = \frac{4000}{xy} \geq 4, xy \leq 1000 \Rightarrow y \leq 1000/x.$$

Also $x \geq 30$ and $y \geq 30$, so the domain of h is

$$D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}.$$



$$(b) h(x, y) = 6xy + 80,000x^{-1} + 64,000y^{-1} \Rightarrow h_x = 6y - 80,000x^{-2}, h_y = 6x - 64,000y^{-2}.$$

$$h_x = 0 \text{ implies } 6x^2y = 80,000 \Rightarrow y = \frac{80,000}{6x^2} \text{ and substituting into } h_y = 0 \text{ gives}$$

$$6x = 64,000 \left(\frac{6x^2}{80,000} \right)^2 \Rightarrow x^3 = \frac{80,000^2}{6 \cdot 64,000} = \frac{50,000}{3}, \text{ so } x = \sqrt[3]{\frac{50,000}{3}} = 10\sqrt[3]{\frac{50}{3}} \Rightarrow y = \frac{80}{\sqrt[3]{60}},$$

and the only critical point of h is $\left(10\sqrt[3]{\frac{50}{3}}, \frac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$ which is not in D . Next we check the

boundary of D . On L_1 : $y = 30$, $h(x, 30) = 180x + 80,000/x + 6400/3$, $30 \leq x \leq \frac{100}{3}$. Since

$$h'(x, 30) = 180 - 80,000/x^2 > 0 \text{ for } 30 \leq x \leq \frac{100}{3}, h(x, 30) \text{ is an increasing function with minimum}$$

$$h(30, 30) = 10,200 \text{ and maximum } h\left(\frac{100}{3}, 30\right) \approx 10,533. \text{ On } L_2: y = 1000/x,$$

$h(x, 1000/x) = 6000 + 64x + 80,000/x$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x, 1000/x) = 64 - 80,000/x^2 < 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h(\frac{100}{3}, 30) \approx 10,533$ and maximum $h(30, \frac{100}{3}) \approx 10,587$. On L_3 : $x = 30$, $h(30, y) = 180y + 64,000/y + 8000/3$, $30 \leq y \leq \frac{100}{3}$. $h'(30, y) = 180 - 64,000/y^2 > 0$ for $30 \leq y \leq \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10,200$ and maximum $h(30, \frac{100}{3}) \approx 10,587$. Thus the absolute minimum of h is $h(30, 30) = 10,200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of h , which gives a local (and absolute) minimum, is approximately

$h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m^3 with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

51. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and

$L = \sqrt{x^2 + y^2 + z^2} \Rightarrow L^2 = x^2 + y^2 + z^2 \Rightarrow z = \sqrt{L^2 - x^2 - y^2}$. Substituting, we have volume $V(x, y) = xy \sqrt{L^2 - x^2 - y^2}$, $x, y > 0$.

$$V_x = xy \cdot \frac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y \sqrt{L^2 - x^2 - y^2} = y \sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}},$$

$$V_y = x \sqrt{L^2 - x^2 - y^2} - \frac{xy^2}{\sqrt{L^2 - x^2 - y^2}},$$

$V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y \Rightarrow y(L^2 - 2x^2 - y^2) = 0 \Rightarrow 2x^2 + y^2 = L^2$ (since $y > 0$), and

$V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \Rightarrow x(L^2 - x^2 - 2y^2) = 0 \Rightarrow x^2 + 2y^2 = L^2$ (since $x > 0$).

Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \Rightarrow 3x^2 = L^2 \Rightarrow x = L/\sqrt{3}$

(since $x > 0$) and then $y = \sqrt{L^2 - 2(L/\sqrt{3})^2} = L/\sqrt{3}$. So the only critical point is $(L/\sqrt{3}, L/\sqrt{3})$ which,

from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is

$$V(L/\sqrt{3}, L/\sqrt{3}) = (L/\sqrt{3})^2 \sqrt{L^2 - (L/\sqrt{3})^2 - (L/\sqrt{3})^2} = L^3/(3\sqrt{3}) \text{ cubic units.}$$

52. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving

$P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \geq 0$, $r \geq 0$, and $q + r \leq 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and

$q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q, r) = 2 - 4q - 2r = 0$ and

$P_r(q, r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have

$2 - 4(1 - 2q) - 2q = 0 \Rightarrow q = \frac{1}{3}$. Then we have one critical point, $(\frac{1}{3}, \frac{1}{3})$, where $P(\frac{1}{3}, \frac{1}{3}) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0, 0 \leq q \leq 1, P(q, r) = P(q, 0) = 2q - 2q^2, 0 \leq q \leq 1$. This represents a parabola with maximum value $P(\frac{1}{2}, 0) = \frac{1}{2}$. On the segment $q = 0, 0 \leq r \leq 1$ we have $P(0, r) = 2r - 2r^2, 0 \leq r \leq 1$. This represents a parabola with maximum value $P(0, \frac{1}{2}) = \frac{1}{2}$. Finally, on the segment $q + r = 1, 0 \leq q \leq 1$, $P(q, r) = P(q, 1 - q) = 2q - 2q^2, 0 \leq q \leq 1$ which has a maximum value of $P(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q, r)$ on D is $\frac{2}{3}$.

53. Note that here the variables are m and b , and $f(m, b) = \sum_{i=1}^n [y_i - (mx_i + b)]^2$. Then

$$f_m = \sum_{i=1}^n -2x_i[y_i - (mx_i + b)] = 0 \text{ implies } \sum_{i=1}^n (x_i y_i - mx_i^2 - bx_i) = 0 \text{ or } \sum_{i=1}^n x_i y_i = m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i$$

$$\text{and } f_b = \sum_{i=1}^n -2[y_i - (mx_i + b)] = 0 \text{ implies } \sum_{i=1}^n y_i = m \sum_{i=1}^n x_i + \sum_{i=1}^n b = m \left(\sum_{i=1}^n x_i \right) + nb. \text{ Thus we have}$$

$$\text{the two desired equations. Now } f_{mm} = \sum_{i=1}^n 2x_i^2, f_{bb} = \sum_{i=1}^n 2 = 2n \text{ and } f_{mb} = \sum_{i=1}^n 2x_i. \text{ And } f_{mm}(m, b) > 0$$

$$\text{always and } D(m, b) = 4n \left(\sum_{i=1}^n x_i^2 \right) - 4 \left(\sum_{i=1}^n x_i \right)^2 = 4 \left[n \left(\sum_{i=1}^n x_i^2 \right) - \left(\sum_{i=1}^n x_i \right)^2 \right] > 0 \text{ always so the}$$

$$\text{solutions of these two equations do indeed minimize } \sum_{i=1}^n d_i^2.$$

54. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron

that passes through the point $(1, 2, 3)$. Writing the equation of the plane as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the volume of the

tetrahedron is given by $V = \frac{abc}{6}$. But $(1, 2, 3)$ must lie on the plane, so we need $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$ (*) and thus can

think of c as a function of a and b . Then $V_a = \frac{b}{6} \left(c + a \frac{\partial c}{\partial a} \right)$ and $V_b = \frac{a}{6} \left(c + b \frac{\partial c}{\partial b} \right)$. Differentiating (*) with

respect to a we get $-a^{-2} - 3c^{-2} \frac{\partial c}{\partial a} = 0 \Rightarrow \frac{\partial c}{\partial a} = \frac{-c^2}{3a^2}$, and differentiating (*) with respect to b gives

$$-2b^{-2} - 3c^{-2} \frac{\partial c}{\partial b} = 0 \Rightarrow \frac{\partial c}{\partial b} = \frac{-2c^2}{3b^2}. \text{ Then } V_a = \frac{b}{6} \left(c + a \frac{-c^2}{3a^2} \right) = 0 \Rightarrow c = 3a, \text{ and}$$

$$V_b = \frac{a}{6} \left(c + b \frac{-2c^2}{3b^2} \right) = 0 \Rightarrow c = \frac{3}{2}b. \text{ Thus } 3a = \frac{3}{2}b \text{ or } b = 2a. \text{ Putting these into (*) gives } \frac{3}{a} = 1 \text{ or } a = 3$$

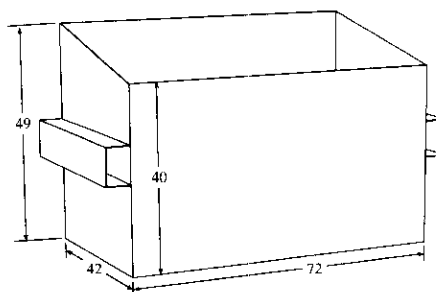
and then $b = 6, c = 9$. Thus the equation of the required plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$ or $6x + 3y + 2z = 18$.

APPLIED PROJECT Designing a Dumpster

Note: The difficulty and results of this project vary widely with the type of container studied. In addition to the variation of basic shapes of containers, dumpsters may include additional constructed parts such as supports, lift pockets, wheels, etc. Also, a CAS or graphing utility may be needed to solve the resulting equations.

Here we present a typical solution for one particular trash dumpster.

1. The basic shape and dimensions (in inches) of an actual trash dumpster are as shown in the figure.



The front and back, as well as both sides, have an extra one-inch-wide flap that is folded under and welded to the base. In addition, the side panels each fold over one inch onto the front and back pieces where they are welded. Each side has a rectangular lift pocket, with cross-section 5 by 8 inches, made of the same material. These are attached with an extra one-inch width of steel on both top and bottom where each pocket is welded to the side sheet. All four sides have a “lip” at the top; the front and back panels have an extra 5 inches of steel at the top which is folded outward in three creases to form a rectangular tube. The edge is then welded back to the main sheet. The two sides form a top lip with separate sheets of steel 5 inches wide, similarly bent into three sides and welded to the main sheets (requiring two welds each). These extend beyond the main side sheets by 1.5 inches at each end in order to join with the lips on the front and back panels. The container has a hinged lid, extra steel supports on the base at each corner, metal “fins” serving as extra support for the side lift pockets, and wheels underneath. The volume of the container is $V = \frac{1}{2}(40 + 49) \times 42 \times 72 = 134,568 \text{ in}^3$ or 77.875 ft^3 .

2. First, we assume that some aspects of the construction do not change with different dimensions, so they may be considered fixed costs. This includes the lid (with hinges), wheels, and extra steel supports. Also, the upper “lip” we previously described extends beyond the side width to connect to the other pieces. We can safely assume that this extra portion, including any associated welds, costs the same regardless of the container’s dimensions, so we will consider just the portion matching the measurement of the side panels in our calculations. We will further assume that the angle of the top of the container should be preserved. Then to compute the variable costs, let x be the width, y the length, and z the height of the front of the container. The back of the container is 9 inches, or $\frac{3}{4}$ ft, taller than the front, so using similar triangles we can say the back panel has height $z + \frac{3}{14}x$. Measuring in feet, we want the volume to remain constant, so $V = \frac{1}{2}(z + z + \frac{3}{14}x)(x)(y) = xyz + \frac{3}{28}x^2y = 77.875$. To determine a function for the variable cost, we first find the area of each sheet of metal needed. The base has area $xy \text{ ft}^2$. The front panel has visible area yz plus $\frac{1}{12}y$ for the portion folded onto the base and $\frac{5}{12}y$ for the steel at the top used to form the lip, so $(yz + \frac{1}{2}y) \text{ ft}^2$ in total. Similarly, the back sheet has area $y(z + \frac{3}{14}x) + \frac{1}{12}y + \frac{5}{12}y = yz + \frac{3}{14}xy + \frac{1}{2}y$.

Each side has visible area $\frac{1}{2}\left[z + \left(z + \frac{3}{14}x\right)\right](x)$, and the sheet includes one-inch flaps folding onto the front and back panels, so with area $\frac{1}{12}z$ and $\frac{1}{12}\left(z + \frac{3}{14}x\right)$, and a one-inch flap to fold onto the base with area $\frac{1}{12}x$. The lift pocket is constructed of a piece of steel 20 inches by x ft (including the 2 extra inches used by the welds). The additional metal used to make the lip at the top of the panel has width 5 inches and length that we can determine using the Pythagorean Theorem: $x^2 + \left(\frac{3}{14}x\right)^2 = \text{length}^2$, so $\text{length} = \frac{\sqrt{205}}{14}x \approx 1.0227x$. Thus the area of steel needed for each side panel is approximately

$$\frac{1}{2}\left[z + \left(z + \frac{3}{14}x\right)\right](x) + \frac{1}{12}z + \frac{1}{12}\left(z + \frac{3}{14}x\right) + \frac{1}{12}x + \frac{5}{3}x + \frac{5}{12}(1.0227x) \approx xz + \frac{3}{28}x^2 + \frac{1}{6}z + 2.194x$$

We also have the following welds:

Weld	Length
Front, back welded to base	$2y$
Sides welded to base	$2x$
Sides welded to front	$2z$
Sides welded to back	$2\left(z + \frac{3}{14}x\right)$
Weld on front and back lip	$2y$
Two welds on each side lip	$4(1.0227x)$
Two welds for each lift pocket	$4x$

Thus the total length of welds needed is

$$2y + 2x + 2z + 2\left(z + \frac{3}{14}x\right) + 2y + 4(1.0227x) + 4x \approx 10.519x + 4y + 4z$$

Finally, the total variable cost is approximately

$$\begin{aligned} 0.90(xy) + 0.70\left[\left(yz + \frac{1}{2}y\right) + \left(yz + \frac{3}{14}xy + \frac{1}{2}y\right) + 2\left(xz + \frac{3}{28}x^2 + \frac{1}{6}z + 2.194x\right)\right] \\ + 0.18(10.519x + 4y + 4z) \\ \approx 1.05xy + 1.4yz + 1.42y + 1.4xz + 0.15x^2 + 0.953z + 4.965x \end{aligned}$$

We would like to minimize this function while keeping volume constant, so since $xyz + \frac{3}{28}x^2y = 77.875$

we can substitute $z = \frac{77.875}{xy} - \frac{3}{28}x$ giving variable cost as a function of x and y :

$$C(x, y) \approx 0.9xy + \frac{109.0}{x} + 1.42y + \frac{109.0}{y} + \frac{74.2}{xy} + 4.86x. \text{ Using a CAS, we solve the system of equations}$$

$C_x(x, y) = 0$ and $C_y(x, y) = 0$; the only critical point within an appropriate domain is approximately $(3.58, 5.29)$.

From the nature of the function C (or from a graph) we can determine that C has an absolute minimum at $(3.58, 5.29)$, and so the minimum cost is attained for $x \approx 3.58$ ft (or 43.0 in), $y \approx 5.29$ ft (or 63.5 in), and

$$z \approx \frac{77.875}{3.58(5.29)} - \frac{3}{28}(3.58) \approx 3.73 \text{ ft (or 44.8 in).}$$

3. The fixed cost aspects of the container which we did not include in our calculations, such as the wheels and lid, don't affect the validity of our results. Some of our other assumptions, however, may influence the accuracy of our findings. We simplified the price of the steel sheets to include cuts and bends, and we simplified the price of welding to include the labor and materials. This may not be accurate for areas of the container, such as the lip and lift pockets, that require several cuts, bends, and welds in a relatively small surface area. Consequently, increasing some dimensions of the container may not increase the cost in the same manner as our computations predict. If we do not assume that the angle of the sloped top of the container must be preserved, it is likely that we could further improve our cost. Finally, our results show that the length of the container should be changed to minimize cost; this may not be possible if the two lift pockets must remain a fixed distance apart for handling by machinery.
4. The minimum variable cost using our values found in Problem 2 is $C(3.58, 5.29) \approx \$96.95$, while the current dimensions give an estimated variable cost of $C(3.5, 6.0) \approx \$97.30$. If we determine that our assumptions and simplifications are acceptable, our work shows that a slight savings can be gained by adjusting the dimensions of the container. However, the difference in cost is modest, and may not justify changes in the manufacturing process.

DISCOVERY PROJECT Quadratic Approximations and Critical Points

$$1. Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2}f_{xx}(a, b)(x - a)^2 \\ + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2}f_{yy}(a, b)(y - b)^2,$$

so

$$Q_x(x, y) = f_x(a, b) + \frac{1}{2}f_{xx}(a, b)(2)(x - a) + f_{xy}(a, b)(y - b) \\ = f_x(a, b) + f_{xx}(a, b)(x - a) + f_{xy}(a, b)(y - b)$$

At (a, b) we have $Q_x(a, b) = f_x(a, b) + f_{xx}(a, b)(a - a) + f_{xy}(a, b)(b - b) = f_x(a, b)$.

Similarly, $Q_y(x, y) = f_y(a, b) + f_{xy}(a, b)(x - a) + f_{yy}(a, b)(y - b) \Rightarrow$

$$Q_y(a, b) = f_y(a, b) + f_{xy}(a, b)(a - a) + f_{yy}(a, b)(b - b) = f_y(a, b).$$

For the second-order partial derivatives we have

$$Q_{xx}(x, y) = \frac{\partial}{\partial x} [f_x(a, b) + f_{xx}(a, b)(x - a) + f_{xy}(a, b)(y - b)] = f_{xx}(a, b)$$

$$\Rightarrow Q_{xx}(a, b) = f_{xx}(a, b)$$

$$Q_{xy}(x, y) = \frac{\partial}{\partial y} [f_x(a, b) + f_{xx}(a, b)(x - a) + f_{xy}(a, b)(y - b)] = f_{xy}(a, b)$$

$$\Rightarrow Q_{xy}(a, b) = f_{xy}(a, b)$$

$$Q_{yy}(x, y) = \frac{\partial}{\partial y} [f_y(a, b) + f_{xy}(a, b)(x - a) + f_{yy}(a, b)(y - b)] = f_{yy}(a, b)$$

$$\Rightarrow Q_{yy}(a, b) = f_{yy}(a, b)$$

2. (a) First we find the partial derivatives and values that will be needed:

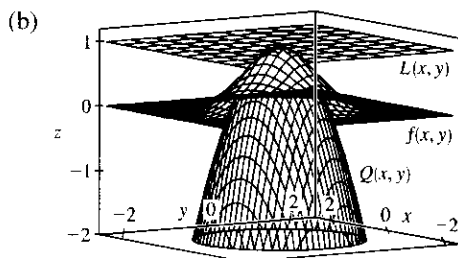
$$\begin{aligned} f(x, y) &= e^{-x^2-y^2} & f(0, 0) &= 1 \\ f_x(x, y) &= -2xe^{-x^2-y^2} & f_x(0, 0) &= 0 \\ f_y(x, y) &= -2ye^{-x^2-y^2} & f_y(0, 0) &= 0 \\ f_{xx}(x, y) &= (4x^2 - 2)e^{-x^2-y^2} & f_{xx}(0, 0) &= -2 \\ f_{xy}(x, y) &= 4xye^{-x^2-y^2} & f_{xy}(0, 0) &= 0 \\ f_{yy}(x, y) &= (4y^2 - 2)e^{-x^2-y^2} & f_{yy}(0, 0) &= -2 \end{aligned}$$

Then the first-degree Taylor polynomial of f at $(0, 0)$ is

$$\begin{aligned} L(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 1 + (0)(x - 0) + (0)(y - 0) \\ &= 1 \end{aligned}$$

The second-degree Taylor polynomial is given by

$$\begin{aligned} Q(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + \frac{1}{2}f_{xx}(0, 0)(x - 0)^2 \\ &\quad + f_{xy}(0, 0)(x - 0)(y - 0) + \frac{1}{2}f_{yy}(0, 0)(y - 0)^2 \\ &= 1 - x^2 - y^2 \end{aligned}$$



As we see from the graph, L approximates f well only for points (x, y) extremely close to the origin. Q is a much better approximation; the shape of its graph looks similar to that of the graph of f near the origin, and the values of Q appear to be good estimates for the values of f within a significant radius of the origin.

3. (a) First we find the partial derivatives and values that will be needed:

$$\begin{aligned} f(x, y) &= xe^y & f(1, 0) &= 1 & f_{xx}(x, y) &= 0 & f_{xx}(1, 0) &= 0 \\ f_x(x, y) &= e^y & f_x(1, 0) &= 1 & f_{xy}(x, y) &= e^y & f_{xy}(1, 0) &= 1 \\ f_y(x, y) &= xe^y & f_y(1, 0) &= 1 & f_{yy}(x, y) &= xe^y & f_{yy}(1, 0) &= 1 \end{aligned}$$

Then the first-degree Taylor polynomial of f at $(1, 0)$ is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + (1)(x - 1) + (1)(y - 0) \\ &= x + y \end{aligned}$$

The second-degree Taylor polynomial is given by

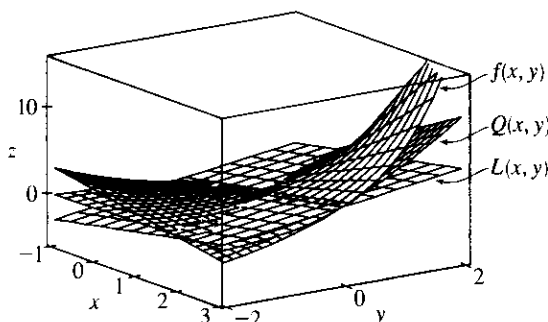
$$\begin{aligned} Q(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) + \frac{1}{2}f_{xx}(1, 0)(x - 1)^2 \\ &\quad + f_{xy}(1, 0)(x - 1)(y - 0) + \frac{1}{2}f_{yy}(1, 0)(y - 0)^2 \\ &= \frac{1}{2}y^2 + x + xy \end{aligned}$$

$$(b) L(0.9, 0.1) = 0.9 + 0.1 = 1.0$$

$$Q(0.9, 0.1) = \frac{1}{2}(0.1)^2 + 0.9 + (0.9)(0.1) = 0.995$$

$$f(0.9, 0.1) = 0.9e^{0.1} \approx 0.9947$$

(c)



As we see from the graph, L and Q both approximate f reasonably well near the point $(1, 0)$. As we venture farther from the point, the graph of Q follows the shape of the graph of f more closely than L .

$$4. (a) f(x, y) = ax^2 + bxy + cy^2 = a \left[x^2 + \frac{b}{a}xy + \frac{c}{a}y^2 \right]$$

$$= a \left[x^2 + \frac{b}{a}xy + \left(\frac{b}{2a}y \right)^2 - \left(\frac{b}{2a}y \right)^2 + \frac{c}{a}y^2 \right]$$

$$= a \left[\left(x + \frac{b}{2a}y \right)^2 - \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2 \right] = a \left[\left(x + \frac{b}{2a}y \right)^2 + \left(\frac{4ac - b^2}{4a^2} \right)y^2 \right]$$

$$(b) \text{ For } D = 4ac - b^2, \text{ from part (a) we have } f(x, y) = a \left[\left(x + \frac{b}{2a}y \right)^2 + \left(\frac{D}{4a^2} \right)y^2 \right]. \text{ If } D > 0,$$

$$\left(\frac{D}{4a^2} \right)y^2 \geq 0 \text{ and } \left(x + \frac{b}{2a}y \right)^2 \geq 0, \text{ so } \left[\left(x + \frac{b}{2a}y \right)^2 + \left(\frac{D}{4a^2} \right)y^2 \right] \geq 0. \text{ Here } a > 0, \text{ thus}$$

$$f(x, y) = a \left[\left(x + \frac{b}{2a}y \right)^2 + \left(\frac{D}{4a^2} \right)y^2 \right] \geq 0. \text{ We know } f(0, 0) = 0, \text{ so } f(0, 0) \leq f(x, y) \text{ for all } (x, y), \text{ and by definition } f \text{ has a local minimum at } (0, 0).$$

$$(c) \text{ As in part (b), } \left[\left(x + \frac{b}{2a}y \right)^2 + \left(\frac{D}{4a^2} \right)y^2 \right] \geq 0, \text{ and since } a < 0 \text{ we have}$$

$$f(x, y) = a \left[\left(x + \frac{b}{2a}y \right)^2 + \left(\frac{D}{4a^2} \right)y^2 \right] \leq 0. \text{ Since } f(0, 0) = 0, \text{ we must have } f(0, 0) \geq f(x, y) \text{ for all } (x, y), \text{ so by definition } f \text{ has a local maximum at } (0, 0).$$

$$(d) f(x, y) = ax^2 + bxy + cy^2, \text{ so } f_x(x, y) = 2ax + by \Rightarrow f_x(0, 0) = 0 \text{ and } f_y(x, y) = bx + 2cy \Rightarrow f_y(0, 0) = 0. \text{ Since } f(0, 0) = 0 \text{ and } f \text{ and its partial derivatives are continuous, we know from Equation 15.4.2 [ET 14.4.2] that the tangent plane to the graph of } f \text{ at } (0, 0) \text{ is the plane } z = 0. \text{ Then } f \text{ has a saddle point at } (0, 0) \text{ if the graph of } f \text{ crosses the tangent plane at } (0, 0), \text{ or equivalently, if some paths to the origin have positive function values while other paths have negative function values. Suppose we approach the origin along the } x\text{-axis; then we have } y = 0 \Rightarrow f(x, 0) = ax^2 \text{ which has the same sign as } a. \text{ We must now find at least}$$

one path to the origin where $f(x, y)$ gives values with sign opposite that of a . Since

$$f(x, y) = a \left[\left(x + \frac{b}{2a} y \right)^2 + \left(\frac{D}{4a^2} \right) y^2 \right], \text{ if we approach the origin along the line } x = -\frac{b}{2a} y, \text{ we have}$$

$$f\left(-\frac{b}{2a} y, y\right) = a \left[\left(-\frac{b}{2a} y + \frac{b}{2a} y \right)^2 + \left(\frac{D}{4a^2} \right) y^2 \right] = \frac{D}{4a} y^2. \text{ Since } D < 0, \text{ these values have signs}$$

opposite that of a . Thus, f has a saddle point at $(0, 0)$.

5. (a) Since the partial derivatives of f exist at $(0, 0)$ and $(0, 0)$ is a critical point, we know $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Then the second-degree Taylor polynomial of f at $(0, 0)$ can be expressed as

$$\begin{aligned} Q(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + \frac{1}{2} f_{xx}(0, 0)(x - 0)^2 \\ &\quad + f_{xy}(0, 0)(x - 0)(y - 0) + \frac{1}{2} f_{yy}(0, 0)(y - 0)^2 \\ &= \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2. \end{aligned}$$

- (b) $Q(x, y) = \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2$ fits the form of the polynomial function in

Problem 4 with $a = \frac{1}{2} f_{xx}(0, 0)$, $b = f_{xy}(0, 0)$, and $c = \frac{1}{2} f_{yy}(0, 0)$. Then we know Q is a paraboloid, and that Q has a local maximum, local minimum, or saddle point at $(0, 0)$. Here,

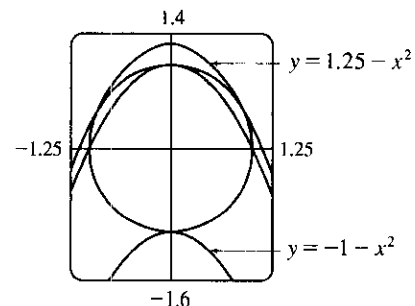
$D = 4ac - b^2 = 4\left(\frac{1}{2} f_{xx}(0, 0)\right)\left(\frac{1}{2} f_{yy}(0, 0)\right) - [f_{xy}(0, 0)]^2 = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2$, and if $D > 0$ with $a = \frac{1}{2} f_{xx}(0, 0) > 0 \Rightarrow f_{xx}(0, 0) > 0$, we know from Problem 4 that Q has a local minimum at $(0, 0)$. Similarly, if $D > 0$ and $a < 0 \Rightarrow f_{xx}(0, 0) < 0$, Q has a local maximum at $(0, 0)$, and if $D < 0$, Q has a saddle point at $(0, 0)$.

- (c) Since $f(x, y) \approx Q(x, y)$ near $(0, 0)$, part (b) suggests that for $D = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2$, if $D > 0$ and $f_{xx}(0, 0) > 0$, f has a local minimum at $(0, 0)$. If $D > 0$ and $f_{xx}(0, 0) < 0$, f has a local maximum at $(0, 0)$, and if $D < 0$, f has a saddle point at $(0, 0)$. Together with the conditions given in part (a), this is precisely the Second Derivatives Test from Section 15.7 [ET 14.7].

15.8 Lagrange Multipliers

ET 14.8

- At the extreme values of f , the level curves of f just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of c which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.
- (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.

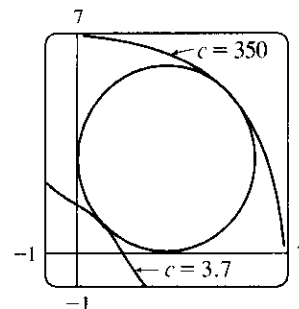


- (b) $\nabla f = \langle 2x, 1 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. So $2x = 2\lambda x \Rightarrow$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm \frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $(\pm \frac{\sqrt{3}}{2}, \frac{1}{2})$. We calculate $f(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}) = \frac{5}{4}$ (the maximum value), $f(0, 1) = 1$, and $f(0, -1) = -1$ (the minimum value). These are our answers from (a).
3. $f(x, y) = x^2 - y^2$, $g(x, y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2x, -2y \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2x = 2\lambda x$ implies $x = 0$ or $\lambda = 1$. If $x = 0$, then $x^2 + y^2 = 1$ implies $y = \pm 1$ and if $\lambda = 1$, then $-2y = 2\lambda y$ implies $y = 0$ and thus $x = \pm 1$. Thus the possible points for the extreme values of f are $(\pm 1, 0)$, $(0, \pm 1)$. But $f(\pm 1, 0) = 1$ while $f(0, \pm 1) = -1$ so the maximum value of f on $x^2 + y^2 = 1$ is $f(\pm 1, 0) = 1$ and the minimum value is $f(0, \pm 1) = -1$.
4. $f(x, y) = 4x + 6y$, $g(x, y) = x^2 + y^2 = 13 \Rightarrow \nabla f = \langle 4, 6 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \frac{2}{\lambda}$ and $y = \frac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\frac{2}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 \Rightarrow 13 = \frac{13}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(2, 3)$, $(-2, -3)$. We compute $f(2, 3) = 26$ and $f(-2, -3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2, 3) = 26$ and the minimum value is $f(-2, -3) = -26$.
5. $f(x, y) = x^2 y$, $g(x, y) = x^2 + 2y^2 = 6 \Rightarrow \nabla f = \langle 2xy, x^2 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 4\lambda y \rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $\lambda = y$. If $x = 0$, then $x^2 = 4\lambda y$ implies $\lambda = 0$ or $y = 0$. However, if $y = 0$ then $g(x, y) = 0$, a contradiction. So $\lambda = 0$ and then $g(x, y) = 6 \Rightarrow y = \pm \sqrt{3}$. If $\lambda = y$, then $x^2 = 4\lambda y$ implies $x^2 = 4y^2$, and so $g(x, y) = 6 \Rightarrow 4y^2 + 2y^2 = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Thus f has possible extreme values at the points $(0, \pm \sqrt{3})$, $(\pm 2, 1)$, and $(\pm 2, -1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2, 1) = 4$ and the minimum to be $f(\pm 2, -1) = -4$.
6. $f(x, y) = x^2 + y^2$, $g(x, y) = x^4 + y^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y \rangle$, $\lambda \nabla g = \langle 4\lambda x^3, 4\lambda y^3 \rangle$. Then $x = 2\lambda x^3$ implies $x = 0$ or $\lambda = \frac{1}{2x^2}$. If $x = 0$, then $x^4 + y^4 = 1$ implies $y = \pm 1$. But $y = 2\lambda y^3$ implies $y = 0$ so $x = \pm 1$ or $\lambda = \frac{1}{2y^2}$ and $x^2 = y^2$ and $2x^4 = 1$ so $x = \pm \frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}})$, with the maximum value of f on $x^4 + y^4 = 1$ being $f(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}) = \frac{2}{\sqrt{2}} = \sqrt{2}$ and the minimum value being $f(0, \pm 1) = f(\pm 1, 0) = 1$.
7. $f(x, y, z) = 2x + 6y + 10z$, $g(x, y, z) = x^2 + y^2 + z^2 = 35 \Rightarrow \nabla f = \langle 2, 6, 10 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$. Then $2\lambda x = 2$, $2\lambda y = 6$, $2\lambda z = 10$ imply $x = \frac{1}{\lambda}$, $y = \frac{3}{\lambda}$, and $z = \frac{5}{\lambda}$. But $35 = x^2 + y^2 + z^2 = \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 \Rightarrow 35 = \frac{35}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(1, 3, 5)$, $(-1, -3, -5)$. The maximum value of f on $x^2 + y^2 + z^2 = 35$ is $f(1, 3, 5) = 70$, and the minimum is $f(-1, -3, -5) = -70$.
8. $f(x, y, z) = 8x - 4z$, $g(x, y, z) = x^2 + 10y^2 + z^2 = 5 \Rightarrow \nabla f = \langle 8, 0, -4 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z \rangle$. Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \frac{4}{\lambda}$, $y = 0$, and $z = -\frac{2}{\lambda}$. But $5 = x^2 + 10y^2 + z^2 = \left(\frac{4}{\lambda}\right)^2 + 10(0)^2 + \left(-\frac{2}{\lambda}\right)^2 \Rightarrow 5 = \frac{20}{\lambda^2} \Rightarrow \lambda = \pm 2$, so f has possible extreme values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

9. $f(x, y, z) = xyz$, $g(x, y, z) = x^2 + 2y^2 + 3z^2 = 6 \Rightarrow \nabla f = \langle yz, xz, xy \rangle$, $\lambda \nabla g = \langle 2\lambda x, 4\lambda y, 6\lambda z \rangle$. Then $\nabla f = \lambda \nabla g$ implies $\lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z)$ or $x^2 = 2y^2$ and $z^2 = \frac{2}{3}y^2$. Thus $x^2 + 2y^2 + 3z^2 = 6$ implies $6y^2 = 6$ or $y = \pm 1$. Then the possible points are $\left(\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.
10. $f(x, y, z) = x^2y^2z^2$, $g(x, y, z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$. Then $\nabla f = \lambda \nabla g$ implies (1) $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or (2) $\lambda = 0$ and one or two (but not three) of the coordinates are 0. If (1) then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.
11. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x^4 + y^4 + z^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3 \rangle$.
Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm \frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f -value of $\sqrt{3}$.
Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and corresponding f value of $\sqrt{2}$.
Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1. Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.
12. $f(x, y, z) = x^4 + y^4 + z^4$, $g(x, y, z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 4x^3, 4y^3, 4z^3 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$.
Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ yielding 8 points each with an f -value of $\frac{1}{3}$.
Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f -value is $\frac{1}{2}$.
Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f -value of 1. Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.
13. $f(x, y, z, t) = x + y + z + t$, $g(x, y, z, t) = x^2 + y^2 + z^2 + t^2 = 1 \Rightarrow \langle 1, 1, 1, 1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t \rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$. Thus the maximum value of f is $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 2$ and the minimum value is $f(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) = -2$.
14. $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$, $g(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2 = 1 \Rightarrow \langle 1, 1, \dots, 1 \rangle = \langle 2\lambda x_1, 2\lambda x_2, \dots, 2\lambda x_n \rangle$, so $\lambda = 1/(2x_1) = 1/(2x_2) = \dots = 1/(2x_n)$ and $x_1 = x_2 = \dots = x_n$. But $x_1^2 + x_2^2 + \dots + x_n^2 = 1$, so $x_i = \pm 1/\sqrt{n}$ for $i = 1, \dots, n$. Thus the maximum value of f is $f(1/\sqrt{n}, 1/\sqrt{n}, \dots, 1/\sqrt{n}) = \sqrt{n}$ and the minimum value is $f(-1/\sqrt{n}, -1/\sqrt{n}, \dots, -1/\sqrt{n}) = -\sqrt{n}$.

15. $f(x, y, z) = x + 2y$, $g(x, y, z) = x + y + z = 1$, $h(x, y, z) = y^2 + z^2 = 4 \Rightarrow \nabla f = \langle 1, 2, 0 \rangle$,
 $\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$ and $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $1 = \lambda$, $2 = \lambda + 2\mu y$ and $0 = \lambda + 2\mu z$ so $\mu y = \frac{1}{2} = -\mu z$ or
 $y = 1/(2\mu)$, $z = -1/(2\mu)$. Thus $x + y + z = 1$ implies $x = 1$ and $y^2 + z^2 = 4$ implies $\mu = \pm \frac{1}{2\sqrt{2}}$. Then the
possible points are $(1, \pm\sqrt{2}, \mp\sqrt{2})$ and the maximum value is $f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}$ and the minimum
value is $f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$.
16. $f(x, y, z) = 3x - y - 3z$, $g(x, y, z) = x + y - z = 0$, $h(x, y, z) = x^2 + 2z^2 = 1 \Rightarrow \nabla f = \langle 3, -1, -3 \rangle$,
 $\lambda \nabla g = \langle \lambda, \lambda, -\lambda \rangle$, $\mu \nabla h = \langle 2\mu x, 0, 4\mu z \rangle$. Then $3 = \lambda + 2\mu x$, $-1 = \lambda$ and $-3 = -\lambda + 4\mu z$, so $\lambda = -1$,
 $\mu z = -1$, $\mu x = 2$. Thus $h(x, y, z) = 1$ implies $\frac{4}{\mu^2} + 2\left(\frac{1}{\mu^2}\right) = 1$ or $\mu = \pm\sqrt{6}$, so $z = \mp \frac{1}{\sqrt{6}}$; $x = \pm \frac{2}{\sqrt{6}}$; and
 $g(x, y, z) = 0$ implies $y = \mp \frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is
 $f\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) = 2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) = -2\sqrt{6}$.
17. $f(x, y, z) = yz + xy$, $g(x, y, z) = xy = 1$, $h(x, y, z) = y^2 + z^2 = 1 \Rightarrow \nabla f = \langle y, x + z, y \rangle$,
 $\lambda \nabla g = \langle \lambda y, \lambda x, 0 \rangle$, $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $y = \lambda y$ implies $\lambda = 1$ [$y \neq 0$ since $g(x, y, z) = 1$],
 $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm \frac{1}{\sqrt{2}}$,
 $z = \pm \frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\pm\sqrt{2}, \pm \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.
Hence the maximum of f subject to the constraints is $f(\pm\sqrt{2}, \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}) = \frac{3}{2}$ and the minimum is
 $f(\pm\sqrt{2}, \pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}) = \frac{1}{2}$.
Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y, z) = yz + 1$ subject
to $y^2 + z^2 = 1$.
18. $f(x, y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \Rightarrow x = 1, y = 0$. Thus $(1, 0)$ is the only
critical point of f , and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x, y) = x^2 + y^2 = 16 \Rightarrow$
 $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then
 $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1, 0) = -7$, $f(4, 0) = 11$, $f(-4, 0) = 43$, and
 $f(-2, \pm 2\sqrt{3}) = 47$. Thus the maximum value of $f(x, y)$ on the disk $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and
the minimum value is $f(1, 0) = -7$.
19. $f(x, y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$,
so the only critical point is $(0, 0)$, and $f(0, 0) = 1$. For the boundary, we use Lagrange multipliers.
 $g(x, y) = x^2 + 4y^2 = 1 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 8\lambda y \rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and
 $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \Rightarrow$
 $x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm \frac{1}{\sqrt{2}}$ and $y = \pm \frac{1}{2\sqrt{2}}$. Now
 $f(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{2\sqrt{2}}) = e^{1/4} \approx 1.284$ and $f(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{2\sqrt{2}}) = e^{-1/4} \approx 0.779$. The former are the maxima on the
region and the latter are the minima.

20. (a) The graphs of $f(x, y) = 3.7$ and $f(x, y) = 350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x, y)$ subject to the constraint
- $$(x - 3)^2 + (y - 3)^2 = 9.$$



- (b) Let $g(x, y) = (x - 3)^2 + (y - 3)^2$. We calculate $f_x(x, y) = 3x^2 + 3y$,

$$f_y(x, y) = 3y^2 + 3x, g_x(x, y) = 2x - 6, \text{ and } g_y(x, y) = 2y - 6,$$

and use a CAS to search for solutions to the equations

$$g(x, y) = (x - 3)^2 + (y - 3)^2 = 9, f_x = \lambda g_x, \text{ and } f_y = \lambda g_y. \text{ The solutions are}$$

$$(x, y) = \left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) \approx (0.879, 0.879) \text{ and } (x, y) = \left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) \approx (5.121, 5.121).$$

$$\text{These give } f\left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} - \frac{243}{2}\sqrt{2} \approx 3.673 \text{ and}$$

$$f\left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} + \frac{243}{2}\sqrt{2} \approx 347.33, \text{ in accordance with part (a).}$$

21. $P(L, K) = bL^\alpha K^{1-\alpha}$, $g(L, K) = mL + nK = p \Rightarrow \nabla P = \langle \alpha bL^{\alpha-1}K^{1-\alpha}, (1-\alpha)bL^\alpha K^{-\alpha} \rangle$, $\lambda \nabla g = \langle \lambda m, \lambda n \rangle$. Then $\alpha b(K/L)^{1-\alpha} = \lambda m$ and $(1-\alpha)b(L/K)^\alpha = \lambda n$ and $mL + nK = p$, so $\alpha b(K/L)^{1-\alpha}/m = (1-\alpha)b(L/K)^\alpha/n$ or $n\alpha/[m(1-\alpha)] = (L/K)^\alpha(L/K)^{1-\alpha}$ or $L = Kn\alpha/[m(1-\alpha)]$. Substituting into $mL + nK = p$ gives $K = (1-\alpha)p/n$ and $L = \alpha p/m$ for the maximum production.

22. $C(L, K) = mL + nK$, $g(L, K) = bL^\alpha K^{1-\alpha} = Q \Rightarrow \nabla C = \langle m, n \rangle$,

$$\lambda \nabla g = \langle \lambda \alpha bL^{\alpha-1}K^{1-\alpha}, \lambda(1-\alpha)bL^\alpha K^{-\alpha} \rangle. \text{ Then } \frac{m}{\alpha b} \left(\frac{L}{K}\right)^{1-\alpha} = \frac{n}{(1-\alpha)b} \left(\frac{K}{L}\right)^\alpha \text{ and}$$

$$bL^\alpha K^{1-\alpha} = Q \Rightarrow \frac{n\alpha}{m(1-\alpha)} = \left(\frac{L}{K}\right)^{1-\alpha} \left(\frac{L}{K}\right)^\alpha \Rightarrow L = \frac{Kn\alpha}{m(1-\alpha)} \text{ and so}$$

$$b \left[\frac{Kn\alpha}{m(1-\alpha)} \right]^\alpha K^{1-\alpha} = Q.$$

$$\text{Hence } K = \frac{Q}{b(n\alpha/[m(1-\alpha)])^\alpha} = \frac{Qm^\alpha(1-\alpha)^\alpha}{bn^\alpha\alpha^\alpha} \text{ and } L = \frac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}} = \frac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$$

minimizes cost.

23. Let the sides of the rectangle be x and y . Then $f(x, y) = xy$, $g(x, y) = 2x + 2y = p \Rightarrow \nabla f(x, y) = \langle y, x \rangle$, $\lambda \nabla g = \langle 2\lambda, 2\lambda \rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

24. Let $f(x, y, z) = s(s-x)(s-y)(s-z)$, $g(x, y, z) = x + y + z$. Then

$$\nabla f = \langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y) \rangle, \lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle. \text{ Thus}$$

(1) $(s-y)(s-z) = (s-x)(s-z)$ and (2) $(s-x)(s-z) = (s-x)(s-y)$. (1) implies $x = y$ while (2) implies $y = z$, so $x = y = z = p/3$ and the triangle with maximum area is equilateral.

25. Let $f(x, y, z) = d^2 = (x-2)^2 + (y-1)^2 + (z+1)^2$, then we want to minimize f subject to the constraint $g(x, y, z) = x + y - z = 1$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-2), 2(y-1), 2(z+1) \rangle = \lambda \langle 1, 1, -1 \rangle$, so $x = (\lambda + 4)/2$, $y = (\lambda + 2)/2$, $z = -(\lambda + 2)/2$. Substituting into the constraint equation gives $\frac{\lambda + 4}{2} + \frac{\lambda + 2}{2} - \frac{\lambda + 2}{2} = 1 \Rightarrow 3\lambda + 8 = 2 \Rightarrow \lambda = -2$, so $x = 1$, $y = 0$, and $z = 0$. This must correspond to a minimum, so the shortest distance is $d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$.

26. Let $f(x, y, z) = d^2 = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$, then we want to minimize f subject to the constraint $g(x, y, z) = x - y + z = 4$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x - 1), 2(y - 2), 2(z - 3) \rangle = \lambda \langle 1, -1, 1 \rangle$, so $x = (\lambda + 2)/2$, $y = (4 - \lambda)/2$, $z = (\lambda + 6)/2$. Substituting into the constraint equation gives $\frac{\lambda + 2}{2} - \frac{4 - \lambda}{2} + \frac{\lambda + 6}{2} = 4 \Rightarrow \lambda = \frac{4}{3}$, so $x = \frac{5}{3}$, $y = \frac{4}{3}$, and $z = \frac{11}{3}$. This must correspond to a minimum, so the point on the plane closest to the point $(1, 2, 3)$ is $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$.

27. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = z^2 - xy - 1 = 0 \Rightarrow \nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle -\lambda y, -\lambda x, 2\lambda z \rangle$. Then $2z = 2\lambda z$ implies $z = 0$ or $\lambda = 1$. If $z = 0$ then $g(x, y, z) = 1$ implies $xy = -1$ or $x = -1/y$. Thus $2x = -\lambda y$ and $2y = -\lambda x$ imply $\lambda = 2/y^2 = 2y^2$ or $y = \pm 1$, $x = \pm 1$. If $\lambda = 1$, then $2x = -y$ and $2y = -x$ imply $x = y = 0$, so $z = \pm 1$. Hence the possible points are $(\pm 1, \mp 1, 0)$, $(0, 0, \pm 1)$ and the minimum value of f is $f(0, 0, \pm 1) = 1$, so the points closest to the origin are $(0, 0, \pm 1)$.

28. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x^2 y^2 z = 1 \Rightarrow \nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle 2\lambda x y^2 z, 2\lambda x^2 y z, \lambda x^2 y^2 \rangle$. Then $\lambda y^2 z = 1$, $\lambda x^2 z = 1$ and $\lambda x^2 y^2 = 2z$ so $y^2 z = x^2 z$ and $x = \pm y$. Also $2z/1 = \lambda x^2 y^2 / (\lambda x^2 z)$ so $2z^2 = y^2$ and $y = \pm \sqrt{2} z$. But $x^2 y^2 z = 1$ implies $z > 0$ and $4z^5 = 1$. Thus the points are $(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5})$, and the minimum distance is attained at each of these.

29. $f(x, y, z) = xyz$, $g(x, y, z) = x + y + z = 100 \Rightarrow \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Then $\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

30. $f(x, y, z) = x^a y^b z^c$, $g(x, y, z) = x + y + z = 100 \Rightarrow \nabla f = \langle ax^{a-1} y^b z^c, bx^a y^{b-1} z^c, cx^a y^b z^{c-1} \rangle = \lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Then $\lambda = ax^{a-1} y^b z^c = bx^a y^{b-1} z^c = cx^a y^b z^{c-1}$ or $ayz = bxz = cxy$. Thus $x = \frac{ay}{b}$, $z = \frac{cy}{b}$, and $\frac{ay}{b} + y + \frac{cy}{b} = 100$ implies that $y = \frac{100b}{a+b+c}$, $x = \frac{100a}{a+b+c}$ and $z = \frac{100c}{a+b+c}$ gives the maximum.

31. If the dimensions are $2x$, $2y$ and $2z$, then $f(x, y, z) = 8xyz$ and $g(x, y, z) = 9x^2 + 36y^2 + 4z^2 = 36 \Rightarrow \nabla f = \langle 8yz, 8xz, 8xy \rangle = \lambda \nabla g = \langle 18\lambda x, 72\lambda y, 8\lambda z \rangle$. Thus $18\lambda x = 8yz$, $72\lambda y = 8xz$, $8\lambda z = 8xy$ so $x^2 = 4y^2$, $z^2 = 9y^2$ and $36y^2 + 36y^2 + 36y^2 = 36$ or $y = \frac{1}{\sqrt{3}}$ ($y > 0$). Thus the volume of the largest such rectangle is $8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}}\right) = 16\sqrt{3}$.

32. $f(x, y, z) = 8xyz$, $g(x, y, z) = b^2 c^2 x + a^2 c^2 y^2 + a^2 b^2 z^2 = a^2 b^2 c^2 \Rightarrow \nabla f = \langle 8yz, 8xz, 8xy \rangle = \lambda \nabla g = \langle 2\lambda b^2 c^2, 2\lambda a^2 c^2 y, 2\lambda a^2 b^2 z \rangle$. Then $4yz = \lambda b^2 c^2$, $4xz = \lambda a^2 c^2 y$, $4xy = \lambda a^2 b^2 z$ imply $\lambda = \frac{4yz}{b^2 c^2} = \frac{4xz}{a^2 c^2 y} = \frac{4xy}{a^2 b^2 z}$ or $\frac{y}{b^2 x} = \frac{x}{a^2 y}$ and $\frac{z}{c^2 y} = \frac{y}{b^2 z}$. Thus $x = \frac{ay}{b}$, $z = \frac{cy}{b}$, and $a^2 c^2 y^2 + c^2 a^2 y^2 + a^2 c^2 y^2 = a^2 b^2 c^2$, or $y = \frac{b}{\sqrt{3}}$, $x = \frac{a}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$ and the volume is $\frac{8}{3\sqrt{3}} abc$.

33. $f(x, y, z) = xyz, g(x, y, z) = x + 2y + 3z = 6 \Rightarrow \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, 2\lambda, 3\lambda \rangle.$

Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$ implies $x = 2y, z = \frac{2}{3}y$. But $2y + 2y + 2y = 6$ so $y = 1, x = 2, z = \frac{2}{3}$ and the volume is $V = \frac{4}{3}$.

34. $f(x, y, z) = xyz, g(x, y, z) = xy + yz + xz = 32 \Rightarrow$

$\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle$. Then (1) $\lambda(y+z) = yz$, (2) $\lambda(x+z) = xz$ and (3) $\lambda(x+y) = xy$. And (1) minus (2) implies $\lambda(y-x) = z(y-x)$ so $x = y$ or $\lambda = z$. If $\lambda = z$, then (1) implies $z(y+z) = yz$ or $z = 0$ which is false. Thus $x = y$. Similarly (2) minus (3) implies $\lambda(z-y) = x(z-y)$ so $y = z$ or $\lambda = x$. As above, $\lambda \neq x$, so $x = y = z$ and $3x^2 = 32$ or $x = y = z = \frac{8}{\sqrt{6}}$ cm.

35. $f(x, y, z) = xyz, g(x, y, z) = 4(x+y+z) = c \Rightarrow \nabla f = \langle yz, xz, xy \rangle, \lambda \nabla g = \langle 4\lambda, 4\lambda, 4\lambda \rangle$. Thus

$4\lambda = yz = xz = xy$ or $x = y = z = \frac{1}{12}c$ are the dimensions giving the maximum volume.

36. $C(x, y, z) = 5xy + 2xz + 2yz, g(x, y, z) = xyz = V \Rightarrow$

$\nabla C = \langle 5y + 2z, 5x + 2z, 2x + 2y \rangle = \lambda \nabla g = \langle \lambda yz, \lambda xz, \lambda xy \rangle$. Then (1) $\lambda yz = 5y + 2z$, (2) $\lambda xz = 5x + 2z$, (3) $\lambda xy = 2(x+y)$ and (4) $xyz = V$. Now (1)–(2) implies $\lambda z(y-x) = 5(y-x)$, so $x = y$ or $\lambda = 5/z$, but z can't be 0, so $x = y$. Then twice (2) minus five times (3) together with $x = y$ implies $\lambda y(2x - 5y) = 2(2z - 5y)$ which gives $z = \frac{5}{2}y$ [again $\lambda \neq 2/y$ or else (3) implies $y = 0$]. Hence $\frac{5}{2}y^3 = V$ and the dimensions which minimize cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}(\frac{5}{2})^{2/3}$ units.

37. If the dimensions of the box are given by x, y , and z , then we need to find the maximum value of $f(x, y, z) = xyz$

$(x, y, z > 0)$ subject to the constraint $L = \sqrt{x^2 + y^2 + z^2}$ or $g(x, y, z) = x^2 + y^2 + z^2 = L^2$. $\nabla f = \lambda \nabla g \Rightarrow$

$$\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle, \text{ so } yz = 2\lambda x \Rightarrow \lambda = \frac{yz}{2x}, xz = 2\lambda y \Rightarrow \lambda = \frac{xz}{2y}, \text{ and } xy = 2\lambda z \Rightarrow$$

$$\lambda = \frac{xy}{2z}. \text{ Thus } \lambda = \frac{yz}{2x} = \frac{xz}{2y} \Rightarrow x^2 = y^2 \text{ [since } z \neq 0] \Rightarrow x = y \text{ and } \lambda = \frac{yz}{2x} = \frac{xy}{2z} \Rightarrow x = z$$

[since $y \neq 0$]. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = L^2 \Rightarrow x^2 = L^2/3 \Rightarrow$

$x = L/\sqrt{3} = y = z$ and the maximum volume is $(L/\sqrt{3})^3 = L^3/(3\sqrt{3})$.

38. Let the dimensions of the box be x, y , and z , so its volume is $f(x, y, z) = xyz$, its surface area is

$g(x, y, z) = xy + yz + xz = 750$ and its total edge length is $h(x, y, z) = x + y + z = 50$. Then

$\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle + \langle \mu, \mu, \mu \rangle$. So (1) $yz = \lambda(y+z) + \mu$,

(2) $xz = \lambda(x+z) + \mu$, and (3) $xy = \lambda(x+y) + \mu$. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$

but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y, x \neq z$. Then (1) minus (2)

implies $z(y-x) = \lambda(y-x)$ or $\lambda = z$, and (1) minus (3) implies $y(z-x) = \lambda(z-x)$ or $\lambda = y$. So $y = z = \lambda$

and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence

$50 - 2\lambda = \frac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \frac{50 \pm 5\sqrt{10}}{3}$, giving the points

$(\frac{1}{3}(50 \mp 10\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10}))$. Thus the minimum of f is

$f(\frac{1}{3}(50 - 10\sqrt{10}), \frac{1}{3}(50 + 5\sqrt{10}), \frac{1}{3}(50 + 5\sqrt{10})) = \frac{1}{27}(87,500 - 2500\sqrt{10})$, and its

maximum is $f(\frac{1}{3}(50 + 10\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10})) = \frac{1}{27}(87,500 + 2500\sqrt{10})$.

Note: If either y or z is the distinct side, then symmetry gives the same result.

39. We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints

$g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$ and

$\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$. Thus we need (1) $2x = \lambda + 2\mu x$, (2) $2y = \lambda + 2\mu y$, (3) $2z = 2\lambda - \mu$,

(4) $x + y + 2z = 2$, and (5) $x^2 + y^2 - z = 0$. From (1) and (2), $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$.

Putting this in (3) gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into (1) says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then (4) and (5) become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then (4) and (5) become $2x + 2z = 2$ and $2x^2 - z = 0$ which

imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$.

The two points to check are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(-1, -1, 2)$: $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

40. (a) Parametric equations for the ellipse are easiest to determine

using cylindrical coordinates. The cone is given by $z = r$, and

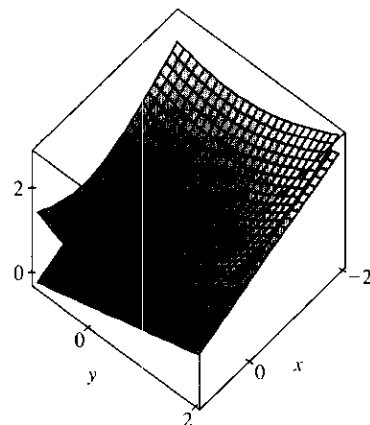
the plane is $4r \cos \theta - 3r \sin \theta + 8z = 5$. Substituting $z = r$

into the plane equation gives $4r \cos \theta - 3r \sin \theta + 8r = 5 \Rightarrow$

$r = \frac{5}{4 \cos \theta - 3 \sin \theta + 8}$. Since $z = r$ on the ellipse,

parametric equations (in cylindrical coordinates) are

$\theta = t, r = z = \frac{5}{4 \cos t - 3 \sin t + 8}, 0 \leq t \leq 2\pi$.



- (b) We need to find the extreme values of $f(x, y, z) = z$ subject to the two constraints

$g(x, y, z) = 4x - 3y + 8z = 5$ and $h(x, y, z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow$

$\langle 0, 0, 1 \rangle = \lambda \langle 4, -3, 8 \rangle + \mu \langle 2x, 2y, -2z \rangle$, so we need (1) $4\lambda + 2\mu x = 0 \Rightarrow x = -\frac{2\lambda}{\mu}$,

(2) $-3\lambda + 2\mu y = 0 \Rightarrow y = \frac{3\lambda}{2\mu}$, (3) $8\lambda - 2\mu z = 1 \Rightarrow z = \frac{8\lambda - 1}{2\mu}$, (4) $4x - 3y + 8z = 5$, and

(5) $x^2 + y^2 = z^2$. [Note that $\mu \neq 0$, else $\lambda = 0$ from (1), but substitution into (3) gives a contradiction.]

Substituting (1), (2), and (3) into (4) gives $4(-\frac{2\lambda}{\mu}) - 3(\frac{3\lambda}{2\mu}) + 8(\frac{8\lambda - 1}{2\mu}) = 5 \Rightarrow \mu = \frac{39\lambda - 8}{10}$ and into (5)

gives $(-\frac{2\lambda}{\mu})^2 + (\frac{3\lambda}{2\mu})^2 = (\frac{8\lambda - 1}{2\mu})^2 \Rightarrow 16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2 \Rightarrow 39\lambda^2 - 16\lambda + 1 = 0 \Rightarrow$

$\lambda = \frac{1}{13}$ or $\lambda = \frac{1}{3}$. If $\lambda = \frac{1}{13}$ then $\mu = -\frac{1}{2}$ and $x = \frac{4}{13}, y = -\frac{3}{13}, z = \frac{5}{13}$. If $\lambda = \frac{1}{3}$ then $\mu = \frac{1}{2}$ and $x = -\frac{4}{3}, y = 1, z = \frac{5}{3}$. Thus the highest point on the ellipse is $(-\frac{4}{3}, 1, \frac{5}{3})$ and the lowest point is $(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13})$.

41. $f(x, y, z) = ye^{x-z}, g(x, y, z) = 9x^2 + 4y^2 + 36z^2 = 36, h(x, y, z) = xy + yz = 1.$

$\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow \langle ye^{x-z}, e^{x-z}, -ye^{x-z} \rangle = \lambda \langle 18x, 8y, 72z \rangle + \mu \langle y, x+z, y \rangle$, so $ye^{x-z} = 18\lambda x + \mu y$, $e^{x-z} = 8\lambda y + \mu(x+z)$, $-ye^{x-z} = 72\lambda z + \mu y$, $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ , and μ (in Maple, use the `allvalues` command), we get 4 real-valued solutions:

$$x \approx 0.222444, \quad y \approx -2.157012, \quad z \approx -0.686049, \quad \lambda \approx -0.200401, \quad \mu \approx 2.108584$$

$$x \approx -1.951921, \quad y \approx -0.545867, \quad z \approx 0.119973, \quad \lambda \approx 0.003141, \quad \mu \approx -0.076238$$

$$x \approx 0.155142, \quad y \approx 0.904622, \quad z \approx 0.950293, \quad \lambda \approx -0.012447, \quad \mu \approx 0.489938$$

$$x \approx 1.138731, \quad y \approx 1.768057, \quad z \approx -0.573138, \quad \lambda \approx 0.317141, \quad \mu \approx 1.862675$$

Substituting these values into f gives $f(0.222444, -2.157012, -0.686049) \approx -5.3506$,

$f(-1.951921, -0.545867, 0.119973) \approx -0.0688$, $f(0.155142, 0.904622, 0.950293) \approx 0.4084$,

$f(1.138731, 1.768057, -0.573138) \approx 9.7938$. Thus the maximum is approximately 9.7938, and the minimum is approximately -5.3506.

42. $f(x, y, z) = x + y + z, g(x, y, z) = x^2 - y^2 - z = 0, h(x, y, z) = x^2 + z^2 = 4.$

$\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow \langle 1, 1, 1 \rangle = \lambda \langle 2x, -2y, -1 \rangle + \mu \langle 2x, 0, 2z \rangle$, so $1 = 2\lambda x + 2\mu x$, $1 = -2\lambda y$,

$1 = -\lambda + 2\mu z$, $x^2 - y^2 = z$, $x^2 + z^2 = 4$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ , and μ , we get 4 real-valued solutions:

$$x \approx -1.652878, \quad y \approx -1.964194, \quad z \approx -1.126052, \quad \lambda \approx 0.254557, \quad \mu \approx -0.557060$$

$$x \approx -1.502800, \quad y \approx 0.968872, \quad z \approx 1.319694, \quad \lambda \approx -0.516064, \quad \mu \approx 0.183352$$

$$x \approx -0.992513, \quad y \approx 1.649677, \quad z \approx -1.736352, \quad \lambda \approx -0.303090, \quad \mu \approx -0.200682$$

$$x \approx 1.895178, \quad y \approx 1.718347, \quad z \approx 0.638984, \quad \lambda \approx -0.290977, \quad \mu \approx 0.554805$$

Substituting these values into f gives $f(-1.652878, -1.964194, -1.126052) \approx -4.7431$,

$f(-1.502800, 0.968872, 1.319694) \approx 0.7858$, $f(-0.992513, 1.649677, -1.736352) \approx -1.0792$,

$f(1.895178, 1.718347, 0.638984) \approx 4.2525$. Thus the maximum is approximately 4.2525, and the minimum is approximately -4.7431.

43. (a) We wish to maximize $f(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to

$$g(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n = c \text{ and } x_i > 0.$$

$$\nabla f = \left\langle \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n), \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n), \dots, \right.$$

$$\left. \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$$

and $\lambda \nabla g = \langle \lambda, \lambda, \dots, \lambda \rangle$, so we need to solve the system of equations

$$\begin{aligned}\frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n) &= \lambda \Rightarrow x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_1 \\ \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n) &= \lambda \Rightarrow x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_2 \\ &\vdots \\ \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) &= \lambda \Rightarrow x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_n\end{aligned}$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus

$$x_1 = x_2 = \cdots = x_n. \text{ But } x_1 + x_2 + \cdots + x_n = c \Rightarrow nx_1 = c \Rightarrow x_1 = \frac{c}{n} = x_2 = x_3 = \cdots = x_n.$$

Then the only point where f can have an extreme value is $\left(\frac{c}{n}, \frac{c}{n}, \dots, \frac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \dots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is $f\left(\frac{c}{n}, \frac{c}{n}, \dots, \frac{c}{n}\right) = \sqrt[n]{\frac{c}{n} \cdot \frac{c}{n} \cdots \frac{c}{n}} = \frac{c}{n}$.

(b) From part (a), $\frac{c}{n}$ is the maximum value of f . Thus $f(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{c}{n}$. But

$x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\frac{c}{n}$, but this can occur only at the point $\left(\frac{c}{n}, \frac{c}{n}, \dots, \frac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \frac{c}{n}$.

44. (a) Let $f(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i=1}^n x_i y_i$, $g(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$, and $h(x_1, \dots, x_n) = \sum_{i=1}^n y_i^2$. Then

$$\nabla f = \nabla \sum_{i=1}^n x_i y_i = \langle y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_n \rangle, \nabla g = \nabla \sum_{i=1}^n x_i^2 = \langle 2x_1, 2x_2, \dots, 2x_n, 0, 0, \dots, 0 \rangle$$

$$\text{and } \nabla h = \nabla \sum_{i=1}^n y_i^2 = \langle 0, 0, \dots, 0, 2y_1, 2y_2, \dots, 2y_n \rangle. \text{ So } \nabla f = \lambda \nabla g + \mu \nabla h \Leftrightarrow y_i = 2\lambda x_i$$

$$\text{and } x_i = 2\mu y_i, 1 \leq i \leq n. \text{ Then } 1 = \sum_{i=1}^n y_i^2 = \sum_{i=1}^n 4\lambda^2 x_i^2 = 4\lambda^2 \sum_{i=1}^n x_i^2 = 4\lambda^2 \Rightarrow \lambda = \pm \frac{1}{2}.$$

If $\lambda = \frac{1}{2}$ then $y_i = 2\left(\frac{1}{2}\right)x_i = x_i, 1 \leq i \leq n$. Thus $\sum_{i=1}^n x_i y_i = \sum_{i=1}^n x_i^2 = 1$. Similarly if $\lambda = -\frac{1}{2}$ we get

$y_i = -x_i$ and $\sum_{i=1}^n x_i y_i = -1$. Similarly we get $\mu = \pm \frac{1}{2}$ giving $y_i = \pm x_i, 1 \leq i \leq n$, and $\sum_{i=1}^n x_i y_i = \pm 1$.

Thus the maximum value of $\sum_{i=1}^n x_i y_i$ is 1.

(b) Here we assume $\sum_{i=1}^n a_i^2 \neq 0$ and $\sum_{i=1}^n b_i^2 \neq 0$. (If $\sum_{i=1}^n a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially

$$\text{true.) } x_i = \frac{a_i}{\sqrt{\sum_{j=1}^n a_j^2}} \Rightarrow \sum x_i^2 = \frac{\sum a_i^2}{\sum a_j^2} = 1, \text{ and } y_i = \frac{b_i}{\sqrt{\sum_{j=1}^n b_j^2}} \Rightarrow \sum y_i^2 = \frac{\sum b_i^2}{\sum b_j^2} = 1. \text{ Therefore,}$$

$$\text{from (a), } \sum x_i y_i = \sum \frac{a_i b_i}{\sqrt{\sum_{j=1}^n a_j^2} \sqrt{\sum_{j=1}^n b_j^2}} \leq 1 \Leftrightarrow \sum a_i b_i \leq \sqrt{\sum_{j=1}^n a_j^2} \sqrt{\sum_{j=1}^n b_j^2}.$$

APPLIED PROJECT Rocket Science

1. Initially the rocket engine has mass $M_r = M_1$ and payload mass $P = M_2 + M_3 + A$. Then the change in velocity resulting from the first stage is $\Delta V_1 = -c \ln \left(1 - \frac{(1-S)M_1}{M_2 + M_3 + A + M_1} \right)$. After the first stage is jettisoned we can consider the rocket engine to have mass $M_r = M_2$ and the payload to have mass $P = M_3 + A$. The resulting change in velocity from the second stage is $\Delta V_2 = -c \ln \left(1 - \frac{(1-S)M_2}{M_3 + A + M_2} \right)$. When only the third stage remains, we have $M_r = M_3$ and $P = A$, so the resulting change in velocity is $\Delta V_3 = -c \ln \left(1 - \frac{(1-S)M_3}{A + M_3} \right)$. Since the rocket started from rest, the final velocity attained is

$$\begin{aligned} v_f &= \Delta V_1 + \Delta V_2 + \Delta V_3 \\ &= -c \ln \left(1 - \frac{(1-S)M_1}{M_2 + M_3 + A + M_1} \right) + (-c) \ln \left(1 - \frac{(1-S)M_2}{M_3 + A + M_2} \right) \\ &\quad + (-c) \ln \left(1 - \frac{(1-S)M_3}{A + M_3} \right) \\ &= -c \left[\ln \left(\frac{M_1 + M_2 + M_3 + A - (1-S)M_1}{M_1 + M_2 + M_3 + A} \right) + \ln \left(\frac{M_2 + M_3 + A - (1-S)M_2}{M_2 + M_3 + A} \right) \right. \\ &\quad \left. + \ln \left(\frac{M_3 + A - (1-S)M_3}{M_3 + A} \right) \right] \\ &= c \left[\ln \left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} \right) + \ln \left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A} \right) + \ln \left(\frac{M_3 + A}{SM_3 + A} \right) \right] \end{aligned}$$

2. Define $N_1 = \frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}$, $N_2 = \frac{M_2 + M_3 + A}{SM_2 + M_3 + A}$, and $N_3 = \frac{M_3 + A}{SM_3 + A}$. Then

$$\begin{aligned} \frac{(1-S)N_1}{1-SN_1} &= \frac{(1-S) \frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}}{1 - S \frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}} \\ &= \frac{(1-S)(M_1 + M_2 + M_3 + A)}{SM_1 + M_2 + M_3 + A - S(M_1 + M_2 + M_3 + A)} \\ &= \frac{(1-S)(M_1 + M_2 + M_3 + A)}{(1-S)(M_2 + M_3 + A)} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} \end{aligned}$$

as desired.

Similarly,

$$\frac{(1-S)N_2}{1-SN_2} = \frac{(1-S)(M_2 + M_3 + A)}{SM_2 + M_3 + A - S(M_2 + M_3 + A)} = \frac{(1-S)(M_2 + M_3 + A)}{(1-S)(M_3 + A)} = \frac{M_2 + M_3 + A}{M_3 + A}$$

and

$$\frac{(1-S)N_3}{1-SN_3} = \frac{(1-S)(M_3 + A)}{SM_3 + A - S(M_3 + A)} = \frac{(1-S)(M_3 + A)}{(1-S)A} = \frac{M_3 + A}{A}$$

Then

$$\begin{aligned}\frac{M+A}{A} &= \frac{M_1 + M_2 + M_3 + A}{A} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} \cdot \frac{M_2 + M_3 + A}{M_3 + A} \cdot \frac{M_3 + A}{A} \\ &= \frac{(1-S)N_1}{1-SN_1} \cdot \frac{(1-S)N_2}{1-SN_2} \cdot \frac{(1-S)N_3}{1-SN_3} = \frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}\end{aligned}$$

3. Since $A > 0$, $M + A$ and consequently $\frac{M+A}{A}$ is minimized for the same values as M . $\ln x$ is a strictly increasing function, so $\ln\left(\frac{M+A}{A}\right)$ must give a minimum for the same values as $\frac{M+A}{A}$ and hence M . We then wish to minimize $\ln\left(\frac{M+A}{A}\right)$ subject to the constraint $c(\ln N_1 + \ln N_2 + \ln N_3) = v_f$. From Problem 2,

$$\begin{aligned}\ln\left(\frac{M+A}{A}\right) &= \ln\left(\frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}\right) \\ &= 3\ln(1-S) + \ln N_1 + \ln N_2 + \ln N_3 - \ln(1-SN_1) - \ln(1-SN_2) - \ln(1-SN_3)\end{aligned}$$

Using the method of Lagrange multipliers, we need to solve $\nabla\left[\ln\left(\frac{M+A}{A}\right)\right] = \lambda \nabla[c(\ln N_1 + \ln N_2 + \ln N_3)]$ with $c(\ln N_1 + \ln N_2 + \ln N_3) = v_f$ in terms of N_1 , N_2 , and N_3 . The resulting system is

$$\begin{aligned}\frac{1}{N_1} + \frac{S}{1-SN_1} &= \lambda \frac{c}{N_1} & \frac{1}{N_2} + \frac{S}{1-SN_2} &= \lambda \frac{c}{N_2} & \frac{1}{N_3} + \frac{S}{1-SN_3} &= \lambda \frac{c}{N_3} \\ c(\ln N_1 + \ln N_2 + \ln N_3) &= v_f\end{aligned}$$

One approach to solving the system is isolating $c\lambda$ in the first three equations which gives

$$\begin{aligned}1 + \frac{SN_1}{1-SN_1} &= c\lambda = 1 + \frac{SN_2}{1-SN_2} = 1 + \frac{SN_3}{1-SN_3} \Rightarrow \frac{N_1}{1-SN_1} = \frac{N_2}{1-SN_2} = \frac{N_3}{1-SN_3} \Rightarrow \\ N_1 &= N_2 = N_3 \text{ (Verify!)}. \text{ This says the fourth equation can be expressed as } c(\ln N_1 + \ln N_1 + \ln N_1) = v_f \Rightarrow \\ 3c \ln N_1 &= v_f \Rightarrow \ln N_1 = \frac{v_f}{3c}. \text{ Thus the minimum mass } M \text{ of the rocket engine is attained for} \\ N_1 &= N_2 = N_3 = e^{v_f/(3c)}.\end{aligned}$$

4. Using the previous results,

$$\begin{aligned}\frac{M+A}{A} &= \frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)} = \frac{(1-S)^3 [e^{v_f/(3c)}]^3}{[1 - Se^{v_f/(3c)}]^3} = \frac{(1-S)^3 e^{v_f/c}}{[1 - Se^{v_f/(3c)}]^3}. \text{ Then} \\ M &= \frac{A(1-S)^3 e^{v_f/c}}{[1 - Se^{v_f/(3c)}]^3} - A.\end{aligned}$$

5. (a) From Problem 4, $M = \frac{A(1-0.2)^3 e^{(17,500/6000)}}{(1-0.2e^{[17,500/(3 \cdot 6000)]})^3} - A \approx 90.4A - A = 89.4A$.

$$\begin{aligned}\text{(b) First, } N_3 &= \frac{M_3 + A}{SM_3 + A} \Rightarrow e^{[17,500/(3 \cdot 6000)]} = \frac{M_3 + A}{0.2M_3 + A} \Rightarrow M_3 = \frac{A(1 - e^{35/36})}{0.2e^{35/36} - 1} \approx 3.49A. \text{ Then} \\ N_2 &= \frac{M_2 + M_3 + A}{SM_2 + M_3 + A} = \frac{M_2 + 3.49A + A}{0.2M_2 + 3.49A + A} \Rightarrow M_2 = \frac{4.49A(1 - e^{35/36})}{0.2e^{35/36} - 1} \approx 15.67A \\ \text{and } N_3 &= \frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} = \frac{M_1 + 15.67A + 3.49A + A}{0.2M_1 + 15.67A + 3.49A + A} \Rightarrow \\ M_1 &= \frac{20.16A(1 - e^{35/36})}{0.2e^{35/36} - 1} \approx 70.36A.\end{aligned}$$

6. As in Problem 5, $N_3 = \frac{M_3 + A}{SM_3 + A} \Rightarrow e^{24.700/(3.6000)} = \frac{M_3 + A}{0.2M_3 + A} \Rightarrow$
 $M_3 = \frac{A(1 - e^{247/180})}{0.2e^{247/180} - 1} \approx 13.9A$, $N_2 = \frac{M_2 + M_3 + A}{SM_2 + M_3 + A} = \frac{M_2 + 13.9A + A}{0.2M_2 + 13.9A + A} \Rightarrow$
 $M_2 = \frac{14.9A(1 - e^{247/180})}{0.2e^{247/180} - 1} \approx 208A$, and $N_3 = \frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} = \frac{M_1 + 208A + 13.9A + A}{0.2M_1 + 208A + 13.9A + A} \Rightarrow$
 $M_1 = \frac{222.9A(1 - e^{247/180})}{0.2e^{247/180} - 1} \approx 3110A$. Here $A = 500$, so the mass of each stage of the rocket engine is
 approximately $M_1 = 3110(500) = 1,550,000$ lb, $M_2 = 208(500) = 104,000$ lb, and
 $M_3 = 13.9(500) = 6950$ lb.

APPLIED PROJECT Hydro-Turbine Optimization

1. We wish to maximize the total energy production for a given total flow, so we can say Q_T is fixed and we want to maximize $KW_1 + KW_2 + KW_3$. Notice each KW_i has a constant factor $(170 - 1.6 \cdot 10^{-6}Q_T^2)$, so to simplify the computations we can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2, Q_3) &= \frac{KW_1 + KW_2 + KW_3}{170 - 1.6 \cdot 10^{-6}Q_T^2} \\ &= (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5}Q_1^2) \\ &\quad + (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5}Q_2^2) \\ &\quad + (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5}Q_3^2) \end{aligned}$$

subject to the constraint $g(Q_1, Q_2, Q_3) = Q_1 + Q_2 + Q_3 = Q_T$. So first we find the values of Q_1, Q_2, Q_3 where $\nabla f(Q_1, Q_2, Q_3) = \lambda \nabla g(Q_1, Q_2, Q_3)$ and $Q_1 + Q_2 + Q_3 = Q_T$ which is equivalent to solving the system

$$0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = \lambda$$

$$0.1358 - 2(4.69 \cdot 10^{-5})Q_2 = \lambda$$

$$0.1380 - 2(3.84 \cdot 10^{-5})Q_3 = \lambda$$

$$Q_1 + Q_2 + Q_3 = Q_T$$

Comparing the first and third equations, we have $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \Rightarrow$
 $Q_1 = -126.2255 + 0.9412Q_3$. From the second and third equations,

$0.1358 - 2(4.69 \cdot 10^{-5})Q_2 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \Rightarrow Q_2 = -23.4542 + 0.8188Q_3$. Substituting
 into $Q_1 + Q_2 + Q_3 = Q_T$ gives $(-126.2255 + 0.9412Q_3) + (-23.4542 + 0.8188Q_3) + Q_3 = Q_T \Rightarrow$
 $2.76Q_3 = Q_T + 149.6797 \Rightarrow Q_3 = 0.3623Q_T + 54.23$. Then

$Q_1 = -126.2255 + 0.9412Q_3 = -126.2255 + 0.9412(0.3623Q_T + 54.23) = 0.3410Q_T - 75.18$ and

$Q_2 = -23.4542 + 0.8188(0.3623Q_T + 54.23) = 0.2967Q_T + 20.95$. As long as we maintain

$250 \leq Q_1 \leq 1110$, $250 \leq Q_2 \leq 1110$, and $250 \leq Q_3 \leq 1225$, we can reason from the nature of the functions KW_i that these values give a maximum of f , and hence a maximum energy production, and not a minimum.

2. From Problem 1, the value of Q_1 that maximizes energy production is $0.3410Q_T - 75.18$, but since $250 \leq Q_1 \leq 1110$, we must have $250 \leq 0.3410Q_T - 75.18 \leq 1110 \Rightarrow 325.18 \leq 0.3410Q_T \leq 1185.18 \Rightarrow 953.6 \leq Q_T \leq 3475.6$. Similarly, $250 \leq Q_2 \leq 1110 \Rightarrow 250 \leq 0.2967Q_T + 20.95 \leq 1110 \Rightarrow 772.0 \leq Q_T \leq 3670.5$, and $250 \leq Q_3 \leq 1225 \Rightarrow 250 \leq 0.3623Q_T + 54.23 \leq 1225 \Rightarrow 540.4 \leq Q_T \leq 3231.5$. Consolidating these results, we see that the values from Problem 1 are applicable only for $953.6 \leq Q_T \leq 3231.5$.
3. If $Q_T = 2500$, the results from Problem 1 show that the maximum energy production occurs for

$$Q_1 = 0.3410Q_T - 75.18 = 0.3410(2500) - 75.18 = 777.3$$

$$Q_2 = 0.2967Q_T + 20.95 = 0.2967(2500) + 20.95 = 762.7$$

$$Q_3 = 0.3623Q_T + 54.23 = 0.3623(2500) + 54.23 = 960.0$$

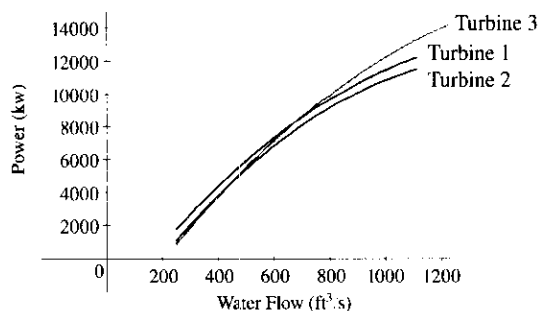
The energy produced for these values is $KW_1 + KW_2 + KW_3 \approx 8915.2 + 8285.1 + 11,211.3 \approx 28,411.6$. We compute the energy production for a nearby distribution, $Q_1 = 770$, $Q_2 = 760$, and $Q_3 = 970$:

$KW_1 + KW_2 + KW_3 \approx 8839.8 + 8257.4 + 11,313.5 = 28,410.7$. For another example, we take $Q_1 = 780$, $Q_2 = 765$, and $Q_3 = 955$: $KW_1 + KW_2 + KW_3 \approx 8942.9 + 8308.8 + 11,159.7 = 28,411.4$. These distributions are both close to the distribution from Problem 1 and both give slightly lower energy productions, suggesting that $Q_1 = 777.3$, $Q_2 = 762.7$, and $Q_3 = 960.0$ is indeed the optimal distribution.

4. First we graph each power function in its domain if all of the flow is directed to that turbine (so $Q_i = Q_T$).

If we use only one turbine, the graph indicates that for a water flow of $1000 \text{ ft}^3/\text{s}$, Turbine 3 produces the most power, approximately $12,200 \text{ kW}$. In comparison, if we use all three turbines, the results of Problem 1 with $Q_T = 1000$ give $Q_1 = 265.8$, $Q_2 = 317.7$, and $Q_3 = 416.5$, resulting in a

total energy production of $KW_1 + KW_2 + KW_3 \approx 8397.4 \text{ kW}$. Here, using only one turbine produces significantly more energy! If the flow is only $600 \text{ ft}^3/\text{s}$, we do not have the option of using all three turbines, as the domain restrictions require a minimum of $250 \text{ ft}^3/\text{s}$ in each turbine. We can use just one turbine, then, and from the graph Turbine 1 produces the most energy for a water flow of 600 ft^3 .



5. If we examine the graph from Problem 4, we see that for water flows above approximately $450 \text{ ft}^3/\text{s}$, Turbine 2 produces the least amount of power. Therefore it seems reasonable to assume that we should distribute the incoming flow of $1500 \text{ ft}^3/\text{s}$ between Turbines 1 and 3. (This can be verified by computing the power produced with the other pairs of turbines for comparison.) So now we wish to maximize $KW_1 + KW_3$ subject to the constraint $Q_1 + Q_3 = Q_T$ where $Q_T = 1500$.

As in Problem 1, we can equivalently maximize

$$\begin{aligned} f(Q_1, Q_3) &= \frac{KW_1 + KW_3}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\ &= (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2) + (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5} Q_3^2) \end{aligned}$$

subject to the constraint $g(Q_1, Q_3) = Q_1 + Q_3 = Q_T$.

Then we solve $\nabla f(Q_1, Q_3) = \lambda \nabla g(Q_1, Q_3) \Rightarrow 0.1277 - 2(4.08 \cdot 10^{-5}) Q_1 = \lambda$ and $0.1380 - 2(3.84 \cdot 10^{-5}) Q_3 = \lambda$, thus $0.1277 - 2(4.08 \cdot 10^{-5}) Q_1 = 0.1380 - 2(3.84 \cdot 10^{-5}) Q_3 \Rightarrow Q_1 = -126.2255 + 0.9412Q_3$. Substituting into $Q_1 + Q_3 = Q_T$ gives $-126.2255 + 0.9412Q_3 + Q_3 = 1500 \Rightarrow Q_3 \approx 837.7$, and then $Q_1 = Q_T - Q_3 \approx 1500 - 837.7 = 662.3$. So we should apportion approximately 662.3 ft³/s to Turbine 1 and the remaining 837.7 ft³/s to Turbine 3. The resulting energy production is $KW_1 + KW_3 \approx 7952.1 + 10,256.2 = 18,208.3$ kW. (We can verify that this is indeed a maximum energy production by checking nearby distributions.) In comparison, if we use all three turbines with $Q_T = 1500$ we get $Q_1 = 436.3$, $Q_2 = 466.0$, and $Q_3 = 597.7$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx 16,538.7$ kW. Clearly, for this flow level it is beneficial to use only two turbines.

6. Note that an incoming flow of 3400 ft³/s is not within the domain we established in Problem 2, so we cannot simply use our previous work to give the optimal distribution. We will need to use all three turbines, due to the capacity limitations of each individual turbine, but 3400 is less than the maximum combined capacity of 3445 ft³/s, so we still must decide how to distribute the flows. From the graph in Problem 4, Turbine 3 produces the most power for the higher flows, so it seems reasonable to use Turbine 3 at its maximum capacity of 1225 and distribute the remaining 2175 ft³/s flow between Turbines 1 and 2. We can again use the technique of Lagrange multipliers to determine the optimal distribution. Following the procedure we used in Problem 5, we wish to maximize $KW_1 + KW_2$ subject to the constraint $Q_1 + Q_2 = Q_T$ where $Q_T = 2175$. We can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2) &= \frac{KW_1 + KW_2}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\ &= (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2) + (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2) \end{aligned}$$

subject to the constraint $g(Q_1, Q_2) = Q_1 + Q_2 = Q_T$. Then we solve $\nabla f(Q_1, Q_2) = \lambda \nabla g(Q_1, Q_2) \Rightarrow$

$$0.1277 - 2(4.08 \cdot 10^{-5}) Q_1 = \lambda \text{ and } 0.1358 - 2(4.69 \cdot 10^{-5}) Q_2 = \lambda, \text{ thus}$$

$$0.1277 - 2(4.08 \cdot 10^{-5}) Q_1 = 0.1358 - 2(4.69 \cdot 10^{-5}) Q_2 \Rightarrow Q_1 = -99.2647 + 1.1495Q_2. \text{ Substituting}$$

into $Q_1 + Q_2 = Q_T$ gives $-99.2647 + 1.1495Q_2 + Q_2 = 2175 \Rightarrow Q_2 \approx 1058.0$, and then $Q_1 \approx 1117.0$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, 1110 and 1225 ft³/s, and the remaining 1065 ft³/s to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.

15 Review

ET 14

CONCEPT CHECK

- (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

(b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy -plane. Also, we can use an arrow diagram such as Figure 1 in Section 15.1 [ET 14.1].
- A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.
- $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f . We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.
- (a) See Definition 15.2.4 [ET 14.2.4].

(b) If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.
- (a) See (2) and (3) in Section 15.3 [ET 14.3].

(b) See "Interpretations of Partial Derivatives" on page 948 [ET 912].

(c) To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x . To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .
- See the statement of Clairaut's Theorem on page 952 [ET 916].
- (a) See (2) in Section 15.4 [ET 14.4].

(b) See (19) and the preceding discussion in Section 15.6 [ET 14.6].
- See (3) and (4) and the accompanying discussion in Section 15.4 [ET 14.4]. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b) . Thus it is the linear function which best approximates f near (a, b) .
- (a) See Definition 15.4.7 [ET 14.4.7].

(b) Use Theorem 15.4.8 [ET 14.4.8].
- See (10) and the associated discussion in Section 15.4 [ET 14.4].
- See (2) and (3) in Section 15.5 [ET 14.5].
- See (7) and the preceding discussion in Section 15.5 [ET 14.5].
- (a) See Definition 15.6.2 [ET 14.6.2]. We can interpret it as the rate of change of f at (x_0, y_0) in the direction of $\mathbf{u}$. Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction $\mathbf{u}$, the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the slope of the tangent line to C at P . (See Figure 5 in Section 15.6 [ET 14.6].)

(b) See Theorem 15.6.3 [ET 14.6.3].
- (a) See (8) and (13) in Section 15.6 [ET 14.6].

(b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ or $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

(c) The gradient vector of a function points in the direction of maximum rate of increase of the function. On a graph of the function, the gradient points in the direction of steepest ascent.

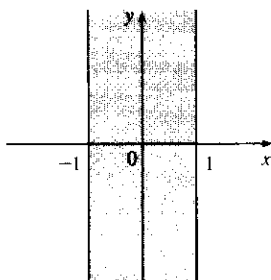
15. (a) f has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) .
 (b) f has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain of f .
 (c) f has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) .
 (d) f has an absolute minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in the domain of f .
 (e) f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.
16. (a) By Theorem 15.7.2 [ET 14.7.2], if f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
 (b) A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.
17. See (3) in Section 15.7 [ET 14.7].
18. (a) See Figure 11 and the accompanying discussion in Section 15.7 [ET 14.7].
 (b) See Theorem 15.7.8 [ET 14.7.8].
 (c) See the procedure outlined in (9) in Section 15.7 [ET 14.7].
19. See the discussion beginning on page 1001 [ET 965]; see the discussion preceding Example 5 in Section 15.8 [ET 14.8].

 TRUE-FALSE QUIZ

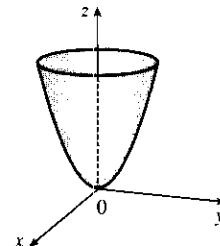
1. True. $f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$ from Equation 15.3.3 [ET 14.3.3]. Let $h = y - b$. As $h \rightarrow 0$, $y \rightarrow b$.
 Then by substituting, we get $f_y(a, b) = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b}$.
2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.
3. False. $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$.
4. True. From Equation 15.6.14 [ET 14.6.14] we get $D_{\mathbf{k}} f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.
5. False. See Example 15.2.3 [ET 14.2.3].
6. False. See Exercise 15.4.42(a) [ET 14.4.42(a)].
7. True. If f has a local minimum and f is differentiable at (a, b) then by Theorem 15.7.2 [ET 14.7.2], $f_x(a, b) = 0$ and $f_y(a, b) = 0$, so $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = \langle 0, 0 \rangle = \mathbf{0}$.
8. False. If f is not continuous at $(2, 5)$, then we can have $\lim_{(x, y) \rightarrow (2, 5)} f(x, y) \neq f(2, 5)$.
 (See Example 15.2.7 [ET 14.2.7].)
9. False. $\nabla f(x, y) = \langle 0, 1/y \rangle$.
10. True. This is part (c) of the Second Derivatives Test (15.7.3 [ET 14.7.3]).
11. True. $\nabla f = \langle \cos x, \cos y \rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos \theta| \leq 1$, so $|\nabla f| \leq \sqrt{2}$. Now $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}} f(x, y)| \leq \sqrt{2} \cdot 1 \cdot 1 = \sqrt{2}$.
12. False. See Exercise 15.7.35 [14.7.35].

EXERCISES

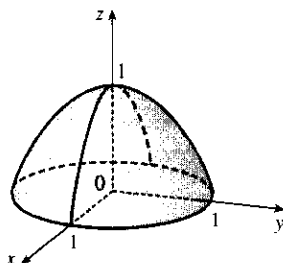
1. The domain of $\sin^{-1} x$ is $-1 \leq x \leq 1$ while the domain of $\tan^{-1} y$ is all real numbers, so the domain of $f(x, y) = \sin^{-1} x + \tan^{-1} y$ is $\{(x, y) \mid -1 \leq x \leq 1\}$.



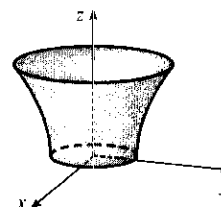
2. $D = \{(x, y, z) \mid z \geq x^2 + y^2\}$, the points on and above the paraboloid $z = x^2 + y^2$.



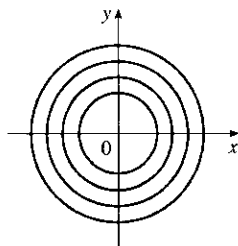
3. $z = f(x, y) = 1 - x^2 - y^2$, a paraboloid with vertex $(0, 0, 1)$.



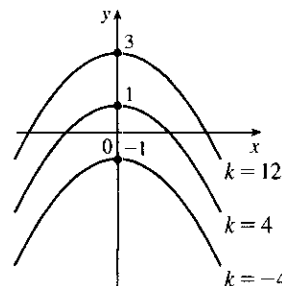
4. $z = f(x, y) = \sqrt{x^2 + y^2 - 1}$, so $z \geq 0$ and $1 = x^2 + y^2 - z^2$. Thus the graph is the upper half of a hyperboloid of one sheet.



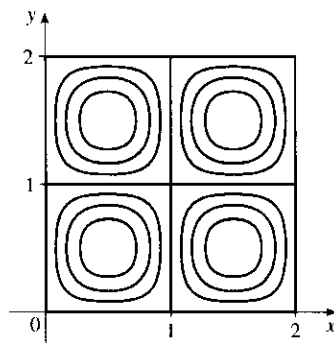
5. Let $k = e^{-c} = e^{-(x^2+y^2)}$ be the level curves. Then $-\ln k = c = x^2 + y^2$, so we have a family of concentric circles.



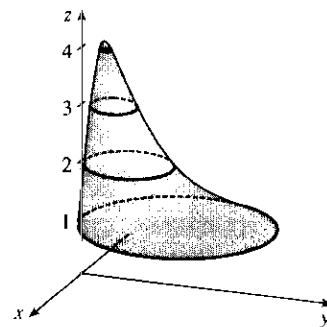
6. $k = x^2 + 4y$ or $4(y - k/4) = -x^2$, a family of parabolas with vertex at $(0, k/4)$.



7.



8.



9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1, 1)$, we use direct substitution to

evaluate the limit: $\lim_{(x,y) \rightarrow (1,1)} \frac{2xy}{x^2 + 2y^2} = \frac{2(1)(1)}{1^2 + 2(1)^2} = \frac{2}{3}$.

10. As $(x, y) \rightarrow (0, 0)$ along the x -axis, $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ along this line. But $f(x, x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x, y) \rightarrow (0, 0)$ along the line $x = y$, $f(x, y) \rightarrow \frac{2}{3}$. Thus the limit doesn't exist.

11. (a) $T_x(6, 4) = \lim_{h \rightarrow 0} \frac{T(6+h, 4) - T(6, 4)}{h}$, so we can approximate $T_x(6, 4)$ by considering $h = \pm 2$ and using the

values given in the table: $T_x(6, 4) \approx \frac{T(8, 4) - T(6, 4)}{2} = \frac{86 - 80}{2} = 3$,

$T_x(6, 4) \approx \frac{T(4, 4) - T(6, 4)}{-2} = \frac{72 - 80}{-2} = 4$. Averaging these values, we estimate $T_x(6, 4)$ to be

approximately $3.5^\circ\text{C}/\text{m}$. Similarly, $T_y(6, 4) = \lim_{h \rightarrow 0} \frac{T(6, 4+h) - T(6, 4)}{h}$, which we can

approximate with $h = \pm 2$: $T_y(6, 4) \approx \frac{T(6, 6) - T(6, 4)}{2} = \frac{75 - 80}{2} = -2.5$,

$T_y(6, 4) \approx \frac{T(6, 2) - T(6, 4)}{-2} = \frac{87 - 80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6, 4)$ to be approximately $-3.0^\circ\text{C}/\text{m}$.

- (b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 15.6.9 [ET 14.6.9],

$D_{\mathbf{u}} T(6, 4) = \nabla T(6, 4) \cdot \mathbf{u} = T_x(6, 4) \frac{1}{\sqrt{2}} + T_y(6, 4) \frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have

$D_{\mathbf{u}} T(6, 4) \approx (3.5) \frac{1}{\sqrt{2}} + (-3.0) \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6, 4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35^\circ\text{C}/\text{m}$.

Alternatively, we can use Definition 15.6.2 [ET 14.6.2]:

$D_{\mathbf{u}} T(6, 4) = \lim_{h \rightarrow 0} \frac{T\left(6 + h \frac{1}{\sqrt{2}}, 4 + h \frac{1}{\sqrt{2}}\right) - T(6, 4)}{h}$, which we can estimate with $h = \pm 2\sqrt{2}$. Then

$D_{\mathbf{u}} T(6, 4) \approx \frac{T(8, 6) - T(6, 4)}{2\sqrt{2}} = \frac{80 - 80}{2\sqrt{2}} = 0$, $D_{\mathbf{u}} T(6, 4) \approx \frac{T(4, 2) - T(6, 4)}{-2\sqrt{2}} = \frac{74 - 80}{-2\sqrt{2}} = \frac{3}{\sqrt{2}}$.

Averaging these values, we have $D_{\mathbf{u}} T(6, 4) \approx \frac{3}{2\sqrt{2}} \approx 1.1^\circ\text{C}/\text{m}$.

$$(c) T_{xy}(x, y) = \frac{\partial}{\partial y} [T_x(x, y)] = \lim_{h \rightarrow 0} \frac{T_x(x, y+h) - T_x(x, y)}{h}, \text{ so } T_{xy}(6, 4) = \lim_{h \rightarrow 0} \frac{T_x(6, 4+h) - T_x(6, 4)}{h}$$

which we can estimate with $h = \pm 2$. We have $T_x(6, 4) \approx 3.5$ from part (a), but we will also need values for $T_x(6, 6)$ and $T_x(6, 2)$. If we use $h = \pm 2$ and the values given in the table, we have

$$T_x(6, 6) \approx \frac{T(8, 6) - T(6, 6)}{2} = \frac{80 - 75}{2} = 2.5, \quad T_x(6, 6) \approx \frac{T(4, 6) - T(6, 6)}{-2} = \frac{68 - 75}{-2} = 3.5.$$

Averaging these values, we estimate $T_x(6, 6) \approx 3.0$. Similarly,

$$T_x(6, 2) \approx \frac{T(8, 2) - T(6, 2)}{2} = \frac{90 - 87}{2} = 1.5, \quad T_x(6, 2) \approx \frac{T(4, 2) - T(6, 2)}{-2} = \frac{74 - 87}{-2} = 6.5.$$

Averaging these values, we estimate $T_x(6, 2) \approx 4.0$. Finally, we estimate $T_{xy}(6, 4)$:

$$T_{xy}(6, 4) \approx \frac{T_x(6, 6) - T_x(6, 4)}{2} = \frac{3.0 - 3.5}{2} = -0.25,$$

$$T_{xy}(6, 4) \approx \frac{T_x(6, 2) - T_x(6, 4)}{-2} = \frac{4.0 - 3.5}{-2} = -0.25. \text{ Averaging these values, we have}$$

$$T_{xy}(6, 4) \approx -0.25.$$

12. From the table, $T(6, 4) = 80$, and from Exercise 11 we estimated $T_x(6, 4) \approx 3.5$ and $T_y(6, 4) \approx -3.0$. The linear approximation then is

$$\begin{aligned} T(x, y) &\approx T(6, 4) + T_x(6, 4)(x - 6) + T_y(6, 4)(y - 4) \approx 80 + 3.5(x - 6) - 3(y - 4) \\ &= 3.5x - 3y + 71 \end{aligned}$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate

$$T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1^\circ\text{C}.$$

$$13. f(x, y) = \sqrt{2x + y^2} \Rightarrow f_x = \frac{1}{2}(2x + y^2)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^2}}, f_y = \frac{1}{2}(2x + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{2x + y^2}}$$

$$14. u = e^{-r} \sin 2\theta \Rightarrow u_r = -e^{-r} \sin 2\theta, u_\theta = 2e^{-r} \cos 2\theta$$

$$15. g(u, v) = u \tan^{-1} v \Rightarrow g_u = \tan^{-1} v, g_v = \frac{u}{1 + v^2}$$

$$16. w = \frac{x}{y - z} \Rightarrow w_x = \frac{1}{y - z}, w_y = x(-1)(y - z)^{-2} = -\frac{x}{(y - z)^2},$$

$$w_z = x(-1)(y - z)^{-2}(-1) = \frac{x}{(y - z)^2}$$

$$17. T(p, q, r) = p \ln(q + e^r) \Rightarrow T_p = \ln(q + e^r), T_q = \frac{p}{q + e^r}, T_r = \frac{pe^r}{q + e^r}$$

18. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D \Rightarrow$
 $\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S - 35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35 - 35) \approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

$$19. f(x, y) = 4x^3 - xy^2 \Rightarrow f_x = 12x^2 - y^2, f_y = -2xy, f_{xx} = 24x, f_{yy} = -2x, \text{ and } f_{xy} = f_{yx} = -2y.$$

20. $z = xe^{-2y} \Rightarrow z_x = e^{-2y}, z_y = -2xe^{-2y}, z_{xx} = 0, z_{yy} = 4xe^{-2y}, \text{ and } z_{xy} = z_{yx} = -2e^{-2y}.$

21. $f(x, y, z) = x^k y^l z^m \Rightarrow f_x = kx^{k-1} y^l z^m, f_y = lx^k y^{l-1} z^m, f_z = mx^k y^l z^{m-1},$
 $f_{xx} = k(k-1)x^{k-2} y^l z^m, f_{yy} = l(l-1)x^k y^{l-2} z^m, f_{zz} = m(m-1)x^k y^l z^{m-2}, f_{xy} = f_{yx} = klx^{k-1} y^{l-1} z^m,$
 $f_{xz} = f_{zx} = kmx^{k-1} y^l z^{m-1}, \text{ and } f_{yz} = f_{zy} = lm x^k y^{l-1} z^{m-1}.$

22. $v = r \cos(s + 2t) \Rightarrow v_r = \cos(s + 2t), v_s = -r \sin(s + 2t), v_t = -2r \sin(s + 2t), v_{rr} = 0,$
 $v_{ss} = -r \cos(s + 2t), v_{tt} = -4r \cos(s + 2t), v_{rs} = v_{sr} = -\sin(s + 2t), v_{rt} = v_{tr} = -2 \sin(s + 2t), \text{ and}$
 $v_{st} = v_{ts} = -2r \cos(s + 2t).$

23. $u = x^y \Rightarrow u_x = yx^{y-1}, u_y = x^y \ln x \text{ and } (x/y)u_x + (\ln x)^{-1}u_y = x^y + x^y = 2u.$

24. $\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \rho_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \rho_{xx} = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}.$

By symmetry, $\rho_{yy} = \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$ and $\rho_{zz} = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}}.$ Thus

$$\rho_{xx} + \rho_{yy} + \rho_{zz} = 2 \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2}{(x^2 + y^2 + z^2)^{1/2}} = \frac{2}{\rho}.$$

25. (a) $z_x = 6x + 2 \Rightarrow z_x(1, -2) = 8 \text{ and } z_y = -2y \Rightarrow z_y(1, -2) = 4,$ so an equation of the tangent plane is $z - 1 = 8(x - 1) + 4(y + 2)$ or $z = 8x + 4y + 1.$

(b) A normal vector to the tangent plane (and the surface) at $(1, -2, 1)$ is $\langle 8, 4, -1 \rangle.$ Then parametric equations for the normal line there are $x = 1 + 8t, y = -2 + 4t, z = 1 - t,$ and symmetric equations are

$$\frac{x-1}{8} = \frac{y+2}{4} = \frac{z-1}{-1}.$$

26. (a) $z_x = e^x \cos y \Rightarrow z_x(0, 0) = 1 \text{ and } z_y = -e^x \sin y \Rightarrow z_y(0, 0) = 0,$ so an equation of the tangent plane is $z - 1 = 1(x - 0) + 0(y - 0)$ or $z = x + 1.$

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1 \rangle.$ Then parametric equations for the normal line there are $x = t, y = 0, z = 1 - t,$ and symmetric equations are $x = 1 - z, y = 0.$

27. (a) Let $F(x, y, z) = x^2 + 2y^2 - 3z^2.$ Then $F_x = 2x, F_y = 4y, F_z = -6z,$ so $F_x(2, -1, 1) = 4,$
 $F_y(2, -1, 1) = -4, F_z(2, -1, 1) = -6.$ From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $4(x - 2) - 4(y + 1) - 6(z - 1) = 0$ or equivalently $2x - 2y - 3z = 3.$

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\frac{x-2}{4} = \frac{y+1}{-4} = \frac{z-1}{-6}.$

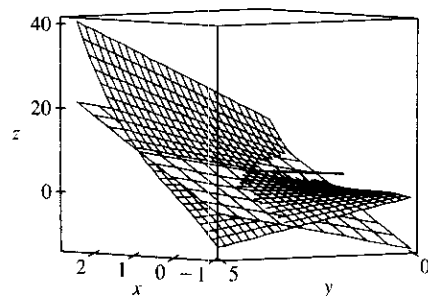
28. (a) Let $F(x, y, z) = xy + yz + zx.$ Then $F_x = y + z, F_y = x + z, F_z = x + y,$ so
 $F_x(1, 1, 1) = F_y(1, 1, 1) = F_z(1, 1, 1) = 2.$ From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $2(x - 1) + 2(y - 1) + 2(z - 1) = 0$ or equivalently $x + y + z = 3.$

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$
or equivalently $x = y = z.$

29. (a) Let $F(x, y, z) = x + 2y + 3z - \sin(xyz).$ Then $F_x = 1 - yz \cos(xyz), F_y = 2 - xz \cos(xyz),$
 $F_z = 3 - xy \cos(xyz),$ so $F_x(2, -1, 0) = 1, F_y(2, -1, 0) = 2, F_z(2, -1, 0) = 5.$ From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $1(x - 2) + 2(y + 1) + 5(z - 0) = 0$ or $x + 2y + 5z = 0.$

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{5}.$

30. Let $f(x, y) = x^3 + 2xy$. Then $f_x(x, y) = 3x^2 + 2y$ and $f_y(x, y) = 2x$, so $f_x(1, 2) = 7$, $f_y(1, 2) = 2$ and an equation of the tangent plane is $z - 5 = 7(x - 1) + 2(y - 2)$ or $7x + 2y - z = 6$. The normal line is given by $\frac{x-1}{7} = \frac{y-2}{2} = \frac{z-5}{-1}$ or $x = 7t + 1$, $y = 2t + 2$, $z = -t + 5$.



31. $F(x, y, z) = x^2 + y^2 + z^2$, $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 2z_0 \rangle = k\langle 2, 1, -3 \rangle$ or $x_0 = k$, $y_0 = \frac{1}{2}k$ and $z_0 = -\frac{3}{2}k$. But $x_0^2 + y_0^2 + z_0^2 = 1$, so $\frac{7}{2}k^2 = 1$ and $k = \pm\sqrt{\frac{2}{7}}$. Hence there are two such points: $(\pm\sqrt{\frac{2}{7}}, \pm\frac{1}{\sqrt{14}}, \mp\frac{3}{\sqrt{14}})$.

32. $z = x^2 \tan^{-1} y \Rightarrow dz = (2x \tan^{-1} y) dx + [x^2/(y^2 + 1)] dy$

33. $f(x, y, z) = x^3 \sqrt{y^2 + z^2} \Rightarrow f_x(x, y, z) = 3x^2 \sqrt{y^2 + z^2}$, $f_y(x, y, z) = \frac{yx^3}{\sqrt{y^2 + z^2}}$, and

$$f_z(x, y, z) = \frac{zx^3}{\sqrt{y^2 + z^2}}, \text{ so } f(2, 3, 4) = 8(5) = 40, f_x(2, 3, 4) = 3(4)\sqrt{25} = 60, f_y(2, 3, 4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5},$$

and $f_z(2, 3, 4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2, 3, 4)$ is

$$\begin{aligned} f(x, y, z) &\approx f(2, 3, 4) + f_x(2, 3, 4)(x - 2) + f_y(2, 3, 4)(y - 3) + f_z(2, 3, 4)(z - 4) \\ &= 40 + 60(x - 2) + \frac{24}{5}(y - 3) + \frac{32}{5}(z - 4) = 60x + \frac{24}{5}y + \frac{32}{5}z - 120 \end{aligned}$$

Then

$$\begin{aligned} (1.98)^3 \sqrt{(3.01)^2 + (3.97)^2} &= f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 \\ &= 38.656 \end{aligned}$$

34. (a) $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = \frac{1}{2}y dx + \frac{1}{2}x dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017 \text{ m}^2$ or 170 cm^2 .

- (b) $z = \sqrt{x^2 + y^2}$, $dz = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026 \text{ m}$ or 0.26 cm .

35. $\frac{dw}{dt} = \frac{1}{2\sqrt{x}} (2e^{2t}) + \frac{2y}{z} (3t^2 + 4) + \frac{-y^2}{z^2} (2t) = e^t + \frac{2y}{z} (3t^2 + 4) - 2t \frac{y^2}{z^2}$

36. $\frac{\partial z}{\partial u} = (-y \sin xy - y \sin x)(2u) + (-x \sin xy + \cos x) = \cos x - 2uy \sin x - (\sin xy)(x + 2uy)$,

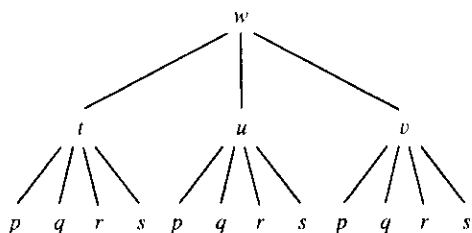
$$\frac{\partial z}{\partial v} = (-y \sin xy - y \sin x)(1) + (-x \sin xy + \cos x)(-2v) = -2v \cos x + (\sin xy)(2vx - y) - y \sin x$$

37. By the Chain Rule, $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1, 2) = 3$ and $y = h(1, 2) = 6$, so

$$\frac{\partial z}{\partial s} = f_x(3, 6)g_s(1, 2) + f_y(3, 6)h_s(1, 2) = (7)(-1) + (8)(-5) = -47. \text{ Similarly, } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}, \text{ so}$$

$$\frac{\partial z}{\partial t} = f_x(3, 6)g_t(1, 2) + f_y(3, 6)h_t(1, 2) = (7)(4) + (8)(10) = 108.$$

38.



Using the tree diagram as a guide, we have

$$\begin{aligned}\frac{\partial w}{\partial p} &= \frac{\partial w}{\partial t} \frac{\partial t}{\partial p} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial p} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial p} \\ \frac{\partial w}{\partial q} &= \frac{\partial w}{\partial t} \frac{\partial t}{\partial q} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial q} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial q} \\ \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial t} \frac{\partial t}{\partial r} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial r} \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial t} \frac{\partial t}{\partial s} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial s}\end{aligned}$$

39. $\frac{\partial z}{\partial x} = 2xf'(x^2 - y^2)$, $\frac{\partial z}{\partial y} = 1 - 2yf'(x^2 - y^2)$ [where $f' = \frac{df}{d(x^2 - y^2)}$]. Then

$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 2xyf'(x^2 - y^2) + x - 2xyf'(x^2 - y^2) = x.$$

40. $A = \frac{1}{2}xy \sin \theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and

$$\frac{dA}{dt} = \frac{1}{2} \left[(y \sin \theta) \frac{dx}{dt} + (x \sin \theta) \frac{dy}{dt} + (xy \cos \theta) \frac{d\theta}{dt} \right]. \text{ So when } x = 40, y = 50 \text{ and } \theta = \frac{\pi}{6},$$

$$\frac{dA}{dt} = \frac{1}{2} [(25)(3) + (20)(-2) + (1000\sqrt{3})(0.05)] = \frac{35 + 50\sqrt{3}}{2} \approx 60.8 \text{ in}^2/\text{s}.$$

41. $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} y + \frac{\partial z}{\partial v} \frac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + \frac{2y}{x^3} \frac{\partial z}{\partial v} + \frac{-y}{x^2} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \\ &= \frac{2y}{x^3} \frac{\partial z}{\partial v} + y \left(\frac{\partial^2 z}{\partial u^2} y + \frac{\partial^2 z}{\partial v \partial u} \frac{-y}{x^2} \right) + \frac{-y}{x^2} \left(\frac{\partial^2 z}{\partial v^2} \frac{-y}{x^2} + \frac{\partial^2 z}{\partial u \partial v} y \right) \\ &= \frac{2y}{x^3} \frac{\partial z}{\partial v} + y^2 \frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2} \frac{\partial^2 z}{\partial u \partial v} + \frac{y^2}{x^4} \frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\frac{\partial z}{\partial y} = x \frac{\partial z}{\partial u} + \frac{1}{x} \frac{\partial z}{\partial v}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= x \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) + \frac{1}{x} \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right) = x \left(\frac{\partial^2 z}{\partial u^2} x + \frac{\partial^2 z}{\partial v \partial u} \frac{1}{x} \right) + \frac{1}{x} \left(\frac{\partial^2 z}{\partial v^2} \frac{1}{x} + \frac{\partial^2 z}{\partial u \partial v} x \right) \\ &= x^2 \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{x^2} \frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Thus

$$\begin{aligned}x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} &= \frac{2y}{x} \frac{\partial z}{\partial v} + x^2 y^2 \frac{\partial^2 z}{\partial u^2} - 2y^2 \frac{\partial^2 z}{\partial u \partial v} + \frac{y^2}{x^2} \frac{\partial^2 z}{\partial v^2} - x^2 y^2 \frac{\partial^2 z}{\partial u^2} - 2y^2 \frac{\partial^2 z}{\partial u \partial v} - \frac{y^2}{x^2} \frac{\partial^2 z}{\partial v^2} \\ &= \frac{2y}{x} \frac{\partial z}{\partial v} - 4y^2 \frac{\partial^2 z}{\partial u \partial v} = 2v \frac{\partial z}{\partial v} - 4uv \frac{\partial^2 z}{\partial u \partial v}\end{aligned}$$

since $y = xv = \frac{uv}{y}$ or $y^2 = uv$.

42. $F(x, y, z) = e^{xyz} - yz^4 - x^2 z^3 = 0$, so $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yze^{xyz} - 2xz^3}{xye^{xyz} - 4yz^3 - 3x^2 z^2} = \frac{2xz^3 - yze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2 z^2}$

$$\text{and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{xyz} - z^4}{xye^{xyz} - 4yz^3 - 3x^2 z^2} = \frac{z^4 - xze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2 z^2}.$$

$$43. \nabla f = \left\langle z^2 \sqrt{y} e^{x\sqrt{y}}, \frac{xz^2 e^{x\sqrt{y}}}{2\sqrt{y}}, 2ze^{x\sqrt{y}} \right\rangle = ze^{x\sqrt{y}} \left\langle z\sqrt{y}, \frac{xz}{2\sqrt{y}}, 2 \right\rangle$$

44. (a) By Theorem 15.6.15 [ET 14.6.15], the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}} f = |\nabla f| \cos \theta$ (see the proof of Theorem 15.6.15 [ET 14.6.15]) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}} f = |\nabla f| \cos \theta = \frac{1}{2} |\nabla f| \Leftrightarrow \cos \theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$.

$$45. \nabla f = \langle 1/\sqrt{x}, -2y \rangle, \nabla f(1, 5) = \langle 1, -10 \rangle, \mathbf{u} = \frac{1}{5} \langle 3, -4 \rangle. \text{ Then } D_{\mathbf{u}} f(1, 5) = \frac{43}{5}.$$

$$46. \nabla f = \langle 2xy + \sqrt{1+z}, x^2, x/(2\sqrt{1+z}) \rangle, \nabla f(1, 2, 3) = \langle 6, 1, \frac{1}{4} \rangle, \text{ and } \mathbf{u} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle. \text{ Then } D_{\mathbf{u}} f(1, 2, 3) = \frac{25}{6}.$$

$$47. \nabla f = \langle 2xy, x^2 + 1/(2\sqrt{y}) \rangle, |\nabla f(2, 1)| = |\langle 4, \frac{9}{2} \rangle|. \text{ Thus the maximum rate of change of } f \text{ at } (2, 1) \text{ is } \frac{\sqrt{145}}{2} \text{ in the direction } \langle 4, \frac{9}{2} \rangle.$$

$$48. \nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy} \rangle, \nabla f(0, 1, 2) = \langle 2, 0, 1 \rangle \text{ is the direction of most rapid increase while the rate is } |\langle 2, 0, 1 \rangle| = \sqrt{5}.$$

49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8} = \frac{5}{8} = 0.625$ knot/mi.

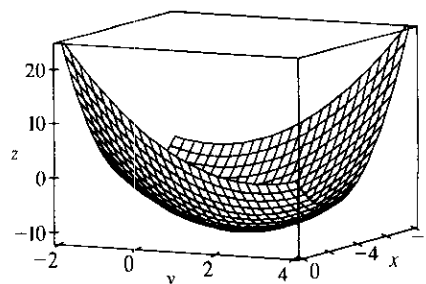
50. The surfaces are $f(x, y, z) = z - 2x^2 + y^2 = 0$ and $g(x, y, z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2, 2, 4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line.

$$\nabla f(x, y, z) = \langle -4x, 2y, 1 \rangle \Rightarrow \nabla f(-2, 2, 4) = \langle 8, 4, 1 \rangle, \nabla g(x, y, z) = \langle 0, 0, 1 \rangle \Rightarrow$$

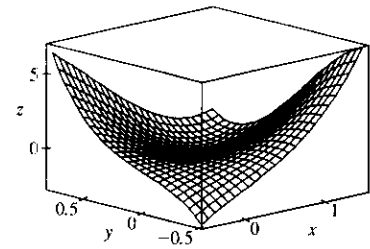
$$\nabla g(-2, 2, 4) = \langle 0, 0, 1 \rangle. \text{ Hence } \mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\mathbf{i} - 8\mathbf{j}. \text{ Thus, parametric equations are:}$$

$$x = -2 + 4t, y = 2 - 8t, \text{ and } z = 4.$$

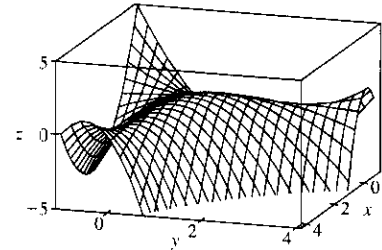
51. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10 \Rightarrow f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4, 1)$ and $f_{xx}(-4, 1) > 0$, $D(-4, 1) = 3 > 0$, so $f(-4, 1) = -11$ is a local minimum.



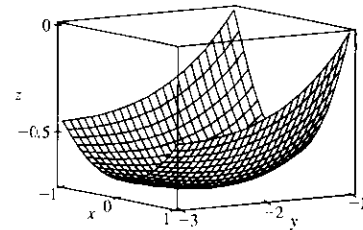
52. $f(x, y) = x^3 - 6xy + 8y^3 \Rightarrow f_x = 3x^2 - 6y, f_y = -6x + 24y^2$,
 $f_{xx} = 6x, f_{yy} = 48y, f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$,
 substituting into $f_y = 0$ implies $6x(x^3 - 1) = 0$, so the critical points
 are $(0, 0)$, $(1, \frac{1}{2})$. $D(0, 0) = -36 < 0$ so $(0, 0)$ is a saddle point while
 $f_{xx}(1, \frac{1}{2}) = 6 > 0$ and $D(1, \frac{1}{2}) = 108 > 0$ so $f(1, \frac{1}{2}) = -1$ is a local
 minimum.



53. $f(x, y) = 3xy - x^2y - xy^2 \Rightarrow f_x = 3y - 2xy - y^2$,
 $f_y = 3x - x^2 - 2xy, f_{xx} = -2y, f_{yy} = -2x, f_{xy} = 3 - 2x - 2y$. Then
 $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting
 into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical
 points are $(0, 0)$, $(3, 0)$, $(0, 3)$ and $(1, 1)$.
 $D(0, 0) = D(3, 0) = D(0, 3) = -9 < 0$ so $(0, 0)$, $(3, 0)$, and $(0, 3)$ are
 saddle points. $D(1, 1) = 3 > 0$ and $f_{xx}(1, 1) = -2 < 0$, so $f(1, 1) = 1$
 is a local maximum.



54. $f(x, y) = (x^2 + y)e^{y/2} \Rightarrow f_x = 2xe^{y/2}, f_y = e^{y/2}(2 + x^2 + y)/2$,
 $f_{xx} = 2e^{y/2}, f_{yy} = e^{y/2}(4 + x^2 + y)/4, f_{xy} = xe^{y/2}$. Then $f_x = 0$
 implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0, -2) > 0$,
 $D(0, -2) = e^{-2} - 0 > 0$ so $f(0, -2) = -2/e$ is a local minimum.

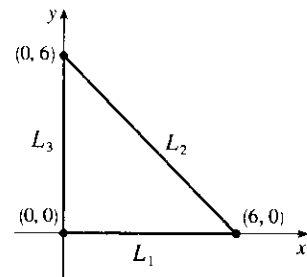


55. First solve inside D . Here $f_x = 4y^2 - 2xy^2 - y^3$,
 $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$,
 but $y = 0$ isn't inside D . Substituting $y = 4 - 2x$ into $f_y = 0$ implies
 $x = 0, x = 2$ or $x = 1$, but $x = 0$ isn't inside D , and when $x = 2, y = 0$
 but $(2, 0)$ isn't inside D . Thus the only critical point inside D is $(1, 2)$ and
 $f(1, 2) = 4$. Secondly we consider the boundary of D .

On L_1 , $f(x, 0) = 0$ and so $f = 0$ on L_1 . On L_2 , $x = -y + 6$ and

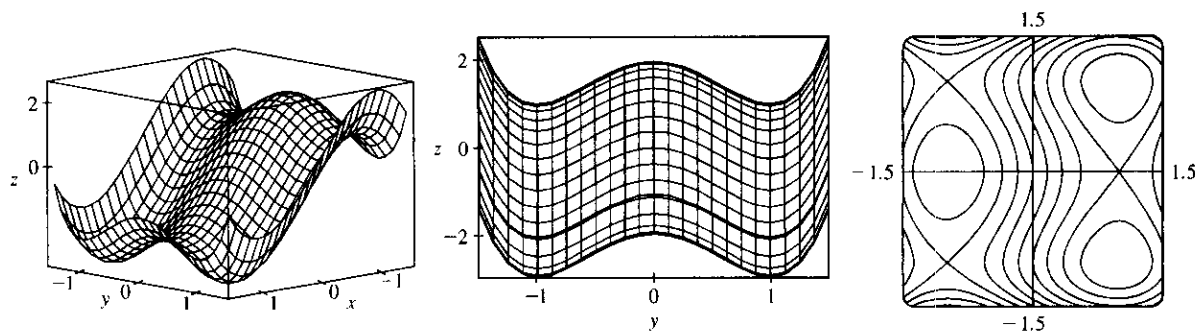
$$f(-y + 6, y) = y^2(6 - y)(-2) = -2(6y^2 - y^3) \text{ which has}$$

critical points at $y = 0$ and $y = 4$. Then $f(6, 0) = 0$ while $f(2, 4) = -64$. On L_3 , $f(0, y) = 0$, so $f = 0$ on L_3 .
 Thus on D the absolute maximum of f is $f(1, 2) = 4$ while the absolute minimum is $f(2, 4) = -64$.



56. Inside D : $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$,
 $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0, 0)$, $(0, \pm 1)$. If
 $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0, 0) = 0$, $f(\pm 1, 0) = e^{-1}$ and
 $f(0, \pm 1) = 2e^{-1}$. On the boundary of D : $x^2 + y^2 = 4$, so $f(x, y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$
 and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0, \pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is
 $f(0, \pm 1) = 2e^{-1}$ and the absolute minimum is $f(0, 0) = 0$.

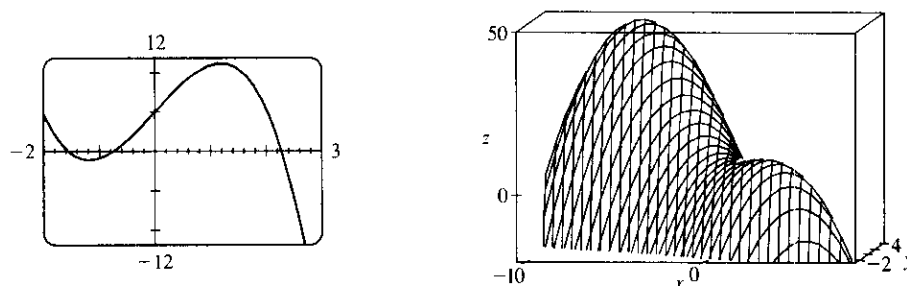
57. $f(x, y) = x^3 - 3x + y^4 - 2y^2$



From the graphs, it appears that f has a local maximum $f(-1, 0) \approx 2$, local minima $f(1, \pm 1) \approx -3$, and saddle points at $(-1, \pm 1)$ and $(1, 0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \Leftrightarrow y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1, 0) = 24 > 0$ and $f_{xx}(-1, 0) = -6 < 0$ indicating a local maximum $f(-1, 0) = 2$; $D(1, \pm 1) = 48 > 0$ and $f_{xx}(1, \pm 1) = 6 > 0$ indicating local minima $f(1, \pm 1) = -3$; and $D(-1, \pm 1) = -48$ and $D(1, 0) = -24$, indicating saddle points.

58. $f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4 \Rightarrow f_x(x, y) = -4x - 8y$, $f_y(x, y) = 10 - 8x - 4y^3$. Now $f_x(x, y) = 0 \Rightarrow x = -2y$, and substituting this into $f_y(x, y) = 0$ gives $10 + 16y - 4y^3 = 0 \Leftrightarrow 5 + 8y - 2y^3 = 0$.



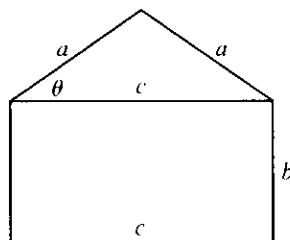
From the first graph, we see that this is true when $y \approx -1.542, -0.717$, or 2.260 . (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x, y)$ has local maxima $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

59. $f(x, y) = x^2y$, $g(x, y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ imply $\lambda = x^2/(2y)$ and $\lambda = y$ if $x \neq 0$ and $y \neq 0$. Hence $x^2 = 2y^2$. Then $x^2 + y^2 = 1$ implies $3y^2 = 1$ so $y = \pm \frac{1}{\sqrt{3}}$ and $x = \pm \sqrt{\frac{2}{3}}$. [Note if $x = 0$ then $x^2 = 2\lambda y$ implies $y = 0$ and $f(0, 0) = 0$.] Thus the possible points are $(\pm \sqrt{\frac{2}{3}}, \pm \frac{1}{\sqrt{3}})$ and the absolute maxima are $f(\pm \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}}$ while the absolute minima are $f(\pm \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}) = -\frac{2}{3\sqrt{3}}$.

60. $f(x, y) = 1/x + 1/y$, $g(x, y) = 1/x^2 + 1/y^2 = 1 \Rightarrow \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$.
Then $-x^{-2} = -2\lambda x^{-3}$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$
implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to
 $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.
61. $f(x, y, z) = xyz$, $g(x, y, z) = x^2 + y^2 + z^2 = 3$, $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x, y ,
or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z} \Rightarrow$
 $2y^2z = 2xz^2 \Rightarrow y^2 = x^2$, and similarly $2yz^2 = 2x^2y \Rightarrow z^2 = x^2$. Substituting into the constraint equation
gives $x^2 + x^2 + x^2 = 3 \Rightarrow x^2 = 1 = y^2 = z^2$. Thus the possible points are
 $(1, 1, \pm 1), (1, -1, \pm 1), (-1, 1, \pm 1), (-1, -1, \pm 1)$. The absolute maximum is
 $f(1, 1, 1) = f(1, -1, -1) = f(-1, 1, -1) = f(-1, -1, 1) = 1$ and the absolute minimum is
 $f(1, 1, -1) = f(1, -1, 1) = f(-1, 1, 1) = f(-1, -1, -1) = -1$.
62. $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $g(x, y, z) = x + y + z = 1$, $h(x, y, z) = x - y + 2z = 2 \Rightarrow$
 $\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and (1) $2x = \lambda + \mu$, (2) $4y = \lambda - \mu$,
(3) $6z = \lambda + 2\mu$, (4) $x + y + z = 1$, (5) $x - y + 2z = 2$. Then six times (1) plus three times (2) plus two times
(3) implies $12(x + y + z) = 11\lambda + 7\mu$, so (4) gives $11\lambda + 7\mu = 12$. Also six times (1) minus three times (2) plus
four times (3) implies $12(x - y + 2z) = 7\lambda + 17\mu$, so (5) gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$,
 $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into (1), (2) and (3) implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$,
 $z = \frac{11}{23}$ giving only one point. Then $f(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}) = \frac{33}{23}$. Now since $(0, 0, 1)$ satisfies both constraints and
 $f(0, 0, 1) = 3 > \frac{33}{23}$, $f(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.
63. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = xy^2z^3 = 2 \Rightarrow$
 $\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle \lambda y^2z^3, 2\lambda xy^2z^3, 3\lambda xy^2z^2 \rangle$. Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so
(1) $2x = \lambda y^2z^3$, (2) $1 = \lambda xz^3$, (3) $2 = 3\lambda xy^2z$. Then (2) and (3) imply $\frac{1}{xz^3} = \frac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so
 $y = \pm z\sqrt{\frac{2}{3}}$. Similarly (1) and (3) imply $\frac{2x}{y^2z^3} = \frac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm \frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z
must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x, y, z) = 2$ implies $\frac{1}{\sqrt{3}}z(\frac{2}{3}z^2)z^3 = 2$ or $z = \pm 3^{1/4}$ and the
possible points are $(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}), (\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4})$. However at each of these points
 f takes on the same value, $2\sqrt{3}$. But $(2, 1, 1)$ also satisfies $g(x, y, z) = 2$ and $f(2, 1, 1) = 6 > 2\sqrt{3}$. Thus f has
an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3 = 2$.
- Alternate solution:* $g(x, y, z) = xy^2z^3 = 2$ implies $y^2 = \frac{2}{xz^3}$, so minimize $f(x, z) = x^2 + \frac{2}{xz^3} + z^2$. Then
 $f_x = 2x - \frac{2}{x^2z^3}$, $f_z = -\frac{6}{xz^4} + 2z$, $f_{xx} = 2 + \frac{4}{x^3z^3}$, $f_{zz} = \frac{24}{xz^5} + 2$ and $f_{xz} = \frac{6}{x^2z^4}$. Now $f_x = 0$ implies
 $2x^3z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_z = 0$ implies $-6x^3 + 2x^{-1} = 0$ or $x = \frac{1}{\sqrt[4]{3}}$, so the two critical
points are $(\pm \frac{1}{\sqrt[4]{3}}, \pm \sqrt[4]{3})$. Then $D\left(\pm \frac{1}{\sqrt[4]{3}}, \pm \sqrt[4]{3}\right) = (2 + 4)(2 + \frac{24}{3}) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and
 $f_{xx}\left(\pm \frac{1}{\sqrt[4]{3}}, \pm \sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \frac{2}{xz^3}$, so the four points closest to the
origin are $(\pm \frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm \sqrt[4]{3}), (\pm \frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm \sqrt[4]{3})$.
64. $V = xyz$, say x is the length and $x + 2y + 2z \leq 108$, $x > 0$, $y > 0$, $z > 0$. First maximize V subject to
 $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy \rangle = \langle \lambda, 2\lambda, 2\lambda \rangle$ implies $2yz = xz$ or $x = 2y$

and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11,664$ cubic units. Since $(104, 1, 1)$ also satisfies $g(x, y, z) = 108$ and $V(104, 1, 1) = 104$ cubic units, $(36, 18, 18)$ gives an absolute maximum of V subject to $g(x, y, z) = 108$. But if $x + 2y + 2z < 108$, there exists $\alpha > 0$ such that $x + 2y + 2z = 108 - \alpha$ and as above $6y = 108 - \alpha$ implies $y = (108 - \alpha)/6 = z$, $x = (108 - \alpha)/3$ with $V = (108 - \alpha)^3/(6^2 \cdot 3) < (108)^3/(6^2 \cdot 3) = 11,664$. Hence we have shown that the maximum of V subject to $g(x, y, z) \leq 108$ is the maximum of V subject to $g(x, y, z) = 108$ (an intuitively obvious fact).

65.



The area of the triangle is $\frac{1}{2}ca \sin \theta$ and the area of the rectangle is bc .

Thus, the area of the whole object is $f(a, b, c) = \frac{1}{2}ca \sin \theta + bc$. The perimeter of the object is $g(a, b, c) = 2a + 2b + c = P$. To simplify

$\sin \theta$ in terms of a , b , and c notice that $a^2 \sin^2 \theta + (\frac{1}{2}c)^2 = a^2 \Rightarrow$

$$\sin \theta = \frac{1}{2a} \sqrt{4a^2 - c^2}. \text{ Thus } f(a, b, c) = \frac{c}{4} \sqrt{4a^2 - c^2} + bc.$$

(Instead of using θ , we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a , b , c , and λ by solving $\nabla f = \lambda \nabla g$ which gives the following equations: (1) $ca(4a^2 - c^2)^{-1/2} = 2\lambda$,

(2) $c = 2\lambda$, (3) $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$, and (4) $2a + 2b + c = P$. From (2), $\lambda = \frac{1}{2}c$ and so (1) produces $ca(4a^2 - c^2)^{-1/2} = c \Rightarrow (4a^2 - c^2)^{1/2} = a \Rightarrow 4a^2 - c^2 = a^2 \Rightarrow$ (5) $c = \sqrt{3}a$.

Similarly, since $(4a^2 - c^2)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, (3) gives $\frac{a}{4} - \frac{c^2}{4a} + b = \frac{c}{2}$, so from (5), $\frac{a}{4} - \frac{3a}{4} + b = \frac{\sqrt{3}a}{2}$

$$\Rightarrow -\frac{a}{2} + \frac{\sqrt{3}a}{2} = -b \Rightarrow (6) \quad b = \frac{a}{2}(1 + \sqrt{3}). \text{ Substituting (5) and (6) into (4) we get:}$$

$$2a + a(1 + \sqrt{3}) + \sqrt{3}a = P \Rightarrow 3a + 2\sqrt{3}a = P \Rightarrow a = \frac{P}{3 + 2\sqrt{3}} = \frac{2\sqrt{3} - 3}{3}P \text{ and thus}$$

$$b = \frac{(2\sqrt{3} - 3)(1 + \sqrt{3})}{6}P = \frac{3 - \sqrt{3}}{6}P \text{ and } c = (2 - \sqrt{3})P.$$

$$66. (a) \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + f(x(t), y(t))\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right)\mathbf{k}$$

(by the Chain Rule). Therefore

$$\begin{aligned} K = \frac{1}{2}m|\mathbf{v}|^2 &= \frac{m}{2} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right)^2 \right] \\ &= \frac{m}{2} \left[(1 + f_x^2) \left(\frac{dx}{dt}\right)^2 + 2f_x f_y \left(\frac{dx}{dt}\right) \left(\frac{dy}{dt}\right) + (1 + f_y^2) \left(\frac{dy}{dt}\right)^2 \right] \end{aligned}$$

$$(b) \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \left[f_{xx} \left(\frac{dx}{dt}\right)^2 + 2f_{xy} \frac{dx}{dt} \frac{dy}{dt} + f_{yy} \left(\frac{dy}{dt}\right)^2 + f_x \frac{d^2x}{dt^2} + f_y \frac{d^2y}{dt^2} \right] \mathbf{k}$$

(c) If $z = x^2 + y^2$, where $x = t \cos t$ and $y = t \sin t$, then $z = f(x, y) = t^2$.

$$\mathbf{r} = t \cos t \mathbf{i} + t \sin t \mathbf{j} + t^2 \mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t) \mathbf{i} + (\sin t + t \cos t) \mathbf{j} + 2t \mathbf{k},$$

$$K = \frac{m}{2} [(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (2t)^2] = \frac{m}{2} (1 + t^2 + 4t^2) = \frac{m}{2} (1 + 5t^2), \text{ and}$$

$\mathbf{a} = (-2 \sin t - t \cos t) \mathbf{i} + (2 \cos t - t \sin t) \mathbf{j} + 2 \mathbf{k}$. Notice that it is easier not to use the formulas in (a) and (b).

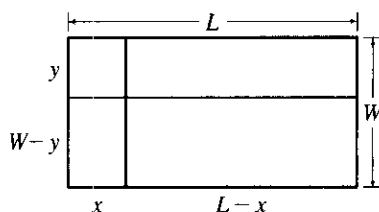
PROBLEMS PLUS

1. The areas of the smaller rectangles are $A_1 = xy$, $A_2 = (L - x)y$,

$$A_3 = (L - x)(W - y), A_4 = x(W - y). \text{ For } 0 \leq x \leq L,$$

$$0 \leq y \leq W, \text{ let}$$

$$\begin{aligned} f(x, y) &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \\ &= x^2 y^2 + (L - x)^2 y^2 + (L - x)^2 (W - y)^2 + x^2 (W - y)^2 \\ &= [x^2 + (L - x)^2][y^2 + (W - y)^2] \end{aligned}$$



Then we need to find the maximum and minimum values of $f(x, y)$. Here

$$f_x(x, y) = [2x - 2(L - x)][y^2 + (W - y)^2] = 0 \Rightarrow 4x - 2L = 0 \text{ or } x = \frac{1}{2}L, \text{ and}$$

$$f_y(x, y) = [x^2 + (L - x)^2][2y - 2(W - y)] = 0 \Rightarrow 4y - 2W = 0 \text{ or } y = \frac{1}{2}W. \text{ Also}$$

$$f_{xx} = 4[y^2 + (W - y)^2], f_{yy} = 4[x^2 + (L - x)^2], \text{ and } f_{xy} = (4x - 2L)(4y - 2W). \text{ Then}$$

$$D = 16[y^2 + (W - y)^2][x^2 + (L - x)^2] - (4x - 2L)^2(4y - 2W)^2. \text{ Thus when } x = \frac{1}{2}L \text{ and } y = \frac{1}{2}W,$$

$D > 0$ and $f_{xx} = 2W^2 > 0$. Thus a minimum of f occurs at $(\frac{1}{2}L, \frac{1}{2}W)$ and this minimum value is

$$f(\frac{1}{2}L, \frac{1}{2}W) = \frac{1}{4}L^2W^2. \text{ There are no other critical points, so the maximum must occur on the boundary. Now}$$

along the width of the rectangle let $g(y) = f(0, y) = f(L, y) = L^2[y^2 + (W - y)^2]$, $0 \leq y \leq W$. Then

$$g'(y) = L^2[2y - 2(W - y)] = 0 \Leftrightarrow y = \frac{1}{2}W. \text{ And } g(\frac{1}{2}) = \frac{1}{2}L^2W^2. \text{ Checking the endpoints, we get}$$

$$g(0) = g(W) = L^2W^2. \text{ Along the length of the rectangle let } h(x) = f(x, 0) = f(x, W) = W^2[x^2 + (L - x)^2],$$

$$0 \leq x \leq L. \text{ By symmetry } h'(x) = 0 \Leftrightarrow x = \frac{1}{2}L \text{ and } h(\frac{1}{2}L) = \frac{1}{2}L^2W^2. \text{ At the endpoints we have}$$

$$h(0) = h(L) = L^2W^2. \text{ Therefore } L^2W^2 \text{ is the maximum value of } f. \text{ This maximum value of } f \text{ occurs when the}$$

“cutting” lines correspond to sides of the rectangle.

2. (a) The level curves of the function $C(x, y) = e^{-(x^2 + 2y^2)/10^4}$

are the curves $e^{-(x^2 + 2y^2)/10^4} = k$ (k is a positive constant).

This equation is equivalent to $x^2 + 2y^2 = K \Rightarrow$

$$\frac{x^2}{(\sqrt{K})^2} + \frac{y^2}{(\sqrt{K/2})^2} = 1, \text{ where } K = -10^4 \ln k,$$

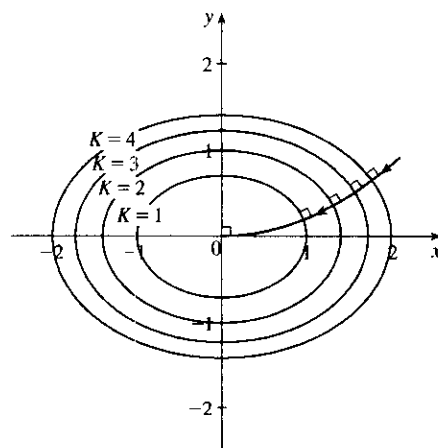
a family of ellipses. We sketch level curves for $K = 1, 2,$

3, and 4. If the shark always swims in the direction of

maximum increase of blood concentration, its direction

at any point would coincide with the gradient vector. Then we know the shark's path is perpendicular to the level

curves it intersects. We sketch one example of such a path.



(b) $\nabla C = -\frac{2}{10^4} e^{-(x^2+2y^2)/10^4} (x\mathbf{i} + 2y\mathbf{j})$. And ∇C points in the direction of most rapid increase in concentration; that is, ∇C is tangent to the most rapid increase curve. If $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a parametrization of the most rapid increase curve, then $\frac{dr}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ is tangent to the curve, so $\frac{dr}{dt} = \lambda \nabla C$
 $\Rightarrow \frac{dx}{dt} = \lambda \left[-\frac{2}{10^4} e^{-(x^2+2y^2)/10^4} \right] x$ and $\frac{dy}{dt} = \lambda \left[-\frac{2}{10^4} e^{-(x^2+2y^2)/10^4} \right] (2y)$. Therefore
 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2 \frac{y}{x} \Rightarrow \frac{dy}{y} = 2 \frac{dx}{x} \Rightarrow \ln |y| = 2 \ln |x|$ so that $y = kx^2$ for some constant k . But
 $y(x_0) = y_0 \Rightarrow y_0 = kx_0^2 \Rightarrow k = y_0/x_0^2$ ($x_0 = 0 \Rightarrow y_0 = 0 \Rightarrow$ the shark is already at the origin, so we can assume $x_0 \neq 0$.) Therefore the path the shark will follow is along the parabola
 $y = y_0(x/x_0)^2$.

3. (a) The area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height (the distance between the two parallel sides) and b_1, b_2 are the lengths of the bases (the parallel sides). From the figure in the text, we see that $h = x \sin \theta$, $b_1 = w - 2x$, and $b_2 = w - 2x + 2x \cos \theta$. Therefore the cross-sectional area of the rain gutter is

$$\begin{aligned} A(x, \theta) &= \frac{1}{2}x \sin \theta [(w - 2x) + (w - 2x + 2x \cos \theta)] = (x \sin \theta)(w - 2x + x \cos \theta) \\ &= wx \sin \theta - 2x^2 \sin \theta + x^2 \sin \theta \cos \theta, \quad 0 < x \leq \frac{1}{2}w, 0 < \theta \leq \frac{\pi}{2} \end{aligned}$$

We look for the critical points of A : $\partial A / \partial x = w \sin \theta - 4x \sin \theta + 2x \sin \theta \cos \theta$ and

$\partial A / \partial \theta = wx \cos \theta - 2x^2 \cos \theta + x^2(\cos^2 \theta - \sin^2 \theta)$, so $\partial A / \partial x = 0 \Leftrightarrow \sin \theta (w - 4x + 2x \cos \theta) = 0$

$\Leftrightarrow \cos \theta = \frac{4x - w}{2x} = 2 - \frac{w}{2x}$ ($0 < \theta \leq \frac{\pi}{2} \Rightarrow \sin \theta > 0$). If, in addition, $\partial A / \partial \theta = 0$, then

$$\begin{aligned} 0 &= wx \cos \theta - 2x^2 \cos \theta + x^2(2 \cos^2 \theta - 1) \\ &= wx \left(2 - \frac{w}{2x} \right) - 2x^2 \left(2 - \frac{w}{2x} \right) + x^2 \left[2 \left(2 - \frac{w}{2x} \right)^2 - 1 \right] \\ &= 2wx - \frac{1}{2}w^2 - 4x^2 + wx + x^2 \left[8 - \frac{4w}{x} + \frac{w^2}{2x^2} - 1 \right] = -wx + 3x^2 = x(3x - w) \end{aligned}$$

Since $x > 0$, we must have $x = \frac{1}{3}w$, in which case $\cos \theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$, $\sin \theta = \frac{\sqrt{3}}{2}$, $k = \frac{\sqrt{3}}{6}w$, $b_1 = \frac{1}{3}w$,

$b_2 = \frac{2}{3}w$, and $A = \frac{\sqrt{3}}{12}w^2$. As in Example 15.7.6 [ET 14.7.6], we can argue from the physical nature of this problem that we have found a local maximum of A . Now checking the boundary of A , let

$g(\theta) = A(w/2, \theta) = \frac{1}{2}w^2 \sin \theta - \frac{1}{2}w^2 \sin \theta + \frac{1}{4}w^2 \sin \theta \cos \theta = \frac{1}{8}w^2 \sin 2\theta$, $0 < \theta \leq \frac{\pi}{2}$. Clearly g is maximized when $\sin 2\theta = 1$ in which case $A = \frac{1}{8}w^2$. Also along the line $\theta = \frac{\pi}{2}$, let

$h(x) = A(x, \frac{\pi}{2}) = wx - 2x^2$, $0 < x < \frac{1}{2}w \Rightarrow h'(x) = w - 4x = 0 \Leftrightarrow x = \frac{1}{4}w$, and

$h(\frac{1}{4}w) = w(\frac{1}{4}w) - 2(\frac{1}{4}w)^2 = \frac{1}{8}w^2$. Since $\frac{1}{8}w^2 < \frac{\sqrt{3}}{12}w^2$, we conclude that the local maximum found earlier was an absolute maximum.

- (b) If the metal were bent into a semi-circular gutter of radius r , we would have $w = \pi r$ and

$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \left(\frac{w}{\pi} \right)^2 = \frac{w^2}{2\pi}$. Since $\frac{w^2}{2\pi} > \frac{\sqrt{3}w^2}{12}$, it would be better to bend the metal into a gutter with a semicircular cross-section.

4. Since $(x + y + z)^r / (x^2 + y^2 + z^2)$ is a rational function with domain $\{(x, y, z) \mid (x, y, z) \neq (0, 0, 0)\}$,

f is continuous on $\mathbb{R}^3$ if and only if $\lim_{(x,y,z) \rightarrow (0,0,0)} f(x, y, z) = f(0, 0, 0) = 0$. Recall that

$$(a + b)^2 \leq 2a^2 + 2b^2 \text{ and a double application of this inequality to } (x + y + z)^2$$

gives $(x + y + z)^2 \leq 4x^2 + 4y^2 + 4z^2 \leq 4(x^2 + y^2 + z^2)$. Now for each r ,

$$|(x + y + z)^r| = (|x + y + z|^2)^{r/2} = [(x + y + z)^2]^{r/2} \leq [4(x^2 + y^2 + z^2)]^{r/2} = 2^r (x^2 + y^2 + z^2)^{r/2}$$

for $(x, y, z) \neq (0, 0, 0)$. Thus

$$|f(x, y, z) - 0| = \left| \frac{(x + y + z)^r}{x^2 + y^2 + z^2} \right| = \frac{|(x + y + z)^r|}{x^2 + y^2 + z^2} \leq 2^r \frac{(x^2 + y^2 + z^2)^{r/2}}{x^2 + y^2 + z^2} = 2^r (x^2 + y^2 + z^2)^{(r/2)-1}$$

for $(x, y, z) \neq (0, 0, 0)$. Thus if $(r/2) - 1 > 0$, that is $r > 2$, then $2^r (x^2 + y^2 + z^2)^{(r/2)-1} \rightarrow 0$ as

$(x, y, z) \rightarrow (0, 0, 0)$ and so $\lim_{(x,y,z) \rightarrow (0,0,0)} (x + y + z)^r / (x^2 + y^2 + z^2) = 0$. Hence for $r > 2$, f is continuous

on $\mathbb{R}^3$. Now if $r \leq 2$, then as $(x, y, z) \rightarrow (0, 0, 0)$ along the x -axis, $f(x, 0, 0) = x^r / x^2 = x^{r-2}$ for $x \neq 0$. So

when $r = 2$, $f(x, y, z) \rightarrow 1 \neq 0$ as $(x, y, z) \rightarrow (0, 0, 0)$ along the x -axis and when $r < 2$ the limit of $f(x, y, z)$ as

$(x, y, z) \rightarrow (0, 0, 0)$ along the x -axis doesn't exist and thus can't be zero. Hence for $r \leq 2$ f isn't continuous at

$(0, 0, 0)$ and thus is not continuous on $\mathbb{R}^3$.

5. Let $g(x, y) = xf\left(\frac{y}{x}\right)$. Then $g_x(x, y) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$ and

$g_y(x, y) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$. Thus the tangent plane at (x_0, y_0, z_0) on the surface has equation

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left[f\left(\frac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\frac{y_0}{x_0}\right) \right] (x - x_0) + f'\left(\frac{y_0}{x_0}\right) (y - y_0) \Rightarrow$$

$$\left[f\left(\frac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\frac{y_0}{x_0}\right) \right] x + \left[f'\left(\frac{y_0}{x_0}\right) \right] y - z = 0. \text{ But any plane whose equation is of the form}$$

$ax + by + cz = 0$ passes through the origin. Thus the origin is the common point of intersection.

6. (a) At $(x_1, y_1, 0)$ the equations of the tangent planes to $z = f(x, y)$ and $z = g(x, y)$ are

$$P_1: z - f(x_1, y_1) = f_x(x_1, y_1)(x - x_1) + f_y(x_1, y_1)(y - y_1)$$

and

$$P_2: z - g(x_1, y_1) = g_x(x_1, y_1)(x - x_1) + g_y(x_1, y_1)(y - y_1)$$

respectively. P_1 intersects the xy -plane in the line given by

$$f_x(x_1, y_1)(x - x_1) + f_y(x_1, y_1)(y - y_1) = -f(x_1, y_1), z = 0; \text{ and } P_2 \text{ intersects the } xy\text{-plane in the line}$$

given by $g_x(x_1, y_1)(x - x_1) + g_y(x_1, y_1)(y - y_1) = -g(x_1, y_1), z = 0$. The point $(x_2, y_2, 0)$ is the point of

intersection of these two lines, since $(x_2, y_2, 0)$ is the point where the line of intersection of the two tangent

planes intersects the xy -plane. Thus (x_2, y_2) is the solution of the simultaneous equations

$$f_x(x_1, y_1)(x_2 - x_1) + f_y(x_1, y_1)(y_2 - y_1) = -f(x_1, y_1)$$

and

$$g_x(x_1, y_1)(x_2 - x_1) + g_y(x_1, y_1)(y_2 - y_1) = -g(x_1, y_1)$$

For simplicity, rewrite $f_x(x_1, y_1)$ as f_x and similarly for f_y, g_x, g_y, f and g and solve the equations

$$(f_x)(x_2 - x_1) + (f_y)(y_2 - y_1) = -f \text{ and } (g_x)(x_2 - x_1) + (g_y)(y_2 - y_1) = -g$$

simultaneously for $(x_2 - x_1)$ and $(y_2 - y_1)$. Then $y_2 - y_1 = \frac{gf_x - fg_x}{g_x f_y - f_x g_y}$ or

$$y_2 = y_1 - \frac{gf_x - fg_x}{f_x g_y - g_x f_y} \text{ and } (f_x)(x_2 - x_1) + \frac{(f_y)(gf_x - fg_x)}{g_x f_y - f_x g_y} = -f \text{ so}$$

$$x_2 - x_1 = \frac{-f - [(f_y)(gf_x - fg_x)/(g_x f_y - f_x g_y)]}{f_x} = \frac{f g_y - f_y g}{g_x f_y - f_x g_y}. \text{ Hence } x_2 = x_1 - \frac{f g_y - f_y g}{f_x g_y - g_x f_y}.$$

(b) Let $f(x, y) = x^x + y^y - 1000$ and $g(x, y) = x^y + y^x - 100$. Then we wish to solve the system of equations

$$f(x, y) = 0, g(x, y) = 0. \text{ Recall } \frac{d}{dx} [x^x] = x^x(1 + \ln x) \text{ (differentiate logarithmically), so}$$

$$f_x(x, y) = x^x(1 + \ln x), f_y(x, y) = y^y(1 + \ln y), g_x(x, y) = yx^{y-1} + y^x \ln y, \text{ and}$$

$g_y(x, y) = x^y \ln x + xy^{x-1}$. Looking at the graph, we estimate the first point of intersection of the curves, and thus the solution to the system, to be approximately $(2.5, 4.5)$. Then following the method of part (a), $x_1 = 2.5$, $y_1 = 4.5$ and

$$x_2 = 2.5 - \frac{f(2.5, 4.5) g_y(2.5, 4.5) - f_y(2.5, 4.5) g(2.5, 4.5)}{f_x(2.5, 4.5) g_y(2.5, 4.5) - f_y(2.5, 4.5) g_x(2.5, 4.5)} \approx 2.447674117$$

$$y_2 = 4.5 - \frac{f_x(2.5, 4.5) g(2.5, 4.5) - f(2.5, 4.5) g_x(2.5, 4.5)}{f_x(2.5, 4.5) g_y(2.5, 4.5) - f_y(2.5, 4.5) g_x(2.5, 4.5)} \approx 4.555657467$$

Continuing this procedure, we arrive at the following values. (If you use a CAS, you may need to increase its computational precision.)

$x_1 = 2.5$	$y_1 = 4.5$
$x_2 = 2.447674117$	$y_2 = 4.555657467$
$x_3 = 2.449614877$	$y_3 = 4.551969333$
$x_4 = 2.449624628$	$y_4 = 4.551951420$
$x_5 = 2.449624628$	$y_5 = 4.551951420$

Thus, to six decimal places, the point of intersection is $(2.449625, 4.551951)$. The second point of intersection can be found similarly, or, by symmetry it is approximately $(4.551951, 2.449625)$.

7. (a) $x = r \cos \theta, y = r \sin \theta, z = z$. Then $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$ and

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= \cos \theta \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y \partial x} \frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z \partial x} \frac{\partial z}{\partial r} \right] + \sin \theta \left[\frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z \partial y} \frac{\partial z}{\partial r} \right] \\ &= \frac{\partial^2 u}{\partial x^2} \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta + 2 \frac{\partial^2 u}{\partial y \partial x} \cos \theta \sin \theta \end{aligned}$$

Similarly $\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x} r \sin \theta + \frac{\partial u}{\partial y} r \cos \theta$ and

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} r^2 \sin^2 \theta + \frac{\partial^2 u}{\partial y^2} r^2 \cos^2 \theta - 2 \frac{\partial^2 u}{\partial y \partial x} r^2 \sin \theta \cos \theta - \frac{\partial u}{\partial x} r \cos \theta - \frac{\partial u}{\partial y} r \sin \theta. \text{ So}$$

$$\begin{aligned} & \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \\ &= \frac{\partial^2 u}{\partial x^2} \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta + 2 \frac{\partial^2 u}{\partial y \partial x} \cos \theta \sin \theta + \frac{\partial u}{\partial x} \frac{\cos \theta}{r} + \frac{\partial u}{\partial y} \frac{\sin \theta}{r} \\ & \quad + \frac{\partial^2 u}{\partial x^2} \sin^2 \theta + \frac{\partial^2 u}{\partial y^2} \cos^2 \theta - 2 \frac{\partial^2 u}{\partial y \partial x} \sin \theta \cos \theta - \frac{\partial u}{\partial x} \frac{\cos \theta}{r} - \frac{\partial u}{\partial y} \frac{\sin \theta}{r} + \frac{\partial^2 u}{\partial z^2} \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \end{aligned}$$

(b) $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$. Then

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \rho} = \frac{\partial u}{\partial x} \sin \phi \cos \theta + \frac{\partial u}{\partial y} \sin \phi \sin \theta + \frac{\partial u}{\partial z} \cos \phi, \text{ and}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \rho^2} &= \sin \phi \cos \theta \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y \partial x} \frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial z \partial x} \frac{\partial z}{\partial \rho} \right] \\ & \quad + \sin \phi \sin \theta \left[\frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial z \partial y} \frac{\partial z}{\partial \rho} \right] \\ & \quad + \cos \phi \left[\frac{\partial^2 u}{\partial z^2} \frac{\partial z}{\partial \rho} + \frac{\partial^2 u}{\partial x \partial z} \frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y \partial z} \frac{\partial y}{\partial \rho} \right] \\ &= 2 \frac{\partial^2 u}{\partial y \partial x} \sin^2 \phi \sin \theta \cos \theta + 2 \frac{\partial^2 u}{\partial z \partial x} \sin \phi \cos \phi \cos \theta + 2 \frac{\partial^2 u}{\partial y \partial z} \sin \phi \cos \phi \sin \theta \\ & \quad + \frac{\partial^2 u}{\partial x^2} \sin^2 \phi \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \phi \sin^2 \theta + \frac{\partial^2 u}{\partial z^2} \cos^2 \phi \end{aligned}$$

$$\text{Similarly } \frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x} \rho \cos \phi \cos \theta + \frac{\partial u}{\partial y} \rho \cos \phi \sin \theta - \frac{\partial u}{\partial z} \rho \sin \phi, \text{ and}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \phi^2} &= 2 \frac{\partial^2 u}{\partial y \partial x} \rho^2 \cos^2 \phi \sin \theta \cos \theta - 2 \frac{\partial^2 u}{\partial x \partial z} \rho^2 \sin \phi \cos \phi \cos \theta \\ & \quad - 2 \frac{\partial^2 u}{\partial y \partial z} \rho^2 \sin \phi \cos \phi \sin \theta + \frac{\partial^2 u}{\partial x^2} \rho^2 \cos^2 \phi \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \rho^2 \cos^2 \phi \sin^2 \theta \\ & \quad + \frac{\partial^2 u}{\partial z^2} \rho^2 \sin^2 \phi - \frac{\partial u}{\partial x} \rho \sin \phi \cos \theta - \frac{\partial u}{\partial y} \rho \sin \phi \sin \theta - \frac{\partial u}{\partial z} \rho \cos \phi \end{aligned}$$

$$\text{And } \frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x} \rho \sin \phi \sin \theta + \frac{\partial u}{\partial y} \rho \sin \phi \cos \theta, \text{ while}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \theta^2} &= -2 \frac{\partial^2 u}{\partial y \partial x} \rho^2 \sin^2 \phi \cos \theta \sin \theta + \frac{\partial^2 u}{\partial x^2} \rho^2 \sin^2 \phi \sin^2 \theta \\ & \quad + \frac{\partial^2 u}{\partial y^2} \rho^2 \sin^2 \phi \cos^2 \theta - \frac{\partial u}{\partial x} \rho \sin \phi \cos \theta - \frac{\partial u}{\partial y} \rho \sin \phi \sin \theta \end{aligned}$$

Therefore

$$\begin{aligned}
 & \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial u}{\partial \rho} + \frac{\cot \phi}{\rho^2} \frac{\partial u}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2} \\
 &= \frac{\partial^2 u}{\partial x^2} [(\sin^2 \phi \cos^2 \theta) + (\cos^2 \phi \cos^2 \theta) + \sin^2 \theta] \\
 &+ \frac{\partial^2 u}{\partial y^2} [(\sin^2 \phi \sin^2 \theta) + (\cos^2 \phi \sin^2 \theta) + \cos^2 \theta] + \frac{\partial^2 u}{\partial z^2} [\cos^2 \phi + \sin^2 \phi] \\
 &+ \frac{\partial u}{\partial x} \left[\frac{2 \sin^2 \phi \cos \theta + \cos^2 \phi \cos \theta - \sin^2 \phi \cos \theta - \cos \theta}{\rho \sin \phi} \right] \\
 &+ \frac{\partial u}{\partial y} \left[\frac{2 \sin^2 \phi \sin \theta + \cos^2 \phi \sin \theta - \sin^2 \phi \sin \theta - \sin \theta}{\rho \sin \phi} \right]
 \end{aligned}$$

But $2 \sin^2 \phi \cos \theta + \cos^2 \phi \cos \theta - \sin^2 \phi \cos \theta - \cos \theta = (\sin^2 \phi + \cos^2 \phi - 1) \cos \theta = 0$ and similarly the coefficient of $\partial u / \partial y$ is 0. Also $\sin^2 \phi \cos^2 \theta + \cos^2 \phi \cos^2 \theta + \sin^2 \theta = \cos^2 \theta (\sin^2 \phi + \cos^2 \phi) + \sin^2 \theta = 1$, and similarly the coefficient of $\partial^2 u / \partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

8. The tangent plane to the surface $xy^2z^2 = 1$, at the point (x_0, y_0, z_0) is

$$y_0^2 z_0^2 (x - x_0) + 2x_0 y_0 z_0^2 (y - y_0) + 2x_0 y_0^2 z_0 (z - z_0) = 0 \Rightarrow$$

$$(y_0^2 z_0^2)x + (2x_0 y_0 z_0^2)y + (2x_0 y_0^2 z_0)z = 5x_0 y_0^2 z_0^2 = 5. \text{ Using the formula derived in}$$

Example 13.5.8 [ET 12.5.8], we find that the distance from $(0, 0, 0)$ to this tangent plane is

$$D(x_0, y_0, z_0) = \frac{|5x_0 y_0^2 z_0^2|}{\sqrt{(y_0^2 z_0^2)^2 + (2x_0 y_0 z_0^2)^2 + (2x_0 y_0^2 z_0)^2}}.$$

When D is a maximum, D^2 is a maximum and $\nabla D^2 = 0$. Dropping the subscripts, let

$$f(x, y, z) = D^2 = \frac{25(xy^2z^2)^2}{y^2z^2 + 4x^2z^2 + 4x^2y^2}. \text{ Now use the fact that for points on the surface } xy^2z^2 = 1 \text{ we have}$$

$$z^2 = \frac{1}{xy^2}, \text{ to get } f(x, y) = D^2 = \frac{25x}{\frac{1}{x} + \frac{4x}{y^2} + 4x^2y^2} = \frac{25x^2y^2}{y^2 + 4x^2 + 4x^3y^4}. \text{ Now } \nabla D^2 = 0 \Rightarrow f_x = 0$$

$$\text{and } f_y = 0. f_x = 0 \Rightarrow \frac{50xy^2(y^2 + 4x^2 + 4x^3y^4) - (8x + 12x^2y^4)(25x^2y^2)}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \Rightarrow$$

$$xy^2(y^2 + 4x^2 + 4x^3y^4) - (4x + 6x^2y^4)x^2y^2 = 0 \Rightarrow xy^4 - 2x^4y^6 = 0 \Rightarrow xy^4(1 - 2x^3y^2) = 0 \Rightarrow$$

$$1 = 2y^2x^3 \text{ (since } x = 0, y = 0 \text{ both give a minimum distance of 0). Also } f_y = 0 \Rightarrow$$

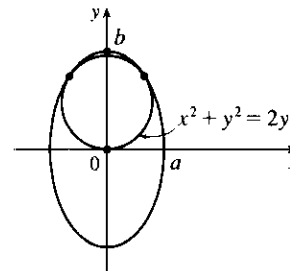
$$\frac{50x^2y(y^2 + 4x^2 + 4x^3y^4) - (2y + 16x^3y^3)25x^2y^2}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \Rightarrow 4x^4y - 4x^5y^5 = 0 \Rightarrow x^4y(1 - xy^4) = 0$$

$$\Rightarrow 1 = xy^4. \text{ Now substituting } x = 1/y^4 \text{ into } 1 = 2y^2x^3, \text{ we get } 1 = 2y^{-10} \Rightarrow y = \pm 2^{1/10} \Rightarrow$$

$$x = 2^{-2/5} \Rightarrow z^2 = \frac{1}{xy^2} = \frac{1}{(2^{-2/5})(2^{1/5})} = 2^{1/5} \Rightarrow z = \pm 2^{1/10}.$$

Therefore the tangent planes that are farthest from the origin are at the four points $(2^{-2/5}, \pm 2^{1/10}, \pm 2^{1/10})$. These points all give a maximum since the minimum distance occurs when $x_0 = 0$ or $y_0 = 0$ in which case $D = 0$. The equations are $(2^{1/5} 2^{1/5})x \pm [(2)(2^{-2/5})(2^{1/10})(2^{1/5})]y \pm [(2)(2^{-2/5})(2^{1/5})(2^{1/10})]z = 5 \Rightarrow (2^{2/5})x \pm (2^{9/10})y \pm (2^{9/10})z = 5$.

9. Since we are minimizing the area of the ellipse, and the circle lies above the x -axis, the ellipse will intersect the circle for only one value of y . This y -value must satisfy both the equation of the circle and the equation of the ellipse. Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow x^2 = \frac{a^2}{b^2}(b^2 - y^2)$. Substituting into



the equation of the circle gives $\frac{a^2}{b^2}(b^2 - y^2) + y^2 - 2y = 0 \Rightarrow$

$\left(\frac{b^2 - a^2}{b^2}\right)y^2 - 2y + a^2 = 0$. In order for there to be only one solution to this quadratic equation, the discriminant must be 0, so $4 - 4a^2 \frac{b^2 - a^2}{b^2} = 0 \Rightarrow b^2 - a^2b^2 + a^4 = 0$. The area of the ellipse is $A(a, b) = \pi ab$, and we minimize this function subject to the constraint $g(a, b) = b^2 - a^2b^2 + a^4 = 0$.

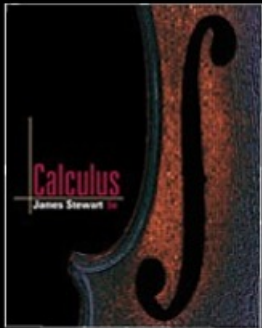
$$\text{Now } \nabla A = \lambda \nabla g \Leftrightarrow \pi b = \lambda(4a^3 - 2ab^2), \pi a = \lambda(2b - 2ba^2) \Rightarrow (1) \lambda = \frac{\pi b}{2a(2a^2 - b^2)},$$

$$(2) \lambda = \frac{\pi a}{2b(1 - a^2)}, (3) b^2 - a^2b^2 + a^4 = 0. \text{ Comparing (1) and (2) gives } \frac{\pi b}{2a(2a^2 - b^2)} = \frac{\pi a}{2b(1 - a^2)} \Rightarrow$$

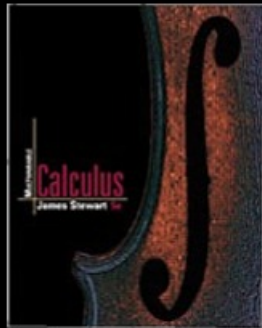
$$2\pi b^2 = 4\pi a^4 \Leftrightarrow a^2 = \frac{1}{\sqrt{2}}b. \text{ Substitute this into (3) to get } b = \frac{3}{\sqrt{2}} \Rightarrow a = \sqrt{\frac{3}{2}}.$$

Chapter 16

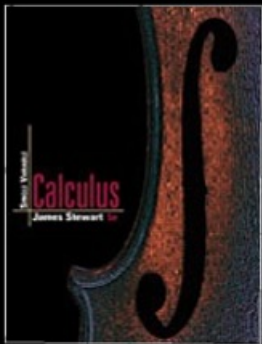
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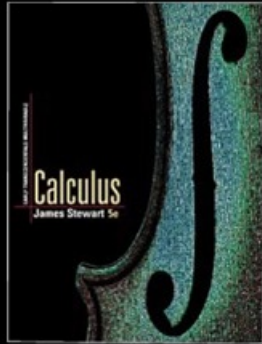
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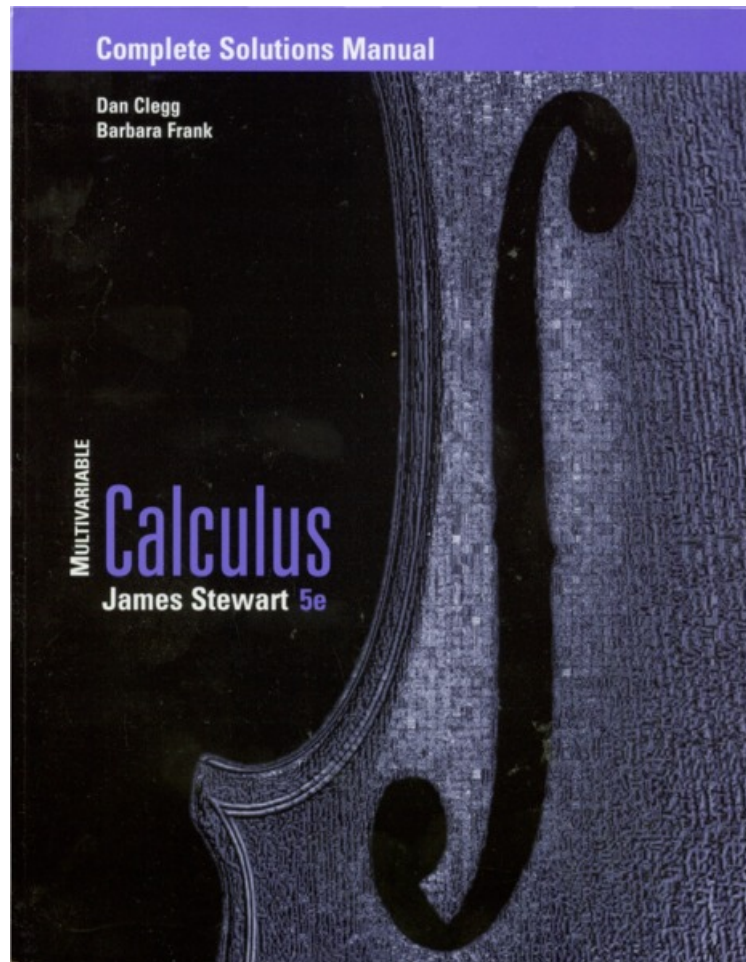
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16 □ MULTIPLE INTEGRALS

□ ET 15

16.1 Double Integrals over Rectangles

ET 15.1

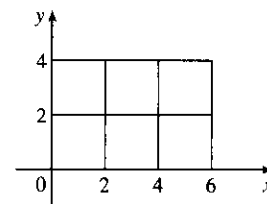
1. (a) The subrectangles are shown in the figure.

The surface is the graph of $f(x, y) = xy$ and $\Delta A = 4$,

so we estimate

$$\begin{aligned} V &\approx \sum_{i=1}^3 \sum_{j=1}^2 f(x_i, y_j) \Delta A \\ &= f(2, 2) \Delta A + f(2, 4) \Delta A + f(4, 2) \Delta A + f(4, 4) \Delta A + f(6, 2) \Delta A + f(6, 4) \Delta A \\ &= 4(4) + 8(4) + 8(4) + 16(4) + 12(4) + 24(4) = 288 \end{aligned}$$

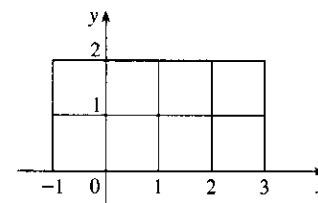
$$\begin{aligned} \text{(b)} \quad V &\approx \sum_{i=1}^3 \sum_{j=1}^2 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &= f(1, 1) \Delta A + f(1, 3) \Delta A + f(3, 1) \Delta A + f(3, 3) \Delta A + f(5, 1) \Delta A + f(5, 3) \Delta A \\ &= 1(4) + 3(4) + 3(4) + 9(4) + 5(4) + 15(4) = 144 \end{aligned}$$



2. The subrectangles are shown in the figure.

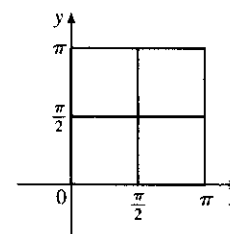
Since $\Delta A = 1$, we estimate

$$\begin{aligned} \iint_R (y^2 - 2x^2) dA &\approx \sum_{i=1}^4 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A \\ &= f(-1, 1) \Delta A + f(-1, 2) \Delta A + f(0, 1) \Delta A + f(0, 2) \Delta A \\ &\quad + f(1, 1) \Delta A + f(1, 2) \Delta A + f(2, 1) \Delta A + f(2, 2) \Delta A \\ &= -1(1) + 2(1) + 1(1) + 4(1) - 1(1) + 2(1) - 7(1) - 4(1) = -4 \end{aligned}$$

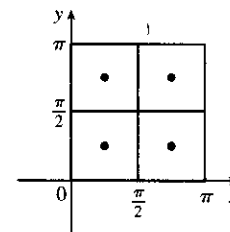


3. (a) The subrectangles are shown in the figure. Since $\Delta A = \pi^2/4$, we estimate

$$\begin{aligned} \iint_R \sin(x + y) dA &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A \\ &= f(0, 0) \Delta A + f(0, \frac{\pi}{2}) \Delta A + f(\frac{\pi}{2}, 0) \Delta A + f(\frac{\pi}{2}, \frac{\pi}{2}) \Delta A \\ &= 0\left(\frac{\pi^2}{4}\right) + 1\left(\frac{\pi^2}{4}\right) + 1\left(\frac{\pi^2}{4}\right) + 0\left(\frac{\pi^2}{4}\right) = \frac{\pi^2}{2} \approx 4.935 \end{aligned}$$



$$\begin{aligned} \text{(b)} \quad \iint_R \sin(x + y) dA &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &= f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) \Delta A + f\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \Delta A \\ &\quad + f\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \Delta A + f\left(\frac{3\pi}{4}, \frac{3\pi}{4}\right) \Delta A \\ &= 1\left(\frac{\pi^2}{4}\right) + 0\left(\frac{\pi^2}{4}\right) + 0\left(\frac{\pi^2}{4}\right) + (-1)\left(\frac{\pi^2}{4}\right) = 0 \end{aligned}$$

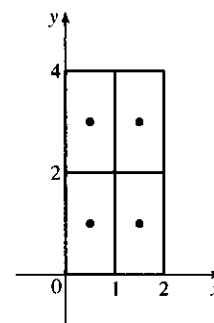
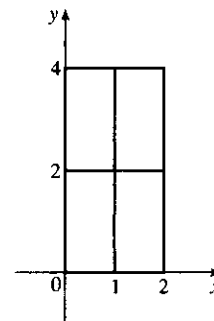


4. (a) The subrectangles are shown in the figure.

The surface is the graph of $f(x, y) = x + 2y^2$ and $\Delta A = 2$, so we estimate

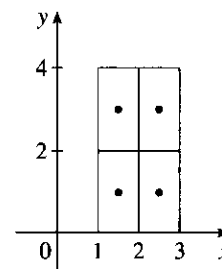
$$\begin{aligned} V &= \iint_R (x + 2y^2) dA \approx \sum_{i=1}^2 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A \\ &= f(1, 0) \Delta A + f(1, 2) \Delta A + f(2, 0) \Delta A + f(2, 2) \Delta A \\ &= 1(2) + 9(2) + 2(2) + 10(2) = 44 \end{aligned}$$

$$\begin{aligned} \text{(b) } V &= \iint_R (x + 2y^2) dA \approx \sum_{i=1}^2 \sum_{j=1}^2 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &= f\left(\frac{1}{2}, 1\right) \Delta A + f\left(\frac{1}{2}, 3\right) \Delta A + f\left(\frac{3}{2}, 1\right) \Delta A + f\left(\frac{3}{2}, 3\right) \Delta A \\ &= \frac{5}{2}(2) + \frac{37}{2}(2) + \frac{7}{2}(2) + \frac{39}{2}(2) = 88 \end{aligned}$$



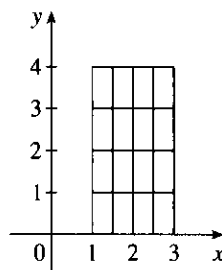
5. (a) Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is
- $\Delta A = 2$
- , so we evaluate
- f
- at each midpoint and estimate

$$\begin{aligned} \iint_R f(x, y) dA &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &= f(1.5, 1) \Delta A + f(1.5, 3) \Delta A \\ &\quad + f(2.5, 1) \Delta A + f(2.5, 3) \Delta A \\ &= 1(2) + (-8)(2) + 5(2) + (-1)(2) = -6 \end{aligned}$$

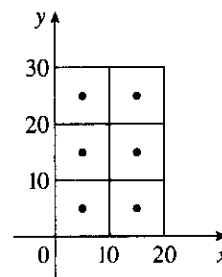


- (b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus we estimate

$$\begin{aligned} \iint_R f(x, y) dA &\approx \sum_{i=1}^4 \sum_{j=1}^4 f(x_i, y_j) \Delta A \\ &= f(1.5, 1) \Delta A + f(1.5, 2) \Delta A + f(1.5, 3) \Delta A + f(1.5, 4) \Delta A \\ &\quad + f(2, 1) \Delta A + f(2, 2) \Delta A + f(2, 3) \Delta A + f(2, 4) \Delta A \\ &\quad + f(2.5, 1) \Delta A + f(2.5, 2) \Delta A + f(2.5, 3) \Delta A + f(2.5, 4) \Delta A \\ &\quad + f(3, 1) \Delta A + f(3, 2) \Delta A + f(3, 3) \Delta A + f(3, 4) \Delta A \\ &= 1\left(\frac{1}{2}\right) + (-4)\left(\frac{1}{2}\right) + (-8)\left(\frac{1}{2}\right) + (-6)\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) + 0\left(\frac{1}{2}\right) + (-5)\left(\frac{1}{2}\right) + (-8)\left(\frac{1}{2}\right) \\ &\quad + 5\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) + (-4)\left(\frac{1}{2}\right) + 8\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) + 0\left(\frac{1}{2}\right) \\ &= -3.5 \end{aligned}$$



6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x, y)$ to be the depth of the water at (x, y) , so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x, y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$\begin{aligned} V &\approx \sum_{i=1}^2 \sum_{j=1}^3 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &= \Delta A [f(5, 5) + f(5, 15) + f(5, 25) + f(15, 5) + f(15, 15) + f(15, 25)] \\ &= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600 \end{aligned}$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$\begin{aligned} V &\approx \sum_{i=1}^4 \sum_{j=1}^6 f(x_i, y_j) \Delta A \\ &= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 \\ &\quad + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4] \\ &= 25(140) = 3500 \end{aligned}$$

So we estimate that the pool contains 3500 ft³ of water.

7. The values of $f(x, y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)
8. From the level curves we see that $f(\frac{1}{2}, \frac{1}{2}) \approx 11$. So, using the Midpoint Rule with only one subrectangle, we get
- $$\iint_R f(x, y) dA \approx 1 \cdot f(\frac{1}{2}, \frac{1}{2}) \approx 11.$$
- Dividing R into four squares of equal size, we get
- $$\iint_R f(x, y) dA \approx \frac{1}{4} [f(\frac{1}{4}, \frac{1}{4}) + f(\frac{1}{4}, \frac{3}{4}) + f(\frac{3}{4}, \frac{1}{4}) + f(\frac{3}{4}, \frac{3}{4})] \approx \frac{1}{4} (11 + 13 + 9.5 + 11) \approx 11.$$
- Using sixteen squares we get the same result. So $\iint_R f(x, y) dA \approx 11$.
9. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

$$\begin{aligned} \iint_R f(x, y) dA &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(\bar{x}_i, \bar{y}_j) \Delta A = \Delta A [f(1, 1) + f(1, 3) + f(3, 1) + f(3, 3)] \\ &\approx 4(27 + 4 + 14 + 17) = 248 \end{aligned}$$

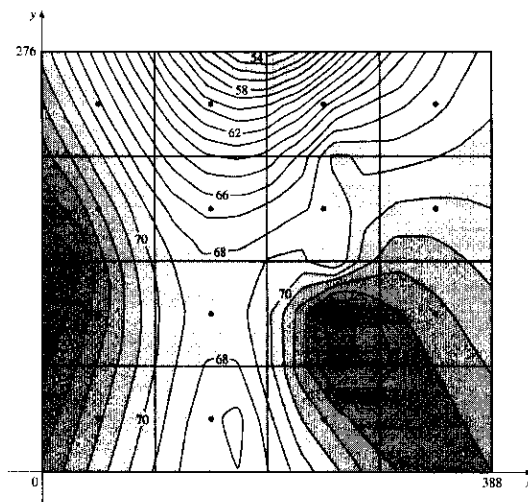
(b) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y) dA \approx \frac{1}{16} (248) = 15.5$

10. As in Example 4, we place the origin at the southwest corner of the state. Then $R = [0, 388] \times [0, 276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x, y)$ to be the temperature at the location (x, y) .

The average temperature is given by

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y) dA = \frac{1}{388 \cdot 276} \iint_R f(x, y) dA$$

We can use the Midpoint Rule with $m = n = 4$ to give a reasonable estimate of the value of the double integral.



Thus, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated. The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\begin{aligned} \iint_R f(x, y) dA &\approx \sum_{i=1}^4 \sum_{j=1}^4 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &\approx \Delta A [72.2 + 73.6 + 72.1 + 68.2 + 67.4 + 68.5 + 66.7 + 60.3 \\ &\quad + 72.0 + 74.9 + 68.4 + 63.7 + 73.2 + 72.3 + 70.3 + 67.7] \\ &= 6693(1111.5) \end{aligned}$$

Therefore, $f_{\text{ave}} \approx \frac{6693 \cdot 1111.5}{388 \cdot 276} \approx 69.5$, so the average temperature in Colorado on May 1, 1996, was approximately 69.5°F .

Alternatively, we can use the Midpoint Rule with $m = n = 2$ which is easier computationally but will most likely be less accurate since we have fewer subrectangles. In this case, $\Delta A = \frac{388}{2} \cdot \frac{276}{2} = 26,772$ and we can use the same grid to estimate the function values at the midpoints of the four subrectangles. Then

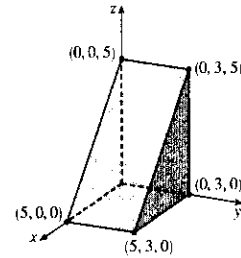
$$\begin{aligned} \iint_R f(x, y) dA &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(\bar{x}_i, \bar{y}_j) \Delta A \approx 26,772 [70.0 + 66.5 + 74.3 + 68.5] \\ &= 26,772 \cdot 279.3 \end{aligned}$$

$$\text{and } f_{\text{ave}} \approx \frac{26,772 \cdot 279.3}{388 \cdot 276} \approx 69.8^\circ\text{F}.$$

11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2, 2] \times [1, 6]$. S is a rectangular solid, thus $\iint_R 3 dA = 4 \cdot 5 \cdot 3 = 60$.

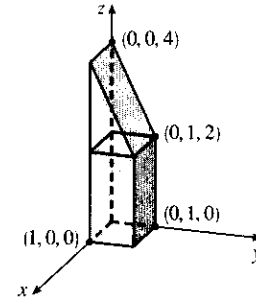
12. $z = 5 - x \geq 0$ for $0 \leq x \leq 5$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 5 - x$ and above the rectangle $[0, 5] \times [0, 3]$. S is a triangular cylinder whose volume is $3(\text{area of triangle}) = 3(\frac{1}{2} \cdot 5 \cdot 5) = 37.5$. Thus,

$$\iint_R (5 - x) dA = 37.5$$

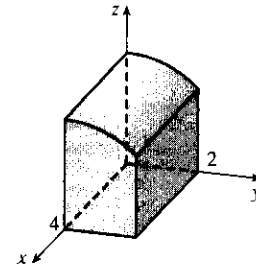


13. $z = f(x, y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0, 1] \times [0, 1] \times [0, 4]$ which lies below the plane $z = 4 - 2y$. So

$$\iint_R (4 - 2y) dA = (1)(1)(2) + \frac{1}{2}(1)(1)(2) = 3$$



14. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x, y) = e^{-x^2 - y^2}$ (calling it `f`), load the `student` package, and then use the command

```
middlesum(middlesum(f, x=0..1, m),
           y=0..1, m);
```

to get the estimate with $n = m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define `f` and then use nested `Sum` commands to calculate the estimates.

n	estimate
1	0.6065
4	0.5694
16	0.5606
64	0.5585
256	0.5579
1024	0.5578

- 16.

n	estimate
1	0.9922
4	0.9262
16	0.8797

n	estimate
64	0.8660
256	0.8625
1024	0.8616

17. If we divide R into mn subrectangles, $\iint_R k \, dA \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$ for any choice of sample points

(x_{ij}^*, y_{ij}^*) . But $f(x_{ij}^*, y_{ij}^*) = k$ always and $\sum_{i=1}^m \sum_{j=1}^n \Delta A = \text{area of } R = (b-a)(d-c)$. Thus, no matter how we

choose the sample points, $\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A = k \sum_{i=1}^m \sum_{j=1}^n \Delta A = k(b-a)(d-c)$ and so

$$\begin{aligned} \iint_R k \, dA &= \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A = \lim_{m,n \rightarrow \infty} k \sum_{i=1}^m \sum_{j=1}^n \Delta A \\ &= \lim_{m,n \rightarrow \infty} k(b-a)(d-c) = k(b-a)(d-c) \end{aligned}$$

18. On R , $0 \leq x+y \leq 2 < \pi$ and $\sin \theta \geq 0$ for $0 \leq \theta \leq \pi$. Thus $f(x, y) = \sin(x+y) \geq 0$ for all $(x, y) \in R$. Since $0 \leq \sin(x+y) \leq 1$, Property (9) gives $\iint_R 0 \, dA \leq \iint_R \sin(x+y) \, dA \leq \iint_R 1 \, dA$, so by Exercise 17 we have $0 \leq \iint_R \sin(x+y) \, dA \leq 1$.

16.2 Iterated Integrals

ET 15.2

$$1. \int_0^3 (2x + 3x^2y) \, dx = [x^2 + x^3y]_{x=0}^{x=3} = (9 + 27y) - (0 + 0) = 9 + 27y,$$

$$\int_0^4 (2x + 3x^2y) \, dy = \left[2xy + 3x^2 \frac{y^2}{2} \right]_{y=0}^{y=4} = \left(8x + 3x^2 \cdot \frac{16}{2} \right) - (0 + 0) = 8x + 24x^2$$

$$2. \int_0^3 \frac{y}{x+2} \, dx = y \ln|x+2| \Big|_{x=0}^{x=3} = y \ln 5 - y \ln 2 = y \ln \frac{5}{2},$$

$$\int_0^4 \frac{y}{x+2} \, dy = \frac{1}{x+2} \left[\frac{y^2}{2} \right]_{y=0}^{y=4} = \frac{1}{x+2} \left(\frac{16}{2} - 0 \right) = \frac{8}{x+2}$$

$$3. \int_1^3 \int_0^1 (1 + 4xy) \, dx \, dy = \int_1^3 [x + 2x^2y]_{x=0}^{x=1} \, dy = \int_1^3 (1 + 2y) \, dy = [y + y^2]_1^3 = (3 + 9) - (1 + 1) = 10$$

$$\begin{aligned} 4. \int_2^4 \int_{-1}^1 (x^2 + y^2) \, dy \, dx &= \int_2^4 [x^2y + \frac{1}{3}y^3]_{y=-1}^{y=1} \, dx = \int_2^4 [(x^2 + \frac{1}{3}) - (-x^2 - \frac{1}{3})] \, dx \\ &= \int_2^4 (2x^2 + \frac{2}{3}) \, dx = [\frac{2}{3}x^3 + \frac{2}{3}x]_2^4 = (\frac{128}{3} + \frac{8}{3}) - (\frac{16}{3} + \frac{4}{3}) = \frac{116}{3} \end{aligned}$$

$$5. \int_0^2 \int_0^{\pi/2} x \sin y \, dy \, dx = \int_0^2 x \, dx \int_0^{\pi/2} \sin y \, dy \quad [\text{as in Example 5}]$$

$$= \left[\frac{x^2}{2} \right]_0^2 [-\cos y]_0^{\pi/2} = (2 - 0)(0 + 1) = 2.$$

$$\begin{aligned} 6. \int_1^4 \int_0^2 (x + \sqrt{y}) \, dx \, dy &= \int_1^4 [\frac{1}{2}x^2 + x\sqrt{y}]_{x=0}^{x=2} \, dy = \int_1^4 (2 + 2\sqrt{y}) \, dy \\ &= \left[2y + 2 \cdot \frac{2}{3}y^{3/2} \right]_1^4 = (8 + \frac{4}{3} \cdot 8) - (2 + \frac{4}{3}) = \frac{46}{3} \end{aligned}$$

$$\begin{aligned}
 7. \int_0^2 \int_0^1 (2x+y)^8 dx dy &= \int_0^2 \left[\frac{1}{2} \frac{(2x+y)^9}{9} \right]_{x=0}^{x=1} dy \quad [\text{substitute } u = 2x+y \Rightarrow dx = \frac{1}{2} du] \\
 &= \frac{1}{18} \int_0^2 [(2+y)^9 - (0+y)^9] dy = \frac{1}{18} \left[\frac{(2+y)^{10}}{10} - \frac{y^{10}}{10} \right]_0^2 \\
 &= \frac{1}{180} [(4^{10} - 2^{10}) - (2^{10} - 0^{10})] = \frac{1,046,528}{180} = \frac{261,632}{45}
 \end{aligned}$$

$$\begin{aligned}
 8. \int_0^1 \int_1^2 \frac{xe^x}{y} dy dx &= \int_0^1 xe^x dx \int_1^2 \frac{1}{y} dy \quad [\text{as in Example 5}] \\
 &= [xe^x - e^x]_0^1 [\ln|y|]_1^2 \quad [\text{by integrating by parts}] \\
 &= [(e - e) - (0 - 1)](\ln 2 - 0) = \ln 2
 \end{aligned}$$

$$\begin{aligned}
 9. \int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy dx &= \int_1^4 \left[x \ln|y| + \frac{1}{x} \cdot \frac{1}{2} y^2 \right]_{y=1}^{y=2} dx = \int_1^4 \left(x \ln 2 + \frac{3}{2x} \right) dx \\
 &= \left[\frac{1}{2} x^2 \ln 2 + \frac{3}{2} \ln|x| \right]_1^4 = 8 \ln 2 + \frac{3}{2} \ln 4 - \frac{1}{2} \ln 2 \\
 &= \frac{15}{2} \ln 2 + 3 \ln 4^{1/2} = \frac{21}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 10. \int_1^2 \int_0^1 (x+y)^{-2} dx dy &= \int_1^2 \left[-(x+y)^{-1} \right]_{x=0}^{x=1} dy = \int_1^2 [y^{-1} - (1+y)^{-1}] dy \\
 &= [\ln y - \ln(1+y)]_1^2 = \ln 2 - \ln 3 - 0 + \ln 2 = \ln \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 11. \int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y} dx dy &= \left(\int_0^{\ln 5} e^{2x} dx \right) \left(\int_0^{\ln 2} e^{-y} dy \right) = \left[\frac{1}{2} e^{2x} \right]_0^{\ln 5} [-e^{-y}]_0^{\ln 2} \\
 &= \left(\frac{25}{2} - \frac{1}{2} \right) \left(-\frac{1}{2} + 1 \right) = 6
 \end{aligned}$$

$$\begin{aligned}
 12. \int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} dy dx &= \int_0^1 \left[x \sqrt{x^2+y^2+1} \right]_{y=0}^{y=1} dx = \int_0^1 x(\sqrt{x^2+2} - \sqrt{x^2+1}) dx \\
 &= \frac{1}{3} \left[(x^2+2)^{3/2} - (x^2+1)^{3/2} \right]_0^1 = \frac{1}{3} \left[(3^{3/2} - 2^{3/2}) - (2^{3/2} - 1) \right] \\
 &= \frac{1}{3} (3\sqrt{3} - 4\sqrt{2} + 1)
 \end{aligned}$$

$$\begin{aligned}
 13. \iint_R (6x^2y^3 - 5y^4) dA &= \int_0^3 \int_0^1 (6x^2y^3 - 5y^4) dy dx = \int_0^3 \left[\frac{3}{2} x^2 y^4 - y^5 \right]_{y=0}^{y=1} dx \\
 &= \int_0^3 \left(\frac{3}{2} x^2 - 1 \right) dx = \left[\frac{1}{2} x^3 - x \right]_0^3 = \frac{27}{2} - 3 = \frac{21}{2}
 \end{aligned}$$

$$\begin{aligned}
 14. \iint_R \cos(x+2y) dA &= \int_0^\pi \int_0^{\pi/2} \cos(x+2y) dy dx \\
 &= \int_0^\pi \left[\frac{1}{2} \sin(x+2y) \right]_{y=0}^{y=\pi/2} dx = \frac{1}{2} \int_0^\pi (\sin(x+\pi) - \sin x) dx \\
 &= \frac{1}{2} [-\cos(x+\pi) + \cos x]_0^\pi = \frac{1}{2} [-\cos 2\pi + \cos \pi - (-\cos \pi + \cos 0)] \\
 &= \frac{1}{2} (-1 - 1 - (1 + 1)) = -2
 \end{aligned}$$

$$\begin{aligned}
 15. \iint_R \frac{xy^2}{x^2+1} dA &= \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} dy dx = \int_0^1 \frac{x}{x^2+1} dx \int_{-3}^3 y^2 dy \\
 &= \left[\frac{1}{2} \ln(x^2+1) \right]_0^1 \left[\frac{1}{3} y^3 \right]_{-3}^3 = \frac{1}{2} (\ln 2 - \ln 1) \cdot \frac{1}{3} (27 + 27) = 9 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 16. \iint_R \frac{1+x^2}{1+y^2} dA &= \int_0^1 \int_0^1 \frac{1+x^2}{1+y^2} dy dx = \int_0^1 (1+x^2) dx \int_0^1 \frac{1}{1+y^2} dy \\
 &= \left[x + \frac{1}{3}x^3 \right]_0^1 \left[\tan^{-1} y \right]_0^1 = \left(1 + \frac{1}{3} - 0 \right) \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 17. \int_0^{\pi/6} \int_0^{\pi/3} x \sin(x+y) dy dx &= \int_0^{\pi/6} \left[-x \cos(x+y) \right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x \cos x - x \cos\left(x + \frac{\pi}{3}\right) \right] dx \\
 &= x \left[\sin x - \sin\left(x + \frac{\pi}{3}\right) \right]_0^{\pi/6} - \int_0^{\pi/6} \left[\sin x - \sin\left(x + \frac{\pi}{3}\right) \right] dx \\
 &\quad \text{[by integrating by parts separately for each term]} \\
 &= \frac{\pi}{6} \left[\frac{1}{2} - 1 \right] - \left[-\cos x + \cos\left(x + \frac{\pi}{3}\right) \right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} + 0 - (-1 + \frac{1}{2}) \right] \\
 &= \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}
 \end{aligned}$$

$$\begin{aligned}
 18. \iint_R \frac{x}{1+xy} dA &= \int_0^1 \int_0^1 \frac{x}{1+xy} dy dx \\
 &= \int_0^1 \left[\ln(1+xy) \right]_{y=0}^{y=1} dx = \int_0^1 [\ln(1+x) - \ln 1] dx \\
 &= \int_0^1 \ln(1+x) dx = [(1+x) \ln(1+x) - x]_0^1 \quad \text{[by integrating by parts]} \\
 &= (2 \ln 2 - 1) - (\ln 1 - 0) = 2 \ln 2 - 1
 \end{aligned}$$

$$\begin{aligned}
 19. \iint_R xy e^{x^2 y} dA &= \int_0^2 \int_0^1 xy e^{x^2 y} dx dy = \int_0^2 \left[\frac{1}{2} e^{x^2 y} \right]_{x=0}^{x=1} dy = \frac{1}{2} \int_0^2 (e^y - 1) dy \\
 &= \frac{1}{2} [e^y - y]_0^2 = \frac{1}{2} [(e^2 - 2) - (1 - 0)] = \frac{1}{2} (e^2 - 3)
 \end{aligned}$$

$$20. \int_0^1 \int_1^2 \frac{x}{x^2 + y^2} dx dy = \int_0^1 \left[\frac{1}{2} \ln(x^2 + y^2) \right]_{x=1}^{x=2} dy = \frac{1}{2} \int_0^1 [\ln(4 + y^2) - \ln(1 + y^2)] dy$$

To evaluate the first term, we integrate by parts with $u = \ln(4 + y^2) \Rightarrow du = \frac{2y}{4 + y^2} dy$ and

$dv = dy \Rightarrow v = y$. Then

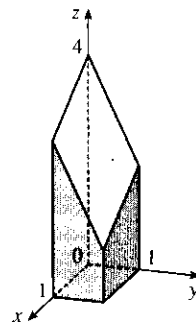
$$\begin{aligned}
 \int \ln(4 + y^2) dy &= y \ln(4 + y^2) - \int \frac{2y^2}{4 + y^2} dy = y \ln(4 + y^2) - \int \left(2 - \frac{8}{4 + y^2} \right) dy \\
 &= y \ln(4 + y^2) - 2y + 8 \cdot \frac{1}{2} \tan^{-1} \left(\frac{y}{2} \right) = y \ln(4 + y^2) - 2y + 4 \tan^{-1} \left(\frac{y}{2} \right)
 \end{aligned}$$

Similarly, $\int \ln(1 + y^2) dy = y \ln(1 + y^2) - 2y + 2 \tan^{-1} y$. Thus,

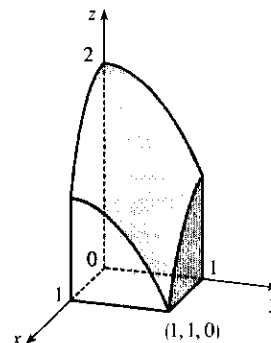
$$\begin{aligned}
 \int_0^1 \int_1^2 \frac{x}{x^2 + y^2} dx dy &= \frac{1}{2} \int_0^1 [\ln(4 + y^2) - \ln(1 + y^2)] dy \\
 &= \frac{1}{2} \left[y \ln(4 + y^2) - 2y + 4 \tan^{-1} \left(\frac{y}{2} \right) - y \ln(1 + y^2) + 2y - 2 \tan^{-1} y \right]_0^1 \\
 &= \frac{1}{2} \left[(\ln 5 + 4 \tan^{-1} \left(\frac{1}{2} \right) - \ln 2 - 2 \tan^{-1} 1) - 0 \right] \\
 &= \frac{1}{2} \left[\ln 5 - \ln 2 + 4 \tan^{-1} \left(\frac{1}{2} \right) - 2 \left(\frac{\pi}{4} \right) \right] = \frac{1}{2} \ln \frac{5}{2} + 2 \tan^{-1} \left(\frac{1}{2} \right) - \frac{\pi}{4}
 \end{aligned}$$

21. $z = f(x, y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0, 1] \times [0, 1]$.



22. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0, 1] \times [0, 1]$.



$$\begin{aligned} 23. V &= \iint_R (12 - 3x - 2y) dA = \int_{-2}^3 \int_0^1 (12 - 3x - 2y) dx dy = \int_{-2}^3 \left[12x - \frac{3}{2}x^2 - 2xy \right]_{x=0}^{x=1} dy \\ &= \int_{-2}^3 \left(\frac{21}{2} - 2y \right) dy = \left[\frac{21}{2}y - y^2 \right]_{-2}^3 = \frac{95}{2} \end{aligned}$$

$$\begin{aligned} 24. V &= \iint_R (4 + x^2 - y^2) dA = \int_{-1}^1 \int_0^2 (4 + x^2 - y^2) dy dx = \int_{-1}^1 \left[4y + x^2y - \frac{1}{3}y^3 \right]_{y=0}^{y=2} dx \\ &= \int_{-1}^1 \left(2x^2 + \frac{16}{3} \right) dx = \left[\frac{2}{3}x^3 + \frac{16}{3}x \right]_{-1}^1 = \frac{2}{3} + \frac{16}{3} + \frac{2}{3} + \frac{16}{3} = 12 \end{aligned}$$

$$\begin{aligned} 25. V &= \int_{-2}^2 \int_{-1}^1 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2 \right) dx dy = 4 \int_0^2 \int_0^1 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2 \right) dx dy \\ &= 4 \int_0^2 \left[x - \frac{1}{12}x^3 - \frac{1}{9}y^2x \right]_{x=0}^{x=1} dy = 4 \int_0^2 \left(\frac{11}{12} - \frac{1}{9}y^2 \right) dy = 4 \left[\frac{11}{12}y - \frac{1}{27}y^3 \right]_0^2 = 4 \cdot \frac{83}{54} = \frac{166}{27} \end{aligned}$$

$$\begin{aligned} 26. V &= \int_{-1}^1 \int_0^\pi (1 + e^x \sin y) dy dx = \int_{-1}^1 \left[y - e^x \cos y \right]_{y=0}^{y=\pi} dx = \int_{-1}^1 (\pi + e^x - 0 + e^x) dx \\ &= \int_{-1}^1 (\pi + 2e^x) dx = \left[\pi x + 2e^x \right]_{-1}^1 = 2\pi + 2e - \frac{2}{e} \end{aligned}$$

27. Here we need the volume of the solid lying under the surface $z = x\sqrt{x^2 + y}$ and above the square $R = [0, 1] \times [0, 1]$ in the xy -plane.

$$\begin{aligned} V &= \int_0^1 \int_0^1 x\sqrt{x^2 + y} dx dy = \int_0^1 \frac{1}{3} \left[(x^2 + y)^{3/2} \right]_{x=0}^{x=1} dy = \frac{1}{3} \int_0^1 \left[(1 + y)^{3/2} - y^{3/2} \right] dy \\ &= \frac{1}{3} \cdot \frac{2}{5} \left[(1 + y)^{5/2} - y^{5/2} \right]_0^1 = \frac{4}{15} (2\sqrt{2} - 1) \end{aligned}$$

28. Here we need the volume of the solid lying under the surface $z = 1 + (x - 1)^2 + 4y^2$ and above the rectangle $R = [0, 3] \times [0, 2]$ in the xy -plane.

$$\begin{aligned} V &= \int_0^3 \int_0^2 [1 + (x - 1)^2 + 4y^2] dy dx = \int_0^3 \left[y + (x - 1)^2y + \frac{4}{3}y^3 \right]_{y=0}^{y=2} dx \\ &= \int_0^3 \left[2 + 2(x - 1)^2 + \frac{32}{3} \right] dx = \left[\frac{38}{3}x + \frac{2}{3}(x - 1)^3 \right]_0^3 = 44 \end{aligned}$$

29. In the first octant, $z \geq 0 \Rightarrow y \leq 3$, so

$$V = \int_0^3 \int_0^2 (9 - y^2) dx dy = \int_0^3 [9x - y^2x]_{x=0}^{x=2} dy = \int_0^3 (18 - 2y^2) dy = \left[18y - \frac{2}{3}y^3 \right]_0^3 = 36$$

35. Let $f(x, y) = \frac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1 \int_0^1 f(x, y) dy dx = \frac{1}{2}$ and $\int_0^1 \int_0^1 f(x, y) dx dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0, 0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

36. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy} , we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1:

$$g_x = \frac{d}{dx} g(x, y) = \frac{d}{dx} \int_a^x \left(\int_c^y f(s, t) dt \right) ds = \int_c^y f(x, t) dt. \text{ Now we use the Fundamental Theorem again: } g_{xy} = \frac{d}{dy} \int_c^y f(x, t) dt = f(x, y).$$

To find g_{yx} , we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s, t) dt ds = \int_c^y \int_a^x f(s, t) dt ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x, y)$. So $g_{xy} = g_{yx} = f(x, y)$.

16.3 Double Integrals over General Regions

ET 15.3

- $\int_0^1 \int_0^{x^2} (x+2y) dy dx = \int_0^1 [xy + y^2]_{y=0}^{y=x^2} dx = \int_0^1 [x(x^2) + (x^2)^2 - 0 - 0] dx$
 $= \int_0^1 (x^3 + x^4) dx = [\frac{1}{4}x^4 + \frac{1}{5}x^5]_0^1 = \frac{9}{20}$
- $\int_1^2 \int_y^2 xy dx dy = \int_1^2 [\frac{1}{2}x^2 y]_{x=y}^{x=2} dy = \int_1^2 \frac{1}{2}y(4 - y^2) dy = \frac{1}{2} \int_1^2 (4y - y^3) dy$
 $= \frac{1}{2} [2y^2 - \frac{1}{4}y^4]_1^2 = \frac{1}{2} (8 - 4 - 2 + \frac{1}{4}) = \frac{9}{8}$
- $\int_0^1 \int_y^{e^y} \sqrt{x} dx dy = \int_0^1 [\frac{2}{3}x^{3/2}]_{x=y}^{x=e^y} dy = \frac{2}{3} \int_0^1 (e^{3y/2} - y^{3/2}) dy = \frac{2}{3} [\frac{2}{3}e^{3y/2} - \frac{2}{5}y^{5/2}]_0^1$
 $= \frac{2}{3} (\frac{2}{3}e^{3/2} - \frac{2}{5} - \frac{2}{3}e^0 + 0) = \frac{4}{9}e^{3/2} - \frac{32}{45}$
- $\int_0^1 \int_x^{2-x} (x^2 - y) dy dx = \int_0^1 [x^2y - \frac{1}{2}y^2]_{y=x}^{y=2-x} dx = \int_0^1 [x^2(2-x) - \frac{1}{2}(2-x)^2 - x^2(x) + \frac{1}{2}x^2] dx$
 $= \int_0^1 (-2x^3 + 2x^2 + 2x - 2) dx = [-\frac{1}{2}x^4 + \frac{2}{3}x^3 + x^2 - 2x]_0^1 = -\frac{5}{6}$
- $\int_0^{\pi/2} \int_0^{\cos \theta} e^{\sin \theta} dr d\theta = \int_0^{\pi/2} [re^{\sin \theta}]_{r=0}^{r=\cos \theta} d\theta = \int_0^{\pi/2} (\cos \theta) e^{\sin \theta} d\theta = e^{\sin \theta} \Big|_0^{\pi/2}$
 $= e^{\sin(\pi/2)} - e^0 = e - 1$
- $\int_0^1 \int_0^v \sqrt{1-v^2} du dv = \int_0^1 [u\sqrt{1-v^2}]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2} dv = -\frac{1}{3}(1-v^2)^{3/2} \Big|_0^1$
 $= -\frac{1}{3}(0-1) = \frac{1}{3}$
- $\iint_D x^3 y^2 dA = \int_0^2 \int_{-x}^x x^3 y^2 dy dx = \int_0^2 [\frac{1}{3}x^3 y^3]_{y=-x}^{y=x} dx = \frac{1}{3} \int_0^2 2x^6 dx$
 $= \frac{2}{3} [\frac{1}{7}x^7]_0^2 = \frac{2}{21} [2^7 - 0] = \frac{256}{21}$

$$\begin{aligned}
 8. \iint_D \frac{4y}{x^3+2} dA &= \int_1^2 \int_0^{2x} \frac{4y}{x^3+2} dy dx = \int_1^2 \left[\frac{2y^2}{x^3+2} \right]_{y=0}^{y=2x} dx = \int_1^2 \frac{8x^2}{x^3+2} dx \\
 &= \frac{8}{3} \ln |x^3+2| \Big|_1^2 = \frac{8}{3} (\ln 10 - \ln 3) = \frac{8}{3} \ln \frac{10}{3}
 \end{aligned}$$

$$\begin{aligned}
 9. \int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1} dy dx &= \int_0^1 \left[\frac{y^2}{x^2+1} \right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1} dx \\
 &= \frac{1}{2} \ln |x^2+1| \Big|_0^1 = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2
 \end{aligned}$$

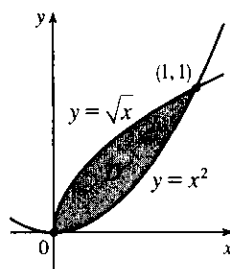
$$10. \int_0^1 \int_0^y e^{y^2} dx dy = \int_0^1 [xe^{y^2}]_{x=0}^{x=y} dy = \int_0^1 ye^{y^2} dy = \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{1}{2} (e - 1)$$

$$11. \int_1^2 \int_y^{y^3} e^{x/y} dx dy = \int_1^2 [ye^{x/y}]_{x=y}^{x=y^3} dy = \int_1^2 (ye^{y^2} - ey) dy = \left[\frac{1}{2} e^{y^2} - \frac{1}{2} ey^2 \right]_1^2 = \frac{1}{2} (e^4 - 4e)$$

$$12. \int_0^1 \int_0^y x \sqrt{y^2 - x^2} dx dy = \int_0^1 \left[-\frac{1}{3} (y^2 - x^2)^{3/2} \right]_{x=0}^{x=y} dy = \frac{1}{3} \int_0^1 y^3 dy = \frac{1}{3} \cdot \frac{1}{4} y^4 \Big|_0^1 = \frac{1}{12}$$

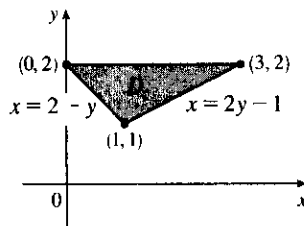
$$13. \int_0^1 \int_0^{x^2} x \cos y dy dx = \int_0^1 [x \sin y]_{y=0}^{y=x^2} dx = \int_0^1 x \sin x^2 dx = -\frac{1}{2} \cos x^2 \Big|_0^1 = \frac{1}{2} (1 - \cos 1)$$

14.



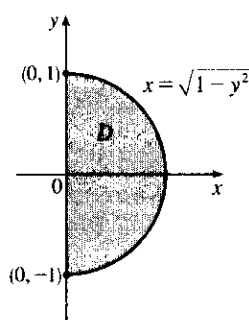
$$\begin{aligned}
 \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y) dy dx &= \int_0^1 \left[xy + \frac{1}{2} y^2 \right]_{y=x^2}^{y=\sqrt{x}} dx \\
 &= \int_0^1 \left(x^{3/2} + \frac{1}{2} x - x^3 - \frac{1}{2} x^4 \right) dx \\
 &= \left[\frac{2}{5} x^{5/2} + \frac{1}{4} x^2 - \frac{1}{4} x^4 - \frac{1}{10} x^5 \right]_0^1 = \frac{3}{10}
 \end{aligned}$$

15.



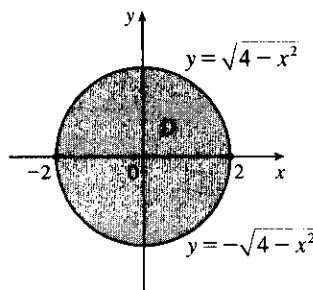
$$\begin{aligned}
 \int_1^2 \int_{2-y}^{2y-1} y^3 dx dy &= \int_1^2 [xy^3]_{x=2-y}^{x=2y-1} dy \\
 &= \int_1^2 [(2y-1) - (2-y)] y^3 dy \\
 &= \int_1^2 (3y^4 - 3y^3) dy = \left[\frac{3}{5} y^5 - \frac{3}{4} y^4 \right]_1^2 \\
 &= \frac{96}{5} - 12 - \frac{3}{5} + \frac{3}{4} = \frac{147}{20}
 \end{aligned}$$

16.



$$\begin{aligned}
 \iint_D xy^2 dA &= \int_{-1}^1 \int_0^{\sqrt{1-y^2}} xy^2 dx dy \\
 &= \int_{-1}^1 y^2 \left[\frac{1}{2} x^2 \right]_{x=0}^{x=\sqrt{1-y^2}} dy = \frac{1}{2} \int_{-1}^1 y^2 (1-y^2) dy \\
 &= \frac{1}{2} \int_{-1}^1 (y^2 - y^4) dy = \frac{1}{2} \left[\frac{1}{3} y^3 - \frac{1}{5} y^5 \right]_{-1}^1 \\
 &= \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5} \right) = \frac{2}{15}
 \end{aligned}$$

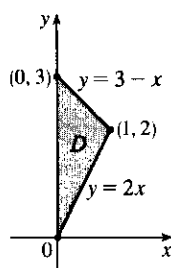
17.



$$\begin{aligned}
 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y) dy dx &= \int_{-2}^2 \left[2xy - \frac{1}{2}y^2 \right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx \\
 &= \int_{-2}^2 \left[2x\sqrt{4-x^2} - \frac{1}{2}(4-x^2) + 2x\sqrt{4-x^2} + \frac{1}{2}(4-x^2) \right] dx \\
 &= \int_{-2}^2 4x\sqrt{4-x^2} dx = -\frac{4}{3}(4-x^2)^{3/2} \Big|_{-2}^2 = 0
 \end{aligned}$$

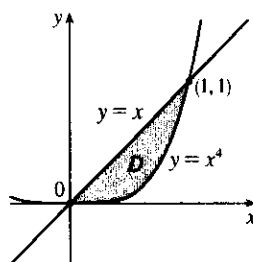
(Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^2 4x\sqrt{4-x^2} dx = 0$.)

18.



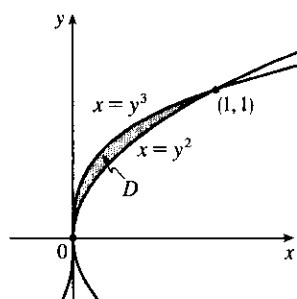
$$\begin{aligned}
 \iint_D 2xy dA &= \int_0^1 \int_{2x}^{3-x} 2xy dy dx = \int_0^1 [xy^2]_{y=2x}^{y=3-x} dx \\
 &= \int_0^1 x[(3-x)^2 - (2x)^2] dx \\
 &= \int_0^1 (-3x^3 - 6x^2 + 9x) dx \\
 &= \left[-\frac{3}{4}x^4 - 2x^3 + \frac{9}{2}x^2 \right]_0^1 = -\frac{3}{4} - 2 + \frac{9}{2} = \frac{7}{4}
 \end{aligned}$$

19.



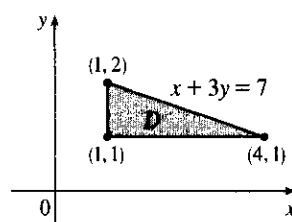
$$\begin{aligned}
 V &= \int_0^1 \int_{x^4}^x (x+2y) dy dx \\
 &= \int_0^1 [xy + y^2]_{y=x^4}^{y=x} dx = \int_0^1 (2x^2 - x^5 - x^8) dx \\
 &= \left[\frac{2}{3}x^3 - \frac{1}{6}x^6 - \frac{1}{9}x^9 \right]_0^1 = \frac{2}{3} - \frac{1}{6} - \frac{1}{9} = \frac{7}{18}
 \end{aligned}$$

20.



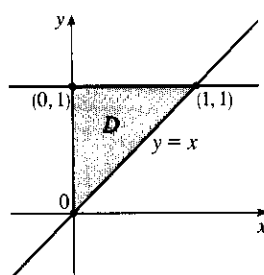
$$\begin{aligned}
 V &= \int_0^1 \int_{y^3}^{y^2} (2x+y^2) dx dy \\
 &= \int_0^1 [x^2 + xy^2]_{x=y^3}^{x=y^2} dy = \int_0^1 (2y^4 - y^6 - y^5) dy \\
 &= \left[\frac{2}{5}y^5 - \frac{1}{7}y^7 - \frac{1}{6}y^6 \right]_0^1 = \frac{19}{210}
 \end{aligned}$$

21.



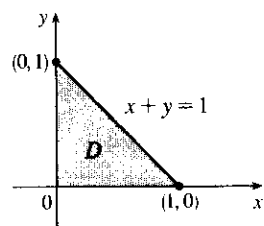
$$\begin{aligned}
 V &= \int_1^2 \int_1^{7-3y} xy dx dy = \int_1^2 \left[\frac{1}{2}x^2 y \right]_{x=1}^{x=7-3y} dy \\
 &= \frac{1}{2} \int_1^2 (48y - 42y^2 + 9y^3) dy \\
 &= \frac{1}{2} \left[24y^2 - 14y^3 + \frac{9}{4}y^4 \right]_1^2 = \frac{31}{8}
 \end{aligned}$$

22.



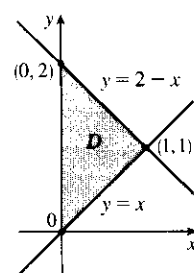
$$\begin{aligned}
 V &= \int_0^1 \int_x^1 (x^2 + 3y^2) dy dx \\
 &= \int_0^1 [x^2 y + y^3]_{y=x}^{y=1} dx = \int_0^1 (x^2 + 1 - 2x^3) dx \\
 &= \left[\frac{1}{3}x^3 + x - \frac{1}{2}x^4 \right]_0^1 = \frac{5}{6}
 \end{aligned}$$

23.



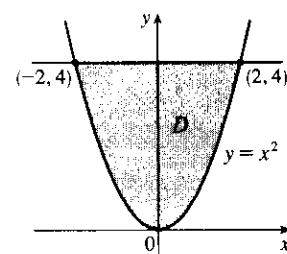
$$\begin{aligned}
 V &= \int_0^1 \int_0^{1-x} (1-x-y) dy dx \\
 &= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1-x} dx \\
 &= \int_0^1 \left[(1-x)^2 - \frac{1}{2}(1-x)^2 \right] dx \\
 &= \int_0^1 \frac{1}{2}(1-x)^2 dx = \left[-\frac{1}{6}(1-x)^3 \right]_0^1 = \frac{1}{6}
 \end{aligned}$$

24.



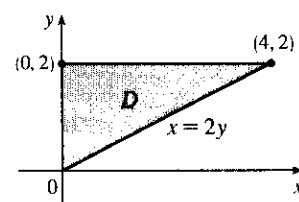
$$\begin{aligned}
 V &= \int_0^1 \int_x^{2-x} x dy dx \\
 &= \int_0^1 x [y]_{y=x}^{y=2-x} dx = \int_0^1 (2x - x^2) dx \\
 &= \left[x^2 - \frac{2}{3}x^3 \right]_0^1 = \frac{1}{3}
 \end{aligned}$$

25.



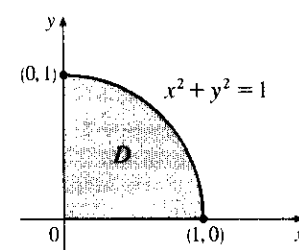
$$\begin{aligned}
 V &= \int_{-2}^2 \int_{x^2}^4 x^2 dy dx \\
 &= \int_{-2}^2 x^2 [y]_{y=x^2}^{y=4} dx = \int_{-2}^2 (4x^2 - x^4) dx \\
 &= \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{-2}^2 = \frac{32}{3} - \frac{32}{5} + \frac{32}{3} - \frac{32}{5} = \frac{128}{15}
 \end{aligned}$$

26.



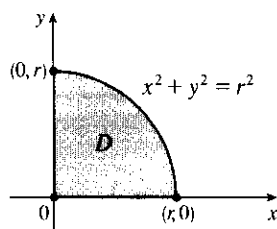
$$\begin{aligned}
 V &= \int_0^2 \int_0^{2y} \sqrt{4-y^2} dx dy \\
 &= \int_0^2 \left[x \sqrt{4-y^2} \right]_{x=0}^{x=2y} dy = \int_0^2 2y \sqrt{4-y^2} dy \\
 &= \left[-\frac{2}{3}(4-y^2)^{3/2} \right]_0^2 = 0 + \frac{16}{3} = \frac{16}{3}
 \end{aligned}$$

27.



$$\begin{aligned}
 V &= \int_0^1 \int_0^{\sqrt{1-x^2}} y dy dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{1-x^2}} dx \\
 &= \int_0^1 \frac{1-x^2}{2} dx = \frac{1}{2} \left[x - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{3}
 \end{aligned}$$

28.

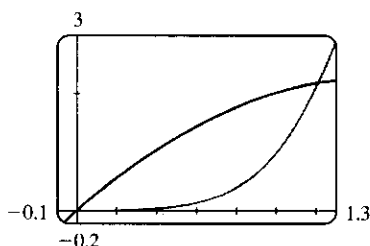


By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

$$\begin{aligned} V_1 &= \int_0^r \int_0^{\sqrt{r^2 - y^2}} \sqrt{r^2 - y^2} \, dx \, dy \\ &= \int_0^r \left[x \sqrt{r^2 - y^2} \right]_{x=0}^{x=\sqrt{r^2 - y^2}} dy \\ &= \int_0^r (r^2 - y^2) \, dy = \left[r^2 y - \frac{1}{3} y^3 \right]_0^r = \frac{2}{3} r^3 \end{aligned}$$

Thus $V = \frac{16}{3} r^3$.

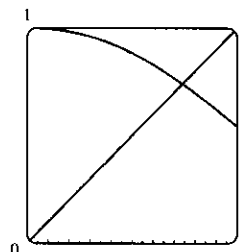
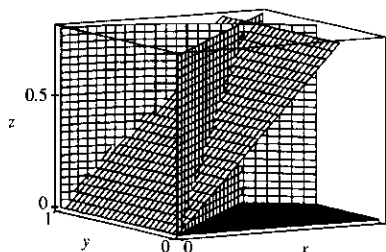
29.



From the graph, it appears that the two curves intersect at $x = 0$ and at $x \approx 1.213$. Thus the desired integral is

$$\begin{aligned} \iint_D x \, dA &\approx \int_0^{1.213} \int_{x^4}^{3x - x^2} x \, dy \, dx = \int_0^{1.213} [xy]_{y=x^4}^{y=3x - x^2} dx \\ &= \int_0^{1.213} (3x^2 - x^3 - x^5) \, dx = \left[x^3 - \frac{1}{4} x^4 - \frac{1}{6} x^6 \right]_0^{1.213} \\ &\approx 0.713 \end{aligned}$$

30.



The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$\begin{aligned} V &\approx \int_0^{0.7391} \int_x^{\cos x} z \, dy \, dx = \int_0^{0.7391} \int_x^{\cos x} x \, dy \, dx \\ &= \int_0^{0.7391} [xy]_{y=x}^{y=\cos x} dx = \int_0^{0.7391} (x \cos x - x^2) \, dx \\ &= \left[\cos x + x \sin x - \frac{1}{3} x^3 \right]_0^{0.7391} \approx 0.1024 \end{aligned}$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391} \int_0^x x \, dy \, dx + \int_{0.7391}^{\pi/2} \int_0^{\cos x} x \, dy \, dx \approx 0.4684$.

31. The two bounding curves $y = 1 - x^2$ and $y = x^2 - 1$ intersect at $(\pm 1, 0)$ with $1 - x^2 \geq x^2 - 1$ on $[-1, 1]$.

Within this region, the plane $z = 2x + 2y + 10$ is above the plane $z = 2 - x - y$, so

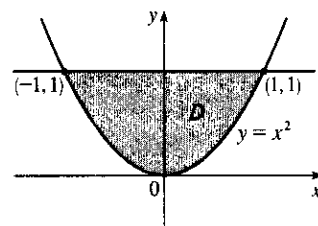
$$\begin{aligned} V &= \int_{-1}^1 \int_{x^2-1}^{1-x^2} (2x + 2y + 10) dy dx - \int_{-1}^1 \int_{x^2-1}^{1-x^2} (2 - x - y) dy dx \\ &= \int_{-1}^1 \int_{x^2-1}^{1-x^2} (2x + 2y + 10 - (2 - x - y)) dy dx = \int_{-1}^1 \int_{x^2-1}^{1-x^2} (3x + 3y + 8) dy dx \\ &= \int_{-1}^1 \left[3xy + \frac{3}{2}y^2 + 8y \right]_{y=x^2-1}^{y=1-x^2} dx \\ &= \int_{-1}^1 \left[3x(1-x^2) + \frac{3}{2}(1-x^2)^2 + 8(1-x^2) - 3x(x^2-1) - \frac{3}{2}(x^2-1)^2 - 8(x^2-1) \right] dx \\ &= \int_{-1}^1 (-6x^3 - 16x^2 + 6x + 16) dx = \left[-\frac{3}{2}x^4 - \frac{16}{3}x^3 + 3x^2 + 16x \right]_{-1}^1 \\ &= -\frac{3}{2} - \frac{16}{3} + 3 + 16 + \frac{3}{2} - \frac{16}{3} - 3 + 16 = \frac{64}{3} \end{aligned}$$

32. The two planes intersect in the line $y = 1, z = 3$, so the region of

integration is the plane region enclosed by the parabola $y = x^2$ and the

line $y = 1$. We have $2 + y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is

bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$\begin{aligned} V &= \int_{-1}^1 \int_{x^2}^1 (2 + y) dy dx - \int_{-1}^1 \int_{x^2}^1 (3y) dy dx = \int_{-1}^1 \int_{x^2}^1 (2 + y - 3y) dy dx = \int_{-1}^1 \int_{x^2}^1 (2 - 2y) dy dx \\ &= \int_{-1}^1 \left[2y - y^2 \right]_{y=x^2}^{y=1} dx = \int_{-1}^1 (1 - 2x^2 + x^4) dx = \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^1 = \frac{16}{15} \end{aligned}$$

33. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$$V = \int_0^2 \int_{x^3-x}^{x^2+x} z dy dx = \int_0^2 \int_{x^3-x}^{x^2+x} (x^3 y^4 + xy^2) dy dx = \frac{13,984,735,616}{14,549,535}$$

34. For $|x| \leq 1$ and $|y| \leq 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \leq x \leq 1$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. So the volume is given by

$$V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [(8 - x^2 - 2y^2) - (2x^2 + y^2)] dy dx = \frac{13\pi}{2} \quad [\text{using a CAS}]$$

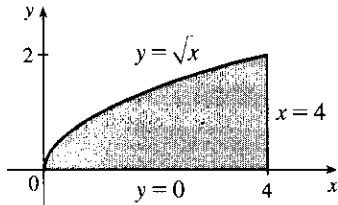
35. The two surfaces intersect in the circle $x^2 + y^2 = 1, z = 0$ and the region of integration is the disk $D: x^2 + y^2 \leq 1$.

Using a CAS, the volume is $\iint_D (1 - x^2 - y^2) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2) dy dx = \frac{\pi}{2}$.

36. The projection onto the xy -plane of the intersection of the two surfaces is the circle $x^2 + y^2 = 2y \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y - 1)^2 = 1$, so the region of integration is given by $-1 \leq x \leq 1$, $1 - \sqrt{1 - x^2} \leq y \leq 1 + \sqrt{1 - x^2}$. In this region, $2y \geq x^2 + y^2$ so, using a CAS, the volume is

$$V = \int_{-1}^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y - (x^2 + y^2)] dy dx = \frac{\pi}{2}.$$

37.



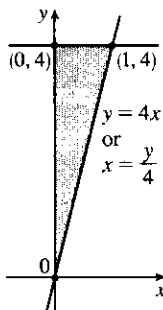
Because the region of integration is

$$D = \{(x, y) \mid 0 \leq y \leq \sqrt{x}, 0 \leq x \leq 4\} \\ = \{(x, y) \mid y^2 \leq x \leq 4, 0 \leq y \leq 2\}$$

we have

$$\int_0^4 \int_0^{\sqrt{x}} f(x, y) dy dx = \iint_D f(x, y) dA = \int_0^2 \int_{y^2}^4 f(x, y) dx dy.$$

38.



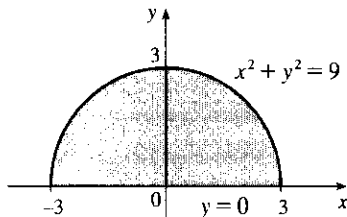
Because the region of integration is

$$D = \{(x, y) \mid 4x \leq y \leq 4, 0 \leq x \leq 1\} \\ = \{(x, y) \mid 0 \leq x \leq \frac{y}{4}, 0 \leq y \leq 4\}$$

we have

$$\int_0^1 \int_{4x}^4 f(x, y) dy dx = \iint_D f(x, y) dA \\ = \int_0^4 \int_0^{y/4} f(x, y) dx dy$$

39.



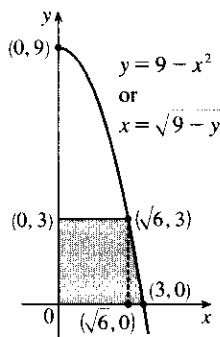
Because the region of integration is

$$D = \{(x, y) \mid -\sqrt{9 - y^2} \leq x \leq \sqrt{9 - y^2}, 0 \leq y \leq 3\} \\ = \{(x, y) \mid 0 \leq y \leq \sqrt{9 - x^2}, -3 \leq x \leq 3\}$$

we have

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y) dx dy = \iint_D f(x, y) dA \\ = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} f(x, y) dy dx$$

40.



To reverse the order, we must break the region into two separate type I

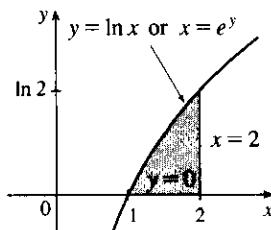
regions. Because the region of integration is

$$D = \{(x, y) \mid 0 \leq x \leq \sqrt{9 - y}, 0 \leq y \leq 3\} \\ = \{(x, y) \mid 0 \leq y \leq 3, 0 \leq x \leq \sqrt{6}\} \cup \{(x, y) \mid 0 \leq y \leq 9 - x^2, \sqrt{6} \leq x \leq 3\}$$

we have

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x, y) dx dy = \iint_D f(x, y) dA \\ = \int_0^{\sqrt{6}} \int_0^{9-x^2} f(x, y) dy dx + \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x, y) dy dx$$

41.



Because the region of integration is

$$D = \{(x, y) \mid 0 \leq y \leq \ln x, 1 \leq x \leq 2\}$$

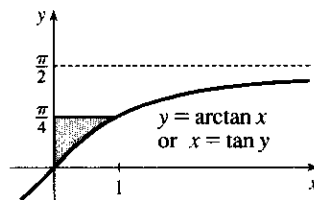
$$= \{(x, y) \mid e^y \leq x \leq 2, 0 \leq y \leq \ln 2\}$$

we have

$$\int_1^2 \int_0^{\ln x} f(x, y) dy dx = \iint_D f(x, y) dA$$

$$= \int_0^{\ln 2} \int_{e^y}^2 f(x, y) dx dy$$

42.



Because the region of integration is

$$D = \{(x, y) \mid \arctan x \leq y \leq \frac{\pi}{4}, 0 \leq x \leq 1\}$$

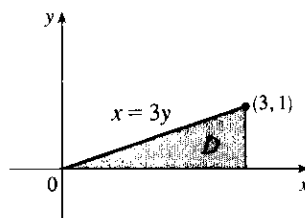
$$= \{(x, y) \mid 0 \leq x \leq \tan y, 0 \leq y \leq \frac{\pi}{4}\}$$

we have

$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y) dy dx = \iint_D f(x, y) dA$$

$$= \int_0^{\pi/4} \int_0^{\tan y} f(x, y) dx dy$$

43.

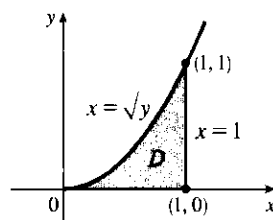


$$\int_0^1 \int_{3y}^3 e^{x^2} dx dy = \int_0^3 \int_0^{x/3} e^{x^2} dy dx$$

$$= \int_0^3 [e^{x^2} y]_{y=0}^{y=x/3} dx = \int_0^3 \left(\frac{x}{3}\right) e^{x^2} dx$$

$$= \frac{1}{6} e^{x^2} \Big|_0^3 = \frac{e^9 - 1}{6}$$

44.

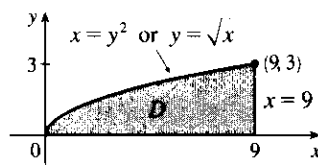


$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} dx dy = \int_0^1 \int_0^{x^2} \sqrt{x^3 + 1} dy dx$$

$$= \int_0^1 [\sqrt{x^3 + 1} y]_{y=0}^{y=x^2} dx = \int_0^1 x^2 \sqrt{x^3 + 1} dx$$

$$= \frac{2}{9} (x^3 + 1)^{3/2} \Big|_0^1 = \frac{2}{9} (2^{3/2} - 1)$$

45.

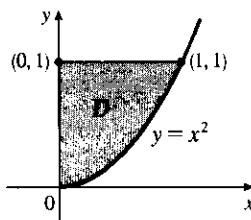


$$\int_0^3 \int_{y^2}^9 y \cos x^2 dx dy = \int_0^9 \int_0^{\sqrt{x}} y \cos x^2 dy dx$$

$$= \int_0^9 \cos x^2 \left[\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx = \int_0^9 \frac{1}{2} x \cos x^2 dx$$

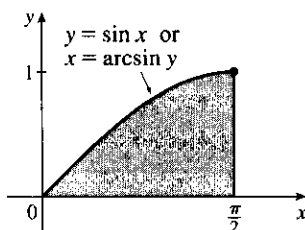
$$= \frac{1}{4} \sin x^2 \Big|_0^9 = \frac{1}{4} \sin 81$$

46.



$$\begin{aligned}
 \int_0^1 \int_{x^2}^1 x^3 \sin(y^3) dy dx &= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3) dx dy \\
 &= \int_0^1 \left[\frac{x^4}{4} \sin(y^3) \right]_{x=0}^{x=\sqrt{y}} dy \\
 &= \int_0^1 \frac{1}{4} y^2 \sin(y^3) dy \\
 &= -\frac{1}{12} \cos(y^3) \Big|_0^1 = -\frac{1}{12} (1 - \cos 1)
 \end{aligned}$$

47.

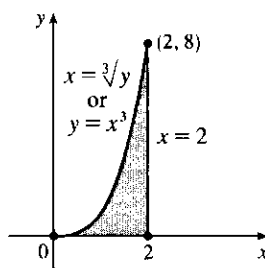


$$\begin{aligned}
 \int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x} dx dy &= \int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1 + \cos^2 x} dy dx \\
 &= \int_0^{\pi/2} \cos x \sqrt{1 + \cos^2 x} [y]_{y=0}^{y=\sin x} dx \\
 &= \int_0^{\pi/2} \cos x \sqrt{1 + \cos^2 x} \sin x dx
 \end{aligned}$$

[Let $u = \cos x$, $du = -\sin x dx$, $dx = du/(-\sin x)$]

$$\begin{aligned}
 &= \int_1^0 -u \sqrt{1 + u^2} du = -\frac{1}{3} (1 + u^2)^{3/2} \Big|_1^0 \\
 &= \frac{1}{3} (\sqrt{8} - 1) = \frac{1}{3} (2\sqrt{2} - 1)
 \end{aligned}$$

48.



$$\begin{aligned}
 \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4} dx dy &= \int_0^2 \int_0^{x^3} e^{x^4} dy dx \\
 &= \int_0^2 e^{x^4} [y]_{y=0}^{y=x^3} dx = \int_0^2 x^3 e^{x^4} dx \\
 &= \frac{1}{4} e^{x^4} \Big|_0^2 = \frac{1}{4} (e^{16} - 1)
 \end{aligned}$$

$$\begin{aligned}
 49. D &= \{(x, y) \mid 0 \leq x \leq 1, -x + 1 \leq y \leq 1\} \cup \{(x, y) \mid -1 \leq x \leq 0, x + 1 \leq y \leq 1\} \\
 &\quad \cup \{(x, y) \mid 0 \leq x \leq 1, -1 \leq y \leq x - 1\} \cup \{(x, y) \mid -1 \leq x \leq 0, -1 \leq y \leq -x - 1\},
 \end{aligned}$$

all type I.

$$\begin{aligned}
 \iint_D x^2 dA &= \int_0^1 \int_{1-x}^1 x^2 dy dx + \int_{-1}^0 \int_{x+1}^1 x^2 dy dx + \int_0^1 \int_{-1}^{x-1} x^2 dy dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2 dy dx \\
 &= 4 \int_0^1 \int_{1-x}^1 x^2 dy dx \quad [\text{by symmetry of the regions and because } f(x, y) = x^2 \geq 0] \\
 &= 4 \int_0^1 x^3 dx = 4 \left[\frac{1}{4} x^4 \right]_0^1 = 1
 \end{aligned}$$

$$50. D = \{(x, y) \mid -1 \leq x \leq 0, -1 \leq y \leq 1 + x^2\} \cup \{(x, y) \mid 0 \leq x \leq 1, \sqrt{x} \leq y \leq 1 + x^2\} \\ \cup \{(x, y) \mid 0 \leq x \leq 1, -1 \leq y \leq -\sqrt{x}\},$$

all type I.

$$\begin{aligned} \iint_D xy \, dA &= \int_{-1}^0 \int_{-1}^{1+x^2} xy \, dy \, dx + \int_0^1 \int_{\sqrt{x}}^{1+x^2} xy \, dy \, dx + \int_0^1 \int_{-1}^{-\sqrt{x}} xy \, dy \, dx \\ &= \int_{-1}^0 \left[\frac{1}{2} xy^2 \right]_{y=-1}^{y=1+x^2} dx + \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=\sqrt{x}}^{y=1+x^2} dx + \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=-1}^{y=-\sqrt{x}} dx \\ &= \int_{-1}^0 \left(x^3 + \frac{1}{2} x^5 \right) dx + \int_0^1 \frac{1}{2} (x^5 + 2x^3 - x^2 + x) dx + \int_0^1 \frac{1}{2} (x^2 - x) dx \\ &= \left[\frac{1}{4} x^4 + \frac{1}{12} x^6 \right]_{-1}^0 + \frac{1}{2} \left[\frac{1}{6} x^6 + \frac{1}{2} x^4 - \frac{1}{3} x^3 + \frac{1}{2} x^2 \right]_0^1 + \frac{1}{2} \left[\frac{1}{3} x^3 - \frac{1}{2} x^2 \right]_0^1 \\ &= -\frac{1}{3} + \frac{5}{12} - \frac{1}{12} = 0 \end{aligned}$$

$$51. \text{ For } D = [0, 1] \times [0, 1], 0 \leq \sqrt{x^3 + y^3} \leq \sqrt{2} \text{ and } A(D) = 1, \text{ so } 0 \leq \iint_D \sqrt{x^3 + y^3} \, dA \leq \sqrt{2}.$$

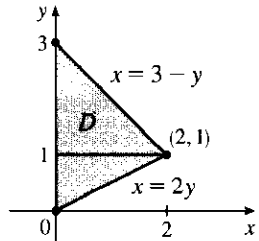
$$52. \text{ Since } D = \{(x, y) \mid x^2 + y^2 \leq \frac{1}{4}\}, 1 = e^0 \leq e^{x^2 + y^2} \leq e^{1/4} \text{ and } A(D) = \frac{\pi}{4}, \text{ so}$$

$$\frac{\pi}{4} \leq \iint_D e^{x^2 + y^2} \, dA \leq (e^{1/4}) \frac{\pi}{4}.$$

$$53. \text{ Since } m \leq f(x, y) \leq M, \iint_D m \, dA \leq \iint_D f(x, y) \, dA \leq \iint_D M \, dA \text{ by (8)} \Rightarrow$$

$$m \iint_D 1 \, dA \leq \iint_D f(x, y) \, dA \leq M \iint_D 1 \, dA \text{ by (7)} \Rightarrow$$

$$mA(D) \leq \iint_D f(x, y) \, dA \leq MA(D) \text{ by (10).}$$

$$54. \iint_D f(x, y) \, dA = \int_0^1 \int_0^{2y} f(x, y) \, dx \, dy + \int_1^3 \int_0^{3-y} f(x, y) \, dx \, dy \\ = \int_0^2 \int_{x/2}^{3-x} f(x, y) \, dy \, dx$$


$$55. \iint_D (x^2 \tan x + y^3 + 4) \, dA = \iint_D x^2 \tan x \, dA + \iint_D y^3 \, dA + \iint_D 4 \, dA. \text{ But } x^2 \tan x \text{ is an odd function of } x \\ \text{and } D \text{ is symmetric with respect to the } y\text{-axis, so } \iint_D x^2 \tan x \, dA = 0. \text{ Similarly, } y^3 \text{ is an odd function of } y \text{ and } D \\ \text{is symmetric with respect to the } x\text{-axis, so } \iint_D y^3 \, dA = 0. \text{ Thus}$$

$$\iint_D (x^2 \tan x + y^3 + 4) \, dA = 4 \iint_D dA = 4(\text{area of } D) = 4 \cdot \pi(\sqrt{2})^2 = 8\pi$$

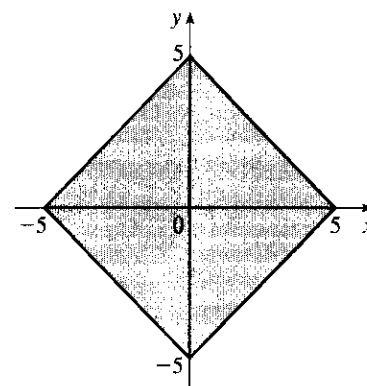
56. First,

$$\iint_D (2 - 3x + 4y) \, dA = \iint_D 2 \, dA - \iint_D 3x \, dA + \iint_D 4y \, dA$$

The region D , shown in the figure, is symmetric with respect to the y -axis and $3x$ is an odd function of x , so $\iint_D 3x \, dA = 0$. Similarly, $4y$ is an odd function of y and D is symmetric with respect to the x -axis, so

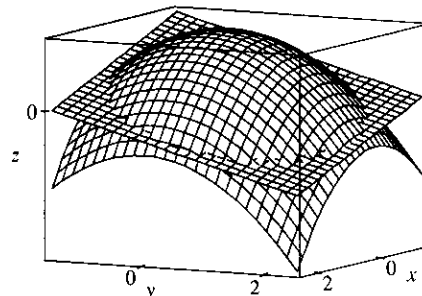
$$\iint_D 4y \, dA = 0. \text{ Then}$$

$$\begin{aligned} \iint_D (2 - 3x + 4y) \, dA &= \iint_D 2 \, dA = 2 \iint_D dA \\ &= 2(\text{area of } D) = 2(50) \\ &= 100 \end{aligned}$$



57. Since $\sqrt{1 - x^2 - y^2} \geq 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2} \, dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1 - x^2 - y^2}$ and above the region D in the xy -plane. $z = \sqrt{1 - x^2 - y^2}$ is equivalent to $x^2 + y^2 + z^2 = 1, z \geq 0$ which meets the xy -plane in the circle $x^2 + y^2 = 1$, the boundary of D . Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2} \left[\frac{4}{3} \pi (1)^3 \right] = \frac{2}{3} \pi$.

58. To find the equations of the boundary curves, we require that the z -values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y, y)`; and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y, y]`. We find that the curves have equations $y = \frac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$.



To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is

$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{(1-\sqrt{13+4x-4x^2})/2}^{(1+\sqrt{13+4x-4x^2})/2} [(4-x^2-y^2) - (1-x-y)] \, dy \, dx = \frac{49\pi}{8}.$$

16.4 Double Integrals in Polar Coordinates

ET 15.4

1. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$.

$$\text{Thus } \iint_R f(x, y) \, dA = \int_0^{2\pi} \int_0^2 f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

2. The region R is more easily described by rectangular coordinates: $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}$.

$$\text{Thus } \iint_R f(x, y) \, dA = \int_0^2 \int_0^{2-x} f(x, y) \, dy \, dx.$$

3. The region R is more easily described by rectangular coordinates: $R = \{(x, y) \mid -2 \leq x \leq 2, x \leq y \leq 2\}$.

$$\text{Thus } \iint_R f(x, y) \, dA = \int_{-2}^2 \int_x^2 f(x, y) \, dy \, dx.$$

4. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 1 \leq r \leq 3, 0 \leq \theta \leq \frac{\pi}{2}\}$.

$$\text{Thus } \iint_R f(x, y) dA = \int_0^{\pi/2} \int_1^3 f(r \cos \theta, r \sin \theta) r dr d\theta.$$

5. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 2 \leq r \leq 5, 0 \leq \theta \leq 2\pi\}$.

$$\text{Thus } \iint_R f(x, y) dA = \int_0^{2\pi} \int_2^5 f(r \cos \theta, r \sin \theta) r dr d\theta.$$

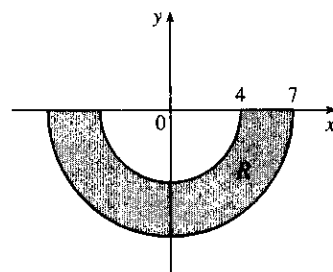
6. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 0 \leq r \leq 2\sqrt{2}, \frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}\}$.

$$\text{Thus } \iint_R f(x, y) dA = \int_{\pi/4}^{5\pi/4} \int_0^{2\sqrt{2}} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

7. The integral $\int_{\pi}^{2\pi} \int_4^7 r dr d\theta$ represents the area of the region

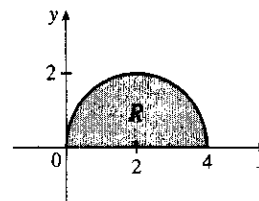
$R = \{(r, \theta) \mid 4 \leq r \leq 7, \pi \leq \theta \leq 2\pi\}$, the lower half of a ring.

$$\begin{aligned} \int_{\pi}^{2\pi} \int_4^7 r dr d\theta &= \left(\int_{\pi}^{2\pi} d\theta \right) \left(\int_4^7 r dr \right) \\ &= [\theta]_{\pi}^{2\pi} \left[\frac{1}{2} r^2 \right]_4^7 = \pi \cdot \frac{1}{2} (49 - 16) = \frac{33\pi}{2} \end{aligned}$$



8. The integral $\int_0^{\pi/2} \int_0^{4 \cos \theta} r dr d\theta$ represents the area of the region

$R = \{(r, \theta) \mid 0 \leq r \leq 4 \cos \theta, 0 \leq \theta \leq \pi/2\}$. Since $r = 4 \cos \theta \Leftrightarrow r^2 = 4r \cos \theta \Leftrightarrow x^2 + y^2 = 4x \Leftrightarrow (x-2)^2 + y^2 = 4$, R is the portion in the first quadrant of a circle of radius 2 with center $(2, 0)$.



$$\begin{aligned} \int_0^{\pi/2} \int_0^{4 \cos \theta} r dr d\theta &= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=4 \cos \theta} d\theta = \int_0^{\pi/2} 8 \cos^2 \theta d\theta \\ &= \int_0^{\pi/2} 4(1 + \cos 2\theta) d\theta = 4 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = 2\pi \end{aligned}$$

9. The disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$. Then

$$\begin{aligned} \iint_D xy dA &= \int_0^{2\pi} \int_0^3 (r \cos \theta)(r \sin \theta) r dr d\theta = \left(\int_0^{2\pi} \sin \theta \cos \theta d\theta \right) \left(\int_0^3 r^3 dr \right) \\ &= \left[\frac{1}{2} \sin^2 \theta \right]_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^3 = 0 \end{aligned}$$

$$\begin{aligned} 10. \iint_R (x+y) dA &= \int_{\pi/2}^{3\pi/2} \int_1^2 (r \cos \theta + r \sin \theta) r dr d\theta = \int_{\pi/2}^{3\pi/2} \int_1^2 r^2 (\cos \theta + \sin \theta) dr d\theta \\ &= \left(\int_{\pi/2}^{3\pi/2} (\cos \theta + \sin \theta) d\theta \right) \left(\int_1^2 r^2 dr \right) = [\sin \theta - \cos \theta]_{\pi/2}^{3\pi/2} \left[\frac{1}{3} r^3 \right]_1^2 \\ &= (-1 - 0 - 1 + 0) \left(\frac{8}{3} - \frac{1}{3} \right) = -\frac{14}{3} \end{aligned}$$

$$\begin{aligned} 11. \iint_R \cos(x^2 + y^2) dA &= \int_0^{\pi} \int_0^3 \cos(r^2) r dr d\theta = \left(\int_0^{\pi} d\theta \right) \left(\int_0^3 r \cos(r^2) dr \right) \\ &= [\theta]_0^{\pi} \left[\frac{1}{2} \sin(r^2) \right]_0^3 = \pi \cdot \frac{1}{2} (\sin 9 - \sin 0) = \frac{\pi}{2} \sin 9 \end{aligned}$$

$$\begin{aligned}
 12. \iint_R \sqrt{4-x^2-y^2} dA &= \int_{-\pi/2}^{\pi/2} \int_0^2 \sqrt{4-r^2} r dr d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta \right) \left(\int_0^2 r \sqrt{4-r^2} dr \right) \\
 &= [\theta]_{-\pi/2}^{\pi/2} \left[-\frac{1}{2} \cdot \frac{2}{3} (4-r^2)^{3/2} \right]_0^2 = \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \left(-\frac{1}{3} (0 - 4^{3/2}) \right) = \frac{8}{3} \pi
 \end{aligned}$$

$$\begin{aligned}
 13. \iint_D e^{-x^2-y^2} dA &= \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r dr d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta \right) \left(\int_0^2 r e^{-r^2} dr \right) \\
 &= [\theta]_{-\pi/2}^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^2 = \pi \left(-\frac{1}{2} \right) (e^{-4} - e^0) = \frac{\pi}{2} (1 - e^{-4})
 \end{aligned}$$

$$\begin{aligned}
 14. \iint_R y e^x dA &= \int_0^{\pi/2} \int_0^5 (r \sin \theta) e^{r \cos \theta} r dr d\theta = \int_0^{\pi/2} \int_0^5 r^2 \sin \theta e^{r \cos \theta} d\theta dr. \text{ First we integrate} \\
 &\int_0^{\pi/2} r^2 \sin \theta e^{r \cos \theta} d\theta: \text{ Let } u = r \cos \theta \Rightarrow du = -r \sin \theta d\theta, \\
 &\text{and } \int_0^{\pi/2} r^2 \sin \theta e^{r \cos \theta} d\theta = \int_{u=r}^{u=0} -r e^u du = -r[e^u - e^r] = r e^r - r. \text{ Then} \\
 &\int_0^5 \int_0^{\pi/2} r^2 \sin \theta e^{r \cos \theta} d\theta dr = \int_0^5 (r e^r - r) dr = [r e^r - e^r - \frac{1}{2} r^2]_0^5 = 4e^5 - \frac{23}{2}, \text{ where we integrated by parts} \\
 &\text{in the first term.}
 \end{aligned}$$

15. R is the region shown in the figure, and can be described

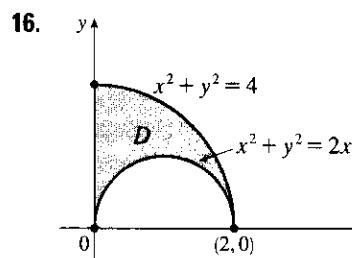
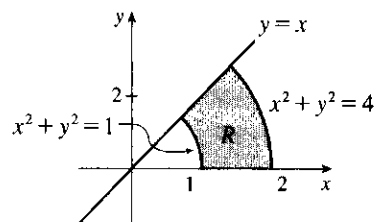
by $R = \{(r, \theta) \mid 0 \leq \theta \leq \pi/4, 1 \leq r \leq 2\}$. Thus

$$\iint_R \arctan(y/x) dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan \theta) r dr d\theta$$

since $y/x = \tan \theta$. Also, $\arctan(\tan \theta) = \theta$ for $0 \leq \theta \leq \pi/4$,

so the integral becomes

$$\int_0^{\pi/4} \int_1^2 \theta r dr d\theta = \int_0^{\pi/4} \theta d\theta \int_1^2 r dr = \left[\frac{1}{2} \theta^2 \right]_0^{\pi/4} \left[\frac{1}{2} r^2 \right]_1^2 = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3}{64} \pi^2.$$

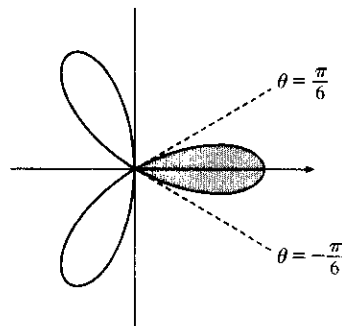


$$\begin{aligned}
 16. \iint_D x dA &= \iint_{\substack{x^2+y^2 \leq 4 \\ x \geq 0, y \geq 0}} x dA - \iint_{\substack{(x-1)^2+y^2 \leq 1 \\ y \geq 0}} x dA \\
 &= \int_0^{\pi/2} \int_0^2 r^2 \cos \theta dr d\theta - \int_0^{\pi/2} \int_0^{2 \cos \theta} r^2 \cos \theta dr d\theta \\
 &= \int_0^{\pi/2} \frac{1}{3} (8 \cos \theta) d\theta - \int_0^{\pi/2} \frac{1}{3} (8 \cos^4 \theta) d\theta \\
 &= \frac{8}{3} - \frac{8}{12} \left[\cos^3 \theta \sin \theta + \frac{3}{2} (\theta + \sin \theta \cos \theta) \right]_0^{\pi/2} \\
 &= \frac{8}{3} - \frac{2}{3} \left[0 + \frac{3}{2} \left(\frac{\pi}{2} \right) \right] = \frac{16-3\pi}{6}
 \end{aligned}$$

17. One loop is given by the region

$D = \{(r, \theta) \mid -\pi/6 \leq \theta \leq \pi/6, 0 \leq r \leq \cos 3\theta\}$, so the area is

$$\begin{aligned}
 \iint_D dA &= \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r dr d\theta = \int_{-\pi/6}^{\pi/6} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=\cos 3\theta} d\theta \\
 &= \int_{-\pi/6}^{\pi/6} \frac{1}{2} \cos^2 3\theta d\theta = 2 \int_0^{\pi/6} \frac{1}{2} \left(\frac{1 + \cos 6\theta}{2} \right) d\theta \\
 &= \frac{1}{2} \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = \frac{\pi}{12}
 \end{aligned}$$

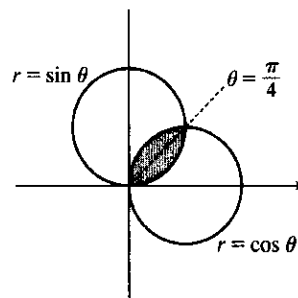


- 18.
- $D = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 4 + 3 \cos \theta\}$
- , so

$$\begin{aligned}
 A(D) &= \iint_D dA = \int_0^{2\pi} \int_0^{4+3\cos\theta} r \, dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=4+3\cos\theta} d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 3 \cos \theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (16 + 24 \cos \theta + 9 \cos^2 \theta) d\theta = \frac{1}{2} \int_0^{2\pi} \left(16 + 24 \cos \theta + 9 \cdot \frac{1+\cos 2\theta}{2} \right) d\theta \\
 &= \frac{1}{2} \left[16\theta + 24 \sin \theta + \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta \right]_0^{2\pi} = \frac{41}{2} \pi
 \end{aligned}$$

19. By symmetry,

$$\begin{aligned}
 A &= 2 \int_0^{\pi/4} \int_0^{\sin\theta} r \, dr \, d\theta = 2 \int_0^{\pi/4} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=\sin\theta} d\theta \\
 &= \int_0^{\pi/4} \sin^2 \theta \, d\theta = \int_0^{\pi/4} \frac{1}{2} (1 - \cos 2\theta) \, d\theta \\
 &= \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/4} \\
 &= \frac{1}{2} \left[\frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} - 0 + \frac{1}{2} \sin 0 \right] = \frac{1}{8} (\pi - 2)
 \end{aligned}$$



- 20.
- $2 = 4 \sin \theta$
- implies that
- $\theta = \frac{\pi}{6}$
- or
- $\frac{5\pi}{6}$
- , so

$$\begin{aligned}
 A &= \int_{\pi/6}^{5\pi/6} \int_2^{4\sin\theta} r \, dr \, d\theta = \int_{\pi/6}^{5\pi/6} \left[\frac{1}{2} r^2 \right]_{r=2}^{r=4\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6} (8 \sin^2 \theta - 2) d\theta \\
 &= \int_{\pi/6}^{5\pi/6} [4(1 - \cos 2\theta) - 2] d\theta = [2\theta - 2 \sin 2\theta]_{\pi/6}^{5\pi/6} = \frac{4\pi}{3} + 2\sqrt{3}.
 \end{aligned}$$

- 21.
- $V = \iiint_{x^2+y^2 \leq 9} (x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^3 (r^2) r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^3 r^3 \, dr = [\theta]_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^3 = 2\pi \left(\frac{81}{4} \right) = \frac{81\pi}{2}$

22. The sphere
- $x^2 + y^2 + z^2 = 16$
- intersects the
- xy
- plane in the circle
- $x^2 + y^2 = 16$
- , so

$$\begin{aligned}
 V &= 2 \iint_{4 \leq x^2+y^2 \leq 16} \sqrt{16 - x^2 - y^2} \, dA \quad [\text{by symmetry}] \\
 &= 2 \int_0^{2\pi} \int_2^4 \sqrt{16 - r^2} \, r \, dr \, d\theta = 2 \int_0^{2\pi} d\theta \int_2^4 r(16 - r^2)^{1/2} \, dr \\
 &= 2 \left[\theta \right]_0^{2\pi} \left[-\frac{1}{3} (16 - r^2)^{3/2} \right]_2^4 = -\frac{2}{3} (2\pi) (0 - 12^{3/2}) = \frac{4\pi}{3} (12\sqrt{12}) = 32\sqrt{3} \pi
 \end{aligned}$$

23. By symmetry,

$$\begin{aligned}
 V &= 2 \iint_{x^2+y^2 \leq a^2} \sqrt{a^2 - x^2 - y^2} \, dA = 2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta = 2 \int_0^{2\pi} d\theta \int_0^a r \sqrt{a^2 - r^2} \, dr \\
 &= 2 \left[\theta \right]_0^{2\pi} \left[-\frac{1}{3} (a^2 - r^2)^{3/2} \right]_0^a = 2(2\pi) \left(0 + \frac{1}{3} a^3 \right) = \frac{4\pi}{3} a^3
 \end{aligned}$$

24. The paraboloid
- $z = 10 - 3x^2 - 3y^2$
- intersects the plane
- $z = 4$
- when
- $4 = 10 - 3x^2 - 3y^2$
- or
- $x^2 + y^2 = 2$
- . So

$$\begin{aligned}
 V &= \iint_{x^2+y^2 \leq 2} [(10 - 3x^2 - 3y^2) - 4] \, dA = \int_0^{2\pi} \int_0^{\sqrt{2}} (6 - 3r^2) r \, dr \, d\theta \\
 &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (6r - 3r^3) \, dr = [\theta]_0^{2\pi} \left[3r^2 - \frac{3}{4} r^4 \right]_0^{\sqrt{2}} = 6\pi
 \end{aligned}$$

25. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So

$$\begin{aligned} V &= \iint_{x^2 + y^2 \leq 1/2} (\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}) dA = \int_0^{2\pi} \int_0^{1/\sqrt{2}} (\sqrt{1 - r^2} - r) r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}} (r\sqrt{1 - r^2} - r^2) dr = [\theta]_0^{2\pi} \left[-\frac{1}{3}(1 - r^2)^{3/2} - \frac{1}{3}r^3 \right]_0^{1/\sqrt{2}} \\ &= 2\pi \left(-\frac{1}{3} \right) \left(\frac{1}{\sqrt{2}} - 1 \right) = \frac{\pi}{3} (2 - \sqrt{2}) \end{aligned}$$

26. The two paraboloids intersect when $3x^2 + 3y^2 = 4 - x^2 - y^2$ or $x^2 + y^2 = 1$. So

$$\begin{aligned} V &= \iint_{x^2 + y^2 \leq 1} [(4 - x^2 - y^2) - 3(x^2 + y^2)] dA = \int_0^{2\pi} \int_0^1 4(1 - r^2) r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (4r - 4r^3) dr = [\theta]_0^{2\pi} [2r^2 - r^4]_0^1 = 2\pi \end{aligned}$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$\begin{aligned} V &= \iint_{x^2 + y^2 \leq 4} [\sqrt{64 - 4x^2 - 4y^2} - (-\sqrt{64 - 4x^2 - 4y^2})] dA \\ &= \iint_{x^2 + y^2 \leq 4} 2\sqrt{64 - 4x^2 - 4y^2} dA = 4 \int_0^{2\pi} \int_0^2 \sqrt{16 - r^2} r dr d\theta \\ &= 4 \int_0^{2\pi} d\theta \int_0^2 r \sqrt{16 - r^2} dr = 4 [\theta]_0^{2\pi} \left[-\frac{1}{3}(16 - r^2)^{3/2} \right]_0^2 \\ &= 8\pi \left(-\frac{1}{3} \right) (12^{3/2} - 16^{2/3}) = \frac{8\pi}{3} (64 - 24\sqrt{3}) \end{aligned}$$

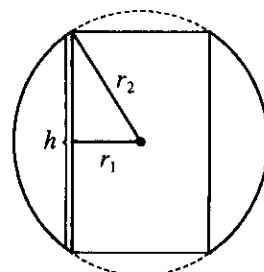
28. (a) Here the region in the xy -plane is the annular region $r_1^2 \leq x^2 + y^2 \leq r_2^2$ and the desired volume is twice that above the xy -plane. Hence

$$\begin{aligned} V &= 2 \iint_{r_1^2 \leq x^2 + y^2 \leq r_2^2} \sqrt{r_2^2 - x^2 - y^2} dA = 2 \int_0^{2\pi} \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2} r dr d\theta \\ &= 2 \int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2} r dr = \frac{4\pi}{3} \left[-(r_2^2 - r^2)^{3/2} \right]_{r_1}^{r_2} = \frac{4\pi}{3} (r_2^2 - r_1^2)^{3/2} \end{aligned}$$

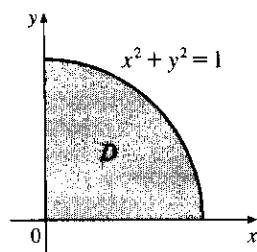
- (b) A cross-sectional cut is shown in the figure.

$$\text{So } r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2 \text{ or } \frac{1}{4}h^2 = r_2^2 - r_1^2.$$

$$\text{Thus the volume in terms of } h \text{ is } V = \frac{4\pi}{3} \left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3.$$

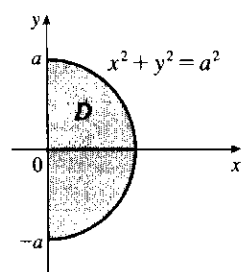


29.



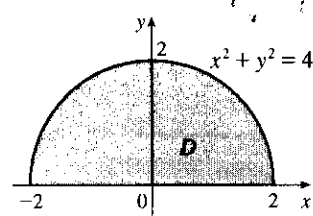
$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2} dy dx &= \int_0^{\pi/2} \int_0^1 e^{r^2} r dr d\theta \\ &= \int_0^{\pi/2} d\theta \int_0^1 r e^{r^2} dr \\ &= [\theta]_0^{\pi/2} \left[\frac{1}{2} e^{r^2} \right]_0^1 = \frac{1}{4} \pi (e - 1)\end{aligned}$$

30.



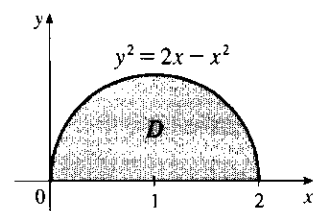
$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \int_0^a (r^2)^{3/2} r dr d\theta &= \int_{-\pi/2}^{\pi/2} d\theta \int_0^a r^4 dr \\ &= [\theta]_{-\pi/2}^{\pi/2} \left[\frac{1}{5} r^5 \right]_0^a \\ &= \frac{1}{5} \pi a^5\end{aligned}$$

31.



$$\begin{aligned}\int_0^{\pi} \int_0^2 (r \cos \theta)^2 (r \sin \theta)^2 r dr d\theta &= \int_0^{\pi} (\sin \theta \cos \theta)^2 d\theta \int_0^2 r^5 dr \\ &= \int_0^{\pi} \left(\frac{1}{2} \sin 2\theta \right)^2 d\theta \int_0^2 r^5 dr \\ &= \frac{1}{4} \left[\frac{1}{2} \theta - \frac{1}{8} \sin 4\theta \right]_0^{\pi} \left[\frac{1}{6} r^6 \right]_0^2 \\ &= \frac{1}{4} \left(\frac{\pi}{2} \right) \left(\frac{64}{6} \right) = \frac{4\pi}{3}\end{aligned}$$

32.



$$\begin{aligned}\int_0^{\pi/2} \int_0^{2 \cos \theta} r^2 dr d\theta &= \int_0^{\pi/2} \left[\frac{1}{3} r^3 \right]_{r=0}^{r=2 \cos \theta} d\theta \\ &= \int_0^{\pi/2} \left(\frac{8}{3} \cos^3 \theta \right) d\theta \\ &= \frac{8}{3} \left[\sin \theta - \frac{1}{3} \sin^3 \theta \right]_0^{\pi/2} = \frac{16}{9}\end{aligned}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x, y)$ to be the depth of the water at (x, y) , then the volume of water in the pool is the volume of the solid that lies above $D = \{(x, y) \mid x^2 + y^2 \leq 400\}$ and below the graph of $f(x, y)$. We can associate north with the positive y -direction, so we are given that the depth is constant in the x -direction and the depth increases linearly in the y -direction from $f(0, -20) = 2$ to $f(0, 20) = 7$. The trace in the yz -plane is a line segment from $(0, -20, 2)$ to $(0, 20, 7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y - 20) \Rightarrow z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x, y)$ is independent of x , $f(x, y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x, y) dA$, which is most conveniently evaluated using polar coordinates. Then

$D = \{(r, \theta) \mid 0 \leq r \leq 20, 0 \leq \theta \leq 2\pi\}$ and substituting $x = r \cos \theta$, $y = r \sin \theta$ the integral becomes

$$\begin{aligned}\int_0^{2\pi} \int_0^{20} \left(\frac{1}{8} r \sin \theta + \frac{9}{2} \right) r dr d\theta &= \int_0^{2\pi} \left[\frac{1}{24} r^3 \sin \theta + \frac{9}{4} r^2 \right]_{r=0}^{r=20} d\theta = \int_0^{2\pi} \left(\frac{1000}{3} \sin \theta + 900 \right) d\theta \\ &= \left[-\frac{1000}{3} \cos \theta + 900\theta \right]_0^{2\pi} = 1800\pi\end{aligned}$$

Thus the pool contains $1800\pi \approx 5655 \text{ ft}^3$ of water.

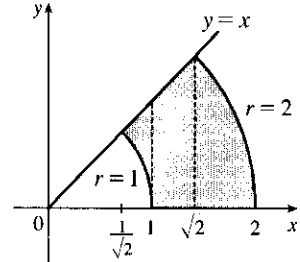
34. (a) The total amount of water supplied each hour to the region within R feet of the sprinkler is

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^R e^{-r} r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^R r e^{-r} \, dr = [\theta]_0^{2\pi} [-r e^{-r} - e^{-r}]_0^R \\ &= 2\pi[-R e^{-R} - e^{-R} + 0 + 1] = 2\pi(1 - R e^{-R} - e^{-R}) \text{ ft}^3 \end{aligned}$$

- (b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is

$$\frac{V}{\text{area of region}} = \frac{V}{\pi R^2} = \frac{2(1 - R e^{-R} - e^{-R})}{R^2} \text{ ft}^3 \text{ (per hour per square foot). See the definition of the average value of a function on page 1022 [ET 986].}$$

$$\begin{aligned} 35. \int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy \, dy \, dx + \int_1^{\sqrt{2}} \int_0^x xy \, dy \, dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy \, dy \, dx \\ = \int_0^{\pi/4} \int_1^2 r^3 \cos \theta \sin \theta \, dr \, d\theta = \int_0^{\pi/4} \left[\frac{r^4}{4} \cos \theta \sin \theta \right]_{r=1}^{r=2} d\theta \\ = \frac{15}{4} \int_0^{\pi/4} \sin \theta \cos \theta \, d\theta = \frac{15}{4} \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/4} = \frac{15}{16} \end{aligned}$$



36. (a) $\iint_{D_a} e^{-(x^2+y^2)} dA = \int_0^{2\pi} \int_0^a r e^{-r^2} \, dr \, d\theta = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^a = \pi(1 - e^{-a^2})$ for each a . Then

$$\lim_{a \rightarrow \infty} \pi(1 - e^{-a^2}) = \pi \text{ since } e^{-a^2} \rightarrow 0 \text{ as } a \rightarrow \infty. \text{ Hence } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dA = \pi.$$

- (b) $\iint_{S_a} e^{-(x^2+y^2)} dA = \int_{-a}^a \int_{-a}^a e^{-x^2} e^{-y^2} \, dx \, dy = \left(\int_{-a}^a e^{-x^2} \, dx \right) \left(\int_{-a}^a e^{-y^2} \, dy \right)$ for each a .

Then, from (a), $\pi = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$, so

$$\pi = \lim_{a \rightarrow \infty} \iint_{S_a} e^{-(x^2+y^2)} dA = \lim_{a \rightarrow \infty} \left(\int_{-a}^a e^{-x^2} \, dx \right) \left(\int_{-a}^a e^{-y^2} \, dy \right) = \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} \, dy \right).$$

To evaluate $\lim_{a \rightarrow \infty} \left(\int_{-a}^a e^{-x^2} \, dx \right) \left(\int_{-a}^a e^{-y^2} \, dy \right)$, we are using the fact that these integrals are bounded. This is true since on $[-1, 1]$, $0 < e^{-x^2} \leq 1$ while on $(-\infty, -1)$, $0 < e^{-x^2} \leq e^x$ and on $(1, \infty)$, $0 < e^{-x^2} < e^{-x}$.

$$\text{Hence } 0 \leq \int_{-\infty}^{\infty} e^{-x^2} \, dx \leq \int_{-\infty}^{-1} e^x \, dx + \int_{-1}^1 1 \, dx + \int_1^{\infty} e^{-x} \, dx = 2(e^{-1} + 1).$$

- (c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} \, dy \right) = \pi$ and y can be replaced by x , $\left(\int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \pi$ implies that

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \pm \sqrt{\pi}. \text{ But } e^{-x^2} \geq 0 \text{ for all } x, \text{ so } \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

- (d) Letting $t = \sqrt{2}x$, $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-t^2/2} \, dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2/2} \, dt$

$$\text{or } \int_{-\infty}^{\infty} e^{-t^2/2} \, dt = \sqrt{2\pi}.$$

37. (a) We integrate by parts with $u = x$ and $dv = x e^{-x^2} \, dx$. Then $du = dx$ and $v = -\frac{1}{2} e^{-x^2}$, so

$$\begin{aligned} \int_0^{\infty} x^2 e^{-x^2} \, dx &= \lim_{t \rightarrow \infty} \int_0^t x^2 e^{-x^2} \, dx = \lim_{t \rightarrow \infty} \left(-\frac{1}{2} x e^{-x^2} \right)_0^t + \int_0^t \frac{1}{2} e^{-x^2} \, dx \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} t e^{-t^2} \right) + \frac{1}{2} \int_0^{\infty} e^{-x^2} \, dx = 0 + \frac{1}{2} \int_0^{\infty} e^{-x^2} \, dx \quad [\text{by l'Hospital's Rule}] \\ &= \frac{1}{4} \int_{-\infty}^{\infty} e^{-x^2} \, dx \quad [\text{since } e^{-x^2} \text{ is an even function}] \\ &= \frac{1}{4} \sqrt{\pi} \quad [\text{by Exercise 36(c)}] \end{aligned}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \Rightarrow dx = 2u du \Rightarrow$

$$\begin{aligned}\int_0^\infty \sqrt{x} e^{-x} dx &= \lim_{t \rightarrow \infty} \int_0^t \sqrt{x} e^{-x} dx = \lim_{t \rightarrow \infty} \int_0^{\sqrt{t}} u e^{-u^2} 2u du = 2 \int_0^\infty u^2 e^{-u^2} du \\ &= 2 \left(\frac{1}{4} \sqrt{\pi} \right) \quad [\text{by part(a)}] = \frac{1}{2} \sqrt{\pi}\end{aligned}$$

16.5 Applications of Double Integrals

ET 15.5

$$\begin{aligned}1. Q &= \iint_D \sigma(x, y) dA = \int_1^3 \int_0^2 (2xy + y^2) dy dx = \int_1^3 \left[xy^2 + \frac{1}{3}y^3 \right]_{y=0}^{y=2} dx \\ &= \int_1^3 \left(4x + \frac{8}{3} \right) dx = \left[2x^2 + \frac{8}{3}x \right]_1^3 = 16 + \frac{16}{3} = \frac{64}{3} \text{ C}\end{aligned}$$

$$\begin{aligned}2. Q &= \iint_D \sigma(x, y) dA = \iint_D (x + y + x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 (r \cos \theta + r \sin \theta + r^2) r dr d\theta \\ &= \int_0^{2\pi} \int_0^2 [r^2(\cos \theta + \sin \theta) + r^3] dr d\theta = \int_0^{2\pi} \left[\frac{1}{3}r^3(\cos \theta + \sin \theta) + \frac{1}{4}r^4 \right]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} \left[\frac{8}{3}(\cos \theta + \sin \theta) + 4 \right] d\theta = \left[\frac{8}{3}(\sin \theta - \cos \theta) + 4\theta \right]_0^{2\pi} = 8\pi \text{ C}\end{aligned}$$

$$\begin{aligned}3. m &= \iint_D \rho(x, y) dA = \int_0^2 \int_{-1}^1 xy^2 dy dx = \int_0^2 x dx \int_{-1}^1 y^2 dy = \left[\frac{1}{2}x^2 \right]_0^2 \left[\frac{1}{3}y^3 \right]_{-1}^1 = 2 \cdot \frac{2}{3} = \frac{4}{3}, \\ \bar{x} &= \frac{1}{m} \iint_D x\rho(x, y) dA = \frac{3}{4} \int_0^2 \int_{-1}^1 x^2 y^2 dy dx = \frac{3}{4} \int_0^2 x^2 dx \int_{-1}^1 y^2 dy = \frac{3}{4} \left[\frac{1}{3}x^3 \right]_0^2 \left[\frac{1}{3}y^3 \right]_{-1}^1 = \frac{3}{4} \cdot \frac{8}{3} \cdot \frac{2}{3} = \frac{4}{3}, \\ \bar{y} &= \frac{1}{m} \iint_D y\rho(x, y) dA = \frac{3}{4} \int_0^2 \int_{-1}^1 xy^3 dy dx = \frac{3}{4} \int_0^2 x dx \int_{-1}^1 y^3 dy = \frac{3}{4} \left[\frac{1}{2}x^2 \right]_0^2 \left[\frac{1}{4}y^4 \right]_{-1}^1 = \frac{3}{4} \cdot 2 \cdot 0 = 0. \\ \text{Hence, } (\bar{x}, \bar{y}) &= \left(\frac{4}{3}, 0 \right).\end{aligned}$$

$$\begin{aligned}4. m &= \iint_D \rho(x, y) dA = \int_0^a \int_0^b cxy dy dx = c \int_0^a x dx \int_0^b y dy = c \left[\frac{1}{2}x^2 \right]_0^a \left[\frac{1}{2}y^2 \right]_0^b = \frac{1}{4}a^2b^2c, \\ M_y &= \iint_D x\rho(x, y) dA = \int_0^a \int_0^b cx^2y dy dx = c \int_0^a x^2 dx \int_0^b y dy = c \left[\frac{1}{3}x^3 \right]_0^a \left[\frac{1}{2}y^2 \right]_0^b = \frac{1}{6}a^3b^2c, \text{ and} \\ M_x &= \iint_D y\rho(x, y) dA = \int_0^a \int_0^b cxy^2 dy dx = c \int_0^a x dx \int_0^b y^2 dy = c \left[\frac{1}{2}x^2 \right]_0^a \left[\frac{1}{3}y^3 \right]_0^b = \frac{1}{6}a^2b^3c. \\ \text{Hence, } (\bar{x}, \bar{y}) &= \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \left(\frac{2}{3}a, \frac{2}{3}b \right).\end{aligned}$$

$$\begin{aligned}5. m &= \int_0^2 \int_{x/2}^{3-x} (x + y) dy dx = \int_0^2 \left[xy + \frac{1}{2}y^2 \right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left[x(3 - \frac{3}{2}x) + \frac{1}{2}(3 - x)^2 - \frac{1}{8}x^2 \right] dx \\ &= \int_0^2 \left(-\frac{9}{8}x^2 + \frac{9}{2} \right) dx = \left[-\frac{9}{8} \left(\frac{1}{3}x^3 \right) + \frac{9}{2}x \right]_0^2 = 6, \\ M_y &= \int_0^2 \int_{x/2}^{3-x} x(x + y) dy dx = \int_0^2 \left[x^2y + \frac{1}{2}xy^2 \right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(\frac{9}{2}x - \frac{9}{8}x^3 \right) dx = \frac{9}{2}, \text{ and} \\ M_x &= \int_0^2 \int_{x/2}^{3-x} xy(x + y) dy dx = \int_0^2 \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3 \right]_{y=x/2}^{y=3-x} dx = \int_0^2 \left(9 - \frac{9}{2}x \right) dx = 9. \\ \text{Hence } m &= 6, (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \left(\frac{3}{4}, \frac{3}{2} \right).\end{aligned}$$

$$\begin{aligned}6. m &= \int_0^1 \int_y^{4-3y} x dx dy = \int_0^1 \left[\frac{1}{2}(4 - 3y)^2 - \frac{1}{2}y^2 \right] dy = \left[-\frac{1}{18}(4 - 3y)^3 - \frac{1}{6}y^3 \right]_0^1 = \frac{10}{3}, \\ M_y &= \int_0^1 \int_y^{4-3y} x^2 dx dy = \int_0^1 \left[\frac{1}{3}(4 - 3y)^3 - \frac{1}{3}y^3 \right] dy = \left[-\frac{1}{36}(4 - 3y)^4 - \frac{1}{12}y^4 \right]_0^1 = 7, \\ M_x &= \int_0^1 \int_y^{4-3y} xy dx dy = \int_0^1 \left[\frac{1}{2}y(4 - 3y)^2 - \frac{1}{2}y^3 \right] dy = \int_0^1 (8y - 12y^2 + 4y^3) dy = 1. \\ \text{Hence } m &= \frac{10}{3}, (\bar{x}, \bar{y}) = (2.1, 0.3).\end{aligned}$$

$$7. m = \int_0^1 \int_0^{e^x} y \, dy \, dx = \int_0^1 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 e^{2x} dx = \frac{1}{4} e^{2x} \Big|_0^1 = \frac{1}{4} (e^2 - 1),$$

$$M_y = \int_0^1 \int_0^{e^x} xy \, dy \, dx = \frac{1}{2} \int_0^1 x e^{2x} dx = \frac{1}{2} \left[\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right]_0^1 = \frac{1}{8} (e^2 + 1), \text{ and}$$

$$M_x = \int_0^1 \int_0^{e^x} y^2 \, dy \, dx = \int_0^1 \left[\frac{1}{3} y^3 \right]_{y=0}^{y=e^x} dx = \frac{1}{3} \int_0^1 e^{3x} dx = \frac{1}{3} \left[\frac{1}{3} e^{3x} \right]_0^1 = \frac{1}{9} (e^3 - 1).$$

$$\text{Hence } m = \frac{1}{4} (e^2 - 1), (\bar{x}, \bar{y}) = \left(\frac{\frac{1}{8} (e^2 + 1)}{\frac{1}{4} (e^2 - 1)}, \frac{\frac{1}{9} (e^3 - 1)}{\frac{1}{4} (e^2 - 1)} \right) = \left(\frac{e^2 + 1}{2(e^2 - 1)}, \frac{4(e^3 - 1)}{9(e^2 - 1)} \right).$$

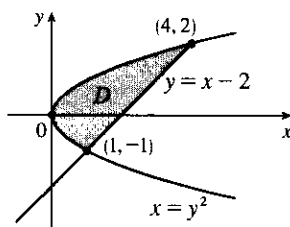
$$8. m = \int_0^1 \int_0^{\sqrt{x}} x \, dy \, dx = \int_0^1 x [y]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{3/2} dx = \left[\frac{2}{5} x^{5/2} \right]_0^1 = \frac{2}{5},$$

$$M_y = \int_0^1 \int_0^{\sqrt{x}} x^2 \, dy \, dx = \int_0^1 x [y]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{5/2} dx = \left[\frac{2}{7} x^{7/2} \right]_0^1 = \frac{2}{7}, \text{ and}$$

$$M_x = \int_0^1 \int_0^{\sqrt{x}} xy \, dy \, dx = \int_0^1 x \left[\frac{1}{2} y^2 \right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2} \int_0^1 x^2 dx = \frac{1}{2} \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1}{6}.$$

$$\text{Hence } m = \frac{2}{5}, (\bar{x}, \bar{y}) = \left(\frac{2/7}{2/5}, \frac{1/6}{2/5} \right) = \left(\frac{5}{7}, \frac{5}{12} \right).$$

$$9. m = \int_{-1}^2 \int_{y^2+2}^{y+2} 3 \, dx \, dy = \int_{-1}^2 (3y + 6 - 3y^2) \, dy = \frac{27}{2},$$



$$M_y = \int_{-1}^2 \int_{y^2+2}^{y+2} 3x \, dx \, dy = \int_{-1}^2 \frac{3}{2} [(y+2)^2 - y^4] \, dy$$

$$= \left[\frac{1}{2} (y+2)^3 - \frac{3}{10} y^5 \right]_{-1}^2 = \frac{108}{5}$$

and

$$M_x = \int_{-1}^2 \int_{y^2+2}^{y+2} 3y \, dx \, dy = \int_{-1}^2 (3y^2 + 6y - 3y^3) \, dy$$

$$= \left[y^3 + 3y^2 - \frac{3}{4} y^4 \right]_{-1}^2 = \frac{27}{4}$$

$$\text{Hence } m = \frac{27}{2}, (\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2} \right).$$

$$10. m = \int_0^{\pi/2} \int_0^{\cos x} x \, dy \, dx = \int_0^{\pi/2} x \cos x \, dx = [x \sin x + \cos x]_0^{\pi/2} = \frac{\pi}{2} - 1,$$

$$M_y = \int_0^{\pi/2} \int_0^{\cos x} x^2 \, dy \, dx = \int_0^{\pi/2} x^2 \cos x \, dx = [x^2 \sin x + 2x \cos x - 2 \sin x]_0^{\pi/2} = \frac{\pi^2}{4} - 2, \text{ and}$$

$$M_x = \int_0^{\pi/2} \int_0^{\cos x} xy \, dy \, dx = \int_0^{\pi/2} \frac{1}{2} x \cos^2 x \, dx = \frac{1}{2} \left[\frac{1}{4} x^2 + \frac{1}{4} x \sin 2x + \frac{1}{8} \cos 2x \right]_0^{\pi/2} = \frac{\pi^2}{32} - \frac{1}{8}.$$

$$\text{Hence } m = \frac{\pi - 2}{2}, (\bar{x}, \bar{y}) = \left(\frac{\pi^2 - 8}{2(\pi - 2)}, \frac{\pi + 2}{16} \right).$$

$$11. \rho(x, y) = ky = kr \sin \theta, m = \int_0^{\pi/2} \int_0^1 kr^2 \sin \theta \, dr \, d\theta = \frac{1}{3} k \int_0^{\pi/2} \sin \theta \, d\theta = \frac{1}{3} k [-\cos \theta]_0^{\pi/2} = \frac{1}{3} k,$$

$$M_y = \int_0^{\pi/2} \int_0^1 kr^3 \sin \theta \cos \theta \, dr \, d\theta = \frac{1}{4} k \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{1}{8} k [-\cos 2\theta]_0^{\pi/2} = \frac{1}{8} k,$$

$$M_x = \int_0^{\pi/2} \int_0^1 kr^3 \sin^2 \theta \, dr \, d\theta = \frac{1}{4} k \int_0^{\pi/2} \sin^2 \theta \, d\theta = \frac{1}{8} k [\theta + \sin 2\theta]_0^{\pi/2} = \frac{\pi}{16} k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(\frac{3}{8}, \frac{3\pi}{16} \right).$$

$$12. \rho(x, y) = k(x^2 + y^2) = kr^2, m = \int_0^{\pi/2} \int_0^1 kr^3 dr d\theta = \frac{\pi}{8}k,$$

$$M_y = \int_0^{\pi/2} \int_0^1 kr^4 \cos \theta dr d\theta = \frac{1}{5}k \int_0^{\pi/2} \cos \theta d\theta = \frac{1}{5}k [\sin \theta]_0^{\pi/2} = \frac{1}{5}k,$$

$$M_x = \int_0^{\pi/2} \int_0^1 kr^4 \sin \theta dr d\theta = \frac{1}{5}k \int_0^{\pi/2} \sin \theta d\theta = \frac{1}{5}k [-\cos \theta]_0^{\pi/2} = \frac{1}{5}k.$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right).$$

13. Placing the vertex opposite the hypotenuse at $(0, 0)$, $\rho(x, y) = k(x^2 + y^2)$. Then

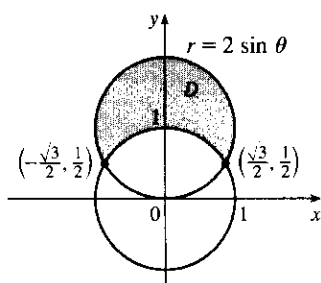
$$\begin{aligned} m &= \int_0^a \int_0^{a-x} k(x^2 + y^2) dy dx = k \int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3 \right] dx \\ &= k \left[\frac{1}{3}ax^3 - \frac{1}{4}x^4 - \frac{1}{12}(a-x)^4 \right]_0^a = \frac{1}{6}ka^4 \end{aligned}$$

By symmetry,

$$\begin{aligned} M_y &= M_x = \int_0^a \int_0^{a-x} ky(x^2 + y^2) dy dx = k \int_0^a \left[\frac{1}{2}(a-x)^2 x^2 + \frac{1}{4}(a-x)^4 \right] dx \\ &= k \left[\frac{1}{6}a^2 x^3 - \frac{1}{4}ax^4 + \frac{1}{10}x^5 - \frac{1}{20}(a-x)^5 \right]_0^a = \frac{1}{15}ka^5 \end{aligned}$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right).$$

14.



$$\rho(x, y) = k/\sqrt{x^2 + y^2} = k/r,$$

$$\begin{aligned} m &= \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} \frac{k}{r} r dr d\theta = k \int_{\pi/6}^{5\pi/6} [(2\sin\theta) - 1] d\theta \\ &= k [-2\cos\theta - \theta]_{\pi/6}^{5\pi/6} = 2k(\sqrt{3} - \frac{\pi}{3}) \end{aligned}$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$\begin{aligned} M_x &= \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} kr \sin\theta dr d\theta = \frac{1}{2}k \int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta) d\theta \\ &= \frac{1}{2}k \left[-3\cos\theta + \frac{4}{3}\cos^3\theta \right]_{\pi/6}^{5\pi/6} = \sqrt{3}k \end{aligned}$$

$$\text{Hence } (\bar{x}, \bar{y}) = \left(0, \frac{3\sqrt{3}}{2(3\sqrt{3} - \pi)}\right).$$

$$\begin{aligned} 15. I_x &= \iint_D y^2 \rho(x, y) dA = \int_0^1 \int_0^{e^x} y^2 \cdot y dy dx = \int_0^1 \left[\frac{1}{4}y^4 \right]_{y=0}^{y=e^x} dx = \frac{1}{4} \int_0^1 e^{4x} dx \\ &= \frac{1}{4} \left[\frac{1}{4}e^{4x} \right]_0^1 = \frac{1}{16}(e^4 - 1), \end{aligned}$$

$$\begin{aligned} I_y &= \iint_D x^2 \rho(x, y) dA = \int_0^1 \int_0^{e^x} x^2 y dy dx = \int_0^1 x^2 \left[\frac{1}{2}y^2 \right]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 x^2 e^{2x} dx \\ &= \frac{1}{2} \left[\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} \right) e^{2x} \right]_0^1 \quad [\text{integrate by parts twice}] \\ &= \frac{1}{8}(e^2 - 1), \end{aligned}$$

$$\text{and } I_0 = I_x + I_y = \frac{1}{16}(e^4 - 1) + \frac{1}{8}(e^2 - 1) = \frac{1}{16}(e^4 + 2e^2 - 3).$$

$$16. I_x = \int_0^{\pi/2} \int_0^1 (r^2 \sin^2 \theta)(kr^2) r dr d\theta = \frac{1}{6}k \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{6}k \left[\frac{1}{4}(2\theta - \sin 2\theta) \right]_0^{\pi/2} = \frac{\pi}{24}k,$$

$$I_y = \int_0^{\pi/2} \int_0^1 (r^2 \cos^2 \theta)(kr^2) r dr d\theta = \frac{1}{6}k \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{1}{6}k \left[\frac{1}{4}(2\theta + \sin 2\theta) \right]_0^{\pi/2} = \frac{\pi}{24}k, \text{ and}$$

$$I_0 = I_x + I_y = \frac{\pi}{12}k.$$

$$17. I_x = \int_{-1}^2 \int_{y^2}^{y+2} 3y^2 dx dy = \int_{-1}^2 (3y^3 + 6y^2 - 3y^4) dy = \left[\frac{3}{4}y^4 + 2y^3 - \frac{3}{5}y^5 \right]_{-1}^2 = \frac{189}{20},$$

$$I_y = \int_{-1}^2 \int_{y^2}^{y+2} 3x^2 dx dy = \int_{-1}^2 [(y+2)^3 - y^6] dy = \left[\frac{1}{4}(y+2)^4 - \frac{1}{7}y^7 \right]_{-1}^2 = \frac{1269}{28}, \text{ and}$$

$$I_0 = I_x + I_y = \frac{1917}{35}.$$

18. If we find the moments of inertia about the x - and y -axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x -axis is given by

$$I_x = \iint_D y^2 \rho(x, y) dA = \int_0^2 \int_0^2 y^2 (1 + 0.1x) dy dx = \int_0^2 (1 + 0.1x) \left[\frac{1}{3}y^3 \right]_{y=0}^{y=2} dx$$

$$= \frac{8}{3} \int_0^2 (1 + 0.1x) dx = \frac{8}{3} \left[x + 0.1 \cdot \frac{1}{2}x^2 \right]_0^2 = \frac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y -axis is given by

$$I_y = \iint_D x^2 \rho(x, y) dA = \int_0^2 \int_0^2 x^2 (1 + 0.1x) dy dx = \int_0^2 x^2 (1 + 0.1x) \left[y \right]_{y=0}^{y=2} dx$$

$$= 2 \int_0^2 (x^2 + 0.1x^3) dx = 2 \left[\frac{1}{3}x^3 + 0.1 \cdot \frac{1}{4}x^4 \right]_0^2 = 2 \left(\frac{8}{3} + 0.4 \right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y -axis.

19. Using a CAS, we find $m = \iint_D \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} xy dy dx = \frac{\pi^2}{8}$. Then

$$\bar{x} = \frac{1}{m} \iint_D x \rho(x, y) dA = \frac{8}{\pi^2} \int_0^\pi \int_0^{\sin x} x^2 y dy dx = \frac{2\pi}{3} - \frac{1}{\pi} \text{ and}$$

$$\bar{y} = \frac{1}{m} \iint_D y \rho(x, y) dA = \frac{8}{\pi^2} \int_0^\pi \int_0^{\sin x} xy^2 dy dx = \frac{16}{9\pi}, \text{ so } (\bar{x}, \bar{y}) = \left(\frac{2\pi}{3} - \frac{1}{\pi}, \frac{16}{9\pi} \right).$$

$$\text{The moments of inertia are } I_x = \iint_D y^2 \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} xy^3 dy dx = \frac{3\pi^2}{64},$$

$$I_y = \iint_D x^2 \rho(x, y) dA = \int_0^\pi \int_0^{\sin x} x^3 y dy dx = \frac{\pi^2}{16}(\pi^2 - 3), \text{ and } I_0 = I_x + I_y = \frac{\pi^2}{64}(4\pi^2 - 9).$$

20. Using a CAS, we find $m = \iint_D \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos \theta} r^2 dr d\theta = \frac{5}{3}\pi$,

$$\bar{x} = \frac{1}{m} \iint_D x \sqrt{x^2 + y^2} dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos \theta} r^3 \cos \theta dr d\theta = \frac{21}{20} \text{ and}$$

$$\bar{y} = \frac{1}{m} \iint_D y \sqrt{x^2 + y^2} dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos \theta} r^3 \sin \theta dr d\theta = 0, \text{ so } (\bar{x}, \bar{y}) = \left(\frac{21}{20}, 0 \right).$$

$$\text{The moments of inertia are } I_x = \iint_D y^2 \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos \theta} r^4 \sin^2 \theta dr d\theta = \frac{33}{40}\pi,$$

$$I_y = \iint_D x^2 \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^{1+\cos \theta} r^4 \cos^2 \theta dr d\theta = \frac{93}{40}\pi, \text{ and } I_0 = I_x + I_y = \frac{63}{20}\pi.$$

21. $I_x = \int_0^a \int_0^a \rho y^2 dx dy = \rho \int_0^a dx \int_0^a y^2 dy = \rho \left[x \right]_0^a \left[\frac{1}{3}y^3 \right]_0^a = \rho a \left(\frac{1}{3}a^3 \right) = \frac{1}{3}\rho a^4 = I_y$ by symmetry, and

$$m = \rho a^2 \text{ since the lamina is homogeneous. Hence } \bar{x}^2 = \frac{I_y}{m} \Rightarrow \bar{x} = \left[\left(\frac{1}{3}\rho a^4 \right) / (\rho a^2) \right]^{1/2} = \frac{1}{\sqrt{3}}a \text{ and}$$

$$\bar{y}^2 = \frac{I_x}{m} \Rightarrow \bar{y} = \frac{1}{\sqrt{3}}a.$$

$$22. m = \int_0^\pi \int_0^{\sin x} \rho \, dy \, dx = \rho \int_0^\pi \sin x \, dx = \rho [-\cos x]_0^\pi = 2\rho,$$

$$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2 \, dy \, dx = \frac{1}{3} \rho \int_0^\pi \sin^3 x \, dx = \frac{1}{3} \rho \int_0^\pi (1 - \cos^2 x) \sin x \, dx \\ = \frac{1}{3} \rho [-\cos x + \frac{1}{3} \cos^3 x]_0^\pi = \frac{4}{9} \rho,$$

$$I_y = \int_0^\pi \int_0^{\sin x} \rho x^2 \, dy \, dx = \rho \int_0^\pi x^2 \sin x \, dx \\ = \rho [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^\pi \quad [\text{by integrating by parts twice}] \\ = \rho(\pi^2 - 4).$$

$$\text{Then } \bar{y}^2 = \frac{I_x}{m} = \frac{2}{9}, \text{ so } \bar{y} = \frac{\sqrt{2}}{3} \text{ and } \bar{x}^2 = \frac{I_y}{m} = \frac{\pi^2 - 4}{2}, \text{ so } \bar{x} = \sqrt{\frac{\pi^2 - 4}{2}}.$$

23. (a) $f(x, y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x, y) \, dA = 1$. Since $f(x, y) = 0$ outside the rectangle $[0, 1] \times [0, 2]$, we can say

$$\iint_{\mathbb{R}^2} f(x, y) \, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx = \int_0^1 \int_0^2 Cx(1+y) \, dy \, dx \\ = C \int_0^1 x \left[y + \frac{1}{2} y^2 \right]_{y=0}^{y=2} dx = C \int_0^1 4x \, dx = C \left[2x^2 \right]_0^1 = 2C$$

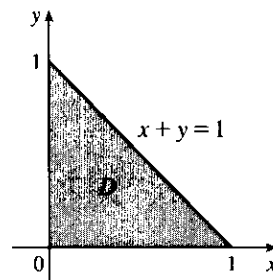
$$\text{Then } 2C = 1 \Rightarrow C = \frac{1}{2}.$$

$$(b) P(X \leq 1, Y \leq 1) = \int_{-\infty}^1 \int_{-\infty}^1 f(x, y) \, dy \, dx = \int_0^1 \int_0^1 \frac{1}{2} x(1+y) \, dy \, dx$$

$$= \int_0^1 \frac{1}{2} x \left[y + \frac{1}{2} y^2 \right]_{y=0}^{y=1} dx = \int_0^1 \frac{1}{2} x \left(\frac{3}{2} \right) dx = \frac{3}{4} \left[\frac{1}{2} x^2 \right]_0^1 = \frac{3}{8} \text{ or } 0.375$$

- (c) $P(X + Y \leq 1) = P((X, Y) \in D)$ where D is the triangular region shown in the figure. Thus

$$P(X + Y \leq 1) = \iint_D f(x, y) \, dA = \int_0^1 \int_0^{1-x} \frac{1}{2} x(1+y) \, dy \, dx \\ = \int_0^1 \frac{1}{2} x \left[y + \frac{1}{2} y^2 \right]_{y=0}^{y=1-x} dx = \int_0^1 \frac{1}{2} x \left(\frac{1}{2} x^2 - 2x + \frac{3}{2} \right) dx \\ = \frac{1}{4} \int_0^1 (x^3 - 4x^2 + 3x) \, dx = \frac{1}{4} \left[\frac{x^4}{4} - 4 \frac{x^3}{3} + 3 \frac{x^2}{2} \right]_0^1 \\ = \frac{5}{48} \approx 0.1042$$



24. (a) $f(x, y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x, y) \, dA = 1$. Here, $f(x, y) = 0$ outside the square $[0, 1] \times [0, 1]$, so $\iint_{\mathbb{R}^2} f(x, y) \, dA = \int_0^1 \int_0^1 4xy \, dy \, dx = \int_0^1 [2xy^2]_{y=0}^{y=1} dx = \int_0^1 2x \, dx = x^2 \Big|_0^1 = 1$. Thus, $f(x, y)$ is a joint density function.

- (b) (i) No restriction is placed on Y , so

$$P(X \geq \frac{1}{2}) = \int_{1/2}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx = \int_{1/2}^1 \int_0^1 4xy \, dy \, dx \\ = \int_{1/2}^1 [2xy^2]_{y=0}^{y=1} dx = \int_{1/2}^1 2x \, dx = x^2 \Big|_{1/2}^1 = \frac{3}{4}$$

$$(ii) P(X \geq \frac{1}{2}, Y \leq \frac{1}{2}) = \int_{1/2}^{\infty} \int_{-\infty}^{1/2} f(x, y) \, dy \, dx = \int_{1/2}^1 \int_0^{1/2} 4xy \, dy \, dx \\ = \int_{1/2}^1 [2xy^2]_{y=0}^{y=1/2} dx = \int_{1/2}^1 \frac{1}{2} x \, dx = \frac{1}{2} \cdot \frac{1}{2} x^2 \Big|_{1/2}^1 = \frac{3}{16}$$

(c) The expected value of X is given by

$$\begin{aligned}\mu_1 &= \iint_{\mathbb{R}^2} x f(x, y) dA = \int_0^1 \int_0^1 x(4xy) dy dx = \int_0^1 2x^2 [y^2]_{y=0}^{y=1} dx \\ &= 2 \int_0^1 x^2 dx = 2 \left[\frac{1}{3} x^3 \right]_0^1 = \frac{2}{3}\end{aligned}$$

The expected value of Y is

$$\begin{aligned}\mu_2 &= \iint_{\mathbb{R}^2} y f(x, y) dA = \int_0^1 \int_0^1 y(4xy) dy dx = \int_0^1 4x \left[\frac{1}{3} y^3 \right]_{y=0}^{y=1} dx \\ &= \frac{4}{3} \int_0^1 x dx = \frac{4}{3} \left[\frac{1}{2} x^2 \right]_0^1 = \frac{2}{3}\end{aligned}$$

25. (a) $f(x, y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x, y) dA = 1$. Here, $f(x, y) = 0$ outside the first quadrant, so

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x, y) dA &= \int_0^\infty \int_0^\infty 0.1 e^{-(0.5x+0.2y)} dy dx = 0.1 \int_0^\infty \int_0^\infty e^{-0.5x} e^{-0.2y} dy dx \\ &= 0.1 \int_0^\infty e^{-0.5x} dx \int_0^\infty e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t e^{-0.2y} dy \\ &= 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} [-5(e^{-0.2t} - 1)] \\ &= (0.1) \cdot (-2)(0 - 1) \cdot (-5)(0 - 1) = 1\end{aligned}$$

Thus $f(x, y)$ is a joint density function.

- (b) (i) No restriction is placed on X , so

$$\begin{aligned}P(Y \geq 1) &= \int_{-\infty}^\infty \int_1^\infty f(x, y) dy dx = \int_0^\infty \int_1^\infty 0.1 e^{-(0.5x+0.2y)} dy dx \\ &= 0.1 \int_0^\infty e^{-0.5x} dx \int_1^\infty e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_1^t e^{-0.2y} dy \\ &= 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_1^t \\ &= 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} [-5(e^{-0.2t} - e^{-0.2})] \\ &= (0.1) \cdot (-2)(0 - 1) \cdot (-5)(0 - e^{-0.2}) = e^{-0.2} \approx 0.8187\end{aligned}$$

$$\begin{aligned}\text{(ii) } P(X \leq 2, Y \leq 4) &= \int_{-\infty}^2 \int_{-\infty}^4 f(x, y) dy dx = \int_0^2 \int_0^4 0.1 e^{-(0.5x+0.2y)} dy dx \\ &= 0.1 \int_0^2 e^{-0.5x} dx \int_0^4 e^{-0.2y} dy = 0.1 [-2e^{-0.5x}]_0^2 [-5e^{-0.2y}]_0^4 \\ &= (0.1) \cdot (-2)(e^{-1} - 1) \cdot (-5)(e^{-0.8} - 1) \\ &= (e^{-1} - 1)(e^{-0.8} - 1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481\end{aligned}$$

- (c) The expected value of X is given by

$$\begin{aligned}\mu_1 &= \iint_{\mathbb{R}^2} x f(x, y) dA = \int_0^\infty \int_0^\infty x [0.1 e^{-(0.5x+0.2y)}] dy dx \\ &= 0.1 \int_0^\infty x e^{-0.5x} dx \int_0^\infty e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t x e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t e^{-0.2y} dy\end{aligned}$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x} dx$ (or we can use Formula 96 in the Table of Integrals):

$$\int x e^{-0.5x} dx = -2x e^{-0.5x} - \int -2e^{-0.5x} dx = -2x e^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}. \text{ Thus}$$

$$\begin{aligned} \mu_1 &= 0.1 \lim_{t \rightarrow \infty} [-2(x+2)e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} (-2)[(t+2)e^{-0.5t} - 2] \lim_{t \rightarrow \infty} (-5)[e^{-0.2t} - 1] \\ &= 0.1(-2) \left(\lim_{t \rightarrow \infty} \frac{t+2}{e^{0.5t}} - 2 \right) (-5)(-1) = 2 \quad [\text{by l'Hospital's Rule}] \end{aligned}$$

The expected value of Y is given by

$$\begin{aligned} \mu_2 &= \iint_{\mathbb{R}^2} y f(x, y) dA = \int_0^\infty \int_0^\infty y [0.1 e^{-(0.5+0.2y)}] dy dx \\ &= 0.1 \int_0^\infty e^{-0.5x} dx \int_0^\infty y e^{-0.2y} dy = 0.1 \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t y e^{-0.2y} dy \end{aligned}$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y} dy$ (or again we can use Formula 96 in the Table of Integrals) which gives

$$\int y e^{-0.2y} dy = -5y e^{-0.2y} + \int 5e^{-0.2y} dy = -5(y+5)e^{-0.2y}. \text{ Then}$$

$$\begin{aligned} \mu_2 &= 0.1 \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5(y+5)e^{-0.2y}]_0^t \\ &= 0.1 \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} (-5)[(t+5)e^{-0.2t} - 5] \\ &= 0.1(-2)(-1) \cdot (-5) \left(\lim_{t \rightarrow \infty} \frac{t+5}{e^{0.2t}} - 5 \right) = 5 \quad [\text{by l'Hospital's Rule}] \end{aligned}$$

26. (a) Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x, y) = \begin{cases} 10^{-6} e^{-(x+y)/1000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned} P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x, y) dy dx \\ &= \int_0^{1000} \int_0^{1000} 10^{-6} e^{-(x+y)/1000} dy dx \\ &= 10^{-6} \int_0^{1000} e^{-x/1000} dx \int_0^{1000} e^{-y/1000} dy \\ &= 10^{-6} \left[-1000 e^{-x/1000} \right]_0^{1000} \left[-1000 e^{-y/1000} \right]_0^{1000} \\ &= (e^{-1} - 1)^2 \approx 0.3996 \end{aligned}$$

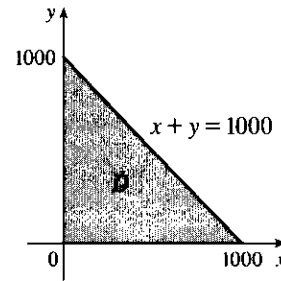
(b) Now we are asked for the probability that the combined

lifetimes of both bulbs is 1000 hours or less. Thus we want

to find $P(X + Y \leq 1000)$, or equivalently $P((X, Y) \in D)$

where D is the triangular region shown in the figure.

Then



$$\begin{aligned} P(X + Y \leq 1000) &= \iint_D f(x, y) \, dA = \int_0^{1000} \int_0^{1000-x} 10^{-6} e^{-(x+y)/1000} \, dy \, dx \\ &= 10^{-6} \int_0^{1000} \left[-1000 e^{-(x+y)/1000} \right]_{y=0}^{y=1000-x} \, dx \\ &= -10^{-3} \int_0^{1000} \left(e^{-1} - e^{-x/1000} \right) \, dx \\ &= -10^{-3} \left[e^{-1} x + 1000 e^{-x/1000} \right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642 \end{aligned}$$

27. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y , then, are $f_1(x) = \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5}$ and

$f_2(y) = \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$$\begin{aligned} f(x, y) &= f_1(x)f_2(y) = \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5} \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02} \\ &= \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} \end{aligned}$$

Then

$$\begin{aligned} P(40 \leq X \leq 50, 20 \leq Y \leq 25) &= \int_{40}^{50} \int_{20}^{25} f(x, y) \, dy \, dx \\ &= \frac{10}{\pi} \int_{40}^{50} \int_{20}^{25} e^{-2(x-45)^2 - 50(y-20)^2} \, dy \, dx \end{aligned}$$

Using a CAS or calculator to evaluate the integral, we get $P(40 \leq X \leq 50, 20 \leq Y \leq 25) \approx 0.500$.

- (b) $P(4(X - 45)^2 + 100(Y - 20)^2 \leq 2) = \iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} \, dA$, where D is the region enclosed by the ellipse $4(x - 45)^2 + 100(y - 20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10} \sqrt{2 - 4(x - 45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at

$(45, 20)$] $\Rightarrow 4(x - 45)^2 = 2 \Rightarrow x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

$$\iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} \, dA = \frac{10}{\pi} \int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}} \int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2 - 50(y-20)^2} \, dy \, dx.$$

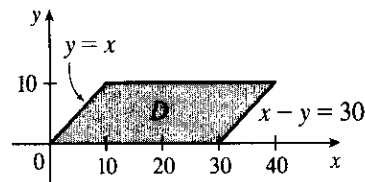
Using a CAS or calculator to evaluate the integral, we get $P(4(X - 45)^2 + 100(Y - 20)^2 \leq 2) \approx 0.632$.

28. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50} e^{-x} y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$. Additionally, Yolanda will wait up to half an

hour but no longer, so they won't meet unless $X - Y \leq 30$. Thus the probability that they meet is $P((X, Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$\begin{aligned} P((X, Y) \in D) &= \iint_D f(x, y) dx dy = \int_0^{10} \int_y^{y+30} \frac{1}{50} e^{-x} y dx dy = \frac{1}{50} \int_0^{10} y [-e^{-x}]_{x=y}^{x=y+30} dy \\ &= \frac{1}{50} \int_0^{10} y (-e^{-(y+30)} + e^{-y}) dy = \frac{1}{50} (1 - e^{-30}) \int_0^{10} y e^{-y} dy \end{aligned}$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

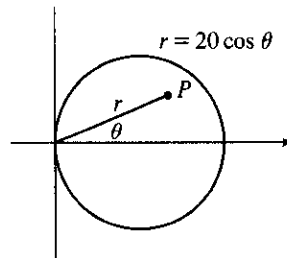
$\frac{1}{50} (1 - e^{-30}) [-(y+1)e^{-y}]_0^{10} = \frac{1}{50} (1 - e^{-30}) (1 - 11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

29. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P , and $k dA$ is the number of infected individuals in an element of area dA , then $f(P, A)k dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA . Integration over D gives the number of infections of the person at A due to all the infected people in D . In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D k f(P, A) dA = k \iint_D \frac{20 - d(P, A)}{20} dA = k \iint_D \left[1 - \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{20} \right] dx dy.$$

- (b) If $A = (0, 0)$, then

$$\begin{aligned} E &= k \iint_D \left[1 - \frac{1}{20} \sqrt{x^2 + y^2} \right] dx dy \\ &= k \int_0^{2\pi} \int_0^{10} \left(1 - \frac{r}{20} \right) r dr d\theta = 2\pi k \left[\frac{r^2}{2} - \frac{r^3}{60} \right]_0^{10} \\ &= 2\pi k \left(50 - \frac{50}{3} \right) = \frac{200}{3} \pi k \approx 209k \end{aligned}$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A . Then the polar equation for the circular boundary of the city becomes $r = 20 \cos \theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned} E &= k \int_{-\pi/2}^{\pi/2} \int_0^{20 \cos \theta} \left(1 - \frac{r}{20} \right) r dr d\theta = k \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{60} \right]_{r=0}^{r=20 \cos \theta} d\theta \\ &= k \int_{-\pi/2}^{\pi/2} \left(200 \cos^2 \theta - \frac{400}{3} \cos^3 \theta \right) d\theta = 200k \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2} + \frac{1}{2} \cos 2\theta - \frac{2}{3} (1 - \sin^2 \theta) \cos \theta \right] d\theta \\ &= 200k \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta - \frac{2}{3} \sin \theta + \frac{2}{3} \cdot \frac{1}{3} \sin^3 \theta \right]_{-\pi/2}^{\pi/2} = 200k \left[\frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9} + \frac{\pi}{4} + 0 - \frac{2}{3} + \frac{2}{9} \right] \\ &= 200k \left(\frac{\pi}{2} - \frac{8}{9} \right) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

16.6 Surface Area

ET 15.6

1. Here $z = f(x, y) = 2 + 3x + 4y$ and D is the rectangle $[0, 5] \times [1, 4]$, so by Formula 2 the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_D \sqrt{3^2 + 4^2 + 1} \, dA = \sqrt{26} \iint_D dA \\ &= \sqrt{26} A(D) = \sqrt{26} (5)(3) = 15\sqrt{26} \end{aligned}$$

2. $z = f(x, y) = 10 - 2x - 5y$ and D is the disk $x^2 + y^2 \leq 9$, so by Formula 2

$$\begin{aligned} A(S) &= \iint_D \sqrt{(-2)^2 + (-5)^2 + 1} \, dA = \sqrt{30} \iint_D dA = \sqrt{30} A(D) \\ &= \sqrt{30} (\pi \cdot 3^2) = 9\sqrt{30} \pi \end{aligned}$$

3. $z = f(x, y) = 6 - 3x - 2y$ which intersects the xy -plane in the line $3x + 2y = 6$, so D is the triangular region given by $\{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3 - \frac{3}{2}x\}$. Thus

$$\begin{aligned} A(S) &= \iint_D \sqrt{(-3)^2 + (-2)^2 + 1} \, dA = \sqrt{14} \iint_D dA = \sqrt{14} A(D) \\ &= \sqrt{14} \left(\frac{1}{2} \cdot 2 \cdot 3 \right) = 3\sqrt{14} \end{aligned}$$

4. $z = f(x, y) = 1 + 3x + 2y^2$ with $0 \leq x \leq 2y$, $0 \leq y \leq 1$. Thus by Formula 2,

$$\begin{aligned} A(S) &= \iint_D \sqrt{(3)^2 + (4y)^2 + 1} \, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2} \, dx \, dy = \int_0^1 \sqrt{10 + 16y^2} [x]_{x=0}^{x=2y} dy \\ &= \int_0^1 2y \sqrt{10 + 16y^2} \, dy = 2 \cdot \frac{1}{32} \cdot \frac{2}{3} (10 + 16y^2)^{3/2} \Big|_0^1 = \frac{1}{24} (26^{3/2} - 10^{3/2}) \end{aligned}$$

5. $y^2 + z^2 = 9 \Rightarrow z = \sqrt{9 - y^2}$. $f_x = 0$, $f_y = -y(9 - y^2)^{-1/2} \Rightarrow$

$$\begin{aligned} A(S) &= \int_0^4 \int_0^2 \sqrt{0^2 + [-y(9 - y^2)^{-1/2}]^2 + 1} \, dy \, dx = \int_0^4 \int_0^2 \sqrt{\frac{y^2}{9 - y^2} + 1} \, dy \, dx \\ &= \int_0^4 \int_0^2 \frac{3}{\sqrt{9 - y^2}} \, dy \, dx = 3 \int_0^4 \left[\sin^{-1} \frac{y}{3} \right]_{y=0}^{y=2} dx = 3 \left[\left(\sin^{-1} \left(\frac{2}{3} \right) \right) x \right]_0^4 = 12 \sin^{-1} \left(\frac{2}{3} \right) \end{aligned}$$

6. $z = f(x, y) = 4 - x^2 - y^2$ and D is the projection of the paraboloid $z = 4 - x^2 - y^2$ onto the xy -plane, that is,

$$D = \{(x, y) \mid x^2 + y^2 \leq 4\}. \text{ So } f_x = -2x, f_y = -2y \Rightarrow$$

$$\begin{aligned} A(S) &= \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1} \, dA = \iint_D \sqrt{4(x^2 + y^2) + 1} \, dA = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \frac{1}{12} (17\sqrt{17} - 1) \, d\theta = \frac{\pi}{6} (17\sqrt{17} - 1) \end{aligned}$$

7. $z = f(x, y) = y^2 - x^2$ with $1 \leq x^2 + y^2 \leq 4$. Then

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_1^2 r \sqrt{1 + 4r^2} \, dr \\ &= [\theta]_0^{2\pi} \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_1^2 = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5}) \end{aligned}$$

8. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2 + 1} dA = \int_0^1 \int_0^1 \sqrt{x+y+1} dy dx \\ &= \int_0^1 \left[\frac{2}{3}(x+y+1)^{3/2} \right]_{y=0}^{y=1} dx = \frac{2}{3} \int_0^1 [(x+2)^{3/2} - (x+1)^{3/2}] dx \\ &= \frac{2}{3} \left[\frac{2}{5}(x+2)^{5/2} - \frac{2}{5}(x+1)^{5/2} \right]_0^1 = \frac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) \\ &= \frac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

9. $z = f(x, y) = xy$ with $0 \leq x^2 + y^2 \leq 1$, so $f_x = y$, $f_y = x \Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{y^2 + x^2 + 1} dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{3}(r^2 + 1)^{3/2} \right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \frac{1}{3}(2\sqrt{2} - 1) d\theta = \frac{2\pi}{3}(2\sqrt{2} - 1) \end{aligned}$$

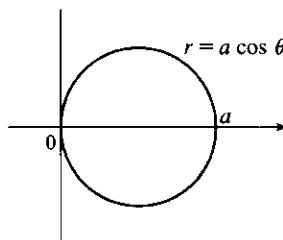
10. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \{(x, y) \mid x^2 + y^2 \leq 3\}$ and

$z = f(x, y) = \sqrt{4 - x^2 - y^2}$. Thus

$$\begin{aligned} A(S) &= \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1} dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1} r dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}} dr d\theta \\ &= \int_0^{2\pi} \left[-2(4 - r^2)^{1/2} \right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi} (-2 + 4) d\theta = 2\theta \Big|_0^{2\pi} = 4\pi \end{aligned}$$

11. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x(a^2 - x^2 - y^2)^{-1/2}$, $z_y = -y(a^2 - x^2 - y^2)^{-1/2}$,

$$\begin{aligned} A(S) &= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1} dA \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{a \cos \theta} \sqrt{\frac{r^2}{a^2 - r^2} + 1} r dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{a \cos \theta} \frac{ar}{\sqrt{a^2 - r^2}} dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \left[-a \sqrt{a^2 - r^2} \right]_{r=0}^{r=a \cos \theta} d\theta \\ &= \int_{-\pi/2}^{\pi/2} -a(\sqrt{a^2 - a^2 \cos^2 \theta} - a) d\theta = 2a^2 \int_0^{\pi/2} (1 - \sqrt{1 - \cos^2 \theta}) d\theta \\ &= 2a^2 \int_0^{\pi/2} d\theta - 2a^2 \int_0^{\pi/2} \sqrt{\sin^2 \theta} d\theta = a^2 \pi - 2a^2 \int_0^{\pi/2} \sin \theta d\theta = a^2(\pi - 2) \end{aligned}$$



12. To find the region D : $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z - 2)^2 = 4$, so $z = 3$ intersects the upper

hemisphere. Thus $(z - 2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3 - 2)^2 = 4$, that is, $D = \{(x, y) \mid x^2 + y^2 \leq 3\}$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + [(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2} dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4 - r^2}} r dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r dr}{\sqrt{4 - r^2}} d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2} \right]_{r=0}^{r=\sqrt{3}} d\theta \\ &= \int_0^{2\pi} (-2 + 4) d\theta = 2\theta \Big|_0^{2\pi} = 4\pi \end{aligned}$$

13. $z = f(x, y) = e^{-x^2 - y^2}$, $f_x = -2xe^{-x^2 - y^2}$, $f_y = -2ye^{-x^2 - y^2}$. Then

$$A(S) = \iint_{x^2 + y^2 \leq 4} \sqrt{(-2xe^{-x^2 - y^2})^2 + (-2ye^{-x^2 - y^2})^2 + 1} dA = \iint_{x^2 + y^2 \leq 4} \sqrt{4(x^2 + y^2)e^{-2(x^2 + y^2)} + 1} dA.$$

Converting to polar coordinates we have

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 e^{-2r^2} + 1} r dr d\theta = \int_0^{2\pi} d\theta \int_0^2 r \sqrt{4r^2 e^{-2r^2} + 1} dr \\ &= 2\pi \int_0^2 r \sqrt{4r^2 e^{-2r^2} + 1} dr \approx 13.9783 \text{ using a calculator.} \end{aligned}$$

14. $z = f(x, y) = \cos(x^2 + y^2)$, $f_x = -2x \sin(x^2 + y^2)$, $f_y = -2y \sin(x^2 + y^2)$.

$$\begin{aligned} A(S) &= \iint_{x^2 + y^2 \leq 1} \sqrt{4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2) + 1} dA \\ &= \iint_{x^2 + y^2 \leq 1} \sqrt{4(x^2 + y^2) \sin^2(x^2 + y^2) + 1} dA \end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 \sin^2(r^2) + 1} r dr d\theta = \int_0^{2\pi} d\theta \int_0^1 r \sqrt{4r^2 \sin^2(r^2) + 1} dr \\ &= 2\pi \int_0^1 r \sqrt{4r^2 \sin^2(r^2) + 1} dr \approx 4.1073 \text{ using a calculator.} \end{aligned}$$

15. (a) The midpoints of the four squares are $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4})$, and $(\frac{3}{4}, \frac{3}{4})$. Here $f(x, y) = x^2 + y^2$, so the Midpoint Rule gives

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1} dA \\ &\approx \frac{1}{4} \left(\sqrt{[2(\frac{1}{4})]^2 + [2(\frac{1}{4})]^2 + 1} + \sqrt{[2(\frac{1}{4})]^2 + [2(\frac{3}{4})]^2 + 1} \right. \\ &\quad \left. + \sqrt{[2(\frac{3}{4})]^2 + [2(\frac{1}{4})]^2 + 1} + \sqrt{[2(\frac{3}{4})]^2 + [2(\frac{3}{4})]^2 + 1} \right) \\ &= \frac{1}{4} \left(\sqrt{\frac{3}{2}} + 2\sqrt{\frac{7}{2}} + \sqrt{\frac{11}{2}} \right) \approx 1.8279 \end{aligned}$$

- (b) A CAS estimates the integral to be

$$A(S) = \iint_D \sqrt{1 + (2x)^2 + (2y)^2} dA = \int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 4y^2} dy dx \approx 1.8616. \text{ This agrees with the Midpoint estimate only in the first decimal place.}$$

16.7 Triple Integrals

ET 15.7

$$\begin{aligned}
 1. \iiint_E xyz^2 dV &= \int_0^1 \int_{-1}^2 \int_0^3 xyz^2 dz dx dy = \int_0^1 \int_{-1}^2 xy \left[\frac{1}{3} z^3 \right]_{z=0}^{z=3} dx dy = \int_0^1 \int_{-1}^2 9xy dx dy \\
 &= \int_0^1 \left[\frac{9}{2} x^2 y \right]_{x=-1}^{x=2} dy = \int_0^1 \frac{27}{2} y dy = \left[\frac{27}{4} y^2 \right]_0^1 = \frac{27}{4}
 \end{aligned}$$

2. There are six different possible orders of integration.

$$\begin{aligned}
 \iiint_E (xz - y^3) dV &= \int_{-1}^1 \int_0^2 \int_0^1 (xz - y^3) dz dy dx = \int_{-1}^1 \int_0^2 \left[\frac{1}{2} xz^2 - y^3 z \right]_{z=0}^{z=1} dy dx \\
 &= \int_{-1}^1 \int_0^2 \left(\frac{1}{2} x - y^3 \right) dy dx = \int_{-1}^1 \left[\frac{1}{2} xy - \frac{1}{4} y^4 \right]_{y=0}^{y=2} dx \\
 &= \int_{-1}^1 (x - 4) dx = \left[\frac{1}{2} x^2 - 4x \right]_{-1}^1 = -8
 \end{aligned}$$

$$\begin{aligned}
 \iiint_E (xz - y^3) dV &= \int_0^2 \int_{-1}^1 \int_0^1 (xz - y^3) dz dx dy = \int_0^2 \int_{-1}^1 \left[\frac{1}{2} xz^2 - y^3 z \right]_{z=0}^{z=1} dx dy \\
 &= \int_0^2 \int_{-1}^1 \left(\frac{1}{2} x - y^3 \right) dx dy = \int_0^2 \left[\frac{1}{4} x^2 - xy^3 \right]_{x=-1}^{x=1} dy \\
 &= \int_0^2 -2y^3 dy = \left[-\frac{1}{2} y^4 \right]_0^2 = -8
 \end{aligned}$$

$$\begin{aligned}
 \iiint_E (xz - y^3) dV &= \int_{-1}^1 \int_0^1 \int_0^2 (xz - y^3) dy dz dx = \int_{-1}^1 \int_0^1 \left[xyz - \frac{1}{4} y^4 \right]_{y=0}^{y=2} dz dx \\
 &= \int_{-1}^1 \int_0^1 (2xz - 4) dz dx = \int_{-1}^1 \left[xz^2 - 4z \right]_{z=0}^{z=1} dx \\
 &= \int_{-1}^1 (x - 4) dx = \left[\frac{1}{2} x^2 - 4x \right]_{-1}^1 = -8
 \end{aligned}$$

$$\begin{aligned}
 \iiint_E (xz - y^3) dV &= \int_0^1 \int_{-1}^1 \int_0^2 (xz - y^3) dy dx dz = \int_0^1 \int_{-1}^1 \left[xyz - \frac{1}{4} y^4 \right]_{y=0}^{y=2} dx dz \\
 &= \int_0^1 \int_{-1}^1 (2xz - 4) dx dz = \int_0^1 \left[x^2 z - 4x \right]_{x=-1}^{x=1} dz \\
 &= \int_0^1 -8 dz = \left[-8z \right]_0^1 = -8
 \end{aligned}$$

$$\begin{aligned}
 \iiint_E (xz - y^3) dV &= \int_0^2 \int_0^1 \int_{-1}^1 (xz - y^3) dx dz dy = \int_0^2 \int_0^1 \left[\frac{1}{2} x^2 z - xy^3 \right]_{x=-1}^{x=1} dz dy \\
 &= \int_0^2 \int_0^1 -2y^3 dz dy = \int_0^2 \left[-2y^3 z \right]_{z=0}^{z=1} dy = \int_0^2 -2y^3 dy = \left[-\frac{1}{2} y^4 \right]_0^2 = -8
 \end{aligned}$$

$$\begin{aligned}
 \iiint_E (xz - y^3) dV &= \int_0^1 \int_0^2 \int_{-1}^1 (xz - y^3) dx dy dz = \int_0^1 \int_0^2 \left[\frac{1}{2} x^2 z - xy^3 \right]_{x=-1}^{x=1} dy dz \\
 &= \int_0^1 \int_0^2 -2y^3 dy dz = \int_0^1 \left[-\frac{1}{2} y^4 \right]_{y=0}^{y=2} dz = \int_0^1 -8 dz = \left[-8z \right]_0^1 = -8
 \end{aligned}$$

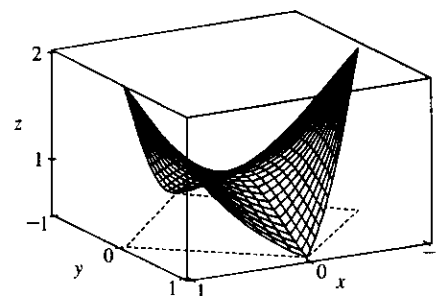
$$\begin{aligned}
 3. \int_0^1 \int_0^z \int_0^{x+z} 6xz dy dx dz &= \int_0^1 \int_0^z [6xyz]_{y=0}^{y=x+z} dx dz = \int_0^1 \int_0^z 6xz(x+z) dx dz \\
 &= \int_0^1 [2x^3 z + 3x^2 z^2]_{x=0}^{x=z} dz = \int_0^1 (2z^4 + 3z^4) dz = \int_0^1 5z^4 dz = \left[z^5 \right]_0^1 = 1
 \end{aligned}$$

$$\begin{aligned}
 4. \int_0^1 \int_x^{2x} \int_0^y 2xyz dz dy dx &= \int_0^1 \int_x^{2x} [xyz^2]_{z=0}^{z=y} dy dx = \int_0^1 \int_x^{2x} xy^3 dy dx \\
 &= \int_0^1 \left[\frac{1}{4} xy^4 \right]_{y=x}^{y=2x} dx = \int_0^1 \frac{15}{4} x^5 dx = \left[\frac{5}{8} x^6 \right]_0^1 = \frac{5}{8}
 \end{aligned}$$

20. Let $f(x, y) = \frac{1+x^2}{1+y^2}$. Then $f_x = \frac{2x}{1+y^2}$,

$$f_y = (1+x^2) \left[-\frac{2y}{(1+y^2)^2} \right] = -\frac{2y(1+x^2)}{(1+y^2)^2}. \text{ We use a CAS}$$

to estimate $\int_{-1}^1 \int_{-(1-|x|)}^{1-|x|} \sqrt{f_x^2 + f_y^2 + 1} dy dx \approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1-|x|) \leq y \leq 1-|x|$ as the y -range in our plot command.



21. Here $z = f(x, y) = ax + by + c$, $f_x(x, y) = a$, $f_y(x, y) = b$, so

$$A(S) = \iint_D \sqrt{a^2 + b^2 + 1} dA = \sqrt{a^2 + b^2 + 1} \iint_D dA = \sqrt{a^2 + b^2 + 1} A(D).$$

22. Let S be the upper hemisphere. Then $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$, so

$$\begin{aligned} A(S) &= \iint_D \sqrt{[-x(a^2 - x^2 - y^2)^{-1/2}]^2 + [-y(a^2 - x^2 - y^2)^{-1/2}]^2 + 1} dA \\ &= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1} dA = \lim_{t \rightarrow a^-} \int_0^{2\pi} \int_0^t \sqrt{\frac{r^2}{a^2 - r^2} + 1} r dr d\theta \\ &= \lim_{t \rightarrow a^-} \int_0^{2\pi} \int_0^t \frac{ar}{\sqrt{a^2 - r^2}} dr d\theta = 2\pi \lim_{t \rightarrow a^-} \left[-a \sqrt{a^2 - r^2} \right]_0^t = 2\pi \lim_{t \rightarrow a^-} -a \left[\sqrt{a^2 - t^2} - a \right] \\ &= 2\pi(-a)(-a) = 2\pi a^2. \text{ Thus the surface area of the entire sphere is } 4\pi a^2. \end{aligned}$$

23. If we project the surface onto the xz -plane, then the surface lies “above” the disk $x^2 + z^2 \leq 25$ in the xz -plane.

We have $y = f(x, z) = x^2 + z^2$ and, adapting Formula 2, the area of the surface is

$$A(S) = \iint_{x^2+z^2 \leq 25} \sqrt{[f_x(x, z)]^2 + [f_z(x, z)]^2 + 1} dA = \iint_{x^2+z^2 \leq 25} \sqrt{4x^2 + 4z^2 + 1} dA$$

Converting to polar coordinates $x = r \cos \theta$, $z = r \sin \theta$ we have

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^5 \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} d\theta \int_0^5 r(4r^2 + 1)^{1/2} dr = [\theta]_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^5 \\ &= \frac{\pi}{6} (101\sqrt{101} - 1) \end{aligned}$$

24. First we find the area of the face of the surface that intersects the positive y -axis. As in Exercise 23, we can project the face onto the xz -plane, so the surface lies “above” the disk $x^2 + z^2 \leq 1$. Then $z = f(x, z) = \sqrt{1 - z^2}$ and the area is

$$\begin{aligned} A(S) &= \iint_{x^2+z^2 \leq 1} \sqrt{[f_x(x, z)]^2 + [f_z(x, z)]^2 + 1} dA = \iint_{x^2+z^2 \leq 1} \sqrt{0 + \left(\frac{-z}{\sqrt{1-z^2}} \right)^2 + 1} dA \\ &= \iint_{x^2+z^2 \leq 1} \sqrt{\frac{z^2}{1-z^2} + 1} dA = \int_{-1}^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}} dx dz \\ &= 4 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}} dx dz \quad [\text{by the symmetry of the surface}] \end{aligned}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t \rightarrow 1^-} 4 \int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}} dx dz = \lim_{t \rightarrow 1^-} 4 \int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}} dz = \lim_{t \rightarrow 1^-} 4 \int_0^t dz = \lim_{t \rightarrow 1^-} 4t = 4.$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

16. (a) With $m = n = 2$ we have four squares with midpoints $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$, and $(\frac{3}{2}, \frac{3}{2})$. Since $z = xy + x^2 + y^2$, the Midpoint Rule gives

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_D \sqrt{(y+2x)^2 + (x+2y)^2 + 1} dA \\ &\approx 1 \left(\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 1} + \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{7}{2}\right)^2 + 1} + \sqrt{\left(\frac{7}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + 1} + \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{2}\right)^2 + 1} \right) \\ &= \frac{\sqrt{22}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{166}}{2} \approx 17.619 \end{aligned}$$

(b) Using a CAS, we have

$$A(S) = \iint_D \sqrt{(y+2x)^2 + (x+2y)^2 + 1} dA = \int_0^2 \int_0^2 \sqrt{1 + (y+2x)^2 + (x+2y)^2} dy dx \approx 17.7165.$$

This is within about 0.1 of the Midpoint Rule estimate.

17. $z = 1 + 2x + 3y + 4y^2$, so

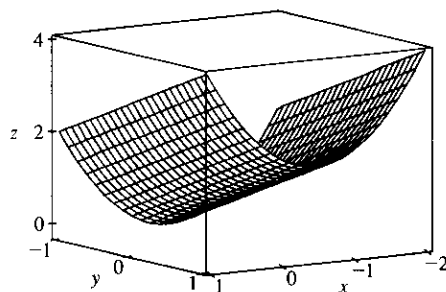
$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3 + 8y)^2} dy dx \\ &= \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2} dy dx. \end{aligned}$$

Using a CAS, we have

$$\begin{aligned} \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2} dy dx &= \frac{45}{8} \sqrt{14} + \frac{15}{16} \ln(11\sqrt{5} + 3\sqrt{14}\sqrt{5}) - \frac{15}{16} \ln(3\sqrt{5} + \sqrt{14}\sqrt{5}) \\ \text{or } \frac{45}{8} \sqrt{14} + \frac{15}{16} \ln \frac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}. \end{aligned}$$

18. $f(x, y) = 1 + x + y + x^2 \Rightarrow f_x = 1 + 2x, f_y = 1$. We use a CAS to calculate the integral

$$\begin{aligned} A(S) &= \int_{-2}^1 \int_{-1}^1 \sqrt{f_x^2 + f_y^2 + 1} dy dx = \int_{-2}^1 \int_{-1}^1 \sqrt{(1+2x)^2 + 2} dy dx = 2 \int_{-2}^1 \sqrt{4x^2 + 4x + 3} dx \text{ and find} \\ \text{that } A(S) &= 3\sqrt{11} + 2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right) \text{ or } A(S) = 3\sqrt{11} + \ln(10 + 3\sqrt{11}). \end{aligned}$$



19. $f(x, y) = 1 + x^2y^2 \Rightarrow f_x = 2xy^2, f_y = 2x^2y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral

$$\begin{aligned} A(S) &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{f_x^2 + f_y^2 + 1} dy dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2y^4 + 4x^4y^2 + 1} dy dx, \text{ and find that} \\ A(S) &\approx 3.3213. \end{aligned}$$

$$\begin{aligned}
 5. \int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} z e^y dx dz dy &= \int_0^3 \int_0^1 [x z e^y]_{x=0}^{x=\sqrt{1-z^2}} dz dy = \int_0^3 \int_0^1 z e^y \sqrt{1-z^2} dz dy \\
 &= \int_0^3 \left[-\frac{1}{3}(1-z^2)^{3/2} e^y \right]_{z=0}^{z=1} dy = \int_0^3 \frac{1}{3} e^y dy = \frac{1}{3} e^y \Big|_0^3 = \frac{1}{3}(e^3 - 1)
 \end{aligned}$$

$$\begin{aligned}
 6. \int_0^1 \int_0^z \int_0^y z e^{-y^2} dx dy dz &= \int_0^1 \int_0^z [x z e^{-y^2}]_{x=0}^{x=y} dy dz = \int_0^1 \int_0^z y z e^{-y^2} dy dz = \int_0^1 \left[-\frac{1}{2} z e^{-y^2} \right]_{y=0}^{y=z} dz \\
 &= \int_0^1 -\frac{1}{2} z (e^{-z^2} - 1) dz = \frac{1}{2} \int_0^1 (z - z e^{-z^2}) dz \\
 &= \frac{1}{2} \left[\frac{1}{2} z^2 + \frac{1}{2} e^{-z^2} \right]_0^1 = \frac{1}{4}(1 + e^{-1} - 0 - 1) = \frac{1}{4e}
 \end{aligned}$$

$$\begin{aligned}
 7. \iiint_E 2x dV &= \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^y 2x dz dx dy = \int_0^2 \int_0^{\sqrt{4-y^2}} [2xz]_{z=0}^{z=y} dx dy = \int_0^2 \int_0^{\sqrt{4-y^2}} 2xy dx dy \\
 &= \int_0^2 [x^2 y]_{x=0}^{x=\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)y dy = [2y^2 - \frac{1}{4}y^4]_0^2 = 4
 \end{aligned}$$

$$\begin{aligned}
 8. \iiint_E yz \cos(x^5) dV &= \int_0^1 \int_0^x \int_x^{2x} yz \cos(x^5) dz dy dx = \int_0^1 \int_0^x \left[\frac{1}{2} y z^2 \cos(x^5) \right]_{z=x}^{z=2x} dy dx \\
 &= \frac{1}{2} \int_0^1 \int_0^x 3x^2 y \cos(x^5) dy dx = \frac{1}{2} \int_0^1 \left[\frac{3}{2} x^2 y^2 \cos(x^5) \right]_{y=0}^{y=x} dx \\
 &= \frac{3}{4} \int_0^1 x^4 \cos(x^5) dx = \frac{3}{4} \left[\frac{1}{5} \sin(x^5) \right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20} \sin 1
 \end{aligned}$$

9. Here $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, 0 \leq z \leq 1+x+y\}$, so

$$\begin{aligned}
 \iiint_E 6xy dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy dz dy dx = \int_0^1 \int_0^{\sqrt{x}} [6xyz]_{z=0}^{z=1+x+y} dy dx \\
 &= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y) dy dx = \int_0^1 [3xy^2 + 3x^2y^2 + 2xy^3]_{y=0}^{y=\sqrt{x}} dx \\
 &= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2}) dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2} \right]_0^1 = \frac{65}{28}
 \end{aligned}$$

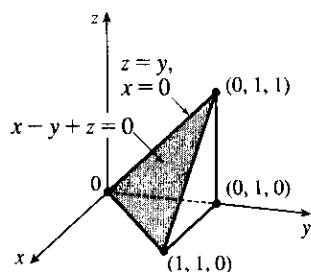
10. Here E is the region in the first octant that lies below the plane $2x + 2y + z = 4$ (and above the region in the xy -plane bounded by the lines $x = 0$, $y = 0$, $x + y = 2$). So

$$\begin{aligned}
 \iiint_E y dV &= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y dz dy dx = \int_0^2 \int_0^{2-x} y(4-2x-2y) dy dx \\
 &= \int_0^2 \int_0^{2-x} (4y - 2xy - 2y^2) dy dx = \int_0^2 \left[2y^2 - xy^2 - \frac{2}{3}y^3 \right]_{y=0}^{y=2-x} dx \\
 &= \int_0^2 \left[2(2-x)^2 - x(2-x)^2 - \frac{2}{3}(2-x)^3 \right] dx \\
 &= \int_0^2 \left[(2-x)(2-x)^2 - \frac{2}{3}(2-x)^3 \right] dx = \frac{1}{3} \int_0^2 (2-x)^3 dx \\
 &= \frac{1}{3} \left[-\frac{1}{4}(2-x)^4 \right]_0^2 = -\frac{1}{12}(0-16) = \frac{4}{3}
 \end{aligned}$$

11. Here E is the region that lies below the plane with x -, y -, and z -intercepts 1, 2, and 3 respectively, that is, below the plane $2z + 6x + 3y = 6$ and above the region in the xy -plane bounded by the lines $x = 0$, $y = 0$ and $6x + 3y = 6$. So

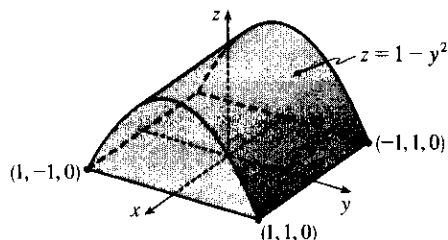
$$\begin{aligned}
 \iiint_E xy dV &= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} xy dz dy dx = \int_0^1 \int_0^{2-2x} (3xy - 3x^2y - \frac{3}{2}xy^2) dy dx \\
 &= \int_0^1 \left[\frac{3}{2}xy^2 - \frac{3}{2}x^2y^2 - \frac{1}{2}xy^3 \right]_{y=0}^{y=2-2x} dx = \int_0^1 (2x - 6x^2 + 6x^3 - 2x^4) dx \\
 &= \left[x^2 - 2x^3 + \frac{3}{2}x^4 - \frac{2}{5}x^5 \right]_0^1 = \frac{1}{10}.
 \end{aligned}$$

12.



$$\begin{aligned}
 \int_0^1 \int_0^y \int_0^{y-z} xz \, dx \, dz \, dy &= \int_0^1 \int_0^y \frac{1}{2} (y-z)^2 z \, dz \, dy \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{2} y^2 z^2 - \frac{2}{3} y z^3 + \frac{1}{4} z^4 \right]_{z=0}^{z=y-y} dy \\
 &= \frac{1}{24} \int_0^1 y^4 \, dy = \frac{1}{24} \left[\frac{1}{5} y^5 \right]_0^1 = \frac{1}{120}
 \end{aligned}$$

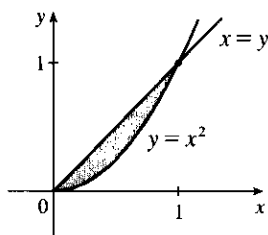
13.



E is the region below the parabolic cylinder $z = 1 - y^2$ and above the square $[-1, 1] \times [-1, 1]$ in the xy -plane.

$$\begin{aligned}
 \iiint_E x^2 e^y \, dV &= \int_{-1}^1 \int_{-1}^1 \int_0^{1-y^2} x^2 e^y \, dz \, dy \, dx \\
 &= \int_{-1}^1 \int_{-1}^1 x^2 e^y (1 - y^2) \, dy \, dx \\
 &= \int_{-1}^1 x^2 \, dx \int_{-1}^1 (e^y - y^2 e^y) \, dy \\
 &= \left[\frac{1}{3} x^3 \right]_{-1}^1 [e^y - (y^2 - 2y + 2)e^y]_{-1}^1 \\
 &\quad \text{[integrate by parts twice]} \\
 &= \frac{1}{3} (2) [e - e - e^{-1} + 5e^{-1}] = \frac{8}{3e}
 \end{aligned}$$

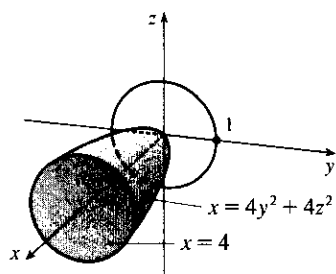
14.



E is the solid above the region shown in the xy -plane and below the plane $z = x$. Thus,

$$\begin{aligned}
 \iiint_E (x + 2y) \, dV &= \int_0^1 \int_{x^2}^x \int_0^x (x + 2y) \, dz \, dy \, dx \\
 &= \int_0^1 \int_{x^2}^x (x^2 + 2yx) \, dy \, dx = \int_0^1 [x^2 y + xy^2]_{y=x^2}^{y=x} dx \\
 &= \int_0^1 (2x^3 - x^4 - x^5) \, dx = \left[\frac{1}{2} x^4 - \frac{1}{5} x^5 - \frac{1}{6} x^6 \right]_0^1 = \frac{2}{15}
 \end{aligned}$$

15.

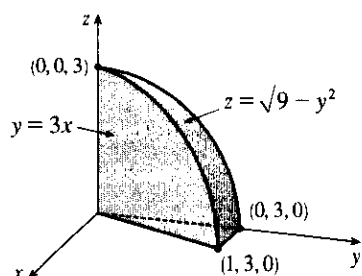


The projection E on the yz -plane is the disk $y^2 + z^2 \leq 1$.

Using polar coordinates $y = r \cos \theta$ and $z = r \sin \theta$, we get

$$\begin{aligned}
 \iiint_E x \, dV &= \iint_D \left[\int_{4y^2+4z^2}^4 x \, dx \right] dA \\
 &= \frac{1}{2} \iint_D [4^2 - (4y^2 + 4z^2)^2] dA = 8 \int_0^{2\pi} \int_0^1 (1 - r^4) r \, dr \, d\theta \\
 &= 8 \int_0^{2\pi} d\theta \int_0^1 (r - r^5) \, dr = 8(2\pi) \left[\frac{1}{2} r^2 - \frac{1}{6} r^6 \right]_0^1 = \frac{16\pi}{3}
 \end{aligned}$$

16.



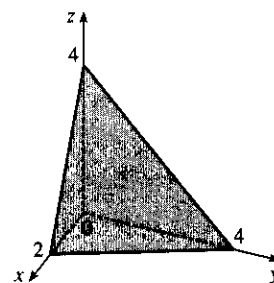
$$\begin{aligned}
 \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx &= \int_0^1 \int_{3x}^3 \frac{1}{2} (9 - y^2) \, dy \, dx \\
 &= \int_0^1 \left[\frac{9}{2} y - \frac{1}{6} y^3 \right]_{y=3x}^{y=3} dx \\
 &= \int_0^1 \left[9 - \frac{27}{2} x + \frac{9}{2} x^3 \right] dx \\
 &= \left[9x - \frac{27}{4} x^2 + \frac{9}{8} x^4 \right]_0^1 = \frac{27}{8}
 \end{aligned}$$

17. The plane $2x + y + z = 4$ intersects the xy -plane when

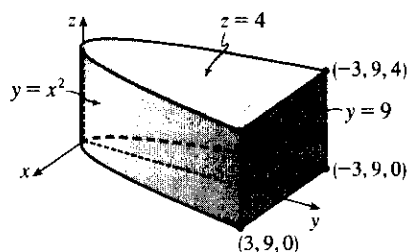
$$2x + y + 0 = 4 \Rightarrow y = 4 - 2x, \text{ so}$$

$$E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 4 - 2x, 0 \leq z \leq 4 - 2x - y\} \text{ and}$$

$$\begin{aligned} V &= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz \, dy \, dx = \int_0^2 \int_0^{4-2x} (4 - 2x - y) \, dy \, dx \\ &= \int_0^2 \left[4y - 2xy - \frac{1}{2}y^2 \right]_{y=0}^{y=4-2x} dx \\ &= \int_0^2 \left[4(4-2x) - 2x(4-2x) - \frac{1}{2}(4-2x)^2 \right] dx \\ &= \int_0^2 (2x^2 - 8x + 8) \, dx = \left[\frac{2}{3}x^3 - 4x^2 + 8x \right]_0^2 = \frac{16}{3} \end{aligned}$$



18.

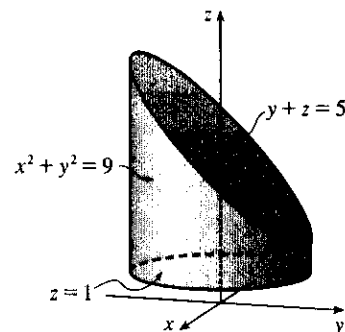


$$\begin{aligned} V &= \iiint_E dV = \int_{-3}^3 \int_{x^2}^9 \int_0^4 dz \, dy \, dx \\ &= 4 \int_{-3}^3 \int_{x^2}^9 dy \, dx = 4 \int_{-3}^3 (9 - x^2) \, dx \\ &= 4 \left[9x - \frac{1}{3}x^3 \right]_{-3}^3 = 4(27 - 9 + 27 - 9) = 144 \end{aligned}$$

$$\begin{aligned} 19. \quad V &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} dz \, dy \, dx = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5 - y - 1) \, dy \, dx = \int_{-3}^3 \left[4y - \frac{1}{2}y^2 \right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx \\ &= \int_{-3}^3 8\sqrt{9-x^2} \, dx = 8 \left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) \right]_{-3}^3 \quad \left[\text{using trigonometric substitution} \right. \\ &= 8 \left[\frac{9}{2}\sin^{-1}(1) - \frac{9}{2}\sin^{-1}(-1) \right] = 36 \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = 36\pi \quad \left. \text{or Formula 30 in the Table of Integrals} \right] \end{aligned}$$

Alternatively, use polar coordinates to evaluate the double integral:

$$\begin{aligned} \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - y) \, dy \, dx &= \int_0^{2\pi} \int_0^3 (4 - r \sin \theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{1}{3}r^3 \sin \theta \right]_{r=0}^{r=3} d\theta \\ &= \int_0^{2\pi} (18 - 9 \sin \theta) \, d\theta \\ &= 18\theta + 9 \cos \theta \Big|_0^{2\pi} = 36\pi \end{aligned}$$



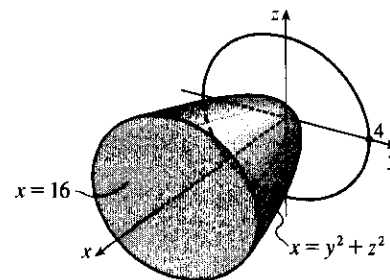
20. The paraboloid $x = y^2 + z^2$ intersects the plane $x = 16$ in the circle $y^2 + z^2 = 16$, $x = 16$.

$$\text{Thus, } E = \{(x, y, z) \mid y^2 + z^2 \leq x \leq 16, y^2 + z^2 \leq 16\}.$$

Let $D = \{(y, z) \mid y^2 + z^2 \leq 16\}$. Then using polar coordinates

$y = r \cos \theta$ and $z = r \sin \theta$, we have

$$\begin{aligned} V &= \iint_D \left(\int_{y^2+z^2}^{16} dx \right) dA = \iint_D (16 - (y^2 + z^2)) \, dA \\ &= \int_0^{2\pi} \int_0^4 (16 - r^2) r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^4 (16r - r^3) \, dr \\ &= \left[\theta \right]_0^{2\pi} \left[8r^2 - \frac{1}{4}r^4 \right]_0^4 = 2\pi(128 - 64) = 128\pi \end{aligned}$$



21. (a) The wedge can be described as the region

$$D = \{(x, y, z) \mid y^2 + z^2 \leq 1, 0 \leq x \leq 1, 0 \leq y \leq x\}$$

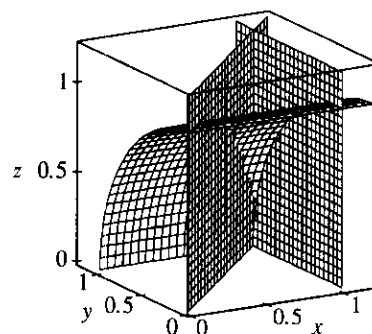
$$= \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq \sqrt{1 - y^2}\}$$

So the integral expressing the volume of the wedge is

$$\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz dy dx.$$

- (b) A CAS gives
- $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz dy dx = \frac{\pi}{4} - \frac{1}{3}$
- .

(Or use Formulas 30 and 87 from the Table of Integrals.)



22. (a) Note that
- $\Delta V = (\frac{1}{2})^3 = \frac{1}{8}$
- , so the Midpoint Rule gives

$$\begin{aligned} \iiint_B f(x, y, z) dV &\approx \frac{1}{8} [f(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) + f(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}) + f(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}) + f(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}) \\ &\quad + f(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}) + f(\frac{3}{4}, \frac{1}{4}, \frac{3}{4}) + f(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}) + f(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})] \\ &= \frac{1}{8} [e^{-3(1/4)^2} + 3e^{-2(1/4)^2 - (3/4)^2} + 3e^{-(1/4)^2 - 2(3/4)^2} + e^{-3(3/4)^2}] \approx 0.42968 \end{aligned}$$

- (b) A CAS estimates the integral to be
- $\iiint_B e^{-x^2-y^2-z^2} dV \approx 0.42$
- . The estimate in part (a) is correct to one decimal place, and is larger than the actual value of the integral.

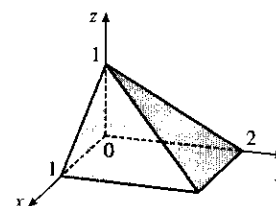
23. Here
- $f(x, y, z) = \frac{1}{\ln(1+x+y+z)}$
- and
- $\Delta V = 2 \cdot 4 \cdot 2 = 16$
- , so the Midpoint Rule gives

$$\begin{aligned} \iiint_B f(x, y, z) dV &\approx \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(\bar{x}_i, \bar{y}_j, \bar{z}_k) \Delta V \\ &= 16 [f(1, 2, 1) + f(1, 2, 3) + f(1, 6, 1) + f(1, 6, 3) \\ &\quad + f(3, 2, 1) + f(3, 2, 3) + f(3, 6, 1) + f(3, 6, 3)] \\ &= 16 \left[\frac{1}{\ln 5} + \frac{1}{\ln 7} + \frac{1}{\ln 9} + \frac{1}{\ln 11} + \frac{1}{\ln 7} + \frac{1}{\ln 9} + \frac{1}{\ln 11} + \frac{1}{\ln 13} \right] \approx 60.533 \end{aligned}$$

24. Here
- $f(x, y, z) = \sin(xy^2z^3)$
- and
- $\Delta V = 2 \cdot 1 \cdot \frac{1}{2} = 1$
- , so the Midpoint Rule gives

$$\begin{aligned} \iiint_B f(x, y, z) dV &\approx \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(\bar{x}_i, \bar{y}_j, \bar{z}_k) \Delta V \\ &= 1 [f(1, \frac{1}{2}, \frac{1}{4}) + f(1, \frac{1}{2}, \frac{3}{4}) + f(1, \frac{3}{2}, \frac{1}{4}) + f(1, \frac{3}{2}, \frac{3}{4}) \\ &\quad + f(3, \frac{1}{2}, \frac{1}{4}) + f(3, \frac{1}{2}, \frac{3}{4}) + f(3, \frac{3}{2}, \frac{1}{4}) + f(3, \frac{3}{2}, \frac{3}{4})] \\ &= \sin \frac{1}{256} + \sin \frac{27}{256} + \sin \frac{9}{256} + \sin \frac{243}{256} + \sin \frac{3}{256} + \sin \frac{81}{256} + \sin \frac{27}{256} + \sin \frac{729}{256} \approx 1.675 \end{aligned}$$

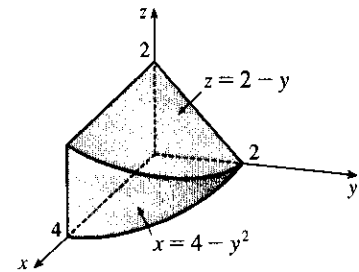
- 25.
- $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq z \leq 1 - x, 0 \leq y \leq 2 - 2z\}$
- ,
-
- the solid bounded by the three coordinate planes and the planes
-
- $z = 1 - x, y = 2 - 2z$
- .



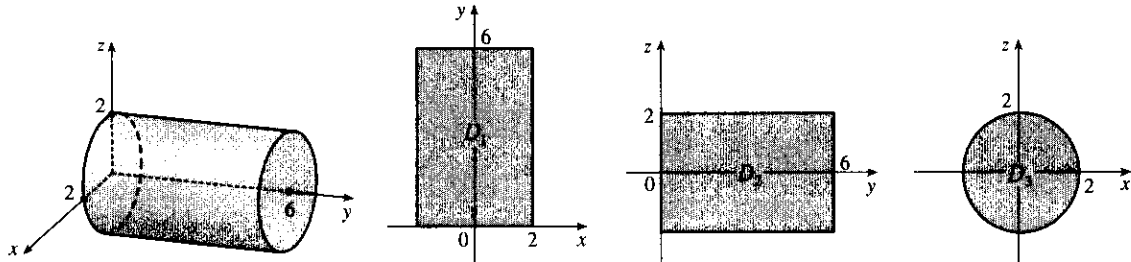
26. $E = \{(x, y, z) \mid 0 \leq y \leq 2, 0 \leq z \leq 2 - y, 0 \leq x \leq 4 - y^2\}$,

the solid bounded by the three coordinate planes, the plane $z = 2 - y$,

and the cylindrical surface $x = 4 - y^2$.



27.



If D_1 , D_2 , D_3 are the projections of E on the xy -, yz -, and xz -planes, then

$$D_1 = \{(x, y) \mid -2 \leq x \leq 2, 0 \leq y \leq 6\}$$

$$D_2 = \{(y, z) \mid -2 \leq z \leq 2, 0 \leq y \leq 6\}$$

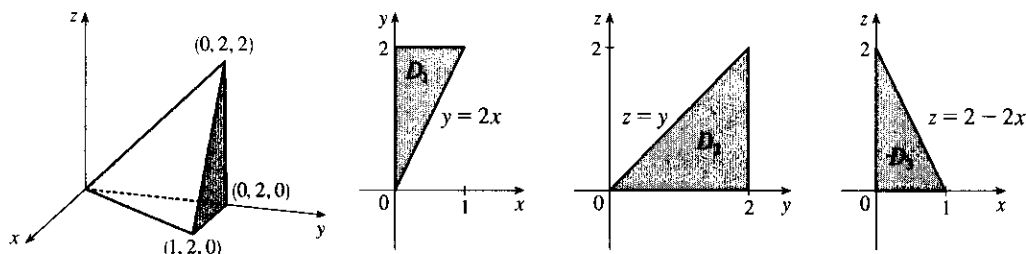
$$D_3 = \{(x, z) \mid x^2 + z^2 \leq 4\}$$

Therefore

$$\begin{aligned} E &= \{(x, y, z) \mid -\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}, -2 \leq x \leq 2, 0 \leq y \leq 6\} \\ &= \{(x, y, z) \mid -\sqrt{4-z^2} \leq x \leq \sqrt{4-z^2}, -2 \leq z \leq 2, 0 \leq y \leq 6\} \end{aligned}$$

$$\begin{aligned} \iiint_E f(x, y, z) dV &= \int_{-2}^2 \int_0^6 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z) dz dy dx = \int_0^6 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z) dz dx dy \\ &= \int_0^6 \int_{-2}^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z) dx dz dy = \int_{-2}^2 \int_0^6 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z) dx dy dz \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^6 f(x, y, z) dy dz dx = \int_{-2}^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_0^6 f(x, y, z) dy dx dz \end{aligned}$$

28.



If D_1 , D_2 , and D_3 are the projections of E on the xy -, yz -, and xz -planes, then

$$D_1 = \{(x, y) \mid 0 \leq x \leq 1, 2x \leq y \leq 2\} = \{(x, y) \mid 0 \leq y \leq 2, 0 \leq x \leq y/2\},$$

$$D_2 = \{(y, z) \mid 0 \leq y \leq 2, 0 \leq z \leq y\} = \{(y, z) \mid 0 \leq z \leq 2, z \leq y \leq 2\}, \text{ and}$$

$$D_3 = \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq 2 - 2x\} = \{(x, z) \mid 0 \leq z \leq 2, 0 \leq x \leq (2 - z)/2\}$$

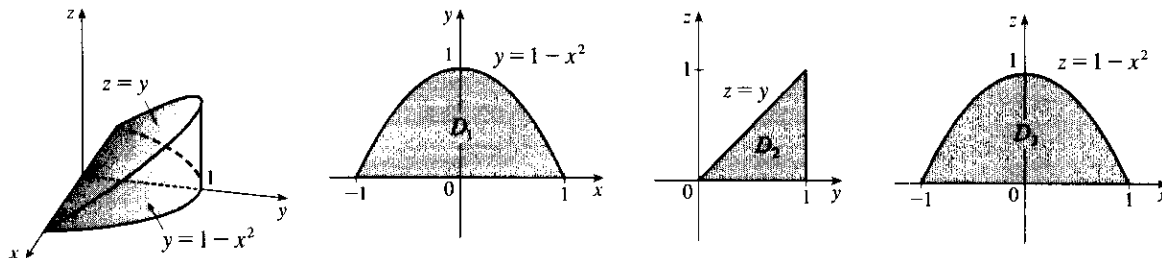
Therefore

$$\begin{aligned} E &= \{(x, y, z) \mid 0 \leq x \leq 1, 2x \leq y \leq 2, 0 \leq z \leq y - 2x\} \\ &= \{(x, y, z) \mid 0 \leq y \leq 2, 0 \leq x \leq y/2, 0 \leq z \leq y - 2x\} \\ &= \{(x, y, z) \mid 0 \leq y \leq 2, 0 \leq z \leq y, 0 \leq x \leq (y - z)/2\} \\ &= \{(x, y, z) \mid 0 \leq z \leq 2, z \leq y \leq 2, 0 \leq x \leq (y - z)/2\} \\ &= \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq z \leq 2 - 2x, z + 2x \leq y \leq 2\} \\ &= \{(x, y, z) \mid 0 \leq z \leq 2, 0 \leq x \leq (2 - z)/2, z + 2x \leq y \leq 2\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x, y, z) dV &= \int_0^1 \int_{2x}^2 \int_0^{y-2x} f(x, y, z) dz dy dx \\ &= \int_0^2 \int_0^{y/2} \int_0^{y-2x} f(x, y, z) dz dx dy \\ &= \int_0^2 \int_0^y \int_0^{(y-z)/2} f(x, y, z) dx dz dy \\ &= \int_0^2 \int_z^2 \int_0^{(y-z)/2} f(x, y, z) dx dy dz \\ &= \int_0^1 \int_0^{2-2x} \int_{z+2x}^2 f(x, y, z) dy dz dx \\ &= \int_0^2 \int_0^{(2-z)/2} \int_{z+2x}^2 f(x, y, z) dy dx dz \end{aligned}$$

29.



If D_1 , D_2 , and D_3 are the projections of E on the xy -, yz -, and xz -planes, then

$$D_1 = \{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2\} = \{(x, y) \mid 0 \leq y \leq 1, -\sqrt{1-y} \leq x \leq \sqrt{1-y}\},$$

$$D_2 = \{(y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y\} = \{(y, z) \mid 0 \leq z \leq 1, z \leq y \leq 1\}, \text{ and}$$

$$D_3 = \{(x, z) \mid -1 \leq x \leq 1, 0 \leq z \leq 1 - x^2\} = \{(x, z) \mid 0 \leq z \leq 1, -\sqrt{1-z} \leq x \leq \sqrt{1-z}\}$$

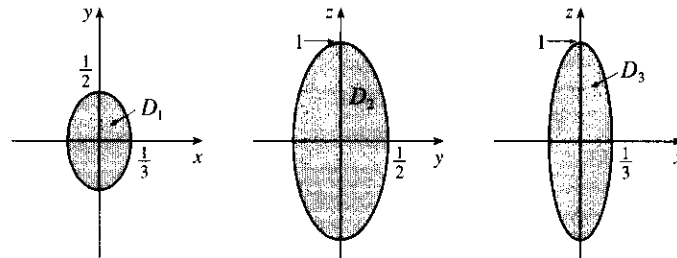
Therefore

$$\begin{aligned} E &= \{(x, y, z) \mid -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2, 0 \leq z \leq y\} \\ &= \{(x, y, z) \mid 0 \leq y \leq 1, -\sqrt{1-y} \leq x \leq \sqrt{1-y}, 0 \leq z \leq y\} \\ &= \{(x, y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y, -\sqrt{1-y} \leq x \leq \sqrt{1-y}\} \\ &= \{(x, y, z) \mid 0 \leq z \leq 1, z \leq y \leq 1, -\sqrt{1-y} \leq x \leq \sqrt{1-y}\} \\ &= \{(x, y, z) \mid -1 \leq x \leq 1, 0 \leq z \leq 1 - x^2, z \leq y \leq 1 - x^2\} \\ &= \{(x, y, z) \mid 0 \leq z \leq 1, -\sqrt{1-z} \leq x \leq \sqrt{1-z}, z \leq y \leq 1 - x^2\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x, y, z) dV &= \int_{-1}^1 \int_0^{1-x^2} \int_0^y f(x, y, z) dz dy dx = \int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \int_0^y f(x, y, z) dz dx dy \\ &= \int_0^1 \int_0^y \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y, z) dx dz dy = \int_0^1 \int_z^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y, z) dx dy dz \\ &= \int_{-1}^1 \int_0^{1-x^2} \int_z^{1-x^2} f(x, y, z) dy dz dx = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_z^{1-x^2} f(x, y, z) dy dx dz \end{aligned}$$

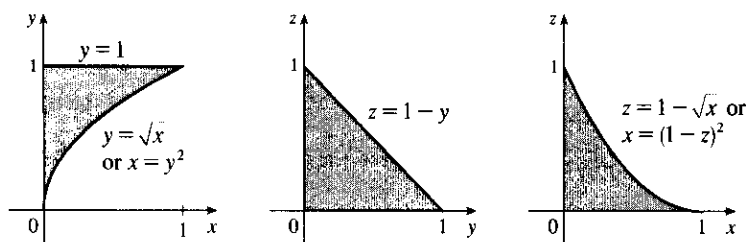
30.



If D_1 , D_2 and D_3 are the projections of E on the xy -, yz -, and xz -planes, then $D_1 = \{(x, y) \mid 9x^2 + 4y^2 \leq 1\}$, $D_2 = \{(y, z) \mid 4y^2 + z^2 \leq 1\}$, $D_3 = \{(x, z) \mid 9x^2 + z^2 \leq 1\}$. Therefore

$$\begin{aligned} \iiint_E f(x, y, z) dV &= \int_{-1/3}^{1/3} \int_{-\sqrt{1-9x^2}/2}^{\sqrt{1-9x^2}/2} \int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x, y, z) dz dy dx \\ &= \int_{-1/2}^{1/2} \int_{-\sqrt{1-4y^2}/3}^{\sqrt{1-4y^2}/3} \int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x, y, z) dz dx dy \\ &= \int_{-1/2}^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x, y, z) dx dz dy \\ &= \int_{-1}^1 \int_{-\sqrt{1-z^2}/2}^{\sqrt{1-z^2}/2} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x, y, z) dx dy dz \\ &= \int_{-1/3}^{1/3} \int_{-\sqrt{1-9x^2}}^{\sqrt{1-9x^2}} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x, y, z) dy dz dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-z^2}/3}^{\sqrt{1-z^2}/3} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x, y, z) dy dx dz \end{aligned}$$

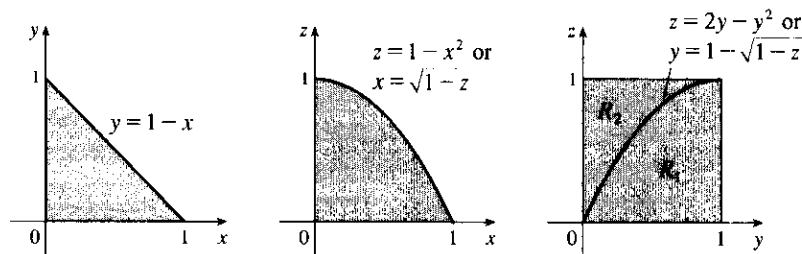
31.



The diagrams show the projections of E on the xy -, yz -, and xz -planes. Therefore

$$\begin{aligned}
 \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx &= \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z) \, dz \, dx \, dy \\
 &= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z) \, dx \, dy \, dz \\
 &= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z) \, dx \, dz \, dy \\
 &= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z) \, dy \, dz \, dx \\
 &= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z) \, dy \, dx \, dz
 \end{aligned}$$

32.



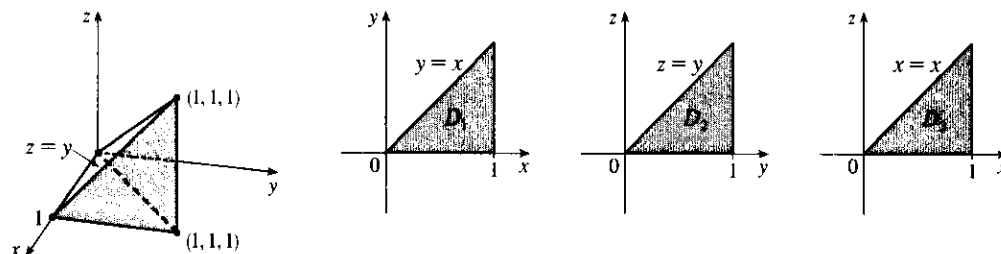
The projections of E onto the xy - and xz -planes are as in the first two diagrams and so

$$\begin{aligned}
 \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx &= \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z) \, dy \, dx \, dz \\
 &= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z) \, dz \, dx \, dy = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z) \, dz \, dy \, dx
 \end{aligned}$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz -plane is $z = 1 - (1 - y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz -plane into two regions as in the third diagram. For (y, z) in R_1 , $0 \leq x \leq 1 - y$ and for (y, z) in R_2 , $0 \leq x \leq \sqrt{1 - z}$, and so the given integral is also equal to

$$\begin{aligned}
 \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x, y, z) \, dx \, dy \, dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x, y, z) \, dx \, dy \, dz \\
 = \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x, y, z) \, dx \, dz \, dy + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x, y, z) \, dx \, dz \, dy.
 \end{aligned}$$

33.



$$\int_0^1 \int_y^1 \int_0^y f(x, y, z) \, dz \, dx \, dy = \iiint_E f(x, y, z) \, dV \text{ where } E = \{(x, y, z) \mid 0 \leq z \leq y, y \leq x \leq 1, 0 \leq y \leq 1\}.$$

If D_1 , D_2 , and D_3 are the projections of E on the xy -, yz - and xz -planes then

$$D_1 = \{(x, y) \mid 0 \leq y \leq 1, y \leq x \leq 1\} = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\},$$

$$D_2 = \{(y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y\} = \{(y, z) \mid 0 \leq z \leq 1, z \leq y \leq 1\}, \text{ and}$$

$$D_3 = \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq x\} = \{(x, z) \mid 0 \leq z \leq 1, z \leq x \leq 1\}.$$

Thus we also have

$$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq y\} = \{(x, y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y, y \leq x \leq 1\}$$

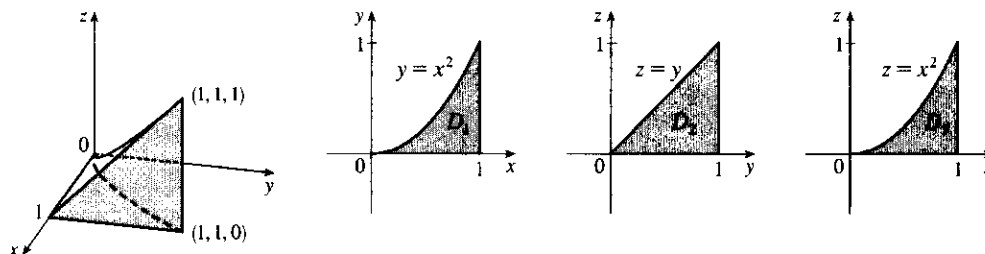
$$= \{(x, y, z) \mid 0 \leq z \leq 1, z \leq y \leq 1, y \leq x \leq 1\} = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq z \leq x, z \leq y \leq x\}$$

$$= \{(x, y, z) \mid 0 \leq z \leq 1, z \leq x \leq 1, z \leq y \leq x\}.$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^y f(x, y, z) dz dx dy &= \int_0^1 \int_0^x \int_0^y f(x, y, z) dz dy dx = \int_0^1 \int_0^y \int_y^1 f(x, y, z) dx dz dy \\ &= \int_0^1 \int_z^1 \int_y^1 f(x, y, z) dx dy dz = \int_0^1 \int_0^x \int_z^x f(x, y, z) dy dz dx \\ &= \int_0^1 \int_z^1 \int_z^x f(x, y, z) dy dx dz \end{aligned}$$

34.



$$\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) dz dy dx = \iiint_E f(x, y, z) dV \text{ where}$$

$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq y\}$. If D_1 , D_2 , D_3 are the projections of E on the

xy -, yz -, and xz -planes, then $D_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\} = \{(x, y) \mid 0 \leq y \leq 1, \sqrt{y} \leq x \leq 1\}$,

$D_2 = \{(y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y\} = \{(y, z) \mid 0 \leq z \leq 1, z \leq y \leq 1\}$,

$D_3 = \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq x^2\} = \{(x, z) \mid 0 \leq z \leq 1, \sqrt{z} \leq x \leq 1\}$. Thus we also have

$$\begin{aligned} E &= \{(x, y, z) \mid 0 \leq y \leq 1, \sqrt{y} \leq x \leq 1, 0 \leq z \leq y\} \\ &= \{(x, y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y, \sqrt{y} \leq x \leq 1\} \\ &= \{(x, y, z) \mid 0 \leq z \leq 1, z \leq y \leq 1, \sqrt{y} \leq x \leq 1\} \\ &= \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq z \leq x^2, z \leq y \leq x^2\} \\ &= \{(x, y, z) \mid 0 \leq z \leq 1, \sqrt{z} \leq x \leq 1, z \leq y \leq x^2\} \end{aligned}$$

Then

$$\begin{aligned}
 \int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) \, dz \, dy \, dx &= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x, y, z) \, dz \, dx \, dy \\
 &= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x, y, z) \, dx \, dz \, dy \\
 &= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x, y, z) \, dx \, dy \, dz \\
 &= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x, y, z) \, dy \, dz \, dx \\
 &= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x, y, z) \, dy \, dx \, dz
 \end{aligned}$$

$$\begin{aligned}
 35. \, m &= \iiint_E \rho(x, y, z) \, dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2 \, dz \, dy \, dx \\
 &= \int_0^1 \int_0^{\sqrt{x}} 2(1+x+y) \, dy \, dx = \int_0^1 [2y + 2xy + y^2]_{y=0}^{y=\sqrt{x}} \, dx \\
 &= \int_0^1 \left(2\sqrt{x} + 2x^{3/2} + x \right) \, dx = \left[\frac{4}{3}x^{3/2} + \frac{4}{5}x^{5/2} + \frac{1}{2}x^2 \right]_0^1 = \frac{79}{30}
 \end{aligned}$$

$$\begin{aligned}
 M_{yz} &= \iiint_E x\rho(x, y, z) \, dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x \, dz \, dy \, dx \\
 &= \int_0^1 \int_0^{\sqrt{x}} 2x(1+x+y) \, dy \, dx = \int_0^1 [2xy + 2x^2y + xy^2]_{y=0}^{y=\sqrt{x}} \, dx \\
 &= \int_0^1 (2x^{3/2} + 2x^{5/2} + x^2) \, dx = \left[\frac{4}{5}x^{5/2} + \frac{4}{7}x^{7/2} + \frac{1}{3}x^3 \right]_0^1 = \frac{179}{105}
 \end{aligned}$$

$$\begin{aligned}
 M_{xz} &= \iiint_E y\rho(x, y, z) \, dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2y \, dz \, dy \, dx \\
 &= \int_0^1 \int_0^{\sqrt{x}} 2y(1+x+y) \, dy \, dx = \int_0^1 [y^2 + xy^2 + \frac{2}{3}y^3]_{y=0}^{y=\sqrt{x}} \, dx \\
 &= \int_0^1 \left(x + x^2 + \frac{2}{3}x^{3/2} \right) \, dx = \left[\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{4}{15}x^{5/2} \right]_0^1 = \frac{11}{10}
 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= \iiint_E z\rho(x, y, z) \, dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2z \, dz \, dy \, dx \\
 &= \int_0^1 \int_0^{\sqrt{x}} [z^2]_{z=0}^{z=1+x+y} \, dy \, dx = \int_0^1 \int_0^{\sqrt{x}} (1+x+y)^2 \, dy \, dx \\
 &= \int_0^1 \int_0^{\sqrt{x}} (1+2x+2y+2xy+x^2+y^2) \, dy \, dx \\
 &= \int_0^1 [y + 2xy + y^2 + xy^2 + x^2y + \frac{1}{3}y^3]_{y=0}^{y=\sqrt{x}} \, dx = \int_0^1 \left(\sqrt{x} + \frac{7}{3}x^{3/2} + x + x^2 + x^{5/2} \right) \, dx \\
 &= \left[\frac{2}{3}x^{3/2} + \frac{14}{15}x^{5/2} + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{2}{7}x^{7/2} \right]_0^1 = \frac{571}{210}
 \end{aligned}$$

Thus the mass is $\frac{79}{30}$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\frac{358}{553}, \frac{33}{79}, \frac{571}{553} \right)$.

$$\begin{aligned}
 36. \, m &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4 \, dx \, dz \, dy = 4 \int_{-1}^1 \int_0^{1-y^2} (1-z) \, dz \, dy \\
 &= 4 \int_{-1}^1 \left[z - \frac{1}{2}z^2 \right]_{z=0}^{z=1-y^2} \, dy = 2 \int_{-1}^1 (1-y^4) \, dy = \frac{16}{5},
 \end{aligned}$$

$$\begin{aligned}
 M_{yz} &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4x \, dx \, dz \, dy = 2 \int_{-1}^1 \int_0^{1-y^2} (1-z)^2 \, dz \, dy = 2 \int_{-1}^1 \left[-\frac{1}{3}(1-z)^3 \right]_{z=0}^{z=1-y^2} \, dy \\
 &= \frac{2}{3} \int_{-1}^1 (1-y^6) \, dy = \left(\frac{4}{3} \right) \left(\frac{6}{7} \right) = \frac{24}{21}
 \end{aligned}$$

$$\begin{aligned}
 M_{xz} &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4y \, dx \, dz \, dy = \int_{-1}^1 \int_0^{1-y^2} 4y(1-z) \, dz \, dy \\
 &= \int_{-1}^1 [4y(1-y^2) - 2y(1-y^2)^2] \, dy = \int_{-1}^1 (2y - 2y^5) \, dy = 0 \quad [\text{the integrand is odd}]
 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4z \, dx \, dz \, dy = \int_{-1}^1 \int_0^{1-y^2} (4z - 4z^2) \, dz \, dy = 2 \int_{-1}^1 [(1-y^2)^2 - \frac{2}{3}(1-y^2)^3] \, dy \\
 &= 2 \int_{-1}^1 [\frac{1}{3} - y^4 + \frac{2}{3}y^6] \, dy = [\frac{4}{3}y - \frac{4}{5}y^5 + \frac{8}{21}y^7]_0^1 = \frac{96}{105} = \frac{32}{35}
 \end{aligned}$$

$$\text{Thus, } (\bar{x}, \bar{y}, \bar{z}) = (\frac{5}{14}, 0, \frac{2}{7})$$

$$\begin{aligned}
 37. \quad m &= \int_0^a \int_0^a \int_0^a (x^2 + y^2 + z^2) \, dx \, dy \, dz = \int_0^a \int_0^a [\frac{1}{3}x^3 + xy^2 + xz^2]_{x=0}^{x=a} \, dy \, dz \\
 &= \int_0^a \int_0^a (\frac{1}{3}a^3 + ay^2 + az^2) \, dy \, dz = \int_0^a [\frac{1}{3}a^3y + \frac{1}{3}ay^3 + ayz^2]_{y=0}^{y=a} \, dz \\
 &= \int_0^a (\frac{2}{3}a^4 + a^2z^2) \, dz = [\frac{2}{3}a^4z + \frac{1}{3}a^2z^3]_0^a = \frac{2}{3}a^5 + \frac{1}{3}a^5 = a^5
 \end{aligned}$$

$$\begin{aligned}
 M_{yz} &= \int_0^a \int_0^a \int_0^a [x^3 + x(y^2 + z^2)] \, dx \, dy \, dz = \int_0^a \int_0^a [\frac{1}{4}a^4 + \frac{1}{2}a^2(y^2 + z^2)] \, dy \, dz \\
 &= \int_0^a (\frac{1}{4}a^5 + \frac{1}{6}a^5 + \frac{1}{2}a^3z^2) \, dz = \frac{1}{4}a^6 + \frac{1}{3}a^6 = \frac{7}{12}a^6 \\
 &= M_{xz} = M_{xy} \text{ by symmetry of } E \text{ and } \rho(x, y, z)
 \end{aligned}$$

$$\text{Hence } (\bar{x}, \bar{y}, \bar{z}) = (\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a).$$

$$\begin{aligned}
 38. \quad m &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} [(1-x)y - y^2] \, dy \, dx \\
 &= \int_0^1 [\frac{1}{2}(1-x)^3 - \frac{1}{3}(1-x)^3] \, dx = \frac{1}{6} \int_0^1 (1-x)^3 \, dx = \frac{1}{24}
 \end{aligned}$$

$$\begin{aligned}
 M_{yz} &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} [(x-x^2)y - xy^2] \, dy \, dx \\
 &= \int_0^1 [\frac{1}{2}x(1-x)^3 - \frac{1}{3}x(1-x)^3] \, dx = \frac{1}{6} \int_0^1 (x - 3x^2 + 3x^3 - x^4) \, dx \\
 &= \frac{1}{6} (\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}) = \frac{1}{120}
 \end{aligned}$$

$$\begin{aligned}
 M_{xz} &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2 \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} [(1-x)y^2 - y^3] \, dy \, dx \\
 &= \int_0^1 [\frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4] \, dx = \frac{1}{12} [-\frac{1}{5}(1-x)^5]_0^1 = \frac{1}{60}
 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} [\frac{1}{2}y(1-x-y)^2] \, dy \, dx \\
 &= \frac{1}{2} \int_0^1 \int_0^{1-x} [(1-x)^2y - 2(1-x)y^2 + y^3] \, dy \, dx \\
 &= \frac{1}{2} \int_0^1 [\frac{1}{2}(1-x)^4 - \frac{2}{3}(1-x)^4 + \frac{1}{4}(1-x)^4] \, dx \\
 &= \frac{1}{24} \int_0^1 (1-x)^4 \, dx = -\frac{1}{24} [\frac{1}{5}(1-x)^5]_0^1 = \frac{1}{120}
 \end{aligned}$$

$$\text{Hence } (\bar{x}, \bar{y}, \bar{z}) = (\frac{1}{5}, \frac{2}{5}, \frac{1}{5}).$$

39. (a) $m = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} \sqrt{x^2 + y^2} \, dz \, dy \, dx$

(b) $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$ where

$$M_{yz} = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} x \sqrt{x^2 + y^2} \, dz \, dy \, dx, \quad M_{xz} = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} y \sqrt{x^2 + y^2} \, dz \, dy \, dx, \text{ and}$$

$$M_{xy} = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} z \sqrt{x^2 + y^2} \, dz \, dy \, dx.$$

(c) $I_z = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} (x^2 + y^2) \sqrt{x^2 + y^2} \, dz \, dy \, dx = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} (x^2 + y^2)^{3/2} \, dz \, dy \, dx$

40. (a) $m = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy$

(b) $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = m^{-1} \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} x \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy,$

$$\bar{y} = m^{-1} \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} y \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy,$$

$$\bar{z} = m^{-1} \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} z \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy$$

(c) $I_z = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2 + y^2)(1 + x + y + z) \, dz \, dx \, dy$

41. (a) $m = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (1 + x + y + z) \, dz \, dy \, dx = \frac{3\pi}{32} + \frac{11}{24}$

(b) $(\bar{x}, \bar{y}, \bar{z}) = \left(m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y x(1 + x + y + z) \, dz \, dy \, dx, \right.$

$$m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y y(1 + x + y + z) \, dz \, dy \, dx, \quad \left. m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y z(1 + x + y + z) \, dz \, dy \, dx \right)$$

$$= \left(\frac{28}{9\pi + 44}, \frac{30\pi + 128}{45\pi + 220}, \frac{45\pi + 208}{135\pi + 660} \right)$$

(c) $I_z = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (x^2 + y^2)(1 + x + y + z) \, dz \, dy \, dx = \frac{68 + 15\pi}{240}$

42. (a) $m = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2) \, dz \, dy \, dx = \frac{56}{5} = 11.2$

(b) $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} x(x^2 + y^2) \, dz \, dy \, dx \approx 0.375,$

$$\bar{y} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} y(x^2 + y^2) \, dz \, dy \, dx = \frac{45\pi}{64} \approx 2.209,$$

$$\bar{z} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z(x^2 + y^2) \, dz \, dy \, dx = \frac{15}{16} = 0.9375.$$

(c) $I_z = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)^2 \, dz \, dy \, dx = \frac{10,464}{175} \approx 59.79$

$$43. I_x = \int_0^L \int_0^L \int_0^L k(y^2 + z^2) dz dy dx = k \int_0^L \int_0^L (Ly^2 + \frac{1}{3}L^3) dy dx = k \int_0^L \frac{2}{3}L^4 dx = \frac{2}{3}kL^5.$$

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

44. Let k be the density. Then

$$\begin{aligned} I_x &= \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} k(y^2 + z^2) dx dy dz = ka \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} (y^2 + z^2) dy dz \\ &= ak \int_{-c/2}^{c/2} \left[\frac{1}{3}y^3 + z^2y \right]_{y=-b/2}^{y=b/2} dz = ak \int_{-c/2}^{c/2} \left(\frac{1}{12}b^3 + bz^2 \right) dz = ak \left[\frac{1}{12}b^3z + \frac{1}{3}bz^3 \right]_{-c/2}^{c/2} \\ &= ak \left(\frac{1}{12}b^3c + \frac{1}{12}bc^3 \right) = \frac{1}{12}kabc(b^2 + c^2) \end{aligned}$$

By symmetry, $I_y = \frac{1}{12}kabc(a^2 + c^2)$ and $I_z = \frac{1}{12}kabc(a^2 + b^2)$.

45. (a) $f(x, y, z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x, y, z) dV = 1$. Here we have

$$\begin{aligned} \iiint_{\mathbb{R}^3} f(x, y, z) dV &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) dz dy dx = \int_0^2 \int_0^2 \int_0^2 Cxyz dz dy dx \\ &= C \int_0^2 x dx \int_0^2 y dy \int_0^2 z dz = C \left[\frac{x^2}{2} \right]_0^2 \left[\frac{y^2}{2} \right]_0^2 \left[\frac{z^2}{2} \right]_0^2 \\ &= 8C \end{aligned}$$

Then we must have $8C = 1 \Rightarrow C = \frac{1}{8}$.

$$\begin{aligned} (b) P(X \leq 1, Y \leq 1, Z \leq 1) &= \int_{-\infty}^1 \int_{-\infty}^1 \int_{-\infty}^1 f(x, y, z) dz dy dx \\ &= \int_0^1 \int_0^1 \int_0^1 \frac{1}{8}xyz dz dy dx = \frac{1}{8} \int_0^1 x dx \int_0^1 y dy \int_0^1 z dz \\ &= \frac{1}{8} \left[\frac{x^2}{2} \right]_0^1 \left[\frac{y^2}{2} \right]_0^1 \left[\frac{z^2}{2} \right]_0^1 = \frac{1}{8} \left(\frac{1}{2} \right)^3 = \frac{1}{64} \end{aligned}$$

(c) $P(X + Y + Z \leq 1) = P((X, Y, Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 1$. The plane $x + y + z = 1$ meets the xy -plane in the line $x + y = 1$, so we have

$$\begin{aligned} P(X + Y + Z \leq 1) &= \iiint_E f(x, y, z) dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{8}xyz dz dy dx \\ &= \frac{1}{8} \int_0^1 \int_0^{1-x} xy \left[\frac{1}{2}z^2 \right]_{z=0}^{z=1-x-y} dy dx \\ &= \frac{1}{16} \int_0^1 \int_0^{1-x} xy(1-x-y)^2 dy dx \\ &= \frac{1}{16} \int_0^1 \int_0^{1-x} [(x^3 - 2x^2 + x)y + (2x^2 - 2x)y^2 + xy^3] dy dx \\ &= \frac{1}{16} \int_0^1 \left[(x^3 - 2x^2 + x)\frac{1}{2}y^2 + (2x^2 - 2x)\frac{1}{3}y^3 + x\left(\frac{1}{4}y^4\right) \right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{192} \int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5) dx = \frac{1}{192} \left(\frac{1}{30} \right) = \frac{1}{5760} \end{aligned}$$

46. (a) $f(x, y, z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x, y, z) dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x, y, z) dV &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) dz dy dx \\&= \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} C e^{-(0.5x+0.2y+0.1z)} dz dy dx \\&= C \int_0^{\infty} e^{-0.5x} dx \int_0^{\infty} e^{-0.2y} dy \int_0^{\infty} e^{-0.1z} dz \\&= C \lim_{t \rightarrow \infty} \int_0^t e^{-0.5x} dx \lim_{t \rightarrow \infty} \int_0^t e^{-0.2y} dy \lim_{t \rightarrow \infty} \int_0^t e^{-0.1z} dz \\&= C \lim_{t \rightarrow \infty} [-2e^{-0.5x}]_0^t \lim_{t \rightarrow \infty} [-5e^{-0.2y}]_0^t \lim_{t \rightarrow \infty} [-10e^{-0.1z}]_0^t \\&= C \lim_{t \rightarrow \infty} [-2(e^{-0.5t} - 1)] \lim_{t \rightarrow \infty} [-5(e^{-0.2t} - 1)] \lim_{t \rightarrow \infty} [-10(e^{-0.1t} - 1)] \\&= C \cdot (-2)(0 - 1) \cdot (-5)(0 - 1) \cdot (-10)(0 - 1) = 100C\end{aligned}$$

So we must have $100C = 1 \Rightarrow C = \frac{1}{100}$.

(b) We have no restriction on Z , so

$$\begin{aligned}P(X \leq 1, Y \leq 1) &= \int_{-\infty}^1 \int_{-\infty}^1 \int_{-\infty}^{\infty} f(x, y, z) dz dy dx \\&= \int_0^1 \int_0^1 \int_0^{\infty} \frac{1}{100} e^{-(0.5x+0.2y+0.1z)} dz dy dx \\&= \frac{1}{100} \int_0^1 e^{-0.5x} dx \int_0^1 e^{-0.2y} dy \int_0^{\infty} e^{-0.1z} dz \\&= \frac{1}{100} [-2e^{-0.5x}]_0^1 [-5e^{-0.2y}]_0^1 \lim_{t \rightarrow \infty} [-10e^{-0.1z}]_0^t \quad [\text{by part (a)}] \\&= \frac{1}{100} (2 - 2e^{-0.5})(5 - 5e^{-0.2})(10) = (1 - e^{-0.5})(1 - e^{-0.2}) \approx 0.07132\end{aligned}$$

$$\begin{aligned}\text{(c) } P(X \leq 1, Y \leq 1, Z \leq 1) &= \int_{-\infty}^1 \int_{-\infty}^1 \int_{-\infty}^1 f(x, y, z) dz dy dx \\&= \int_0^1 \int_0^1 \int_0^1 \frac{1}{100} e^{-(0.5x+0.2y+0.1z)} dz dy dx \\&= \frac{1}{100} \int_0^1 e^{-0.5x} dx \int_0^1 e^{-0.2y} dy \int_0^1 e^{-0.1z} dz \\&= \frac{1}{100} [-2e^{-0.5x}]_0^1 [-5e^{-0.2y}]_0^1 [-10e^{-0.1z}]_0^1 \\&= (1 - e^{-0.5})(1 - e^{-0.2})(1 - e^{-0.1}) \approx 0.006787\end{aligned}$$

47. $V(E) = L^3$,

$$\begin{aligned}f_{\text{ave}} &= \frac{1}{L^3} \int_0^L \int_0^L \int_0^L xyz dx dy dz = \frac{1}{L^3} \int_0^L x dx \int_0^L y dy \int_0^L z dz \\&= \frac{1}{L^3} \left[\frac{x^2}{2} \right]_0^L \left[\frac{y^2}{2} \right]_0^L \left[\frac{z^2}{2} \right]_0^L = \frac{1}{L^3} \frac{L^2}{2} \frac{L^2}{2} \frac{L^2}{2} = \frac{L^3}{8}\end{aligned}$$

$$\begin{aligned}48. V(E) &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} dz dy dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2) dy dx \\&= \int_0^{2\pi} \int_0^1 (1-r^2) r dr d\theta = \int_0^{2\pi} d\theta \int_0^1 (r-r^3) dr = 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1 = \frac{\pi}{2}.\end{aligned}$$

Then

$$\begin{aligned}
 f_{\text{ave}} &= \frac{1}{\pi/2} \iiint_E (x^2 z + y^2 z) dV = \frac{2}{\pi} \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} (x^2 + y^2) z dz dy dx \\
 &= \frac{2}{\pi} \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2) \cdot \frac{1}{2} (1 - x^2 - y^2)^2 dy dx = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^2 (1 - r^2)^2 r dr d\theta \\
 &= \frac{1}{\pi} \int_0^{2\pi} d\theta \int_0^1 (r^3 - 2r^5 + r^7) dr = \frac{1}{\pi} (2\pi) \left[\frac{1}{4} r^4 - \frac{2}{6} r^6 + \frac{1}{8} r^8 \right]_0^1 \\
 &= 2 \left(\frac{1}{24} \right) = \frac{1}{12}
 \end{aligned}$$

49. The triple integral will attain its maximum when the integrand $1 - x^2 - 2y^2 - 3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E , and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E . So we require that $x^2 + 2y^2 + 3z^2 \leq 1$. This describes the region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

DISCOVERY PROJECT Volumes of Hyperspheres

In this project we use V_n to denote the n -dimensional volume of an n -dimensional hypersphere.

1. The interior of the circle is the set of points $\{(x, y) \mid -r \leq y \leq r, -\sqrt{r^2 - y^2} \leq x \leq \sqrt{r^2 - y^2}\}$. So, substituting $y = r \sin \theta$ and then using Formula 64 to evaluate the integral, we get

$$\begin{aligned}
 V_2 &= \int_{-r}^r \int_{-\sqrt{r^2 - y^2}}^{\sqrt{r^2 - y^2}} dx dy = \int_{-r}^r 2 \sqrt{r^2 - y^2} dy = \int_{-\pi/2}^{\pi/2} 2r \sqrt{1 - \sin^2 \theta} (r \cos \theta d\theta) \\
 &= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = 2r^2 \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = 2r^2 \left(\frac{\pi}{2} \right) = \pi r^2
 \end{aligned}$$

2. The region of integration is

$\{(x, y, z) \mid -r \leq z \leq r, -\sqrt{r^2 - z^2} \leq y \leq \sqrt{r^2 - z^2}, -\sqrt{r^2 - z^2 - y^2} \leq x \leq \sqrt{r^2 - z^2 - y^2}\}$. Substituting $y = \sqrt{r^2 - z^2} \sin \theta$ and using Formula 64 to integrate $\cos^2 \theta$, we get

$$\begin{aligned}
 V_3 &= \int_{-r}^r \int_{-\sqrt{r^2 - z^2}}^{\sqrt{r^2 - z^2}} \int_{-\sqrt{r^2 - z^2 - y^2}}^{\sqrt{r^2 - z^2 - y^2}} dx dy dz = \int_{-r}^r \int_{-\sqrt{r^2 - z^2}}^{\sqrt{r^2 - z^2}} 2 \sqrt{r^2 - z^2 - y^2} dy dz \\
 &= \int_{-r}^r \int_{-\pi/2}^{\pi/2} 2 \sqrt{r^2 - z^2} \sqrt{1 - \sin^2 \theta} (\sqrt{r^2 - z^2} \cos \theta d\theta) dz \\
 &= 2 \left[\int_{-r}^r (r^2 - z^2) dz \right] \left[\int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta \right] = 2 \left(\frac{4r^3}{3} \right) \left(\frac{\pi}{2} \right) = \frac{4\pi r^3}{3}
 \end{aligned}$$

3. Here we substitute $y = \sqrt{r^2 - w^2 - z^2} \sin \theta$ and, later, $w = r \sin \phi$. Because $\int_{-\pi/2}^{\pi/2} \cos^p \theta d\theta$ seems to occur frequently in these calculations, it is useful to find a general formula for that integral. From Exercises 43 and 44 in Section 8.1 [ET 7.1], we have

$$\int_0^{\pi/2} \sin^{2k} x dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k} \frac{\pi}{2} \quad \text{and} \quad \int_0^{\pi/2} \sin^{2k+1} x dx = \frac{2 \cdot 4 \cdot 6 \cdots 2k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$$

and from the symmetry of the sine and cosine functions, we can conclude that

$$\int_{-\pi/2}^{\pi/2} \cos^{2k} x \, dx = 2 \int_0^{\pi/2} \sin^{2k} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)\pi}{2 \cdot 4 \cdot 6 \cdots 2k} \quad (1)$$

$$\int_{-\pi/2}^{\pi/2} \cos^{2k+1} x \, dx = 2 \int_0^{\pi/2} \sin^{2k+1} x \, dx = \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdots 2k}{1 \cdot 3 \cdot 5 \cdots (2k+1)} \quad (2)$$

Thus

$$\begin{aligned} V_4 &= \int_{-r}^r \int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}} \int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}} \int_{-\sqrt{r^2-w^2-z^2-y^2}}^{\sqrt{r^2-w^2-z^2-y^2}} dx \, dy \, dz \, dw \\ &= 2 \int_{-r}^r \int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}} \int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}} \sqrt{r^2-w^2-z^2-y^2} \, dy \, dz \, dw \\ &= 2 \int_{-r}^r \int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}} \int_{-\pi/2}^{\pi/2} (r^2-w^2-z^2) \cos^2 \theta \, d\theta \, dz \, dw \\ &= 2 \left[\int_{-r}^r \int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}} (r^2-w^2-z^2) \, dz \, dw \right] \left[\int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta \right] \\ &= 2 \left(\frac{\pi}{2} \right) \left[\int_{-r}^r \frac{4}{3} (r^2-w^2)^{3/2} \, dw \right] = \pi \left(\frac{4}{3} \right) \int_{-\pi/2}^{\pi/2} r^4 \cos^4 \phi \, d\phi = \frac{4\pi}{3} r^4 \cdot \frac{1 \cdot 3 \cdot \pi}{2 \cdot 4} = \frac{\pi^2 r^4}{2} \end{aligned}$$

4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2} \cos \theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

$$\begin{aligned} V_n &= \int_{-r}^r \int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}} \cdots \int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}} \int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1 \, dx_2 \cdots dx_{n-1} \, dx_n \\ &= 2 \left[\int_{-\pi/2}^{\pi/2} \cos^2 \theta_2 \, d\theta_2 \right] \left[\int_{-\pi/2}^{\pi/2} \cos^3 \theta_3 \, d\theta_3 \right] \cdots \left[\int_{-\pi/2}^{\pi/2} \cos^{n-1} \theta_{n-1} \, d\theta_{n-1} \right] \left[\int_{-\pi/2}^{\pi/2} \cos^n \theta_n \, d\theta_n \right] r^n \\ &= \begin{cases} \left[2 \cdot \frac{\pi}{2} \right] \left[\frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{1 \cdot 3 \pi}{2 \cdot 4} \right] \left[\frac{2 \cdot 2 \cdot 4}{1 \cdot 3 \cdot 5} \cdot \frac{1 \cdot 3 \cdot 5 \pi}{2 \cdot 4 \cdot 6} \right] \cdots \left[\frac{2 \cdots (n-2)}{1 \cdots (n-1)} \cdot \frac{1 \cdots (n-1) \pi}{2 \cdots n} \right] r^n & n \text{ even} \\ 2 \left[\frac{\pi}{2} \cdot \frac{2 \cdot 2}{1 \cdot 3} \right] \left[\frac{1 \cdot 3 \pi}{2 \cdot 4} \cdot \frac{2 \cdot 2 \cdot 4}{1 \cdot 3 \cdot 5} \right] \cdots \left[\frac{1 \cdots (n-2) \pi}{2 \cdots (n-1)} \cdot \frac{2 \cdots (n-1)}{1 \cdots n} \right] r^n & n \text{ odd} \end{cases} \end{aligned}$$

By canceling within each set of brackets, we find that

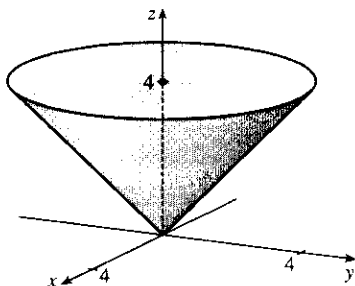
$$V_n = \begin{cases} \frac{2\pi}{2} \cdot \frac{2\pi}{4} \cdot \frac{2\pi}{6} \cdots \frac{2\pi}{n} r^n = \frac{(2\pi)^{n/2}}{2 \cdot 4 \cdot 6 \cdots n} r^n = \frac{\pi^{n/2}}{(\frac{1}{2}n)!} r^n & n \text{ even} \\ 2 \cdot \frac{2\pi}{3} \cdot \frac{2\pi}{5} \cdot \frac{2\pi}{7} \cdots \frac{2\pi}{n} r^n = \frac{2(2\pi)^{(n-1)/2}}{3 \cdot 5 \cdot 7 \cdots n} r^n = \frac{2^n [\frac{1}{2}(n-1)]! \pi^{(n-1)/2}}{n!} r^n & n \text{ odd} \end{cases}$$

16.8 Triple Integrals in Cylindrical and Spherical Coordinates

ET 15.8

1. The region of integration is given in cylindrical coordinates by

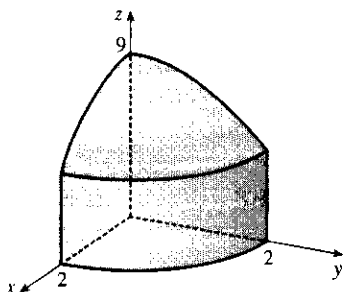
$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 4, r \leq z \leq 4\}$. This represents the solid region bounded below by the cone $z = r$ and above by the horizontal plane $z = 4$.



$$\begin{aligned} \int_0^4 \int_0^{2\pi} \int_r^4 r \, dz \, d\theta \, dr &= \int_0^4 \int_0^{2\pi} [rz]_{z=r}^{z=4} \, d\theta \, dr \\ &= \int_0^4 \int_0^{2\pi} r(4-r) \, d\theta \, dr \\ &= \int_0^4 (4r - r^2) \, dr \int_0^{2\pi} d\theta \\ &= \left[2r^2 - \frac{1}{3}r^3 \right]_0^4 \left[\theta \right]_0^{2\pi} \\ &= \left(32 - \frac{64}{3} \right) (2\pi) = \frac{64\pi}{3} \end{aligned}$$

2. The region of integration is given in cylindrical coordinates by

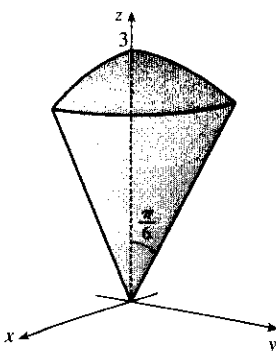
$E = \{(r, \theta, z) \mid 0 \leq \theta \leq \pi/2, 0 \leq r \leq 2, 0 \leq z \leq 9 - r^2\}$. This represents the solid region in the first octant enclosed by the circular cylinder $r = 2$, bounded above by $z = 9 - r^2$, a circular paraboloid, and bounded below by the xy -plane.



$$\begin{aligned} \int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta &= \int_0^{\pi/2} \int_0^2 [rz]_{z=0}^{z=9-r^2} \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^2 r(9-r^2) \, dr \, d\theta \\ &= \int_0^{\pi/2} d\theta \int_0^2 (9r - r^3) \, dr \\ &= \left[\theta \right]_0^{\pi/2} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^2 \\ &= \frac{\pi}{2} (18 - 4) = 7\pi \end{aligned}$$

3. The region of integration is given in spherical coordinates by

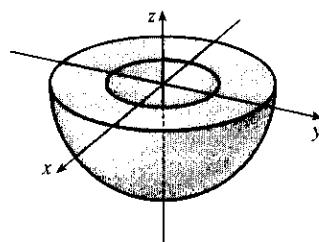
$E = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 3, 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.



$$\begin{aligned} \int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/6} \sin \phi \, d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2 \, d\rho \\ &= [-\cos \phi]_0^{\pi/6} \left[\theta \right]_0^{\pi/2} \left[\frac{1}{3}\rho^3 \right]_0^3 \\ &= \left(1 - \frac{\sqrt{3}}{2} \right) \left(\frac{\pi}{2} \right) (9) \\ &= \frac{9\pi}{4} (2 - \sqrt{3}) \end{aligned}$$

4. The region of integration is given in spherical coordinates by

$E = \{(\rho, \theta, \phi) \mid 1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \pi/2 \leq \phi \leq \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy -plane.



$$\begin{aligned} \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta &= \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin \phi \, d\phi \int_1^2 \rho^2 \, d\rho \\ &= [\theta]_0^{2\pi} [-\cos \phi]_{\pi/2}^{\pi} \left[\frac{1}{3}\rho^3\right]_1^2 \\ &= 2\pi(1)\left(\frac{7}{3}\right) = \frac{14\pi}{3} \end{aligned}$$

5. The solid E is most conveniently described if we use cylindrical

coordinates: $E = \{(r, \theta, z) \mid 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 3, 0 \leq z \leq 2\}$. Then

$$\iiint_E f(x, y, z) \, dV = \int_0^{\pi/2} \int_0^3 \int_0^2 f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta.$$

6. The solid E is most conveniently described if we use spherical coordinates:

$E = \{(\rho, \theta, \phi) \mid 1 \leq \rho \leq 2, \frac{\pi}{2} \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}\}$. Then

$$\iiint_E f(x, y, z) \, dV = \int_0^{\pi/2} \int_{\pi/2}^{2\pi} \int_1^2 f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$

7. In cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 4, -5 \leq z \leq 4\}$. So

$$\begin{aligned} \iiint_E \sqrt{x^2 + y^2} \, dV &= \int_0^{2\pi} \int_0^4 \int_{-5}^4 \sqrt{r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2 \, dr \int_{-5}^4 dz \\ &= [\theta]_0^{2\pi} \left[\frac{1}{3}r^3\right]_0^4 [z]_{-5}^4 = (2\pi)\left(\frac{64}{3}\right)(9) = 384\pi \end{aligned}$$

8. The paraboloid $z = 1 - x^2 - y^2$ intersects the xy -plane in the circle $x^2 + y^2 = r^2 = 1$ or $r = 1$, so in cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 1, 0 \leq z \leq 1 - r^2\}$. Thus

$$\begin{aligned} \iiint_E (x^3 + xy^2) \, dV &= \int_0^{\pi/2} \int_0^1 \int_0^{1-r^2} (r^3 \cos^3 \theta + r^3 \cos \theta \sin^2 \theta) r \, dz \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^1 \int_0^{1-r^2} r^4 \cos \theta \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 r^4 \cos \theta [z]_{z=0}^{z=1-r^2} \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^1 r^4 (1 - r^2) \cos \theta \, dr \, d\theta = \int_0^{\pi/2} \cos \theta \left[\frac{1}{5}r^5 - \frac{1}{7}r^7\right]_{r=0}^{r=1} \, d\theta \\ &= \int_0^{\pi/2} \frac{2}{35} \cos \theta \, d\theta = \frac{2}{35} [\sin \theta]_0^{\pi/2} = \frac{2}{35} \end{aligned}$$

9. In cylindrical coordinates E is bounded by the paraboloid $z = 1 + r^2$, the cylinder $r^2 = 5$ or $r = \sqrt{5}$, and the xy -plane, so E is given by $\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{5}, 0 \leq z \leq 1 + r^2\}$. Thus

$$\begin{aligned} \iiint_E e^z \, dV &= \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{5}} r [e^z]_{z=0}^{z=1+r^2} \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{5}} r(e^{1+r^2} - 1) \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} \left(re^{1+r^2} - r\right) \, dr = 2\pi \left[\frac{1}{2}e^{1+r^2} - \frac{1}{2}r^2\right]_0^{\sqrt{5}} = \pi(e^6 - e - 5) \end{aligned}$$

10. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r \cos \theta + r \sin \theta + 3$ and the cylinders $r = 2$ and $r = 3$, so E is given by $\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 2 \leq r \leq 3, 0 \leq z \leq r \cos \theta + r \sin \theta + 3\}$. Thus

$$\begin{aligned} \iiint_E x \, dV &= \int_0^{2\pi} \int_2^3 \int_0^{r \cos \theta + r \sin \theta + 3} (r \cos \theta) r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_2^3 (r^2 \cos \theta) \Big|_{z=0}^{z=r \cos \theta + r \sin \theta + 3} dr \, d\theta = \int_0^{2\pi} \int_2^3 (r^2 \cos \theta)(r \cos \theta + r \sin \theta + 3) dr \, d\theta \\ &= \int_0^{2\pi} \int_2^3 (r^3 (\cos^2 \theta + \cos \theta \sin \theta) + 3r^2 \cos \theta) dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{4} r^4 (\cos^2 \theta + \cos \theta \sin \theta) + r^3 \cos \theta \right]_{r=2}^{r=3} d\theta \\ &= \int_0^{2\pi} \left[\left(\frac{81}{4} - \frac{16}{4} \right) (\cos^2 \theta + \cos \theta \sin \theta) + (27 - 8) \cos \theta \right] d\theta \\ &= \int_0^{2\pi} \left(\frac{65}{4} \left(\frac{1}{2} (1 + \cos 2\theta) + \cos \theta \sin \theta \right) + 19 \cos \theta \right) d\theta \\ &= \left[\frac{65}{8} \theta + \frac{65}{16} \sin 2\theta + \frac{65}{8} \sin^2 \theta + 19 \sin \theta \right]_0^{2\pi} = \frac{65}{4} \pi \end{aligned}$$

11. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So $E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 0 \leq z \leq 2r\}$ and

$$\begin{aligned} \iiint_E x^2 \, dV &= \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2 \theta \, r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 [r^3 \cos^2 \theta z]_{z=0}^{z=2r} dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 2r^4 \cos^2 \theta \, dr \, d\theta = \int_0^{2\pi} \left[\frac{2}{5} r^5 \cos^2 \theta \right]_{r=0}^{r=1} d\theta = \frac{2}{5} \int_0^{2\pi} \cos^2 \theta \, d\theta \\ &= \frac{2}{5} \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{5} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi} = \frac{2\pi}{5} \end{aligned}$$

12. In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}\}$. Thus the volume is

$$\begin{aligned} \iiint_E dV &= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r \sqrt{4-r^2} \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 2r \sqrt{4-r^2} \, dr = 2\pi \left[-\frac{2}{3} (4-r^2)^{3/2} \right]_0^1 = \frac{4}{3} \pi (8 - 3^{3/2}) \end{aligned}$$

13. (a) The paraboloids intersect when $x^2 + y^2 = 36 - 3x^2 - 3y^2 \Rightarrow x^2 + y^2 = 9$, so the region of integration is $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$. Then, in cylindrical coordinates, $E = \{(r, \theta, z) \mid r^2 \leq z \leq 36 - 3r^2, 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$ and

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^3 (36r - 4r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} [18r^2 - r^4]_{r=0}^{r=3} d\theta = \int_0^{2\pi} 81 \, d\theta = 162\pi \end{aligned}$$

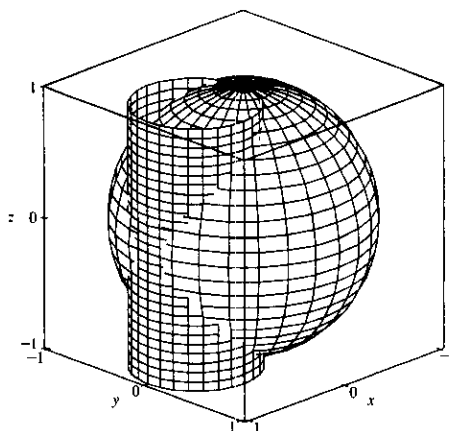
- (b) For constant density K , $m = KV = 162\pi K$ from part (a). Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} (zK) r \, dz \, dr \, d\theta = K \int_0^{2\pi} \int_0^3 r \left[\frac{1}{2} z^2 \right]_{z=r^2}^{z=36-3r^2} dr \, d\theta \\ &= \frac{K}{2} \int_0^{2\pi} \int_0^3 r ((36 - 3r^2)^2 - r^4) dr \, d\theta = \frac{K}{2} \int_0^{2\pi} d\theta \int_0^3 (8r^5 - 216r^3 + 1296r) dr \\ &= \frac{K}{2} (2\pi) \left[\frac{8}{6} r^6 - \frac{216}{4} r^4 + \frac{1296}{2} r^2 \right]_0^3 = \pi K (2430) = 2430\pi K \end{aligned}$$

$$\text{Thus } (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(0, 0, \frac{2430\pi K}{162\pi K} \right) = (0, 0, 15).$$

$$14. (a) V = \int_{-\pi/2}^{\pi/2} \int_0^{a \cos \theta} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta \quad (b)$$

$$\begin{aligned} &= 4 \int_0^{\pi/2} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta \\ &= 4 \int_0^{\pi/2} \int_0^{a \cos \theta} r \sqrt{a^2-r^2} \, dr \, d\theta \\ &= -\frac{4}{3} \int_0^{\pi/2} \left[(a^2-r^2)^{3/2} \right]_{r=0}^{r=a \cos \theta} d\theta \\ &= -\frac{4}{3} \int_0^{\pi/2} \left[(a^2-a^2 \cos^2 \theta)^{3/2} - a^3 \right] d\theta \\ &= -\frac{4}{3} \int_0^{\pi/2} \left[(a^2 \sin^2 \theta)^{3/2} - a^3 \right] d\theta \\ &= -\frac{4}{3} \int_0^{\pi/2} (a^3 \sin^3 \theta - a^3) d\theta \\ &= -\frac{4a^3}{3} \int_0^{\pi/2} [\sin \theta (1 - \cos^2 \theta) - 1] d\theta \\ &= -\frac{4a^3}{3} \left[-\cos \theta + \frac{1}{3} \cos^3 \theta - \theta \right]_0^{\pi/2} = -\frac{4a^3}{3} \left(-\frac{\pi}{2} + \frac{2}{3} \right) = \frac{2}{9} a^3 (3\pi - 4) \end{aligned}$$



To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

```
sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d([cos(theta),theta,z],
theta=0..2*Pi,z=-1..1,coords=cylindrical):
with(plots): display3d({sphere,cylinder});
```

In Mathematica, we can use

```
sphere=SphericalPlot3D[1,{theta,0,2Pi},{phi,0,Pi}]
cylinder=ParametricPlot3D[{Sin[theta],Cos[theta],z},
{theta,0,2Pi},{z,-1,1}]
Show[{sphere,cylinder}]
```

15. The paraboloid $z = 4x^2 + 4y^2$ intersects the plane $z = a$ when $a = 4x^2 + 4y^2$ or $x^2 + y^2 = \frac{1}{4}a$. So, in cylindrical coordinates, $E = \{(r, \theta, z) \mid 0 \leq r \leq \frac{1}{2}\sqrt{a}, 0 \leq \theta \leq 2\pi, 4r^2 \leq z \leq a\}$. Thus

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^{\sqrt{a}/2} \int_{4r^2}^a K r \, dz \, dr \, d\theta = K \int_0^{2\pi} \int_0^{\sqrt{a}/2} (ar - 4r^3) \, dr \, d\theta \\ &= K \int_0^{2\pi} \left[\frac{1}{2} ar^2 - r^4 \right]_{r=0}^{r=\sqrt{a}/2} d\theta = K \int_0^{2\pi} \frac{1}{16} a^2 \, d\theta = \frac{1}{8} a^2 \pi K \end{aligned}$$

Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^{\sqrt{a}/2} \int_{4r^2}^a K r z \, dz \, dr \, d\theta = K \int_0^{2\pi} \int_0^{\sqrt{a}/2} \left(\frac{1}{2} a^2 r - 8r^5 \right) \, dr \, d\theta \\ &= K \int_0^{2\pi} \left[\frac{1}{4} a^2 r^2 - \frac{8}{6} r^6 \right]_{r=0}^{r=\sqrt{a}/2} d\theta = K \int_0^{2\pi} \frac{1}{24} a^3 \, d\theta = \frac{1}{12} a^3 \pi K \end{aligned}$$

Hence $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{2}{3}a)$.

16. Since density is proportional to the distance from the z -axis, we can say $\rho(x, y, z) = K \sqrt{x^2 + y^2}$. Then

$$\begin{aligned} m &= 2 \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} K r^2 \, dz \, dr \, d\theta = 2K \int_0^{2\pi} \int_0^a r^2 \sqrt{a^2-r^2} \, dr \, d\theta \\ &= 2K \int_0^{2\pi} \left[\frac{1}{8} r(2r^2 - a^2) \sqrt{a^2-r^2} + \frac{1}{8} a^4 \sin^{-1}(r/a) \right]_{r=0}^r d\theta \\ &= 2K \int_0^{2\pi} \left[\left(\frac{1}{8} a^4 \right) \left(\frac{\pi}{2} \right) \right] d\theta = \frac{1}{4} a^4 \pi^2 K. \end{aligned}$$

17. In spherical coordinates, B is represented by $\{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$. Thus

$$\begin{aligned}\iiint_B (x^2 + y^2 + z^2) dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2) \rho^2 \sin \phi d\rho d\theta d\phi = \int_0^\pi \sin \phi d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^4 d\rho \\ &= [-\cos \phi]_0^\pi [\theta]_0^{2\pi} \left[\frac{1}{5}\rho^5\right]_0^1 = (2)(2\pi)\left(\frac{1}{5}\right) = \frac{4\pi}{5}\end{aligned}$$

18. In spherical coordinates, H is represented by $\{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}\}$. Thus

$$\begin{aligned}\iiint_H (x^2 + y^2) dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin^2 \phi) \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^3 \phi d\phi \int_0^1 \rho^4 d\rho \\ &= [\theta]_0^{2\pi} \left[-\cos \phi + \frac{1}{3} \cos^3 \phi\right]_0^{\pi/2} \left[\frac{1}{5}\rho^5\right]_0^1 = \frac{4\pi}{15}\end{aligned}$$

19. In spherical coordinates, E is represented by $\{(\rho, \theta, \phi) \mid 1 \leq \rho \leq 2, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}\}$. Thus

$$\begin{aligned}\iiint_E z dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 (\rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi \\ &= \int_0^{\pi/2} \cos \phi \sin \phi d\phi \int_0^{\pi/2} d\theta \int_1^2 \rho^3 d\rho = \left[\frac{1}{2} \sin^2 \phi\right]_0^{\pi/2} [\theta]_0^{\pi/2} \left[\frac{1}{4}\rho^4\right]_1^2 \\ &= \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\left(\frac{15}{4}\right) = \frac{15\pi}{16}\end{aligned}$$

20. $\iiint_E e^{\sqrt{x^2+y^2+z^2}} dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 e^{\sqrt{\rho^2}} \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^2 e^\rho \sin \phi d\rho d\phi d\theta$
 $= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin \phi d\phi \int_0^3 \rho^2 e^\rho d\rho = [\theta]_0^{\pi/2} [-\cos \phi]_0^{\pi/2} [(\rho^2 - 2\rho + 2)e^\rho]_0^3$
 [integrate by parts twice]
 $= \frac{\pi}{2}(0+1)(5e^3 - 2) = \frac{\pi}{2}(5e^3 - 2)$

21. $\iiint_E x^2 dV = \int_0^\pi \int_0^\pi \int_3^4 (\rho \sin \phi \cos \theta)^2 \rho^2 \sin \phi d\rho d\phi d\theta$
 $= \int_0^\pi \cos^2 \theta d\theta \int_0^\pi \sin^3 \phi d\phi \int_3^4 \rho^4 d\rho$
 $= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi \left[-\frac{1}{3}(2 + \sin^2 \phi) \cos \phi\right]_0^\pi \left[\frac{1}{5}\rho^5\right]_3^4$
 $= \left(\frac{\pi}{2}\right)\left(\frac{2}{3} + \frac{2}{3}\right)\frac{1}{5}(4^5 - 3^5) = \frac{1562}{15}\pi$

22. $\iiint_E xyz dV = \int_0^{\pi/3} \int_0^{2\pi} \int_2^4 (\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi$
 $= \int_0^{\pi/3} \sin^3 \phi \cos \phi d\phi \int_0^{2\pi} \sin \theta \cos \theta d\theta \int_2^4 \rho^5 d\rho = \left[\frac{1}{4} \sin^4 \phi\right]_0^{\pi/3} \left[\frac{1}{2} \sin^2 \theta\right]_0^{2\pi} \left[\frac{1}{6}\rho^6\right]_2^4 = 0$

23. Since $\rho = 4 \cos \phi$ implies $\rho^2 = 4\rho \cos \phi$, the equation is that of a sphere of radius 2 with center at $(0, 0, 2)$. Thus

$$\begin{aligned}V &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4 \cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4 \cos \phi} \sin \phi d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \left(\frac{64}{3} \cos^3 \phi\right) \sin \phi d\phi d\theta = \int_0^{2\pi} \left[-\frac{16}{3} \cos^4 \phi\right]_{\phi=0}^{\phi=\pi/3} d\theta \\ &= \int_0^{2\pi} -\frac{16}{3} \left(\frac{1}{16} - 1\right) d\theta = 5\theta \Big|_0^{2\pi} = 10\pi\end{aligned}$$

24. In spherical coordinates, the sphere $x^2 + y^2 + z^2 = 4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2 + y^2}$ is represented by $\phi = \frac{\pi}{4}$. Thus, the solid is given by $\{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}\}$ and

$$\begin{aligned}V &= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin \phi d\rho d\theta d\phi = \int_{\pi/4}^{\pi/2} \sin \phi d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 d\rho \\ &= [-\cos \phi]_{\pi/4}^{\pi/2} [\theta]_0^{2\pi} \left[\frac{1}{3}\rho^3\right]_0^2 = \left(\frac{\sqrt{2}}{2}\right)(2\pi)\left(\frac{8}{3}\right) = \frac{8\sqrt{2}\pi}{3}\end{aligned}$$

25. By the symmetry of the region, $M_{xy} = 0$ and $M_{yz} = 0$. Assuming constant density K ,

$$\begin{aligned} m &= \iiint_E KV = K \int_0^\pi \int_0^\pi \int_3^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = K \int_0^\pi d\theta \int_0^\pi \sin \phi \, d\phi \int_3^4 \rho^2 \, d\rho \\ &= K\pi [-\cos \phi]_0^\pi \left[\frac{1}{3}\rho^3\right]_3^4 = 2K\pi \cdot \frac{37}{3} = \frac{74}{3}\pi K \end{aligned}$$

and

$$\begin{aligned} M_{xz} &= \iiint_E y K \, dV = K \int_0^\pi \int_0^\pi \int_3^4 (\rho \sin \phi \sin \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= K \int_0^\pi \sin \theta \, d\theta \int_0^\pi \sin^2 \phi \, d\phi \int_3^4 \rho^3 \, d\rho \\ &= K [-\cos \theta]_0^\pi \left[\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\right]_0^\pi \left[\frac{1}{4}\rho^4\right]_3^4 \\ &= K(2)\left(\frac{\pi}{2}\right)\frac{1}{4}(256 - 81) = \frac{175}{4}\pi K \end{aligned}$$

Thus the centroid is $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(0, \frac{175\pi K/4}{74\pi K/3}, 0\right) = \left(0, \frac{525}{296}, 0\right)$.

26. (a) Placing the center of the base at $(0, 0, 0)$, $\rho(x, y, z) = K\sqrt{x^2 + y^2 + z^2}$ is the density function. So

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^3 \sin \phi \, d\rho \, d\phi \, d\theta = K \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \phi \, d\phi \int_0^a \rho^3 \, d\rho \\ &= K[\theta]_0^{2\pi} [-\cos \phi]_0^{\pi/2} \left[\frac{1}{4}\rho^4\right]_0^a = K(2\pi)(1)\left(\frac{1}{4}a^4\right) = \frac{1}{2}\pi Ka^4 \end{aligned}$$

- (b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^4 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = K \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \phi \cos \phi \, d\phi \int_0^a \rho^4 \, d\rho \\ &= K[\theta]_0^{2\pi} \left[\frac{1}{2}\sin^2 \phi\right]_0^{\pi/2} \left[\frac{1}{5}\rho^5\right]_0^a = K(2\pi)\left(\frac{1}{2}\right)\left(\frac{1}{5}a^5\right) = \frac{1}{5}\pi Ka^5 \end{aligned}$$

Hence $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{2}{5}a\right)$.

$$\begin{aligned} (c) \, I_z &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho^3 \sin \phi)(\rho^2 \sin^2 \phi) \, d\rho \, d\phi \, d\theta = K \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^3 \phi \, d\phi \int_0^a \rho^5 \, d\rho \\ &= K[\theta]_0^{2\pi} \left[-\cos \phi + \frac{1}{3}\cos^3 \phi\right]_0^{\pi/2} \left[\frac{1}{6}\rho^6\right]_0^a = K(2\pi)\left(\frac{2}{3}\right)\left(\frac{1}{6}a^6\right) = \frac{2}{9}\pi Ka^6 \end{aligned}$$

27. (a) The density function is $\rho(x, y, z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$.

Then $M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = \frac{1}{2}\pi Ka^4 \int_0^{\pi/2} \sin \phi \cos \phi \, d\phi = \frac{1}{8}\pi Ka^4$. But the mass is $K(\text{volume of the hemisphere}) = \frac{2}{3}\pi Ka^3$, so the centroid is $\left(0, 0, \frac{3}{8}a\right)$.

- (b) Place the center of the base at $(0, 0, 0)$; the density function is $\rho(x, y, z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x :

$$\begin{aligned} I_x &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho^2 \sin \phi) \rho^2 (\sin^2 \phi \sin^2 \theta + \cos^2 \phi) \, d\rho \, d\phi \, d\theta \\ &= K \int_0^{2\pi} \int_0^{\pi/2} (\sin^3 \phi \sin^2 \theta + \sin \phi \cos^2 \phi) \left(\frac{1}{5}a^5\right) \, d\phi \, d\theta \\ &= \frac{1}{5}Ka^5 \int_0^{2\pi} \left[\sin^2 \theta \left(-\cos \phi + \frac{1}{3}\cos^3 \phi\right) + \left(-\frac{1}{3}\cos^3 \phi\right)\right]_{\phi=0}^{\phi=\pi/2} \, d\theta \\ &= \frac{1}{5}Ka^5 \int_0^{2\pi} \left[\frac{2}{3}\sin^2 \theta + \frac{1}{3}\right] \, d\theta = \frac{1}{5}Ka^5 \left[\frac{2}{3}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right) + \frac{1}{3}\theta\right]_0^{2\pi} \\ &= \frac{1}{5}Ka^5 \left[\frac{2}{3}(\pi - 0) + \frac{1}{3}(2\pi - 0)\right] = \frac{4}{15}Ka^5\pi \end{aligned}$$

28. Place the center of the base at $(0, 0, 0)$, then the density is $\rho(x, y, z) = Kz$, K a constant. Then

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 2\pi K \int_0^{\pi/2} \cos \phi \sin \phi \cdot \frac{1}{4} a^4 \, d\phi \\ &= \frac{1}{2} \pi K a^4 \left[-\frac{1}{4} \cos 2\phi \right]_0^{\pi/2} = \frac{\pi}{4} K a^4 \end{aligned}$$

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^4 \cos^2 \phi \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{2}{5} \pi K a^5 \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \\ &= \frac{2}{5} \pi K a^5 \left[-\frac{1}{3} \cos^3 \theta \right]_0^{\pi/2} = \frac{2}{15} \pi K a^5 \end{aligned}$$

Hence $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{8}{15}a)$.

29. In spherical coordinates $z = \sqrt{x^2 + y^2}$ becomes $\cos \phi = \sin \phi$ or $\phi = \frac{\pi}{4}$. Then

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin \phi \, d\phi \int_0^1 \rho^2 \, d\rho = \frac{1}{3} \pi (2 - \sqrt{2}),$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = 2\pi \left[-\frac{1}{4} \cos 2\phi \right]_0^{\pi/4} \left(\frac{1}{4} \right) = \frac{\pi}{8} \text{ and by symmetry } M_{yz} = M_{xz} = 0.$$

$$\text{Hence } (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3}{8(2 - \sqrt{2})} \right).$$

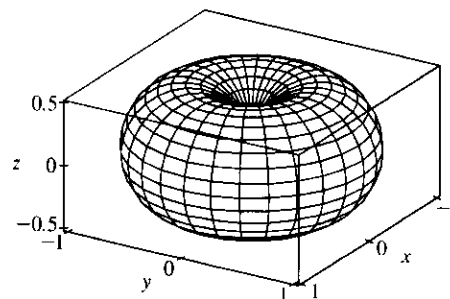
30. Place the center of the sphere at $(0, 0, 0)$, let the diameter of intersection be along the z -axis, one of the planes be the xz -plane and the other be the plane whose angle with the xz -plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by $V = \int_0^{\pi/6} \int_0^\pi \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/6} d\theta \int_0^\pi \sin \phi \, d\phi \int_0^a \rho^2 \, d\rho = \frac{\pi}{6} (2) \left(\frac{1}{3} a^3 \right) = \frac{1}{9} \pi a^3$.
31. In cylindrical coordinates the paraboloid is given by $z = r^2$ and the plane by $z = 2r \sin \theta$ and they intersect in the circle $r = 2 \sin \theta$. Then $\iiint_E z \, dV = \int_0^\pi \int_0^{2 \sin \theta} \int_{r^2}^{2r \sin \theta} r z \, dz \, dr \, d\theta = \frac{5\pi}{6}$ [using a CAS].
32. (a) The region enclosed by the torus is $\{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi, 0 \leq \rho \leq \sin \phi\}$, so its volume is

$$V = \int_0^{2\pi} \int_0^\pi \int_0^{\sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^\pi \frac{1}{3} \sin^4 \phi \, d\phi = \frac{2}{3} \pi \left[\frac{3}{8} \phi - \frac{1}{4} \sin 2\phi + \frac{1}{16} \sin 4\phi \right]_0^\pi = \frac{1}{4} \pi^2.$$

- (b) In Maple, we can plot the torus using the

`plots[sphereplot]` command, or with the `coords=spherical` option in a regular plot command.

In Mathematica, use `ParametricPlot3d`.



33. The region E of integration is the region above the paraboloid $z = x^2 + y^2$, or $z = r^2$, and below the paraboloid $z = 2 - x^2 - y^2$, or $z = 2 - r^2$. Also, we have $-1 \leq x \leq 1$ with $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ which describes the unit circle in the xy -plane. Thus,

$$\begin{aligned} \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{3/2} dz dy dx &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} (r^2)^{3/2} r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 [r^4 z]_{z=r^2}^{z=2-r^2} dr d\theta = \int_0^{2\pi} \int_0^1 (2r^4 - r^6 - r^6) dr d\theta = \int_0^{2\pi} \left(\frac{2}{5} - \frac{2}{7}\right) d\theta = \frac{8\pi}{35} \end{aligned}$$

34. The region E of integration is the region above the paraboloid $z = x^2 + y^2 = r^2$ and below the cone $z = \sqrt{x^2 + y^2} = r$. Also, we have $0 \leq y \leq 1$, $0 \leq x \leq \sqrt{1-y^2}$ which is equivalent to $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq 1$.

Thus

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz dz dx dy &= \int_0^{\pi/2} \int_0^1 \int_{r^2}^r r^2 \cos \theta \sin \theta z r dz dr d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \cos \theta \sin \theta [z^2]_{z=r^2}^{z=r} dr d\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^1 (r^5 - r^7) \cos \theta \sin \theta dr d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \left[\frac{1}{6}r^6 - \frac{1}{8}r^8\right]_{r=0}^{r=1} \cos \theta \sin \theta d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{1}{24} \cos \theta \sin \theta d\theta \\ &= \frac{1}{48} \int_0^{\pi/2} \frac{1}{2} \sin 2\theta d\theta = \frac{1}{96} \left[-\frac{1}{2} \cos 2\theta\right]_0^{\pi/2} = \frac{1}{96} \end{aligned}$$

35. The region of integration E is the top half of the sphere $x^2 + y^2 + z^2 = 9$. So

$$\begin{aligned} \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} z \sqrt{x^2+y^2+z^2} dz dy dx &= \iiint_E z \sqrt{x^2+y^2+z^2} dV \\ &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (\rho^2 \cos \phi) (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos \phi \sin \phi d\phi \int_0^3 \rho^4 d\rho \\ &= [\theta]_0^{2\pi} \left[\frac{1}{2} \sin^2 \phi\right]_0^{\pi/2} \left[\frac{1}{5} \rho^5\right]_0^3 = (2\pi) \left(\frac{1}{2}\right) \left(\frac{243}{5}\right) = \frac{243}{5} \pi \end{aligned}$$

36. The region of integration E is the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 18$ in the first octant. Because E is in the first octant we have $0 \leq \theta \leq \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4) and so $0 \leq \phi \leq \frac{\pi}{4}$. Also $0 \leq \rho \leq \sqrt{18} = 3\sqrt{2}$. So the integral becomes

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{3\sqrt{2}} \rho^4 \sin \phi d\rho d\phi d\theta &= \int_0^{\pi/2} d\theta \int_0^{\pi/4} \sin \phi d\phi \int_0^{3\sqrt{2}} \rho^4 d\rho \\ &= [\theta]_0^{\pi/2} [-\cos \phi]_0^{\pi/4} \left[\frac{1}{5} \rho^5\right]_0^{3\sqrt{2}} \\ &= \left(\frac{\pi}{2}\right) \left(1 - \frac{\sqrt{2}}{2}\right) \left(\frac{972\sqrt{2}}{5}\right) = 486\pi \left(\frac{\sqrt{2}-1}{5}\right) \end{aligned}$$

37. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5} \sin 6\theta \sin 5\phi$, it can be described in spherical coordinates as

$E = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1 + \frac{1}{5} \sin 6\theta \sin 5\phi, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$. Its volume is given by

$$V(E) = \iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_0^{1 + (\sin 6\theta \sin 5\phi)/5} \rho^2 \sin \phi d\rho d\theta d\phi = \frac{136\pi}{99} \text{ [using a CAS].}$$

38. The given integral is equal to

$\lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^\pi \int_0^R \rho e^{-\rho^2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \lim_{R \rightarrow \infty} \left(\int_0^{2\pi} d\theta \right) \left(\int_0^\pi \sin \phi \, d\phi \right) \left(\int_0^R \rho^3 e^{-\rho^2} \, d\rho \right)$. Now use integration by parts with $u = \rho^2$, $dv = \rho e^{-\rho^2} \, d\rho$ to get

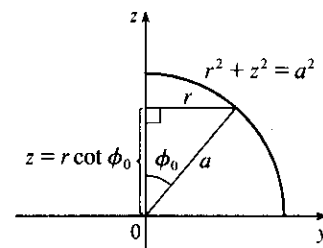
$$\begin{aligned} \lim_{R \rightarrow \infty} 2\pi(2) \left(\rho^2 \left(-\frac{1}{2}\right) e^{-\rho^2} \right)_0^R - \int_0^R 2\rho \left(-\frac{1}{2}\right) e^{-\rho^2} \, d\rho &= \lim_{R \rightarrow \infty} 4\pi \left(-\frac{1}{2} R^2 e^{-R^2} + \left[-\frac{1}{2} e^{-\rho^2} \right]_0^R \right) \\ &= 4\pi \lim_{R \rightarrow \infty} \left[-\frac{1}{2} R^2 e^{-R^2} - \frac{1}{2} e^{-R^2} + \frac{1}{2} \right] \\ &= 4\pi \left(\frac{1}{2} \right) = 2\pi \end{aligned}$$

(Note that $R^2 e^{-R^2} \rightarrow 0$ as $R \rightarrow \infty$ by l'Hospital's Rule.)

39. (a) From the diagram, $z = r \cot \phi_0$ to $z = \sqrt{a^2 - r^2}$, $r = 0$ to

$r = a \sin \phi_0$ (or use $a^2 - r^2 = r^2 \cot^2 \phi_0$). Thus

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{a \sin \phi_0} \int_{r \cot \phi_0}^{\sqrt{a^2 - r^2}} r \, dz \, dr \, d\theta \\ &= 2\pi \int_0^{a \sin \phi_0} (r \sqrt{a^2 - r^2} - r^2 \cot \phi_0) \, dr \\ &= \frac{2\pi}{3} \left[-(a^2 - r^2)^{3/2} - r^3 \cot \phi_0 \right]_0^{a \sin \phi_0} \\ &= \frac{2\pi}{3} \left[-(a^2 - a^2 \sin^2 \phi_0)^{3/2} - a^3 \sin^3 \phi_0 \cot \phi_0 + a^3 \right] \\ &= \frac{2}{3} \pi a^3 [1 - (\cos^3 \phi_0 + \sin^2 \phi_0 \cos \phi_0)] = \frac{2}{3} \pi a^3 (1 - \cos \phi_0) \end{aligned}$$



(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$.

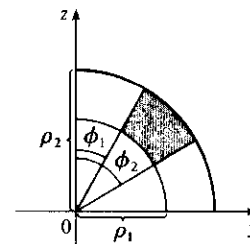
Letting

V_{ij} = volume of the region bounded by the sphere of radius ρ_i
and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)

and letting V be the volume of the wedge, we have

$$\begin{aligned} V &= (V_{22} - V_{21}) - (V_{12} - V_{11}) \\ &= \frac{1}{3}(\theta_2 - \theta_1) [\rho_2^3(1 - \cos \phi_2) - \rho_2^3(1 - \cos \phi_1) - \rho_1^3(1 - \cos \phi_2) + \rho_1^3(1 - \cos \phi_1)] \\ &= \frac{1}{3}(\theta_2 - \theta_1) [(\rho_2^3 - \rho_1^3)(1 - \cos \phi_2) - (\rho_2^3 - \rho_1^3)(1 - \cos \phi_1)] \\ &= \frac{1}{3}(\theta_2 - \theta_1) [(\rho_2^3 - \rho_1^3)(\cos \phi_1 - \cos \phi_2)] \end{aligned}$$

Or: Show that $V = \int_{\theta_1}^{\theta_2} \int_{\rho_1 \sin \phi_1}^{\rho_2 \sin \phi_2} \int_{r \cot \phi_2}^{r \cot \phi_1} r \, dz \, dr \, d\theta$.



(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \leq \tilde{\rho} \leq \rho_2$ such that

$f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_2^3 - \rho_1^3 = 3\tilde{\rho}^2 \Delta\rho$. Similarly there exists $\tilde{\phi}$ with $\phi_1 \leq \tilde{\phi} \leq \phi_2$ such that

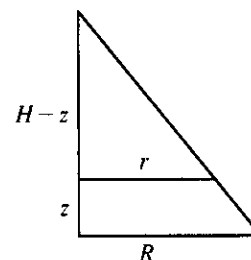
$\cos \phi_2 - \cos \phi_1 = (-\sin \tilde{\phi}) \Delta\phi$. Substituting into the result from (b) gives

$$\Delta V = (\tilde{\rho}^2 \Delta\rho)(\theta_2 - \theta_1)(\sin \tilde{\phi}) \Delta\phi = \tilde{\rho}^2 \sin \tilde{\phi} \Delta\rho \Delta\phi \Delta\theta.$$

40. (a) The mountain comprises a solid conical region C . The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\Delta V$. Summing over the whole mountain we get
- $$W = \iiint_C h(P)g(P)\Delta V.$$

- (b) Here C is a solid right circular cone with radius $R = 62,000$ ft, height $H = 12,400$ ft, and density $g(P) = 200 \text{ lb/ft}^3$ at all points P in C . We use cylindrical coordinates:

$$\begin{aligned} W &= \int_0^{2\pi} \int_0^H \int_0^{R(1-z/H)} z \cdot 200r \, dr \, dz \, d\theta \\ &= 2\pi \int_0^H 200z \left[\frac{1}{2}r^2 \right]_{r=0}^{r=R(1-z/H)} dz \\ &= 400\pi \int_0^H z \frac{R^2}{2} \left(1 - \frac{z}{H} \right)^2 dz \\ &= 200\pi R^2 \int_0^H \left(z - \frac{2z^2}{H} + \frac{z^3}{H^2} \right) dz \\ &= 200\pi R^2 \left[\frac{z^2}{2} - \frac{2z^3}{3H} + \frac{z^4}{4H^2} \right]_0^H \\ &= 200\pi R^2 \left(\frac{H^2}{2} - \frac{2H^2}{3} + \frac{H^2}{4} \right) = \frac{50}{3}\pi R^2 H^2 \\ &= \frac{50}{3}\pi (62,000)^2 (12,400)^2 \approx 3.1 \times 10^{19} \text{ ft}\cdot\text{lb} \end{aligned}$$



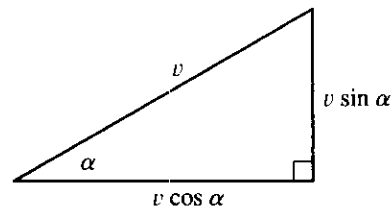
$$\frac{r}{R} = \frac{H-z}{H} = 1 - \frac{z}{H}$$

APPLIED PROJECT Roller Derby

1. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m + I/r^2)v^2$, so $v^2 = \frac{2mgh}{m + I/r^2} = \frac{2gh}{1 + I^*}$.

2. The vertical component of the speed is $v \sin \alpha$, so

$$\frac{dy}{dt} = \sqrt{\frac{2gy}{1 + I^*}} \sin \alpha = \sqrt{\frac{2g}{1 + I^*}} \sin \alpha \sqrt{y}.$$



3. Solving the separable differential equation, we get $\frac{dy}{\sqrt{y}} = \sqrt{\frac{2g}{1 + I^*}} \sin \alpha \, dt \Rightarrow$

$$2\sqrt{y} = \sqrt{\frac{2g}{1 + I^*}} (\sin \alpha)t + C. \text{ But } y = 0 \text{ when } t = 0, \text{ so } C = 0 \text{ and we have } 2\sqrt{y} = \sqrt{\frac{2g}{1 + I^*}} (\sin \alpha)t.$$

$$\text{Solving for } t \text{ when } y = h \text{ gives } T = \frac{2\sqrt{h}}{\sin \alpha} \sqrt{\frac{1 + I^*}{2g}} = \sqrt{\frac{2h(1 + I^*)}{g \sin^2 \alpha}}.$$

4. Assume that the length of each cylinder is ℓ . Then the density of the solid cylinder is $\frac{m}{\pi r^2 \ell}$, and from Formulas 16.7.16 [ET 15.7.16], its moment of inertia (using cylindrical coordinates) is

$$I_z = \iiint \frac{m}{\pi r^2 \ell} (x^2 + y^2) dV = \int_0^\ell \int_0^{2\pi} \int_0^r \frac{m}{\pi r^2 \ell} R^2 R \, dR \, d\theta \, dz = \frac{m}{\pi r^2 \ell} 2\pi \ell \left[\frac{1}{4} R^4 \right]_0^r = \frac{mr^2}{2}$$

$$\text{and so } I^* = \frac{I_z}{mr^2} = \frac{1}{2}.$$

For the hollow cylinder, we consider its entire mass to lie a distance r from the axis of rotation, so $x^2 + y^2 = r^2$ is a constant. We express the density in terms of mass per unit area as $\rho = \frac{m}{2\pi r\ell}$, and then the moment of inertia is calculated as a double integral:

$$I_z = \iint (x^2 + y^2) \frac{m}{2\pi r\ell} dA = \frac{mr^2}{2\pi r\ell} \iint dA = mr^2$$

$$\text{so } I^* = \frac{I_z}{mr^2} = 1.$$

5. The volume of such a ball is $\frac{4}{3}\pi(r^3 - a^3) = \frac{4}{3}\pi r(1 - b^3)$, and so its density is $\frac{m}{\frac{4}{3}\pi r^3(1 - b^3)}$.

Using Formula 16.8.4 [ET 15.8.4], we get

$$\begin{aligned} I_z &= \iiint (x^2 + y^2) \frac{m}{\frac{4}{3}\pi r^3(1 - b^3)} dV \\ &= \frac{m}{\frac{4}{3}\pi r^3(1 - b^3)} \int_a^r \int_0^{2\pi} \int_0^\pi (\rho^2 \sin^2 \phi)(\rho^2 \sin \phi) d\phi d\theta d\rho \\ &= \frac{m}{\frac{4}{3}\pi r^3(1 - b^3)} \cdot 2\pi \left[-\frac{(2 + \sin^2 \phi) \cos \phi}{3} \right]_0^\pi \left[\frac{\rho^5}{5} \right]_a^r \quad [\text{from the Table of Integrals}] \\ &= \frac{m}{\frac{4}{3}\pi r^3(1 - b^3)} \cdot 2\pi \cdot \frac{4}{3} \cdot \frac{r^5 - a^5}{5} \\ &= \frac{2mr^5(1 - b^5)}{5r^3(1 - b^3)} = \frac{2(1 - b^5)mr^2}{5(1 - b^3)} \end{aligned}$$

$$\text{Therefore } I^* = \frac{2(1 - b^5)}{5(1 - b^3)}.$$

Since a represents the inner radius, $a \rightarrow 0$ corresponds to a solid ball, and $a \rightarrow r$ corresponds to a hollow ball.

6. For a solid ball, $a \rightarrow 0 \Rightarrow b \rightarrow 0$, so $I^* = \lim_{b \rightarrow 0} \frac{2(1 - b^5)}{5(1 - b^3)} = \frac{2}{5}$. For a hollow ball, $a \rightarrow r \Rightarrow b \rightarrow 1$, so

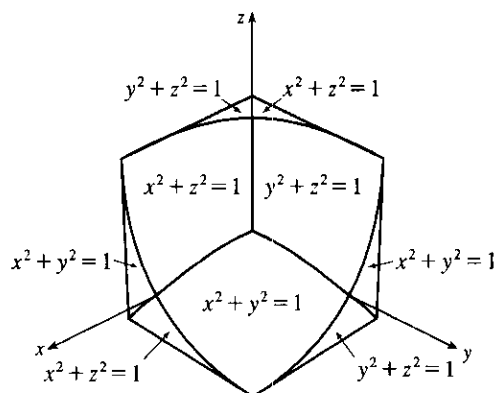
$$I^* = \lim_{b \rightarrow 1} \frac{2(1 - b^5)}{5(1 - b^3)} = \frac{2}{5} \lim_{b \rightarrow 1} \frac{-5b^4}{-3b^2} = \frac{2}{5} \left(\frac{5}{3} \right) = \frac{2}{3} \quad [\text{by l'Hospital's Rule}]$$

$$\text{Note: We could instead have calculated } I^* = \lim_{b \rightarrow 1} \frac{2(1 - b)(1 + b + b^2 + b^3 + b^4)}{5(1 - b)(1 + b + b^2)} = \frac{2 \cdot 5}{5 \cdot 3} = \frac{2}{3}.$$

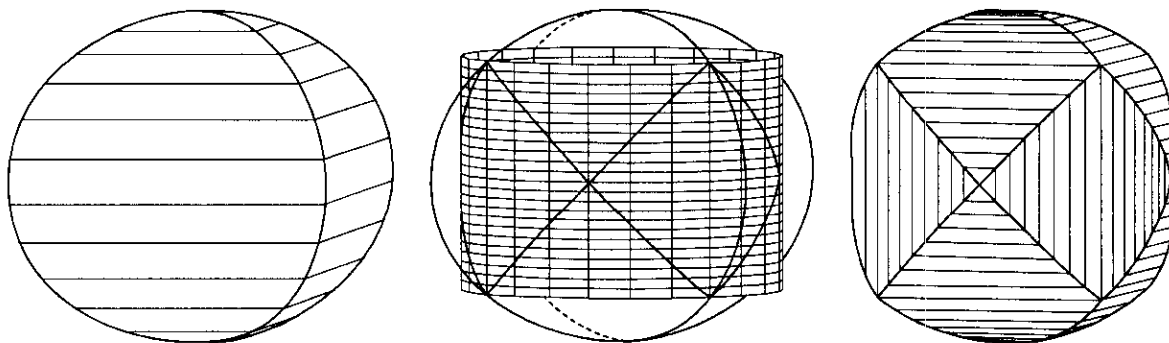
Thus the objects finish in the following order: solid ball ($I^* = \frac{2}{5}$), solid cylinder ($I^* = \frac{1}{2}$), hollow ball ($I^* = \frac{2}{3}$), hollow cylinder ($I^* = 1$).

DISCOVERY PROJECT The Intersection of Three Cylinders

1. The three cylinders in the illustration in the text can be visualized as representing the surfaces $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Then we sketch the solid of intersection with the coordinate axes and equations indicated. To be more precise, we start by finding the bounding curves of the solid (shown in the first graph below) enclosed by the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$: $x = \pm y = \pm\sqrt{1 - z^2}$ are the symmetric



equations, and these can be expressed parametrically as $x = s$, $y = \pm s$, $z = \pm\sqrt{1 - s^2}$, $-1 \leq s \leq 1$. Now the cylinder $x^2 + y^2 = 1$ intersects these curves at the eight points $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$. The resulting solid has twelve curved faces bounded by “edges” which are arcs of circles, as shown in the third diagram. Each cylinder defines four of the twelve faces.



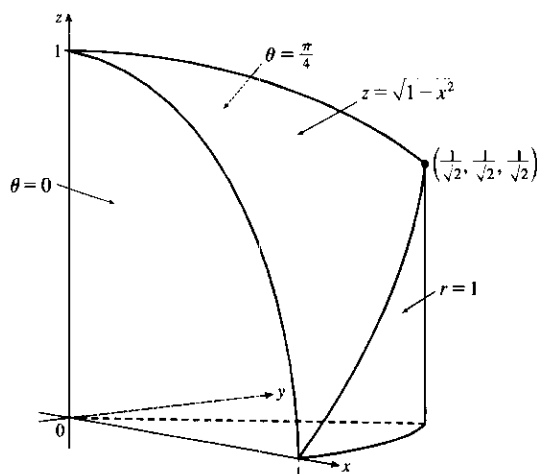
2. To find the volume, we split the solid into sixteen congruent pieces, one of which lies in the part of the first octant with $0 \leq \theta \leq \frac{\pi}{4}$. (Naturally, we use cylindrical coordinates!)

This piece is described by

$$\{(r, \theta, z) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq z \leq \sqrt{1 - x^2}\},$$

and so, substituting $x = r \cos \theta$, the volume of the entire solid is

$$\begin{aligned} V &= 16 \int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{1-x^2}} r \, dz \, dr \, d\theta \\ &= 16 \int_0^{\pi/4} \int_0^1 r \sqrt{1-r^2 \cos^2 \theta} \, dr \, d\theta \\ &= 16 - 8\sqrt{2} \approx 4.6863 \end{aligned}$$

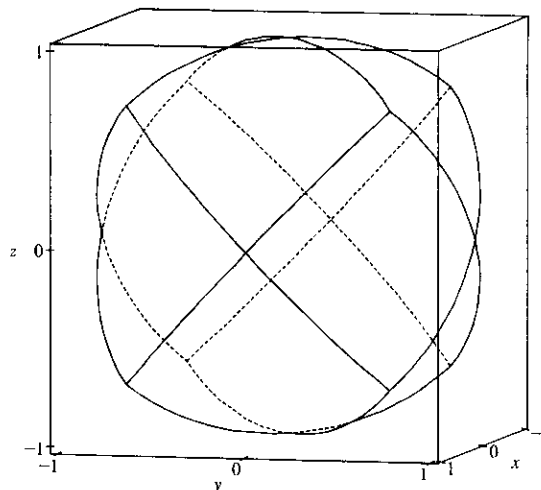


3. To graph the edges of the solid, we use parametrized curves similar to those found in Problem 1 for the intersection of two cylinders. We must restrict the parameter intervals so that each arc extends exactly to the desired vertex. One possible set of parametric equations (with all sign choices allowed) is

$$x = r, y = \pm r, z = \pm\sqrt{1-r^2}, -\frac{1}{\sqrt{2}} \leq r \leq \frac{1}{\sqrt{2}};$$

$$x = \pm s, y = \pm\sqrt{1-s^2}, z = s, -\frac{1}{\sqrt{2}} \leq s \leq \frac{1}{\sqrt{2}};$$

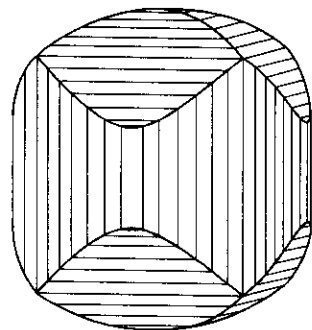
$$x = \pm\sqrt{1-t^2}, y = t, z = \pm t, -\frac{1}{\sqrt{2}} \leq t \leq \frac{1}{\sqrt{2}}.$$



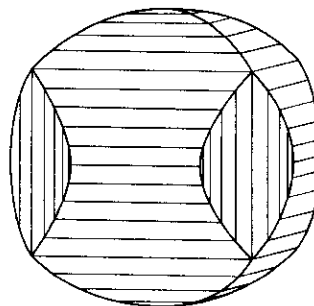
4. Let the three cylinders be $x^2 + y^2 = a^2$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

If $a < 1$, then the four faces defined by the cylinder $x^2 + y^2 = 1$ in Problem 1 collapse into a single face, as in the first graph. If $1 < a < \sqrt{2}$, then each pair of vertically opposed faces, defined by one of the other two cylinders, collapse into a single face, as in the second graph. If $a \geq \sqrt{2}$, then the vertical cylinder encloses the solid of intersection of the other two cylinders completely, so the solid of intersection coincides with the solid of intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, as illustrated in Problem 1.

If we were to vary b or c instead of a , we would get solids with the same shape, but differently oriented.



$$a = 0.95, b = c = 1$$



$$a = 1.1, b = c = 1$$

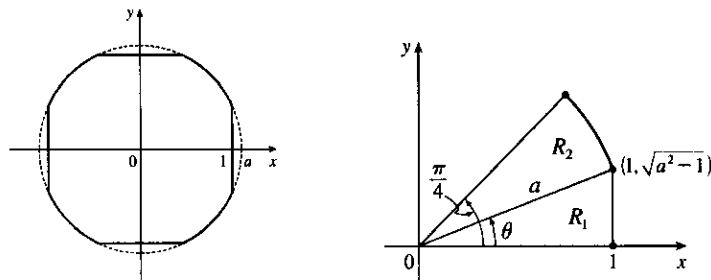
5. If $a < 1$, the solid looks similar to the first graph in Problem 4. As in Problem 2, we split the solid into sixteen congruent pieces, one of which can be described as the solid above the polar region

$\{(r, \theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq \frac{\pi}{4}\}$ in the xy -plane and below the surface $z = \sqrt{1-x^2} = \sqrt{1-r^2 \cos^2 \theta}$. Thus, the total volume is

$$V = 16 \int_0^{\pi/4} \int_0^a \sqrt{1-r^2 \cos^2 \theta} r dr d\theta$$

If $a > 1$ and $a < \sqrt{2}$, we have a solid similar to the second graph in Problem 4. Its intersection with the xy -plane is graphed below. Again we split the solid into sixteen congruent pieces, one of which is the solid above the region

shown in the second figure and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2 \cos^2 \theta}$.



We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = 1/a$. R_1 is a right triangle, so $\cos \theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}(\frac{1}{a})$ in polar coordinates, or $y = \sqrt{a^2 - 1}x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2 , we find the total volume of the solid to be

$$V = 16 \left[\int_0^1 \int_0^{\sqrt{a^2-1}x} \sqrt{1-x^2} dy dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1-r^2 \cos^2 \theta} r dr d\theta \right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 16.6.24 [ET 15.6.24]. Its volume is

$$V = 16 \int_0^1 \int_0^x \sqrt{1-x^2} dy dx$$

16.9 Change of Variables in Multiple Integrals

ET 15.9

1. $x = u + 4v, y = 3u - 2v.$

The Jacobian is $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = 1(-2) - 4(3) = -14.$

2. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4uv - (-4uv) = 8uv$

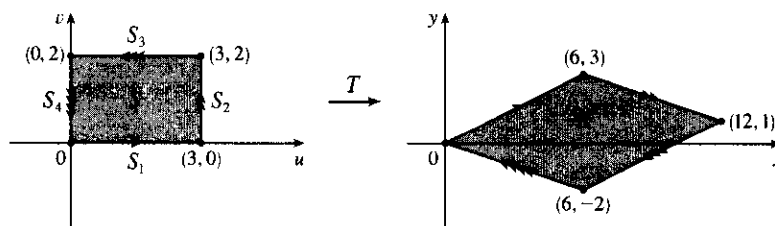
3. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{(u+v)^2} & -\frac{u}{(u+v)^2} \\ -\frac{v}{(u-v)^2} & \frac{u}{(u-v)^2} \end{vmatrix} = \frac{uv}{(u+v)^2(u-v)^2} - \frac{uv}{(u+v)^2(u-v)^2} = 0$

4. $\frac{\partial(x, y)}{\partial(\alpha, \beta)} = \begin{vmatrix} \partial x / \partial \alpha & \partial x / \partial \beta \\ \partial y / \partial \alpha & \partial y / \partial \beta \end{vmatrix} = \begin{vmatrix} \sin \beta & \alpha \cos \beta \\ \cos \beta & -\alpha \sin \beta \end{vmatrix} = -\alpha \sin^2 \beta - \alpha \cos^2 \beta = -\alpha$

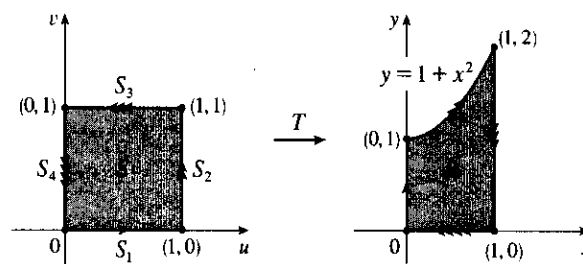
$$\begin{aligned}
 5. \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \begin{vmatrix} \partial x / \partial u & \partial x / \partial v & \partial x / \partial w \\ \partial y / \partial u & \partial y / \partial v & \partial y / \partial w \\ \partial z / \partial u & \partial z / \partial v & \partial z / \partial w \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix} \\
 &= v \begin{vmatrix} w & v \\ 0 & u \end{vmatrix} - u \begin{vmatrix} 0 & v \\ w & u \end{vmatrix} + 0 \begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix} = v(uw - 0) - u(0 - vw) = 2uvw
 \end{aligned}$$

$$\begin{aligned}
 6. \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \begin{vmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{vmatrix} = e^{u+v+w} \begin{vmatrix} e^{u-v} & -e^{u-v} \\ e^{u+v} & e^{u+v} \\ e^{u+v} & e^{u+v} \end{vmatrix} \\
 &= e^{u+v+w} (e^{u-v} e^{u+v} + e^{u-v} e^{u+v}) = e^{u+v+w} (2e^{2u}) = 2e^{3u+v+w}
 \end{aligned}$$

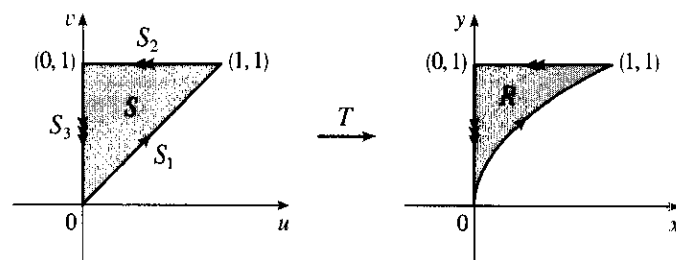
7. The transformation maps the boundary of S to the boundary of the image R , so we first look at side S_1 in the uv -plane. S_1 is described by $v = 0$ ($0 \leq u \leq 3$), so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u , we have $x = 2y$, $0 \leq x \leq 6$. S_2 is the line segment $u = 3$, $0 \leq v \leq 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \Rightarrow x = 6 + 3(3 - y) = 15 - 3y$, $6 \leq x \leq 12$. S_3 is the line segment $v = 2$, $0 \leq u \leq 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \Rightarrow x = 2y + 10$, $6 \leq x \leq 12$. Finally, S_4 is the segment $u = 0$, $0 \leq v \leq 2$, so $x = 3v$ and $y = -v \Rightarrow x = -3y$, $0 \leq x \leq 6$. The image of set S is the region R shown in the xy -plane, a parallelogram bounded by these four segments.



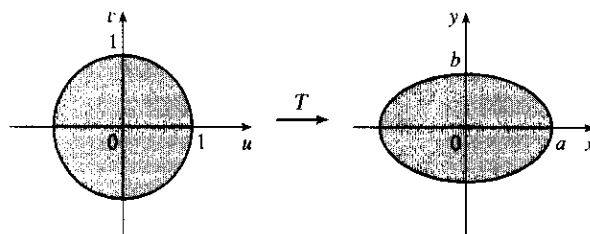
8. S_1 is the line segment $v = 0$, $0 \leq u \leq 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \leq u \leq 1$, the image is the line segment $x = 0$, $0 \leq y \leq 1$. S_2 is the segment $u = 1$, $0 \leq v \leq 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \leq x \leq 1$. S_3 is the segment $v = 1$, $0 \leq u \leq 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \leq y \leq 2$. S_4 is described by $u = 0$, $0 \leq v \leq 1$, so $0 \leq x = v \leq 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \leq x \leq 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x -axis, and the lines $x = 0$, $x = 1$.



9. S_1 is the line segment $u = v$, $0 \leq u \leq 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \leq u \leq 1$, the image is the portion of the parabola $x = y^2$, $0 \leq y \leq 1$. S_2 is the segment $v = 1$, $0 \leq u \leq 1$, thus $y = v = 1$ and $x = u^2$, so $0 \leq x \leq 1$. The image is the line segment $y = 1$, $0 \leq x \leq 1$. S_3 is the segment $u = 0$, $0 \leq v \leq 1$, so $x = u^2 = 0$ and $y = v \Rightarrow 0 \leq y \leq 1$. The image is the segment $x = 0$, $0 \leq y \leq 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y -axis, and the line $y = 1$.



10. Substituting $u = \frac{x}{a}$, $v = \frac{y}{b}$ into $u^2 + v^2 \leq 1$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, so the image of $u^2 + v^2 \leq 1$ is the elliptical region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$.



11. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv -plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \Rightarrow v = 0$; the line through $(2, 1)$ and $(1, 2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \Rightarrow u + v = 1$; the line through $(0, 0)$ and $(1, 2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \Rightarrow u = 0$. Thus S is the triangle $0 \leq v \leq 1 - u$, $0 \leq u \leq 1$ in the uv -plane and

$$\begin{aligned} \iint_R (x - 3y) dA &= \int_0^1 \int_0^{1-u} (-u - 5v) |3| dv du \\ &= -3 \int_0^1 \left[uv + \frac{5}{2}v^2 \right]_{v=0}^{v=1-u} du = -3 \int_0^1 \left(u - u^2 + \frac{5}{2}(1-u)^2 \right) du \\ &= -3 \left[\frac{1}{2}u^2 - \frac{1}{3}u^3 - \frac{5}{6}(1-u)^3 \right]_0^1 = -3 \left(\frac{1}{2} - \frac{1}{3} + \frac{5}{6} \right) = -3 \end{aligned}$$

12. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \frac{1}{4}$, $4x + 8y = 4 \cdot \frac{1}{4}(u + v) + 8 \cdot \frac{1}{4}(v - 3u) = 3v - 5u$. R is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, R is the image of

the rectangle enclosed by the lines $u = -4$, $u = 4$, $v = 0$, and $v = 8$. Thus

$$\begin{aligned}\iint_R (4x + 8y) dA &= \int_{-4}^4 \int_0^8 (3v - 5u) \left| \frac{1}{4} \right| dv du = \frac{1}{4} \int_{-4}^4 \left[\frac{3}{2} v^2 - 5uv \right]_{v=0}^{v=8} du \\ &= \frac{1}{4} \int_{-4}^4 (96 - 40u) du = \frac{1}{4} [96u - 20u^2]_{-4}^4 = 192\end{aligned}$$

13. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2 + 4y^2 \leq 36$ is the image of the disk $u^2 + v^2 \leq 1$. Thus

$$\begin{aligned}\iint_R x^2 dA &= \iint_{u^2+v^2 \leq 1} (4u^2)(6) du dv = \int_0^{2\pi} \int_0^1 (24r^2 \cos^2 \theta) r dr d\theta \\ &= 24 \int_0^{2\pi} \cos^2 \theta d\theta \int_0^1 r^3 dr = 24 \left[\frac{1}{2} x + \frac{1}{4} \sin 2x \right]_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^1 \\ &= 24(\pi) \left(\frac{1}{4} \right) = 6\pi\end{aligned}$$

14. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \frac{4}{\sqrt{3}}$, $x^2 - xy + y^2 = 2u^2 + 2v^2$ and the planar ellipse $x^2 - xy + y^2 \leq 2$

is the image of the disk $u^2 + v^2 \leq 1$. Thus

$$\iint_R (x^2 - xy + y^2) dA = \iint_{u^2+v^2 \leq 1} (2u^2 + 2v^2) \left(\frac{4}{\sqrt{3}} du dv \right) = \int_0^{2\pi} \int_0^1 \frac{8}{\sqrt{3}} r^3 dr d\theta = \frac{4\pi}{\sqrt{3}}$$

15. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

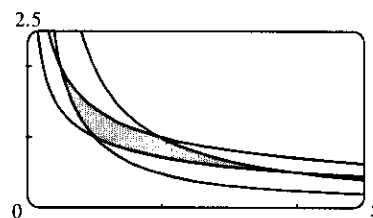
$$\begin{aligned}\iint_R xy dA &= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \left(\frac{1}{v} \right) dv du = \int_1^3 u (\ln \sqrt{3u} - \ln \sqrt{u}) du \\ &= \int_1^3 u \ln \sqrt{3} du = 4 \ln \sqrt{3} = 2 \ln 3\end{aligned}$$

16. Here $y = \frac{v}{u}$, $x = \frac{u^2}{v}$ so $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \frac{1}{v}$

and R is the image of the square with vertices $(1, 1)$, $(2, 1)$, $(2, 2)$,

and $(1, 2)$. So

$$\iint_R y^2 dA = \int_1^2 \int_1^2 \frac{v^2}{u^2} \left(\frac{1}{v} \right) du dv = \int_1^2 \frac{v}{2} dv = \frac{3}{4}$$



17. (a) $\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \frac{x}{a}$, $v = \frac{y}{b}$, $w = \frac{z}{c}$ the solid enclosed by the ellipsoid is the

image of the ball $u^2 + v^2 + w^2 \leq 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2 \leq 1} abc du dv dw = (abc)(\text{volume of the ball}) = \frac{4}{3}\pi abc$$

- (b) If we approximate the surface of Earth by the ellipsoid $\frac{x^2}{6378^2} + \frac{y^2}{6378^2} + \frac{z^2}{6356^2} = 1$, then we can estimate the volume of Earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is
- $$\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12} \text{ km}^3.$$

18. $\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \leq 1$.

Now $x^2 y = (a^2 u^2)(bv)$, so

$$\begin{aligned} \iiint_E x^2 y \, dV &= \iiint_{u^2+v^2+w^2 \leq 1} (a^2 b u^2 v)(abc) \, du \, dv \, dw \\ &= \int_0^{2\pi} \int_0^\pi \int_0^1 (a^3 b^2 c)(\rho^2 \sin^2 \phi \cos^2 \theta)(\rho \sin \phi \sin \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= a^3 b^2 c \int_0^{2\pi} \int_0^\pi \int_0^1 (\rho^5 \sin^4 \phi \cos^2 \theta \sin \theta) \, d\rho \, d\phi \, d\theta \\ &= a^3 b^2 c \int_0^{2\pi} \cos^2 \theta \sin \theta \, d\theta \int_0^\pi \sin^4 \phi \, d\phi \int_0^1 \rho^5 \, d\rho \\ &= 0 \quad \text{since} \quad \int_0^{2\pi} \cos^2 \theta \sin \theta \, d\theta = 0 \end{aligned}$$

19. Letting $u = x - 2y$ and $v = 3x - y$, we have $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$. Then

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \frac{1}{5} \text{ and } R \text{ is the image of the rectangle enclosed by the lines } u = 0, u = 4, v = 1, \text{ and } \\ v &= 8. \text{ Thus } \iint_R \frac{x-2y}{3x-y} \, dA = \int_0^4 \int_1^8 \frac{u}{v} \left| \frac{1}{5} \right| dv \, du = \frac{1}{5} \int_0^4 u \, du \int_1^8 \frac{1}{v} \, dv = \frac{1}{5} \left[\frac{1}{2} u^2 \right]_0^4 [\ln |v|]_1^8 = \frac{8}{5} \ln 8. \end{aligned}$$

20. Letting $u = x + y$ and $v = x - y$, we have $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2} \text{ and } R \text{ is the image of the rectangle enclosed by the lines } u = 0, u = 3, v = 0, \\ \text{and } v &= 2. \text{ Thus} \end{aligned}$$

$$\begin{aligned} \iint_R (x+y) e^{x^2-y^2} \, dA &= \int_0^3 \int_0^2 u e^{uv} \left| -\frac{1}{2} \right| dv \, du = \frac{1}{2} \int_0^3 [e^{uv}]_{v=0}^{v=2} du = \frac{1}{2} \int_0^3 (e^{2u} - 1) du \\ &= \frac{1}{2} \left[\frac{1}{2} e^{2u} - u \right]_0^3 = \frac{1}{2} \left(\frac{1}{2} e^6 - 3 - \frac{1}{2} \right) = \frac{1}{4} (e^6 - 7) \end{aligned}$$

21. Letting $u = y - x$, $v = y + x$, we have $y = \frac{1}{2}(u + v)$, $x = \frac{1}{2}(v - u)$. Then $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{2}$ and

R is the image of the trapezoidal region with vertices $(-1, 1)$, $(-2, 2)$, $(2, 2)$, and $(1, 1)$. Thus

$$\begin{aligned} \iint_R \cos \frac{y-x}{y+x} \, dA &= \int_1^2 \int_{-v}^v \cos \frac{u}{v} \left| -\frac{1}{2} \right| du \, dv = \frac{1}{2} \int_1^2 \left[v \sin \frac{u}{v} \right]_{u=-v}^{u=v} dv \\ &= \frac{1}{2} \int_1^2 2v \sin(1) \, dv = \frac{3}{2} \sin 1 \end{aligned}$$

22. Letting $u = 3x$, $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{6}$ and R is the image of the quarter-disk D given by $u^2 + v^2 \leq 1$, $u \geq 0$, $v \geq 0$. Thus

$$\begin{aligned}\iint_R \sin(9x^2 + 4y^2) dA &= \iint_D \frac{1}{6} \sin(u^2 + v^2) du dv = \int_0^{\pi/2} \int_0^1 \frac{1}{6} \sin(r^2) r dr d\theta \\ &= \frac{\pi}{12} \left[-\frac{1}{2} \cos r^2 \right]_0^1 = \frac{\pi}{24} (1 - \cos 1)\end{aligned}$$

23. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \Rightarrow y = \frac{1}{2}(u + v)$ and

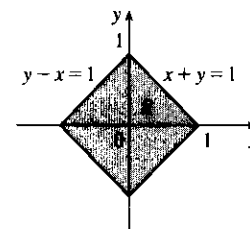
$$u - v = 2x \Rightarrow x = \frac{1}{2}(u - v). \quad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Now $|u| = |x + y| \leq |x| + |y| \leq 1 \Rightarrow -1 \leq u \leq 1$, and

$|v| = |-x + y| \leq |x| + |y| \leq 1 \Rightarrow -1 \leq v \leq 1$. R is the image of

the square region with vertices $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$.

$$\text{So } \iint_R e^{x+y} dA = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 e^u du dv = \frac{1}{2} [e^u]_{-1}^1 [v]_{-1}^1 = e - e^{-1}.$$



24. Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\frac{\partial(x, y)}{\partial(u, v)} = 1$ and R is the image under T of the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. Thus

$$\iint_R f(x + y) dA = \int_0^1 \int_0^u f(u) dv du = \int_0^1 f(u) [v]_{v=0}^{v=u} du = \int_0^1 u f(u) du \quad \text{as desired.}$$

16 Review

ET 15

CONCEPT CHECK

- (a) A double Riemann sum of f is $\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$, where ΔA is the area of each subrectangle and (x_{ij}^*, y_{ij}^*) is a sample point in each subrectangle. If $f(x, y) \geq 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f .

(b) $\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$

(c) If $f(x, y) \geq 0$, $\iint_R f(x, y) dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$. If f takes on both positive and negative values, $\iint_R f(x, y) dA$ is the difference of the volume above R but below the surface $z = f(x, y)$ and the volume below R but above the surface $z = f(x, y)$.

(d) We usually evaluate $\iint_R f(x, y) dA$ as an iterated integral according to Fubini's Theorem (see Theorem 16.2.4 [ET 15.2.4]).

(e) The Midpoint Rule for Double Integrals says that we approximate the double integral $\iint_R f(x, y) dA$ by the double Riemann sum $\sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A$ where the sample points $(\bar{x}_i, \bar{y}_j)$ are the centers of the subrectangles.

(f) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y) dA$ where $A(R)$ is the area of R .

2. (a) See (1) and (2) and the accompanying discussion in Section 16.3 [ET 15.3].
 (b) See (3) and the accompanying discussion in Section 16.3 [ET 15.3].
 (c) See (5) and the preceding discussion in Section 16.3 [ET 15.3].
 (d) See (6)–(11) in Section 16.3 [ET 15.3].
3. We may want to change from rectangular to polar coordinates in a double integral if the region R of integration is more easily described in polar coordinates. To accomplish this, we use

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$
 where R is given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$.
4. (a) $m = \iint_D \rho(x, y) dA$
 (b) $M_x = \iint_D y\rho(x, y) dA$, $M_y = \iint_D x\rho(x, y) dA$
 (c) The center of mass is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$.
 (d) $I_x = \iint_D y^2 \rho(x, y) dA$, $I_y = \iint_D x^2 \rho(x, y) dA$, $I_0 = \iint_D (x^2 + y^2) \rho(x, y) dA$
5. (a) $P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y) dy dx$
 (b) $f(x, y) \geq 0$ and $\iint_{\mathbb{R}^2} f(x, y) dA = 1$.
 (c) The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} xf(x, y) dA$; the expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} yf(x, y) dA$.
6. $A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA$
7. (a) $\iiint_B f(x, y, z) dV = \lim_{l, m, n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$
 (b) We usually evaluate $\iiint_B f(x, y, z) dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals (see Theorem 16.7.4 [ET 15.7.4]).
 (c) See the paragraph following Example 16.7.1 [ET 15.7.1].
 (d) See (5) and (6) and the accompanying discussion in Section 16.7 [ET 15.7].
 (e) See (10) and the accompanying discussion in Section 16.7 [ET 15.7].
 (f) See (11) and the preceding discussion in Section 16.7 [ET 15.7].
8. (a) $m = \iiint_E \rho(x, y, z) dV$
 (b) $M_{yz} = \iiint_E x\rho(x, y, z) dV$, $M_{xz} = \iiint_E y\rho(x, y, z) dV$, $M_{xy} = \iiint_E z\rho(x, y, z) dV$.
 (c) The center of mass is $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = \frac{M_{yz}}{m}$, $\bar{y} = \frac{M_{xz}}{m}$, and $\bar{z} = \frac{M_{xy}}{m}$.
 (d) $I_x = \iiint_E (y^2 + z^2) \rho(x, y, z) dV$, $I_y = \iiint_E (x^2 + z^2) \rho(x, y, z) dV$, $I_z = \iiint_E (x^2 + y^2) \rho(x, y, z) dV$.
9. (a) See Formula 16.8.2 [ET 15.8.2] and the accompanying discussion.
 (b) See Formula 16.8.4 [ET 15.8.4] and the accompanying discussion.
 (c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

10. (a) $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 16.9 [ET 15.9].

(c) See (13) and the accompanying discussion in Section 16.9 [ET 15.9].

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.
2. False. $\int_0^1 \int_0^x \sqrt{x+y^2} dy dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1 \int_y^1 \sqrt{x+y^2} dx dy$.
3. True. See the discussion following Example 4 on page 1029 [ET page 993].
4. $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin y dx dy = \left(\int_0^1 e^{x^2} dx \right) \left(\int_{-1}^1 e^{y^2} \sin y dy \right) = \left(\int_0^1 e^{x^2} dx \right) (0) = 0$, since $e^{y^2} \sin y$ is an odd function. Therefore the statement is true.
5. True:

$$\iint_D \sqrt{4-x^2-y^2} dA = \text{the volume under the surface } x^2+y^2+z^2=4 \text{ and above the } xy\text{-plane}$$

$$= \frac{1}{2} (\text{the volume of the sphere } x^2+y^2+z^2=4) = \frac{1}{2} \cdot \frac{4}{3} \pi (2)^3 = \frac{16}{3} \pi$$
6. This statement is true because in the given region, $(x^2 + \sqrt{y}) \sin(x^2 y^2) \leq (1+2)(1) = 3$, so

$$\int_1^4 \int_0^1 (x^2 + \sqrt{y}) \sin(x^2 y^2) dx dy \leq \int_1^4 \int_0^1 3 dA = 3A(D) = 3(3) = 9.$$
7. The volume enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$ is, in cylindrical coordinates,

$$V = \int_0^{2\pi} \int_0^2 \int_r^2 r dz dr d\theta \neq \int_0^{2\pi} \int_0^2 \int_r^2 dz dr d\theta$$
, so the assertion is false.
8. True. The moment of inertia about the z -axis of a solid E with constant density k is

$$I_z = \iiint_E (x^2 + y^2) \rho(x, y, z) dV = \iiint_E (kr^2) r dz dr d\theta = \iiint_E kr^3 dz dr d\theta.$$

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x, y) dA$ by a Riemann sum with $m = n = 3$ and the sample points the upper right corners of each square, so

$$\begin{aligned} \iint_R f(x, y) dA &\approx \sum_{i=1}^3 \sum_{j=1}^3 f(x_i, y_j) \Delta A \\ &= \Delta A [f(1, 1) + f(1, 2) + f(1, 3) + f(2, 1) + f(2, 2) \\ &\quad + f(2, 3) + f(3, 1) + f(3, 2) + f(3, 3)] \end{aligned}$$

Using the contour lines to estimate the function values, we have

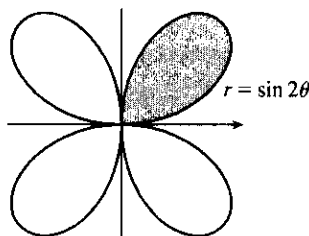
$$\iint_R f(x, y) dA \approx 1[2.7 + 4.7 + 8.0 + 4.7 + 6.7 + 10.0 + 6.7 + 8.6 + 11.9] \approx 64.0$$

2. As in Exercise 1, we have $m = n = 3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\begin{aligned}\iint_R f(x, y) dA &\approx \sum_{i=1}^3 \sum_{j=1}^3 f(\bar{x}_i, \bar{y}_j) \Delta A \\ &= \Delta A [f(0.5, 0.5) + (0.5, 1.5) + (0.5, 2.5) + (1.5, 0.5) + f(1.5, 1.5) \\ &\quad + f(1.5, 2.5) + (2.5, 0.5) + f(2.5, 1.5) + f(2.5, 2.5)] \\ &\approx 1[1.2 + 2.5 + 5.0 + 3.2 + 4.5 + 7.1 + 5.2 + 6.5 + 9.0] = 44.2\end{aligned}$$

3. $\int_1^2 \int_0^2 (y + 2xe^y) dx dy = \int_1^2 [xy + x^2 e^y]_{x=0}^{x=2} dy = \int_1^2 (2y + 4e^y) dy = [y^2 + 4e^y]_1^2$
 $= 4 + 4e^2 - 1 - 4e = 4e^2 - 4e + 3$
4. $\int_0^1 \int_0^1 ye^{xy} dx dy = \int_0^1 [e^{xy}]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1) dy = [e^y - y]_0^1 = e - 2$
5. $\int_0^1 \int_0^x \cos(x^2) dy dx = \int_0^1 [\cos(x^2)y]_{y=0}^{y=x} dx = \int_0^1 x \cos(x^2) dx = \frac{1}{2} \sin(x^2) \Big|_0^1 = \frac{1}{2} \sin 1$
6. $\int_0^1 \int_x^{e^x} 3xy^2 dy dx = \int_0^1 [xy^3]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x} - x^4) dx$
 $= \frac{1}{3} xe^{3x} \Big|_0^1 - \int_0^1 \frac{1}{3} e^{3x} dx - [\frac{1}{5} x^5]_0^1$ [integrating by parts in the first term]
 $= \frac{1}{3} e^3 - [\frac{1}{9} e^{3x}]_0^1 - \frac{1}{5} = \frac{2}{9} e^3 - \frac{4}{45}$
7. $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin x dz dy dx = \int_0^\pi \int_0^1 [(y \sin x)z]_{z=0}^{z=\sqrt{1-y^2}} dy dx = \int_0^\pi \int_0^1 y \sqrt{1-y^2} \sin x dy dx$
 $= \int_0^\pi \left[-\frac{1}{3} (1-y^2)^{3/2} \sin x \right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3} \sin x dx = -\frac{1}{3} \cos x \Big|_0^\pi = \frac{2}{3}$
8. $\int_0^1 \int_0^y \int_x^1 6xyz dz dx dy = \int_0^1 \int_0^y [3xyz^2]_{z=x}^{z=1} dx dy = \int_0^1 \int_0^y (3xy - 3x^3y) dx dy$
 $= \int_0^1 [\frac{3}{2} x^2 y - \frac{3}{4} x^4 y]_{x=0}^{x=y} dy = \int_0^1 (\frac{3}{2} y^3 - \frac{3}{4} y^5) dy = [\frac{3}{8} y^4 - \frac{1}{8} y^6]_0^1 = \frac{1}{4}$
9. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 2 \leq r \leq 4, 0 \leq \theta \leq \pi\}$. Thus
 $\iint_R f(x, y) dA = \int_0^\pi \int_2^4 f(r \cos \theta, r \sin \theta) r dr d\theta$.
10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x -axis. So using rectangular coordinates, we can say $R = \{(x, y) \mid y - 4 \leq x \leq 4 - y, 0 \leq y \leq 4\}$ and $\iint_R f(x, y) dA = \int_0^4 \int_{y-4}^{4-y} f(x, y) dx dy$.

11.

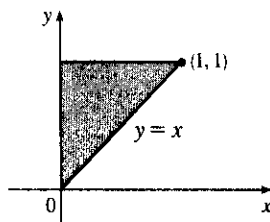


The region whose area is given by $\int_0^{\pi/2} \int_0^{\sin 2\theta} r dr d\theta$ is

$\{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq \sin 2\theta\}$, which is the region contained in the loop in the first quadrant of the four-leaved rose $r = \sin 2\theta$.

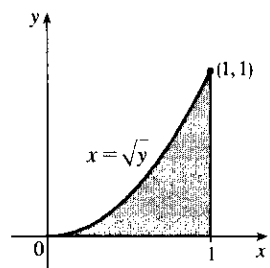
12. The solid is $\{(\rho, \theta, \phi) \mid 1 \leq \rho \leq 2, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}\}$ which is the region in the first octant on or between the two spheres $\rho = 1$ and $\rho = 2$.

13.



$$\begin{aligned}\int_0^1 \int_x^1 \cos(y^2) dy dx &= \int_0^1 \int_0^y \cos(y^2) dx dy \\ &= \int_0^1 \cos(y^2) [x]_{x=0}^{x=y} dy = \int_0^1 y \cos(y^2) dy \\ &= \left[\frac{1}{2} \sin(y^2) \right]_0^1 = \frac{1}{2} \sin 1\end{aligned}$$

14.

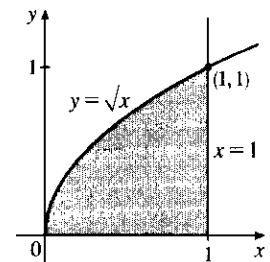


$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dx dy &= \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} dy dx \\ &= \int_0^1 \frac{e^{x^2}}{x^3} \left[\frac{1}{2} y^2 \right]_{y=0}^{y=x^2} dx = \int_0^1 \frac{1}{2} x e^{x^2} dx \\ &= \left[\frac{1}{4} e^{x^2} \right]_0^1 = \frac{1}{4} (e - 1)\end{aligned}$$

15. $\iint_R ye^{xy} dA = \int_0^3 \int_0^2 ye^{xy} dx dy = \int_0^3 [e^{xy}]_{x=0}^{x=2} dy = \int_0^3 (e^{2y} - 1) dy = \left[\frac{1}{2} e^{2y} - y \right]_0^3$
 $= \frac{1}{2} e^6 - 3 - \frac{1}{2} = \frac{1}{2} e^6 - \frac{7}{2}$

16. $\iint_D xy dA = \int_0^1 \int_{y^2}^{y+2} xy dx dy = \int_0^1 y \left[\frac{1}{2} x^2 \right]_{x=y^2}^{x=y+2} dy = \frac{1}{2} \int_0^1 y((y+2)^2 - y^4) dy$
 $= \frac{1}{2} \int_0^1 (y^3 + 4y^2 + 4y - y^5) dy = \frac{1}{2} \left[\frac{1}{4} y^4 + \frac{4}{3} y^3 + 2y^2 - \frac{1}{6} y^6 \right]_0^1 = \frac{41}{24}$

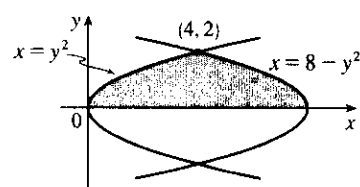
17.



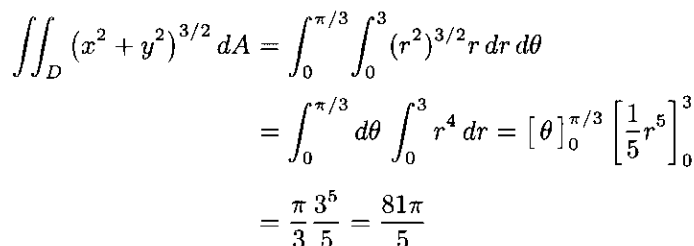
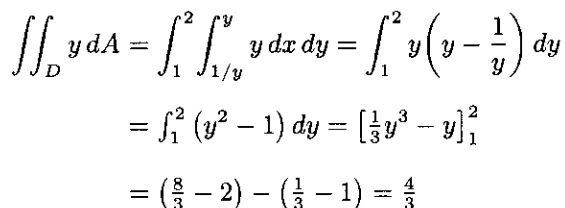
$$\begin{aligned}\iint_D \frac{y}{1+x^2} dA &= \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dy dx = \int_0^1 \frac{1}{1+x^2} \left[\frac{1}{2} y^2 \right]_{y=0}^{y=\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^1 \frac{x}{1+x^2} dx = \left[\frac{1}{4} \ln(1+x^2) \right]_0^1 = \frac{1}{4} \ln 2\end{aligned}$$

18. $\iint_D \frac{1}{1+x^2} dA = \int_0^1 \int_x^1 \frac{1}{1+x^2} dy dx = \int_0^1 \frac{1}{1+x^2} [y]_{y=x}^{y=1} dx = \int_0^1 \frac{1-x}{1+x^2} dx$
 $= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2} \right) dx = \left[\tan^{-1} x - \frac{1}{2} \ln(1+x^2) \right]_0^1$
 $= \tan^{-1} 1 - \frac{1}{2} \ln 2 - (\tan^{-1} 0 - \frac{1}{2} \ln 1) = \frac{\pi}{4} - \frac{1}{2} \ln 2$

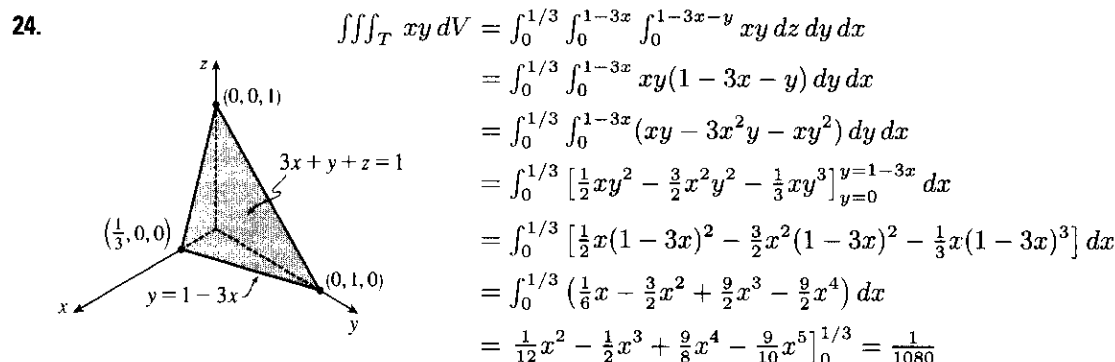
19.



$$\begin{aligned}\iint_D y dA &= \int_0^2 \int_{y^2}^{8-y^2} y dx dy \\ &= \int_0^2 y [x]_{x=y^2}^{x=8-y^2} dy = \int_0^2 y(8-y^2-y^2) dy \\ &= \int_0^2 (8y-2y^3) dy = \left[4y^2 - \frac{1}{2} y^4 \right]_0^2 = 8\end{aligned}$$



23. $\iiint_E xy \, dV = \int_0^3 \int_0^x \int_0^{x+y} xy \, dz \, dy \, dx = \int_0^3 \int_0^x xy [z]_{z=0}^{z=x+y} \, dy \, dx = \int_0^3 \int_0^x xy(x+y) \, dy \, dx$
 $= \int_0^3 \int_0^x (x^2y + xy^2) \, dy \, dx = \int_0^3 [\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3]_{y=0}^{y=x} \, dx = \int_0^3 (\frac{1}{2}x^4 + \frac{1}{3}x^4) \, dx$
 $= \frac{5}{6} \int_0^3 x^4 \, dx = [\frac{1}{6}x^5]_0^3 = \frac{81}{2} = 40.5$



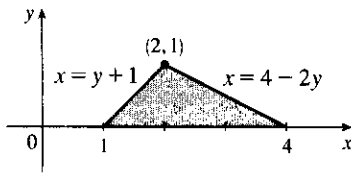
26. $\iiint_E z \, dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{2-y} z \, dx \, dz \, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} (2-y)z \, dz \, dy = \int_0^1 \frac{1}{2}(2-y)(1-y^2) \, dy$
 $= \int_0^1 \frac{1}{2}(2-y-2y^2+y^3) \, dy = \frac{13}{24}$

27. $\iiint_E yz \, dV = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_0^y yz \, dz \, dy \, dx = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \frac{1}{2} y^3 \, dy \, dx = \int_0^\pi \int_0^2 \frac{1}{2} r^3 (\sin^3 \theta) r \, dr \, d\theta$
 $= \frac{16}{5} \int_0^\pi \sin^3 \theta \, d\theta = \frac{16}{5} \left[-\cos \theta + \frac{1}{3} \cos^3 \theta \right]_0^\pi = \frac{64}{15}$

$$\begin{aligned} 28. \iiint_H z^3 \sqrt{x^2 + y^2 + z^2} dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^3 \cos^3 \phi) \rho (\rho^2 \sin \phi) d\rho d\phi d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^3 \phi \sin \phi d\phi \int_0^1 \rho^6 d\rho = 2\pi \left[-\frac{1}{4} \cos^4 \phi \right]_0^{\pi/2} \left(\frac{1}{7} \right) = \frac{\pi}{14} \end{aligned}$$

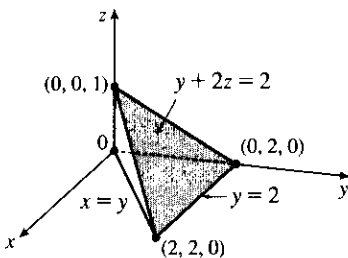
$$29. V = \int_0^2 \int_1^4 (x^2 + 4y^2) dy dx = \int_0^2 \left[x^2 y + \frac{4}{3} y^3 \right]_{y=1}^{y=4} dx = \int_0^2 (3x^2 + 84) dx = 176$$

30.



$$\begin{aligned} V &= \int_0^1 \int_{y+1}^{4-2y} \int_0^{x^2+y} dz dx dy = \int_0^1 \int_{y+1}^{4-2y} x^2 y dx dy \\ &= \int_0^1 \frac{1}{3} [(4-2y)^3 y - (y+1)^3 y] dy \\ &= \int_0^1 3(-y^4 + 5y^3 - 11y^2 + 7y) dy \\ &= 3\left(-\frac{1}{5} + \frac{5}{4} - \frac{11}{3} + \frac{7}{2}\right) = \frac{53}{20} \end{aligned}$$

31.



$$\begin{aligned} V &= \int_0^2 \int_0^y \int_0^{(2-y)/2} dz dx dy \\ &= \int_0^2 \int_0^y \left(1 - \frac{1}{2}y\right) dx dy \\ &= \int_0^2 \left(y - \frac{1}{2}y^2\right) dy = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} 32. V &= \int_0^{2\pi} \int_0^2 \int_0^{3-r\sin\theta} r dz dr d\theta = \int_0^{2\pi} \int_0^2 (3r - r^2 \sin \theta) dr d\theta = \int_0^{2\pi} \left[6 - \frac{8}{3} \sin \theta\right] d\theta \\ &= 6\theta \Big|_0^{2\pi} + 0 = 12\pi \end{aligned}$$

33. Using the wedge above the plane $z = 0$ and below the plane $z = mx$ and noting that we have the same volume for $m < 0$ as for $m > 0$ (so use $m > 0$), we have

$$\begin{aligned} V &= 2 \int_0^{a/3} \int_0^{\sqrt{a^2-9y^2}} mx dx dy = 2 \int_0^{a/3} \frac{1}{2} m(a^2 - 9y^2) dy = m[a^2 y - 3y^3]_0^{a/3} \\ &= m\left(\frac{1}{3}a^3 - \frac{1}{9}a^3\right) = \frac{2}{9}ma^3 \end{aligned}$$

34. The paraboloid and the half-cone intersect when $x^2 + y^2 = \sqrt{x^2 + y^2}$, that is when $x^2 + y^2 = 1$ or 0. So

$$\begin{aligned} V &= \iint_{x^2+y^2 \leq 1} \int_{\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} dz dA = \int_0^{2\pi} \int_0^1 \int_{r^2}^r r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (r^2 - r^3) dr d\theta = \int_0^{2\pi} \left(\frac{1}{3} - \frac{1}{4}\right) d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6} \end{aligned}$$

$$35. (a) m = \int_0^1 \int_0^{1-y^2} y dx dy = \int_0^1 (y - y^3) dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\begin{aligned} (b) M_y &= \int_0^1 \int_0^{1-y^2} xy dx dy = \int_0^1 \frac{1}{2} y(1 - y^2)^2 dy = -\frac{1}{12}(1 - y^2)^3 \Big|_0^1 = \frac{1}{12}, \\ M_x &= \int_0^1 \int_0^{1-y^2} y^2 dx dy = \int_0^1 (y^2 - y^4) dy = \frac{2}{15}. \text{ Hence } (\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{8}{15}\right). \end{aligned}$$

$$\begin{aligned} (c) I_x &= \int_0^1 \int_0^{1-y^2} y^3 dx dy = \int_0^1 (y^3 - y^5) dy = \frac{1}{12}, \\ I_y &= \int_0^1 \int_0^{1-y^2} yx^2 dx dy = \int_0^1 \frac{1}{3} y(1 - y^2)^3 dy = -\frac{1}{24}(1 - y^2)^4 \Big|_0^1 = \frac{1}{24}, \\ I_0 &= I_x + I_y = \frac{1}{8}, \bar{\bar{y}}^2 = \frac{1/12}{1/4} = \frac{1}{3} \Rightarrow \bar{\bar{y}} = \frac{1}{\sqrt{3}}, \text{ and } \bar{\bar{x}}^2 = \frac{1/24}{1/4} = \frac{1}{6} \Rightarrow \bar{\bar{x}} = \frac{1}{\sqrt{6}}. \end{aligned}$$

36. (a)
- $m = \frac{1}{4}\pi K a^2$
- where
- K
- is constant,

$$M_y = \iint_{x^2+y^2 \leq a^2} Kx \, dA = K \int_0^{\pi/2} \int_0^a r^2 \cos \theta \, dr \, d\theta = \frac{1}{3} K a^3 \int_0^{\pi/2} \cos \theta \, d\theta = \frac{1}{3} a^3 K, \text{ and}$$

$$M_x = K \int_0^{\pi/2} \int_0^a r^2 \sin \theta \, dr \, d\theta = \frac{1}{3} a^3 K \quad [\text{by symmetry } M_y = M_x].$$

$$\text{Hence the centroid is } (\bar{x}, \bar{y}) = \left(\frac{4}{3\pi}a, \frac{4}{3\pi}a\right).$$

$$(b) \, m = \int_0^{\pi/2} \int_0^a r^4 \cos \theta \sin^2 \theta \, dr \, d\theta = \left[\frac{1}{5} \sin^3 \theta\right]_0^{\pi/2} \left(\frac{1}{5}a^5\right) = \frac{1}{15}a^5,$$

$$M_y = \int_0^{\pi/2} \int_0^a r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta = \frac{1}{8} \left[\theta - \frac{1}{4} \sin 4\theta\right]_0^{\pi/2} \left(\frac{1}{6}a^6\right) = \frac{1}{96}\pi a^6, \text{ and}$$

$$M_x = \int_0^{\pi/2} \int_0^a r^5 \cos \theta \sin^3 \theta \, dr \, d\theta = \left[\frac{1}{4} \sin^4 \theta\right]_0^{\pi/2} \left(\frac{1}{6}a^6\right) = \frac{1}{24}a^6. \text{ Hence } (\bar{x}, \bar{y}) = \left(\frac{5}{32}\pi a, \frac{5}{8}a\right).$$

37. (a) The equation of the cone with the suggested orientation is
- $(h-z) = \frac{h}{a}\sqrt{x^2+y^2}$
- ,
- $0 \leq z \leq h$
- . Then

$$V = \frac{1}{3}\pi a^2 h \text{ is the volume of one frustum of a cone; by symmetry } M_{yz} = M_{xz} = 0; \text{ and}$$

$$\begin{aligned} M_{xy} &= \iiint_{x^2+y^2 \leq a^2} \int_0^{h-(h/a)\sqrt{x^2+y^2}} z \, dz \, dA = \int_0^{2\pi} \int_0^a \int_0^{(h/a)(a-r)} r z \, dz \, dr \, d\theta \\ &= \pi \int_0^a r \frac{h^2}{a^2} (a-r)^2 \, dr = \frac{\pi h^2}{a^2} \int_0^a (a^2 r - 2ar^2 + r^3) \, dr = \frac{\pi h^2}{a^2} \left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{\pi h^2 a^2}{12} \end{aligned}$$

$$\text{Hence the centroid is } (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{1}{4}h\right).$$

$$(b) \, I_z = \int_0^{2\pi} \int_0^a \int_0^{(h/a)(a-r)} r^3 \, dz \, dr \, d\theta = 2\pi \int_0^a \frac{h}{a} (ar^3 - r^4) \, dr = \frac{2\pi h}{a} \left(\frac{a^5}{4} - \frac{a^5}{5}\right) = \frac{\pi a^4 h}{10}$$

- 38.
- $1 \leq z^2 \leq 4 \Rightarrow 1/a^2 \leq x^2 + y^2 \leq 4/a^2$
- . Let
- $D = \{(x, y) \mid 1/a^2 \leq x^2 + y^2 \leq 4/a^2\}$
- .

$$z = f(x, y) = a\sqrt{x^2 + y^2}, \text{ so } f_x(x, y) = ax(x^2 + y^2)^{-1/2}, f_y(x, y) = ay(x^2 + y^2)^{-1/2}, \text{ and}$$

$$\begin{aligned} A(S) &= \iint_D \sqrt{\frac{a^2 x^2 + a^2 y^2}{x^2 + y^2} + 1} \, dA = \iint_D \sqrt{a^2 + 1} \, dA = \sqrt{a^2 + 1} A(D) \\ &= \sqrt{a^2 + 1} \left[\pi \left(\frac{2}{a}\right)^2 - \pi \left(\frac{1}{a}\right)^2 \right] = \frac{3\pi}{a^2} \sqrt{a^2 + 1} \end{aligned}$$

39. Let
- D
- represent the given triangle; then
- D
- can be described as the area enclosed by the
- x
- and
- y
- axes and the line
- $y = 2 - 2x$
- , or equivalently
- $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x\}$
- . We want to find the surface area of the part of the graph of
- $z = x^2 + y$
- that lies over
- D
- , so using Equation 16.6.3 [ET 15.6.3] we have

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \iint_D \sqrt{1 + (2x)^2 + (1)^2} \, dA \\ &= \int_0^1 \int_0^{2-2x} \sqrt{2+4x^2} \, dy \, dx = \int_0^1 \sqrt{2+4x^2} [y]_{y=0}^{y=2-2x} \, dx = \int_0^1 (2-2x) \sqrt{2+4x^2} \, dx \\ &= \int_0^1 2\sqrt{2+4x^2} \, dx - \int_0^1 2x\sqrt{2+4x^2} \, dx \end{aligned}$$

Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2 \, dx$, we have

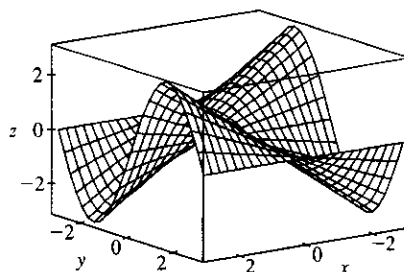
$\int 2\sqrt{2+4x^2} \, dx = x\sqrt{2+4x^2} + \ln(2x + \sqrt{2+4x^2})$. If we substitute $u = 2 + 4x^2$ in the second integral,

then $du = 8x dx$ and $\int 2x \sqrt{2 + 4x^2} dx = \frac{1}{4} \int \sqrt{u} du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} = \frac{1}{6} (2 + 4x^2)^{3/2}$. Thus

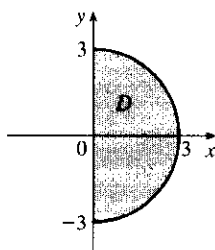
$$\begin{aligned} A(S) &= \left[x \sqrt{2 + 4x^2} + \ln(2x + \sqrt{2 + 4x^2}) - \frac{1}{6} (2 + 4x^2)^{3/2} \right]_0^1 \\ &= \sqrt{6} + \ln(2 + \sqrt{6}) - \frac{1}{6} (6)^{3/2} - \ln \sqrt{2} + \frac{\sqrt{2}}{3} = \ln \frac{2 + \sqrt{6}}{\sqrt{2}} + \frac{\sqrt{2}}{3} \\ &= \ln(\sqrt{2} + \sqrt{3}) + \frac{\sqrt{2}}{3} \approx 1.6176 \end{aligned}$$

40. Using Formula 16.6.3 [ET 15.6.3] with $\partial z / \partial x = \sin y$, $\partial z / \partial y = x \cos y$, we get

$$S = \int_{-\pi}^{\pi} \int_{-3}^3 \sqrt{\sin^2 y + x^2 \cos^2 y + 1} dx dy \approx 62.9714.$$



41.



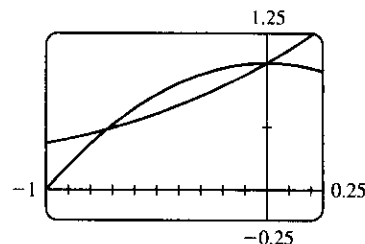
$$\begin{aligned} \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2) dy dx &= \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x(x^2 + y^2) dy dx \\ &= \int_{-\pi/2}^{\pi/2} \int_0^3 (r \cos \theta)(r^2) r dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \cos \theta d\theta \int_0^3 r^4 dr \\ &= [\sin \theta]_{-\pi/2}^{\pi/2} \left[\frac{1}{5} r^5 \right]_0^3 = 2 \cdot \frac{1}{5} (243) = \frac{486}{5} = 97.2 \end{aligned}$$

42. The region of integration is the solid hemisphere $x^2 + y^2 + z^2 \leq 4$, $x \geq 0$.

$$\begin{aligned} \int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} dz dx dy \\ = \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^2 (\rho \sin \phi \sin \theta)^2 (\sqrt{\rho^2}) \rho^2 \sin \phi d\rho d\phi d\theta = \int_{-\pi/2}^{\pi/2} \sin^2 \theta d\theta \int_0^{\pi} \sin^3 \phi d\phi \int_0^2 \rho^5 d\rho \\ = \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/2} \left[-\frac{1}{3} (2 + \sin^2 \phi) \cos \phi \right]_0^{\pi} \left[\frac{1}{6} \rho^6 \right]_0^2 = \left(\frac{\pi}{2} \right) \left(\frac{2}{3} + \frac{2}{3} \right) \left(\frac{32}{3} \right) = \frac{64}{9} \pi \end{aligned}$$

43. From the graph, it appears that $1 - x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1 - x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is

$$\begin{aligned} \iint_D y^2 dA &\approx \int_{-0.71}^0 \int_{e^x}^{1-x^2} y^2 dy dx \\ &= \frac{1}{3} \int_{-0.71}^0 [(1-x^2)^3 - e^{3x}] dx \\ &= \frac{1}{3} \left[x - x^3 + \frac{3}{5} x^5 - \frac{1}{7} x^7 - \frac{1}{3} e^{3x} \right]_{-0.71}^0 \approx 0.0512 \end{aligned}$$



44. Let the tetrahedron be called T . The front face of T is given by the plane $x + \frac{1}{2}y + \frac{1}{3}z = 1$, or

$z = 3 - 3x - \frac{3}{2}y$, which intersects the xy -plane in the line $y = 2 - 2x$. So the total mass is

$$m = \iiint_T \rho(x, y, z) dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} (x^2 + y^2 + z^2) dz dy dx = \frac{7}{5}. \text{ The center of mass is}$$

$$\begin{aligned} (\bar{x}, \bar{y}, \bar{z}) &= (m^{-1} \iiint_T x\rho(x, y, z) dV, m^{-1} \iiint_T y\rho(x, y, z) dV, m^{-1} \iiint_T z\rho(x, y, z) dV) \\ &= \left(\frac{4}{21}, \frac{11}{21}, \frac{8}{7}\right) \end{aligned}$$

45. (a) $f(x, y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x, y) dA = 1$. Since $f(x, y) = 0$ outside the rectangle $[0, 3] \times [0, 2]$, we can say

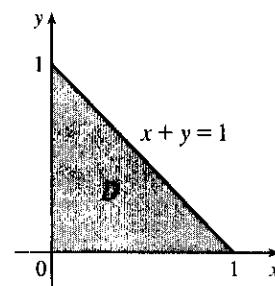
$$\begin{aligned} \iint_{\mathbb{R}^2} f(x, y) dA &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = \int_0^3 \int_0^2 C(x + y) dy dx \\ &= C \int_0^3 \left[xy + \frac{1}{2}y^2\right]_{y=0}^{y=2} dx = C \int_0^3 (2x + 2) dx = C[x^2 + 2x]_0^3 = 15C \end{aligned}$$

$$\text{Then } 15C = 1 \Rightarrow C = \frac{1}{15}.$$

$$\begin{aligned} \text{(b) } P(X \leq 2, Y \geq 1) &= \int_{-\infty}^2 \int_1^{\infty} f(x, y) dy dx = \int_0^2 \int_1^2 \frac{1}{15}(x + y) dy dx = \frac{1}{15} \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2} dx \\ &= \frac{1}{15} \int_0^2 \left(x + \frac{3}{2}\right) dx = \frac{1}{15} \left[\frac{1}{2}x^2 + \frac{3}{2}x\right]_0^2 = \frac{1}{3} \end{aligned}$$

(c) $P(X + Y \leq 1) = P((X, Y) \in D)$ where D is the triangular region shown in the figure. Thus

$$\begin{aligned} P(X + Y \leq 1) &= \iint_D f(x, y) dA = \int_0^1 \int_0^{1-x} \frac{1}{15}(x + y) dy dx \\ &= \frac{1}{15} \int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{15} \int_0^1 \left[x(1-x) + \frac{1}{2}(1-x)^2\right] dx \\ &= \frac{1}{30} \int_0^1 (1 - x^2) dx = \frac{1}{30} \left[x - \frac{1}{3}x^3\right]_0^1 = \frac{1}{45} \end{aligned}$$



46. Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800}e^{-t/800} & \text{if } t \geq 0 \end{cases}$$

If X , Y , and Z are the lifetimes of the individual bulbs, then X , Y , and Z are independent, so the joint density function is the product of the individual density functions:

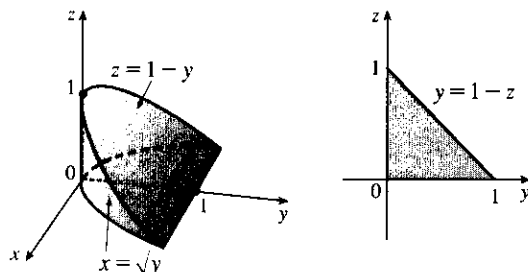
$$f(x, y, z) = \begin{cases} \frac{1}{800^3}e^{-(x+y+z)/800} & \text{if } x \geq 0, y \geq 0, z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P(X + Y + Z \leq 1000)$, or equivalently $P((X, Y, Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane

$x + y + z = 1000$. The plane $x + y + z = 1000$ meets the xy -plane in the line $x + y = 1000$, so we have

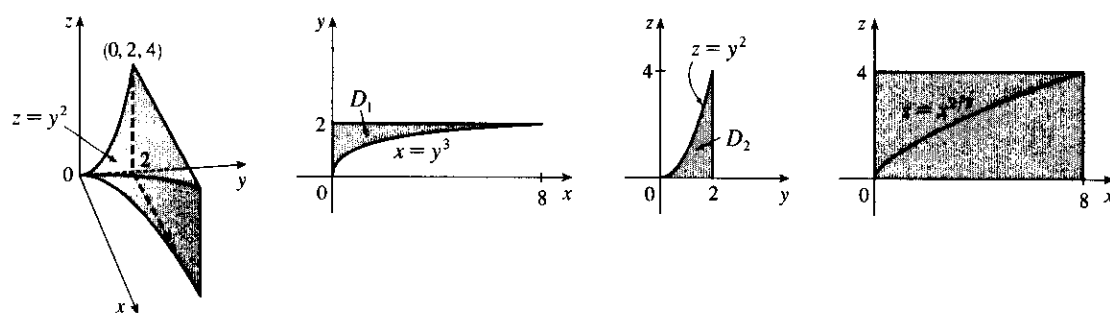
$$\begin{aligned} P(X + Y + Z \leq 1000) &= \iiint_E f(x, y, z) dV = \int_0^{1000} \int_0^{1000-x} \int_0^{1000-x-y} \frac{1}{800^3} e^{-(x+y+z)/800} dz dy dx \\ &= \frac{1}{800^3} \int_0^{1000} \int_0^{1000-x} -800 \left[e^{-(x+y+z)/800} \right]_{z=0}^{z=1000-x-y} dy dx \\ &= \frac{-1}{800^2} \int_0^{1000} \int_0^{1000-x} [e^{-5/4} - e^{-(x+y)/800}] dy dx \\ &= \frac{-1}{800^2} \int_0^{1000} \left[e^{-5/4} y + 800 e^{-(x+y)/800} \right]_{y=0}^{y=1000-x} dx \\ &= \frac{-1}{800^2} \int_0^{1000} [e^{-5/4}(1800 - x) - 800 e^{-x/800}] dx \\ &= \frac{-1}{800^2} \left[-\frac{1}{2} e^{-5/4} (1800 - x)^2 + 800^2 e^{-x/800} \right]_0^{1000} \\ &= \frac{-1}{800^2} \left[-\frac{1}{2} e^{-5/4} (800)^2 + 800^2 e^{-5/4} + \frac{1}{2} e^{-5/4} (1800)^2 - 800^2 \right] \\ &= 1 - \frac{97}{32} e^{-5/4} \approx 0.1315 \end{aligned}$$

47.



$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) dz dy dx = \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z) dx dy dz$$

48.



$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z) dz dx dy = \iiint_E f(x, y, z) dV \text{ where}$$

$E = \{(x, y, z) \mid 0 \leq y \leq 2, 0 \leq x \leq y^3, 0 \leq z \leq y^2\}$. If D_1 , D_2 , and D_3 are the projections of E on the xy -,

yz -, and xz -planes, then $D_1 = \{(x, y) \mid 0 \leq y \leq 2, 0 \leq x \leq y^3\} = \{(x, y) \mid 0 \leq x \leq 8, \sqrt[3]{x} \leq y \leq 2\}$,

$D_2 = \{(y, z) \mid 0 \leq z \leq 4, \sqrt{z} \leq y \leq 2\} = \{(y, z) \mid 0 \leq y \leq 2, 0 \leq z \leq y^2\}$,

$D_3 = \{(x, z) \mid 0 \leq x \leq 8, 0 \leq z \leq 4\}$.

Therefore we have

$$\begin{aligned}
 \int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z) \, dz \, dx \, dy &= \int_0^8 \int_{\sqrt[3]{x}}^2 \int_0^{y^2} f(x, y, z) \, dz \, dy \, dx \\
 &= \int_0^4 \int_{\sqrt{z}}^2 \int_0^{y^3} f(x, y, z) \, dx \, dy \, dz = \int_0^2 \int_0^{y^2} \int_0^{y^3} f(x, y, z) \, dx \, dz \, dy \\
 &= \int_0^8 \int_0^{x^{2/3}} \int_{\sqrt[3]{x}}^2 f(x, y, z) \, dy \, dz \, dx + \int_0^8 \int_{x^{2/3}}^4 \int_{\sqrt{z}}^2 f(x, y, z) \, dy \, dz \, dx \\
 &= \int_0^4 \int_0^{z^{3/2}} \int_{\sqrt{z}}^2 f(x, y, z) \, dy \, dx \, dz + \int_0^4 \int_{z^{3/2}}^8 \int_{\sqrt[3]{x}}^2 f(x, y, z) \, dy \, dx \, dz
 \end{aligned}$$

49. Since $u = x - y$ and $v = x + y$, $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(v - u)$.

$$\text{Thus } \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2} \text{ and } \iint_R \frac{x - y}{x + y} \, dA = \int_2^4 \int_{-2}^0 \frac{u}{v} \left(\frac{1}{2} \right) \, du \, dv = - \int_2^4 \frac{dv}{v} = -\ln 2.$$

$$50. \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw, \text{ so}$$

$$\begin{aligned}
 V &= \iiint_E dV = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \, dw \, dv \, du = \int_0^1 \int_0^{1-u} 4uv(1-u-v)^2 \, dv \, du \\
 &= \int_0^1 \int_0^{1-u} [4u(1-u)^2v - 8u(1-u)v^2 + 4uv^3] \, dv \, du \\
 &= \int_0^1 [2u(1-u)^4 - \frac{8}{3}u(1-u)^4 + u(1-u)^4] \, du = \int_0^1 \frac{1}{3}u(1-u)^4 \, du \\
 &= \int_0^1 \frac{1}{3}[(1-u)^4 - (1-u)^5] \, du = \frac{1}{3} \left[-\frac{1}{5}(1-u)^5 + \frac{1}{6}(1-u)^6 \right]_0^1 = \frac{1}{3} \left(-\frac{1}{6} + \frac{1}{5} \right) = \frac{1}{90}
 \end{aligned}$$

51. Let $u = y - x$ and $v = y + x$ so $x = y - u = (v - x) - u \Rightarrow x = \frac{1}{2}(v - u)$ and

$$y = v - \frac{1}{2}(v - u) = \frac{1}{2}(v + u). \quad \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| = \left| -\frac{1}{2} \left(\frac{1}{2} \right) - \frac{1}{2} \left(\frac{1}{2} \right) \right| = \left| -\frac{1}{2} \right| = \frac{1}{2}. \quad R \text{ is the}$$

image under this transformation of the square with vertices $(u, v) = (0, 0)$, $(-2, 0)$, $(0, 2)$, and $(-2, 2)$. So

$$\begin{aligned}
 \iint_R xy \, dA &= \int_0^2 \int_{-2}^0 \frac{v^2 - u^2}{4} \left(\frac{1}{2} \right) \, du \, dv = \frac{1}{8} \int_0^2 \left[v^2u - \frac{1}{3}u^3 \right]_{u=-2}^{u=0} \, dv = \frac{1}{8} \int_0^2 \left(2v^2 - \frac{8}{3} \right) \, dv \\
 &= \frac{1}{8} \left[\frac{2}{3}v^3 - \frac{8}{3}v \right]_0^2 = 0
 \end{aligned}$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x -axis.

52. By the Extreme Value Theorem (15.7.8 [ET 14.7.8]), f has an absolute minimum value m and an absolute maximum value M in D . Then by Property 16.3.11 [ET 15.3.11], $mA(D) \leq \iint_D f(x, y) \, dA \leq MA(D)$.

Dividing through by the positive number $A(D)$, we get $m \leq \frac{1}{A(D)} \iint_D f(x, y) dA \leq M$. This says that the average value of f over D lies between m and M . But f is continuous on D and takes on the values m and M , and so by the Intermediate Value Theorem must take on all values between m and M . Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x, y) dA$ or equivalently $\iint_D f(x, y) dA = f(x_0, y_0) A(D)$.

53. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for Double

Integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \frac{1}{\pi r^2} \iint_{D_r} f(x, y) dA$. But $\lim_{r \rightarrow 0^+} (x_r, y_r) = (a, b)$,

so $\lim_{r \rightarrow 0^+} \frac{1}{\pi r^2} \iint_{D_r} f(x, y) dA = \lim_{r \rightarrow 0^+} f(x_r, y_r) = f(a, b)$ by the continuity of f .

$$\begin{aligned} 54. (a) \iint_D \frac{1}{(x^2 + y^2)^{n/2}} dA &= \int_0^{2\pi} \int_r^R \frac{1}{(t^2)^{n/2}} t dt d\theta = 2\pi \int_r^R t^{1-n} dt \\ &= \begin{cases} \left[\frac{2\pi}{2-n} t^{2-n} \right]_r^R = \frac{2\pi}{2-n} (R^{2-n} - r^{2-n}) & \text{if } n \neq 2 \\ 2\pi \ln(R/r) & \text{if } n = 2 \end{cases} \end{aligned}$$

(b) The integral in part (a) has a limit as $r \rightarrow 0^+$ for all values of n such that $2 - n > 0 \Leftrightarrow n < 2$.

$$\begin{aligned} (c) \iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}} dV &= \int_r^R \int_0^\pi \int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}} \rho^2 \sin \phi d\theta d\phi d\rho \\ &= 2\pi \int_r^R \int_0^\pi \rho^{2-n} \sin \phi d\phi d\rho \\ &= \begin{cases} \left[\frac{4\pi}{3-n} \rho^{3-n} \right]_r^R = \frac{4\pi}{3-n} (R^{3-n} - r^{3-n}) & \text{if } n \neq 3 \\ 4\pi \ln(R/r) & \text{if } n = 3 \end{cases} \end{aligned}$$

(d) As $r \rightarrow 0^+$, the above integral has a limit, provided that $3 - n > 0 \Leftrightarrow n < 3$.

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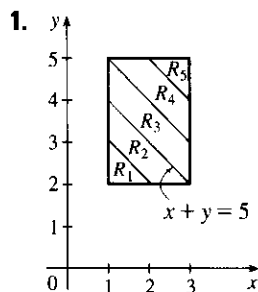
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□ PROBLEMS PLUS



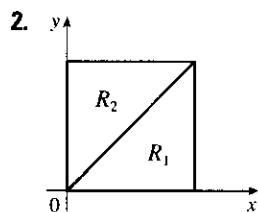
Let $R = \bigcup_{i=1}^5 R_i$, where

$$R_i = \{(x, y) \mid x + y \geq i + 2, x + y < i + 3, 1 \leq x \leq 3, 2 \leq y \leq 5\}.$$

$$\iint_R [x + y] dA = \sum_{i=1}^5 \iint_{R_i} [x + y] dA = \sum_{i=1}^5 [x + y] \iint_{R_i} dA, \text{ since}$$

$[x + y] = \text{constant} = i + 2$ for $(x, y) \in R_i$. Therefore

$$\begin{aligned} \iint_R [x + y] dA &= \sum_{i=1}^5 (i + 2) [A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\frac{1}{2}\right) + 4\left(\frac{3}{2}\right) + 5(2) + 6\left(\frac{3}{2}\right) + 7\left(\frac{1}{2}\right) = 30 \end{aligned}$$



Let $R = \{(x, y) \mid 0 \leq x, y \leq 1\}$. For $x, y \in R$, $\max\{x^2, y^2\} = x^2$ if $x \geq y$,

and $\max\{x^2, y^2\} = y^2$ if $x \leq y$. Therefore we divide R into two regions:

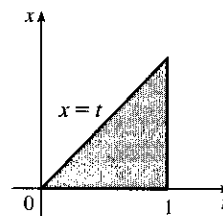
$R = R_1 \cup R_2$, where $R_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ and

$R_2 = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$. Now $\max\{x^2, y^2\} = x^2$ for

$(x, y) \in R_1$, and $\max\{x^2, y^2\} = y^2$ for $(x, y) \in R_2 \Rightarrow$

$$\begin{aligned} \int_0^1 \int_0^1 e^{\max\{x^2, y^2\}} dy dx &= \iint_R e^{\max\{x^2, y^2\}} dA = \iint_{R_1} e^{\max\{x^2, y^2\}} dA + \iint_{R_2} e^{\max\{x^2, y^2\}} dA \\ &= \int_0^1 \int_0^x e^{x^2} dy dx + \int_0^1 \int_0^y e^{y^2} dx dy = \int_0^1 x e^{x^2} dx + \int_0^1 y e^{y^2} dy \\ &= e^{x^2} \Big|_0^1 = e - 1 \end{aligned}$$

$$\begin{aligned} 3. f_{\text{ave}} &= \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{1-0} \int_0^1 \left[\int_x^1 \cos(t^2) dt \right] dx \\ &= \int_0^1 \int_x^1 \cos(t^2) dt dx \\ &= \int_0^1 \int_0^t \cos(t^2) dx dt \quad [\text{changing the order of integration}] \\ &= \int_0^1 t \cos(t^2) dt = \frac{1}{2} \sin(t^2) \Big|_0^1 = \frac{1}{2} \sin 1 \end{aligned}$$



4. Let $u = \mathbf{a} \cdot \mathbf{r}$, $v = \mathbf{b} \cdot \mathbf{r}$, $w = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$. Under this change of variables, E corresponds to the rectangular box $0 \leq u \leq \alpha$, $0 \leq v \leq \beta$, $0 \leq w \leq \gamma$. So, by Formula 16.9.13 [ET 15.9.13],

$$\int_0^\gamma \int_0^\beta \int_0^\alpha uvw du dv dw = \iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r}) \left| \frac{\partial(u, v, w)}{\partial(x, y, z)} \right| dV. \text{ But}$$

$$\left| \frac{\partial(u, v, w)}{\partial(x, y, z)} \right| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}| \Rightarrow$$

$$\begin{aligned} \iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r}) dV &= \frac{1}{|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|} \int_0^\gamma \int_0^\beta \int_0^\alpha uvw du dv dw \\ &= \frac{1}{|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|} \left(\frac{\alpha^2}{2} \right) \left(\frac{\beta^2}{2} \right) \left(\frac{\gamma^2}{2} \right) = \frac{(\alpha\beta\gamma)^2}{8|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|} \end{aligned}$$

5. Since $|xy| < 1$, except at $(1, 1)$, the formula for the sum of a geometric series gives $\frac{1}{1-xy} = \sum_{n=0}^{\infty} (xy)^n$, so

$$\begin{aligned} \int_0^1 \int_0^1 \frac{1}{1-xy} dx dy &= \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n dx dy = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 (xy)^n dx dy = \sum_{n=0}^{\infty} \left[\int_0^1 x^n dx \right] \left[\int_0^1 y^n dy \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

6. Let $x = \frac{u-v}{\sqrt{2}}$ and $y = \frac{u+v}{\sqrt{2}}$. We know the region of integration in the xy -plane, so to find its image in the

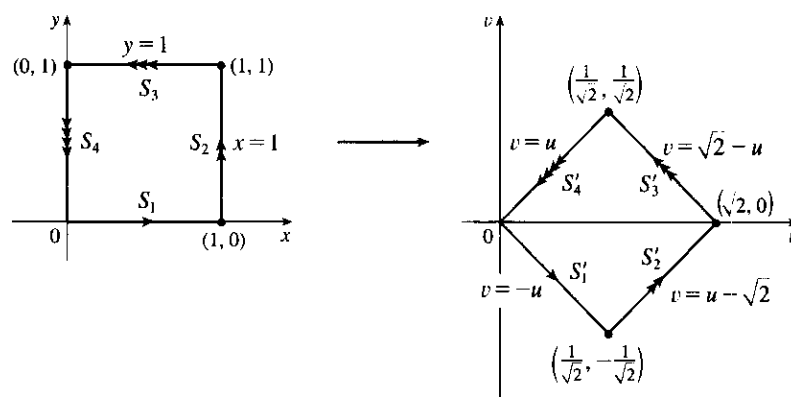
uv -plane we get u and v in terms of x and y , and then use the methods of Section 16.9 [ET 15.9].

$x + y = \frac{u-v}{\sqrt{2}} + \frac{u+v}{\sqrt{2}} = \sqrt{2}u$, so $u = \frac{x+y}{\sqrt{2}}$, and similarly $v = \frac{y-x}{\sqrt{2}}$. S_1 is given by $y = 0$, $0 \leq x \leq 1$, so

from the equations derived above, the image of S_1 is S'_1 : $u = \frac{1}{\sqrt{2}}x$, $v = -\frac{1}{\sqrt{2}}x$, $0 \leq x \leq 1$, that is, $v = -u$,

$0 \leq u \leq \frac{1}{\sqrt{2}}$. Similarly, the image of S_2 is S'_2 : $v = u - \sqrt{2}$, $\frac{1}{\sqrt{2}} \leq u \leq \sqrt{2}$, the image of S_3 is S'_3 : $v = \sqrt{2} - u$,

$\frac{1}{\sqrt{2}} \leq u \leq \sqrt{2}$, and the image of S_4 is S'_4 : $v = u$, $0 \leq u \leq \frac{1}{\sqrt{2}}$.



The Jacobian of the transformation is $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$. From the diagram, we

see that we must evaluate two integrals: one over the region $\{(u, v) \mid 0 \leq u \leq \frac{1}{\sqrt{2}}, -u \leq v \leq u\}$ and the other

over $\left\{ (u, v) \mid \frac{1}{\sqrt{2}} \leq u \leq \sqrt{2}, -\sqrt{2} + u \leq v \leq \sqrt{2} - u \right\}$. So

$$\begin{aligned} \int_0^1 \int_0^1 \frac{dx dy}{1-xy} &= \int_0^{\sqrt{2}/2} \int_{-u}^u \frac{dv du}{1 - \left[\frac{1}{\sqrt{2}}(u+v) \right] \left[\frac{1}{\sqrt{2}}(u-v) \right]} \\ &\quad + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{dv du}{1 - \left[\frac{1}{\sqrt{2}}(u+v) \right] \left[\frac{1}{\sqrt{2}}(u-v) \right]} \\ &= \int_0^{\sqrt{2}/2} \int_{-u}^u \frac{2 dv du}{2-u^2+v^2} + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{2 dv du}{2-u^2+v^2} \\ &= 2 \left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2}-u^2} \left[\arctan \frac{v}{\sqrt{2}-u^2} \right]_{-u}^u du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2}-u^2} \left[\arctan \frac{v}{\sqrt{2}-u^2} \right]_{-\sqrt{2}+u}^{\sqrt{2}-u} du \right] \\ &= 4 \left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2}-u^2} \arctan \frac{u}{\sqrt{2}-u^2} du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2}-u^2} \arctan \frac{\sqrt{2}-u}{\sqrt{2}-u^2} du \right] \end{aligned}$$

Now let $u = \sqrt{2} \sin \theta$, so $du = \sqrt{2} \cos \theta d\theta$ and the limits change to 0 and $\frac{\pi}{6}$ (in the first integral) and $\frac{\pi}{6}$ and $\frac{\pi}{2}$ (in the second integral). Continuing:

$$\begin{aligned} \int_0^1 \int_0^1 \frac{dx dy}{1-xy} &= 4 \left[\int_0^{\pi/6} \frac{1}{\sqrt{2}-2\sin^2 \theta} \arctan \left(\frac{\sqrt{2} \sin \theta}{\sqrt{2}-2\sin^2 \theta} \right) (\sqrt{2} \cos \theta d\theta) \right. \\ &\quad \left. + \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2}-2\sin^2 \theta} \arctan \left(\frac{\sqrt{2}-\sqrt{2} \sin \theta}{\sqrt{2}-2\sin^2 \theta} \right) (\sqrt{2} \cos \theta d\theta) \right] \\ &= 4 \left[\int_0^{\pi/6} \frac{\sqrt{2} \cos \theta}{\sqrt{2} \cos \theta} \arctan \left(\frac{\sqrt{2} \sin \theta}{\sqrt{2} \cos \theta} \right) d\theta + \int_{\pi/6}^{\pi/2} \frac{\sqrt{2} \cos \theta}{\sqrt{2} \cos \theta} \arctan \left(\frac{\sqrt{2}(1-\sin \theta)}{\sqrt{2} \cos \theta} \right) d\theta \right] \\ &= 4 \left[\int_0^{\pi/6} \arctan(\tan \theta) d\theta + \int_{\pi/6}^{\pi/2} \arctan \left(\frac{1-\sin \theta}{\cos \theta} \right) d\theta \right] \end{aligned}$$

But (following the hint)

$$\begin{aligned} \frac{1-\sin \theta}{\cos \theta} &= \frac{1-\cos(\frac{\pi}{2}-\theta)}{\sin(\frac{\pi}{2}-\theta)} = \frac{1-[1-2\sin^2(\frac{1}{2}(\frac{\pi}{2}-\theta))]}{2\sin(\frac{1}{2}(\frac{\pi}{2}-\theta))\cos(\frac{1}{2}(\frac{\pi}{2}-\theta))} \quad [\text{half-angle formulas}] \\ &= \frac{2\sin^2(\frac{1}{2}(\frac{\pi}{2}-\theta))}{2\sin(\frac{1}{2}(\frac{\pi}{2}-\theta))\cos(\frac{1}{2}(\frac{\pi}{2}-\theta))} = \tan(\frac{1}{2}(\frac{\pi}{2}-\theta)) \end{aligned}$$

Continuing:

$$\begin{aligned} \int_0^1 \int_0^1 \frac{dx dy}{1-xy} &= 4 \left[\int_0^{\pi/6} \arctan(\tan \theta) d\theta + \int_{\pi/6}^{\pi/2} \arctan(\tan(\frac{1}{2}(\frac{\pi}{2}-\theta))) d\theta \right] \\ &= 4 \left[\int_0^{\pi/6} \theta d\theta + \int_{\pi/6}^{\pi/2} \left[\frac{1}{2}(\frac{\pi}{2}-\theta) \right] d\theta \right] \\ &= 4 \left(\left[\frac{\theta^2}{2} \right]_0^{\pi/6} + \left[\frac{\pi\theta}{4} - \frac{\theta^2}{4} \right]_{\pi/6}^{\pi/2} \right) = 4 \left(\frac{3\pi^2}{72} \right) = \frac{\pi^2}{6} \end{aligned}$$

7. (a) Since $|xyz| < 1$ except at $(1, 1, 1)$, the formula for the sum of a geometric series gives $\frac{1}{1 - xyz} = \sum_{n=0}^{\infty} (xyz)^n$, so

$$\begin{aligned} \int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyz} dx dy dz &= \int_0^1 \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xyz)^n dx dy dz = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \int_0^1 (xyz)^n dx dy dz \\ &= \sum_{n=0}^{\infty} \left[\int_0^1 x^n dx \right] \left[\int_0^1 y^n dy \right] \left[\int_0^1 z^n dz \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{n+1} \cdot \frac{1}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3} \end{aligned}$$

- (b) Since $|-xyz| < 1$, except at $(1, 1, 1)$, the formula for the sum of a geometric series gives

$$\begin{aligned} \frac{1}{1 + xyz} &= \sum_{n=0}^{\infty} (-xyz)^n, \text{ so} \\ \int_0^1 \int_0^1 \int_0^1 \frac{1}{1 + xyz} dx dy dz &= \int_0^1 \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (-xyz)^n dx dy dz \\ &= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \int_0^1 (-xyz)^n dx dy dz \\ &= \sum_{n=0}^{\infty} (-1)^n \left[\int_0^1 x^n dx \right] \left[\int_0^1 y^n dy \right] \left[\int_0^1 z^n dz \right] \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \cdot \frac{1}{n+1} \cdot \frac{1}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \end{aligned}$$

To evaluate this sum, we first write out a few terms: $s = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} \approx 0.8998$. Notice that $a_7 = \frac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 12.5 [ET 11.5], we have $|s - s_6| \leq a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

8.
$$\begin{aligned} \int_0^{\infty} \frac{\arctan \pi x - \arctan x}{x} dx &= \int_0^{\infty} \left[\frac{\arctan yx}{x} \right]_{y=1}^{y=\pi} dx = \int_0^{\infty} \int_1^{\pi} \frac{1}{1 + y^2 x^2} dy dx \\ &= \int_1^{\pi} \int_0^{\infty} \frac{1}{1 + y^2 x^2} dx dy = \int_1^{\pi} \lim_{t \rightarrow \infty} \left[\frac{\arctan yx}{y} \right]_{x=0}^{x=t} dy \\ &= \int_1^{\pi} \frac{\pi}{2y} dy = \frac{\pi}{2} [\ln y]_1^{\pi} = \frac{\pi}{2} \ln \pi \end{aligned}$$

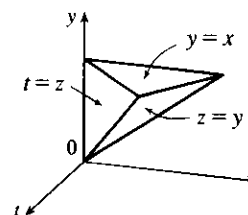
9. $\int_0^x \int_0^y \int_0^z f(t) dt dz dy = \iiint_E f(t) dV$, where

$$E = \{(t, z, y) \mid 0 \leq t \leq z, 0 \leq z \leq y, 0 \leq y \leq x\}.$$

If we let D be the projection of E on the yt -plane then

$$D = \{(y, t) \mid 0 \leq t \leq x, t \leq y \leq x\}.$$

And we see from the diagram that $E = \{(t, z, y) \mid t \leq z \leq y, t \leq y \leq x, 0 \leq t \leq x\}$. So



$$\begin{aligned} \int_0^x \int_0^y \int_0^z f(t) dt dz dy &= \int_0^x \int_t^x \int_t^y f(t) dz dy dt = \int_0^x \left[\int_t^x (y-t) f(t) dy \right] dt \\ &= \int_0^x \left[\left(\frac{1}{2} y^2 - ty \right) \right]_{y=t}^{y=x} dt = \int_0^x \left[\frac{1}{2} x^2 - tx - \frac{1}{2} t^2 + t^2 \right] f(t) dt \\ &= \int_0^x \left[\frac{1}{2} x^2 - tx + \frac{1}{2} t^2 \right] f(t) dt = \int_0^x \left(\frac{1}{2} x^2 - 2tx + t^2 \right) f(t) dt \\ &= \frac{1}{2} \int_0^x (x-t)^2 f(t) dt \end{aligned}$$

10. (a) Consider a polar division of the disk, similar to that in Figure 16.4.4 [ET 15.4.4], where

$0 = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_n = 2\pi$, $0 = r_1 < r_2 < \dots < r_m = R$, and where the polar subrectangle R_{ij} , as well as r_i^* , θ_j^* , Δr and $\Delta \theta$ are the same as in that figure. Thus $\Delta A_i = r_i^* \Delta r \Delta \theta$. The mass of R_{ij} is $\rho \Delta A_i$,

and its distance from m is $s_{ij} \approx \sqrt{(r_i^*)^2 + d^2}$. According to Newton's Law of Gravitation, the force of attraction experienced by m due to this polar subrectangle is in the direction from m towards R_{ij} and has

magnitude $\frac{Gm\rho \Delta A_i}{s_{ij}^2}$. The symmetry of the lamina with respect to the x - and y -axes and the position of m are

such that all horizontal components of the gravitational force cancel, so that the total force is simply in the z -direction. Thus, we need only be concerned with the components of this vertical force; that is,

$\frac{Gm\rho \Delta A_i}{s_{ij}^2} \sin \alpha$, where α is the angle between the origin, r_i^* and the mass m . Thus $\sin \alpha = \frac{d}{s_{ij}}$ and the

previous result becomes $\frac{Gm\rho d \Delta A_i}{s_{ij}^3}$. The total attractive force is just the Riemann sum

$$\sum_{i=1}^m \sum_{j=1}^n \frac{Gm\rho d \Delta A_i}{s_{ij}^3} = \sum_{i=1}^m \sum_{j=1}^n \frac{Gm\rho d (r_i^*) \Delta r \Delta \theta}{[(r_i^*)^2 + d^2]^{3/2}} \text{ which becomes } \int_0^R \int_0^{2\pi} \frac{Gm\rho d}{(r^2 + d^2)^{3/2}} r d\theta dr$$

as $m \rightarrow \infty$ and $n \rightarrow \infty$. Therefore,

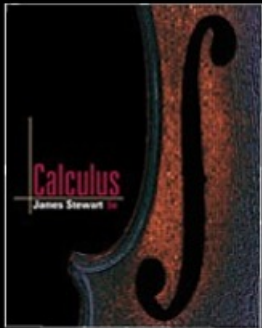
$$F = 2\pi Gm\rho d \int_0^R \frac{r}{(r^2 + d^2)^{3/2}} dr = 2\pi Gm\rho d \left[-\frac{1}{\sqrt{r^2 + d^2}} \right]_0^R = 2\pi Gm\rho d \left(\frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}} \right)$$

- (b) This is just the result of part (a) in the limit as $R \rightarrow \infty$. In this case $\frac{1}{\sqrt{R^2 + d^2}} \rightarrow 0$, and we are left with

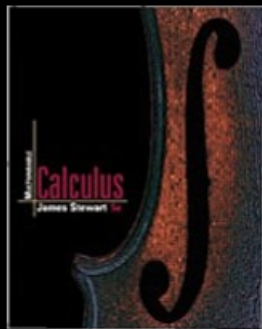
$$F = 2\pi Gm\rho d \left(\frac{1}{d} - 0 \right) = 2\pi Gm\rho.$$

Chapter 17

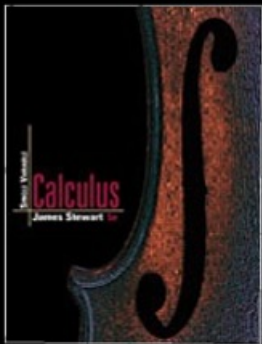
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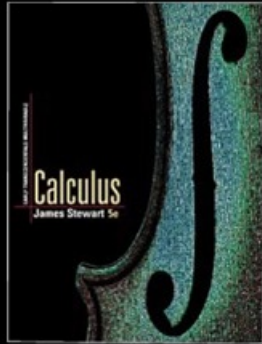
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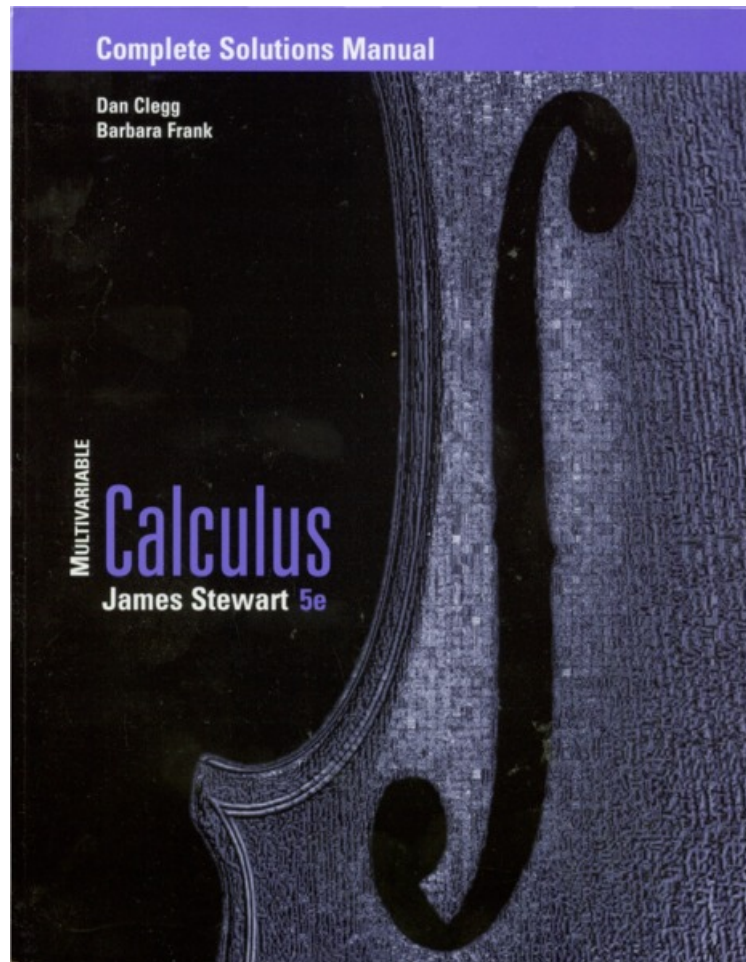
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17 □ VECTOR CALCULUS

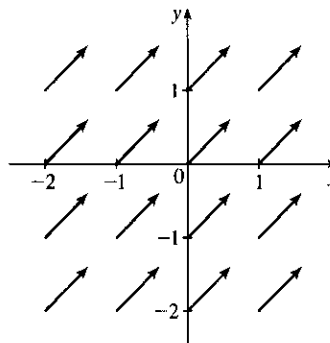
□ ET 16

17.1 Vector Fields

ET 16.1

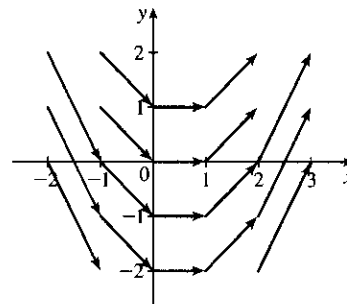
1. $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$

All vectors in this field are identical, with length $\frac{1}{\sqrt{2}}$ and direction parallel to the line $y = x$.



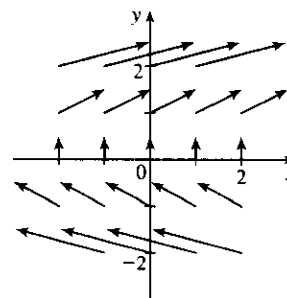
2. $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$

The length of the vector $\mathbf{i} + x\mathbf{j}$ is $\sqrt{1 + x^2}$. Vectors are tangent to parabolas opening about the y -axis.



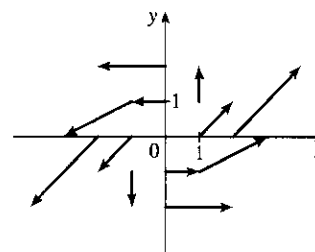
3. $\mathbf{F}(x, y) = y\mathbf{i} + \frac{1}{2}\mathbf{j}$

The length of the vector $y\mathbf{i} + \frac{1}{2}\mathbf{j}$ is $\sqrt{y^2 + \frac{1}{4}}$. Vectors are tangent to parabolas opening about the x -axis.



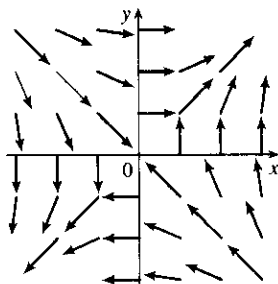
4. $\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$

The length of the vector $(x - y)\mathbf{i} + x\mathbf{j}$ is $\sqrt{(x - y)^2 + x^2}$. Vectors along the line $y = x$ are vertical.



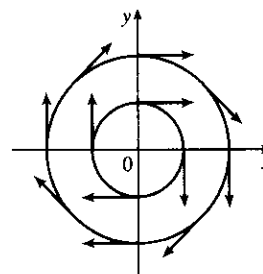
$$5. \mathbf{F}(x, y) = \frac{y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$$

The length of the vector $\frac{y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$ is 1.



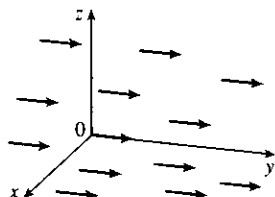
$$6. \mathbf{F}(x, y) = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$$

All the vectors $\mathbf{F}(x, y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.



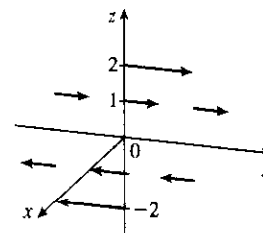
$$7. \mathbf{F}(x, y, z) = \mathbf{j}$$

All vectors in this field are parallel to the y -axis and have length 1.



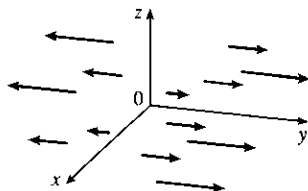
$$8. \mathbf{F}(x, y, z) = z\mathbf{j}$$

At each point (x, y, z) , $\mathbf{F}(x, y, z)$ is a vector of length $|z|$. For $z > 0$, all point in the direction of the positive y -axis while for $z < 0$, all are in the direction of the negative y -axis.



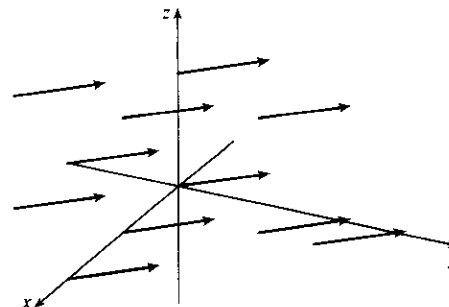
$$9. \mathbf{F}(x, y, z) = y\mathbf{j}$$

The length of $\mathbf{F}(x, y, z)$ is $|y|$. No vectors emanate from the xz -plane since $y = 0$ there. In each plane $y = b$, all the vectors are identical.



$$10. \mathbf{F}(x, y, z) = \mathbf{j} - \mathbf{i}$$

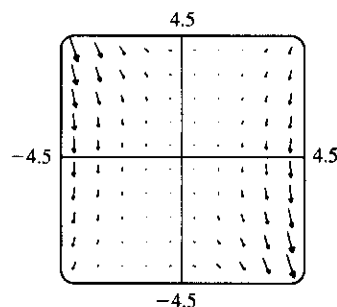
All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xy -plane.



11. $\mathbf{F}(x, y) = \langle y, x \rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x - and y -components, in the second quadrant all vectors have positive x -components and negative y -components, in the third quadrant all vectors have negative x - and y -components, and in the fourth quadrant all vectors have negative x -components and positive y -components. In addition, the vectors get shorter as we approach the origin.

12. $\mathbf{F}(x, y) = \langle 1, \sin y \rangle$ corresponds to graph IV since the x -component of each vector is constant, the vectors are independent of x (vectors along horizontal lines are identical), and the vector field appears to repeat the same pattern vertically.
13. $\mathbf{F}(x, y) = \langle x - 2, x + 1 \rangle$ corresponds to graph I since the vectors are independent of y (vectors along vertical lines are identical) and, as we move to the right, both the x - and the y -components get larger.
14. $\mathbf{F}(x, y) = \langle y, 1/x \rangle$ corresponds to graph III. As in Exercise 11, all the vectors in the first quadrant have positive x - and y -components, in the second quadrant all vectors have positive x -components and negative y -components, in the third quadrant all vectors have negative x - and y -components, and in the fourth quadrant all vectors have negative x -components and positive y -components. Also, the vectors become longer as we approach the y -axis.
15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.
16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + z\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy -plane point generally upward while the vectors below the xy -plane point generally downward.
17. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 3\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy -plane is $x\mathbf{i} + y\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z -components are all 3.
18. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x, y, z)$ has the same length and direction as the position vector of the point (x, y, z) , and therefore the vectors all point directly away from the origin.

19.

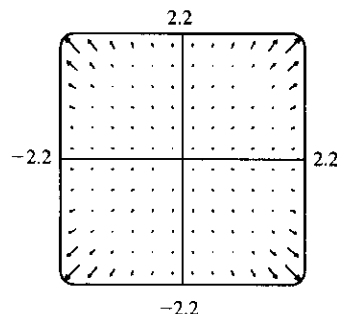


The vector field seems to have very short vectors near the line $y = 2x$.

For $\mathbf{F}(x, y) = \langle 0, 0 \rangle$ we must have $y^2 - 2xy = 0$ and $3xy - 6x^2 = 0$.

The first equation holds if $y = 0$ or $y = 2x$, and the second holds if $x = 0$ or $y = 2x$. So both equations hold [and thus $\mathbf{F}(x, y) = \mathbf{0}$] along the line $y = 2x$.

20.



From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|x| = 2$ and near the origin. Note that $\mathbf{F}(\mathbf{x}) = \mathbf{0} \Leftrightarrow r(r - 2) = 0 \Leftrightarrow r = 0$ or 2 , so as we suspected, $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ for $|x| = 2$ and for $|x| = 0$. Note that where $r^2 - r < 0$, the vectors point towards the origin, and where $r^2 - r > 0$, they point away from the origin.

$$21. \nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \frac{1}{x + 2y}\mathbf{i} + \frac{2}{x + 2y}\mathbf{j}$$

$$22. \nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = [x^\alpha(-\beta e^{-\beta x}) + \alpha x^{\alpha-1}e^{-\beta x}]\mathbf{i} + 0\mathbf{j} = (\alpha - \beta x)x^{\alpha-1}e^{-\beta x}\mathbf{i}$$

$$23. \nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \\ = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k}$$

$$24. \nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

$$= \left(\cos \frac{y}{z}\right)\mathbf{i} - x\left(\sin \frac{y}{z}\right)\left(\frac{1}{z}\right)\mathbf{j} - x\left(\sin \frac{y}{z}\right)\left(-\frac{y}{z^2}\right)\mathbf{k}$$

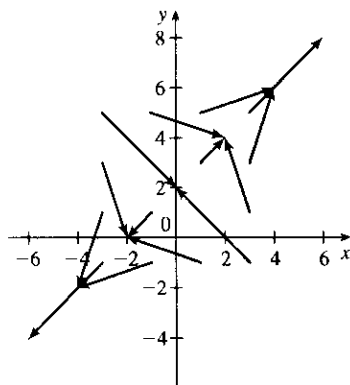
$$= \left(\cos \frac{y}{z}\right)\mathbf{i} - \frac{x}{z}\left(\sin \frac{y}{z}\right)\mathbf{j} + \frac{xy}{z^2}\left(\sin \frac{y}{z}\right)\mathbf{k}$$

$$25. f(x, y) = xy - 2x \Rightarrow$$

$$\nabla f(x, y) = (y - 2)\mathbf{i} + x\mathbf{j}.$$

The length of $\nabla f(x, y)$ is $\sqrt{(y - 2)^2 + x^2}$ and

$\nabla f(x, y)$ terminates on the line $y = x + 2$ at the point $(x + y - 2, x + y)$.

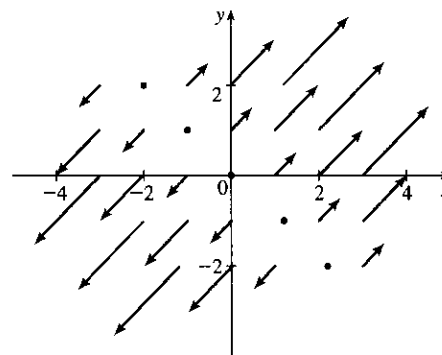


$$26. f(x, y) = \frac{1}{4}(x + y)^2 \Rightarrow$$

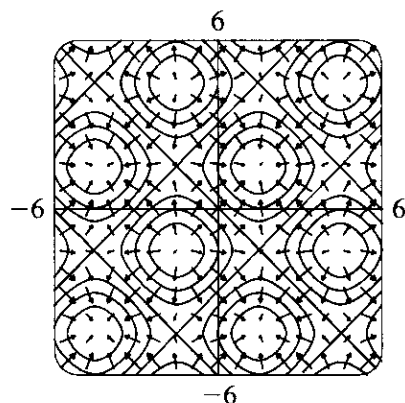
$$\nabla f(x, y) = \frac{1}{2}(x + y)\mathbf{i} + \frac{1}{2}(x + y)\mathbf{j}.$$

The length of $\nabla f(x, y)$ is

$\sqrt{\frac{1}{2}(x + y)^2} = \frac{1}{\sqrt{2}}|x + y|$. The vectors are perpendicular to the line $y = -x$ and point away from the line, with length that increases as the distance from the line $y = -x$ increases.

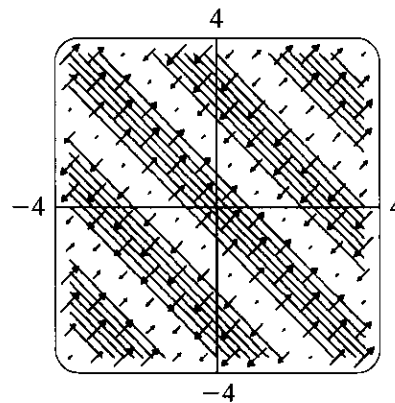


27. We graph ∇f along with a contour map of f .



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

28. We graph ∇f along with a contour map of f .

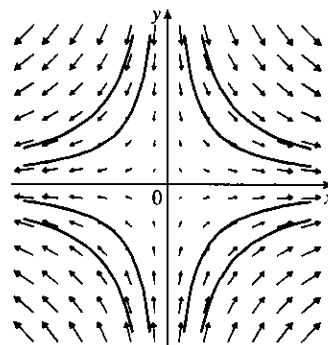


The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

29. $f(x, y) = xy \Rightarrow \nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$. In the first quadrant, both components of each vector are positive, while in the third quadrant both components are negative. However, in the second quadrant each vector's x -component is positive while its y -component is negative (and vice versa in the fourth quadrant). Thus, ∇f is graph IV.
30. $f(x, y) = x^2 - y^2 \Rightarrow \nabla f(x, y) = 2x\mathbf{i} - 2y\mathbf{j}$. In the first quadrant, the x -component of each vector is positive while the y -component is negative. The other three quadrants are similar, where the x -component of each vector has the same sign as the x -value of its initial point, and the y -component has sign opposite that of the y -value of the initial point. Thus, ∇f is graph III.
31. $f(x, y) = x^2 + y^2 \Rightarrow \nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$. Thus, each vector $\nabla f(x, y)$ has the same direction and twice the length of the position vector of the point (x, y) , so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.
32. $f(x, y) = \sqrt{x^2 + y^2} \Rightarrow \nabla f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$. Then

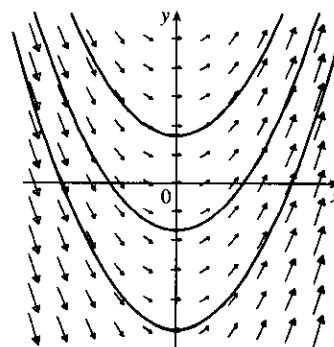
$|\nabla f(x, y)| = \frac{1}{\sqrt{x^2 + y^2}} \sqrt{x^2 + y^2} = 1$, so all vectors are unit vectors. In addition, each vector $\nabla f(x, y)$ has the same direction as the position vector of the point (x, y) , so the vectors all point directly away from the origin. Hence, ∇f is graph I.

33. (a) We sketch the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



- (b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\mathbf{i} + y'(t)\mathbf{j} = x\mathbf{i} - y\mathbf{j} \Rightarrow dx/dt = x, dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \Rightarrow dx/x = dt \Rightarrow \ln|x| = t + C \Rightarrow x = \pm e^{t+C} = Ae^t$ for some constant A , and $dy/dt = -y \Rightarrow dy/y = -dt \Rightarrow \ln|y| = -t + K \Rightarrow y = \pm e^{-t+K} = Be^{-t}$ for some constant B . Therefore $xy = Ae^t Be^{-t} = AB = \text{constant}$. If the flow line passes through $(1, 1)$ then $(1)(1) = \text{constant} = 1 \Rightarrow xy = 1 \Rightarrow y = 1/x, x > 0$.

34. (a) We sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be parabolas.



- (b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\mathbf{i} + y'(t)\mathbf{j} = \mathbf{i} + x\mathbf{j} \Rightarrow \frac{dx}{dt} = 1, \frac{dy}{dt} = x$. Thus $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x}{1} = x$.
- (c) From part (b), $dy/dx = x$. Integrating, we have $y = \frac{1}{2}x^2 + c$. Since the particle starts at the origin, we know $(0, 0)$ is on the curve, so $0 = 0 + c \Rightarrow c = 0$ and the path the particle follows is $y = \frac{1}{2}x^2$.

17.2 Line Integrals

ET 16.2

1. $x = t^2$ and $y = t, 0 \leq t \leq 2$, so by Formula 3

$$\begin{aligned}\int_C y \, ds &= \int_0^2 t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^2 t \sqrt{(2t)^2 + (1)^2} \, dt \\ &= \int_0^2 t \sqrt{4t^2 + 1} \, dt = \frac{1}{12} (4t^2 + 1)^{3/2} \Big|_0^2 = \frac{1}{12} (17\sqrt{17} - 1)\end{aligned}$$

2. $\int_C \frac{y}{x} \, ds = \int_{1/2}^1 \frac{t^3}{t^4} \sqrt{(4t^3)^2 + (3t^2)^2} \, dt = \int_{1/2}^1 \frac{1}{t} \sqrt{16t^6 + 9t^4} \, dt = \int_{1/2}^1 t \sqrt{16t^2 + 9} \, dt$
 $= \frac{1}{48} (16t^2 + 9)^{3/2} \Big|_{1/2}^1 = \frac{1}{48} (25^{3/2} - 13^{3/2}) = \frac{1}{48} (125 - 13\sqrt{13})$

3. Parametric equations for C are $x = 4 \cos t, y = 4 \sin t, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Then

$$\begin{aligned}\int_C xy^4 \, ds &= \int_{-\pi/2}^{\pi/2} (4 \cos t)(4 \sin t)^4 \sqrt{(-4 \sin t)^2 + (4 \cos t)^2} \, dt \\ &= \int_{-\pi/2}^{\pi/2} 4^5 \cos t \sin^4 t \sqrt{16(\sin^2 t + \cos^2 t)} \, dt \\ &= 4^5 \int_{-\pi/2}^{\pi/2} (\sin^4 t \cos t)(4) \, dt = (4)^6 \left[\frac{1}{5} \sin^5 t \right]_{-\pi/2}^{\pi/2} = \frac{2 \cdot 4^6}{5} = 1638.4\end{aligned}$$

4. Parametric equations for C are $x = 1 + 3t, y = 2 + 5t, 0 \leq t \leq 1$. Then

$$\int_C ye^x \, ds = \int_0^1 (2 + 5t) e^{1+3t} \sqrt{3^2 + 5^2} \, dt = \sqrt{34} \int_0^1 (2 + 5t) e^{1+3t} \, dt$$

Integrating by parts with $u = 2 + 5t \Rightarrow du = 5 \, dt, dv = e^{1+3t} \Rightarrow v = \frac{1}{3} e^{1+3t}$ gives

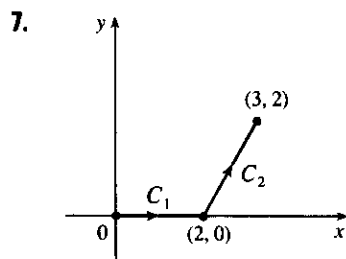
$$\begin{aligned}\int_C ye^x \, ds &= \sqrt{34} \left[\frac{1}{3} (2 + 5t) e^{1+3t} - \frac{5}{9} e^{1+3t} \right]_0^1 \\ &= \sqrt{34} \left[\left(\frac{7}{3} - \frac{5}{9} \right) e^4 - \left(\frac{2}{3} - \frac{5}{9} \right) e \right] = \frac{\sqrt{34}}{9} (16e^4 - e)\end{aligned}$$

5. If we choose x as the parameter, parametric equations for C are $x = x, y = x^2$ for $1 \leq x \leq 3$ and

$$\begin{aligned}\int_C (xy + \ln x) \, dy &= \int_1^3 (x \cdot x^2 + \ln x) 2x \, dx = \int_1^3 2(x^4 + x \ln x) \, dx \\ &= 2 \left[\frac{1}{5} x^5 + \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right]_1^3 \quad (\text{by integrating by parts in the second term}) \\ &= 2 \left(\frac{243}{5} + \frac{9}{2} \ln 3 - \frac{9}{4} - \frac{1}{5} + \frac{1}{4} \right) = \frac{464}{5} + 9 \ln 3\end{aligned}$$

6. Choosing y as the parameter, we have $x = e^y, y = y, 0 \leq y \leq 1$. Then

$$\int_C xe^y \, dx = \int_0^1 e^y (e^y) e^y \, dy = \int_0^1 e^{3y} \, dy = \frac{1}{3} e^{3y} \Big|_0^1 = \frac{1}{3} (e^3 - 1).$$



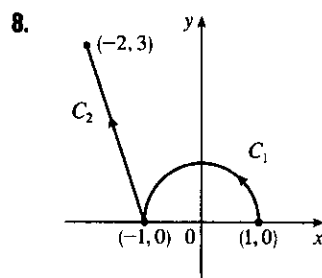
$$C = C_1 + C_2$$

$$\text{On } C_1: x = x, y = 0 \Rightarrow dy = 0 dx, 0 \leq x \leq 2.$$

$$\text{On } C_2: x = x, y = 2x - 4 \Rightarrow dy = 2 dx, 2 \leq x \leq 3.$$

Then

$$\begin{aligned} \int_C xy \, dx + (x - y) \, dy &= \int_{C_1} xy \, dx + (x - y) \, dy + \int_{C_2} xy \, dx + (x - y) \, dy \\ &= \int_0^2 (0 + 0) \, dx + \int_2^3 [(2x^2 - 4x) + (-x + 4)(2)] \, dx \\ &= \int_2^3 (2x^2 - 6x + 8) \, dx = \frac{17}{3} \end{aligned}$$



$$C = C_1 + C_2$$

$$\text{On } C_1: x = \cos t \Rightarrow dx = -\sin t \, dt, y = \sin t \Rightarrow$$

$$dy = \cos t \, dt, 0 \leq t \leq \pi.$$

$$\text{On } C_2: x = -1 - t \Rightarrow dx = -dt, y = 3t \Rightarrow$$

$$dy = 3 \, dt, 0 \leq t \leq 1.$$

$$\begin{aligned} \text{Then } \int_C \sin x \, dx + \cos y \, dy &= \int_{C_1} \sin x \, dx + \cos y \, dy + \int_{C_2} \sin x \, dx + \cos y \, dy \\ &= \int_0^\pi \sin(\cos t)(-\sin t \, dt) + \cos(\sin t) \cos t \, dt \\ &\quad + \int_0^1 \sin(-1 - t)(-dt) + \cos(3t)(3 \, dt) \\ &= [-\cos(\cos t) + \sin(\sin t)]_0^\pi + [-\cos(-1 - t) + \sin(3t)]_0^1 \\ &= -\cos(\cos \pi) + \sin(\sin \pi) + \cos(\cos 0) - \sin(\sin 0) \\ &\quad - \cos(-2) + \sin(3) + \cos(-1) - \sin(0) \\ &= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2) + \sin 3 + \cos(-1) \\ &= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3 \end{aligned}$$

where we have used the identity $\cos(-\theta) = \cos \theta$.

9. $x = 4 \sin t, y = 4 \cos t, z = 3t, 0 \leq t \leq \frac{\pi}{2}$. Then by Formula 9,

$$\begin{aligned} \int_C xy^3 \, ds &= \int_0^{\pi/2} (4 \sin t)(4 \cos t)^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \\ &= \int_0^{\pi/2} 4^4 \cos^3 t \sin t \sqrt{(4 \cos t)^2 + (-4 \sin t)^2 + (3)^2} \, dt \\ &= \int_0^{\pi/2} 256 \cos^3 t \sin t \sqrt{16(\cos^2 t + \sin^2 t) + 9} \, dt \\ &= 1280 \int_0^{\pi/2} \cos^3 t \sin t \, dt = -320 \cos^4 t \Big|_0^{\pi/2} = 320 \end{aligned}$$

10. Parametric equations for C are $x = 4t$, $y = 6 - 5t$, $z = -1 + 6t$, $0 \leq t \leq 1$. Then

$$\begin{aligned}\int_C x^2 z \, ds &= \int_0^1 (4t)^2 (6t - 1) \sqrt{4^2 + (-5)^2 + 6^2} \, dt = \sqrt{77} \int_0^1 (96t^3 - 16t^2) \, dt \\ &= \sqrt{77} \left[96 \cdot \frac{t^4}{4} - 16 \cdot \frac{t^3}{3} \right]_0^1 = \frac{56}{3} \sqrt{77}\end{aligned}$$

11. Parametric equations for C are $x = t$, $y = 2t$, $z = 3t$, $0 \leq t \leq 1$. Then

$$\begin{aligned}\int_C x e^{yz} \, ds &= \int_0^1 t e^{(2t)(3t)} \sqrt{1^2 + 2^2 + 3^2} \, dt = \sqrt{14} \int_0^1 t e^{6t^2} \, dt \\ &= \sqrt{14} \left[\frac{1}{12} e^{6t^2} \right]_0^1 = \frac{\sqrt{14}}{12} (e^6 - 1)\end{aligned}$$

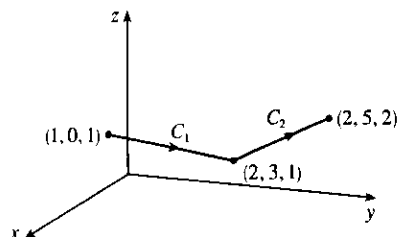
12. $\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$. Then

$$\begin{aligned}\int_C (2x + 9z) \, ds &= \int_0^1 (2t + 9t^3) \sqrt{1 + 4t^2 + 9t^4} \, dt \quad [\text{let } u = 1 + 4t^2 + 9t^4 \Rightarrow \frac{1}{4} du = (2t + 9t^3) \, dt] \\ &= \int_1^{14} \frac{1}{4} \sqrt{u} \, du = \frac{1}{6} u^{3/2} \Big|_1^{14} = \frac{1}{6} (14^{3/2} - 1)\end{aligned}$$

13. $\int_C x^2 y \sqrt{z} \, dz = \int_0^1 (t^3)^2 (t) \sqrt{t^2} \cdot 2t \, dt = \int_0^1 2t^9 \, dt = \frac{1}{5} t^{10} \Big|_0^1 = \frac{1}{5}$

14. $\int_C z \, dx + x \, dy + y \, dz = \int_0^1 t^2 \cdot 2t \, dt + t^2 \cdot 3t^2 \, dt + t^3 \cdot 2t \, dt = \int_0^1 (2t^3 + 5t^4) \, dt$
 $= \left[\frac{1}{2} t^4 + t^5 \right]_0^1 = \frac{1}{2} + 1 = \frac{3}{2}$

15.

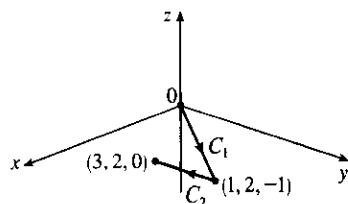


On C_1 : $x = 1 + t \Rightarrow dx = dt$, $y = 3t \Rightarrow dy = 3 \, dt$, $z = 1$
 $\Rightarrow dz = 0 \, dt$, $0 \leq t \leq 1$.

On C_2 : $x = 2 \Rightarrow dx = 0 \, dt$, $y = 3 + 2t \Rightarrow dy = 2 \, dt$, $z = 1 + t \Rightarrow dz = dt$, $0 \leq t \leq 1$.

Then $\int_C (x + yz) \, dx + 2x \, dy + xyz \, dz$
 $= \int_{C_1} (x + yz) \, dx + 2x \, dy + xyz \, dz + \int_{C_2} (x + yz) \, dx + 2x \, dy + xyz \, dz$
 $= \int_0^1 (1 + t + (3t)(1)) \, dt + 2(1 + t) \cdot 3 \, dt + (1 + t)(3t)(1) \cdot 0 \, dt$
 $+ \int_0^1 (2 + (3 + 2t)(1 + t)) \cdot 0 \, dt + 2(2) \cdot 2 \, dt + (2)(3 + 2t)(1 + t) \, dt$
 $= \int_0^1 (10t + 7) \, dt + \int_0^1 (4t^2 + 10t + 14) \, dt$
 $= \left[5t^2 + 7t \right]_0^1 + \left[\frac{4}{3} t^3 + 5t^2 + 14t \right]_0^1 = 12 + \frac{61}{3} = \frac{97}{3}$

16.



On C_1 : $x = t \Rightarrow dx = dt$, $y = 2t \Rightarrow dy = 2 \, dt$, $z = -t$
 $\Rightarrow dz = -dt$, $0 \leq t \leq 1$.

On C_2 : $x = 1 + 2t \Rightarrow dx = 2 \, dt$, $y = 2 \Rightarrow dy = 0 \, dt$, $z = -1 + t \Rightarrow dz = dt$, $0 \leq t \leq 1$.

$$\begin{aligned}
 \text{Then } \int_C x^2 dx + y^2 dy + z^2 dz &= \int_{C_1} x^2 dx + y^2 dy + z^2 dz + \int_{C_2} x^2 dx + y^2 dy + z^2 dz \\
 &= \int_0^1 t^2 dt + (2t)^2 \cdot 2 dt + (-t)^2(-dt) + \int_0^1 (1+2t)^2 \cdot 2 dt + 2^2 \cdot 0 dt + (-1+t)^2 dt \\
 &= \int_0^1 8t^2 dt + \int_0^1 (9t^2 + 6t + 3) dt = \left[\frac{8}{3}t^3 \right]_0^1 + [3t^3 + 3t^2 + 3t]_0^1 = \frac{35}{3}
 \end{aligned}$$

17. (a) Along the line $x = -3$, the vectors of $\mathbf{F}$ have positive y -components, so since the path goes upward, the integrand $\mathbf{F} \cdot \mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T} ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F} \cdot \mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T} ds$ is negative.

18. Vectors starting on C_1 point in roughly the same direction as C_1 , so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is positive.

Then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T} ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2 , while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T} ds$ to be negative.

19. $\mathbf{r}(t) = t^2 \mathbf{i} - t^3 \mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) = (t^2)^2(-t^3)^3 \mathbf{i} - (-t^3) \sqrt{t^2} \mathbf{j} = -t^{13} \mathbf{i} + t^4 \mathbf{j}$ and $\mathbf{r}'(t) = 2t \mathbf{i} - 3t^2 \mathbf{j}$.

$$\text{Thus } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^1 (-2t^{14} - 3t^6) dt = \left[-\frac{2}{15}t^{15} - \frac{3}{7}t^7 \right]_0^1 = -\frac{59}{105}.$$

20. $\mathbf{F}(\mathbf{r}(t)) = (t^2)(t^3) \mathbf{i} + (t)(t^3) \mathbf{j} + (t)(t^2) \mathbf{k} = t^5 \mathbf{i} + t^4 \mathbf{j} + t^3 \mathbf{k}$, $\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}$.

$$\text{Thus } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^2 (t^5 + 2t^5 + 3t^5) dt = t^6 \Big|_0^2 = 64.$$

21. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle \sin t^3, \cos(-t^2), t^4 \rangle \cdot \langle 3t^2, -2t, 1 \rangle dt$

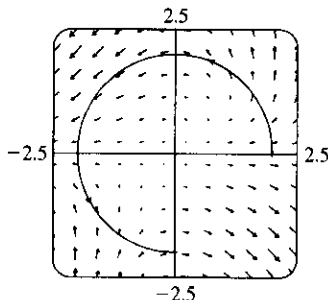
$$= \int_0^1 (3t^2 \sin t^3 - 2t \cos t^2 + t^4) dt = \left[-\cos t^3 - \sin t^2 + \frac{1}{5}t^5 \right]_0^1 = \frac{6}{5} - \cos 1 - \sin 1$$

22. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \langle \cos t, \sin t, -t \rangle \cdot \langle 1, \cos t, -\sin t \rangle dt = \int_0^\pi (\cos t + \sin t \cos t + t \sin t) dt$

$$= \left[\sin t + \frac{1}{2} \sin^2 t + (\sin t - t \cos t) \right]_0^\pi = \pi$$

23. We graph $\mathbf{F}(x, y) = (x - y) \mathbf{i} + xy \mathbf{j}$ and the curve C . We see that most of the vectors starting on C point in roughly the same direction as C , so for these portions of C the tangential component $\mathbf{F} \cdot \mathbf{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$ to be positive.

To verify, we evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$, $0 \leq t \leq \frac{3\pi}{2}$, so $\mathbf{F}(\mathbf{r}(t)) = (2 \cos t - 2 \sin t) \mathbf{i} + 4 \cos t \sin t \mathbf{j}$ and $\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$. Then

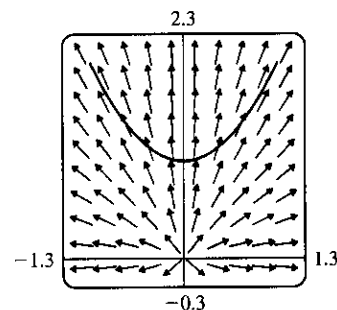


$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
 &= \int_0^{3\pi/2} [-2 \sin t(2 \cos t - 2 \sin t) + 2 \cos t(4 \cos t \sin t)] dt \\
 &= 4 \int_0^{3\pi/2} (\sin^2 t - \sin t \cos t + 2 \sin t \cos^2 t) dt \\
 &= 3\pi + \frac{2}{3} \quad [\text{using a CAS}]
 \end{aligned}$$

24. We graph $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j}$ and the curve C .

In the first quadrant, each vector starting on C points in roughly the same direction as C , so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is positive.

In the second quadrant, each vector starting on C points in roughly the direction opposite to C , so $\mathbf{F} \cdot \mathbf{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants



counteract each other, so it seems reasonable to guess that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$ is zero. To verify, we evaluate

$\int_C \mathbf{F} \cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = t\mathbf{i} + (1+t^2)\mathbf{j}$, $-1 \leq t \leq 1$, so

$$\mathbf{F}(\mathbf{r}(t)) = \frac{t}{\sqrt{t^2 + (1+t^2)^2}} \mathbf{i} + \frac{1+t^2}{\sqrt{t^2 + (1+t^2)^2}} \mathbf{j} \text{ and } \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}. \text{ Then}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_{-1}^1 \left(\frac{t}{\sqrt{t^2 + (1+t^2)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2 + (1+t^2)^2}} \right) dt \\ &= \int_{-1}^1 \frac{t(3+2t^2)}{\sqrt{t^4 + 3t^2 + 1}} dt = 0 \quad [\text{since the integrand is an odd function}] \end{aligned}$$

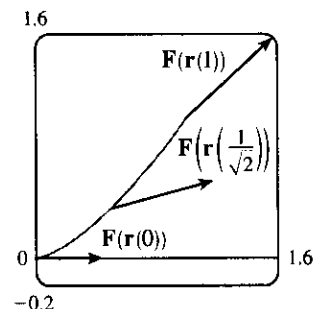
25. (a) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle e^{t^2-1}, t^5 \rangle \cdot \langle 2t, 3t^2 \rangle dt = \int_0^1 (2te^{t^2-1} + 3t^7) dt = \left[e^{t^2-1} + \frac{3}{8}t^8 \right]_0^1 = \frac{11}{8} - 1/e$

(b) $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{F}(\mathbf{r}(0)) = \langle e^{-1}, 0 \rangle$;

$$\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}} \right\rangle, \mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}} \right\rangle;$$

$$\mathbf{r}(1) = \langle 1, 1 \rangle, \mathbf{F}(\mathbf{r}(1)) = \langle 1, 1 \rangle.$$

In order to generate the graph with Maple, we use the PLOT command (not to be confused with the plot command) to define each of the vectors. For example,

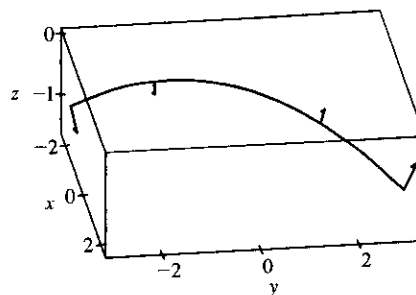
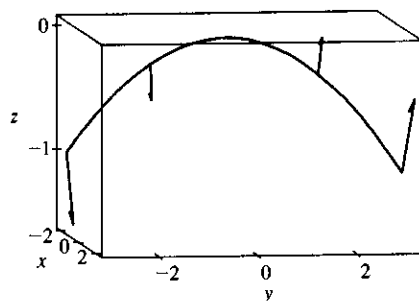


$$v1 := \text{PLOT}(\text{CURVES}([[0, 0], [\text{evalf}(1/\exp(1)), 0]]));$$

generates the vector from the vector field at the point $(0, 0)$ (but without an arrowhead) and gives it the name $v1$. To show everything on the same screen, we use the display command. In Mathematica, we use ListPlot (with the PlotJoined $\rightarrow$ True option) to generate the vectors, and then Show to show everything on the same screen.

26. (a) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^1 \langle 2t, t^2, 3t \rangle \cdot \langle 2, 3, -2t \rangle dt = \int_{-1}^1 (4t + 3t^2 - 6t^2) dt$
 $= [2t^2 - t^3]_{-1}^1 = -2$

- (b) Now $\mathbf{F}(\mathbf{r}(t)) = \langle 2t, t^2, 3t \rangle$, so $\mathbf{F}(\mathbf{r}(-1)) = \langle -2, 1, -3 \rangle$, $\mathbf{F}(\mathbf{r}(-\frac{1}{2})) = \langle -1, \frac{1}{4}, -\frac{3}{2} \rangle$, $\mathbf{F}(\mathbf{r}(\frac{1}{2})) = \langle 1, \frac{1}{4}, \frac{3}{2} \rangle$, and $\mathbf{F}(\mathbf{r}(1)) = \langle 2, 1, 3 \rangle$.



27. The part of the astroid that lies in the quadrant is parametrized by $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq \frac{\pi}{2}$.

Now $\frac{dx}{dt} = 3\cos^2 t(-\sin t)$ and $\frac{dy}{dt} = 3\sin^2 t \cos t$, so

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\cos t \sin t \sqrt{\cos^2 t + \sin^2 t} = 3\cos t \sin t.$$

$$\text{Therefore } \int_C x^3 y^5 ds = \int_0^{\pi/2} \cos^9 t \sin^{15} t (3\cos t \sin t) dt = \frac{945}{16,777,216} \pi.$$

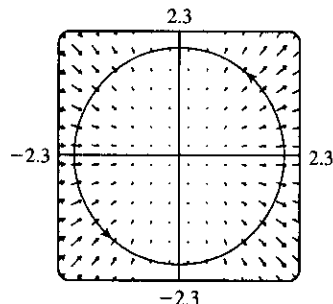
28. We parametrize the line as $\mathbf{r}(t) = \langle 1, 2, 1 \rangle + t(\langle 6, 4, 5 \rangle - \langle 1, 2, 1 \rangle) = (1 + 5t)\mathbf{i} + (2 + 2t)\mathbf{j} + (1 + 4t)\mathbf{k}$, $0 \leq t \leq 1$. Using a CAS, we calculate

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left\langle (1 + 5t)^4 e^{2+2t}, \ln(1 + 4t), \sqrt{(2 + 2t)^2 + (1 + 4t)^2} \right\rangle \cdot \langle 5, 2, 4 \rangle dt \\ &= \frac{5235e^4}{4} - \frac{6285e^2}{4} + \frac{9\sqrt{5} \sinh^{-1}\left(\frac{14}{3}\right)}{25} - \frac{9\sqrt{5} \sinh^{-1}\left(\frac{4}{3}\right)}{25} + \frac{5 \ln 5}{2} + \frac{14\sqrt{41}}{5} - \frac{4\sqrt{5}}{5} - 2 \\ &= \frac{5235e^4}{4} - \frac{6285e^2}{4} - \frac{18\sqrt{5} \ln 3}{25} + \frac{9\sqrt{5} \ln(14 + \sqrt{205})}{25} + \frac{5 \ln 5}{2} + \frac{14\sqrt{41} - 4\sqrt{5}}{5} - 2 \end{aligned}$$

The first answer is the one given by Maple. The two answers are equivalent by Equation 7.6.3 [ET 3.9.3].

29. A calculator or CAS gives $\int_C x \sin y ds = \int_1^2 \ln t \sin(e^{-t}) \sqrt{(1/t)^2 + (-e^{-t})^2} dt \approx 0.052$.
30. (a) We parametrize the circle C as $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$. So $\mathbf{F}(\mathbf{r}(t)) = \langle 4 \cos^2 t, 4 \cos t \sin t \rangle$, $\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t \rangle$, and $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-8 \cos^2 t \sin t + 8 \cos^2 t \sin t) dt = 0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C , $\mathbf{F} \cdot \mathbf{T} = 0$, and so certainly $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

31. We use the parametrization $x = 2 \cos t$, $y = 2 \sin t$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

$$\text{Then } ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} dt = 2 dt, \text{ so}$$

$$m = \int_C k ds = 2k \int_{-\pi/2}^{\pi/2} dt = 2k(\pi), \quad \bar{x} = \frac{1}{2\pi k} \int_C xk ds = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (2 \cos t) 2 dt = \frac{1}{2\pi} [4 \sin t]_{-\pi/2}^{\pi/2} = \frac{4}{\pi},$$

$$\bar{y} = \frac{1}{2\pi k} \int_C yk ds = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (2 \sin t) 2 dt = 0. \text{ Hence } (\bar{x}, \bar{y}) = \left(\frac{4}{\pi}, 0\right).$$

32. We use the parametrization $x = r \cos t$, $y = r \sin t$, $0 \leq t \leq \frac{\pi}{2}$.

$$\text{Then } ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt = r dt, \text{ so}$$

$$m = \int_C (x + y) ds = \int_0^{\pi/2} (r \cos t + r \sin t) r dt = r^2 [\sin t - \cos t]_0^{\pi/2} = 2r^2,$$

$$\begin{aligned} \bar{x} &= \frac{1}{2r^2} \int_C x(x + y) ds = \frac{1}{2r^2} \int_0^{\pi/2} (r^2 \cos^2 t + r^2 \cos t \sin t) r dt = \frac{r}{2} \left[\frac{t}{2} + \frac{\sin 2t}{4} - \frac{\cos 2t}{4} \right]_0^{\pi/2} \\ &= \frac{r(\pi + 2)}{8}, \text{ and} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{2r^2} \int_C y(x + y) ds = \frac{1}{2r^2} \int_0^{\pi/2} (r^2 \sin t \cos t + r^2 \sin^2 t) r dt \\ &= \frac{r}{2} \left[-\frac{\cos 2t}{4} + \frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{\pi/2} = \frac{r(\pi + 2)}{8}. \end{aligned}$$

$$\text{Therefore } (\bar{x}, \bar{y}) = \left(\frac{r(\pi + 2)}{8}, \frac{r(\pi + 2)}{8} \right).$$

33. (a) $\bar{x} = \frac{1}{m} \int_C x \rho(x, y, z) ds$, $\bar{y} = \frac{1}{m} \int_C y \rho(x, y, z) ds$, $\bar{z} = \frac{1}{m} \int_C z \rho(x, y, z) ds$ where $m = \int_C \rho(x, y, z) ds$.

$$(b) m = \int_C k ds = k \int_0^{2\pi} \sqrt{4 \sin^2 t + 4 \cos^2 t + 9} dt = k \sqrt{13} \int_0^{2\pi} dt = 2\pi k \sqrt{13},$$

$$\bar{x} = \frac{1}{2\pi k \sqrt{13}} \int_0^{2\pi} k 2 \sqrt{13} \sin t dt = 0, \quad \bar{y} = \frac{1}{2\pi k \sqrt{13}} \int_0^{2\pi} k 2 \sqrt{13} \cos t dt = 0,$$

$$\bar{z} = \frac{1}{2\pi k \sqrt{13}} \int_0^{2\pi} (k \sqrt{13}) (3t) dt = \frac{3}{2\pi} (2\pi^2) = 3\pi. \text{ Hence } (\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3\pi).$$

34. $m = \int_C (x^2 + y^2 + z^2) ds = \int_0^{2\pi} (t^2 + 1) \sqrt{(1)^2 + (-\sin t)^2 + (\cos t)^2} dt = \int_0^{2\pi} (t^2 + 1) \sqrt{2} dt$
 $= \sqrt{2} \left(\frac{8}{3} \pi^3 + 2\pi \right),$

$$\bar{x} = \frac{1}{\sqrt{2} \left(\frac{8}{3} \pi^3 + 2\pi \right)} \int_0^{2\pi} \sqrt{2} (t^3 + t) dt = \frac{4\pi^4 + 2\pi^2}{\frac{8}{3} \pi^3 + 2\pi} = \frac{3\pi(2\pi^2 + 1)}{4\pi^2 + 3},$$

$$\bar{y} = \frac{3}{2\sqrt{2}\pi(4\pi^2 + 3)} \int_0^{2\pi} (\sqrt{2} \cos t) (t^2 + 1) dt = 0, \text{ and}$$

$$\bar{z} = \frac{3}{2\sqrt{2}\pi(4\pi^2 + 3)} \int_0^{2\pi} (\sqrt{2} \sin t) (t^2 + 1) dt = 0. \text{ Hence } (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{3\pi(2\pi^2 + 1)}{4\pi^2 + 3}, 0, 0 \right).$$

35. From Example 3, $\rho(x, y) = k(1 - y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \leq t \leq \pi \Rightarrow$

$$\begin{aligned} I_x &= \int_C y^2 \rho(x, y) ds = \int_0^\pi \sin^2 t [k(1 - \sin t)] dt = k \int_0^\pi (\sin^2 t - \sin^3 t) dt \\ &= \frac{1}{2} k \int_0^\pi (1 - \cos 2t) dt - k \int_0^\pi (1 - \cos^2 t) \sin t dt \quad \left[\begin{array}{l} \text{Let } u = \cos t, du = -\sin t dt \\ \text{in the second integral} \end{array} \right] \\ &= k \left[\frac{\pi}{2} + \int_1^{-1} (1 - u^2) du \right] = k \left(\frac{\pi}{2} - \frac{4}{3} \right) \end{aligned}$$

$$\begin{aligned} I_y &= \int_C x^2 \rho(x, y) ds = k \int_0^\pi \cos^2 t (1 - \sin t) dt = \frac{k}{2} \int_0^\pi (1 + \cos 2t) dt - k \int_0^\pi \cos^2 t \sin t dt \\ &= k \left(\frac{\pi}{2} - \frac{2}{3} \right), \text{ using the same substitution as above.} \end{aligned}$$

36. The wire is given as $x = 2 \sin t$, $y = 2 \cos t$, $z = 3t$, $0 \leq t \leq 2\pi$ with $\rho(x, y, z) = k$. Then

$$ds = \sqrt{(2 \cos t)^2 + (-2 \sin t)^2 + 3^2} = \sqrt{4(\cos^2 t + \sin^2 t) + 9} = \sqrt{13} \text{ and}$$

$$\begin{aligned} I_x &= \int_C (y^2 + z^2) \rho(x, y, z) ds = \int_0^{2\pi} (4 \cos^2 t + 9t^2)(k) \sqrt{13} dt = \sqrt{13} k \left[4 \left(\frac{1}{2} t + \frac{1}{4} \sin 2t \right) + 3t^3 \right]_0^{2\pi} \\ &= \sqrt{13} k (4\pi + 24\pi^3) = 4\sqrt{13} \pi k (1 + 6\pi^2) \end{aligned}$$

$$\begin{aligned} I_y &= \int_C (x^2 + z^2) \rho(x, y, z) ds = \int_0^{2\pi} (4 \sin^2 t + 9t^2)(k) \sqrt{13} dt = \sqrt{13} k \left[4 \left(\frac{1}{2} t - \frac{1}{4} \sin 2t \right) + 3t^3 \right]_0^{2\pi} \\ &= \sqrt{13} k (4\pi + 24\pi^3) = 4\sqrt{13} \pi k (1 + 6\pi^2) \end{aligned}$$

$$I_z = \int_C (x^2 + y^2) \rho(x, y, z) ds = \int_0^{2\pi} (4 \sin^2 t + 4 \cos^2 t)(k) \sqrt{13} dt = 4\sqrt{13} k \int_0^{2\pi} dt = 8\pi \sqrt{13} k$$

37. $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle t - \sin t, 3 - \cos t \rangle \cdot \langle 1 - \cos t, \sin t \rangle dt$
 $= \int_0^{2\pi} (t - t \cos t - \sin t + \sin t \cos t + 3 \sin t - \sin t \cos t) dt$
 $= \int_0^{2\pi} (t - t \cos t + 2 \sin t) dt = \left[\frac{1}{2} t^2 - (t \sin t + \cos t) - 2 \cos t \right]_0^{2\pi} \quad \left[\begin{array}{l} \text{by integrating by parts} \\ \text{in the second term} \end{array} \right]$
 $= 2\pi^2$

38. $x = x$, $y = x^2$, $-1 \leq x \leq 2$,

$$\begin{aligned} W &= \int_{-1}^2 \langle x \sin x^2, x^2 \rangle \cdot \langle 1, 2x \rangle dx = \int_{-1}^2 (x \sin x^2 + 2x^3) dx = \left[-\frac{1}{2} \cos x^2 + \frac{1}{2} x^4 \right]_{-1}^2 \\ &= \frac{1}{2} (15 + \cos 1 - \cos 4) \end{aligned}$$

39. $\mathbf{r}(t) = \langle 1 + 2t, 4t, 2t \rangle$, $0 \leq t \leq 1$,

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 6t, 1 + 4t, 1 + 6t \rangle \cdot \langle 2, 4, 2 \rangle dt = \int_0^1 (12t + 4(1 + 4t) + 2(1 + 6t)) dt \\ &= \int_0^1 (40t + 6) dt = [20t^2 + 6t]_0^1 = 26 \end{aligned}$$

40. $\mathbf{r}(t) = 2\mathbf{i} + t\mathbf{j} + 5t\mathbf{k}$, $0 \leq t \leq 1$. Therefore

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{K \langle 2, t, 5t \rangle}{(4 + 26t^2)^{3/2}} \cdot \langle 0, 1, 5 \rangle dt = K \int_0^1 \frac{26t}{(4 + 26t^2)^{3/2}} dt \\ &= K \left[-(4 + 26t^2)^{-1/2} \right]_0^1 = K \left(\frac{1}{2} - \frac{1}{\sqrt{30}} \right) \end{aligned}$$

41. Let
- $\mathbf{F} = 185\mathbf{k}$
- . To parametrize the staircase, let

$$x = 20 \cos t, y = 20 \sin t, z = \frac{90}{6\pi}t = \frac{15}{\pi}t, 0 \leq t \leq 6\pi \Rightarrow$$

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \langle 0, 0, 185 \rangle \cdot \langle -20 \sin t, 20 \cos t, \frac{15}{\pi} \rangle dt = (185) \frac{15}{\pi} \int_0^{6\pi} dt = (185)(90) \\ &\approx 1.67 \times 10^4 \text{ ft}\cdot\text{lb} \end{aligned}$$

42. This time
- m
- is a function of
- t
- :
- $m = 185 - \frac{9}{6\pi}t = 185 - \frac{3}{2\pi}t$
- . So let
- $\mathbf{F} = (185 - \frac{3}{2\pi}t)\mathbf{k}$
- . To parametrize the staircase, let
- $x = 20 \cos t, y = 20 \sin t, z = \frac{90}{6\pi}t = \frac{15}{\pi}t, 0 \leq t \leq 6\pi$
- . Therefore

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \langle 0, 0, 185 - \frac{3}{2\pi}t \rangle \cdot \langle -20 \sin t, 20 \cos t, \frac{15}{\pi} \rangle dt = \frac{15}{\pi} \int_0^{6\pi} (185 - \frac{3}{2\pi}t) dt \\ &= \frac{15}{\pi} [185t - \frac{3}{4\pi}t^2]_0^{6\pi} = 90(185 - \frac{9}{2}) \approx 1.62 \times 10^4 \text{ ft}\cdot\text{lb} \end{aligned}$$

43. (a)
- $\mathbf{r}(t) = \langle \cos t, \sin t \rangle, 0 \leq t \leq 2\pi$
- , and let
- $\mathbf{F} = \langle a, b \rangle$
- . Then

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle a, b \rangle \cdot \langle -\sin t, \cos t \rangle dt \\ &= \int_0^{2\pi} (-a \sin t + b \cos t) dt = [a \cos t + b \sin t]_0^{2\pi} \\ &= a + 0 - a + 0 = 0 \end{aligned}$$

(b) Yes. $\mathbf{F}(x, y) = k\mathbf{x} = \langle kx, ky \rangle$ and

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle k \cos t, k \sin t \rangle \cdot \langle -\sin t, \cos t \rangle dt \\ &= \int_0^{2\pi} (-k \sin t \cos t + k \sin t \cos t) dt = \int_0^{2\pi} 0 dt = 0 \end{aligned}$$

44. Consider the base of the fence in the
- xy
- plane, centered at the origin, with the height given by
- $z = h(x, y)$
- . The fence can be graphed using the parametric equations

$$x = 10 \cos u, y = 10 \sin u,$$

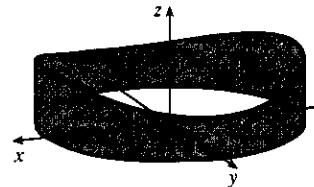
$$\begin{aligned} z &= v[4 + 0.01((10 \cos u)^2 - (10 \sin u)^2)] \\ &= v(4 + \cos^2 u - \sin^2 u) \\ &= v(4 + \cos 2u), 0 \leq u \leq 2\pi, 0 \leq v \leq 1. \end{aligned}$$

The area of the fence is $\int_C h(x, y) ds$ where C , the base of the fence, is given by $x = 10 \cos t$, $y = 10 \sin t, 0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \int_C h(x, y) ds &= \int_0^{2\pi} [4 + 0.01((10 \cos t)^2 - (10 \sin t)^2)] \sqrt{(-10 \sin t)^2 + (10 \cos t)^2} dt \\ &= \int_0^{2\pi} (4 + \cos 2t) \sqrt{100} dt = 10[4t + \frac{1}{2} \sin 2t]_0^{2\pi} \\ &= 10(8\pi) = 80\pi \text{ m}^2 \end{aligned}$$

If we paint both sides of the fence, the total surface area to cover is $160\pi \text{ m}^2$, and since 1 L of paint covers 100 m^2 , we require $\frac{160\pi}{100} = 1.6\pi \approx 5.03 \text{ L}$ of paint.

45. The work done in moving the object is
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$
- . We can approximate this integral by dividing
- C
- into 7 segments of equal length
- $\Delta s = 2$
- and approximating
- $\mathbf{F} \cdot \mathbf{T}$
- , that is, the tangential component of force, at a



point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F} \cdot \mathbf{T}$ is the scalar projection of each force vector onto C . If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is $\int_C \mathbf{F} \cdot \mathbf{T} \, ds \approx \sum_{i=1}^7 [\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)] \Delta s = [2 + 2 + 2 + 2 + 1 + 1 + 1](2) = 22$. Thus, we estimate the work done to be approximately 22 J.

46. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}| \mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle $C: x = r \cos \theta, y = r \sin \theta$. Thus $\mathbf{B} = |\mathbf{B}| \langle -\sin \theta, \cos \theta \rangle$. Then

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}| \langle -\sin \theta, \cos \theta \rangle \cdot \langle -r \sin \theta, r \cos \theta \rangle d\theta = \int_0^{2\pi} |\mathbf{B}| r \, d\theta = 2\pi r |\mathbf{B}|. \quad (\text{Note that } |\mathbf{B}| \text{ here is the magnitude of the field at a distance } r \text{ from the wire's center.})$$

But by Ampere's Law $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I / (2\pi r)$.

17.3 The Fundamental Theorem for Line Integrals

ET 16.3

- C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C . From the graph, this is $50 - 10 = 40$.
- C is represented by the vector function $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^3 + t)\mathbf{j}$, $0 \leq t \leq 1$, so $\mathbf{r}'(t) = 2t\mathbf{i} + (3t^2 + 1)\mathbf{j}$. Since $3t^2 + 1 \neq 0$, we have $\mathbf{r}'(t) \neq \mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2, 2) - f(1, 0) = 9 - 3 = 6$.
- $\partial(6x + 5y)/\partial y = 5 = \partial(5x + 4y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x, y) = 6x + 5y$ and $f_y(x, y) = 5x + 4y$. But $f_x(x, y) = 6x + 5y$ implies $f(x, y) = 3x^2 + 5xy + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x, y) = 5x + g'(y)$. Thus $5x + 4y = 5x + g'(y)$ so $g'(y) = 4y$ and $g(y) = 2y^2 + K$ where K is a constant. Hence $f(x, y) = 3x^2 + 5xy + 2y^2 + K$ is a potential function for $\mathbf{F}$.
- $\partial(x^3 + 4xy)/\partial y = 4x$, $\partial(4xy - y^3)/\partial x = 4y$. Since these are not equal, $\mathbf{F}$ is not conservative.
- $\partial(xe^y)/\partial y = xe^y$, $\partial(ye^x)/\partial x = ye^x$. Since these are not equal, $\mathbf{F}$ is not conservative.
- $\partial(e^y)/\partial y = e^y = \partial(xe^y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = e^y$ implies $f(x, y) = xe^y + g(y)$ and $f_y(x, y) = xe^y + g'(y)$. But $f_y(x, y) = xe^y$ so $g'(y) = 0 \Rightarrow g(y) = K$. Then $f(x, y) = xe^y + K$ is a potential function for $\mathbf{F}$.
- $\partial(2x \cos y - y \cos x)/\partial y = -2x \sin y - \cos x = \partial(-x^2 \sin y - \sin x)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = 2x \cos y - y \cos x$ implies $f(x, y) = x^2 \cos y - y \sin x + g(y)$ and $f_y(x, y) = -x^2 \sin y - \sin x + g'(y)$. But $f_y(x, y) = -x^2 \sin y - \sin x$ so $g'(y) = 0 \Rightarrow g(y) = K$. Then $f(x, y) = x^2 \cos y - y \sin x + K$ is a potential function for $\mathbf{F}$.
- $\partial(1 + 2xy + \ln x)/\partial y = 2x = \partial(x^2)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x, y) \mid x > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = 1 + 2xy + \ln x$ implies $f(x, y) = x + x^2y + x \ln x - x + g(y)$ and $f_y(x, y) = x^2 + g'(y)$. But $f_y(x, y) = x^2$ so $g'(y) = 0 \Rightarrow g(y) = K$. Then $f(x, y) = x^2y + x \ln x + K$ is a potential function for $\mathbf{F}$.

9. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x \cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = ye^x + \sin y$ implies $f(x, y) = ye^x + x \sin y + g(y)$ and $f_y(x, y) = e^x + x \cos y + g'(y)$. But $f_y(x, y) = e^x + x \cos y$ so $g'(y) = K$ and $f(x, y) = ye^x + x \sin y + K$ is a potential function for $\mathbf{F}$.
10. $\frac{\partial(xy \cosh xy + \sinh xy)}{\partial y} = x^2 y \sinh xy + x \cosh xy + x \cosh xy = x^2 y \sinh xy + 2x \cosh xy = \frac{\partial(x^2 \cosh xy)}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Thus $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = xy \cosh xy + \sinh xy$ implies $f(x, y) = x \sinh xy + g(y) \Rightarrow f_y(x, y) = x^2 \cosh xy + g'(y)$. But $f_y(x, y) = x^2 \cosh xy$ so $g'(y) = K$ and $f(x, y) = x \sinh xy + K$ is a potential function for $\mathbf{F}$.
11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\frac{\partial}{\partial y} 2xy = 2x = \frac{\partial}{\partial x} (x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C . Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.
- (b) We first find a potential function f , so that $\nabla f = \mathbf{F}$. We know $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Integrating $f_x(x, y)$ with respect to x , we have $f(x, y) = x^2 y + g(y)$. Differentiating both sides with respect to y gives $f_y(x, y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \Rightarrow g'(y) = 0 \Rightarrow g(y) = K$, a constant. Thus $f(x, y) = x^2 y + K$. All three curves start at $(1, 2)$ and end at $(3, 2)$, so by Theorem 2, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 2) - f(1, 2) = 18 - 2 = 16$ for each curve.
12. (a) $f_x(x, y) = y$ implies $f(x, y) = xy + g(y)$ and $f_y(x, y) = x + g'(y)$. But $f_y(x, y) = x + 2y$ so $g'(y) = 2y \Rightarrow g(y) = y^2 + K$. We can take $K = 0$, so $f(x, y) = xy + y^2$.
- (b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 1) - f(0, 1) = 3 - 1 = 2$.
13. (a) $f_x(x, y) = x^3 y^4$ implies $f(x, y) = \frac{1}{4} x^4 y^4 + g(y)$ and $f_y(x, y) = x^4 y^3 + g'(y)$. But $f_y(x, y) = x^4 y^3$ so $g'(y) = 0 \Rightarrow g(y) = K$, a constant. We can take $K = 0$, so $f(x, y) = \frac{1}{4} x^4 y^4$.
- (b) The initial point of C is $\mathbf{r}(0) = (0, 1)$ and the terminal point is $\mathbf{r}(1) = (1, 2)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2) - f(0, 1) = 4 - 0 = 4$.
14. (a) $f_x(x, y) = y^2/(1+x^2)$ implies $f(x, y) = y^2 \arctan x + g(y) \Rightarrow f_y(x, y) = 2y \arctan x + g'(y)$. But $f_y(x, y) = 2y \arctan x$ so $g'(y) = 0 \Rightarrow g(y) = K$. We can take $K = 0$, so $f(x, y) = y^2 \arctan x$.
- (b) The initial point of C is $\mathbf{r}(0) = (0, 0)$ and the terminal point is $\mathbf{r}(1) = (1, 2)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2) - f(0, 0) = 4 \arctan 1 - 0 = 4 \cdot \frac{\pi}{4} = \pi$.
15. (a) $f_x(x, y, z) = yz$ implies $f(x, y, z) = xyz + g(y, z)$ and so $f_y(x, y, z) = xz + g_y(y, z)$. But $f_y(x, y, z) = xz$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = xyz + h(z)$ and $f_z(x, y, z) = xy + h'(z)$. But $f_z(x, y, z) = xy + 2z$, so $h'(z) = 2z \Rightarrow h(z) = z^2 + K$. Hence $f(x, y, z) = xyz + z^2$ (taking $K = 0$).
- (b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 6, 3) - f(1, 0, -2) = 81 - 4 = 77$.
16. (a) $f_x(x, y, z) = 2xz + y^2$ implies $f(x, y, z) = x^2 z + xy^2 + g(y, z)$ and so $f_y(x, y, z) = 2xy + g_y(y, z)$. But $f_y(x, y, z) = 2xy$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = x^2 z + xy^2 + h(z)$ and $f_z(x, y, z) = x^2 + h'(z)$. But $f_z(x, y, z) = x^2 + 3z^2$, so $h'(z) = 3z^2 \Rightarrow h(z) = z^3 + K$. Hence $f(x, y, z) = x^2 z + xy^2 + z^3$ (taking $K = 0$).
- (b) $t = 0$ corresponds to the point $(0, 1, -1)$ and $t = 1$ corresponds to $(1, 2, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, -1) = 6 - (-1) = 7$.

17. (a) $f_x(x, y, z) = y^2 \cos z$ implies $f(x, y, z) = xy^2 \cos z + g(y, z)$ and so $f_y(x, y, z) = 2xy \cos z + g_y(y, z)$. But $f_y(x, y, z) = 2xy \cos z$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = xy^2 \cos z + h(z)$ and $f_z(x, y, z) = -xy^2 \sin z + h'(z)$. But $f_z(x, y, z) = -xy^2 \sin z$, so $h'(z) = 0 \Rightarrow h(z) = K$. Hence $f(x, y, z) = xy^2 \cos z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}(\pi) = \langle \pi^2, 0, \pi \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi^2, 0, \pi) - f(0, 0, 0) = 0 - 0 = 0$.

18. (a) $f_x(x, y, z) = e^y$ implies $f(x, y, z) = xe^y + g(y, z)$ and so $f_y(x, y, z) = xe^y + g_y(y, z)$. But $f_y(x, y, z) = xe^y$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = xe^y + h(z)$ and $f_z(x, y, z) = 0 + h'(z)$. But $f_z(x, y, z) = (z+1)e^z$, so $h'(z) = (z+1)e^z \Rightarrow h(z) = ze^z + K$ (using integration by parts). Hence $f(x, y, z) = xe^y + ze^z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}(1) = \langle 1, 1, 1 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = 2e - 0 = 2e$.

19. Here $\mathbf{F}(x, y) = \tan y \mathbf{i} + x \sec^2 y \mathbf{j}$. Then $f(x, y) = x \tan y$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C \tan y dx + x \sec^2 y dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2, \frac{\pi}{4}) - f(1, 0) = 2 \tan \frac{\pi}{4} - \tan 0 = 2$.

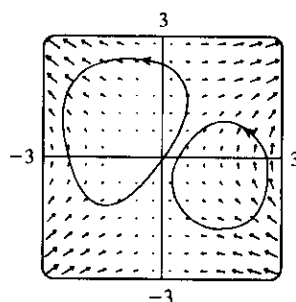
20. Here $\mathbf{F}(x, y) = (1 - ye^{-x}) \mathbf{i} + e^{-x} \mathbf{j}$. Then $f(x, y) = x + ye^{-x}$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C (1 - ye^{-x}) dx + e^{-x} dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2) - f(0, 1) = (1 + 2e^{-1}) - 1 = 2/e$.

21. $\mathbf{F}(x, y) = 2y^{3/2} \mathbf{i} + 3x\sqrt{y} \mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(2y^{3/2})/\partial y = 3\sqrt{y} = \partial(3x\sqrt{y})/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x, y) = 2y^{3/2} \Rightarrow f(x, y) = 2xy^{3/2} + g(y) \Rightarrow f_y(x, y) = 3xy^{1/2} + g'(y)$. But $f_y(x, y) = 3x\sqrt{y}$ so $g'(y) = 0$ or $g(y) = K$. We can take $K = 0 \Rightarrow f(x, y) = 2xy^{3/2}$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 4) - f(1, 1) = 2(2)(8) - 2(1) = 30$.

22. $\mathbf{F}(x, y) = \frac{y^2}{x^2} \mathbf{i} - \frac{2y}{x} \mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\frac{\partial}{\partial y} \left(\frac{y^2}{x^2} \right) = \frac{2y}{x^2} = \frac{\partial}{\partial x} \left(-\frac{2y}{x} \right)$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = y^2/x^2 \Rightarrow f(x, y) = -y^2/x + g(y) \Rightarrow f_y = -2y/x + g'(y) \Rightarrow g'(y) = 0$, so we can take $f(x, y) = -y^2/x$ as a potential function for $\mathbf{F}$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(4, -2) - f(1, 1) = -[(-2)^2/4] + (1/1) = 0$.

23. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C , $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be some circle centered at the origin, oriented counterclockwise. All of the field vectors along C oppose motion along C , so the integral around C will be negative. Therefore the field is not conservative.

24.

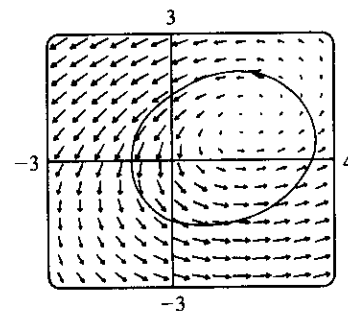


From the graph, it appears that $\mathbf{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate

$$\frac{\partial}{\partial y} (2xy + \sin y) = 2x + \cos y, \quad \frac{\partial}{\partial x} (x^2 + x \cos y) = 2x + \cos y$$

Thus $\mathbf{F}$ is conservative, by Theorem 6.

25. From the graph, it appears that $\mathbf{F}$ is not conservative. For example, any closed curve containing the point $(2, 1)$ seems to have many field vectors pointing counterclockwise along it, and none pointing clockwise. So along this path the integral $\int \mathbf{F} \cdot d\mathbf{r} \neq 0$. To confirm our guess, we calculate



$$\frac{\partial}{\partial y} \left(\frac{x-2y}{\sqrt{1+x^2+y^2}} \right) = (x-2y) \left[\frac{-y}{(1+x^2+y^2)^{3/2}} \right] - \frac{2}{\sqrt{1+x^2+y^2}} = \frac{-2-2x^2-xy}{(1+x^2+y^2)^{3/2}},$$

$$\frac{\partial}{\partial x} \left(\frac{x-2}{\sqrt{1+x^2+y^2}} \right) = (x-2) \left[\frac{-x}{(1+x^2+y^2)^{3/2}} \right] + \frac{1}{\sqrt{1+x^2+y^2}} = \frac{1+y^2+2x}{(1+x^2+y^2)^{3/2}}.$$

These are not equal, so the field is not conservative, by Theorem 5.

26. $\nabla f(x, y) = \cos(x-2y)\mathbf{i} - 2\cos(x-2y)\mathbf{j}$

- (a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0, 0) = \sin 0 = 0$ and $f(\pi, \pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0, 0)$ to (π, π) ; that is, $\mathbf{r}(t) = \pi t \mathbf{i} + \pi t \mathbf{j}$, $0 \leq t \leq 1$.
- (b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0, 0) = \sin 0 = 0$ and $f(\frac{\pi}{2}, 0) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2}t \mathbf{i}$, $0 \leq t \leq 1$, the straight line from $(0, 0)$ to $(\frac{\pi}{2}, 0)$.

27. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P , Q and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P / \partial y = f_{xy} = f_{yx} = \partial Q / \partial x$, $\partial P / \partial z = f_{xz} = f_{zx} = \partial R / \partial x$, and $\partial Q / \partial z = f_{yz} = f_{zy} = \partial R / \partial y$.

28. Here $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$. Then using the notation of Exercise 27, $\partial P / \partial z = 0$ while $\partial R / \partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

29. $D = \{(x, y) \mid x > 0, y > 0\}$ = the first quadrant (excluding the axes).

- (a) D is open because around every point in D we can put a disk that lies in D .
- (b) D is connected because the straight line segment joining any two points in D lies in D .
- (c) D is simply-connected because it's connected and has no holes.

30. $D = \{(x, y) \mid x \neq 0\}$ consists of all points in the xy -plane except for those on the y -axis.

- (a) D is open.
- (b) Points on opposite sides of the y -axis cannot be joined by a path that lies in D , so D is not connected.
- (c) D is not simply-connected because it is not connected.

31. $D = \{(x, y) \mid 1 < x^2 + y^2 < 4\}$ = the annular region between the circles with center $(0, 0)$ and radii 1 and 2.

- (a) D is open.
- (b) D is connected.
- (c) D is not simply-connected. For example, $x^2 + y^2 = (1.5)^2$ is simple and closed and lies within D but encloses points that are not in D . (Or we can say, D has a hole, so is not simply-connected.)

32. $D = \{(x, y) \mid x^2 + y^2 \leq 1 \text{ or } 4 \leq x^2 + y^2 \leq 9\}$ = the points on or inside the circle $x^2 + y^2 = 1$, together with the points on or between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.
- (a) D is not open because, for instance, no disk with center $(0, 2)$ lies entirely within D .
- (b) D is not connected because, for example, $(0, 0)$ and $(0, 2.5)$ lie in D but cannot be joined by a path that lies entirely in D .
- (c) D is not simply-connected because, for example, $x^2 + y^2 = 9$ is a simple closed curve in D but encloses points that are not in D .

33. (a) $P = -\frac{y}{x^2 + y^2}$, $\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$ and $Q = \frac{x}{x^2 + y^2}$, $\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$. Thus $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

(b) $C_1: x = \cos t, y = \sin t, 0 \leq t \leq \pi$, $C_2: x = \cos t, y = \sin t, t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t} dt = \int_0^\pi dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^\pi dt = -\pi$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi \text{ where } C_3 \text{ is the circle } x^2 + y^2 = 1, \text{ and apply the contrapositive of Theorem 3.)$$

This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

34. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 17.1 [ET 16.1].) Hence $\mathbf{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = -\frac{c}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} + \frac{c}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = c\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

(b) In this case, $c = -(mMG) \Rightarrow$

$$\begin{aligned} W &= -mMG \left(\frac{1}{1.52 \times 10^8} - \frac{1}{1.47 \times 10^8} \right) \\ &= -(5.97 \times 10^{24})(1.99 \times 10^{30})(6.67 \times 10^{-11})(-2.2377 \times 10^{-10}) \approx 1.77 \times 10^{35} \text{ J} \end{aligned}$$

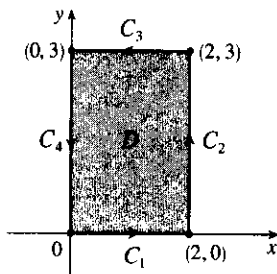
(c) In this case, $c = \epsilon qQ \Rightarrow$

$$W = \epsilon qQ \left(\frac{1}{10^{-12}} - \frac{1}{5 \times 10^{-13}} \right) = (8.985 \times 10^{10})(1)(-1.6 \times 10^{-19})(-10^{12}) \approx 1.4 \times 10^4 \text{ J}.$$

17.4 Green's Theorem

ET 16.4

1. (a)



$$C_1: x = t \Rightarrow dx = dt, y = 0 \Rightarrow dy = 0 dt, 0 \leq t \leq 2.$$

$$C_2: x = 2 \Rightarrow dx = 0 dt, y = t \Rightarrow dy = dt, 0 \leq t \leq 3.$$

$$C_3: x = 2 - t \Rightarrow dx = -dt, y = 3 \Rightarrow dy = 0 dt, 0 \leq t \leq 2.$$

$$C_4: x = 0 \Rightarrow dx = 0 dt, y = 3 - t \Rightarrow dy = -dt, 0 \leq t \leq 3.$$

$$\begin{aligned} \text{Thus } \oint_C xy^2 dx + x^3 dy &= \oint_{C_1 + C_2 + C_3 + C_4} xy^2 dx + x^3 dy \\ &= \int_0^2 0 dt + \int_0^3 8 dt + \int_0^2 -9(2-t) dt + \int_0^3 0 dt \\ &= 0 + 24 - 18 + 0 = 6 \end{aligned}$$

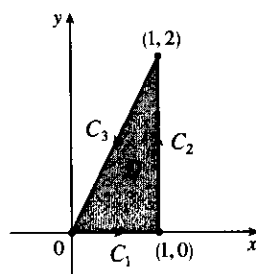
$$\begin{aligned} \text{(b)} \oint_C xy^2 dx + x^3 dy &= \iint_D \left[\frac{\partial}{\partial x} (x^3) - \frac{\partial}{\partial y} (xy^2) \right] dA = \int_0^2 \int_0^3 (3x^2 - 2xy) dy dx \\ &= \int_0^2 (9x^2 - 9x) dx = 24 - 18 = 6 \end{aligned}$$

2. (a) $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$. Then

$$\oint_C y dx - x dy = \int_0^{2\pi} [\sin t(-\sin t) - \cos t(\cos t)] dt = -\int_0^{2\pi} dt = -2\pi.$$

$$\text{(b)} \oint_C y dx - x dy = \iint_D \left[\frac{\partial}{\partial x} (-x) - \frac{\partial}{\partial y} (y) \right] dA = -2 \iint_D dA = -2A(D) = -2\pi(1)^2 = -2\pi$$

3. (a)



$$C_1: x = t \Rightarrow dx = dt, y = 0 \Rightarrow dy = 0 dt, 0 \leq t \leq 1.$$

$$C_2: x = 1 \Rightarrow dx = 0 dt, y = t \Rightarrow dy = dt, 0 \leq t \leq 2.$$

$$C_3: x = 1 - t \Rightarrow dx = -dt, y = 2 - 2t \Rightarrow dy = -2 dt, 0 \leq t \leq 1.$$

$$\begin{aligned} \text{Thus } \oint_C xy dx + x^2 y^3 dy &= \oint_{C_1+C_2+C_3} xy dx + x^2 y^3 dy \\ &= \int_0^1 0 dt + \int_0^2 t^3 dt + \int_0^1 [-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3] dt \\ &= 0 + \left[\frac{1}{4} t^4 \right]_0^2 + \left[\frac{2}{3} (1-t)^3 + \frac{8}{3} (1-t)^6 \right]_0^1 = 4 - \frac{10}{3} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{(b)} \oint_C xy dx + x^2 y^3 dy &= \iint_D \left[\frac{\partial}{\partial x} (x^2 y^3) - \frac{\partial}{\partial y} (xy) \right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x) dy dx \\ &= \int_0^1 \left[\frac{1}{2} xy^4 - xy \right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2) dx = \frac{4}{3} - \frac{2}{3} = \frac{2}{3} \end{aligned}$$

4. (a) $C_1: x = 0 \Rightarrow dx = 0 dt, y = 1 - t \Rightarrow$

$$dy = -dt, 0 \leq t \leq 1$$

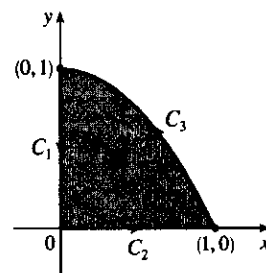
$$C_2: x = t \Rightarrow dx = dt, y = 0 \Rightarrow dy = 0 dt, 0 \leq t \leq 1$$

$$C_3: x = 1 - t \Rightarrow dx = -dt, y = 1 - (1 - t)^2 = 2t - t^2 \Rightarrow$$

$$dy = (2 - 2t) dt, 0 \leq t \leq 1$$

Thus

$$\begin{aligned} \oint_C x dx + y dy &= \oint_{C_1+C_2+C_3} x dx + y dy \\ &= \int_0^1 (0 dt + (1-t)(-dt)) + \int_0^1 (t dt + 0 dt) + \int_0^1 ((1-t)(-dt) + (2t - t^2)(2 - 2t) dt) \\ &= \left[\frac{1}{2} t^2 - t \right]_0^1 + \left[\frac{1}{2} t^2 \right]_0^1 + \left[\frac{1}{2} t^4 - 2t^3 + \frac{5}{2} t^2 - t \right]_0^1 \\ &= -\frac{1}{2} + \frac{1}{2} + \left(\frac{1}{2} - 2 + \frac{5}{2} - 1 \right) = 0 \end{aligned}$$



$$\text{(b)} \oint_C x dx + y dy = \iint_D \left[\frac{\partial}{\partial x} (y) - \frac{\partial}{\partial y} (x) \right] dA = \iint_D 0 dA = 0$$

5. We can parametrize C as $x = \cos \theta$, $y = \sin \theta$, $0 \leq \theta \leq 2\pi$. Then the line integral is

$$\oint_C P dx + Q dy = \int_0^{2\pi} \cos^4 \theta \sin^5 \theta (-\sin \theta) d\theta + \int_0^{2\pi} (-\cos^7 \theta \sin^6 \theta) \cos \theta d\theta = -\frac{29\pi}{1024}, \text{ according to a CAS.}$$

The double integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-7x^6 y^6 - 5x^4 y^4) dy dx = -\frac{29\pi}{1024},$$

verifying Green's Theorem in this case.

6. Since $y = x^2$ along the first part of C and $y = x$ along the second part, the line integral is

$$\begin{aligned}\oint_C P dx + Q dy &= \int_0^1 [x^4 \sin x + x^2 \sin(x^2)(2x)] dx + \int_1^0 (x^2 \sin x + x^2 \sin x) dx \\ &= -16 \cos 1 - 23 \sin 1 + 28\end{aligned}$$

according to a CAS. The double integral is

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_{x^2}^x (2x \sin y - 2y \sin x) dy dx = -16 \cos 1 - 23 \sin 1 + 28$$

7. The region D enclosed by C is $[0, 1] \times [0, 1]$, so

$$\begin{aligned}\int_C e^y dx + 2xe^y dy &= \iint_D \left[\frac{\partial}{\partial x} (2xe^y) - \frac{\partial}{\partial y} (e^y) \right] dA = \int_0^1 \int_0^1 (2e^y - e^y) dy dx \\ &= \int_0^1 dx \int_0^1 e^y dy = (1)(e^1 - e^0) = e - 1\end{aligned}$$

8. The region D enclosed by C is given by $\{(x, y) \mid 0 \leq x \leq 1, 3x \leq y \leq 3\}$, so

$$\begin{aligned}\int_C x^2 y^2 dx + 4xy^3 dy &= \iint_D \left[\frac{\partial}{\partial x} (4xy^3) - \frac{\partial}{\partial y} (x^2 y^2) \right] dA = \int_0^1 \int_{3x}^3 (4y^3 - 2x^2 y) dy dx \\ &= \int_0^1 [y^4 - x^2 y^2]_{y=3x}^{y=3} dx = \int_0^1 (81 - 9x^2 - 72x^4) dx = 81 - 3 - \frac{72}{5} = \frac{318}{5}\end{aligned}$$

9. $\int_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy = \iint_D \left[\frac{\partial}{\partial x} (2x + \cos y^2) - \frac{\partial}{\partial y} (y + e^{\sqrt{x}}) \right] dA$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1) dx dy = \int_0^1 (y^{1/2} - y^2) dy = \frac{1}{3}$$

10. $\int_C xe^{-2x} dx + (x^4 + 2x^2 y^2) dy = \iint_D \left[\frac{\partial}{\partial x} (x^4 + 2x^2 y^2) - \frac{\partial}{\partial y} (xe^{-2x}) \right] dA = \iint_D (4x^3 + 4xy^2 - 0) dA$

$$= 4 \iint_D x(x^2 + y^2) dA = 4 \int_0^{2\pi} \int_1^2 (r \cos \theta)(r^2) r dr d\theta$$

$$= 4 \int_0^{2\pi} \cos \theta d\theta \int_1^2 r^4 dr = 4 [\sin \theta]_0^{2\pi} \left[\frac{1}{5} r^5 \right]_1^2 = 0$$

11. $\int_C y^3 dx - x^3 dy = \iint_D \left[\frac{\partial}{\partial x} (-x^3) - \frac{\partial}{\partial y} (y^3) \right] dA = \iint_D (-3x^2 - 3y^2) dA = \int_0^{2\pi} \int_0^2 (-3r^2) r dr d\theta$

$$= -3 \int_0^{2\pi} d\theta \int_0^2 r^3 dr = -3(2\pi)(4) = -24\pi$$

12. $\int_C \sin y dx + x \cos y dy = \iint_D \left[\frac{\partial}{\partial x} (x \cos y) - \frac{\partial}{\partial y} (\sin y) \right] dA = \iint_D (\cos y - \cos y) dA = \iint_D 0 dA = 0$

13. $\mathbf{F}(x, y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle$ and the region D enclosed by C is given by

$\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \sin x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= - \int_{-C} (\sqrt{x} + y^3) dx + (x^2 + \sqrt{y}) dy = - \iint_D \left[\frac{\partial}{\partial x} (x^2 + \sqrt{y}) - \frac{\partial}{\partial y} (\sqrt{x} + y^3) \right] dA \\ &= - \int_0^\pi \int_0^{\sin x} (2x - 3y^2) dy dx = - \int_0^\pi [2xy - y^3]_{y=0}^{y=\sin x} dx \\ &= - \int_0^\pi (2x \sin x - \sin^3 x) dx = - \int_0^\pi (2x \sin x - (1 - \cos^2 x) \sin x) dx \\ &= - \left[2 \sin x - 2x \cos x + \cos x - \frac{1}{3} \cos^3 x \right]_0^\pi \quad [\text{integrate by parts in the first term}] \\ &= - (2\pi - 2 + \frac{2}{3}) = \frac{4}{3} - 2\pi\end{aligned}$$

14. $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$ and the region D enclosed by C is given by

$\{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= - \int_{-C} (y^2 \cos x) dx + (x^2 + 2y \sin x) dy \\ &= - \iint_D \left[\frac{\partial}{\partial x} (x^2 + 2y \sin x) - \frac{\partial}{\partial y} (y^2 \cos x) \right] dA \\ &= - \iint_D (2x + 2y \cos x - 2y \cos x) dA = - \int_0^2 \int_0^{3x} 2x dy dx \\ &= - \int_0^2 2x [y]_{y=0}^{y=3x} dx = - \int_0^2 6x^2 dx = - 2x^3 \Big|_0^2 = -16\end{aligned}$$

15. $\mathbf{F}(x, y) = \langle e^x + x^2 y, e^y - xy^2 \rangle$ and the region D enclosed by C is the disk $x^2 + y^2 \leq 25$.

C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= - \int_{-C} (e^x + x^2 y) dx + (e^y - xy^2) dy \\ &= - \iint_D \left[\frac{\partial}{\partial x} (e^y - xy^2) - \frac{\partial}{\partial y} (e^x + x^2 y) \right] dA = - \iint_D (-y^2 - x^2) dA \\ &= \iint_D (x^2 + y^2) dA = \int_0^{2\pi} \int_0^5 (r^2) r dr d\theta = \int_0^{2\pi} d\theta \int_0^5 r^3 dr = 2\pi \left[\frac{1}{4} r^4 \right]_0^5 = \frac{625}{2} \pi\end{aligned}$$

16. $\mathbf{F}(x, y) = \left\langle y - \ln(x^2 + y^2), 2 \tan^{-1}\left(\frac{y}{x}\right) \right\rangle$ and the region D enclosed by C is the disk with radius 1 centered at $(2, 3)$. C is oriented positively, so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (y - \ln(x^2 + y^2)) dx + \left(2 \tan^{-1}\left(\frac{y}{x}\right) \right) dy \\ &= \iint_D \left[\frac{\partial}{\partial x} \left(2 \tan^{-1}\left(\frac{y}{x}\right) \right) - \frac{\partial}{\partial y} (y - \ln(x^2 + y^2)) \right] dA \\ &= \iint_D \left[2 \left(\frac{-yx^{-2}}{1 + (y/x)^2} \right) - \left(1 - \frac{2y}{x^2 + y^2} \right) \right] dA = \iint_D \left[-\frac{2y}{x^2 + y^2} - 1 + \frac{2y}{x^2 + y^2} \right] dA \\ &= - \iint_D dA = -(\text{area of } D) = -\pi\end{aligned}$$

17. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x(x+y) dx + xy^2 dy = \iint_D (y^2 - x) dy dx$ where C is the path described in the question and D is the triangle bounded by C . So

$$\begin{aligned}W &= \int_0^1 \int_0^{1-x} (y^2 - x) dy dx = \int_0^1 \left[\frac{1}{3} y^3 - xy \right]_{y=0}^{y=1-x} dx = \int_0^1 \left(\frac{1}{3} (1-x)^3 - x(1-x) \right) dx \\ &= \left[-\frac{1}{12} (1-x)^4 - \frac{1}{2} x^2 + \frac{1}{3} x^3 \right]_0^1 = \left(-\frac{1}{2} + \frac{1}{3} \right) - \left(-\frac{1}{12} \right) = -\frac{1}{12}\end{aligned}$$

18. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x dx + (x^3 + 3xy^2) dy = \iint_D (3x^2 + 3y^2 - 0) dA$, where D is the semicircular region bounded by C . Converting to polar coordinates, we have

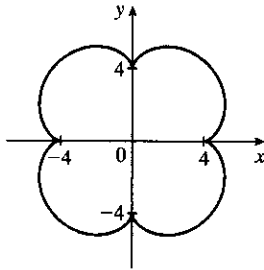
$$W = 3 \int_0^2 \int_0^\pi r^2 \cdot r d\theta dr = 3\pi \left[\frac{1}{4} r^4 \right]_0^2 = 12\pi.$$

19. Let C_1 be the arch of the cycloid from $(0, 0)$ to $(2\pi, 0)$, which corresponds to $0 \leq t \leq 2\pi$, and let C_2 be the segment from $(2\pi, 0)$ to $(0, 0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \leq t \leq 2\pi$. Then $C = C_1 \cup C_2$ is traversed

clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$\begin{aligned} A &= -\oint_{-C} y \, dx = \int_{C_1} y \, dx + \int_{C_2} y \, dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt + \int_0^{2\pi} 0 \, (-dt) \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt + 0 = \left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t \right]_0^{2\pi} = 3\pi \end{aligned}$$

20.



$$\begin{aligned} A &= \oint_C x \, dy = \int_0^{2\pi} (5\cos t - \cos 5t)(5\cos t - 5\cos 5t) \, dt \\ &= \int_0^{2\pi} (25\cos^2 t - 30\cos t \cos 5t + 5\cos^2 5t) \, dt \\ &= \left[25\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right) - 30\left(\frac{1}{8}\sin 4t + \frac{1}{12}\sin 6t\right) + 5\left(\frac{1}{2}t + \frac{1}{20}\sin 10t\right) \right]_0^{2\pi} \\ &\quad \text{[Use Formula 80 in the Table of Integrals]} \\ &= 30\pi \end{aligned}$$

21. (a) Using Equation 17.2.8 [ET 16.2.8], we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \leq t \leq 1$. Then $dx = (x_2 - x_1) \, dt$ and $dy = (y_2 - y_1) \, dt$, so

$$\begin{aligned} \int_C x \, dy - y \, dx &= \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1) \, dt + [(1-t)y_1 + ty_2](x_2 - x_1) \, dt \\ &= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)]) \, dt \\ &= \int_0^1 (x_1y_2 - x_2y_1) \, dt = x_1y_2 - x_2y_1 \end{aligned}$$

- (b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \dots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1) . From (5), $\frac{1}{2} \int_C x \, dy - y \, dx = \iint_D dA$, where D is the polygon bounded by C . Therefore

$$\begin{aligned} \text{area of polygon} &= A(D) = \iint_D dA = \frac{1}{2} \int_C x \, dy - y \, dx \\ &= \frac{1}{2} \left(\int_{C_1} x \, dy - y \, dx + \int_{C_2} x \, dy - y \, dx + \cdots + \int_{C_{n-1}} x \, dy - y \, dx + \int_{C_n} x \, dy - y \, dx \right) \end{aligned}$$

To evaluate these integrals we use the formula from (a) to get

$$A(D) = \frac{1}{2} [(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)].$$

- (c) $A = \frac{1}{2} [(0 \cdot 1 - 2 \cdot 0) + (2 \cdot 3 - 1 \cdot 1) + (1 \cdot 2 - 0 \cdot 3) + (0 \cdot 1 - (-1) \cdot 2) + (-1 \cdot 0 - 0 \cdot 1)]$
 $= \frac{1}{2} (0 + 5 + 2 + 2) = \frac{9}{2}$

22. By Green's Theorem, $\frac{1}{2A} \oint_C x^2 \, dy = \frac{1}{2A} \iint_D 2x \, dA = \frac{1}{A} \iint_D x \, dA = \bar{x}$ and
 $-\frac{1}{2A} \oint_C y^2 \, dx = -\frac{1}{2A} \iint_D (-2y) \, dA = \frac{1}{A} \iint_D y \, dA = \bar{y}$.

23. Here $A = \frac{1}{2}(1)(1) = \frac{1}{2}$ and $C = C_1 + C_2 + C_3$, where C_1 : $x = x, y = 0, 0 \leq x \leq 1$;

C_2 : $x = x, y = 1 - x, x = 1$ to $x = 0$; and C_3 : $x = 0, y = 1$ to $y = 0$. Then

$$\bar{x} = \frac{1}{2A} \int_C x^2 \, dy = \int_{C_1} x^2 \, dy + \int_{C_2} x^2 \, dy + \int_{C_3} x^2 \, dy = 0 + \int_1^0 (x^2)(-dx) + 0 = \frac{1}{3}.$$
 Similarly,

$$\bar{y} = -\frac{1}{2A} \int_C y^2 \, dx = \int_{C_1} y^2 \, dx + \int_{C_2} y^2 \, dx + \int_{C_3} y^2 \, dx = 0 + \int_1^0 (1-x)^2(-dx) + 0 = \frac{1}{3}.$$

Therefore $(\bar{x}, \bar{y}) = (\frac{1}{3}, \frac{1}{3})$.

24. $A = \frac{\pi a^2}{2}$ so $\bar{x} = \frac{1}{\pi a^2} \oint_C x^2 dy$ and $\bar{y} = -\frac{1}{\pi a^2} \oint_C y^2 dx$.

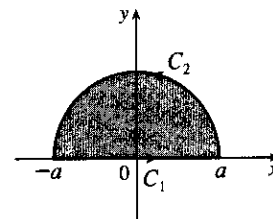
Orienting the semicircular region as in the figure,

$$\begin{aligned}\bar{x} &= \frac{1}{\pi a^2} \oint_{C_1+C_2} x^2 dy \\ &= \frac{1}{\pi a^2} \left[0 + \int_0^\pi (a^2 \cos^2 t)(a \cos t) dt \right] = 0\end{aligned}$$

and

$$\bar{y} = -\frac{1}{\pi a^2} \left[\int_{-a}^a 0 dx + \int_0^\pi (a^2 \sin^2 t)(-a \sin t) dt \right] = \frac{a}{\pi} \int_0^\pi \sin^3 t dt = \frac{a}{\pi} \left[-\cos t + \frac{1}{3}(\cos^3 t) \right]_0^\pi = \frac{4a}{3\pi}.$$

Thus $(\bar{x}, \bar{y}) = (0, \frac{4a}{3\pi})$.



25. By Green's Theorem, $-\frac{1}{3}\rho \oint_C y^3 dx = -\frac{1}{3}\rho \iint_D (-3y^2) dA = \iint_D y^2 \rho dA = I_x$ and

$$\frac{1}{3}\rho \oint_C x^3 dy = \frac{1}{3}\rho \iint_D (3x^2) dA = \iint_D x^2 \rho dA = I_y.$$

26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$\begin{aligned}I_y &= \frac{1}{3}\rho \oint_C x^3 dy = \frac{1}{3}\rho \int_0^{2\pi} (a^4 \cos^4 t) dt = \frac{1}{3}a^4 \rho \int_0^{2\pi} \left[\frac{3}{8} + \frac{1}{2} \cos 2t + \frac{1}{8} \cos 4t \right] dt \\ &= \frac{1}{3}a^4 \rho \cdot \frac{3(2\pi)}{8} = \frac{1}{4}\pi a^4 \rho\end{aligned}$$

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D . Thus $P = -y/(x^2 + y^2)$ and $Q = x/(x^2 + y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) [ET 16.3.33(a)], $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 0 dA = 0$.

28. We express D as a type II region: $D = \{(x, y) \mid f_1(y) \leq x \leq f_2(y), c \leq y \leq d\}$ where f_1 and f_2 are continuous

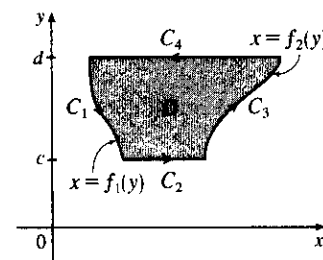
functions. Then $\iint_D \frac{\partial Q}{\partial x} dA = \int_c^d \int_{f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x} dx dy = \int_c^d [Q(f_2(y), y) - Q(f_1(y), y)] dy$ by

the Fundamental Theorem of Calculus. But referring to the figure,

$$\oint_C Q dy = \oint_{C_1+C_2+C_3+C_4} Q dy. \text{ Then } \int_{C_1} Q dy = \int_a^c Q(f_1(y), y) dy,$$

$$\int_{C_2} Q dy = \int_{C_4} Q dy = 0, \text{ and } \int_{C_3} Q dy = \int_c^d Q(f_2(y), y) dy. \text{ Hence}$$

$$\oint_C Q dy = \int_c^d [Q(f_2(y), y) - Q(f_1(y), y)] dy = \iint_D (\partial Q/\partial x) dA.$$



29. Using the first part of (5), we have that $\iint_R dx dy = A(R) = \int_{\partial R} x dy$. But $x = g(u, v)$, and

$$dy = \frac{\partial h}{\partial u} du + \frac{\partial h}{\partial v} dv, \text{ and we orient } \partial S \text{ by taking the positive direction to be that which corresponds, under the}$$

mapping, to the positive direction along ∂R , so

$$\begin{aligned}\int_{\partial R} x \, dy &= \int_{\partial S} g(u, v) \left(\frac{\partial h}{\partial u} du + \frac{\partial h}{\partial v} dv \right) = \int_{\partial S} g(u, v) \frac{\partial h}{\partial u} du + g(u, v) \frac{\partial h}{\partial v} dv \\&= \pm \iint_S \left[\frac{\partial}{\partial u} \left(g(u, v) \frac{\partial h}{\partial v} \right) - \frac{\partial}{\partial v} \left(g(u, v) \frac{\partial h}{\partial u} \right) \right] dA \quad [\text{using Green's Theorem in the } uv\text{-plane}] \\&= \pm \iint_S \left(\frac{\partial g}{\partial u} \frac{\partial h}{\partial v} + g(u, v) \frac{\partial^2 h}{\partial u \partial v} - \frac{\partial g}{\partial v} \frac{\partial h}{\partial u} - g(u, v) \frac{\partial^2 h}{\partial v \partial u} \right) dA \quad [\text{using the Chain Rule}] \\&= \pm \iint_S \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) dA \quad [\text{by the equality of mixed partials}] = \pm \iint_S \frac{\partial(x, y)}{\partial(u, v)} du \, dv\end{aligned}$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of

$$\frac{\partial(x, y)}{\partial(u, v)}. \text{ Therefore } A(R) = \iint_R dx \, dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv.$$

17.5 Curl and Divergence

ET 16.5

$$\begin{aligned}1. \text{ (a) } \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 0 & -x^2y \end{vmatrix} = (-x^2 - 0)\mathbf{i} - (-2xy - xy)\mathbf{j} + (0 - xz)\mathbf{k} \\&= -x^2\mathbf{i} + 3xy\mathbf{j} - xz\mathbf{k}\end{aligned}$$

$$\text{(b) } \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-x^2y) = yz + 0 + 0 = yz$$

$$\begin{aligned}2. \text{ (a) } \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = (xz^2 - xy^2)\mathbf{i} - (yz^2 - x^2y)\mathbf{j} + (y^2z - x^2z)\mathbf{k} \\&= x(z^2 - y^2)\mathbf{i} + y(x^2 - z^2)\mathbf{j} + z(y^2 - x^2)\mathbf{k}\end{aligned}$$

$$\text{(b) } \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\begin{aligned}3. \text{ (a) } \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x + yz & xy - \sqrt{z} \end{vmatrix} = (x - y)\mathbf{i} - (y - 0)\mathbf{j} + (1 - 0)\mathbf{k} \\&= (x - y)\mathbf{i} - y\mathbf{j} + \mathbf{k}\end{aligned}$$

$$\text{(b) } \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(x + yz) + \frac{\partial}{\partial z}(xy - \sqrt{z}) = z - \frac{1}{2\sqrt{z}}$$

$$\begin{aligned}
 4. (a) \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \cos xz & -\sin xy \end{vmatrix} \\
 &= (-x \cos xy + x \sin xz) \mathbf{i} - (-y \cos xy - 0) \mathbf{j} + (-z \sin xz - 0) \mathbf{k} \\
 &= x(\sin xz - \cos xy) \mathbf{i} + y \cos xy \mathbf{j} - z \sin xz \mathbf{k}
 \end{aligned}$$

$$(b) \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(\cos xz) + \frac{\partial}{\partial z}(-\sin xy) = 0 + 0 + 0 = 0$$

$$5. (a) \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y & e^x \cos y & z \end{vmatrix} = (0 - 0) \mathbf{i} - (0 - 0) \mathbf{j} + (e^x \cos y - e^x \cos y) \mathbf{k} = \mathbf{0}$$

$$(b) \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(z) = e^x \sin y - e^x \sin y + 1 = 1$$

$$\begin{aligned}
 6. (a) \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2 + y^2 + z^2} & \frac{y}{x^2 + y^2 + z^2} & \frac{z}{x^2 + y^2 + z^2} \end{vmatrix} \\
 &= \frac{1}{(x^2 + y^2 + z^2)^2} [(-2yz + 2yz) \mathbf{i} - (-2xz + 2xz) \mathbf{j} + (-2xy + 2xy) \mathbf{k}] = \mathbf{0}
 \end{aligned}$$

$$\begin{aligned}
 (b) \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{x^2 + y^2 + z^2} \right) \\
 &= \frac{x^2 + y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^2} + \frac{x^2 + y^2 + z^2 - 2y^2}{(x^2 + y^2 + z^2)^2} + \frac{x^2 + y^2 + z^2 - 2z^2}{(x^2 + y^2 + z^2)^2} \\
 &= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}
 \end{aligned}$$

$$\begin{aligned}
 7. (a) \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln x & \ln(xy) & \ln(xyz) \end{vmatrix} = \left(\frac{xz}{xyz} - 0 \right) \mathbf{i} - \left(\frac{yz}{xyz} - 0 \right) \mathbf{j} + \left(\frac{y}{xy} - 0 \right) \mathbf{k} \\
 &= \left\langle \frac{1}{y}, -\frac{1}{x}, \frac{1}{x} \right\rangle
 \end{aligned}$$

$$(b) \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(\ln x) + \frac{\partial}{\partial y}(\ln(xy)) + \frac{\partial}{\partial z}(\ln(xyz)) = \frac{1}{x} + \frac{x}{xy} + \frac{xy}{xyz} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\begin{aligned}
 8. (a) \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{-y} & xz & ze^y \end{vmatrix} = (ze^y - x) \mathbf{i} - (0 - 0) \mathbf{j} + (z - xe^{-y}(-1)) \mathbf{k} \\
 &= \langle ze^y - x, 0, z + xe^{-y} \rangle
 \end{aligned}$$

$$(b) \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xe^{-y}) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(ze^y) = e^{-y} + 0 + e^y = e^y + e^{-y}$$

9. If the vector field is $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then we know $R = 0$. In addition, the x -component of each vector of $\mathbf{F}$ is 0, so $P = 0$, hence $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\frac{\partial Q}{\partial y} < 0$, but

Q doesn't change in the x - or z -directions, so $\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial z} = 0$.

$$(a) \operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + \frac{\partial Q}{\partial y} + 0 < 0$$

$$(b) \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} = (0 - 0) \mathbf{i} + (0 - 0) \mathbf{j} + (0 - 0) \mathbf{k} = \mathbf{0}$$

10. If the vector field is $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then we know $R = 0$. In addition, P and Q don't vary in the z -direction, so $\frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial z} = \frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0$. As x increases, the x -component of each vector of $\mathbf{F}$ increases while the

y -component remains constant, so $\frac{\partial P}{\partial x} > 0$ and $\frac{\partial Q}{\partial x} = 0$. Similarly, as y increases, the y -component of each vector increases while the x -component remains constant, so $\frac{\partial Q}{\partial y} > 0$ and $\frac{\partial P}{\partial y} = 0$.

$$(a) \operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + 0 > 0$$

$$(b) \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} = (0 - 0) \mathbf{i} + (0 - 0) \mathbf{j} + (0 - 0) \mathbf{k} = \mathbf{0}$$

11. If the vector field is $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then we know $R = 0$. In addition, the y -component of each vector of $\mathbf{F}$ is 0, so $Q = 0$, hence $\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial z} = 0$. P increases as y increases, so $\frac{\partial P}{\partial y} > 0$, but P

doesn't change in the x - or z -directions, so $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial z} = 0$.

$$(a) \operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + 0 + 0 = 0$$

$$(b) \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ = (0 - 0) \mathbf{i} + (0 - 0) \mathbf{j} + \left(0 - \frac{\partial P}{\partial y} \right) \mathbf{k} = -\frac{\partial P}{\partial y} \mathbf{k}$$

Since $\frac{\partial P}{\partial y} > 0$, $-\frac{\partial P}{\partial y} \mathbf{k}$ is a vector pointing in the negative z -direction.

12. (a) $\operatorname{curl} f = \nabla \times f$ is meaningless because f is a scalar field.
 (b) $\operatorname{grad} f$ is a vector field.
 (c) $\operatorname{div} \mathbf{F}$ is a scalar field.
 (d) $\operatorname{curl} (\operatorname{grad} f)$ is a vector field.
 (e) $\operatorname{grad} \mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.
 (f) $\operatorname{grad} (\operatorname{div} \mathbf{F})$ is a vector field.
 (g) $\operatorname{div} (\operatorname{grad} f)$ is a scalar field.
 (h) $\operatorname{grad} (\operatorname{div} f)$ is meaningless because f is a scalar field.

- (i) $\text{curl}(\text{curl } \mathbf{F})$ is a vector field.
 (j) $\text{div}(\text{div } \mathbf{F})$ is meaningless because $\text{div } \mathbf{F}$ is a scalar field.
 (k) $(\text{grad } f) \times (\text{div } \mathbf{F})$ is meaningless because $\text{div } \mathbf{F}$ is a scalar field.
 (l) $\text{div}(\text{curl}(\text{grad } f))$ is a scalar field.

$$13. \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x - x)\mathbf{i} - (y - y)\mathbf{j} + (z - z)\mathbf{k} = \mathbf{0}$$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions which have continuous partial derivatives, so by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_x(x, y, z) = yz$ implies $f(x, y, z) = xyz + g(y, z)$ and $f_y(x, y, z) = xz + g_y(y, z)$. But $f_y(x, y, z) = xz$, so $g(y, z) = h(z)$ and $f(x, y, z) = xyz + h(z)$. Thus $f_z(x, y, z) = xy + h'(z)$ but $f_z(x, y, z) = xy$ so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x, y, z) = xyz + K$.

$$14. \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z^2 & \cos y & 2xz \end{vmatrix} = (0 - 0)\mathbf{i} - (2z - 6z)\mathbf{j} + (0 - 0)\mathbf{k} = 4z\mathbf{j} \neq \mathbf{0},$$

so $\mathbf{F}$ is not conservative.

$$15. \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 + 2yz & y^2 \end{vmatrix} = (2y - 2y)\mathbf{i} - (0 - 0)\mathbf{j} + (2x - 2x)\mathbf{k} = \mathbf{0}, \mathbf{F} \text{ is defined on all}$$

of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = 2xy$ implies $f(x, y, z) = x^2y + g(y, z)$ and $f_y(x, y, z) = x^2 + g_y(y, z)$. But $f_y(x, y, z) = x^2 + 2yz$, so $g(y, z) = y^2z + h(z)$ and $f(x, y, z) = x^2y + y^2z + h(z)$. Thus $f_z(x, y, z) = y^2 + h'(z)$ but $f_z(x, y, z) = y^2$ so $h(z) = K$ and $f(x, y, z) = x^2y + y^2z + K$.

$$16. \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & 1 & xe^z \end{vmatrix} = (0 - 0)\mathbf{i} - (e^z - e^z)\mathbf{j} + (0 - 0)\mathbf{k} = \mathbf{0} \text{ and } \mathbf{F} \text{ is defined on all of}$$

$\mathbb{R}^3$ with component functions that have continuous partial derivatives, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = e^z$ implies $f(x, y, z) = xe^z + g(y, z) \Rightarrow f_y(x, y, z) = g_y(y, z)$. But $f_y(x, y, z) = 1$, so $g(y, z) = y + h(z)$ and $f(x, y, z) = xe^z + y + h(z)$. Thus $f_z(x, y, z) = xe^z + h'(z)$ but $f_z(x, y, z) = xe^z$, so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x, y, z) = xe^z + y + K$.

$$17. \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{-x} & e^{-x} & 2z \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (-e^{-x} - e^{-x})\mathbf{k} = -2e^{-x}\mathbf{k} \neq \mathbf{0},$$

so $\mathbf{F}$ is not conservative.

$$18. \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y \cos xy & x \cos xy & -\sin z \end{vmatrix}$$

$$= (0-0)\mathbf{i} - (0-0)\mathbf{j} + [(-xy \sin xy + \cos xy) - (-xy \sin xy + \cos xy)]\mathbf{k} = \mathbf{0}$$

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = y \cos xy$ implies $f(x, y, z) = \sin xy + g(y, z) \Rightarrow f_y(x, y, z) = x \cos xy + g_y(y, z)$. But $f_y(x, y, z) = x \cos xy$, so $g(y, z) = h(z)$ and $f(x, y, z) = \sin xy + h(z)$. Thus $f_z(x, y, z) = h'(z)$ but $f_z(x, y, z) = -\sin z$ so $h(z) = \cos z + K$ and a potential function for $\mathbf{F}$ is $f(x, y, z) = \sin xy + \cos z + K$.

19. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl} \mathbf{G}) = y^2 + z^2 + x^2 \neq 0$, which contradicts Theorem 11.

20. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl} \mathbf{G}) = xz \neq 0$ which contradicts Theorem 11.

$$21. \operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}.$$

Hence $\mathbf{F} = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$ is irrotational.

$$22. \operatorname{div} \mathbf{F} = \frac{\partial(f(y, z))}{\partial x} + \frac{\partial(g(x, z))}{\partial y} + \frac{\partial(h(x, y))}{\partial z} = 0 \text{ so } \mathbf{F} \text{ is incompressible.}$$

For Exercises 23–29, let $\mathbf{F}(x, y, z) = P_1\mathbf{i} + Q_1\mathbf{j} + R_1\mathbf{k}$ and $\mathbf{G}(x, y, z) = P_2\mathbf{i} + Q_2\mathbf{j} + R_2\mathbf{k}$.

$$23. \operatorname{div}(\mathbf{F} + \mathbf{G}) = \frac{\partial(P_1 + P_2)}{\partial x} + \frac{\partial(Q_1 + Q_2)}{\partial y} + \frac{\partial(R_1 + R_2)}{\partial z}$$

$$= \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z} \right) + \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z} \right) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$$

$$24. \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G} = \left[\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right) \mathbf{k} \right]$$

$$+ \left[\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y} \right) \mathbf{k} \right]$$

$$= \left[\frac{\partial(R_1 + R_2)}{\partial y} - \frac{\partial(Q_1 + Q_2)}{\partial z} \right] \mathbf{i} + \left[\frac{\partial(P_1 + P_2)}{\partial z} - \frac{\partial(R_1 + R_2)}{\partial x} \right] \mathbf{j}$$

$$+ \left[\frac{\partial(Q_1 + Q_2)}{\partial x} - \frac{\partial(P_1 + P_2)}{\partial y} \right] \mathbf{k} = \operatorname{curl}(\mathbf{F} + \mathbf{G})$$

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$$\begin{aligned} 25. \operatorname{div}(f\mathbf{F}) &= \frac{\partial(fP_1)}{\partial x} + \frac{\partial(fQ_1)}{\partial y} + \frac{\partial(fR_1)}{\partial z} \\ &= \left(f \frac{\partial P_1}{\partial x} + P_1 \frac{\partial f}{\partial x}\right) + \left(f \frac{\partial Q_1}{\partial y} + Q_1 \frac{\partial f}{\partial y}\right) + \left(f \frac{\partial R_1}{\partial z} + R_1 \frac{\partial f}{\partial z}\right) \\ &= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1 \rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla f \end{aligned}$$

$$\begin{aligned} 26. \operatorname{curl}(f\mathbf{F}) &= \left[\frac{\partial(fR_1)}{\partial y} - \frac{\partial(fQ_1)}{\partial z}\right] \mathbf{i} + \left[\frac{\partial(fP_1)}{\partial z} - \frac{\partial(fR_1)}{\partial x}\right] \mathbf{j} + \left[\frac{\partial(fQ_1)}{\partial x} - \frac{\partial(fP_1)}{\partial y}\right] \mathbf{k} \\ &= \left[f \frac{\partial R_1}{\partial y} + R_1 \frac{\partial f}{\partial y} - f \frac{\partial Q_1}{\partial z} - Q_1 \frac{\partial f}{\partial z}\right] \mathbf{i} + \left[f \frac{\partial P_1}{\partial z} + P_1 \frac{\partial f}{\partial z} - f \frac{\partial R_1}{\partial x} - R_1 \frac{\partial f}{\partial x}\right] \mathbf{j} \\ &\quad + \left[f \frac{\partial Q_1}{\partial x} + Q_1 \frac{\partial f}{\partial x} - f \frac{\partial P_1}{\partial y} - P_1 \frac{\partial f}{\partial y}\right] \mathbf{k} \\ &= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right] \mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right] \mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right] \mathbf{k} \\ &\quad + \left[R_1 \frac{\partial f}{\partial y} - Q_1 \frac{\partial f}{\partial z}\right] \mathbf{i} + \left[P_1 \frac{\partial f}{\partial z} - R_1 \frac{\partial f}{\partial x}\right] \mathbf{j} + \left[Q_1 \frac{\partial f}{\partial x} - P_1 \frac{\partial f}{\partial y}\right] \mathbf{k} \\ &= f \operatorname{curl} \mathbf{F} + (\nabla f) \times \mathbf{F} \end{aligned}$$

$$\begin{aligned} 27. \operatorname{div}(\mathbf{F} \times \mathbf{G}) &= \nabla \cdot (\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \frac{\partial}{\partial x} \begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \frac{\partial}{\partial y} \begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \frac{\partial}{\partial z} \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix} \\ &= \left[Q_1 \frac{\partial R_2}{\partial x} + R_2 \frac{\partial Q_1}{\partial x} - Q_2 \frac{\partial R_1}{\partial x} - R_1 \frac{\partial Q_2}{\partial x}\right] \\ &\quad - \left[P_1 \frac{\partial R_2}{\partial y} + R_2 \frac{\partial P_1}{\partial y} - P_2 \frac{\partial R_1}{\partial y} - R_1 \frac{\partial P_2}{\partial y}\right] \\ &\quad + \left[P_1 \frac{\partial Q_2}{\partial z} + Q_2 \frac{\partial P_1}{\partial z} - P_2 \frac{\partial Q_1}{\partial z} - Q_1 \frac{\partial P_2}{\partial z}\right] \\ &= \left[P_2 \left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2 \left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2 \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right] \\ &\quad - \left[P_1 \left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1 \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1 \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right] \\ &= \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G} \end{aligned}$$

$$28. \operatorname{div}(\nabla f \times \nabla g) = \nabla g \cdot \operatorname{curl}(\nabla f) - \nabla f \cdot \operatorname{curl}(\nabla g) \text{ [by Exercise 27]} = 0 \text{ (by Theorem 3)}$$

$$\begin{aligned}
 29. \operatorname{curl} \operatorname{curl} \mathbf{F} &= \nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix} \\
 &= \left(\frac{\partial^2 Q_1}{\partial y \partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z \partial x} \right) \mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z \partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x \partial y} \right) \mathbf{j} \\
 &\quad + \left(\frac{\partial^2 P_1}{\partial x \partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y \partial z} \right) \mathbf{k}
 \end{aligned}$$

Now let's consider $\operatorname{grad} \operatorname{div} \mathbf{F} - \nabla^2 \mathbf{F}$ and compare with the above.

(Note that $\nabla^2 \mathbf{F}$ is defined on page 1130 [ET 1094].)

$$\begin{aligned}
 \operatorname{grad} \operatorname{div} \mathbf{F} - \nabla^2 \mathbf{F} &= \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} \right) \mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y \partial z} \right) \mathbf{j} \right. \\
 &\quad \left. + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} + \frac{\partial^2 R_1}{\partial z^2} \right) \mathbf{k} \right] \\
 &\quad - \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2} \right) \mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2} \right) \mathbf{j} \right. \\
 &\quad \left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2} \right) \mathbf{k} \right] \\
 &= \left(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} \right) \mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 R_1}{\partial y \partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2} \right) \mathbf{j} \\
 &\quad + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} \right) \mathbf{k}
 \end{aligned}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have $\operatorname{curl} \operatorname{curl} \mathbf{F} = \operatorname{grad} \operatorname{div} \mathbf{F} - \nabla^2 \mathbf{F}$ as desired.

$$30. (a) \nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) = 1 + 1 + 1 = 3$$

$$\begin{aligned}
 (b) \nabla \cdot (r\mathbf{r}) &= \nabla \cdot \sqrt{x^2 + y^2 + z^2} (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \\
 &= \left(\frac{x^2}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{x^2 + y^2 + z^2} \right) + \left(\frac{y^2}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{x^2 + y^2 + z^2} \right) \\
 &\quad + \left(\frac{z^2}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{x^2 + y^2 + z^2} \right) \\
 &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} (4x^2 + 4y^2 + 4z^2) = 4\sqrt{x^2 + y^2 + z^2} = 4r
 \end{aligned}$$

Another method:

By Exercise 25, $\nabla \cdot (r\mathbf{r}) = \operatorname{div}(r\mathbf{r}) = r \operatorname{div} \mathbf{r} + \mathbf{r} \cdot \nabla r = 3r + \mathbf{r} \cdot \frac{\mathbf{r}}{r}$ [see Exercise 31(a) below] $= 4r$.

$$\begin{aligned}
 \text{(c) } \nabla^2 r^3 &= \nabla^2 (x^2 + y^2 + z^2)^{3/2} \\
 &= \frac{\partial}{\partial x} \left[\frac{3}{2} (x^2 + y^2 + z^2)^{1/2} (2x) \right] + \frac{\partial}{\partial y} \left[\frac{3}{2} (x^2 + y^2 + z^2)^{1/2} (2y) \right] \\
 &\quad + \frac{\partial}{\partial z} \left[\frac{3}{2} (x^2 + y^2 + z^2)^{1/2} (2z) \right] \\
 &= 3 \left[\frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} (2x)(x) + (x^2 + y^2 + z^2)^{1/2} \right] \\
 &\quad + 3 \left[\frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} (2y)(y) + (x^2 + y^2 + z^2)^{1/2} \right] \\
 &\quad + 3 \left[\frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} (2z)(z) + (x^2 + y^2 + z^2)^{1/2} \right] \\
 &= 3(x^2 + y^2 + z^2)^{-1/2} (4x^2 + 4y^2 + 4z^2) = 12(x^2 + y^2 + z^2)^{1/2} \\
 &= 12r
 \end{aligned}$$

Another method: $\frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{3/2} = 3x \sqrt{x^2 + y^2 + z^2} \Rightarrow \nabla r^3 = 3r(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3r\mathbf{r}$,
 so $\nabla^2 r^3 = \nabla \cdot \nabla r^3 = \nabla \cdot (3r\mathbf{r}) = 3(4r) = 12r$ by part (b).

$$\begin{aligned}
 31. \text{ (a) } \nabla r &= \nabla \sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \mathbf{k} \\
 &= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{r}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \nabla \times \mathbf{r} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
 &= \left[\frac{\partial}{\partial y} (z) - \frac{\partial}{\partial z} (y) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} (x) - \frac{\partial}{\partial x} (z) \right] \mathbf{j} + \left[\frac{\partial}{\partial x} (y) - \frac{\partial}{\partial y} (x) \right] \mathbf{k} = \mathbf{0}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } \nabla \left(\frac{1}{r} \right) &= \nabla \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \\
 &= -\frac{1}{2\sqrt{x^2 + y^2 + z^2}} (2x) \mathbf{i} - \frac{1}{2\sqrt{x^2 + y^2 + z^2}} (2y) \mathbf{j} - \frac{1}{2\sqrt{x^2 + y^2 + z^2}} (2z) \mathbf{k} \\
 &= -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d) } \nabla \ln r &= \nabla \ln(x^2 + y^2 + z^2)^{1/2} = \frac{1}{2} \nabla \ln(x^2 + y^2 + z^2) \\
 &= \frac{x}{x^2 + y^2 + z^2} \mathbf{i} + \frac{y}{x^2 + y^2 + z^2} \mathbf{j} + \frac{z}{x^2 + y^2 + z^2} \mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{r}}{r^2}
 \end{aligned}$$

32. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2 + y^2 + z^2)^{p/2}} \mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^{p/2}} \mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^{p/2}} \mathbf{k}$$

Then $\frac{\partial}{\partial x} \frac{x}{(x^2 + y^2 + z^2)^{p/2}} = \frac{(x^2 + y^2 + z^2) - px^2}{(x^2 + y^2 + z^2)^{1+p/2}} = \frac{r^2 - px^2}{r^{p+2}}$. Similarly,

$$\frac{\partial}{\partial y} \frac{y}{(x^2 + y^2 + z^2)^{p/2}} = \frac{r^2 - py^2}{r^{p+2}} \quad \text{and} \quad \frac{\partial}{\partial z} \frac{z}{(x^2 + y^2 + z^2)^{p/2}} = \frac{r^2 - pz^2}{r^{p+2}}. \text{ Thus}$$

$$\begin{aligned} \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} &= \frac{r^2 - px^2}{r^{p+2}} + \frac{r^2 - py^2}{r^{p+2}} + \frac{r^2 - pz^2}{r^{p+2}} = \frac{3r^2 - px^2 - py^2 - pz^2}{r^{p+2}} \\ &= \frac{3r^2 - p(x^2 + y^2 + z^2)}{r^{p+2}} = \frac{3r^2 - pr^2}{r^{p+2}} = \frac{3-p}{r^p} \end{aligned}$$

Consequently, if $p = 3$ we have $\operatorname{div} \mathbf{F} = 0$.

33. By (13), $\oint_C f(\nabla g) \cdot \mathbf{n} \, ds = \iint_D \operatorname{div}(f\nabla g) \, dA = \iint_D [f \operatorname{div}(\nabla g) + \nabla g \cdot \nabla f] \, dA$ by Exercise 25. But $\operatorname{div}(\nabla g) = \nabla^2 g$. Hence $\iint_D f \nabla^2 g \, dA = \oint_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla g \cdot \nabla f \, dA$.

34. By Exercise 33, $\iint_D f \nabla^2 g \, dA = \oint_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla g \cdot \nabla f \, dA$ and $\iint_D g \nabla^2 f \, dA = \oint_C g(\nabla f) \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$. Hence

$$\begin{aligned} \iint_D (f \nabla^2 g - g \nabla^2 f) \, dA &= \oint_C [f(\nabla g) \cdot \mathbf{n} - g(\nabla f) \cdot \mathbf{n}] \, ds + \iint_D (\nabla f \cdot \nabla g - \nabla g \cdot \nabla f) \, dA \\ &= \oint_C [f \nabla g - g \nabla f] \cdot \mathbf{n} \, ds \end{aligned}$$

35. (a) We know that $\omega = v/d$, and from the diagram $\sin \theta = d/r \Rightarrow v = d\omega = (\sin \theta)r\omega = |\mathbf{w} \times \mathbf{r}|$. But $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, so that $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.

(b) From (a), $\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0 \cdot z - \omega y)\mathbf{i} + (\omega x - 0 \cdot z)\mathbf{j} + (0 \cdot y - x \cdot 0)\mathbf{k} = -\omega y\mathbf{i} + \omega x\mathbf{j}$

(c) $\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

$$\begin{aligned} &= \left[\frac{\partial}{\partial y} (0) - \frac{\partial}{\partial z} (\omega x) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} (-\omega y) - \frac{\partial}{\partial x} (0) \right] \mathbf{j} + \left[\frac{\partial}{\partial x} (\omega x) - \frac{\partial}{\partial y} (-\omega y) \right] \mathbf{k} \\ &= [\omega - (-\omega)] \mathbf{k} = 2\omega \mathbf{k} = 2\mathbf{w} \end{aligned}$$

36. Let $\mathbf{H} = \langle h_1, h_2, h_3 \rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$.

$$\begin{aligned}
 \text{(a) } \nabla \times (\nabla \times \mathbf{E}) &= \nabla \times (\text{curl } \mathbf{E}) = \nabla \times \left(-\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{1}{c} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix} \\
 &= -\frac{1}{c} \left[\left(\frac{\partial^2 h_3}{\partial y \partial t} - \frac{\partial^2 h_2}{\partial z \partial t} \right) \mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z \partial t} - \frac{\partial^2 h_3}{\partial x \partial t} \right) \mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x \partial t} - \frac{\partial^2 h_1}{\partial y \partial t} \right) \mathbf{k} \right] \\
 &= -\frac{1}{c} \frac{\partial}{\partial t} \left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right) \mathbf{k} \right] \\
 &\quad \text{[assuming that the partial derivatives are continuous} \\
 &\quad \text{so that the order of differentiation does not matter]} \\
 &= -\frac{1}{c} \frac{\partial}{\partial t} \text{curl } \mathbf{H} = -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \nabla \times (\nabla \times \mathbf{H}) &= \nabla \times (\text{curl } \mathbf{H}) = \nabla \times \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right) = \frac{1}{c} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t \end{vmatrix} \\
 &= \frac{1}{c} \left[\left(\frac{\partial^2 E_3}{\partial y \partial t} - \frac{\partial^2 E_2}{\partial z \partial t} \right) \mathbf{i} + \left(\frac{\partial^2 E_1}{\partial z \partial t} - \frac{\partial^2 E_3}{\partial x \partial t} \right) \mathbf{j} + \left(\frac{\partial^2 E_2}{\partial x \partial t} - \frac{\partial^2 E_1}{\partial y \partial t} \right) \mathbf{k} \right] \\
 &= \frac{1}{c} \frac{\partial}{\partial t} \left[\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} \right) \mathbf{k} \right] \\
 &\quad \text{[assuming that the partial derivatives are continuous} \\
 &\quad \text{so that the order of differentiation does not matter]} \\
 &= \frac{1}{c} \frac{\partial}{\partial t} \text{curl } \mathbf{E} = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}
 \end{aligned}$$

(c) Using Exercise 29, we have that $\text{curl curl } \mathbf{E} = \text{grad div } \mathbf{E} - \nabla^2 \mathbf{E} \Rightarrow$

$$\nabla^2 \mathbf{E} = \text{grad div } \mathbf{E} - \text{curl curl } \mathbf{E} = \text{grad } 0 + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{[from part (a)]} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2 \mathbf{H} = \text{grad div } \mathbf{H} - \text{curl curl } \mathbf{H} = \text{grad } 0 + \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \text{[using part (b)]} = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}.$

37. For any continuous function f on $\mathbb{R}^3$, define a vector field $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0 \rangle$ where

$$g(x, y, z) = \int_0^x f(t, y, z) dt. \text{ Then}$$

$$\text{div } \mathbf{G} = \frac{\partial}{\partial x} (g(x, y, z)) + \frac{\partial}{\partial y} (0) + \frac{\partial}{\partial z} (0) = \frac{\partial}{\partial x} \int_0^x f(t, y, z) dt = f(x, y, z) \text{ by the Fundamental Theorem of}$$

Calculus. Thus every continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

17.6 Parametric Surfaces and Their Areas

ET 16.6

1. $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k}$, so the corresponding parametric equations for the surface are

$x = u \cos v, y = u \sin v, z = u^2$. For any point (x, y, z) on the surface, we have

$x^2 + y^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 = z$. Since no restrictions are placed on the parameters, the surface is $z = x^2 + y^2$, which we recognize as a circular paraboloid opening upward whose axis is the z -axis.

2. $\mathbf{r}(u, v) = (1 + 2u) \mathbf{i} + (-u + 3v) \mathbf{j} + (2 + 4u + 5v) \mathbf{k} = \langle 1, 0, 2 \rangle + u \langle 2, -1, 4 \rangle + v \langle 0, 3, 5 \rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(1, 0, 2)$ and containing vectors $\mathbf{a} = \langle 2, -1, 4 \rangle$ and $\mathbf{b} = \langle 0, 3, 5 \rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix} = -17\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$$

and an equation of the plane is $-17(x - 1) - 10(y - 0) + 6(z - 2) = 0$ or $-17x - 10y + 6z = -5$.

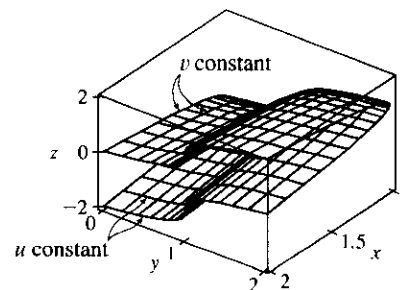
3. $\mathbf{r}(x, \theta) = \langle x, \cos \theta, \sin \theta \rangle$, so the corresponding parametric equations for the surface are

$x = x, y = \cos \theta, z = \sin \theta$. For any point (x, y, z) on the surface, we have $y^2 + z^2 = \cos^2 \theta + \sin^2 \theta = 1$, so any vertical trace in $x = k$ is the circle $y^2 + z^2 = 1, x = k$. Since $x = x$ with no restriction, the surface is a circular cylinder with radius 1 whose axis is the x -axis.

4. $\mathbf{r}(x, \theta) = \langle x, x \cos \theta, x \sin \theta \rangle$, so the corresponding parametric equations for the surface are $x = x, y = x \cos \theta, z = x \sin \theta$. For any point (x, y, z) on the surface, we have $y^2 + z^2 = x^2 \cos^2 \theta + x^2 \sin^2 \theta = x^2$. With $x = x$ and no restrictions on the parameters, the surface is $x^2 = y^2 + z^2$, which we recognize as a circular cone whose axis is the x -axis.

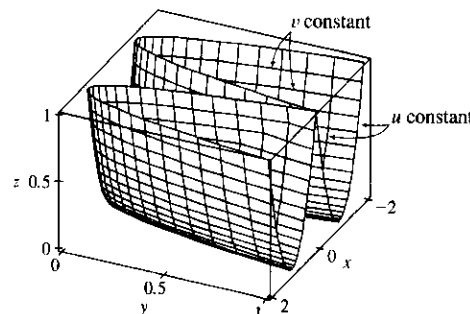
5. $\mathbf{r}(u, v) = \langle u^2 + 1, v^3 + 1, u + v \rangle, -1 \leq u \leq 1, -1 \leq v \leq 1$.

The surface has parametric equations $x = u^2 + 1, y = v^3 + 1, z = u + v, -1 \leq u \leq 1, -1 \leq v \leq 1$. If we keep u constant at $u_0, x = u_0^2 + 1$, a constant, so the corresponding grid curves must be the curves parallel to the yz -plane. If v is constant, we have $y = v_0^3 + 1$, a constant, so these grid curves are the curves parallel to the xz -plane.



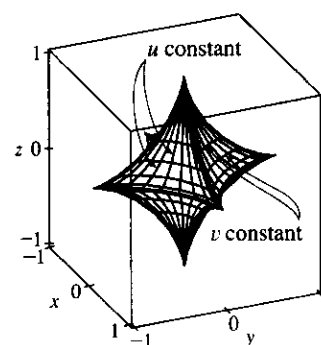
6. $\mathbf{r}(u, v) = \langle u + v, u^2, v^2 \rangle, -1 \leq u \leq 1, -1 \leq v \leq 1$.

The surface has parametric equations $x = u + v, y = u^2, z = v^2, -1 \leq u \leq 1, -1 \leq v \leq 1$. If $u = u_0$ is constant, $y = u_0^2 = \text{constant}$, so the corresponding grid curves are the curves parallel to the xz -plane. If $v = v_0$ is constant, $z = v_0^2 = \text{constant}$, so the corresponding grid curves are the curves parallel to the xy -plane.



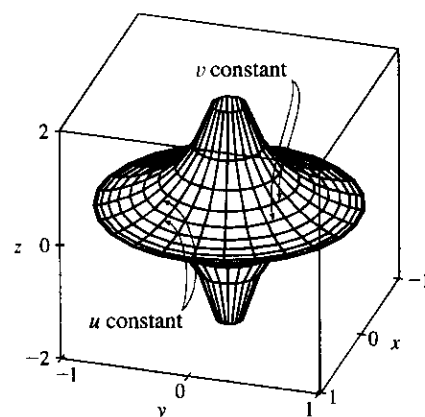
7. $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$.

The surface has parametric equations $x = \cos^3 u \cos^3 v$, $y = \sin^3 u \cos^3 v$, $z = \sin^3 v$, $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$. Note that if $v = v_0$ is constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curves must be the curves parallel to the xy -plane. The vertically oriented grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos^3 u_0 \cos^3 v$, $y = \sin^3 u_0 \cos^3 v$, $z = \sin^3 v$. These curves lie in vertical planes that contain the z -axis.



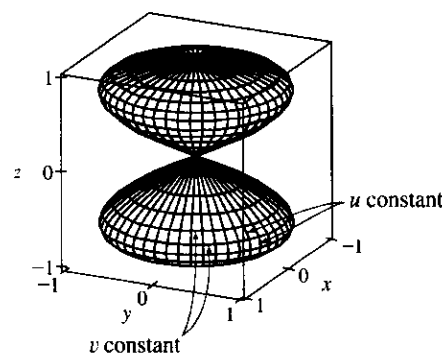
8. $\mathbf{r}(u, v) = \langle \cos u \sin v, \sin u \sin v, \cos v + \ln \tan(v/2) \rangle$.

The surface has parametric equations $x = \cos u \sin v$, $y = \sin u \sin v$, $z = \cos v + \ln \tan(v/2)$, $0 \leq u \leq 2\pi$, $0.1 \leq v \leq 6.2$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin v_0$, $y = \sin u \sin v_0$, $z = \cos v_0 + \ln \tan(v_0/2)$ which represent a circle of radius $\sin v_0$ in the plane $z = \cos v_0 + \ln \tan(v_0/2)$. So the circular grid curves we see lying horizontally are the grid curves with v constant. The vertically oriented grid curves correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin v$, $y = \sin u_0 \sin v$, $z = \cos v + \ln \tan(v/2)$. These curves lie in vertical planes that contain the z -axis.



9. $x = \cos u \sin 2v$, $y = \sin u \sin 2v$, $z = \sin v$.

The complete graph of the surface is given by the parametric domain $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin 2v_0$, $y = \sin u \sin 2v_0$, $z = \sin v_0$ which represent a circle of radius $\sin 2v_0$ in the plane $z = \sin v_0$. So the circular grid curves we see lying horizontally are the grid curves which have v constant. The vertical grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin 2v$ and $y = \sin u_0 \sin 2v$ with $z = \sin v$ which has a "figure-eight" shape.

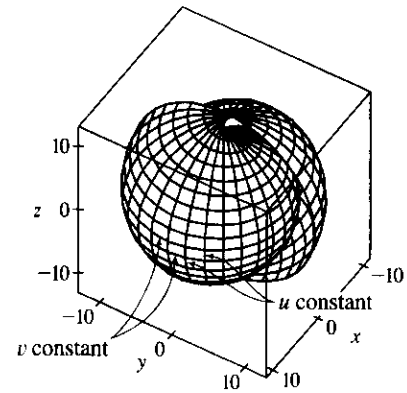


10. $x = u \sin u \cos v, y = u \cos u \cos v, z = u \sin v.$

We graph the portion of the surface with parametric domain

$$0 \leq u \leq 4\pi, 0 \leq v \leq 2\pi.$$

Note that if $v = v_0$ is constant, the parametric equations become $x = u \sin u \cos v_0, y = u \cos u \cos v_0, z = u \sin v_0$. The equations for x and y show that the projections onto the xy -plane give a spiral shape, so the corresponding grid curves are the almost-horizontal spiral curves we see. The vertical grid curves, which look approximately circular, correspond to $u = u_0$ being held constant, giving $x = u_0 \sin u_0 \cos v, y = u_0 \cos u_0 \cos v, z = u_0 \sin v$.



11. $\mathbf{r}(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j} + u \mathbf{k}$. The parametric equations for the surface are $x = \cos v, y = \sin v, z = u$. Then $x^2 + y^2 = \cos^2 v + \sin^2 v = 1$ and $z = u$ with no restriction on u , so we have a circular cylinder, graph IV. The grid curves with u constant are the horizontal circles we see in the plane $z = u$. If v is constant, both x and y are constant with z free to vary, so the corresponding grid curves are the lines on the cylinder parallel to the z -axis.

12. $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u \mathbf{k}$. The parametric equations for the surface are $x = u \cos v, y = u \sin v, z = u$. Then $x^2 + y^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 = z^2$, which represents the equation of a cone with axis the z -axis, graph V. The grid curves with u constant are the horizontal circles we see, corresponding to the equations $x^2 + y^2 = u^2$ in the plane $z = u$. If v is constant, x, y, z are each scalar multiples of u , corresponding to the straight line grid curves through the origin.

13. $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$. The parametric equations for the surface are $x = u \cos v, y = u \sin v, z = v$. We look at the grid curves first; if we fix v , then x and y parametrize a straight line in the plane $z = v$ which intersects the z -axis. If u is held constant, the projection onto the xy -plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $x = u^3, y = u \sin v, z = u \cos v$. Then $y^2 + z^2 = u^2 \sin^2 v + u^2 \cos^2 v = u^2$, so if u is held constant, each grid curve is a circle of radius u in the plane $x = u^3$. The graph then must be graph III. If v is held constant, so $v = v_0$, we have $y = u \sin v_0$ and $z = u \cos v_0$. Then $y = (\tan v_0)z$, so the grid curves we see running lengthwise along the surface in the planes $y = kz$ correspond to keeping v constant.

15. $x = (u - \sin u) \cos v, y = (1 - \cos u) \sin v, z = u$. If u is held constant, x and y give an equation of an ellipse in the plane $z = u$, thus the grid curves are horizontally oriented ellipses. Note that when $u = 0$, the “ellipse” is the single point $(0, 0, 0)$, and when $u = \pi$, we have $y = 0$ while x ranges from $-\pi$ to π , a line segment parallel to the x -axis in the plane $z = \pi$. This is the upper “seam” we see in graph II. When v is held constant, $z = u$ is free to vary, so the corresponding grid curves are the curves we see running up and down along the surface.

16. $x = (1 - u)(3 + \cos v) \cos 4\pi u, y = (1 - u)(3 + \cos v) \sin 4\pi u, z = 3u + (1 - u) \sin v$. These equations correspond to graph VI: when $u = 0$, then $x = 3 + \cos v, y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz -plane centered at $(3, 0, 0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2} \cos v, y = 0$, and $z = \frac{3}{2} + \frac{1}{2} \sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz -plane centered at $(\frac{3}{2}, 0, \frac{3}{2})$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. From Example 3, parametric equations for the plane through the point $(1, 2, -3)$ that contains the vectors

$\mathbf{a} = \langle 1, 1, -1 \rangle$ and $\mathbf{b} = \langle 1, -1, 1 \rangle$ are $x = 1 + u(1) + v(1) = 1 + u + v$, $y = 2 + u(1) + v(-1) = 2 + u - v$,
 $z = -3 + u(-1) + v(1) = -3 - u + v$.

18. Solving the equation for z gives $z^2 = 1 - 2x^2 - 4y^2 \Rightarrow z = -\sqrt{1 - 2x^2 - 4y^2}$ (since we want the lower half of the ellipsoid). If we let x and y be the parameters, parametric equations are $x = x$, $y = y$,

$$z = -\sqrt{1 - 2x^2 - 4y^2}.$$

Alternate solution: The equation can be rewritten as $\frac{x^2}{(1/\sqrt{2})^2} + \frac{y^2}{(1/2)^2} + z^2 = 1$, and if we let $x = \frac{1}{\sqrt{2}} u \cos v$

and $y = \frac{1}{2} u \sin v$, then $z = -\sqrt{1 - 2x^2 - 4y^2} = -\sqrt{1 - u^2 \cos^2 v - u^2 \sin^2 v} = -\sqrt{1 - u^2}$, where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

19. Solving the equation for y gives $y^2 = 1 - x^2 + z^2 \Rightarrow y = \sqrt{1 - x^2 + z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $y \geq 0$.) If we let x and z be the parameters, parametric equations are $x = x$, $z = z$, $y = \sqrt{1 - x^2 + z^2}$.

20. $x = 4 - y^2 - 2z^2$, $y = y$, $z = z$ where $y^2 + 2z^2 \leq 4$ since $x \geq 0$. Then the associated vector equation is
 $\mathbf{r}(y, z) = (4 - y^2 - 2z^2)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

21. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \leq 2$.

Alternate solution: Using spherical coordinates, $x = 2 \sin \phi \cos \theta$, $y = 2 \sin \phi \sin \theta$, $z = 2 \cos \phi$ where $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$.

22. In spherical coordinates, parametric equations are $x = 4 \sin \phi \cos \theta$, $y = 4 \sin \phi \sin \theta$, $z = 4 \cos \phi$. The intersection of the sphere with the plane $z = 2$ corresponds to $z = 4 \cos \phi = 2 \Rightarrow \cos \phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$. By symmetry, the intersection of the sphere with the plane $z = -2$ corresponds to $\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Thus the surface is described by $0 \leq \theta \leq 2\pi$, $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$.

23. Parametric equations are $x = x$, $y = 4 \cos \theta$, $z = 4 \sin \theta$, $0 \leq x \leq 5$, $0 \leq \theta \leq 2\pi$.

24. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \leq x^2 + y^2 \leq 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ , we have $x = s \cos \theta$, $y = s \sin \theta$, $z = 3 + s \cos \theta$ where $0 \leq s \leq 1$ and $0 \leq \theta \leq 2\pi$.

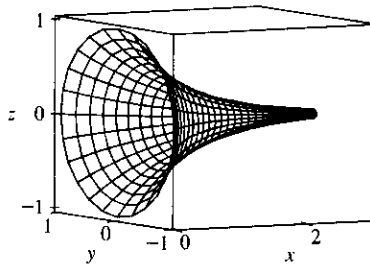
25. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x -axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \leq x \leq 5$, $y \leq 0$ to obtain the portion shown.

To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3 \cos v$, $z = 3 \sin v$ with the parameter

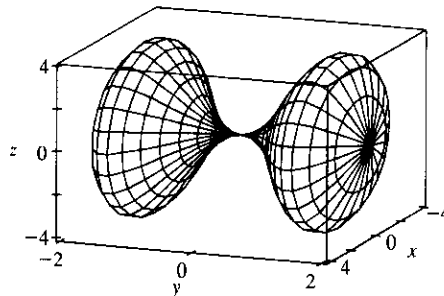
domain $0 \leq u \leq 5$, $\frac{\pi}{2} \leq v \leq \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \leq x \leq 5$ and $-3 \leq z \leq 3$.

26. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho = 1$, and imposing the restrictions $\frac{\pi}{2} \leq \theta \leq 2\pi$, $\frac{\pi}{4} \leq \phi \leq \pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x = \sin \phi \cos \theta$, $y = \sin \phi \sin \theta$, $z = \cos \phi$, $\frac{\pi}{2} \leq \theta \leq 2\pi$, $\frac{\pi}{4} \leq \phi \leq \pi$.

27. Using Equations 3, we have the parametrization $x = x$, $y = e^{-x} \cos \theta$, $z = e^{-x} \sin \theta$, $0 \leq x \leq 3$, $0 \leq \theta \leq 2\pi$.

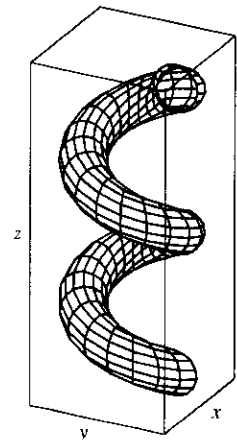


28. Letting θ be the angle of rotation about the y -axis, we have the parametrization $x = (4y^2 - y^4) \cos \theta$, $y = y$, $z = (4y^2 - y^4) \sin \theta$, $-2 \leq y \leq 2$, $0 \leq \theta \leq 2\pi$.



29. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations

$x = (2 + \sin v) \sin u$, $y = (2 + \sin v) \cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy -plane, given by $x = (2 + \sin v) \sin u$, $y = (2 + \sin v) \cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v . The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations

$$x = (2 + \sin v) \cos 2u, y = (2 + \sin v) \sin 2u, z = u + \cos v.$$

From the graph, it appears that the number of coils in the surface doubles within the same parametric

domain. We can verify this observation by noting that the projection of the spiral

grid curves onto the xy -plane, given by $x = (2 + \sin v) \cos 2u$,

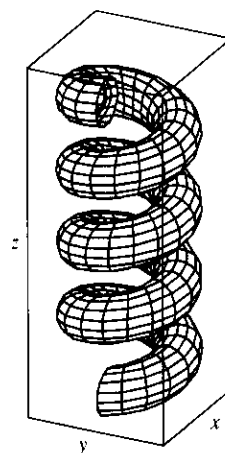
$y = (2 + \sin v) \sin 2u, z = 0$ (where v is constant), complete circular revolutions

for $0 \leq u \leq \pi$ while the original surface requires $0 \leq u \leq 2\pi$ for a complete

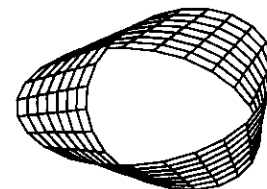
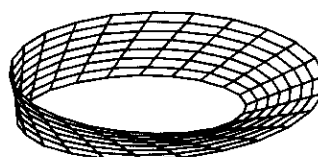
revolution. Thus, the new surface winds around twice as fast as the original

surface, and since the equation for z is identical in both surfaces, we observe twice

as many circular coils in the same z -interval.



30. First we graph the surface as viewed from the front, then from two additional viewpoints.

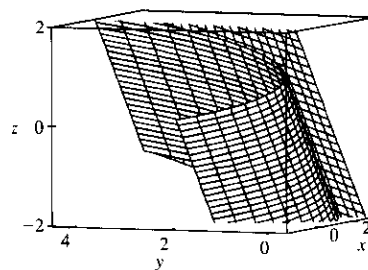


The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 17.7 [ET 16.7].)

31. $\mathbf{r}(u, v) = (u + v)\mathbf{i} + 3u^2\mathbf{j} + (u - v)\mathbf{k}$.

$$\mathbf{r}_u = \mathbf{i} + 6u\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_v = \mathbf{i} - \mathbf{k}, \text{ so}$$

$\mathbf{r}_u \times \mathbf{r}_v = -6u\mathbf{i} + 2\mathbf{j} - 6u\mathbf{k}$. Since the point $(2, 3, 0)$ corresponds to $u = 1, v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.



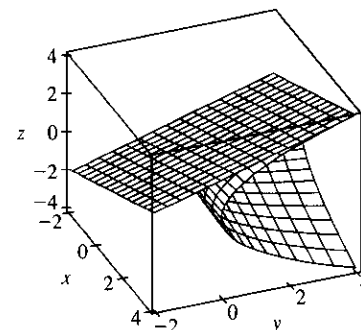
32. $\mathbf{r}(u, v) = u^2\mathbf{i} + v^2\mathbf{j} + uv\mathbf{k} \Rightarrow \mathbf{r}(1, 1) = (1, 1, 1)$.

$\mathbf{r}_u = 2u\mathbf{i} + v\mathbf{k}$ and $\mathbf{r}_v = 2v\mathbf{j} + u\mathbf{k}$, so a normal vector to the surface at the point $(1, 1, 1)$ is

$$\mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = (2\mathbf{i} + \mathbf{k}) \times (2\mathbf{j} + \mathbf{k}) = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}.$$

Thus an equation of the tangent plane at the point $(1, 1, 1)$ is

$$-2(x - 1) - 2(y - 1) + 4(z - 1) = 0 \text{ or } x + y - 2z = 0.$$



$$33. \mathbf{r}(u, v) = u^2 \mathbf{i} + 2u \sin v \mathbf{j} + u \cos v \mathbf{k} \Rightarrow \mathbf{r}(1, 0) = (1, 0, 1).$$

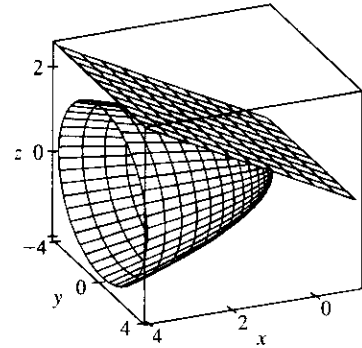
$$\mathbf{r}_u = 2u \mathbf{i} + 2 \sin v \mathbf{j} + \cos v \mathbf{k} \text{ and } \mathbf{r}_v = 2u \cos v \mathbf{j} - u \sin v \mathbf{k},$$

so a normal vector to the surface at the point $(1, 0, 1)$ is

$$\mathbf{r}_u(1, 0) \times \mathbf{r}_v(1, 0) = (2\mathbf{i} + \mathbf{k}) \times (2\mathbf{j}) = -2\mathbf{i} + 4\mathbf{k}.$$

Thus an equation of the tangent plane at $(1, 0, 1)$ is

$$-2(x - 1) + 0(y - 0) + 4(z - 1) = 0 \text{ or } -x + 2z = 1.$$



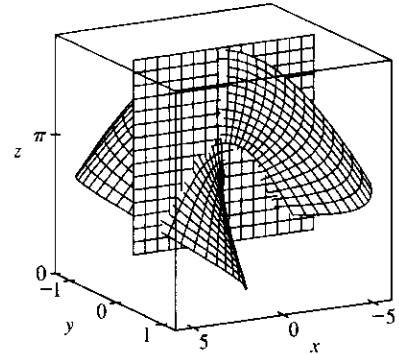
$$34. \mathbf{r}(u, v) = uv \mathbf{i} + u \sin v \mathbf{j} + v \cos u \mathbf{k} \Rightarrow \mathbf{r}(0, \pi) = (0, 0, \pi).$$

$$\mathbf{r}_u = v \mathbf{i} + \sin v \mathbf{j} - v \sin u \mathbf{k} \text{ and } \mathbf{r}_v = u \mathbf{i} + u \cos v \mathbf{j} + \cos u \mathbf{k}, \text{ so}$$

a normal vector to the surface at the point $(0, 0, \pi)$ is

$$\mathbf{r}_u(0, \pi) \times \mathbf{r}_v(0, \pi) = (\pi \mathbf{i}) \times (\mathbf{k}) = -\pi \mathbf{j}. \text{ Thus an equation of the}$$

tangent plane is $-\pi(y - 0) = 0$ or $y = 0$.



35. Here $z = f(x, y) = 4 - x - 2y$ and D is the disk $x^2 + y^2 \leq 4$. Thus, by Formula 9,

$$A(S) = \iint_D \sqrt{1 + (-1)^2 + (-2)^2} dA = \sqrt{6} \iint_D dA = \sqrt{6} A(D) = 4\sqrt{6}\pi$$

36. $\mathbf{r}_u = \langle 0, 1, -5 \rangle$, $\mathbf{r}_v = \langle 1, -2, 1 \rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle -9, -5, -1 \rangle$. Then by Definition 6,

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA = \int_0^1 \int_0^1 |\langle -9, -5, -1 \rangle| du dv = \sqrt{107} \int_0^1 du \int_0^1 dv = \sqrt{107}$$

37. $z = f(x, y) = xy$ with $0 \leq x^2 + y^2 \leq 1$, so $f_x = y$, $f_y = x \Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + y^2 + x^2} dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{3} (r^2 + 1)^{3/2} \right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \frac{1}{3} (2\sqrt{2} - 1) d\theta = \frac{2\pi}{3} (2\sqrt{2} - 1) \end{aligned}$$

38. $z = f(x, y) = 1 + 3x + 2y^2$ with $0 \leq x \leq 2y$, $0 \leq y \leq 1$. Thus, by Formula 9,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 3^2 + (4y)^2} dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2} dx dy = \int_0^1 2y \sqrt{10 + 16y^2} dy \\ &= \frac{1}{16} \cdot \frac{2}{3} (10 + 16y^2)^{3/2} \Big|_0^1 = \frac{1}{24} (26^{3/2} - 10^{3/2}) \end{aligned}$$

39. $z = f(x, y) = y^2 - x^2$ with $1 \leq x^2 + y^2 \leq 4$. Then

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 4x^2 + 4y^2} dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_1^2 \sqrt{1 + 4r^2} r dr \\ &= [\theta]_0^{2\pi} \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_1^2 = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5}) \end{aligned}$$

40. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \leq y^2 + z^2 \leq 9$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\mathbf{i} + \mathbf{j}) \times (2z\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$.

Note: In general, if $x = f(y, z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} - \frac{\partial f}{\partial z}\mathbf{k}$, and

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} dA. \text{ Then}$$

$$\begin{aligned} A(S) &= \iint_{0 \leq y^2 + z^2 \leq 9} \sqrt{1 + 4y^2 + 4z^2} dA = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^3 r \sqrt{1 + 4r^2} dr = 2\pi \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_0^3 = \frac{\pi}{6} (37\sqrt{37} - 1) \end{aligned}$$

41. A parametric representation of the surface is $x = x$, $y = 4x + z^2$, $z = z$ with $0 \leq x \leq 1$, $0 \leq z \leq 1$.

Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 4\mathbf{j}) \times (2z\mathbf{j} + \mathbf{k}) = 4\mathbf{i} - \mathbf{j} + 2z\mathbf{k}$.

Note: In general, if $y = f(x, z)$ then $\mathbf{r}_x \times \mathbf{r}_z = \frac{\partial f}{\partial x}\mathbf{i} - \mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ and

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} dA. \text{ Then}$$

$$\begin{aligned} A(S) &= \int_0^1 \int_0^1 \sqrt{17 + 4z^2} dx dz = \int_0^1 \sqrt{17 + 4z^2} dz \\ &= \frac{1}{2} \left(z \sqrt{17 + 4z^2} + \frac{17}{2} \ln \left| 2z + \sqrt{4z^2 + 17} \right| \right) \Big|_0^1 = \frac{\sqrt{21}}{2} + \frac{17}{4} \left[\ln(2 + \sqrt{21}) - \ln \sqrt{17} \right] \end{aligned}$$

42. Let S_1 be that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$ by symmetry.

On S_1 , $z = \sqrt{a^2 - x^2}$ so $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}$. Hence

$$A(S_1) = \iint_{0 \leq x^2 + y^2 \leq a^2} \frac{a}{\sqrt{a^2 - x^2}} dA = \int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2}} dy dx = \int_{-a}^a 2a dx = 4a^2.$$

Thus $A(S) = 8a^2$.

Alternate solution: If $A(S_2)$ is the surface area in the first octant, then $A(S) = 8A(S_2)$. A parametric representation of the surface in the first octant is $x = a \sin \theta$, $y = y$, $z = a \cos \theta$ (θ being the angle in the xz -plane measured from the positive z -axis), where $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq y \leq a \cos \theta$. The restrictions on y follow from:

$x^2 + y^2 \leq a^2$ or $a^2 \sin^2 \theta + y^2 \leq a^2$ so $y^2 \leq a^2(1 - \sin^2 \theta)$; thus in the first octant $0 \leq y \leq a \cos \theta$. Then

$$\mathbf{r}_y \times \mathbf{r}_\theta = \langle -a \sin \theta, 0, -a \cos \theta \rangle \text{ and } A(S_2) = \int_0^{\pi/2} \int_0^{a \cos \theta} a dy d\theta = \int_0^{\pi/2} a^2 \cos \theta d\theta = a^2.$$

Hence $A(S) = 8a^2$.

43. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z = 0$. Then

$A(S) = 2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x = a \sin \phi \cos \theta$, $y = a \sin \phi \sin \theta$,

$z = a \cos \phi$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi$. For D , $0 \leq \phi \leq \frac{\pi}{2}$ and for each fixed ϕ , $(x - \frac{1}{2}a)^2 + y^2 \leq (\frac{1}{2}a)^2$ or $[a \sin \phi \cos \theta - \frac{1}{2}a]^2 + a^2 \sin^2 \phi \sin^2 \theta \leq (a/2)^2$ implies $a^2 \sin^2 \phi - a^2 \sin \phi \cos \theta \leq 0$ or $\sin \phi (\sin \phi - \cos \theta) \leq 0$. But $0 \leq \phi \leq \frac{\pi}{2}$, so $\cos \theta \geq \sin \phi$ or $\sin(\frac{\pi}{2} + \theta) \geq \sin \phi$ or $\phi - \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} - \phi$. Hence $D = \{(\phi, \theta) \mid 0 \leq \phi \leq \frac{\pi}{2}, \phi - \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} - \phi\}$. Then

$$\begin{aligned} A(S_1) &= \int_0^{\pi/2} \int_{\phi - (\pi/2)}^{(\pi/2) - \phi} a^2 \sin \phi \, d\theta \, d\phi = a^2 \int_0^{\pi/2} (\pi - 2\phi) \sin \phi \, d\phi \\ &= a^2 [(-\pi \cos \phi) - 2(-\phi \cos \phi + \sin \phi)]_0^{\pi/2} = a^2(\pi - 2) \end{aligned}$$

Thus $A(S) = 2a^2(\pi - 2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x = x$, $y = y$,

$z = \sqrt{a^2 - x^2 - y^2}$. Then $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$ and

$$\begin{aligned} A(S_1) &= \iint_{0 \leq (x - (a/2))^2 + y^2 \leq (a/2)^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{a \cos \theta} \frac{a}{\sqrt{a^2 - r^2}} r \, dr \, d\theta \\ &= \int_{-\pi/2}^{\pi/2} -a(a^2 - r^2)^{1/2} \Big|_{r=0}^{r=a \cos \theta} \, d\theta = \int_{-\pi/2}^{\pi/2} a^2[1 - (1 - \cos^2 \theta)^{1/2}] \, d\theta \\ &= \int_{-\pi/2}^{\pi/2} a^2(1 - |\sin \theta|) \, d\theta = 2a^2 \int_0^{\pi/2} (1 - \sin \theta) \, d\theta = 2a^2(\frac{\pi}{2} - 1) \end{aligned}$$

Thus $A(S) = 4a^2(\frac{\pi}{2} - 1) = 2a^2(\pi - 2)$.

Notes:

- (1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D .
- (2) In the alternate solution, you can avoid having to use $|\sin \theta|$ by working in the first octant and then multiplying by 4. However, if you set up S_1 as above and arrived at $A(S_1) = a^2\pi$, you now see your error.

44. $\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle$, $\mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle$. Then

$$\begin{aligned} A(S) &= \int_0^\pi \int_0^1 \sqrt{1 + u^2} \, du \, dv = \int_0^\pi dv \int_0^1 \sqrt{1 + u^2} \, du \\ &= \pi \left[\frac{u}{2} \sqrt{u^2 + 1} + \frac{1}{2} \ln |u + \sqrt{u^2 + 1}| \right]_0^1 = \frac{\pi}{2} \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right] \end{aligned}$$

45. $\mathbf{r}_u = \langle v, 1, 1 \rangle$, $\mathbf{r}_v = \langle u, 1, -1 \rangle$ and $\mathbf{r}_u \times \mathbf{r}_v = \langle -2, u + v, v - u \rangle$. Then

$$\begin{aligned} A(S) &= \iint_{u^2 + v^2 \leq 1} \sqrt{4 + 2u^2 + 2v^2} \, dA = \int_0^{2\pi} \int_0^1 r \sqrt{4 + 2r^2} \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^1 r \sqrt{4 + 2r^2} \, dr \\ &= 2\pi \left[\frac{1}{6} (4 + 2r^2)^{3/2} \right]_0^1 = \frac{\pi}{3} (6\sqrt{6} - 8) = \pi \left(2\sqrt{6} - \frac{8}{3} \right) \end{aligned}$$

- 46.
- $z = f(x, y) = \cos(x^2 + y^2)$
- with
- $x^2 + y^2 \leq 1$
- .

$$\begin{aligned}
 A(S) &= \iint_D \sqrt{1 + (-2x \sin(x^2 + y^2))^2 + (-2y \sin(x^2 + y^2))^2} dA \\
 &= \iint_D \sqrt{1 + 4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2)} dA \\
 &= \iint_D \sqrt{1 + 4(x^2 + y^2) \sin^2(x^2 + y^2)} dA \\
 &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2 \sin^2(r^2)} r dr d\theta = \int_0^{2\pi} d\theta \int_0^1 r \sqrt{1 + 4r^2 \sin^2(r^2)} dr \\
 &= 2\pi \int_0^1 r \sqrt{1 + 4r^2 \sin^2(r^2)} dr \approx 4.1073
 \end{aligned}$$

- 47.
- $z = f(x, y) = e^{-x^2 - y^2}$
- with
- $x^2 + y^2 \leq 4$
- .

$$\begin{aligned}
 A(S) &= \iint_D \sqrt{1 + (-2xe^{-x^2 - y^2})^2 + (-2ye^{-x^2 - y^2})^2} dA \\
 &= \iint_D \sqrt{1 + 4(x^2 + y^2)e^{-2(x^2 + y^2)}} dA \\
 &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2 e^{-2r^2}} r dr d\theta = \int_0^{2\pi} d\theta \int_0^2 r \sqrt{1 + 4r^2 e^{-2r^2}} dr \\
 &= 2\pi \int_0^2 r \sqrt{1 + 4r^2 e^{-2r^2}} dr \approx 13.9783
 \end{aligned}$$

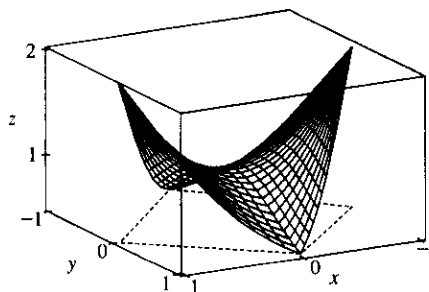
48. Let
- $f(x, y) = \frac{1 + x^2}{1 + y^2}$
- . Then
- $f_x = \frac{2x}{1 + y^2}$
- ,

$$f_y = (1 + x^2) \left[-\frac{2y}{(1 + y^2)^2} \right] = -\frac{2y(1 + x^2)}{(1 + y^2)^2}.$$

We use a CAS to estimate

$$\int_{-1}^1 \int_{-(1-|x|)}^{1-|x|} \sqrt{1 + f_x^2 + f_y^2} dy dx \approx 2.6959.$$

In order to graph only the part of the surface above the square, we use $-(1 - |x|) \leq y \leq 1 - |x|$ as the y -range in our plot command.



49. (a) The midpoints of the four squares are
- $(\frac{1}{4}, \frac{1}{4})$
- ,
- $(\frac{1}{4}, \frac{3}{4})$
- ,
- $(\frac{3}{4}, \frac{1}{4})$
- , and
- $(\frac{3}{4}, \frac{3}{4})$
- ; the derivatives of the function

$f(x, y) = x^2 + y^2$ are $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, so the Midpoint Rule gives

$$\begin{aligned}
 A(S) &= \int_0^1 \int_0^1 \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dy dx \\
 &\approx \frac{1}{4} \left(\sqrt{[2(\frac{1}{4})]^2 + [2(\frac{1}{4})]^2 + 1} + \sqrt{[2(\frac{1}{4})]^2 + [2(\frac{3}{4})]^2 + 1} \right. \\
 &\quad \left. + \sqrt{[2(\frac{3}{4})]^2 + [2(\frac{1}{4})]^2 + 1} + \sqrt{[2(\frac{3}{4})]^2 + [2(\frac{3}{4})]^2 + 1} \right) \\
 &= \frac{1}{4} \left(\sqrt{\frac{3}{2}} + 2\sqrt{\frac{7}{2}} + \sqrt{\frac{11}{2}} \right) \approx 1.8279
 \end{aligned}$$

(b) A CAS estimates the integral to be

$$A(S) = \int_0^1 \int_0^1 \sqrt{f_x^2 + f_y^2 + 1} dy dx = \int_0^1 \int_0^1 \sqrt{4x^2 + 4y^2 + 1} dy dx \approx 1.8616. \text{ This agrees with the Midpoint estimate only in the first decimal place.}$$

50. $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$, so $\mathbf{r}_u = \langle -3 \cos^2 u \sin u \cos^3 v, 3 \sin^2 u \cos u \cos^3 v, 0 \rangle$,
 $\mathbf{r}_v = \langle -3 \cos^3 u \cos^2 v \sin v, -3 \sin^3 u \cos^2 v \sin v, 3 \sin^2 v \cos v \rangle$, and
 $\mathbf{r}_u \times \mathbf{r}_v = \langle 9 \cos u \sin^2 u \cos^4 v \sin^2 v, 9 \cos^2 u \sin u \cos^4 v \sin^2 v, 9 \cos^2 u \sin^2 u \cos^5 v \sin v \rangle$. Then

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= 9 \sqrt{\cos^2 u \sin^4 u \cos^8 v \sin^4 v + \cos^4 u \sin^2 u \cos^8 v \sin^4 v + \cos^4 u \sin^4 u \cos^{10} v \sin^2 v} \\ &= 9 \sqrt{\cos^2 u \sin^2 u \cos^8 v \sin^2 v (\sin^2 v + \cos^2 u \sin^2 u \cos^2 v)} \\ &= 9 \cos^4 v |\cos u \sin u \sin v| \sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v} \end{aligned}$$

Using a CAS, we have

$$A(S) = \int_0^\pi \int_0^{2\pi} 9 \cos^4 v |\cos u \sin u \sin v| \sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v} dv du \approx 4.4506.$$

51. $z = 1 + 2x + 3y + 4y^2$, so

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3 + 8y)^2} dy dx \\ &= \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2} dy dx. \end{aligned}$$

Using a CAS, we have

$$\begin{aligned} \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2} dy dx &= \frac{45}{8} \sqrt{14} + \frac{15}{16} \ln(11\sqrt{5} + 3\sqrt{14}\sqrt{5}) - \frac{15}{16} \ln(3\sqrt{5} + \sqrt{14}\sqrt{5}) \\ \text{or } \frac{45}{8} \sqrt{14} + \frac{15}{16} \ln \frac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}. \end{aligned}$$

52. (a) $\mathbf{r}_u = a \cos v \mathbf{i} + b \sin v \mathbf{j} + 2u \mathbf{k}$, $\mathbf{r}_v = -a \sin v \mathbf{i} + b \cos v \mathbf{j} + 0 \mathbf{k}$, and

$$\mathbf{r}_u \times \mathbf{r}_v = -2bu^2 \cos v \mathbf{i} - 2au^2 \sin v \mathbf{j} + abu \mathbf{k}.$$

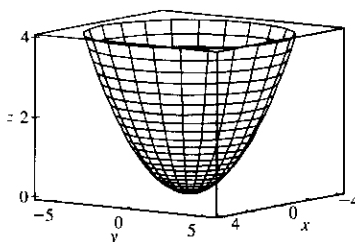
$$A(S) = \int_0^{2\pi} \int_0^2 |\mathbf{r}_u \times \mathbf{r}_v| du dv = \int_0^{2\pi} \int_0^2 \sqrt{4b^2 u^4 \cos^2 v + 4a^2 u^4 \sin^2 v + a^2 b^2 u^2} du dv$$

- (b) $x^2 = a^2 u^2 \cos^2 v$, $y^2 = b^2 u^2 \sin^2 v$, $z = u^2 \Rightarrow x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid.

To find D , notice that $0 \leq u \leq 2 \Rightarrow 0 \leq z \leq 4 \Rightarrow 0 \leq x^2/a^2 + y^2/b^2 \leq 4$. Therefore, using Formula 9,

$$\text{we have } A(S) = \int_{-2a}^{2a} \int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2} dy dx.$$

(c)



- (d) We substitute $a = 2$, $b = 3$ in the integral in part (a) to get

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^2 2u \sqrt{9u^2 \cos^2 v + 4u^2 \sin^2 v + 9} du dv. \text{ We} \\ &\text{use a CAS to estimate the integral accurate to four decimal} \\ &\text{places. To speed up the calculation, we can set Digits:=7;} \\ &\text{(in Maple) or use the approximation command N (in} \\ &\text{Mathematica). We find that } A(S) \approx 115.6596. \end{aligned}$$

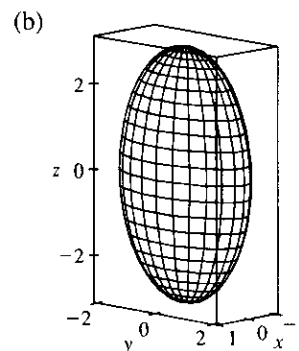
53. (a)
- $x = a \sin u \cos v, y = b \sin u \sin v, z = c \cos u \Rightarrow$

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= (\sin u \cos v)^2 + (\sin u \sin v)^2 + (\cos u)^2 \\ &= \sin^2 u + \cos^2 u = 1\end{aligned}$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.

- (c) From the parametric equations (with $a = 1, b = 2$, and $c = 3$), we calculate $\mathbf{r}_u = \cos u \cos v \mathbf{i} + 2 \cos u \sin v \mathbf{j} - 3 \sin u \mathbf{k}$ and $\mathbf{r}_v = -\sin u \sin v \mathbf{i} + 2 \sin u \cos v \mathbf{j}$. So $\mathbf{r}_u \times \mathbf{r}_v = 6 \sin^2 u \cos v \mathbf{i} + 3 \sin^2 u \sin v \mathbf{j} + 2 \sin u \cos u \mathbf{k}$, and the surface area is given by

$$\begin{aligned}A(S) &= \int_0^{2\pi} \int_0^\pi |\mathbf{r}_u \times \mathbf{r}_v| du dv \\ &= \int_0^{2\pi} \int_0^\pi \sqrt{36 \sin^4 u \cos^2 v + 9 \sin^4 u \sin^2 v + 4 \cos^2 u \sin^2 u} du dv\end{aligned}$$



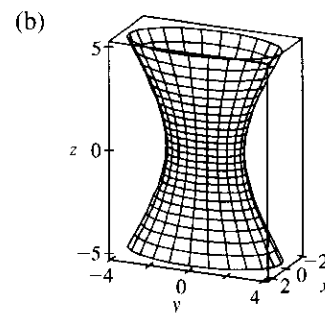
54. (a)
- $x = a \cosh u \cos v, y = b \cosh u \sin v, z = c \sinh u \Rightarrow$

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} &= \cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v - \sinh^2 u \\ &= \cosh^2 u - \sinh^2 u = 1\end{aligned}$$

and the parametric equations represent a hyperboloid of one sheet.

- (c) $\mathbf{r}_u = \sinh u \cos v \mathbf{i} + 2 \sinh u \sin v \mathbf{j} + 3 \cosh u \mathbf{k}$ and $\mathbf{r}_v = -\cosh u \sin v \mathbf{i} + 2 \cosh u \cos v \mathbf{j}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6 \cosh^2 u \cos v \mathbf{i} - 3 \cosh^2 u \sin v \mathbf{j} + 2 \cosh u \sinh u \mathbf{k}$. We integrate between $u = \sinh^{-1}(-1) = -\ln(1 + \sqrt{2})$ and $u = \sinh^{-1} 1 = \ln(1 + \sqrt{2})$, since then z varies between -3 and 3 , as desired. So the surface area is

$$\begin{aligned}A(S) &= \int_0^{2\pi} \int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} |\mathbf{r}_u \times \mathbf{r}_v| du dv \\ &= \int_0^{2\pi} \int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} \sqrt{36 \cosh^4 u \cos^2 v + 9 \cosh^4 u \sin^2 v + 4 \cosh^2 u \sinh^2 u} du dv\end{aligned}$$



55. $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$, so $\mathbf{r}_u = \langle -3 \cos^2 u \sin u \cos^3 v, 3 \sin^2 u \cos u \cos^3 v, 0 \rangle$, $\mathbf{r}_v = \langle -3 \cos^3 u \cos^2 v \sin v, -3 \sin^3 u \cos^2 v \sin v, 3 \sin^2 v \cos v \rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 9 \cos u \sin^2 u \cos^4 v \sin^2 v, 9 \cos^2 u \sin u \cos^4 v \sin^2 v, 9 \cos^2 u \sin^2 u \cos^5 v \sin v \rangle$. Then

$$\begin{aligned}|\mathbf{r}_u \times \mathbf{r}_v| &= 9 \sqrt{\cos^2 u \sin^4 u \cos^8 v \sin^4 v + \cos^4 u \sin^2 u \cos^8 v \sin^4 v + \cos^4 u \sin^4 u \cos^{10} v \sin^2 v} \\ &= 9 \sqrt{\cos^2 u \sin^2 u \cos^8 v \sin^2 v (\sin^2 v + \cos^2 u \sin^2 u \cos^2 v)} \\ &= 9 \cos^4 v |\cos u \sin u \sin v| \sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v}\end{aligned}$$

Using a CAS, we have $A(S) = \int_0^\pi \int_0^{2\pi} 9 \cos^4 v |\cos u \sin u \sin v| \sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v} dv du \approx 4.4506$.

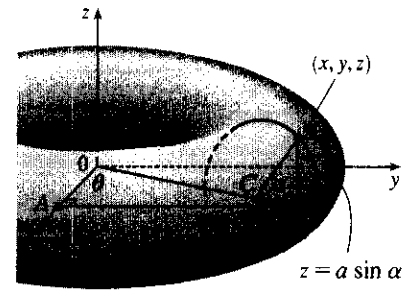
56. (a) Here $z = a \sin \alpha$, $y = |AB|$, and $x = |OA|$. But

$$|OB| = |OC| + |CB| = b + a \cos \alpha \text{ and } \sin \theta = \frac{|AB|}{|OB|} \text{ so that}$$

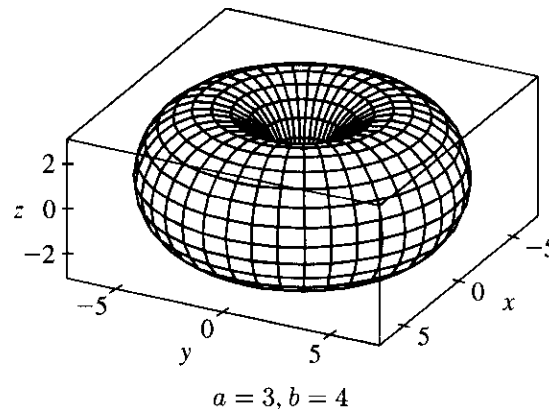
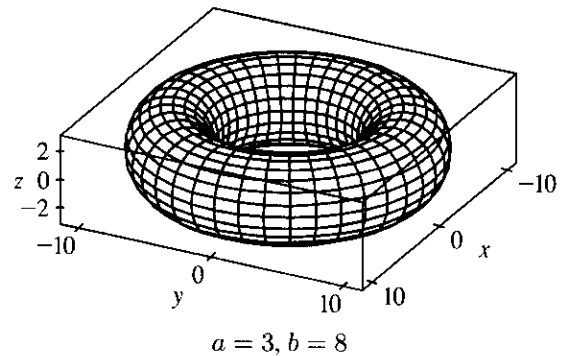
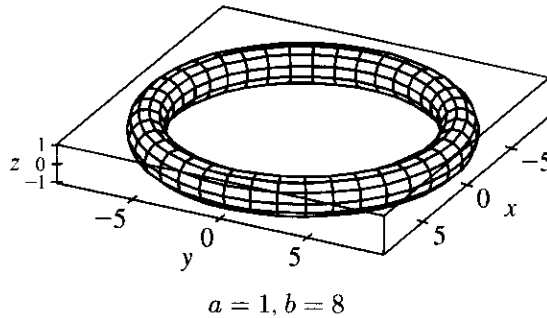
$$y = |OB| \sin \theta = (b + a \cos \alpha) \sin \theta. \text{ Similarly } \cos \theta = \frac{|OA|}{|OB|} \text{ so}$$

$$x = (b + a \cos \alpha) \cos \theta. \text{ Hence a parametric representation for the torus is } x = b \cos \theta + a \cos \alpha \cos \theta, y = b \sin \theta + a \cos \alpha \sin \theta,$$

$$z = a \sin \alpha, \text{ where } 0 \leq \alpha \leq 2\pi, 0 \leq \theta \leq 2\pi.$$



(b)



(c) $x = b \cos \theta + a \cos \alpha \cos \theta$, $y = b \sin \theta + a \cos \alpha \sin \theta$, $z = a \sin \alpha$, so

$$\mathbf{r}_\alpha = \langle -a \sin \alpha \cos \theta, -a \sin \alpha \sin \theta, a \cos \alpha \rangle, \mathbf{r}_\theta = \langle -(b + a \cos \alpha) \sin \theta, (b + a \cos \alpha) \cos \theta, 0 \rangle \text{ and}$$

$$\begin{aligned} \mathbf{r}_\alpha \times \mathbf{r}_\theta &= (-ab \cos \alpha \cos \theta - a^2 \cos \alpha \cos^2 \theta) \mathbf{i} + (-ab \sin \alpha \cos \theta - a^2 \sin \alpha \cos^2 \theta) \mathbf{j} \\ &\quad + (-ab \cos^2 \alpha \sin \theta - a^2 \cos^2 \alpha \sin \theta \cos \theta - ab \sin^2 \alpha \sin \theta - a^2 \sin^2 \alpha \sin \theta \cos \theta) \mathbf{k} \\ &= -a(b + a \cos \alpha) [(\cos \theta \cos \alpha) \mathbf{i} + (\sin \theta \cos \alpha) \mathbf{j} + (\sin \alpha) \mathbf{k}] \end{aligned}$$

$$\text{Then } |\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a \cos \alpha) \sqrt{\cos^2 \theta \cos^2 \alpha + \sin^2 \theta \cos^2 \alpha + \sin^2 \alpha} = a(b + a \cos \alpha).$$

Note: $b > a$, $-1 \leq \cos \alpha \leq 1$ so $|b + a \cos \alpha| = b + a \cos \alpha$. Hence

$$A(S) = \int_0^{2\pi} \int_0^{2\pi} a(b + a \cos \alpha) d\alpha d\theta = 2\pi [ab\alpha + a^2 \sin \alpha]_0^{2\pi} = 4\pi^2 ab.$$

17.7 Surface Integrals

ET 16.7

1. Each face of the cube has surface area $2^2 = 4$, and the points P_{ij}^* are the points where the cube intersects the coordinate axes. Here, $f(x, y, z) = \sqrt{x^2 + 2y^2 + 3z^2}$, so by Definition 1,

$$\begin{aligned}\iint_S f(x, y, z) dS &\approx [f(1, 0, 0)](4) + [f(-1, 0, 0)](4) + [f(0, 1, 0)](4) + [f(0, -1, 0)](4) \\ &\quad + [f(0, 0, 1)](4) + [f(0, 0, -1)](4) \\ &= 4(1 + 1 + 2\sqrt{2} + 2\sqrt{3}) = 8(1 + \sqrt{2} + \sqrt{3}) \approx 33.170\end{aligned}$$

2. Each quarter-cylinder has surface area $\frac{1}{4}[2\pi(1)(2)] = \pi$, and the top and bottom disks have surface area $\pi(1)^2 = \pi$. We can take $(0, 0, 1)$ as a sample point in the top disk, $(0, 0, -1)$ in the bottom disk, and $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ in the four quarter-cylinders. Then $\iint_S f(x, y, z) dS$ can be approximated by the Riemann sum
- $$\begin{aligned}f(1, 0, 0)(\pi) + f(-1, 0, 0)(\pi) + f(0, 1, 0)(\pi) + f(0, -1, 0)(\pi) + f(0, 0, 1)(\pi) + f(0, 0, -1)(\pi) \\ = (2 + 2 + 3 + 3 + 4 + 4)\pi = 18\pi \approx 56.5.\end{aligned}$$

3. We can use the xz - and yz -planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S = \frac{1}{8}(4\pi(\sqrt{50})^2) = 25\pi$. Then $(\pm 3, \pm 4, 5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\begin{aligned}\iint_H f(x, y, z) dS &\approx f(3, 4, 5) \Delta S + f(3, -4, 5) \Delta S + f(-3, 4, 5) \Delta S + f(-3, -4, 5) \Delta S \\ &= (7 + 8 + 9 + 12)(25\pi) = 900\pi \approx 2827\end{aligned}$$

4. On the surface, $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2}) = g(2) = -5$. So since the area of a sphere is $4\pi r^2$,
- $$\iint_S f(x, y, z) dS = \iint_S g(2) dS = -5 \iint_S dS = -5[4\pi(2)^2] = -80\pi.$$
5. $z = 1 + 2x + 3y$ so $\frac{\partial z}{\partial x} = 2$ and $\frac{\partial z}{\partial y} = 3$. Then by Formula 2,

$$\begin{aligned}\iint_S x^2 y z dS &= \iint_D x^2 y z \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA \\ &= \int_0^3 \int_0^2 x^2 y (1 + 2x + 3y) \sqrt{4 + 9 + 1} dy dx \\ &= \sqrt{14} \int_0^3 \int_0^2 (x^2 y + 2x^3 y + 3x^2 y^2) dy dx \\ &= \sqrt{14} \int_0^3 \left[\frac{1}{2} x^2 y^2 + x^3 y^2 + x^2 y^3 \right]_{y=0}^{y=2} dx \\ &= \sqrt{14} \int_0^3 (10x^2 + 4x^3) dx = \sqrt{14} \left[\frac{10}{3} x^3 + x^4 \right]_0^3 = 171\sqrt{14}\end{aligned}$$

6. S is the region in the plane $2x + y + z = 2$ or $z = 2 - 2x - y$ over $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x\}$. Thus

$$\begin{aligned}\iint_S xy dS &= \iint_D xy \sqrt{(-2)^2 + (-1)^2 + 1} dA \\ &= \sqrt{6} \int_0^1 \int_0^{2-2x} xy dy dx = \sqrt{6} \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{y=2-2x} dx \\ &= \frac{\sqrt{6}}{2} \int_0^1 (4x - 8x^2 + 4x^3) dx = \frac{\sqrt{6}}{2} \left(2 - \frac{8}{3} + 1 \right) = \frac{\sqrt{6}}{6}\end{aligned}$$

7. S is the part of the plane $z = 1 - x - y$ over the region $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$. Thus

$$\begin{aligned}\iint_S yz \, dS &= \iint_D y(1 - x - y) \sqrt{(-1)^2 + (-1)^2 + 1} \, dA \\ &= \sqrt{3} \int_0^1 \int_0^{1-x} (y - xy - y^2) \, dy \, dx = \sqrt{3} \int_0^1 \left[\frac{1}{2}y^2 - \frac{1}{2}xy^2 - \frac{1}{3}y^3 \right]_{y=0}^{y=1-x} dx \\ &= \sqrt{3} \int_0^1 \frac{1}{6}(1-x)^3 \, dx = -\frac{\sqrt{3}}{24}(1-x)^4 \Big|_0^1 = \frac{\sqrt{3}}{24}\end{aligned}$$

8. $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ and

$$\begin{aligned}\iint_S y \, dS &= \iint_D y \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2 + 1} \, dA = \int_0^1 \int_0^1 y \sqrt{x+y+1} \, dx \, dy \\ &= \int_0^1 y \left[\frac{2}{3}(x+y+1)^{3/2} \right]_{x=0}^{x=1} dy = \int_0^1 \frac{2}{3}y \left[(y+2)^{3/2} - (y+1)^{3/2} \right] dy\end{aligned}$$

Substituting $u = y + 2$ in the first term and $t = y + 1$ in the second, we have

$$\begin{aligned}\iint_S y \, dS &= \frac{2}{3} \int_2^3 (u-2)u^{3/2} \, du - \frac{2}{3} \int_1^2 (t-1)t^{3/2} \, dt \\ &= \frac{2}{3} \left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} \right]_2^3 - \frac{2}{3} \left[\frac{2}{7}t^{7/2} - \frac{2}{5}t^{5/2} \right]_1^2 \\ &= \frac{2}{3} \left[\frac{2}{7}(3^{7/2} - 2^{7/2}) - \frac{4}{5}(3^{5/2} - 2^{5/2}) - \frac{2}{7}(2^{7/2} - 1) + \frac{2}{5}(2^{5/2} - 1) \right] \\ &= \frac{2}{3} \left(\frac{18}{35}\sqrt{3} + \frac{8}{35}\sqrt{2} - \frac{4}{35} \right) = \frac{4}{105} (9\sqrt{3} + 4\sqrt{2} - 2)\end{aligned}$$

9. S is the portion of the cone $z^2 = x^2 + y^2$ for $1 \leq z \leq 3$, or equivalently, S is the part of the surface $z = \sqrt{x^2 + y^2}$ over the region $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 9\}$. Thus

$$\begin{aligned}\iint_S x^2 z^2 \, dS &= \iint_D x^2(x^2 + y^2) \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1} \, dA \\ &= \iint_D x^2(x^2 + y^2) \sqrt{\frac{x^2 + y^2}{x^2 + y^2} + 1} \, dA = \iint_D \sqrt{2} x^2(x^2 + y^2) \, dA \\ &= \sqrt{2} \int_0^{2\pi} \int_1^3 (r \cos \theta)^2 (r^2) r \, dr \, d\theta = \sqrt{2} \int_0^{2\pi} \cos^2 \theta \, d\theta \int_1^3 r^5 \, dr \\ &= \sqrt{2} \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} \left[\frac{1}{6}r^6 \right]_1^3 = \sqrt{2} (\pi) \cdot \frac{1}{6} (3^6 - 1) = \frac{364\sqrt{2}}{3} \pi\end{aligned}$$

10. Using y and z as parameters, we have $\mathbf{r}(y, z) = (y + 2z^2)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Then $\mathbf{r}_y \times \mathbf{r}_z = (\mathbf{i} + \mathbf{j}) \times (4z\mathbf{i} + \mathbf{k}) = \mathbf{i} - \mathbf{j} - 4z\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{2 + 16z^2}$. Thus

$$\begin{aligned}\iint_S z \, dS &= \int_0^1 \int_0^1 z \sqrt{2 + 16z^2} \, dy \, dz = \int_0^1 z \sqrt{2 + 16z^2} \, dz = \left[\frac{1}{32} \cdot \frac{2}{3} (2 + 16z^2)^{3/2} \right]_0^1 \\ &= \frac{1}{48} (18^{3/2} - 2^{3/2}) = \frac{13}{12} \sqrt{2}\end{aligned}$$

11. Using x and z as parameters, we have $\mathbf{r}(x, z) = x\mathbf{i} + (x^2 + z^2)\mathbf{j} + z\mathbf{k}$, $x^2 + z^2 \leq 4$. Then

$$\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\mathbf{j}) \times (2z\mathbf{j} + \mathbf{k}) = 2x\mathbf{i} - \mathbf{j} + 2z\mathbf{k} \text{ and } |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2 + 1 + 4z^2} = \sqrt{1 + 4(x^2 + z^2)}.$$

Thus

$$\begin{aligned}
 \iint_S y \, dS &= \iint_{x^2+z^2 \leq 4} (x^2+z^2) \sqrt{1+4(x^2+z^2)} \, dA = \int_0^{2\pi} \int_0^2 r^2 \sqrt{1+4r^2} \, r \, dr \, d\theta \\
 &= \int_0^{2\pi} d\theta \int_0^2 r^2 \sqrt{1+4r^2} \, r \, dr = 2\pi \int_0^2 r^2 \sqrt{1+4r^2} \, r \, dr \\
 &\quad [\text{let } u = 1+4r^2 \Rightarrow r^2 = \frac{1}{4}(u-1) \text{ and } \frac{1}{8} du = r \, dr] \\
 &= 2\pi \int_1^{17} \frac{1}{4}(u-1) \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{16} \pi \int_1^{17} (u^{3/2} - u^{1/2}) \, du \\
 &= \frac{1}{16} \pi \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^{17} = \frac{1}{16} \pi \left[\frac{2}{5} (17)^{5/2} - \frac{2}{3} (17)^{3/2} - \frac{2}{5} + \frac{2}{3} \right] = \frac{\pi}{60} (391\sqrt{17} + 1)
 \end{aligned}$$

12. Here S consists of three surfaces: S_1 , the lateral surface of the cylinder; S_2 , the front formed by the plane $x+y=2$; and the back, S_3 , in the plane $y=0$. On S_1 : using cylindrical coordinates,

$$\begin{aligned}
 \mathbf{r}(\theta, y) &= \sin \theta \mathbf{i} + y \mathbf{j} + \cos \theta \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq y \leq 2 - \sin \theta, \quad |\mathbf{r}_\theta \times \mathbf{r}_y| = 1 \text{ and} \\
 \iint_{S_1} xy \, dS &= \int_0^{2\pi} \int_0^{2-\sin \theta} (\sin \theta) y \, dy \, d\theta = \int_0^{2\pi} \left[2 \sin \theta - 2 \sin^2 \theta + \frac{1}{2} \sin^3 \theta \right] d\theta = -2\pi.
 \end{aligned}$$

On S_2 : $\mathbf{r}(x, z) = x \mathbf{i} + (2-x) \mathbf{j} + z \mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = |-\mathbf{i} - \mathbf{j}| = \sqrt{2}$, where $x^2 + z^2 \leq 1$ and

$$\begin{aligned}
 \iint_{S_2} xy \, dS &= \iint_{x^2+z^2 \leq 1} x(2-x) \sqrt{2} \, dA = \int_0^{2\pi} \int_0^1 \sqrt{2} (2r \sin \theta - r^2 \sin^2 \theta) r \, dr \, d\theta \\
 &= \sqrt{2} \int_0^{2\pi} \left[\frac{2}{3} \sin \theta - \frac{1}{4} \sin^2 \theta \right] d\theta = -\frac{\sqrt{2}}{4} \pi
 \end{aligned}$$

On S_3 : $y=0$ so $\iint_{S_3} xy \, dS = 0$. Hence $\iint_S xy \, dS = -2\pi - \frac{\sqrt{2}}{4} \pi = -\frac{1}{4} (8 + \sqrt{2}) \pi$.

13. Using spherical coordinates and Example 17.6.10 [ET 16.6.10] we have

$$\begin{aligned}
 \mathbf{r}(\phi, \theta) &= 2 \sin \phi \cos \theta \mathbf{i} + 2 \sin \phi \sin \theta \mathbf{j} + 2 \cos \phi \mathbf{k} \text{ and } |\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4 \sin \phi. \text{ Then} \\
 \iint_S (x^2 z + y^2 z) \, dS &= \int_0^{2\pi} \int_0^{\pi/2} (4 \sin^2 \phi) (2 \cos \phi) (4 \sin \phi) \, d\phi \, d\theta = 16\pi \sin^4 \phi \Big|_0^{\pi/2} = 16\pi.
 \end{aligned}$$

14. Using spherical coordinates, $\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$, $0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \theta \leq 2\pi$, and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin \phi$ (see Example 17.6.10 [ET 16.6.10]). Then

$$\iint_S xyz \, dS = \int_0^{2\pi} \int_0^{\pi/4} (\sin^3 \phi \cos \phi \cos \theta \sin \theta) \, d\phi \, d\theta = 0 \text{ since } \int_0^{2\pi} \cos \theta \sin \theta \, d\theta = 0.$$

15. Using cylindrical coordinates, we have $\mathbf{r}(\theta, z) = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j} + z \mathbf{k}$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 2$, and $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$.

$$\iint_S (x^2 y + z^2) \, dS = \int_0^{2\pi} \int_0^2 (27 \cos^2 \theta \sin \theta + z^2) 3 \, dz \, d\theta = \int_0^{2\pi} (162 \cos^2 \theta \sin \theta + 8) \, d\theta = 16\pi$$

16. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1 : $\mathbf{r}(\theta, z) = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j} + z \mathbf{k}$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 2$, $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$,

$$\iint_{S_1} (x^2 + y^2 + z^2) \, dS = \int_0^{2\pi} \int_0^2 (9 + z^2) 3 \, dz \, d\theta = 2\pi(54 + 8) = 124\pi.$$

On S_2 : $\mathbf{r}(\theta, r) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 2 \mathbf{k}$, $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$$\iint_{S_2} (x^2 + y^2 + z^2) \, dS = \int_0^{2\pi} \int_0^3 (r^2 + 4) r \, dr \, d\theta = 2\pi \left(\frac{81}{4} + 18 \right) = \frac{153}{2} \pi.$$

On S_3 : $\mathbf{r}(\theta, r) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}$, $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$$\iint_{S_3} (x^2 + y^2 + z^2) \, dS = \int_0^{2\pi} \int_0^3 (r^2 + 0) r \, dr \, d\theta = 2\pi \left(\frac{81}{4} \right) = \frac{81}{2} \pi.$$

Hence $\iint_S (x^2 + y^2 + z^2) \, dS = 124\pi + \frac{153}{2} \pi + \frac{81}{2} \pi = 241\pi$.

17. $\mathbf{r}(u, v) = u^2 \mathbf{i} + u \sin v \mathbf{j} + u \cos v \mathbf{k}$, $0 \leq u \leq 1$, $0 \leq v \leq \pi/2$ and

$$\mathbf{r}_u \times \mathbf{r}_v = (2u \mathbf{i} + \sin v \mathbf{j} + \cos v \mathbf{k}) \times (u \cos v \mathbf{j} - u \sin v \mathbf{k}) = -u \mathbf{i} + 2u^2 \sin v \mathbf{j} + 2u^2 \cos v \mathbf{k} \text{ and}$$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2 + 4u^4 \sin^2 v + 4u^4 \cos^2 v} = \sqrt{u^2 + 4u^4(\sin^2 v + \cos^2 v)} = u\sqrt{1 + 4u^2}$ (since $u \geq 0$). Then

$$\begin{aligned} \iint_S yz \, dS &= \int_0^{\pi/2} \int_0^1 (u \sin v)(u \cos v) \cdot u\sqrt{1 + 4u^2} \, du \, dv = \int_0^1 u^3 \sqrt{1 + 4u^2} \, du \int_0^{\pi/2} \sin v \cos v \, dv \\ &\quad \left[\text{let } t = 1 + 4u^2 \Rightarrow u^2 = \frac{1}{4}(t - 1) \text{ and } \frac{1}{8} dt = u \, du \right] \\ &= \int_1^5 \frac{1}{8} \cdot \frac{1}{4}(t - 1) \sqrt{t} \, dt \int_0^{\pi/2} \sin v \cos v \, dv = \frac{1}{32} \int_1^5 \left(t^{3/2} - \sqrt{t} \right) dt \int_0^{\pi/2} \sin v \cos v \, dv \\ &= \frac{1}{32} \left[\frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right]_1^5 \left[\frac{1}{2} \sin^2 v \right]_0^{\pi/2} = \frac{1}{32} \left(\frac{2}{5}(5)^{5/2} - \frac{2}{3}(5)^{3/2} - \frac{2}{5} + \frac{2}{3} \right) \cdot \frac{1}{2}(1 - 0) \\ &= \frac{5}{48} \sqrt{5} + \frac{1}{240} \end{aligned}$$

18. $\mathbf{r}_u = \cos v \mathbf{i} + \sin v \mathbf{j}$, $\mathbf{r}_v = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \sin v \mathbf{i} - \cos v \mathbf{j} + u \mathbf{k} \Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1 + u^2}$, so $\iint_S \sqrt{1 + x^2 + y^2} \, dS = \int_0^\pi \int_0^1 \sqrt{1 + u^2} \sqrt{1 + u^2} \, du \, dv = \frac{4}{3}\pi$.

19. $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$, $z = g(x, y) = 4 - x^2 - y^2$, and D is the square $[0, 1] \times [0, 1]$, so by Equation 8

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-xy(-2x) - yz(-2y) + zx] \, dA \\ &= \int_0^1 \int_0^1 [2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2)] \, dy \, dx \\ &= \int_0^1 \left(\frac{1}{3}x^2 + \frac{11}{3}x - x^3 + \frac{34}{15} \right) dx = \frac{713}{180} \end{aligned}$$

20. $\mathbf{F}(x, y, z) = xy \mathbf{i} + 4x^2 \mathbf{j} + yz \mathbf{k}$, $z = g(x, y) = xe^y$, and D is the square $[0, 1] \times [0, 1]$, so by Equation 8

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-xy(e^y) - 4x^2(xe^y) + yz] \, dA = \int_0^1 \int_0^1 (-xye^y - 4x^3e^y + xye^y) \, dy \, dx \\ &= \int_0^1 [-4x^3e^y]_{y=0}^{y=1} dx = (e - 1) \int_0^1 (-4x^3) dx = 1 - e \end{aligned}$$

21. $\mathbf{F}(x, y, z) = xze^y \mathbf{i} - xze^y \mathbf{j} + z \mathbf{k}$, $z = g(x, y) = 1 - x - y$, and $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$.

Since S has downward orientation, we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= - \iint_D [-xze^y(-1) - (-xze^y)(-1) + z] \, dA = - \int_0^1 \int_0^{1-x} (1 - x - y) \, dy \, dx \\ &= - \int_0^1 \left(\frac{1}{2}x^2 - x + \frac{1}{2} \right) dx = -\frac{1}{6} \end{aligned}$$

22. $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z^4 \mathbf{k}$, $z = g(x, y) = \sqrt{x^2 + y^2}$, and D is the disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$. Since S has downward orientation, we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= - \iint_D \left[-x \left(\frac{x}{\sqrt{x^2 + y^2}} \right) - y \left(\frac{y}{\sqrt{x^2 + y^2}} \right) + z^4 \right] \, dA \\ &= - \iint_D \left[\frac{-x^2 - y^2}{\sqrt{x^2 + y^2}} + \left(\sqrt{x^2 + y^2} \right)^4 \right] \, dA = - \int_0^{2\pi} \int_0^1 \left(\frac{-r^2}{r} + r^4 \right) r \, dr \, d\theta \\ &= - \int_0^{2\pi} d\theta \int_0^1 (r^5 - r^2) \, dr = -2\pi \left(\frac{1}{6} - \frac{1}{3} \right) = \frac{\pi}{3} \end{aligned}$$

23. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$, $z = g(x, y) = \sqrt{4 - x^2 - y^2}$ and D is the quarter disk

$\{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq \sqrt{4 - x^2}\}$. S has downward orientation, so by Formula 8,

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= - \iint_D \left[-x \cdot \frac{1}{2}(4 - x^2 - y^2)^{-1/2}(-2x) - (-z) \cdot \frac{1}{2}(4 - x^2 - y^2)^{-1/2}(-2y) + y \right] dA \\ &= - \iint_D \left(\frac{x^2}{\sqrt{4 - x^2 - y^2}} - \sqrt{4 - x^2 - y^2} \cdot \frac{y}{\sqrt{4 - x^2 - y^2}} + y \right) dA \\ &= - \iint_D x^2(4 - (x^2 + y^2))^{-1/2} dA = - \int_0^{\pi/2} \int_0^2 (r \cos \theta)^2 (4 - r^2)^{-1/2} r dr d\theta \\ &= - \int_0^{\pi/2} \cos^2 \theta d\theta \int_0^2 r^3 (4 - r^2)^{-1/2} dr \\ &\quad \left[\text{let } u = 4 - r^2 \Rightarrow r^2 = 4 - u \text{ and } -\frac{1}{2} du = r dr \right] \\ &= - \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta \int_4^0 -\frac{1}{2}(4 - u)(u)^{-1/2} du \\ &= - \left[\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} \left(-\frac{1}{2} \right) \left[8\sqrt{u} - \frac{2}{3}u^{3/2} \right]_4^0 \\ &= -\frac{\pi}{4} \left(-\frac{1}{2} \right) \left(-16 + \frac{16}{3} \right) = -\frac{4}{3}\pi \end{aligned}$$

24. $\mathbf{F}(x, y, z) = xz\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

Using spherical coordinates, S is given by $x = 5 \sin \phi \cos \theta$, $y = 5 \sin \phi \sin \theta$, $z = 5 \cos \phi$, $0 \leq \theta \leq \pi$,

$0 \leq \phi \leq \pi$. $\mathbf{F}(\mathbf{r}(\phi, \theta)) = (5 \sin \phi \cos \theta)(5 \cos \phi)\mathbf{i} + (5 \sin \phi \cos \theta)\mathbf{j} + (5 \sin \phi \sin \theta)\mathbf{k}$ and

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 25 \sin^2 \phi \cos \theta \mathbf{i} + 25 \sin^2 \phi \sin \theta \mathbf{j} + 25 \cos \phi \sin \phi \mathbf{k}$, so

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 625 \sin^3 \phi \cos \phi \cos^2 \theta + 125 \sin^3 \phi \cos \phi \sin \theta + 125 \sin^2 \phi \cos \phi \sin \theta$$

Then

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D [\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)] dA \\ &= \int_0^\pi \int_0^\pi (625 \sin^3 \phi \cos \phi \cos^2 \theta + 125 \sin^3 \phi \cos \phi \sin \theta + 125 \sin^2 \phi \cos \phi \sin \theta) d\theta d\phi \\ &= 125 \int_0^\pi \left[5 \sin^3 \phi \cos \phi \left(\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta \right) + \sin^3 \phi \left(\frac{1}{2} \sin^2 \theta \right) + \sin^2 \phi \cos \phi (-\cos \theta) \right]_{\theta=0}^{\theta=\pi} d\phi \\ &= 125 \int_0^\pi \left(\frac{5}{2} \pi \sin^3 \phi \cos \phi + 2 \sin^2 \phi \cos \phi \right) d\phi \\ &= 125 \left[\frac{5}{2} \pi \cdot \frac{1}{4} \sin^4 \phi + 2 \cdot \frac{1}{3} \sin^3 \phi \right]_0^\pi = 0 \end{aligned}$$

25. Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$ and S_2 the disk $x^2 + z^2 \leq 1$, $y = 1$. Since S is a closed surface, we use the outward orientation. On S_1 : $\mathbf{F}(\mathbf{r}(x, z)) = (x^2 + z^2)\mathbf{j} - z\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\mathbf{i} - \mathbf{j} + 2z\mathbf{k}$ (since the $\mathbf{j}$ -component must be negative on S_1). Then

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2 + z^2 \leq 1} [-(x^2 + z^2) - 2z^2] dA = - \int_0^{2\pi} \int_0^1 (r^2 + 2r^2 \cos^2 \theta) r dr d\theta \\ &= - \int_0^{2\pi} \frac{1}{4} (1 + 2 \cos^2 \theta) d\theta = - \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = -\pi \end{aligned}$$

On S_2 : $\mathbf{F}(\mathbf{r}(x, z)) = \mathbf{j} - z\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = \mathbf{j}$. Then $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2 + z^2 \leq 1} (1) dA = \pi$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = -\pi + \pi = 0$.

26. Here S consists of three surfaces: S_1 , the lateral surface of the cylinder; S_2 , the front formed by the plane $x + y = 2$; and the back, S_3 , in the plane $y = 0$.

On S_1 : $\mathbf{F}(\mathbf{r}(\theta, y)) = \sin \theta \mathbf{i} + y \mathbf{j} + 5 \mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin \theta \mathbf{i} + \cos \theta \mathbf{k} \Rightarrow$

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{2-\sin \theta} (\sin^2 \theta + 5 \cos \theta) dy d\theta \\ &= \int_0^{2\pi} (2 \sin^2 \theta + 10 \cos \theta - \sin^3 \theta - 5 \sin \theta \cos \theta) d\theta = 2\pi\end{aligned}$$

On S_2 : $\mathbf{F}(\mathbf{r}(x, z)) = x \mathbf{i} + (2 - x) \mathbf{j} + 5 \mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = \mathbf{i} + \mathbf{j}$.

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2 + z^2 \leq 1} [x + (2 - x)] dA = 2\pi$$

On S_3 : $\mathbf{F}(\mathbf{r}(x, z)) = x \mathbf{i} + 5 \mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ so $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = 0$. Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi$.

27. Here S consists of the six faces of the cube as labeled in the figure. On S_1 :

$\mathbf{F} = \mathbf{i} + 2y \mathbf{j} + 3z \mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 dy dz = 4$;

S_2 : $\mathbf{F} = x \mathbf{i} + 2 \mathbf{j} + 3z \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_z = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 2 dx dz = 8$;

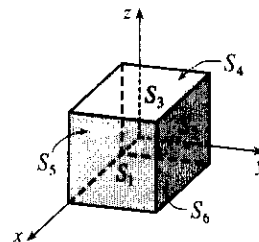
S_3 : $\mathbf{F} = x \mathbf{i} + 2y \mathbf{j} + 3 \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 3 dx dy = 12$;

S_4 : $\mathbf{F} = -\mathbf{i} + 2y \mathbf{j} + 3z \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 4$;

S_5 : $\mathbf{F} = x \mathbf{i} - 2 \mathbf{j} + 3z \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = 8$;

S_6 : $\mathbf{F} = x \mathbf{i} + 2y \mathbf{j} - 3 \mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 3 dx dy = 12$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^6 \iint_{S_i} \mathbf{F} \cdot d\mathbf{S} = 48$.



28. $\mathbf{r}_u = \cos v \mathbf{i} + \sin v \mathbf{j}$, $\mathbf{r}_v = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \sin v \mathbf{i} - \cos v \mathbf{j} + u \mathbf{k}$ and $\mathbf{F}(\mathbf{r}(u, v)) = u \sin v \mathbf{i} + u \cos v \mathbf{j} + v^2 \mathbf{k}$. Then

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^\pi \int_0^1 (u \sin^2 v - u \cos^2 v + uv^2) du dv = \int_0^\pi \int_0^1 (-u \cos 2v + uv^2) du dv \\ &= \int_0^\pi \left[-\frac{1}{2} \cos 2v + \frac{1}{2} v^2 \right] dv = \frac{1}{6} \pi^3.\end{aligned}$$

29. $z = xy \Rightarrow \partial z / \partial x = y$, $\partial z / \partial y = x$, so by Formula 2, a CAS gives

$$\iint_S xyz dS = \int_0^1 \int_0^1 xy(xy) \sqrt{y^2 + x^2 + 1} dx dy \approx 0.1642.$$

30. As in Exercise 29, we use a CAS to calculate

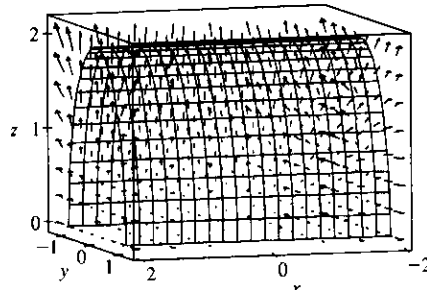
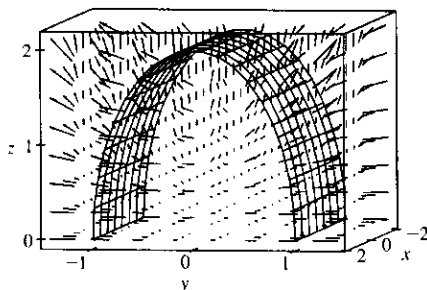
$$\begin{aligned}\iint_S x^2 yz dS &= \int_0^1 \int_0^1 x^2 y(xy) \sqrt{y^2 + x^2 + 1} dx dy \\ &= \frac{1}{60} \sqrt{3} - \frac{1}{12} \ln(1 + \sqrt{3}) - \frac{1}{192} \ln(\sqrt{2} + 1) + \frac{317}{2880} \sqrt{2} + \frac{1}{24} \ln 2\end{aligned}$$

31. We use Formula 2 with $z = 3 - 2x^2 - y^2 \Rightarrow \partial z / \partial x = -4x$, $\partial z / \partial y = -2y$. The boundaries of the region $3 - 2x^2 - y^2 \geq 0$ are $-\sqrt{\frac{3}{2}} \leq x \leq \sqrt{\frac{3}{2}}$ and $-\sqrt{3 - 2x^2} \leq y \leq \sqrt{3 - 2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation takes a very long time) to calculate

$$\iint_S x^2 y^2 z^2 dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}} \int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2 y^2 (3 - 2x^2 - y^2)^2 \sqrt{16x^2 + 4y^2 + 1} dy dx \approx 3.4895$$

32. The flux of $\mathbf{F}$ across S is given by $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$. Now on S , $z = g(x, y) = 2\sqrt{1-y^2}$, so $\partial g/\partial x = 0$ and $\partial g/\partial y = -2y(1-y^2)^{-1/2}$. Therefore, by (8),

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_{-2}^2 \int_{-1}^1 \left(-x^2 y \left[-2y(1-y^2)^{-1/2} \right] + \left[2\sqrt{1-y^2} \right]^2 e^{x/5} \right) dy \, dx \\ &= \frac{1}{3} (16\pi + 80e^{2/5} - 80e^{-2/5})\end{aligned}$$



33. If S is given by $y = h(x, z)$, then S is also the level surface $f(x, y, z) = y - h(x, z) = 0$.

$\mathbf{n} = \frac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \frac{-h_x \mathbf{i} + \mathbf{j} - h_z \mathbf{k}}{\sqrt{h_x^2 + 1 + h_z^2}}$, and $-\mathbf{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (8), using Formula 2 to evaluate

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \\ &= \iint_D (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \frac{\frac{\partial h}{\partial x} \mathbf{i} - \mathbf{j} + \frac{\partial h}{\partial z} \mathbf{k}}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2}} \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2} \, dA\end{aligned}$$

where D is the projection of $f(x, y, z)$ onto the xz -plane. Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P \frac{\partial h}{\partial x} - Q + R \frac{\partial h}{\partial z} \right) dA$$

34. If S is given by $x = k(y, z)$, then S is also the level surface $f(x, y, z) = x - k(y, z) = 0$.

$\mathbf{n} = \frac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \frac{\mathbf{i} - k_y \mathbf{j} - k_z \mathbf{k}}{\sqrt{1 + k_y^2 + k_z^2}}$, and since the x -component is positive this is the unit normal that points forward. Now we proceed as in the derivation of (8), using Formula 2 for

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \\ &= \iint_D (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \frac{\mathbf{i} - \frac{\partial k}{\partial y} \mathbf{j} - \frac{\partial k}{\partial z} \mathbf{k}}{\sqrt{1 + \left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2}} \sqrt{1 + \left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2} \, dA\end{aligned}$$

where D is the projection of $f(x, y, z)$ onto the yz -plane. Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P - Q \frac{\partial k}{\partial y} - R \frac{\partial k}{\partial z} \right) dA$$

35. $m = \iint_S K \, dS = K \cdot 4\pi(\frac{1}{2}a^2) = 2\pi a^2 K$; by symmetry $M_{xz} = M_{yz} = 0$, and
 $M_{xy} = \iint_S zK \, dS = K \int_0^{2\pi} \int_0^{\pi/2} (a \cos \phi)(a^2 \sin \phi) \, d\phi \, d\theta = 2\pi K a^3 \left[-\frac{1}{4} \cos 2\phi\right]_0^{\pi/2} = \pi K a^3$.
Hence $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{1}{2}a)$.

36. S is given by $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \sqrt{x^2 + y^2}\mathbf{k}$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \frac{x^2 + y^2}{x^2 + y^2}} = \sqrt{2}$ so

$$\begin{aligned} m &= \iint_S (10 - \sqrt{x^2 + y^2}) \, dS = \iint_{1 \leq x^2 + y^2 \leq 16} (10 - \sqrt{x^2 + y^2}) \sqrt{2} \, dA \\ &= \int_0^{2\pi} \int_1^4 \sqrt{2} (10 - r) r \, dr \, d\theta = 2\pi \sqrt{2} \left[5r^2 - \frac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\pi \end{aligned}$$

37. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z) \, dS$

$$\begin{aligned} \text{(b) } I_z &= \iint_S (x^2 + y^2) \left(10 - \sqrt{x^2 + y^2}\right) \, dS = \iint_{1 \leq x^2 + y^2 \leq 16} (x^2 + y^2) \left(10 - \sqrt{x^2 + y^2}\right) \sqrt{2} \, dA \\ &= \int_0^{2\pi} \int_1^4 \sqrt{2} (10r^3 - r^4) \, dr \, d\theta = 2\sqrt{2}\pi \left(\frac{4329}{10}\right) = \frac{4329}{5}\sqrt{2}\pi \end{aligned}$$

38. S is given by $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \sqrt{x^2 + y^2}\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$.

$$\begin{aligned} \text{(a) } m &= \iint_S k \, dS = k \iint_{0 \leq x^2 + y^2 \leq a^2} \sqrt{2} \, dS = \sqrt{2} a^2 k \pi; \text{ by symmetry } M_{xz} = M_{yz} = 0, \text{ and} \\ M_{xy} &= \iint_S zk \, dS = k \int_0^{2\pi} \int_0^a \sqrt{2} r^2 \, dr \, d\theta = \frac{2}{3} \sqrt{2} a^3 k \pi. \text{ Hence } (\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{2}{3}a). \end{aligned}$$

$$\text{(b) } I_z = \iint_S (x^2 + y^2)k \, dS = \int_0^{2\pi} \int_0^a \sqrt{2} k r^3 \, dr \, d\theta = 2\pi \sqrt{2} k \left(\frac{1}{4}a^4\right) = \frac{\sqrt{2}}{2} \pi k a^4.$$

39. $\rho(x, y, z) = 1200$, $\mathbf{V} = y\mathbf{i} + \mathbf{j} + z\mathbf{k}$, $\mathbf{F} = \rho\mathbf{V} = (1200)(y\mathbf{i} + \mathbf{j} + z\mathbf{k})$. S is given by

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left[9 - \frac{1}{4}(x^2 + y^2)\right]\mathbf{k}, \quad 0 \leq x^2 + y^2 \leq 36 \text{ and } \mathbf{r}_x \times \mathbf{r}_y = \frac{1}{2}x\mathbf{i} + \frac{1}{2}y\mathbf{j} + \mathbf{k}.$$

Thus the rate of flow is given by

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{0 \leq x^2 + y^2 \leq 36} (1200) \left(\frac{1}{2}xy + \frac{1}{2}y + \left[9 - \frac{1}{4}(x^2 + y^2)\right]\right) \, dA \\ &= 1200 \int_0^6 \int_0^{2\pi} \left[\frac{1}{2}r^2 \sin \theta \cos \theta + \frac{1}{2}r \sin \theta + 9 - \frac{1}{4}r^2\right] r \, d\theta \, dr \\ &= 1200 \int_0^6 2\pi \left(9r - \frac{1}{4}r^3\right) \, dr = (1200)(2\pi)(81) = 194,400\pi \end{aligned}$$

40. $\rho(x, y, z) = 1500$, $\mathbf{F} = \rho\mathbf{V} = (1500)(-y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k})$. S is given by

$$\mathbf{r}(\phi, \theta) = 5 \sin \phi \cos \theta \mathbf{i} + 5 \sin \phi \sin \theta \mathbf{j} + 5 \cos \phi \mathbf{k}, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi, \text{ and}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 25 \sin^2 \phi \cos \theta \mathbf{i} + 25 \sin^2 \phi \sin \theta \mathbf{j} + 25 \sin \phi \cos \phi \mathbf{k}. \text{ Thus the rate of outward flow is}$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= 1500 \int_0^{2\pi} \int_0^\pi (-125 \sin^3 \phi \sin \theta \cos \theta + 125 \sin^3 \phi \sin \theta \cos \theta + 250 \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\ &= (3000\pi)(250) \left(-\frac{1}{3} \cos^3 \phi\right) \Big|_0^\pi = 500,000\pi. \end{aligned}$$

41. S consists of the hemisphere S_1 given by $z = \sqrt{a^2 - x^2 - y^2}$ and the disk S_2 given by $0 \leq x^2 + y^2 \leq a^2$, $z = 0$. On S_1 : $\mathbf{E} = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + 2a \cos \phi \mathbf{k}$, $\mathbf{T}_\phi \times \mathbf{T}_\theta = a^2 \sin^2 \phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k}$. Thus

$$\begin{aligned} \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi/2} (a^3 \sin^3 \phi + 2a^3 \sin \phi \cos^2 \phi) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (a^3 \sin \phi + a^3 \sin \phi \cos^2 \phi) d\phi d\theta = (2\pi)a^3 \left(1 + \frac{1}{3}\right) = \frac{8}{3}\pi a^3 \end{aligned}$$

On S_2 : $\mathbf{E} = x \mathbf{i} + y \mathbf{j}$, and $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ so $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 0$.

Hence the total charge is $q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{8}{3}\pi a^3 \epsilon_0$.

42. Referring to the figure in Exercise 27, on

$$S_1: \mathbf{E} = \mathbf{i} + y \mathbf{j} + z \mathbf{k}, \mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} \text{ and } \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 dy dz = 4;$$

$$S_2: \mathbf{E} = x \mathbf{i} + \mathbf{j} + z \mathbf{k}, \mathbf{r}_z \times \mathbf{r}_x = \mathbf{j} \text{ and } \iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 dx dz = 4;$$

$$S_3: \mathbf{E} = x \mathbf{i} + y \mathbf{j} + \mathbf{k}, \mathbf{r}_x \times \mathbf{r}_y = \mathbf{k} \text{ and } \iint_{S_3} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 dx dy = 4;$$

$$S_4: \mathbf{E} = -\mathbf{i} + y \mathbf{j} + z \mathbf{k}, \mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i} \text{ and } \iint_{S_4} \mathbf{E} \cdot d\mathbf{S} = 4.$$

Similarly $\iint_{S_5} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_6} \mathbf{E} \cdot d\mathbf{S} = 4$. Hence $q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 \sum_{i=1}^6 \iint_{S_i} \mathbf{E} \cdot d\mathbf{S} = 24\epsilon_0$.

43. $K \nabla u = 6.5(4y \mathbf{j} + 4z \mathbf{k})$. S is given by $\mathbf{r}(x, \theta) = x \mathbf{i} + \sqrt{6} \cos \theta \mathbf{j} + \sqrt{6} \sin \theta \mathbf{k}$ and since we want the inward heat flow, we use $\mathbf{r}_x \times \mathbf{r}_\theta = -\sqrt{6} \cos \theta \mathbf{j} - \sqrt{6} \sin \theta \mathbf{k}$. Then the rate of heat flow inward is given by
- $$\iint_S (-K \nabla u) \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^4 -(6.5)(-24) dx d\theta = (2\pi)(156)(4) = 1248\pi.$$

44. $u(x, y, z) = c/\sqrt{x^2 + y^2 + z^2}$,

$$\begin{aligned} \mathbf{F} &= -K \nabla u = -K \left[-\frac{cx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} - \frac{cy}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} - \frac{cz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \right] \\ &= \frac{cK}{(x^2 + y^2 + z^2)^{3/2}} (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \end{aligned}$$

and the outward unit normal is $\mathbf{n} = \frac{1}{a} (x \mathbf{i} + y \mathbf{j} + z \mathbf{k})$.

Thus $\mathbf{F} \cdot \mathbf{n} = \frac{cK}{a(x^2 + y^2 + z^2)^{3/2}} (x^2 + y^2 + z^2)$, but on S , $x^2 + y^2 + z^2 = a^2$ so $\mathbf{F} \cdot \mathbf{n} = \frac{cK}{a^2}$. Hence the rate of

heat flow across S is $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{cK}{a^2} \iint_S dS = \frac{cK}{a^2} (4\pi a^2) = 4\pi Kc$.

17.8 Stokes' Theorem

ET 16.8

- Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2 + y^2 = 4, z = 0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_P \text{curl } \mathbf{F} \cdot d\mathbf{S}$ (where C is the boundary curve).
- The plane $z = 5$ intersects the paraboloid $z = 9 - x^2 - y^2$ in the circle $x^2 + y^2 = 4, z = 5$. This boundary curve C is oriented in the counterclockwise direction, so the vector equation is $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 5 \mathbf{k}$, $0 \leq t \leq 2\pi$. Then $\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 10 \sin t \mathbf{i} + 10 \cos t \mathbf{j} + 4 \cos t \sin t \mathbf{k}$, and by Stokes' Theorem,

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} (-20 \sin^2 t + 20 \cos^2 t) dt \\ &= 20 \int_0^{2\pi} \cos 2t dt = 0 \end{aligned}$$

- The boundary curve C is the circle $x^2 + y^2 = 4, z = 0$ oriented in the counterclockwise direction. The vector equation is $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$, so $\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$ and $\mathbf{F}(\mathbf{r}(t)) = (2 \cos t)^2 e^{(2 \sin t)(0)} \mathbf{i} + (2 \sin t)^2 e^{(2 \cos t)(0)} \mathbf{j} + (0)^2 e^{(2 \cos t)(2 \sin t)} \mathbf{k} = 4 \cos^2 t \mathbf{i} + 4 \sin^2 t \mathbf{j}$. Then, by Stokes' Theorem,

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} (-8 \cos^2 t \sin t + 8 \sin^2 t \cos t) dt \\ &= 8 \left[\frac{1}{3} \cos^3 t + \frac{1}{3} \sin^3 t \right]_0^{2\pi} = 0 \end{aligned}$$

- The boundary curve C is the circle $x^2 + z^2 = 9, y = 3$ with vector equation $\mathbf{r}(t) = 3 \sin t \mathbf{i} + 3 \mathbf{j} + 3 \cos t \mathbf{k}$, $0 \leq t \leq 2\pi$ which gives the positive orientation. Then $\mathbf{F}(\mathbf{r}(t)) = 729 \sin^2 t \cos t \mathbf{i} + \sin(27 \sin t \cos t) \mathbf{j} + 27 \sin t \cos t \mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 2187 \sin^2 t \cos^2 t - 81 \sin^2 t \cos t$. Thus

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} (2187 \sin^2 t \cos^2 t - 81 \sin^2 t \cos t) dt = \int_0^{2\pi} \left(2187 \left(\frac{1}{2} \sin 2t \right)^2 - 81 \sin^2 t \cos t \right) dt \\ &= \left[\frac{2187}{4} \left(\frac{1}{2} t - \frac{1}{8} \sin 4t \right) - 81 \cdot \frac{1}{3} \sin^3 t \right]_0^{2\pi} = \frac{2187}{4} (\pi) - 0 = \frac{2187}{4} \pi \end{aligned}$$

- C is the square in the plane $z = -1$. By (3), $\iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\text{curl } \mathbf{F} = x^2 z \mathbf{i} + (xy - 2xyz) \mathbf{j} + (y - xz) \mathbf{k}$. For S_2 , we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\text{curl } \mathbf{F} \cdot \mathbf{n} = y - xz = x + y$ on S_2 , where $z = -1$. Thus

$$\iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 (x + y) dx dy = 0 \text{ so } \iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.$$

6. The boundary curve C is the unit circle in the yz -plane. By Equation 3,

$$\iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} \text{ where } S_1 \text{ is the original hemisphere and } S_2 \text{ is the disk}$$

$$y^2 + z^2 \leq 1, x = 0. \text{ curl } \mathbf{F} = (x - x^2)\mathbf{i} - (y + e^{xy} \sin z)\mathbf{j} + (2xz - xe^{xy} \cos z)\mathbf{k}, \text{ and for } S_2 \text{ we choose } \mathbf{n} = \mathbf{i}$$

so that C has the same orientation for both surfaces. Then $\text{curl } \mathbf{F} \cdot \mathbf{n} = x - x^2$ on S_2 , where $x = 0$. Thus

$$\iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_{y^2+z^2 \leq 1} (x - x^2) dA = \iint_{y^2+z^2 \leq 1} 0 dA = 0.$$

Alternatively, we can evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$: C with positive orientation is given by $\mathbf{r}(t) = \langle 0, \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$, and

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \left\langle e^{0(\cos t)} \cos(\sin t), (0)^2(\sin t), (0)(\cos t) \right\rangle \cdot \langle 0, -\sin t, \cos t \rangle dt \\ &= \int_0^{2\pi} 0 dt = 0 \end{aligned}$$

7. $\text{curl } \mathbf{F} = -2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}$ and we take the surface S to be the planar region enclosed by C , so S is the portion of the plane $x + y + z = 1$ over $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 17.7.8 [ET 16.7.8], we have $z = g(x, y) = 1 - x - y$, $P = -2z$, $Q = -2x$, $R = -2y$, and

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D [-(-2z)(-1) - (-2x)(-1) + (-2y)] dA \\ &= \int_0^1 \int_0^{1-x} (-2) dy dx = -2 \int_0^1 (1-x) dx = -1 \end{aligned}$$

8. $\text{curl } \mathbf{F} = e^x \mathbf{k}$ and S is the portion of the plane $2x + y + 2z = 2$ over $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x\}$. We orient S upward and use Equation 17.7.8 [ET 16.7.8] with $z = g(x, y) = 1 - x - \frac{1}{2}y$:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (0 + 0 + e^x) dA = \int_0^1 \int_0^{2-2x} e^x dy dx \\ &= \int_0^1 (2-2x)e^x dx = [(2-2x)e^x + 2e^x]_0^1 \quad [\text{by integrating by parts}] \\ &= 2e - 4 \end{aligned}$$

9. $\text{curl } \mathbf{F} = (xe^{xy} - 2x)\mathbf{i} - (ye^{xy} - y)\mathbf{j} + (2z - z)\mathbf{k}$ and we take S to be the disk $x^2 + y^2 \leq 16$, $z = 5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\mathbf{n} = \mathbf{k}$ and $\text{curl } \mathbf{F} \cdot \mathbf{n} = 2z - z$ on S , where $z = 5$. Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS = \iint_S (2z - z) dS = \iint_S (10 - 5) dS = 5(\text{area of } S) = 5(\pi \cdot 4^2) = 80\pi$$

10. S is the part of the surface $z = 1 - x^2 - y^2$ in the first octant. $\text{curl } \mathbf{F} = 2y\mathbf{i} - 2x\mathbf{j}$.

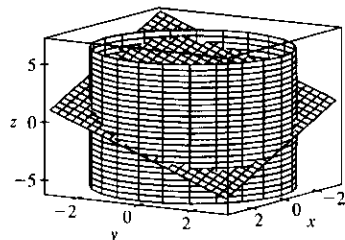
Using Equation 17.7.8 [ET 16.7.8] with $g(x, y) = 1 - x^2 - y^2$, $P = 2y$, $Q = -2x$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D [-2y(-2x) + (2x)(-2y)] dA = \iint_D 0 dA = 0.$$

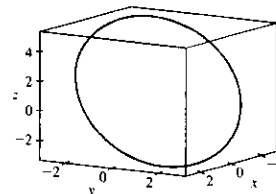
11. (a) The curve of intersection is an ellipse in the plane $x + y + z = 1$ with unit normal $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\text{curl } \mathbf{F} = x^2 \mathbf{j} + y^2 \mathbf{k}$ and $\text{curl } \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(x^2 + y^2)$. Then

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \frac{1}{\sqrt{3}}(x^2 + y^2) dS = \iint_{x^2 + y^2 \leq 9} (x^2 + y^2) dx dy \\ &= \int_0^{2\pi} \int_0^3 r^3 dr d\theta = 2\pi \left(\frac{81}{4}\right) = \frac{81\pi}{2}\end{aligned}$$

(b)



- (c) One possible parametrization is $x = 3 \cos t$, $y = 3 \sin t$, $z = 1 - 3 \cos t - 3 \sin t$, $0 \leq t \leq 2\pi$.

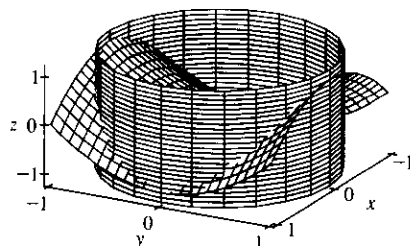


12. (a) S is the part of the surface $z = y^2 - x^2$ that lies above the unit disk D .

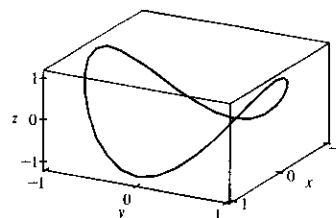
$\text{curl } \mathbf{F} = x \mathbf{i} - y \mathbf{j} + (x^2 - x^2) \mathbf{k} = x \mathbf{i} - y \mathbf{j}$. Using Equation 17.7.8 [ET 16.7.8] with $g(x, y) = y^2 - x^2$, $P = x$, $Q = -y$, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D [-x(-2x) - (-y)(2y)] dA = 2 \iint_D (x^2 + y^2) dA \\ &= 2 \int_0^{2\pi} \int_0^1 r^2 r dr d\theta = 2(2\pi) \left[\frac{1}{4} r^4\right]_0^1 = \pi\end{aligned}$$

(b)



- (c) One possible set of parametric equations is $x = \cos t$, $y = \sin t$, $z = \sin^2 t - \cos^2 t$, $0 \leq t \leq 2\pi$.



13. The boundary curve C is the circle $x^2 + y^2 = 1$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \mathbf{k}$, $0 \leq t \leq 2\pi$, and then $\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = \sin^2 t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \cos^2 t - \sin^3 t$, and

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos^2 t - \sin^3 t) dt = \int_0^{2\pi} \frac{1}{2}(1 + \cos 2t) dt - \int_0^{2\pi} (1 - \cos^2 t) \sin t dt \\ &= \frac{1}{2} \left[t + \frac{1}{2} \sin 2t\right]_0^{2\pi} - \left[-\cos t + \frac{1}{3} \cos^3 t\right]_0^{2\pi} = \pi\end{aligned}$$

Now $\text{curl } \mathbf{F} = (1 - 2y) \mathbf{k}$, and the projection D of S on the xy -plane is the disk $x^2 + y^2 \leq 1$, so by

Equation 17.7.8 [ET 16.7.8] with $z = g(x, y) = x^2 + y^2$ we have

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 - 2y) dA = \int_0^{2\pi} \int_0^1 (1 - 2r \sin \theta) r dr d\theta = \int_0^{2\pi} \left(\frac{1}{2} - \frac{2}{3} \sin \theta\right) d\theta = \pi$$

14. The plane intersects the coordinate axes at $x = 1$, $y = z = 2$ so the boundary curve C consists of the three line segments C_1 : $\mathbf{r}_1(t) = (1-t)\mathbf{i} + 2t\mathbf{j}$, $0 \leq t \leq 1$, C_2 : $\mathbf{r}_2(t) = (2-2t)\mathbf{j} + 2t\mathbf{k}$, $0 \leq t \leq 1$, C_3 : $\mathbf{r}_3(t) = t\mathbf{i} + (2-2t)\mathbf{k}$, $0 \leq t \leq 1$. Then

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 [(1-t)\mathbf{i} + 2t\mathbf{j}] \cdot (-\mathbf{i} + 2\mathbf{j}) dt + \int_0^1 [(2-2t)\mathbf{j}] \cdot (-2\mathbf{j} + 2\mathbf{k}) dt + \int_0^1 (t\mathbf{i}) \cdot (\mathbf{i} - 2\mathbf{k}) dt \\ &= \int_0^1 (5t-1) dt + \int_0^1 (4t-4) dt + \int_0^1 t dt = \frac{3}{2} - 2 + \frac{1}{2} = 0\end{aligned}$$

Now $\text{curl } \mathbf{F} = xz\mathbf{i} - yz\mathbf{j}$, so by Equation 17.7.8 [ET 16.7.8] with $z = g(x, y) = 2 - 2x - y$ we have

$$\begin{aligned}\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-x(2-2x-y)(-2) + y(2-2x-y)(-1)] dA \\ &= \int_0^1 \int_0^{2-2x} (4x - 4x^2 - 2y + y^2) dy dx \\ &= \int_0^1 [4x(2-2x) - 4x^2(2-2x) - (2-2x)^2 + \frac{1}{3}(2-2x)^3] dx \\ &= \int_0^1 \left(\frac{16}{3}x^3 - 12x^2 + 8x - \frac{4}{3}\right) dx = \left[\frac{4}{3}x^4 - 4x^3 + 4x^2 - \frac{4}{3}x\right]_0^1 = 0\end{aligned}$$

15. The boundary curve C is the circle $x^2 + z^2 = 1$, $y = 0$ oriented in the counterclockwise direction as viewed from the positive y -axis. Then C can be described by $\mathbf{r}(t) = \cos t\mathbf{i} - \sin t\mathbf{k}$, $0 \leq t \leq 2\pi$, and $\mathbf{r}'(t) = -\sin t\mathbf{i} - \cos t\mathbf{k}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -\sin t\mathbf{j} + \cos t\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\cos^2 t$, and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -\cos^2 t dt = -\frac{1}{2}t - \frac{1}{4}\sin 2t \Big|_0^{2\pi} = -\pi$$

Now $\text{curl } \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, and S can be parametrized (see Example 17.6.10 [ET 16.6.10]) by

$$\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \pi. \quad \text{Then}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2 \phi \cos \theta \mathbf{i} + \sin^2 \phi \sin \theta \mathbf{j} + \sin \phi \cos \phi \mathbf{k} \quad \text{and}$$

$$\begin{aligned}\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2+z^2 \leq 1} \text{curl } \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) dA \\ &= \int_0^\pi \int_0^\pi (-\sin^2 \phi \cos \theta - \sin^2 \phi \sin \theta - \sin \phi \cos \phi) d\theta d\phi \\ &= \int_0^\pi (-2\sin^2 \phi - \pi \sin \phi \cos \phi) d\phi = \left[\frac{1}{2}\sin 2\phi - \phi - \frac{\pi}{2}\sin^2 \phi\right]_0^\pi = -\pi\end{aligned}$$

16. The components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$, and both the curve C and the surface S meet the requirements of Stokes' Theorem. If there is a vector field $\mathbf{G}$ where $\mathbf{F} = \text{curl } \mathbf{G}$, then Stokes' Theorem says $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{G} \cdot d\mathbf{S}$ depends only on the values of $\mathbf{G}$ on C , and hence is independent of the choice of S . By Theorem 17.5.11 [ET 16.5.11], $\text{div } \text{curl } \mathbf{G} = 0$, so $\text{div } \mathbf{F} = 0 \Leftrightarrow$
- $$(3ax^2 - 3z^2) + (x^2 + 3by^2) + (3cz^2) = 0 \Leftrightarrow (3a+1)x^2 + 3by^2 + (3c-3)z^2 = 0 \Leftrightarrow$$
- $$a = -\frac{1}{3}, b = 0, c = 1.$$

$$17. \operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^x + z^2 & y^y + x^2 & z^z + y^2 \end{vmatrix} = 2y \mathbf{i} + 2z \mathbf{j} + 2x \mathbf{k} \text{ and } W = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

To parametrize the surface, let $x = 2 \cos \theta \sin \phi$, $y = 2 \sin \theta \sin \phi$, $z = 2 \cos \phi$, so that

$$\mathbf{r}(\phi, \theta) = 2 \sin \phi \cos \theta \mathbf{i} + 2 \sin \phi \sin \theta \mathbf{j} + 2 \cos \phi \mathbf{k}, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \text{ and}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4 \sin^2 \phi \cos \theta \mathbf{i} + 4 \sin^2 \phi \sin \theta \mathbf{j} + 4 \sin \phi \cos \phi \mathbf{k}. \text{ Then}$$

$$\operatorname{curl} \mathbf{F}(\mathbf{r}(\phi, \theta)) = 4 \sin \phi \sin \theta \mathbf{i} + 4 \cos \phi \mathbf{j} + 4 \sin \phi \cos \theta \mathbf{k}, \text{ and}$$

$$\operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16 \sin^3 \phi \sin \theta \cos \theta + 16 \cos \phi \sin^2 \phi \sin \theta + 16 \sin^2 \phi \cos \phi \cos \theta. \text{ Therefore}$$

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) dA \\ &= 16 \left[\int_0^{\pi/2} \sin \theta \cos \theta d\theta \right] \left[\int_0^{\pi/2} \sin^3 \phi d\phi \right] + 16 \left[\int_0^{\pi/2} \sin \theta d\theta \right] \left[\int_0^{\pi/2} \sin^2 \phi \cos \phi d\phi \right] \\ &\quad + 16 \left[\int_0^{\pi/2} \cos \theta d\theta \right] \left[\int_0^{\pi/2} \sin^2 \phi \cos \phi d\phi \right] \\ &= 8 \left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} + 16(1) \left[\frac{1}{3} \sin^3 \phi \right]_0^{\pi/2} + 16(1) \left[\frac{1}{3} \sin^3 \phi \right]_0^{\pi/2} \\ &= 8 \left[0 + 1 + 0 - \frac{1}{3} \right] + 16 \left(\frac{1}{3} \right) + 16 \left(\frac{1}{3} \right) = \frac{16}{3} + \frac{16}{3} + \frac{16}{3} = 16 \end{aligned}$$

$$18. \int_C (y + \sin x) dx + (z^2 + \cos y) dy + x^3 dz = \int_C \mathbf{F} \cdot d\mathbf{r}, \text{ where } \mathbf{F}(x, y, z) = (y + \sin x) \mathbf{i} + (z^2 + \cos y) \mathbf{j} + x^3 \mathbf{k}$$

$\Rightarrow \operatorname{curl} \mathbf{F} = -2z \mathbf{i} - 3x^2 \mathbf{j} - \mathbf{k}$. Since $\sin 2t = 2 \sin t \cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C . Then the projection of S onto the xy -plane is the unit disk D ($x^2 + y^2 \leq 1$). C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 17.7.8 [ET 16.7.8] with $g(x, y) = 2xy$, $P = -2(2xy) = -4xy$, $Q = -3x^2$, $R = -1$, we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= - \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = - \iint_D [-(-4xy)(2y) - (-3x^2)(2x) - 1] dA \\ &= - \iint_D (8xy^2 + 6x^3 - 1) dA = - \int_0^{2\pi} \int_0^1 (8r^3 \cos \theta \sin^2 \theta + 6r^3 \cos^3 \theta - 1) r dr d\theta \\ &= - \int_0^{2\pi} \left(\frac{8}{5} \cos \theta \sin^2 \theta + \frac{6}{5} \cos^3 \theta - \frac{1}{2} \right) r dr d\theta \\ &= - \left[\frac{8}{15} \sin^3 \theta + \frac{6}{5} (\sin \theta - \frac{1}{3} \sin^3 \theta) - \frac{1}{2} \theta \right]_0^{2\pi} = \pi \end{aligned}$$

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S . Then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{H_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} + \iint_{H_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ so $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ as desired.

20. (a) By Exercise 17.5.26 [ET 16.5.26], $\operatorname{curl}(f \nabla g) = f \operatorname{curl}(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ since $\operatorname{curl}(\nabla g) = 0$. Hence by Stokes' Theorem $\int_C (f \nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$.

(b) As in (a), $\operatorname{curl}(f \nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f \nabla f) \cdot d\mathbf{r} = \iint_S [\operatorname{curl}(f \nabla f)] \cdot d\mathbf{S} = 0$.

(c) As in part (a),

$$\begin{aligned}\operatorname{curl}(f\nabla g + g\nabla f) &= \operatorname{curl}(f\nabla g) + \operatorname{curl}(g\nabla f) \quad (\text{by Exercise 17.5.24 [ET 16.5.24]}) \\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} \quad [\text{since } \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})]\end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g + g\nabla f) \cdot d\mathbf{S} = 0$.

17.9 The Divergence Theorem

ET 16.9

- The vectors that end near P_1 are longer than the vectors that start near P_1 , so the net flow is inward near P_1 and $\operatorname{div} \mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2 , so the net flow is outward near P_2 and $\operatorname{div} \mathbf{F}(P_2)$ is positive.
- (a) The vectors that end near P_1 are shorter than the vectors that start near P_1 , so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2 , so the net flow is inward and P_2 is a sink.
(b) $\mathbf{F}(x, y) = \langle x, y^2 \rangle \Rightarrow \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = 1 + 2y$. The y -value at P_1 is positive, so $\operatorname{div} \mathbf{F} = 1 + 2y$ is positive, thus P_1 is a source. At P_2 , $y < -1$, so $\operatorname{div} \mathbf{F} = 1 + 2y$ is negative, and P_2 is a sink.

- $\operatorname{div} \mathbf{F} = 3 + x + 2x = 3 + 3x$, so

$$\iiint_E \operatorname{div} \mathbf{F} \, dV = \int_0^1 \int_0^1 \int_0^1 (3x + 3) \, dx \, dy \, dz = \frac{9}{2} \quad (\text{notice the triple integral is three times the volume of the cube plus three times } \bar{x}).$$

To compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, on S_1 : $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$, and

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} 3 \, dS = 3;$$

$$S_2: \mathbf{F} = 3x\mathbf{i} + x\mathbf{j} + 2xz\mathbf{k}, \mathbf{n} = \mathbf{j} \text{ and } \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} x \, dS = \frac{1}{2};$$

$$S_3: \mathbf{F} = 3x\mathbf{i} + xy\mathbf{j} + 2x\mathbf{k}, \mathbf{n} = \mathbf{k} \text{ and } \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_3} 2x \, dS = 1;$$

$$S_4: \mathbf{F} = \mathbf{0}, \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 0; S_5: \mathbf{F} = 3x\mathbf{i} + 2x\mathbf{k}, \mathbf{n} = -\mathbf{j} \text{ and } \iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_5} 0 \, dS = 0;$$

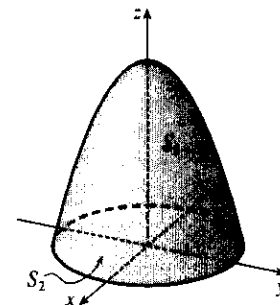
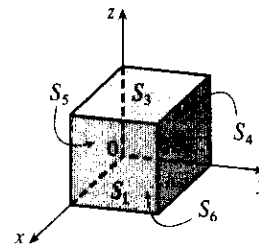
$$S_6: \mathbf{F} = 3x\mathbf{i} + xy\mathbf{j}, \mathbf{n} = -\mathbf{k} \text{ and } \iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_6} 0 \, dS = 0. \text{ Thus } \iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{9}{2}.$$

- $\operatorname{div} \mathbf{F} = 2x + x + 1 = 3x + 1$ so

$$\begin{aligned}\iiint_E \operatorname{div} \mathbf{F} \, dV &= \iiint_E (3x + 1) \, dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3r \cos \theta + 1) r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 r(3r \cos \theta + 1)(4 - r^2) \, d\theta \, dr \\ &= \int_0^{2\pi} r(4 - r^2) [3r \sin \theta + \theta]_{\theta=0}^{\theta=2\pi} \, dr \\ &= 2\pi \int_0^2 (4r - r^3) \, dr = 2\pi [2r^2 - \frac{1}{4}r^4]_0^2 \\ &= 2\pi(8 - 4) = 8\pi\end{aligned}$$

On S_1 : The surface is $z = 4 - x^2 - y^2$, $x^2 + y^2 \leq 4$, with upward orientation, and $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + (4 - x^2 - y^2)\mathbf{k}$. Then

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-(x^2)(-2x) - (xy)(-2y) + (4 - x^2 - y^2)] \, dA \\ &= \iint_D [2x(x^2 + y^2) + 4 - (x^2 + y^2)] \, dA = \int_0^{2\pi} \int_0^2 (2r \cos \theta \cdot r^2 + 4 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} [\frac{2}{5}r^5 \cos \theta + 2r^2 - \frac{1}{4}r^4]_{r=0}^{r=2} \, d\theta = \int_0^{2\pi} (\frac{64}{5} \cos \theta + 4) \, d\theta \\ &= [\frac{64}{5} \sin \theta + 4\theta]_0^{2\pi} = 8\pi\end{aligned}$$



On S_2 : The surface is $z = 0$ with downward orientation, so $\mathbf{F} = x^2 \mathbf{i} + xy \mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_2} 0 \, dS = 0.$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 8\pi$.

5. $\operatorname{div} \mathbf{F} = x + y + z$, so

$$\begin{aligned} \iiint_E \operatorname{div} \mathbf{F} \, dV &= \int_0^{2\pi} \int_0^1 \int_0^1 (r \cos \theta + r \sin \theta + z) r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (r^2 \cos \theta + r^2 \sin \theta + \tfrac{1}{2}r) \, dr \, d\theta \\ &= \int_0^{2\pi} (\tfrac{1}{3} \cos \theta + \tfrac{1}{3} \sin \theta + \tfrac{1}{4}) \, d\theta = \tfrac{1}{4}(2\pi) = \tfrac{\pi}{2} \end{aligned}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. On S_1 , $z = 1$, $\mathbf{n} = \mathbf{k}$, and

$\mathbf{F} = xy \mathbf{i} + y \mathbf{j} + x \mathbf{k}$, so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} x \, dS = \int_0^{2\pi} \int_0^1 (r \cos \theta) r \, dr \, d\theta = [\sin \theta]_0^{2\pi} \left[\tfrac{1}{3} r^3 \right]_0^1 = 0.$$

On S_2 , $z = 0$, $\mathbf{n} = -\mathbf{k}$, and $\mathbf{F} = xy \mathbf{i}$ so $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} 0 \, dS = 0$. S_3 is given by $\mathbf{r}(\theta, z) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + z \mathbf{k}$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 1$. Then $\mathbf{r}_\theta \times \mathbf{r}_z = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ and

$$\begin{aligned} \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) \, dA = \int_0^{2\pi} \int_0^1 (\cos^2 \theta \sin \theta + z \sin^2 \theta) \, dz \, d\theta \\ &= \int_0^{2\pi} (\cos^2 \theta \sin \theta + \tfrac{1}{2} \sin^2 \theta) \, d\theta = \left[-\tfrac{1}{3} \cos^3 \theta + \tfrac{1}{4} (\theta - \tfrac{1}{2} \sin 2\theta) \right]_0^{2\pi} = \tfrac{\pi}{2} \end{aligned}$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0 + 0 + \tfrac{\pi}{2} = \tfrac{\pi}{2}$.

6. $\operatorname{div} \mathbf{F} = 1 + 1 + 1 = 3$, so $\iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 3 \, dV = 3(\text{volume of ball}) = 3(\tfrac{4}{3}\pi) = 4\pi$. To find

$\iint_S \mathbf{F} \cdot d\mathbf{S}$ we use spherical coordinates. S is the unit sphere, represented by

$\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$. Then

$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2 \phi \cos \theta \mathbf{i} + \sin^2 \phi \sin \theta \mathbf{j} + \sin \phi \cos \phi \mathbf{k}$ (see Example 17.6.10 [ET 16.6.10]) and

$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$. Thus

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \, dA = \int_0^{2\pi} \int_0^\pi (\sin^3 \phi \cos^2 \theta + \sin^3 \phi \sin^2 \theta + \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \, d\phi = (2\pi)(2) = 4\pi \end{aligned}$$

7. $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(yz^2) = e^x \sin y - e^x \sin y + 2yz = 2yz$, so by the Divergence Theorem,

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV = \int_0^1 \int_0^1 \int_0^2 2yz \, dz \, dy \, dx = 2 \int_0^1 dx \int_0^1 y \, dy \int_0^1 z \, dz \\ &= 2 \left[x \right]_0^1 \left[\tfrac{1}{2} y^2 \right]_0^1 \left[\tfrac{1}{2} z^2 \right]_0^2 = 2 \end{aligned}$$

8. $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2 z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(xz^4) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$, so by the Divergence Theorem,

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV = \int_{-1}^1 \int_{-2}^2 \int_{-3}^3 8xz^3 \, dz \, dy \, dx = 8 \int_{-1}^1 x \, dx \int_{-2}^2 dy \int_{-3}^3 z^3 \, dz \\ &= 8 \left[\tfrac{1}{2} x^2 \right]_{-1}^1 \left[y \right]_{-2}^2 \left[\tfrac{1}{4} z^4 \right]_{-3}^3 = 0 \end{aligned}$$

9. $\operatorname{div} \mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r \cos \theta$, $z = r \sin \theta$, $x = x$ we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E (3y^2 + 3z^2) \, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^1 (3r^2 \cos^2 \theta + 3r^2 \sin^2 \theta) r \, dx \, dr \, d\theta \\ &= 3 \int_0^{2\pi} d\theta \int_0^1 r^3 \, dr \int_{-1}^1 dx = 3(2\pi) \left(\tfrac{1}{4} \right) (2) = \tfrac{9\pi}{2} \end{aligned}$$

10. $\operatorname{div} \mathbf{F} = 3x^2 y - 2x^2 y - x^2 y = 0$, so $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 0 \, dV = 0$.

11. $\operatorname{div} \mathbf{F} = y \sin z + 0 - y \sin z = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 0 \, dV = 0$.

- 12.
- $\operatorname{div} \mathbf{F} = 2xy + 2xy + 2xy = 6xy$
- , so

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E 6xy \, dV \\
 &= \int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 6xy \, dz \, dx \, dy = \int_0^1 \int_0^{2-2y} 6xy(2-x-2y) \, dx \, dy \\
 &= \int_0^1 \int_0^{2-2y} (12xy - 6x^2y - 12xy^2) \, dx \, dy = \int_0^1 [6x^2y - 2x^3y - 6x^2y^2]_{x=0}^{x=2-2y} \, dy \\
 &= \int_0^1 y(2-2y)^3 \, dy = \left[-\frac{8}{5}y^5 + 6y^4 - 8y^3 + 4y^2\right]_0^1 = \frac{2}{5}
 \end{aligned}$$

- 13.
- $\operatorname{div} \mathbf{F} = y^2 + 0 + x^2 = x^2 + y^2$
- so

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^3(4-r^2) \, dr \, d\theta \\
 &= \int_0^{2\pi} d\theta \int_0^2 (4r^3 - r^5) \, dr = 2\pi \left[r^4 - \frac{1}{6}r^6\right]_0^2 = \frac{32}{3}\pi
 \end{aligned}$$

- 14.
- $\operatorname{div} \mathbf{F} = 4x^3 + 4xy^2$
- so

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E 4x(x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^1 \int_0^{r \cos \theta + 2} (4r^3 \cos \theta) r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 (4r^5 \cos^2 \theta + 8r^4 \cos \theta) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{2}{3} \cos^2 \theta + \frac{8}{5} \cos \theta\right) d\theta = \frac{2}{3}\pi
 \end{aligned}$$

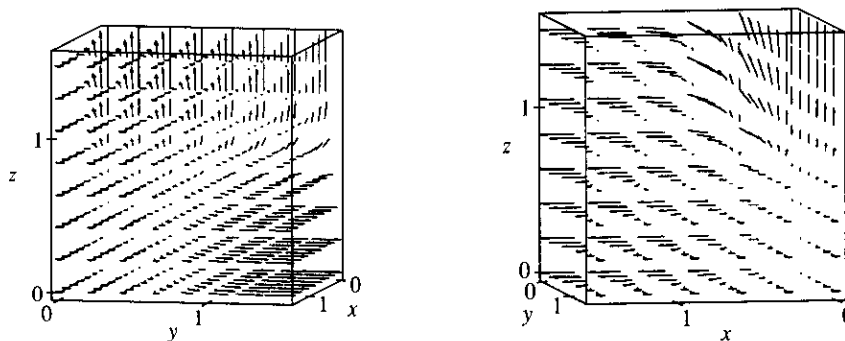
- 15.
- $\operatorname{div} \mathbf{F} = 12x^2z + 12y^2z + 12z^3$
- so

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E 12z(x^2 + y^2 + z^2) \, dV = \int_0^{2\pi} \int_0^\pi \int_0^R 12(\rho \cos \phi)(\rho^2)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= 12 \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \cos \phi \, d\phi \int_0^R \rho^5 \, d\rho = 12(2\pi) \left[\frac{1}{2} \sin^2 \phi\right]_0^\pi \left[\frac{1}{6} \rho^6\right]_0^R = 0
 \end{aligned}$$

- 16.
- $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + 1) \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 3(\rho^2 \sin^2 \phi + 1) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
-
- $= 2\pi \int_0^{\pi/2} \left[\frac{93}{5} \sin^3 \phi + 7 \sin \phi\right] d\phi = 2\pi \left[\frac{93}{5}(-\cos \phi + \frac{1}{3} \cos^3 \phi) - 7 \cos \phi\right]_0^{\pi/2} = \frac{194}{5}\pi$

- 17.
- $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \sqrt{3-x^2} \, dV = \int_{-1}^1 \int_{-1}^1 \int_0^{2-x^4-y^4} \sqrt{3-x^2} \, dz \, dy \, dx = \frac{341}{60} \sqrt{2} + \frac{81}{20} \sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

- 18.



By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of the cube is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} [\cos x \cos^2 y + 3 \sin^2 y \cos y \cos^4 z + 5 \sin^4 z \cos z \cos^6 x] \, dz \, dy \, dx = \frac{19}{64}\pi^2.$$

19. For
- S_1
- we have
- $\mathbf{n} = -\mathbf{k}$
- , so
- $\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot (-\mathbf{k}) = -x^2z - y^2 = -y^2$
- (since
- $z = 0$
- on
- S_1
-). So if
- D
- is the unit disk, we get
- $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D (-y^2) \, dA = -\int_0^{2\pi} \int_0^1 r^2 (\sin^2 \theta) r \, dr \, d\theta = -\frac{1}{4}\pi$
- . Now since
- S_2
- is closed, we can use the Divergence Theorem. Since

$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}(\frac{1}{3}y^3 + \tan z) + \frac{\partial}{\partial z}(x^2z + y^2) = z^2 + y^2 + x^2$, we use spherical coordinates to get

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{2}{5}\pi. \text{ Finally}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi.$$

20. As in the hint to Exercise 19, we create a closed surface $S_2 = S \cup S_1$, where S is the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2 + y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F} \cdot (-\mathbf{k}) = -z = -1$ on S_1 . So

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS = \iint_{S_1} (-1) dS = -A(S_1) = -\pi. \text{ Let } E \text{ be the region bounded by } S_2. \text{ Then}$$

$$\begin{aligned} \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 1 dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r dz d\theta dr = \int_0^1 \int_0^{2\pi} (r - r^3) d\theta dr \\ &= (2\pi) \frac{1}{4} = \frac{\pi}{2} \end{aligned}$$

Thus the flux of $\mathbf{F}$ across S is $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

21. Since $\frac{\mathbf{x}}{|\mathbf{x}|^3} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$ and $\frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) = \frac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{5/2}}$ with similar expressions for $\frac{\partial}{\partial y} \left(\frac{y}{(x^2 + y^2 + z^2)^{3/2}} \right)$ and $\frac{\partial}{\partial z} \left(\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right)$, we have
- $$\operatorname{div} \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} \right) = \frac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0, \text{ except at } (0, 0, 0) \text{ where it is undefined.}$$

22. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_S (2x + 2y + z^2) dS$, so $\mathbf{F} \cdot \mathbf{n} = 2x + 2y + z^2$.

But for S , $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Thus $\mathbf{F} = 2\mathbf{i} + 2\mathbf{j} + z\mathbf{k}$ and $\operatorname{div} \mathbf{F} = 1$. If

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}, \text{ then } \iint_S (2x + 2y + z^2) dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi.$$

23. $\iint_S \mathbf{a} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{a} dV = 0$ since $\operatorname{div} \mathbf{a} = 0$.

24. $\frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \iiint_E \operatorname{div} \mathbf{F} dV = \frac{1}{3} \iiint_E 3 dV = V(E)$

25. $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F}) dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

26. $\iint_S D_{\mathbf{n}} f dS = \iint_S (\nabla f \cdot \mathbf{n}) dS = \iiint_E \operatorname{div}(\nabla f) dV = \iiint_E \nabla^2 f dV$

27. $\iint_S (f \nabla g) \cdot \mathbf{n} dS = \iiint_E \operatorname{div}(f \nabla g) dV = \iiint_E (f \nabla^2 g + \nabla g \cdot \nabla f) dV$ by Exercise 17.5.25 [ET 16.5.25].

28. $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} dS = \iiint_E [(f \nabla^2 g + \nabla g \cdot \nabla f) - (g \nabla^2 f + \nabla f \cdot \nabla g)] dV$ [by Exercise 27].

But $\nabla g \cdot \nabla f = \nabla f \cdot \nabla g$, so that $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} dS = \iiint_E (f \nabla^2 g - g \nabla^2 f) dV$.

29. If $\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1 \mathbf{i} + fc_2 \mathbf{j} + fc_3 \mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \operatorname{div} f\mathbf{c} = \frac{\partial f}{\partial x} c_1 + \frac{\partial f}{\partial y} c_2 + \frac{\partial f}{\partial z} c_3 = \nabla f \cdot \mathbf{c} \text{ and the Divergence Theorem says}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \nabla f \cdot \mathbf{c} dV. \text{ In particular, if } \mathbf{c} = \mathbf{i} \text{ then}$$

$$\iint_S f \mathbf{i} \cdot \mathbf{n} dS = \iiint_E \nabla f \cdot \mathbf{i} dV \Rightarrow \iint_S f n_1 dS = \iiint_E \frac{\partial f}{\partial x} dV \text{ (where } \mathbf{n} = n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k}).$$

Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\iint_S f n_2 dS = \iiint_E \frac{\partial f}{\partial y} dV$, and $\mathbf{c} = \mathbf{k}$ gives $\iint_S f n_3 dS = \iiint_E \frac{\partial f}{\partial z} dV$. Then

$$\begin{aligned} \iint_S f \mathbf{n} dS &= (\iint_S f n_1 dS) \mathbf{i} + (\iint_S f n_2 dS) \mathbf{j} + (\iint_S f n_3 dS) \mathbf{k} \\ &= \left(\iiint_E \frac{\partial f}{\partial x} dV \right) \mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y} dV \right) \mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z} dV \right) \mathbf{k} \\ &= \iiint_E \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) dV = \iiint_E \nabla f dV \end{aligned}$$

as desired.

30. By Exercise 29, $\iint_S p \mathbf{n} \, dS = \iiint_E \nabla p \, dV$, so

$$\begin{aligned} \mathbf{F} &= - \iint_S p \mathbf{n} \, dS = - \iiint_E \nabla p \, dV = - \iiint_E \nabla(\rho g z) \, dV = - \iiint_E (\rho g \mathbf{k}) \, dV \\ &= -\rho g \left(\iiint_E dV \right) \mathbf{k} = -\rho g V(E) \mathbf{k} \end{aligned}$$

But the weight of the displaced liquid is volume $\times$ density $\times g = \rho g V(E)$, thus $\mathbf{F} = -W\mathbf{k}$ as desired.

17 Review

ET 16

CONCEPT CHECK

- See Definitions 1 and 2 in Section 17.1 [ET 16.1]. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.
- (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f .
(b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.
- (a) See Definition 17.2.2 [ET 16.2.2].
(b) We normally evaluate the line integral using Formula 17.2.3 [ET 16.2.3].
(c) The mass is $m = \int_C \rho(x, y) \, ds$, and the center of mass is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{m} \int_C x \rho(x, y) \, ds$,
 $\bar{y} = \frac{1}{m} \int_C y \rho(x, y) \, ds$.
(d) See (5) and (6) in Section 17.2 [ET 16.2] for plane curves; we have similar definitions when C is a space curve (see the equation preceding (10) in Section 17.2 [ET 16.2]).
(e) For plane curves, see Equations 17.2.7 [ET 16.2.7]. We have similar results for space curves (see the equation preceding (10) in Section 17.2 [ET 16.2]).
- (a) See Definition 17.2.13 [ET 16.2.13].
(b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C .
(c) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P \, dx + Q \, dy + R \, dz$
- See Theorem 17.3.2 [ET 16.3.2].
- (a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.
(b) See Theorem 17.3.4 [ET 16.3.4].
- See the statement of Green's Theorem on page 1119 [ET 1083].
- See Equations 17.4.5 [ET 16.4.5].
- (a) $\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} = \nabla \times \mathbf{F}$
(b) $\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$
(c) For $\text{curl } \mathbf{F}$, see the discussion accompanying Figure 1 on page 1129 [ET 1093] as well as Figure 6 and the accompanying discussion on page 1160 [ET 1124]. For $\text{div } \mathbf{F}$, see the discussion following Example 5 on page 1130 [ET 1094] as well as the discussion preceding (8) on page 1167 [ET 1131].
- See Theorem 17.3.6 [ET 16.3.6]; see Theorem 17.5.4 [ET 16.5.4].

11. (a) See (1) and (2) and the accompanying discussion in Section 17.6 [ET 16.6]; See Figure 4 and the accompanying discussion on page 1135 [ET 1099].
(b) See Definition 17.6.6 [ET 16.6.6].
(c) See Equation 17.6.9 [ET 16.6.9].
12. (a) See (1) in Section 17.7 [ET 16.7].
(b) We normally evaluate the surface integral using Formula 17.7.3 [ET 16.7.3].
(c) See Formula 17.7.2 [ET 16.7.2].
(d) The mass is $m = \iint_S \rho(x, y, z) dS$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = \frac{1}{m} \iint_S x\rho(x, y, z) dS$, $\bar{y} = \frac{1}{m} \iint_S y\rho(x, y, z) dS$, $\bar{z} = \frac{1}{m} \iint_S z\rho(x, y, z) dS$.
13. (a) See Figures 7 and 8 and the accompanying discussion in Section 17.7 [ET 16.7]. A Möbius strip is a nonorientable surface; see Figures 5 and 6 and the accompanying discussion on page 1149 [ET 1113].
(b) See Definition 17.7.7 [ET 16.7.7].
(c) See Formula 17.7.9 [ET 16.7.9].
(d) See Formula 17.7.8 [ET 16.7.8].
14. See the statement of Stokes' Theorem on page 1157 [ET 1121].
15. See the statement of the Divergence Theorem on page 1163 [ET 1127].
16. In each theorem, we have an integral of a “derivative” over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

TRUE-FALSE QUIZ

1. False; $\operatorname{div} \mathbf{F}$ is a scalar field.
2. True. (See Definition 17.5.1 [ET 16.5.1].)
3. True, by Theorem 17.5.3 [ET 16.5.3] and the fact that $\operatorname{div} \mathbf{0} = 0$.
4. True, by Theorem 17.3.2 [ET 16.3.2].
5. False. See Exercise 17.3.33 [ET 16.3.33]. (But the assertion is true if D is simply-connected; see Theorem 17.3.6 [ET 16.3.6].)
6. False. See the discussion accompanying Figure 8 on page 1103 [ET 1067].
7. True. Apply the Divergence Theorem and use the fact that $\operatorname{div} \mathbf{F} = 0$.
8. False by Theorem 17.5.11 [ET 16.5.11], because if it were true, then $\operatorname{div} \operatorname{curl} \mathbf{F} = 3 \neq 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C , so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$ is negative.
- (b) The vectors that end near P are shorter than the vectors that start near P , so the net flow is outward near P and $\operatorname{div} \mathbf{F}(P)$ is positive.

- 2.** We can parametrize C by $x = x, y = x^2, 0 \leq x \leq 1$ so

$$\int_C x \, ds = \int_0^1 x \sqrt{1 + (2x)^2} \, dx = \left. \frac{1}{12} (1 + 4x^2)^{3/2} \right|_0^1 = \frac{1}{12} (5\sqrt{5} - 1).$$

$$\begin{aligned} 3. \int_C x^3 z \, ds &= \int_0^{\pi/2} (2 \sin t)^3 (2 \cos t) \sqrt{(2 \cos t)^2 + (1)^2 + (-2 \sin t)^2} \, dt = \int_0^{\pi/2} (16 \sin^3 t \cos t) \sqrt{5} \, dt \\ &= 4\sqrt{5} \sin^4 t \Big|_0^{\pi/2} = 4\sqrt{5} \end{aligned}$$

$$4. \int_C xy \, dx + y \, dy = \int_0^{\pi/2} (x \sin x + \sin x \cos x) \, dx = -x \cos x + \sin x - \frac{1}{4} \cos 2x \Big|_0^{\pi/2} = \frac{3}{2}$$

$$5. x = \cos t \Rightarrow dx = -\sin t \, dt, y = \sin t \Rightarrow dy = \cos t \, dt, 0 \leq t \leq 2\pi \text{ and}$$

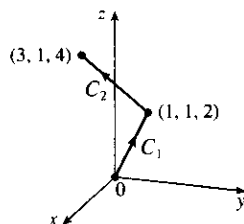
$$\int_C x^3 y \, dx - x \, dy = \int_0^{2\pi} (-\cos^3 t \sin^2 t - \cos^2 t) \, dt = \int_0^{2\pi} (-\cos^3 t \sin^2 t - \cos^2 t) \, dt = -\pi$$

Or: Since C is a simple closed curve, apply Green's Theorem giving

$$\iint_{x^2+y^2 \leq 1} (-1 - x^3) \, dA = \int_0^1 \int_0^{2\pi} (-r - r^4 \cos^3 \theta) \, d\theta = -\pi.$$

$$\begin{aligned} 6. \int_C \sqrt{xy} \, dx + e^y \, dy + xz \, dz &= \int_0^1 \left(\sqrt{t^4 \cdot t^2} \cdot 4t^3 + e^{t^2} \cdot 2t + t^4 \cdot t^3 \cdot 3t^2 \right) dt \\ &= \int_0^1 \left(4t^6 + 2te^{t^2} + 3t^9 \right) dt = \left[\frac{4}{7}t^7 + e^{t^2} + \frac{3}{10}t^{10} \right]_0^1 \\ &= e - \frac{9}{70} \end{aligned}$$

7.



$$C_1: x = t, y = t, z = 2t, 0 \leq t \leq 1;$$

$$C_2: x = 1 + 2t, y = 1, z = 2 + 2t, 0 \leq t \leq 1.$$

Then

$$\int_C y \, dx + z \, dy + x \, dz = \int_0^1 5t \, dt + \int_0^1 (4 + 4t) \, dt = \frac{17}{2}$$

$$8. \mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\mathbf{i} + (\sin^2 t)\mathbf{j}, \mathbf{r}'(t) = \cos t\mathbf{i} + \mathbf{j} \text{ and}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi ((1+t) \sin t \cos t + \sin^2 t) \, dt = \int_0^\pi \left(\frac{1}{2}(1+t) \sin 2t + \sin^2 t \right) dt \\ &= \left[\frac{1}{2} \left((1+t) \left(-\frac{1}{2} \cos 2t \right) + \frac{1}{4} \sin 2t \right) + \frac{1}{2}t - \frac{1}{4} \sin 2t \right]_0^\pi = \frac{\pi}{4}. \end{aligned}$$

$$9. \mathbf{F}(\mathbf{r}(t)) = e^{-t}\mathbf{i} + t^2(-t)\mathbf{j} + (t^2 + t^3)\mathbf{k}, \mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k} \text{ and}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2te^{-t} - 3t^5 - (t^2 + t^3)) \, dt = \left[-2te^{-t} - 2e^{-t} - \frac{1}{2}t^6 - \frac{1}{3}t^3 - \frac{1}{4}t^4 \right]_0^1 = \frac{11}{12} - \frac{4}{e}.$$

$$10. (a) C: x = 3 - 3t, y = \frac{\pi}{2}t, z = 3t, 0 \leq t \leq 1. \text{ Then}$$

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [3t\mathbf{i} + (3-3t)\mathbf{j} + \frac{\pi}{2}t\mathbf{k}] \cdot [-3\mathbf{i} + \frac{\pi}{2}\mathbf{j} + 3\mathbf{k}] \, dt = \int_0^1 [-9t + \frac{3\pi}{2}] \, dt \\ &= \frac{1}{2}(3\pi - 9) \end{aligned}$$

$$(b) W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (3 \sin t \mathbf{i} + 3 \cos t \mathbf{j} + t \mathbf{k}) \cdot (-3 \sin t \mathbf{i} + \mathbf{j} + 3 \cos t \mathbf{k}) \, dt$$

$$= \int_0^{\pi/2} (-9 \sin^2 t + 3 \cos t + 3t \cos t) \, dt$$

$$= \left[-\frac{9}{2}(t - \sin t \cos t) + 3 \sin t + 3(t \sin t + \cos t) \right]_0^{\pi/2} = -\frac{9\pi}{4} + 3 + \frac{3\pi}{2} - 3 = -\frac{3\pi}{4}$$

$$11. \frac{\partial}{\partial y} [(1+xy)e^{xy}] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x} [e^y + x^2e^{xy}] \text{ and the domain of } \mathbf{F} \text{ is } \mathbb{R}^2, \text{ so } \mathbf{F} \text{ is conservative. Thus}$$

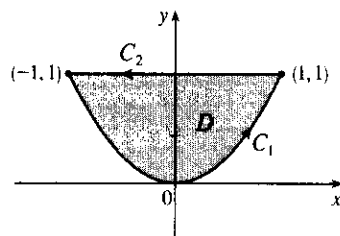
there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x, y) = e^y + x^2e^{xy}$ implies $f(x, y) = e^y + xe^{xy} + g(x)$ and

then $f_x(x, y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x, y) = (1+xy)e^{xy}$, so $g'(x) = 0 \Rightarrow$

$g(x) = K$. Thus $f(x, y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

12. $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\text{curl } \mathbf{F} = (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (\cos y - \cos y)\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 17.5.4 [ET 16.5.4]. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = \sin y$ implies $f(x, y, z) = x \sin y + g(y, z)$ and then $f_y(x, y, z) = x \cos y + g_y(y, z)$. But $f_y(x, y, z) = x \cos y$, so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Then $f(x, y, z) = x \sin y + h(z)$ implies $f_z(x, y, z) = h'(z)$. But $f_z(x, y, z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x, y, z) = x \sin y + \cos z + K$.
13. Since $\frac{\partial}{\partial y}(4x^3y^2 - 2xy^3) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}(2x^4y - 3x^2y^2 + 4y^3)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x, y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0, 1)$ and $t = 1$ corresponds to $(1, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 1) = 1 - 1 = 0$.
14. Here $\text{curl } \mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore $f(x, y, z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 0, 3) - f(0, 2, 0) = 4 - 2 = 2$.

15.



$$C_1: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, -1 \leq t \leq 1;$$

$$C_2: \mathbf{r}(t) = -t\mathbf{i} + \mathbf{j}, -1 \leq t \leq 1.$$

Then

$$\begin{aligned} \int_C xy^2 dx - x^2y dy &= \int_{-1}^1 (t^5 - 2t^5) dt + \int_{-1}^1 t dt \\ &= \left[-\frac{1}{6}t^6\right]_{-1}^1 + \left[\frac{1}{2}t^2\right]_{-1}^1 = 0 \end{aligned}$$

Using Green's Theorem, we have

$$\begin{aligned} \int_C xy^2 dx - x^2y dy &= \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2) \right] dA = \iint_D (-2xy - 2xy) dA \\ &= \int_{-1}^1 \int_{x^2}^1 -4xy dy dx = \int_{-1}^1 [-2xy^2]_{y=x^2}^{y=1} dx \\ &= \int_{-1}^1 (2x^5 - 2x) dx = \left[\frac{2}{6}x^6 - x^2\right]_{-1}^1 = 0 \end{aligned}$$

$$\begin{aligned} 16. \int_C \sqrt{1+x^3} dx + 2xy dy &= \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(\sqrt{1+x^3}) \right] dA = \int_0^1 \int_0^{3x} (2y - 0) dy dx \\ &= \int_0^1 9x^2 dx = 3x^3 \Big|_0^1 = 3 \end{aligned}$$

$$\begin{aligned} 17. \int_C x^2y dx - xy^2 dy &= \iint_{x^2+y^2 \leq 4} \left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y) \right] dA \\ &= \iint_{x^2+y^2 \leq 4} (-y^2 - x^2) dA = -\int_0^{2\pi} \int_0^2 r^3 dr d\theta = -8\pi \end{aligned}$$

$$\begin{aligned} 18. \text{curl } \mathbf{F} &= (0 - e^{-y} \cos z)\mathbf{i} - (e^{-z} \cos x - 0)\mathbf{j} + (0 - e^{-x} \cos y)\mathbf{k} = -e^{-y} \cos z \mathbf{i} - e^{-z} \cos x \mathbf{j} - e^{-x} \cos y \mathbf{k}, \\ \text{div } \mathbf{F} &= -e^{-x} \sin y - e^{-y} \sin z - e^{-z} \sin x \end{aligned}$$

19. If we assume there is such a vector field $\mathbf{G}$, then $\text{div}(\text{curl } \mathbf{G}) = 2 + 3z - 2xz$. But $\text{div}(\text{curl } \mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1 \mathbf{i} + Q_1 \mathbf{j} + R_1 \mathbf{k}$ and $\mathbf{G} = P_2 \mathbf{i} + Q_2 \mathbf{j} + R_2 \mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F}$$

$$= \left[P_1 \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z} \right) \mathbf{i} + Q_1 \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z} \right) \mathbf{j} + R_1 \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z} \right) \mathbf{k} \right] \\ - \left[P_2 \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z} \right) \mathbf{i} + Q_2 \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z} \right) \mathbf{j} + R_2 \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z} \right) \mathbf{k} \right]$$

and

$$(\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G} = \left[\left(P_2 \frac{\partial P_1}{\partial x} + Q_2 \frac{\partial P_1}{\partial y} + R_2 \frac{\partial P_1}{\partial z} \right) \mathbf{i} + \left(P_2 \frac{\partial Q_1}{\partial x} + Q_2 \frac{\partial Q_1}{\partial y} + R_2 \frac{\partial Q_1}{\partial z} \right) \mathbf{j} \right. \\ \left. + \left(P_2 \frac{\partial R_1}{\partial x} + Q_2 \frac{\partial R_1}{\partial y} + R_2 \frac{\partial R_1}{\partial z} \right) \mathbf{k} \right] \\ - \left[\left(P_1 \frac{\partial P_2}{\partial x} + Q_1 \frac{\partial P_2}{\partial y} + R_1 \frac{\partial P_2}{\partial z} \right) \mathbf{i} + \left(P_1 \frac{\partial Q_2}{\partial x} + Q_1 \frac{\partial Q_2}{\partial y} + R_1 \frac{\partial Q_2}{\partial z} \right) \mathbf{j} \right. \\ \left. + \left(P_1 \frac{\partial R_2}{\partial x} + Q_1 \frac{\partial R_2}{\partial y} + R_1 \frac{\partial R_2}{\partial z} \right) \mathbf{k} \right]$$

Hence

$$\mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G}$$

$$= \left[\left(P_1 \frac{\partial Q_2}{\partial y} + Q_2 \frac{\partial P_1}{\partial x} \right) - \left(P_2 \frac{\partial Q_1}{\partial y} + Q_1 \frac{\partial P_2}{\partial x} \right) \right. \\ \left. - \left(P_2 \frac{\partial R_1}{\partial z} + R_1 \frac{\partial P_2}{\partial z} \right) + \left(P_1 \frac{\partial R_2}{\partial z} + R_2 \frac{\partial P_1}{\partial z} \right) \right] \mathbf{i} \\ + \left[\left(Q_1 \frac{\partial R_2}{\partial z} + R_2 \frac{\partial Q_1}{\partial z} \right) - \left(Q_2 \frac{\partial R_1}{\partial z} + R_1 \frac{\partial Q_2}{\partial z} \right) \right. \\ \left. - \left(P_1 \frac{\partial Q_2}{\partial x} + Q_2 \frac{\partial P_1}{\partial x} \right) + \left(P_2 \frac{\partial Q_1}{\partial x} + Q_1 \frac{\partial P_2}{\partial x} \right) \right] \mathbf{j} \\ + \left[\left(P_2 \frac{\partial R_1}{\partial x} + R_1 \frac{\partial P_2}{\partial x} \right) - \left(P_1 \frac{\partial R_2}{\partial x} + R_2 \frac{\partial P_1}{\partial x} \right) \right. \\ \left. - \left(Q_1 \frac{\partial R_2}{\partial y} + R_2 \frac{\partial Q_1}{\partial y} \right) + \left(Q_2 \frac{\partial R_1}{\partial y} + R_1 \frac{\partial Q_2}{\partial y} \right) \right] \mathbf{k} \\ = \left[\frac{\partial}{\partial y} (P_1 Q_2 - P_2 Q_1) - \frac{\partial}{\partial z} (P_2 R_1 - P_1 R_2) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} (Q_1 R_2 - Q_2 R_1) - \frac{\partial}{\partial x} (P_1 Q_2 - P_2 Q_1) \right] \mathbf{j} \\ + \left[\frac{\partial}{\partial x} (P_2 R_1 - P_1 R_2) - \frac{\partial}{\partial y} (Q_1 R_2 - Q_2 R_1) \right] \mathbf{k} \\ = \operatorname{curl}(\mathbf{F} \times \mathbf{G})$$

21. For any piecewise-smooth simple closed plane curve C bounding a region D , we can apply Green's Theorem to

$$\mathbf{F}(x, y) = f(x)\mathbf{i} + g(y)\mathbf{j} \text{ to get } \int_C f(x) dx + g(y) dy = \iint_D \left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x) \right] dA = \iint_D 0 dA = 0.$$

$$\begin{aligned} 22. \nabla^2(fg) &= \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2} \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} g + f \frac{\partial g}{\partial z} \right) \quad [\text{Product Rule}] \\ &= \frac{\partial^2 f}{\partial x^2} g + 2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} g + 2 \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} \\ &\quad + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} g + 2 \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} + f \frac{\partial^2 g}{\partial z^2} \quad [\text{Product Rule}] \\ &= f \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right) + g \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) + 2 \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \\ &= f \nabla^2 g + g \nabla^2 f + 2 \nabla f \cdot \nabla g \end{aligned}$$

Another method: Using the rules in Exercises 15.6.37(b) [ET 14.6.37(b)] and 17.5.25 [ET 16.5.25], we have

$$\begin{aligned} \nabla^2(fg) &= \nabla \cdot \nabla(fg) = \nabla \cdot (g \nabla f + f \nabla g) = \nabla g \cdot \nabla f + g \nabla \cdot \nabla f + \nabla f \cdot \nabla g + f \nabla \cdot \nabla g \\ &= g \nabla^2 f + f \nabla^2 g + 2 \nabla f \cdot \nabla g \end{aligned}$$

23. $\nabla^2 f = 0$ means that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y \mathbf{i} - f_x \mathbf{j}$ and C is any closed path in D , then applying Green's Theorem, we get

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C f_y dx - f_x dy = \iint_D \left[\frac{\partial}{\partial x} (-f_x) - \frac{\partial}{\partial y} (f_y) \right] dA = - \iint_D (f_{xx} + f_{yy}) dA \\ &= - \iint_D 0 dA = 0 \end{aligned}$$

Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

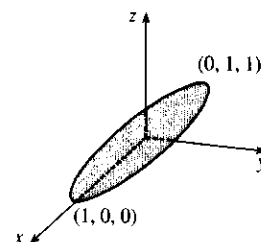
24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$.

But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.

- (b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$:

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y \cot z & -y^2 \csc^2 z \end{vmatrix} \\ &= \langle -2y \csc^2 z - (-2y \csc^2 z), 0, 4xe^{2y} - 4xe^{2y} \rangle = \mathbf{0} \end{aligned}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.



25. $z = f(x, y) = x^2 + 2y$ with $0 \leq x \leq 1$, $0 \leq y \leq 2x$. Thus

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 4x^2 + 4} dA = \int_0^1 \int_0^{2x} \sqrt{5 + 4x^2} dy dx = \int_0^1 2x \sqrt{5 + 4x^2} dx \\ &= \left. \frac{1}{6} (5 + 4x^2)^{3/2} \right|_0^1 = \frac{1}{6} (27 - 5\sqrt{5}). \end{aligned}$$

26. (a) $\mathbf{r}_u = -v\mathbf{j} + 2u\mathbf{k}$, $\mathbf{r}_v = 2v\mathbf{i} - u\mathbf{j}$ and (b)

$$\mathbf{r}_u \times \mathbf{r}_v = 2u^2\mathbf{i} + 4uv\mathbf{j} + 2v^2\mathbf{k}.$$

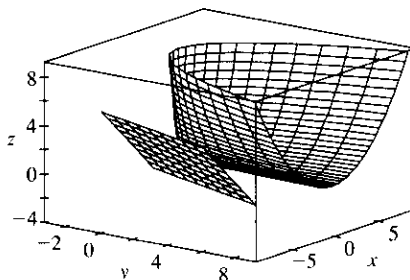
Since the point $(4, -2, 1)$ corresponds to $u = 1$, $v = 2$

(or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for

both), a normal vector to the surface at $(4, -2, 1)$ is

$$2\mathbf{i} + 8\mathbf{j} + 8\mathbf{k} \text{ and an equation of the tangent plane}$$

$$\text{is } 2x + 8y + 8z = 0 \text{ or } x + 4y + 4z = 0.$$



- (c) By Definition 17.6.6 [ET 16.6.6], the area of S is given by

$$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2} dv du = 2 \int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4} dv du.$$

- (d) By Equation 17.7.9 [ET 16.7.9], the surface integral is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1 + (v^2)^2}, \frac{(v^2)^2}{1 + (-uv)^2}, \frac{(-uv)^2}{1 + (u^2)^2} \right\rangle \cdot \langle 2u^2, 4uv, 2v^2 \rangle dv du \\ &= \int_0^3 \int_{-3}^3 \left(\frac{2u^6}{1 + v^4} + \frac{4uv^5}{1 + u^2v^2} + \frac{2u^2v^4}{1 + u^4} \right) dv du \approx 1524.0190 \end{aligned}$$

27. $z = f(x, y) = x^2 + y^2$ with $0 \leq x^2 + y^2 \leq 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\begin{aligned} \iint_S z dS &= \iint_{x^2 + y^2 \leq 4} (x^2 + y^2) \sqrt{4x^2 + 4y^2 + 1} dA = \int_0^{2\pi} \int_0^2 r^3 s \sqrt{1 + 4r^2} dr d\theta \\ &= \frac{1}{60} \pi (391 \sqrt{17} + 1) \end{aligned}$$

(Substitute $u = 1 + 4r^2$ and use tables.)

28. $z = f(x, y) = 4 + x + y$ with $0 \leq x^2 + y^2 \leq 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\begin{aligned} \iint_S (x^2 z + y^2 z) dS &= \iint_{x^2 + y^2 \leq 4} (x^2 + y^2)(4 + x + y) \sqrt{3} dA \\ &= \int_0^{2\pi} \int_0^2 \sqrt{3} r^3 (4 + r \cos \theta + r \sin \theta) d\theta dr = \int_0^2 8\pi \sqrt{3} r^3 dr = 32\pi \sqrt{3} \end{aligned}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (z - 2) dV = \iiint_E z dV - 2 \iiint_E dV = m\bar{z} - 2\left(\frac{4}{3}\pi 2^3\right) = -\frac{64}{3}\pi.$$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi, \theta)) = 4 \sin \phi \cos \theta \cos \phi \mathbf{i} - 4 \sin \phi \sin \theta \mathbf{j} + 6 \sin \phi \cos \theta \mathbf{k}$,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4 \sin^2 \phi \cos \theta \mathbf{i} + 4 \sin^2 \phi \sin \theta \mathbf{j} + 4 \sin \phi \cos \phi \mathbf{k}, \text{ and}$$

$$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16 \sin^3 \phi \cos^2 \theta \cos \phi - 16 \sin^3 \phi \sin^2 \theta + 24 \sin^2 \phi \cos \phi \cos \theta. \text{ Then}$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^\pi (16 \sin^3 \phi \cos \phi \cos^2 \theta - 16 \sin^3 \phi \sin^2 \theta + 24 \sin^2 \phi \cos \phi \cos \theta) d\phi d\theta \\ &= \int_0^{2\pi} \frac{4}{3} (-16 \sin^2 \theta) d\theta = -\frac{64}{3}\pi \end{aligned}$$

30. $z = f(x, y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ (because of upward orientation) and

$$\mathbf{F}(\mathbf{r}(x, y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2. \text{ Then}$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2+y^2 \leq 1} (-2x^3 - 2xy^2 + x^2 + y^2) dA \\ &= \int_0^1 \int_0^{2\pi} (-2r^3 \cos^3 \theta - 2r^3 \cos \theta \sin^2 \theta + r^2) r dr d\theta = \int_0^1 r^3 (2\pi) dr = \frac{\pi}{2} \end{aligned}$$

31. Since $\text{curl } \mathbf{F} = \mathbf{0}$, $\iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = 0$. We parametrize C : $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$ and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t \sin t + \sin^2 t \cos t) dt = \left[\frac{1}{3} \cos^3 t + \frac{1}{3} \sin^3 t \right]_0^{2\pi} = 0.$$

32. $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C : $\mathbf{r}(t) = 2\cos t \mathbf{i} + 2\sin t \mathbf{j} + \mathbf{k}$, $0 \leq t \leq 2\pi$, so $\mathbf{r}'(t) = -2\sin t \mathbf{i} + 2\cos t \mathbf{j}$,

$$\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t \sin t \mathbf{i} + 2\sin t \mathbf{j} + e^{4\cos t \sin t} \mathbf{k}, \text{ and } \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t \sin^2 t + 4\sin t \cos t. \text{ Thus}$$

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-16\cos^2 t \sin^2 t + 4\sin t \cos t) dt \\ &= \left[-16\left(-\frac{1}{4}\sin t \cos^3 t + \frac{1}{16}\sin 2t + \frac{1}{8}t\right) + 2\sin^2 t \right]_0^{2\pi} = -4\pi \end{aligned}$$

33. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (-y\mathbf{i} - z\mathbf{j} - x\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) dA \\ &= \iint_D (-1) dA = -(\text{area of } D) = -\frac{1}{2} \end{aligned}$$

34. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2) dV = \int_0^{2\pi} \int_0^1 \int_0^1 (3r^2 + 3z^2) r dz dr d\theta = 2\pi \int_0^1 (6r^3 + 8r) dr = 11\pi$

35. $\iiint_E \text{div } \mathbf{F} dV = \iiint_{x^2+y^2+z^2 \leq 1} 3 dV = 3(\text{volume of sphere}) = 4\pi$. Then

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3 \phi \cos^2 \theta + \sin^3 \phi \sin^2 \theta + \sin \phi \cos^2 \phi = \sin \phi \text{ and}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta = (2\pi)(2) = 4\pi.$$

36. Here we must use Equation 17.9.6 [ET 16.9.6] since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2 . Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1) dS + \iiint_E \text{div } \mathbf{F} dV. \text{ But } \mathbf{F} = \mathbf{r}/|\mathbf{r}|^3, \text{ so}$$

$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \cdot (|\mathbf{r}|^{-3} \mathbf{r}) = |\mathbf{r}|^{-3} (\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot (\nabla |\mathbf{r}|^{-3}) = |\mathbf{r}|^{-3} (3) + \mathbf{r} \cdot (-3|\mathbf{r}|^{-4}) (\mathbf{r} |\mathbf{r}|^{-1}) = 0. \text{ (Here we have} \\ &\text{used Exercises 17.5.30(a) [ET 16.5.30(a)] and 17.5.31(a) [ET 16.5.31(a)].) And } \mathbf{F} \cdot \mathbf{n}_1 = \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1 \end{aligned}$$

$$\text{on } S_1. \text{ Thus } \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS = \iint_{S_1} dS + \iiint_E 0 dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi.$$

37. Because $\text{curl } \mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, and if $f(x, y, z) = x^3 yz - 3xy + z^2$, then $\nabla f = \mathbf{F}$. Hence

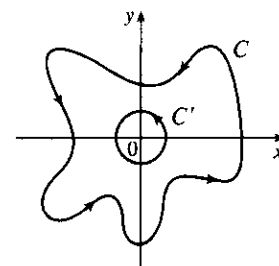
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, 3, 0) - f(0, 0, 2) = 0 - 4 = -4.$$

38. Let C' be the circle with center at the origin and radius a as in the figure.

Let D be the region bounded by C and C' . Then D 's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0, \text{ so}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= - \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} \left[\frac{2a^3 \cos^3 t + 2a^3 \cos t \sin^2 t - 2a \sin t}{a^2} (-a \sin t) \right. \\ &\quad \left. + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \sin t + 2a \cos t}{a^2} (a \cos t) \right] dt \\ &= \int_0^{2\pi} \frac{2a^2}{a^2} dt = 4\pi \end{aligned}$$



39. By the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = 3(\text{volume of } E) = 3(8 - 1) = 21$.
40. The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F}) dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

□ PROBLEMS PLUS

1. Let S_1 be the portion of $\Omega(S)$ between $S(a)$ and S , and let ∂S_1 be its boundary. Also let S_L be the lateral surface of S_1 [that is, the surface of S_1 except S and $S(a)$]. Applying the Divergence Theorem we have

$$\iint_{\partial S_1} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iiint_{S_1} \nabla \cdot \frac{\mathbf{r}}{r^3} dV. \text{ But}$$

$$\begin{aligned} \nabla \cdot \frac{\mathbf{r}}{r^3} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle \\ &= \frac{(x^2 + y^2 + z^2 - 3x^2) + (x^2 + y^2 + z^2 - 3y^2) + (x^2 + y^2 + z^2 - 3z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0 \end{aligned}$$

$$\Rightarrow \iint_{\partial S_1} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iiint_{S_1} 0 dV = 0. \text{ On the other hand, notice that for the surfaces of } \partial S_1 \text{ other than } S(a) \text{ and } S, \mathbf{r} \cdot \mathbf{n} = 0 \Rightarrow$$

$$\begin{aligned} 0 &= \iint_{\partial S_1} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS + \iint_{S(a)} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS + \iint_{S_L} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS \\ &= \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS + \iint_{S(a)} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS \end{aligned}$$

$$\Rightarrow \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = - \iint_{S(a)} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS. \text{ Notice that on } S(a), r = a \Rightarrow \mathbf{n} = -\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{a} \text{ and } \mathbf{r} \cdot \mathbf{r} = r^2 = a^2,$$

$$\text{so that } - \iint_{S(a)} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \iint_{S(a)} \frac{\mathbf{r} \cdot \mathbf{r}}{a^4} dS = \iint_{S(a)} \frac{a^2}{a^4} dS = \frac{1}{a^2} \iint_{S(a)} dS = \frac{\text{area of } S(a)}{a^2} = |\Omega(S)|.$$

$$\text{Therefore } |\Omega(S)| = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS.$$

2. By Green's Theorem

$$\int_C (y^3 - y) dx - 2x^3 dy = \iint_D \left[\frac{\partial(-2x^3)}{\partial x} - \frac{\partial(y^3 - y)}{\partial y} \right] dA = \iint_D (1 - 6x^2 - 3y^2) dA$$

Notice that for $6x^2 + 3y^2 > 1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2 + 3y^2 = 1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2 + 3y^2 = 1$.

3. The given line integral $\frac{1}{2} \int_C (bz - cy) dx + (cx - az) dy + (ay - bx) dz$ can be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} = \frac{1}{2}(bz - cy)\mathbf{i} + \frac{1}{2}(cx - az)\mathbf{j} + \frac{1}{2}(ay - bx)\mathbf{k}$. Then define S to be the planar interior of C , so S is an oriented, smooth surface. Stokes' Theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS. \text{ Now}$$

$$\begin{aligned} \text{curl } \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ &= \left(\frac{1}{2}a + \frac{1}{2}a \right) \mathbf{i} + \left(\frac{1}{2}b + \frac{1}{2}b \right) \mathbf{j} + \left(\frac{1}{2}c + \frac{1}{2}c \right) \mathbf{k} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{n} \end{aligned}$$

so $\text{curl } \mathbf{F} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{n} = |\mathbf{n}|^2 = 1$, hence $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS = \iint_S dS$ which is simply the surface area of S . Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} \int_C (bz - cy) dx + (cx - az) dy + (ay - bx) dz$ is the plane area enclosed by C .

4. (a) First we place the piston on coordinate axes so the top of the cylinder is at the origin and $x(t) \geq 0$ is the distance from the top of the cylinder to the piston at time t . Let C_1 be the curve traced out by the piston during one four-stroke cycle, so C_1 is given by $\mathbf{r}(t) = x(t)\mathbf{i}$, $a \leq t \leq b$. (Thus, the curve lies on the positive x -axis and reverses direction several times.) The force on the piston is $AP(t)\mathbf{i}$, where A is the area of the top of the piston and $P(t)$ is the pressure in the cylinder at time t . As in Section 17.2 [ET 16.2], the work done on the piston is $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b AP(t)\mathbf{i} \cdot x'(t)\mathbf{i} dt = \int_a^b AP(t)x'(t) dt$. Here, the volume of the cylinder at time t is $V(t) = Ax(t) \Rightarrow V'(t) = Ax'(t) \Rightarrow \int_a^b AP(t)x'(t) dt = \int_a^b P(t)V'(t) dt$. Since the curve C in the PV -plane corresponds to the values of P and V at time t , $a \leq t \leq b$, we have

$$W = \int_a^b AP(t)x'(t) dt = \int_a^b P(t)V'(t) dt = \int_C P dV$$

Another method: If we divide the time interval $[a, b]$ into n subintervals of equal length Δt , the amount of work done on the piston in the i th time interval is approximately $AP(t_i)[x(t_i) - x(t_{i-1})]$. Thus we estimate the total work done during one cycle to be $\sum_{i=1}^n AP(t_i)[x(t_i) - x(t_{i-1})]$. If we allow $n \rightarrow \infty$, we have

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n AP(t_i)[x(t_i) - x(t_{i-1})] = \lim_{n \rightarrow \infty} \sum_{i=1}^n P(t_i)[Ax(t_i) - Ax(t_{i-1})] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n P(t_i)[V(t_i) - V(t_{i-1})] = \int_C P dV \end{aligned}$$

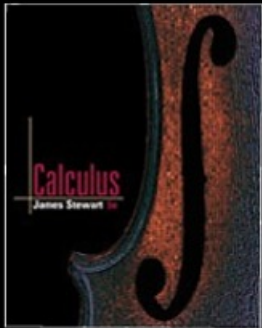
- (b) Let C_L be the lower loop of the curve C and C_U the upper loop. Then $C = C_L \cup C_U$. C_L is positively oriented, so from Formula 17.4.5 [ET 16.4.5] we know the area of the lower loop in the PV -plane is given by $-\oint_{C_L} P dV$. C_U is negatively oriented, so the area of the upper loop is given by $-\left(-\oint_{C_U} P dV\right) = \oint_{C_U} P dV$. From part (a),

$$W = \int_C P dV = \int_{C_L \cup C_U} P dV = \oint_{C_L} P dV + \oint_{C_U} P dV = \oint_{C_U} P dV - \left(-\oint_{C_L} P dV\right),$$

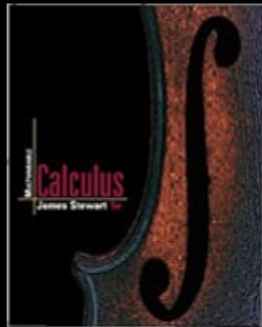
the difference of the areas enclosed by the two loops of C .

Chapter 18

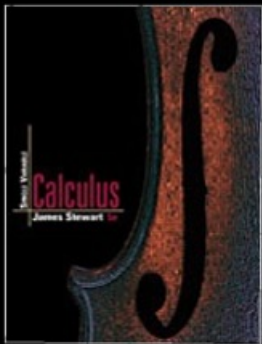
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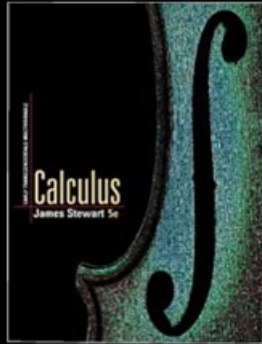
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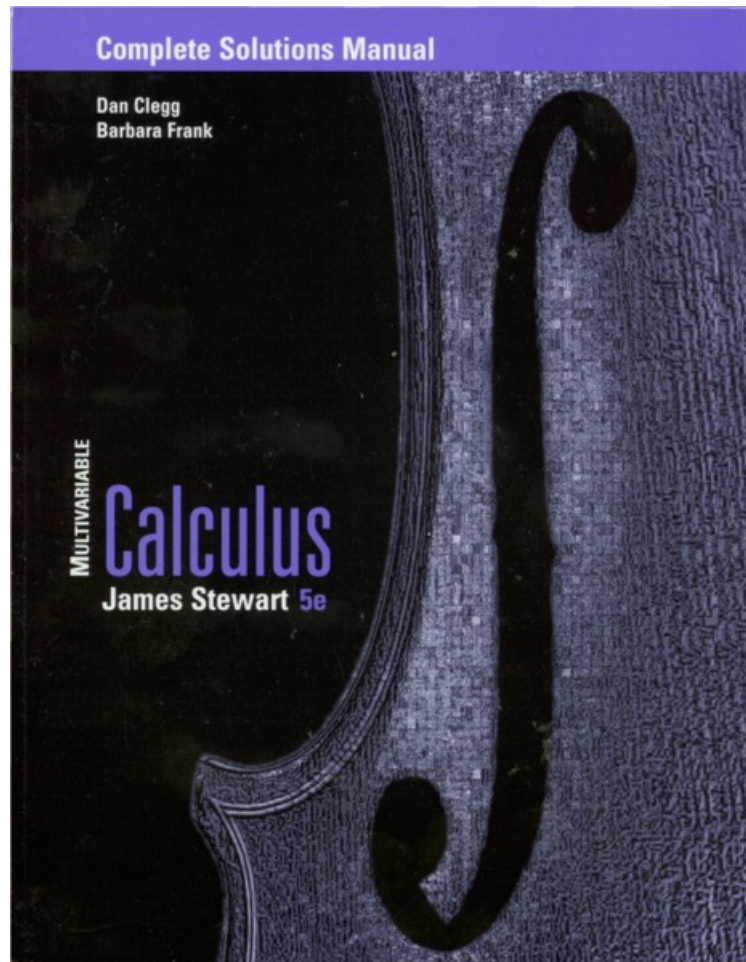
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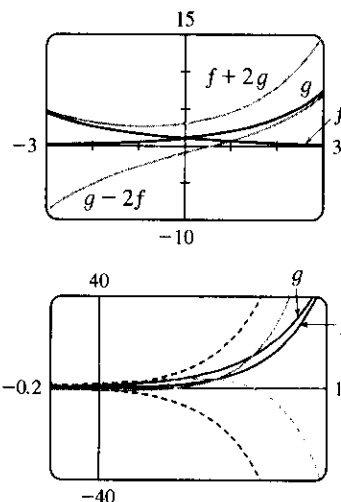
18 □ SECOND-ORDER DIFFERENTIAL EQUATIONS □ ET 17

18.1 Second-Order Linear Equations

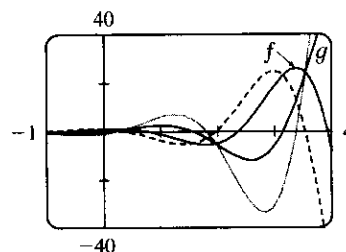
ET 17.1

- The auxiliary equation is $r^2 - 6r + 8 = 0 \Rightarrow (r - 4)(r - 2) = 0 \Rightarrow r = 4, r = 2$. Then by (8) the general solution is $y = c_1 e^{4x} + c_2 e^{2x}$.
- The auxiliary equation is $r^2 - 4r + 8 = 0 \Rightarrow r = 2 \pm 2i$. Then by (11) the general solution is $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$.
- The auxiliary equation is $r^2 + 8r + 41 = 0 \Rightarrow r = -4 \pm 5i$. Then by (11) the general solution is $y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$.
- The auxiliary equation is $2r^2 - r - 1 = (2r + 1)(r - 1) = 0 \Rightarrow r = 1, r = -\frac{1}{2}$. Then the general solution is $y = c_1 e^x + c_2 e^{-x/2}$.
- The auxiliary equation is $r^2 - 2r + 1 = (r - 1)^2 = 0 \Rightarrow r = 1$. Then by (10), the general solution is $y = c_1 e^x + c_2 x e^x$.
- The auxiliary equation is $3r^2 - 5r = r(3r - 5) = 0 \Rightarrow r = 0, r = \frac{5}{3}$, so $y = c_1 + c_2 e^{5x/3}$.
- The auxiliary equation is $4r^2 + 1 = 0 \Rightarrow r = \pm \frac{1}{2}i$, so $y = c_1 \cos(\frac{1}{2}x) + c_2 \sin(\frac{1}{2}x)$.
- The auxiliary equation is $16r^2 + 24r + 9 = (4r + 3)^2 = 0 \Rightarrow r = -\frac{3}{4}$, so $y = c_1 e^{-3x/4} + c_2 x e^{-3x/4}$.
- The auxiliary equation is $4r^2 + r = r(4r + 1) = 0 \Rightarrow r = 0, r = -\frac{1}{4}$, so $y = c_1 + c_2 e^{-x/4}$.
- The auxiliary equation is $9r^2 + 4 = 0 \Rightarrow r = \pm \frac{2}{3}i$, so $y = c_1 \cos(\frac{2}{3}x) + c_2 \sin(\frac{2}{3}x)$.
- The auxiliary equation is $r^2 - 2r - 1 = 0 \Rightarrow r = 1 \pm \sqrt{2}$, so $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$.
- The auxiliary equation is $r^2 - 6r + 4 = 0 \Rightarrow r = 3 \pm \sqrt{5}$, so $y = c_1 e^{(3+\sqrt{5})t} + c_2 e^{(3-\sqrt{5})t}$.
- The auxiliary equation is $r^2 + r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so $y = e^{-t/2} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right]$.
- $6r^2 - r - 2 = (2r + 1)(3r - 2) = 0$ so
 $y = c_1 e^{-x/2} + c_2 e^{2x/3}$. The solutions $(c_1, c_2) = (0, 1), (1, 0), (-2, 1)$ are shown. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x \rightarrow \pm\infty$.

- $r^2 - 8r + 16 = (r - 4)^2 = 0$ so $y = c_1 e^{4x} + c_2 x e^{4x}$.
 The graphs are all asymptotic to the x -axis as $x \rightarrow -\infty$, and as $x \rightarrow \infty$ the solutions tend to $\pm\infty$.



16. $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and the solution is $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Graphs for $(c_1, c_2) = (1, 0), (0, 1), (1, -1), (-1, 2)$ are shown. The solutions are all asymptotic to the x -axis as $x \rightarrow -\infty$ and they all oscillate. The amplitudes of the oscillations become arbitrarily large as $x \rightarrow \infty$ and arbitrarily small as $x \rightarrow -\infty$.



17. $2r^2 + 5r + 3 = (2r + 3)(r + 1) = 0$, so $r = -\frac{3}{2}, r = -1$ and the general solution is $y = c_1 e^{-3x/2} + c_2 e^{-x}$. Then $y(0) = 3 \Rightarrow c_1 + c_2 = 3$ and $y'(0) = -4 \Rightarrow -\frac{3}{2}c_1 - c_2 = -4$, so $c_1 = 2$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = 2e^{-3x/2} + e^{-x}$.
18. $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}i$ and the general solution is $y = e^{0x}(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)) = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = 3 \Rightarrow c_2 = \sqrt{3}$, so the solution to the initial-value problem is $y = \cos(\sqrt{3}x) + \sqrt{3} \sin(\sqrt{3}x)$.
19. $4r^2 - 4r + 1 = (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = -1.5 \Rightarrow \frac{1}{2}c_1 + c_2 = -1.5$, so $c_2 = -2$ and the solution to the initial-value problem is $y = e^{x/2} - 2x e^{x/2}$.
20. $2r^2 + 5r - 3 = (2r - 1)(r + 3) = 0 \Rightarrow r = \frac{1}{2}, r = -3$ and the general solution is $y = c_1 e^{x/2} + c_2 e^{-3x}$. Then $1 = y(0) = c_1 + c_2$ and $4 = y'(0) = \frac{1}{2}c_1 - 3c_2$ so $c_1 = 2, c_2 = -1$ and the solution to the initial-value problem is $y = 2e^{x/2} - e^{-3x}$.
21. $r^2 + 16 = 0 \Rightarrow r = \pm 4i$ and the general solution is $y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x) = c_1 \cos 4x + c_2 \sin 4x$. Then $y(\frac{\pi}{4}) = -3 \Rightarrow -c_1 = -3 \Rightarrow c_1 = 3$ and $y'(\frac{\pi}{4}) = 4 \Rightarrow -4c_2 = 4 \Rightarrow c_2 = -1$, so the solution to the initial-value problem is $y = 3 \cos 4x - \sin 4x$.
22. $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and the general solution is $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Then $0 = y(\pi) = e^\pi(c_1 + 0) \Rightarrow c_1 = 0$ and $2 = y'(\pi) = (c_1 + 2c_2)e^\pi \Rightarrow c_2 = 1/e^\pi$ and the solution to the initial-value problem is $y = \frac{e^x}{e^\pi} \sin 2x = e^{x-\pi} \sin 2x$.
23. $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$ and the general solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $1 = y'(0) = c_2 - c_1 \Rightarrow c_2 = 3$ and the solution to the initial-value problem is $y = e^{-x}(2 \cos x + 3 \sin x)$.
24. $r^2 + 12r + 36 = (r + 6)^2 = 0 \Rightarrow r = -6$ and the general solution is $y = c_1 e^{-6x} + c_2 x e^{-6x}$. Then $0 = y(1) = c_1 e^{-6} + c_2 e^{-6} \Rightarrow c_1 + c_2 = 0$ and $1 = y'(1) = -6c_1 e^{-6} - 5c_2 e^{-6} \Rightarrow 6c_1 + 5c_2 = -e^6$, so $c_1 = -e^6$ and $c_2 = e^6$. The solution to the initial-value problem is $y = -e^6 e^{-6x} + e^6 x e^{-6x} = (x - 1)e^{6-6x}$.
25. $4r^2 + 1 = 0 \Rightarrow r = \pm \frac{1}{2}i$ and the general solution is $y = c_1 \cos(\frac{1}{2}x) + c_2 \sin(\frac{1}{2}x)$. Then $3 = y(0) = c_1$ and $-4 = y(\pi) = c_2$, so the solution of the boundary-value problem is $y = 3 \cos(\frac{1}{2}x) - 4 \sin(\frac{1}{2}x)$.
26. $r^2 + 2r = r(2 + r) = 0 \Rightarrow r = 0, r = -2$ and the general solution is $y = c_1 + c_2 e^{-2x}$. Then $1 = y(0) = c_1 + c_2$ and $2 = y(1) = c_1 + c_2 e^{-2}$ so $c_2 = \frac{e^2}{1 - e^2}, c_1 = \frac{1 - 2e^2}{1 - e^2}$. The solution of the boundary-value problem is $y = \frac{1 - 2e^2}{1 - e^2} + \frac{e^2}{1 - e^2} \cdot e^{-2x}$.

27. $r^2 - 3r + 2 = (r - 2)(r - 1) = 0 \Rightarrow r = 1, r = 2$ and the general solution is $y = c_1 e^x + c_2 e^{2x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(3) = c_1 e^3 + c_2 e^6$ so $c_2 = 1/(1 - e^3)$ and $c_1 = e^3/(e^3 - 1)$. The solution of the boundary-value problem is $y = \frac{e^{x+3}}{e^3 - 1} + \frac{e^{2x}}{1 - e^3}$.
28. $r^2 + 100 = 0 \Rightarrow r = \pm 10i$ and the general solution is $y = c_1 \cos 10x + c_2 \sin 10x$. But $2 = y(0) = c_1$ and $5 = y(\pi) = c_1$, so there is no solution.
29. $r^2 - 6r + 25 = 0 \Rightarrow r = 3 \pm 4i$ and the general solution is $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $2 = y(\pi) = c_1 e^{3\pi} \Rightarrow c_1 = 2/e^{3\pi}$, so there is no solution.
30. $r^2 - 6r + 9 = (r - 3)^2 = 0 \Rightarrow r = 3$ and the general solution is $y = c_1 e^{3x} + c_2 x e^{3x}$. Then $1 = y(0) = c_1$ and $0 = y(1) = c_1 e^3 + c_2 e^3 \Rightarrow c_2 = -1$. The solution of the boundary-value problem is $y = e^{3x} - x e^{3x}$.
31. $r^2 + 4r + 13 = 0 \Rightarrow r = -2 \pm 3i$ and the general solution is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$. But $2 = y(0) = c_1$ and $1 = y(\frac{\pi}{2}) = e^{-\pi}(-c_2)$, so the solution to the boundary-value problem is $y = e^{-2x}(2 \cos 3x - e^\pi \sin 3x)$.
32. $9r^2 - 18r + 10 = 0 \Rightarrow r = 1 \pm \frac{1}{3}i$ and the general solution is $y = e^x(c_1 \cos \frac{x}{3} + c_2 \sin \frac{x}{3})$. Then $0 = y(0) = c_1$ and $1 = y(\pi) = e^\pi(\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2) \Rightarrow c_2 = \frac{2}{\sqrt{3}e^\pi}$. The solution of the boundary-value problem is $y = \frac{2e^x}{\sqrt{3}e^\pi} \sin\left(\frac{x}{3}\right) = \frac{2}{\sqrt{3}} e^{x-\pi} \sin\left(\frac{x}{3}\right)$.
33. (a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \Rightarrow y'' = 0$ which has an auxiliary equation $r^2 = 0 \Rightarrow r = 0 \Rightarrow y = c_1 + c_2 x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \Rightarrow c_1 = c_2 = 0$. Thus, $y = 0$.
- Case 2* ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \Rightarrow r = \pm \sqrt{-\lambda}$ (distinct and real since $\lambda < 0$) $\Rightarrow y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1 + c_2$ (*) and $0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$ (†).
- Multiplying (*) by $e^{\sqrt{-\lambda}L}$ and subtracting (†) gives $c_2(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}) = 0 \Rightarrow c_2 = 0$ and thus $c_1 = 0$ from (*). Thus, $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.
- (b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda} \Rightarrow y = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin \sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin \sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi$ where n is an integer $\Rightarrow \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where n is an integer.
34. The auxiliary equation is $ar^2 + br + c = 0$. If $b^2 - 4ac > 0$, then any solution is of the form $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ where $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$. But a, b , and c are all positive so both r_1 and r_2 are negative and $\lim_{x \rightarrow \infty} y(x) = 0$. If $b^2 - 4ac = 0$, then any solution is of the form $y(x) = c_1 e^{rx} + c_2 x e^{rx}$ where $r = -b/(2a) < 0$ since a, b are positive. Hence $\lim_{x \rightarrow \infty} y(x) = 0$. Finally if $b^2 - 4ac < 0$, then any solution is of the form $y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ where $\alpha = -b/(2a) < 0$ since a and b are positive. Thus $\lim_{x \rightarrow \infty} y(x) = 0$.

18.2 Nonhomogeneous Linear Equations

ET 17.2

1. The auxiliary equation is $r^2 + 3r + 2 = (r + 2)(r + 1) = 0$, so the complementary solution is

$y_c(x) = c_1 e^{-2x} + c_2 e^{-x}$. We try the particular solution $y_p(x) = Ax^2 + Bx + C$, so $y'_p = 2Ax + B$ and $y''_p = 2A$. Substituting into the differential equation, we have $(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$ or $2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$. Comparing coefficients gives $2A = 1$, $6A + 2B = 0$, and $2A + 3B + 2C = 0$, so $A = \frac{1}{2}$, $B = -\frac{3}{2}$, and $C = \frac{7}{4}$. Thus the general solution is

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}.$$

2. The auxiliary equation is $r^2 + 9 = 0$ with roots $r = \pm 3i$, so the complementary solution is

$y_c(x) = c_1 \cos(3x) + c_2 \sin(3x)$. Try the particular solution $y_p(x) = Ae^{3x}$, so $y'_p = 3Ae^{3x}$ and $y''_p = 9Ae^{3x}$. Substitution into the differential equation gives $9Ae^{3x} + 9(Ae^{3x}) = e^{3x}$ or $18Ae^{3x} = e^{3x}$. Thus $A = \frac{1}{18}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{18}e^{3x}$.

3. The auxiliary equation is $r^2 - 2r = r(r - 2) = 0$, so the complementary solution is $y_c(x) = c_1 + c_2 e^{2x}$. Try the particular solution $y_p(x) = A \cos 4x + B \sin 4x$, so $y'_p = -4A \sin 4x + 4B \cos 4x$

and $y''_p = -16A \cos 4x - 16B \sin 4x$. Substitution into the differential equation

gives $(-16A \cos 4x - 16B \sin 4x) - 2(-4A \sin 4x + 4B \cos 4x) = \sin 4x \Rightarrow$

$(-16A - 8B) \cos 4x + (8A - 16B) \sin 4x = \sin 4x$. Then $-16A - 8B = 0$ and $8A - 16B = 1 \Rightarrow A = \frac{1}{40}$

and $B = -\frac{1}{20}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^{2x} + \frac{1}{40} \cos 4x - \frac{1}{20} \sin 4x$.

4. The auxiliary equation is $r^2 + 6r + 9 = (r + 3)^2 = 0$, so the complementary solution is

$y_c(x) = c_1 e^{-3x} + c_2 x e^{-3x}$. Try the particular solution $y_p(x) = Ax + B$, so $y'_p = A$ and $y''_p = 0$. Substitution into the differential equation gives $0 + 6A + 9(Ax + B) = 1 + x$ or $(9A)x + (6A + 9B) = 1 + x$. Comparing coefficients, we have $9A = 1$ and $6A + 9B = 1$, so $A = \frac{1}{9}$ and $B = \frac{1}{27}$. Thus the general solution is

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{1}{9}x + \frac{1}{27}.$$

5. The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is

$y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{-x}$, so $y'_p = -Ae^{-x}$ and $y''_p = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \Rightarrow 10Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{10}$. Thus the general solution is

$$y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}.$$

6. $y_c(x) = e^{-x}(c_1 x + c_2)$. Try $y_p(x) = x^2(Ax + B)e^{-x}$ so that no term in y_p is a solution of

the complementary equation. Then $y'_p = [-Ax^3 + (3A - B)x^2 + 2Bx]e^{-x}$,

$y''_p = [Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B]e^{-x}$ and substitution gives

$[Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B] + 2[-Ax^3 + (3A - B)x^2 + 2Bx] + (Ax^3 + Bx^2) = x \Rightarrow$

$6Ax + 2B = x$. So $y_p(x) = x^2(\frac{1}{6}x)e^{-x}$ and the general solution is $y(x) = e^{-x}(c_1 x + c_2) + \frac{1}{6}x^3 e^{-x}$.

7. The auxiliary equation is $r^2 + 1 = 0$ with roots $r = \pm i$, so the complementary solution is

$y_c(x) = c_1 \cos x + c_2 \sin x$. For $y'' + y = e^x$ try $y_{p1}(x) = Ae^x$. Then $y'_{p1} = y''_{p1} = Ae^x$ and substitution gives $Ae^x + Ae^x = e^x \Rightarrow A = \frac{1}{2}$, so $y_{p1}(x) = \frac{1}{2}e^x$. For $y'' + y = x^3$ try $y_{p2}(x) = Ax^3 + Bx^2 + Cx + D$.

Then $y'_{p2} = 3Ax^2 + 2Bx + C$ and $y''_{p2} = 6Ax + 2B$. Substituting, we have

$6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$, so $A = 1$, $B = 0$, $6A + C = 0 \Rightarrow C = -6$, and $2B + D = 0$

$\Rightarrow D = 0$. Thus $y_{p2}(x) = x^3 - 6x$ and the general solution is

$y(x) = y_c(x) + y_{p1}(x) + y_{p2}(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x + x^3 - 6x$. But $2 = y(0) = c_1 + \frac{1}{2} \Rightarrow c_1 = \frac{3}{2}$

and $0 = y'(0) = c_2 + \frac{1}{2} - 6 \Rightarrow c_2 = \frac{11}{2}$. Thus the solution to the initial-value problem is

$$y(x) = \frac{3}{2} \cos x + \frac{11}{2} \sin x + \frac{1}{2}e^x + x^3 - 6x.$$

8. The auxiliary equation is $r^2 - 4 = 0$ with roots $r = \pm 2$, so the complementary

solution is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. Try $y_p(x) = e^x(A \cos x + B \sin x)$, so

$y'_p = e^x(A \cos x + B \sin x + B \cos x - A \sin x)$ and $y''_p = e^x(2B \cos x - 2A \sin x)$. Substitution gives

$$e^x(2B \cos x - 2A \sin x) - 4e^x(A \cos x + B \sin x) = e^x \cos x \Rightarrow$$

$(2B - 4A)e^x \cos x + (-2A - 4B)e^x \sin x = e^x \cos x \Rightarrow A = -\frac{1}{5}$, $B = \frac{1}{10}$. Thus the general solution is

$y(x) = c_1 e^{2x} + c_2 e^{-2x} + e^x(-\frac{1}{5} \cos x + \frac{1}{10} \sin x)$. But $1 = y(0) = c_1 + c_2 - \frac{1}{5}$ and

$2 = y'(0) = 2c_1 - 2c_2 - \frac{1}{10}$. Then $c_1 = \frac{9}{8}$, $c_2 = \frac{3}{40}$, and the solution to the initial-value problem is

$$y(x) = \frac{9}{8}e^{2x} + \frac{3}{40}e^{-2x} + e^x(-\frac{1}{5} \cos x + \frac{1}{10} \sin x).$$

9. The auxiliary equation is $r^2 - r = 0$ with roots $r = 0$, $r = 1$ so the complementary solution is $y_c(x) = c_1 + c_2 e^x$.

Try $y_p(x) = x(Ax + B)e^x$ so that no term in y_p is a solution of the complementary equation. Then

$y'_p = (Ax^2 + (2A + B)x + B)e^x$ and $y''_p = (Ax^2 + (4A + B)x + (2A + 2B))e^x$. Substitution into the

differential equation gives $(Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \Rightarrow$

$(2Ax + (2A + B))e^x = xe^x \Rightarrow A = \frac{1}{2}$, $B = -1$. Thus $y_p(x) = (\frac{1}{2}x^2 - x)e^x$ and the general solution is

$y(x) = c_1 + c_2 e^x + (\frac{1}{2}x^2 - x)e^x$. But $2 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_2 - 1$, so $c_2 = 2$ and $c_1 = 0$. The

solution to the initial-value problem is $y(x) = 2e^x + (\frac{1}{2}x^2 - x)e^x = e^x(\frac{1}{2}x^2 - x + 2)$.

10. $y_c(x) = c_1 e^x + c_2 e^{-2x}$. For $y'' + y' - 2y = x$ try $y_{p1}(x) = Ax + B$. Then $y'_{p1} = A$, $y''_{p1} = 0$, and substitution gives $0 + A - 2(Ax + B) = x \Rightarrow A = -\frac{1}{2}$, $B = -\frac{1}{4}$, so $y_{p1}(x) = -\frac{1}{2}x - \frac{1}{4}$. For $y'' + y' - 2y = \sin 2x$ try

$y_{p2}(x) = A \cos 2x + B \sin 2x$. Then $y'_{p2} = -2A \sin 2x + 2B \cos 2x$, $y''_{p2} = -4A \cos 2x - 4B \sin 2x$, and

substitution gives $(-4A \cos 2x - 4B \sin 2x) + (-2A \sin 2x + 2B \cos 2x) - 2(A \cos 2x + B \sin 2x) = \sin 2x$

$\Rightarrow A = -\frac{1}{20}$, $B = -\frac{3}{20}$. Thus $y_{p2}(x) = -\frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x$ and the general solution is

$y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x$. But $1 = y(0) = c_1 + c_2 - \frac{1}{4} - \frac{1}{20}$ and

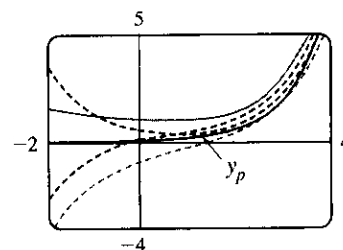
$0 = y'(0) = c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} \Rightarrow c_1 = \frac{17}{15}$ and $c_2 = \frac{1}{6}$. Thus the solution to the initial-value problem is

$$y(x) = \frac{17}{15}e^x + \frac{1}{6}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x.$$

11. $y_c(x) = c_1 e^{-x/4} + c_2 e^{-x}$. Try $y_p(x) = Ae^x$. Then

$10Ae^x = e^x$, so $A = \frac{1}{10}$ and the general solution is

$y(x) = c_1 e^{-x/4} + c_2 e^{-x} + \frac{1}{10} e^x$. The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as $x \rightarrow -\infty$), they all approach either ∞ or $-\infty$ as $x \rightarrow -\infty$. As $x \rightarrow \infty$, all solutions are asymptotic to $y_p = \frac{1}{10} e^x$.



12. The auxiliary equation is $(2r + 1)(r + 1) = 0$, so $r = -1, -\frac{1}{2}$ and $y_c(x) = c_1 e^{-x} + c_2 e^{-x/2}$. For

$2y'' + 3y' + y = 1$, try $y_{p1}(x) = A$; substituting gives $y_{p1}(x) = 1$. For $2y'' + 3y' + y = \cos 2x$ try

$$y_{p2} = A \cos 2x + B \sin 2x \Rightarrow y'_{p2} = -2A \sin 2x + 2B \cos 2x, y''_{p2} = -4A \cos 2x - 4B \sin 2x.$$

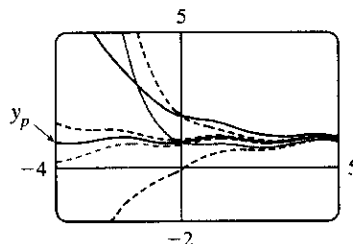
Substituting into the differential equation gives $\cos 2x = (6B - 7A) \cos 2x + (-7B - 6A) \sin 2x$.

Then solving the equations $6B - 7A = 1$ and $-7B - 6A = 0$ gives $A = -\frac{7}{85}$,

$B = \frac{6}{85}$. Thus, $y_{p2}(x) = -\frac{7}{85} \cos 2x + \frac{6}{85} \sin 2x$ and the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-x/2} + 1 - \frac{7}{85} \cos 2x + \frac{6}{85} \sin 2x.$$

The graph shows $y_p = y_{p1} + y_{p2}$ and several other solutions. Notice that all solutions are asymptotic to y_p as $x \rightarrow \infty$.



13. Here $y_c(x) = c_1 \cos 3x + c_2 \sin 3x$. For $y'' + 9y = e^{2x}$ try $y_{p1}(x) = Ae^{2x}$ and for $y'' + 9y = x^2 \sin x$ try

$$y_{p2}(x) = (Bx^2 + Cx + D) \cos x + (Ex^2 + Fx + G) \sin x. \text{ Thus a trial solution is}$$

$$y_p(x) = y_{p1}(x) + y_{p2}(x) = Ae^{2x} + (Bx^2 + Cx + D) \cos x + (Ex^2 + Fx + G) \sin x.$$

14. Since $y_c(x) = c_1 + c_2 e^{-9x}$, try $y_p(x) = (Ax + B)e^{-x} \cos \pi x + (Cx + D)e^{-x} \sin \pi x$.

15. Here $y_c(x) = c_1 + c_2 e^{-9x}$. For $y'' + 9y' = 1$ try $y_{p1}(x) = Ax$ (since $y = A$ is a solution to the complementary equation) and for $y'' + 9y' = xe^{9x}$ try $y_{p2}(x) = (Bx + C)e^{9x}$.

16. Since $y_c(x) = c_1 e^x + c_2 e^{-4x}$ try $y_p(x) = x(Ax^3 + Bx^2 + Cx + D)e^x$ so that no term of $y_p(x)$ satisfies the complementary equation.

17. Since $y_c(x) = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$ we try

$$y_p(x) = x(Ax^2 + Bx + C)e^{-x} \cos 3x + x(Dx^2 + Ex + F)e^{-x} \sin 3x \text{ (so that no term of } y_p \text{ is a solution of the complementary equation).}$$

18. Here $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. For $y'' + 4y = e^{3x}$ try $y_{p1}(x) = Ae^{3x}$ and for $y'' + 4y = x \sin 2x$ try $y_{p2}(x) = x(Bx + C) \cos 2x + x(Dx + E) \sin 2x$ (so that no term of y_{p2} is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u'_1 = -\frac{Gy_2}{a(y_1y'_2 - y_2y'_1)} \quad \text{and} \quad u'_2 = \frac{Gy_1}{a(y_1y'_2 - y_2y'_1)}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

19. (a) The complementary solution is $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. A particular solution is of the form

$y_p(x) = Ax + B$. Thus, $4Ax + 4B = x \Rightarrow A = \frac{1}{4}$ and $B = 0 \Rightarrow y_p(x) = \frac{1}{4}x$. Thus, the general solution is $y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$.

- (b) In (a), $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$, so set $y_1 = \cos 2x$, $y_2 = \sin 2x$. Then

$$y_1y'_2 - y_2y'_1 = 2\cos^2 2x + 2\sin^2 2x = 2 \text{ so } u'_1 = -\frac{1}{2}x \sin 2x \Rightarrow$$

$$u_1(x) = -\frac{1}{2} \int x \sin 2x \, dx = -\frac{1}{4}(-x \cos 2x + \frac{1}{2} \sin 2x) \text{ [by parts] and } u'_2 = \frac{1}{2}x \cos 2x$$

$$\Rightarrow u_2(x) = \frac{1}{2} \int x \cos 2x \, dx = \frac{1}{4}(x \sin 2x + \frac{1}{2} \cos 2x) \text{ [by parts]. Hence}$$

$$y_p(x) = -\frac{1}{4}(-x \cos 2x + \frac{1}{2} \sin 2x) \cos 2x + \frac{1}{4}(x \sin 2x + \frac{1}{2} \cos 2x) \sin 2x = \frac{1}{4}x. \text{ Thus}$$

$$y(x) = y_c(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x.$$

20. (a) Here $r^2 - 3r + 2 = 0 \Rightarrow r = 1$ or 2 and $y_c(x) = c_1 e^{2x} + c_2 e^x$. We try a particular solution of the form

$$y_p(x) = A \cos x + B \sin x \Rightarrow y'_p = -A \sin x + B \cos x \text{ and } y''_p = -A \cos x - B \sin x. \text{ Then the equation}$$

$$y'' - 3y' + 2y = \sin x \text{ becomes } (A - 3B) \cos x + (B + 3A) \sin x = \sin x \Rightarrow A - 3B = 0 \text{ and}$$

$$B + 3A = 1 \Rightarrow A = \frac{3}{10} \text{ and } B = \frac{1}{10}. \text{ Thus, } y_p(x) = \frac{3}{10} \cos x + \frac{1}{10} \sin x. \text{ Therefore, the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^x + \frac{3}{10} \cos x + \frac{1}{10} \sin x.$$

- (b) From (a) we know that $y_c(x) = c_1 e^{2x} + c_2 e^x$. Setting $y_1 = e^{2x}$, $y_2 = e^x$, we have

$$y_1y'_2 - y_2y'_1 = e^{3x} - 2e^{3x} = -e^{3x}. \text{ Thus } u'_1 = -\frac{\sin x e^x}{-e^{3x}} = \sin x e^{-2x} \text{ and}$$

$$u'_2 = \frac{\sin x e^{2x}}{-e^{3x}} = -\sin x e^{-x}. \text{ Then } u_1(x) = \int e^{-2x} \sin x \, dx = \frac{1}{5}e^{-2x}(-2 \sin x - \cos x) \text{ [by}$$

$$\text{parts] and } u_2(x) = -\int e^{-x} \sin x \, dx = -\frac{1}{2}e^{-x}(-\sin x - \cos x). \text{ Thus}$$

$$y_p(x) = \frac{1}{5}(-2 \sin x - \cos x) + \frac{1}{2}(\sin x + \cos x) = \frac{1}{10} \sin x + \frac{3}{10} \cos x \text{ and the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^x + \frac{1}{10} \sin x + \frac{3}{10} \cos x.$$

21. (a) $r^2 - r = r(r - 1) = 0 \Rightarrow r = 0, 1$, so the complementary solution is $y_c(x) = c_1 e^x + c_2 x e^x$. A particular

$$\text{solution is of the form } y_p(x) = Ae^{2x}. \text{ Thus } 4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \Rightarrow Ae^{2x} = e^{2x} \Rightarrow A = 1$$

$$\Rightarrow y_p(x) = e^{2x}. \text{ So a general solution is } y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}.$$

- (b) From (a), $y_c(x) = c_1 e^x + c_2 x e^x$, so set $y_1 = e^x$, $y_2 = x e^x$. Then, $y_1y'_2 - y_2y'_1 = e^{2x}(1 + x) - x e^{2x} = e^{2x}$

$$\text{and so } u'_1 = -x e^x \Rightarrow u_1(x) = -\int x e^x \, dx = -(x - 1)e^x \text{ [by parts] and } u'_2 = e^x \Rightarrow$$

$$u_2(x) = \int e^x \, dx = e^x. \text{ Hence } y_p(x) = (1 - x)e^{2x} + x e^{2x} = e^{2x} \text{ and the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}.$$

22. (a) Here $r^2 - 2r + 1 = (r - 1)^2 = 0 \Rightarrow r = 1$ and $y_c(x) = c_1 + c_2 x$ and so we try a particular solution of

$$\text{the form } y_p(x) = A x e^x. \text{ Thus, after calculating the necessary derivatives, we get } y'' - y' = e^x \Rightarrow$$

$$A e^x(2 + x) - A e^x(1 + x) = e^x \Rightarrow A = 1. \text{ Thus } y_p(x) = x e^x \text{ and the general solution is}$$

$$y(x) = c_1 + c_2 x + x e^x.$$

(b) From (a) we know that $y_c(x) = c_1 + c_2 e^x$, so setting $y_1 = 1$, $y_2 = e^x$, then $y_1 y_2' - y_2 y_1' = e^x - 0 = e^x$. Thus $u_1' = -e^{2x}/e^x = -e^x$ and $u_2' = e^x/e^x = 1$. Then $u_1(x) = -\int e^x dx = -e^x$ and $u_2(x) = x$. Thus $y_p(x) = -e^x + x e^x$ and the general solution is $y(x) = c_1 + c_2 e^x - e^x + x e^x = c_1 + c_3 e^x + x e^x$.

23. As in Example 6, $y_c(x) = c_1 \sin x + c_2 \cos x$, so set $y_1 = \sin x$, $y_2 = \cos x$. Then

$$y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1, \text{ so } u_1' = -\frac{\sec x \cos x}{-1} = 1 \Rightarrow u_1(x) = x \text{ and}$$

$$u_2' = \frac{\sec x \sin x}{-1} = -\tan x \Rightarrow u_2(x) = -\int \tan x dx = \ln |\cos x| = \ln(\cos x) \text{ on } 0 < x < \frac{\pi}{2}. \text{ Hence}$$

$$y_p(x) = x \sin x + \cos x \ln(\cos x) \text{ and the general solution is } y(x) = (c_1 + x) \sin x + [c_2 + \ln(\cos x)] \cos x.$$

24. Setting $y_1 = \sin x$, $y_2 = \cos x$, then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$. Thus $u_1' = -\frac{\cot x \cos x}{-1} = \frac{\cos^2 x}{\sin x}$

$$\text{and } u_2' = \frac{\cot x \sin x}{-1} = -\cos x. \text{ Then } u_1(x) = \int \frac{\cos^2 x}{\sin x} dx = \int (\csc x - \sin x) dx = \ln(\csc x - \cot x) + \cos x$$

$$\text{and } u_2(x) = -\sin x. \text{ Thus } y_p(x) = [\cos x + \ln(\csc x - \cot x)] \sin x + (-\sin x)(\cos x) \text{ and the general solution is } y(x) = c_1 \sin x + c_2 \cos x + \sin x \ln(\csc x - \cot x).$$

25. $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1 y_2' - y_2 y_1' = e^{3x}$. So $u_1' = \frac{-e^{2x}}{(1+e^{-x})e^{3x}} = -\frac{e^{-x}}{1+e^{-x}}$ and

$$u_1(x) = \int -\frac{e^{-x}}{1+e^{-x}} dx = \ln(1+e^{-x}). \quad u_2' = \frac{e^x}{(1+e^{-x})e^{3x}} = \frac{e^x}{e^{3x}+e^{2x}} \text{ so}$$

$$u_2(x) = \int \frac{e^x}{e^{3x}+e^{2x}} dx = \ln\left(\frac{e^x+1}{e^x}\right) - e^{-x} = \ln(1+e^{-x}) - e^{-x}. \text{ Hence}$$

$$y_p(x) = e^x \ln(1+e^{-x}) + e^{2x}[\ln(1+e^{-x}) - e^{-x}] \text{ and the general solution is}$$

$$y(x) = [c_1 + \ln(1+e^{-x})]e^x + [c_2 - e^{-x} + \ln(1+e^{-x})]e^{2x}.$$

26. $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and $y_1 y_2' - y_2 y_1' = -e^{-3x}$. So $u_1' = -\frac{(\sin e^x)e^{-2x}}{-e^{-3x}} = e^x \sin e^x$ and

$$u_2' = \frac{(\sin e^x)e^{-x}}{-e^{-3x}} = -e^{2x} \sin e^x. \text{ Hence } u_1(x) = \int e^x \sin e^x dx = -\cos e^x \text{ and}$$

$$u_2(x) = \int -e^{2x} \sin e^x dx = e^x \cos e^x - \sin e^x. \text{ Then } y_p(x) = -e^{-x} \cos e^x - e^{-2x}[\sin e^x - e^x \cos e^x] \text{ and the general solution is } y(x) = (c_1 - \cos e^x)e^{-x} + [c_2 - \sin e^x + e^x \cos e^x]e^{-2x}.$$

27. $y_1 = e^{-x}$, $y_2 = e^x$ and $y_1 y_2' - y_2 y_1' = 2$. So $u_1' = -\frac{e^x}{2x}$, $u_2' = \frac{e^{-x}}{2x}$ and

$$y_p(x) = -e^{-x} \int \frac{e^x}{2x} dx + e^x \int \frac{e^{-x}}{2x} dx. \text{ Hence the general solution is}$$

$$y(x) = \left(c_1 - \int \frac{e^x}{2x} dx\right)e^{-x} + \left(c_2 + \int \frac{e^{-x}}{2x} dx\right)e^x.$$

28. $y_1 = e^{-2x}$, $y_2 = x e^{-2x}$ and $y_1 y_2' - y_2 y_1' = e^{-4x}$. Then $u_1' = \frac{-e^{-2x} x e^{-2x}}{x^3 e^{-4x}} = -\frac{1}{x^2}$ so $u_1(x) = x^{-1}$ and

$$u_2' = \frac{e^{-2x} e^{-2x}}{x^3 e^{-4x}} = \frac{1}{x^3} \text{ so } u_2(x) = -\frac{1}{2x^2}. \text{ Thus } y_p(x) = \frac{e^{-2x}}{x} - \frac{x e^{-2x}}{2x^2} = \frac{e^{-2x}}{2x} \text{ and the general solution is}$$

$$y(x) = e^{-2x}[c_1 + c_2 x + 1/(2x)].$$

18.3 Applications of Second-Order Differential Equations

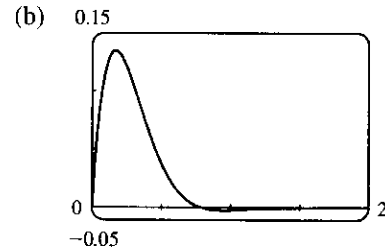
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1. By Hooke's Law $k(0.6) = 20$ so $k = \frac{100}{3}$ is the spring constant and the differential equation is $3x'' + \frac{100}{3}x = 0$. The general solution is $x(t) = c_1 \cos(\frac{10}{3}t) + c_2 \sin(\frac{10}{3}t)$. But $0 = x(0) = c_1$ and $1.2 = x'(0) = \frac{10}{3}c_2$, so the position of the mass after t seconds is $x(t) = 0.36 \sin(\frac{10}{3}t)$.

2. $k(0.3) = 24.3$ or $k = 81$ is the spring constant and the resulting initial-value problem is $4x'' + 81x = 0$, $x(0) = -0.5$ (since compressed), $x'(0) = 0$. The general solution is $x(t) = c_1 \cos(\frac{9}{2}t) + c_2 \sin(\frac{9}{2}t)$. But $-0.2 = x(0) = c_1$ and $0 = x'(0) = \frac{9}{2}c_2$. Thus the position is given by $x(t) = -0.2 \cos(4.5t)$.

3. $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$. The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

4. (a) The differential equation is $3x'' + 30x' + 123x = 0$ with general solution $x(t) = e^{-5t}(c_1 \cos 4t + c_2 \sin 4t)$. Then $0 = x(0) = c_1$ and $2 = x'(0) = 4c_2$, so the position is given by $x(t) = \frac{1}{2}e^{-5t} \sin 4t$.



5. For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

6. For critical damping we need $c^2 = 4mk$ or $c = 2\sqrt{mk} = 2\sqrt{3 \cdot 123} = 6\sqrt{41}$.

7. We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is

$\frac{d^2x}{dt^2} + c \frac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$. If $c = 10$, we have two complex roots

$r = -5 \pm 5\sqrt{3}i$, so the motion is underdamped and the solution is $x = e^{-5t}[c_1 \cos(5\sqrt{3}t) + c_2 \sin(5\sqrt{3}t)]$.

Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = 5\sqrt{3}c_2 - 5c_1 \Rightarrow c_2 = -\frac{1}{10\sqrt{3}}$, so

$x = e^{-5t}[-0.1 \cos(5\sqrt{3}t) - \frac{1}{10\sqrt{3}} \sin(5\sqrt{3}t)]$. If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is $x = e^{-15t/2}[c_1 \cos(\frac{5\sqrt{7}}{2}t) + c_2 \sin(\frac{5\sqrt{7}}{2}t)]$, so

$-0.1 = x(0) = c_1$ and $0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}c_1 \Rightarrow c_2 = -\frac{3}{10\sqrt{7}}$. Thus

$x = e^{-15t/2}[-0.1 \cos(\frac{5\sqrt{7}}{2}t) - \frac{3}{10\sqrt{7}} \sin(\frac{5\sqrt{7}}{2}t)]$. For $c = 20$, we have equal roots $r_1 = r_2 = -10$,

so the oscillation is critically damped and the solution is $x = (c_1 + c_2t)e^{-10t}$. Then $-0.1 = x(0) = c_1$ and

$0 = x'(0) = -10c_1 + c_2 \Rightarrow c_2 = -1$, so $x = (-0.1 - t)e^{-10t}$. If $c = 25$ the auxiliary equation has roots

$r_1 = -5$, $r_2 = -20$, so we have overdamping and the solution is $x = c_1 e^{-5t} + c_2 e^{-20t}$. Then

$-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -5c_1 - 20c_2 \Rightarrow c_1 = -\frac{2}{15}$ and $c_2 = \frac{1}{30}$.

$x_p(t) = A \cos \omega_0 t + B \sin \omega_0 t$. Then we need

$$(m)(-\omega_0^2)(A \cos \omega_0 t + B \sin \omega_0 t) + k(A \cos \omega_0 t + B \sin \omega_0 t) = F_0 \cos \omega_0 t \text{ or } A(k - m\omega_0^2) = F_0 \text{ and}$$

$$B(k - m\omega_0^2) = 0. \text{ Hence } B = 0 \text{ and } A = \frac{F_0}{k - m\omega_0^2} = \frac{F_0}{m(\omega^2 - \omega_0^2)} \text{ since } \omega^2 = \frac{k}{m}. \text{ Thus the motion of the}$$

$$\text{mass is given by } x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t.$$

10. As in Exercise 9, $x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t) = t(A \cos \omega t + B \sin \omega t)$. Then we need

$$m(2\omega B - \omega^2 At) \cos \omega t - m(2\omega A + \omega^2 Bt) \sin \omega t + kAt \cos \omega t + kBt \sin \omega t = F_0 \cos \omega t \text{ or } 2m\omega B = F_0 \text{ and } -2m\omega A = 0 \text{ (noting } -m\omega^2 A + kA = 0 \text{ and } -m\omega^2 B + kB = 0 \text{ since } \omega^2 = k/m). \text{ Hence the general solution is}$$

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + [F_0 t / (2m\omega)] \sin \omega t.$$

11. From Equation 6, $x(t) = f(t) + g(t)$ where $f(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and $g(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$. Then f is periodic, with period $\frac{2\pi}{\omega}$, and if $\omega \neq \omega_0$, g is periodic with period $\frac{2\pi}{\omega_0}$. If $\frac{\omega}{\omega_0}$ is a rational number, then we can say $\frac{\omega}{\omega_0} = \frac{a}{b} \Rightarrow a = \frac{b\omega}{\omega_0}$ where a and b are non-zero integers. Then

$$\begin{aligned} x\left(t + a \cdot \frac{2\pi}{\omega}\right) &= f\left(t + a \cdot \frac{2\pi}{\omega}\right) + g\left(t + a \cdot \frac{2\pi}{\omega}\right) = f(t) + g\left(t + \frac{b\omega}{\omega_0} \cdot \frac{2\pi}{\omega}\right) \\ &= f(t) + g\left(t + b \cdot \frac{2\pi}{\omega_0}\right) = f(t) + g(t) = x(t) \end{aligned}$$

so $x(t)$ is periodic.

12. (a) The graph of $x = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$ has a t -intercept when $c_1 e^{r_1 t} + c_2 t e^{r_1 t} = 0 \Leftrightarrow e^{r_1 t}(c_1 + c_2 t) = 0 \Leftrightarrow c_1 = -c_2 t$. Since $t > 0$, x has a t -intercept if and only if c_1 and c_2 have opposite signs.

(b) For $t > 0$, the graph of x crosses the t -axis when $c_1 e^{r_1 t} + c_2 t e^{r_2 t} = 0 \Leftrightarrow c_2 t e^{r_2 t} = -c_1 e^{r_1 t} \Leftrightarrow$

$$c_2 = -c_1 \frac{e^{r_1 t}}{e^{r_2 t}} = -c_1 e^{(r_1 - r_2)t}. \text{ But } r_1 > r_2 \Rightarrow r_1 - r_2 > 0 \text{ and since } t > 0, e^{(r_1 - r_2)t} > 1. \text{ Thus}$$

$$|c_2| = |c_1| e^{(r_1 - r_2)t} > |c_1|, \text{ and the graph of } x \text{ can cross the } t\text{-axis only if } |c_2| > |c_1|.$$

13. Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then

$$Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) \text{ and try } Q_p(t) = A \Rightarrow 500A = 12 \text{ or } A = \frac{3}{125}.$$

$$\text{The general solution is } Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) + \frac{3}{125}. \text{ But } 0 = Q(0) = c_1 + \frac{3}{125} \text{ and}$$

$$Q'(t) = I(t) = e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t] \text{ but } 0 = Q'(0) = -10c_1 + 20c_2. \text{ Thus the charge is } Q(t) = -\frac{1}{250}e^{-10t}(6 \cos 20t + 3 \sin 20t) + \frac{3}{125} \text{ and the current is } I(t) = e^{-10t}\left(\frac{3}{5}\right) \sin 20t.$$

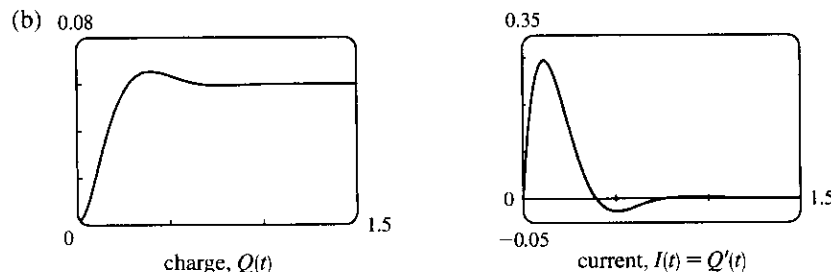
14. (a) Here the initial-value problem for the charge is $2Q'' + 24Q' + 200Q = 12$ with $Q(0) = 0.001$ and $Q'(0) = 0$.

$$\text{Then } Q_c(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) \text{ and try } Q_p(t) = A \Rightarrow A = \frac{3}{50} \text{ and the general solution is}$$

$$Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) + \frac{3}{50}. \text{ But } 0.001 = Q(0) = c_1 + \frac{3}{50} \text{ so } c_1 = -0.059. \text{ Also}$$

$$Q'(t) = I(t) = e^{-6t}[(-6c_1 + 8c_2) \cos 8t + (-6c_2 - 8c_1) \sin 8t] \text{ and } 0 = Q'(0) = -6c_1 + 8c_2 \text{ so}$$

$c_2 = -0.04425$. Hence the charge is $Q(t) = -e^{-6t}(0.059 \cos 8t + 0.04425 \sin 8t) + \frac{3}{50}$ and the current is $I(t) = e^{-6t}(0.7375) \sin 8t$.



15. As in Exercise 13, $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ but $E(t) = 12 \sin 10t$ so try

$Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives

$$(-100A + 200B + 500A) \cos 10t + (-100B - 200A + 500B) \sin 10t = 12 \sin 10t \Rightarrow 400A + 200B = 0$$

and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is

$$Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t. \text{ But } 0 = Q(0) = c_1 - \frac{3}{250} \text{ so } c_1 = \frac{3}{250}.$$

Also $Q'(t) = \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t + e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t]$ and

$$0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2 \text{ so } c_2 = -\frac{3}{500}. \text{ Hence the charge is given by}$$

$$Q(t) = e^{-10t} \left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t \right] - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t.$$

16. (a) As in Exercise 14, $Q_c(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t)$ but try $Q_p(t) = A \cos 10t + B \sin 10t$.

Substituting into the differential equation gives

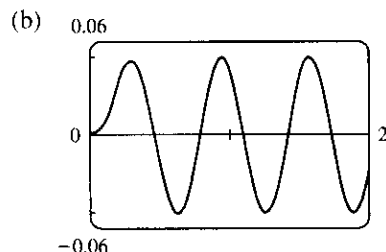
$$(-200A + 240B + 200A) \cos 10t + (-200B - 240A + 200B) \sin 10t = 12 \sin 10t, \text{ so } B = 0 \text{ and}$$

$$A = -\frac{1}{20}. \text{ Hence, the general solution is } Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{1}{20} \cos 10t. \text{ But}$$

$$0.001 = Q(0) = c_1 - \frac{1}{20}, Q'(t) = e^{-6t}[(-6c_1 + 8c_2) \cos 8t + (-6c_2 - 8c_1) \sin 8t] - \frac{1}{2} \sin 10t \text{ and}$$

$$0 = Q'(0) = -6c_1 + 8c_2, \text{ so } c_1 = 0.051 \text{ and } c_2 = 0.03825. \text{ Thus the charge is given by}$$

$$Q(t) = e^{-6t}(0.051 \cos 8t + 0.03825 \sin 8t) - \frac{1}{20} \cos 10t.$$



17. $x(t) = A \cos(\omega t + \delta) \Leftrightarrow x(t) = A[\cos \omega t \cos \delta - \sin \omega t \sin \delta] \Leftrightarrow x(t) = A\left(\frac{c_1}{A} \cos \omega t + \frac{c_2}{A} \sin \omega t\right)$

where $\cos \delta = c_1/A$ and $\sin \delta = -c_2/A \Leftrightarrow x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. (Note that $\cos^2 \delta + \sin^2 \delta = 1 \Rightarrow c_1^2 + c_2^2 = A^2$.)

18. (a) We approximate $\sin \theta$ by θ and, with $L = 1$ and $g = 9.8$, the differential equation becomes $\frac{d^2\theta}{dt^2} + 9.8\theta = 0$.

The auxiliary equation is $r^2 + 9.8 = 0 \Rightarrow r = \pm\sqrt{9.8}i$, so the general solution is

$$\theta(t) = c_1 \cos(\sqrt{9.8}t) + c_2 \sin(\sqrt{9.8}t). \text{ Then } 0.2 = \theta(0) = c_1 \text{ and } 1 = \theta'(0) = \sqrt{9.8}c_2 \Rightarrow c_2 = \frac{1}{\sqrt{9.8}},$$

so the equation is $\theta(t) = 0.2 \cos(\sqrt{9.8}t) + \frac{1}{\sqrt{9.8}} \sin(\sqrt{9.8}t)$.

- (b) $\theta'(t) = -0.2\sqrt{9.8}\sin(\sqrt{9.8}t) + \cos(\sqrt{9.8}t) = 0$ or $\tan(\sqrt{9.8}t) = \frac{5}{\sqrt{9.8}}$, so the critical numbers are

$$t = \frac{1}{\sqrt{9.8}} \tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right) + \frac{\pi}{\sqrt{9.8}} n \text{ (} n \text{ any integer). The maximum angle from the vertical is}$$

$$\theta\left(\frac{1}{\sqrt{9.8}} \tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)\right) \approx 0.377 \text{ radians (or about } 21.7^\circ).$$

- (c) From part (b), the critical numbers of $\theta(t)$ are spaced $\frac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values is $2\left(\frac{\pi}{\sqrt{9.8}}\right)$. Thus the period of the pendulum is $\frac{2\pi}{\sqrt{9.8}} \approx 2.007$ seconds.

- (d) $\theta(t) = 0 \Rightarrow 0.2 \cos(\sqrt{9.8}t) + \frac{1}{\sqrt{9.8}} \sin(\sqrt{9.8}t) = 0 \Rightarrow \tan(\sqrt{9.8}t) = -0.2\sqrt{9.8} \Rightarrow$

$$t = \frac{1}{\sqrt{9.8}} [\tan^{-1}(-0.2\sqrt{9.8}) + \pi] \approx 0.825 \text{ seconds.}$$

- (e) $\theta'(0.825) \approx -1.180$ rad/s.

18.4 Series Solutions

ET 17.4

1. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and the given equation, $y' - y = 0$, becomes

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0. \text{ Replacing } n \text{ by } n+1 \text{ in the first sum gives } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0,$$

so $\sum_{n=0}^{\infty} [(n+1)c_{n+1} - c_n] x^n = 0$. Equating coefficients gives $(n+1)c_{n+1} - c_n = 0$, so the recursion relation is

$$c_{n+1} = \frac{c_n}{n+1}, \quad n = 0, 1, 2, \dots. \text{ Then } c_1 = c_0, c_2 = \frac{1}{2}c_1 = \frac{c_0}{2}, c_3 = \frac{1}{3}c_2 = \frac{1}{3} \cdot \frac{1}{2}c_0 = \frac{c_0}{3!}, c_4 = \frac{1}{4}c_3 = \frac{c_0}{4!}, \text{ and}$$

in general, $c_n = \frac{c_0}{n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x$$

2. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = xy \Rightarrow y' - xy = 0 \Rightarrow \sum_{n=1}^{\infty} n c_n x^{n-1} - x \sum_{n=0}^{\infty} c_n x^n = 0$ or

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0. \text{ Replacing } n \text{ with } n+1 \text{ in the first sum and } n \text{ with } n-1 \text{ in the second}$$

$$\text{gives } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0 \text{ or } c_1 + \sum_{n=1}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0. \text{ Thus,}$$

$c_1 + \sum_{n=1}^{\infty} [(n+1)c_{n+1} - c_{n-1}]x^n = 0$. Equating coefficients gives $c_1 = 0$ and $(n+1)c_{n+1} - c_{n-1} = 0$. Thus, the recursion relation is $c_{n+1} = \frac{c_{n-1}}{n+1}$, $n = 1, 2, \dots$. But $c_1 = 0$, so $c_3 = 0$ and $c_5 = 0$ and in general $c_{2n+1} = 0$. Also, $c_2 = \frac{c_0}{2}$, $c_4 = \frac{c_2}{4} = \frac{c_0}{4 \cdot 2} = \frac{c_0}{2^2 \cdot 2!}$, $c_6 = \frac{c_4}{6} = \frac{c_0}{6 \cdot 4 \cdot 2} = \frac{c_0}{2^3 \cdot 3!}$ and in general $c_{2n} = \frac{c_0}{2^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{c_0}{2^n \cdot n!} x^{2n} = c_0 \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = c_0 e^{x^2/2}$$

3. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n$ and

$$-x^2 y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2} x^n. \text{ Hence, the equation } y' = x^2 y \text{ becomes}$$

$$\sum_{n=0}^{\infty} (n+1)c_{n+1}x^n - \sum_{n=2}^{\infty} c_{n-2}x^n = 0 \text{ or } c_1 + 2c_2x + \sum_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}]x^n = 0. \text{ Equating coefficients}$$

gives $c_1 = c_2 = 0$ and $c_{n+1} = \frac{c_{n-2}}{n+1}$ for $n = 2, 3, \dots$. But $c_1 = 0$, so $c_4 = 0$ and $c_7 = 0$ and in general

$$c_{3n+1} = 0. \text{ Similarly } c_2 = 0 \text{ so } c_{3n+2} = 0. \text{ Finally } c_3 = \frac{c_0}{3}, c_6 = \frac{c_3}{6} = \frac{c_0}{6 \cdot 3} = \frac{c_0}{3^2 \cdot 2!},$$

$$c_9 = \frac{c_6}{9} = \frac{c_0}{9 \cdot 6 \cdot 3} = \frac{c_0}{3^3 \cdot 3!}, \dots, \text{ and } c_{3n} = \frac{c_0}{3^n \cdot n!}. \text{ Thus, the solution is}$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}$$

4. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n$. Then the differential

$$\text{equation becomes } (x-3) \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} (n+1)c_{n+1}x^{n+1} - 3 \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$$

$$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} [(n+2)c_n - 3(n+1)c_{n+1}]x^n = 0$$

(since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the

$$\text{recursion relation is } c_{n+1} = \frac{(n+2)c_n}{3(n+1)}, n = 0, 1, 2, \dots. \text{ Then } c_1 = \frac{2c_0}{3}, c_2 = \frac{3c_1}{3(2)} = \frac{3c_0}{3^2},$$

$$c_3 = \frac{4c_2}{3(3)} = \frac{4c_0}{3^3}, c_4 = \frac{5c_3}{3(4)} = \frac{5c_0}{3^4}, \text{ and in general, } c_n = \frac{(n+1)c_0}{3^n}. \text{ Thus the solution is}$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n. \left[\text{Note that } c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n = \frac{9c_0}{(3-x)^2} \text{ for } |x| < 3. \right]$$

5. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation becomes $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + n c_n + c_n] x^n$ (since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives $(n+2)(n+1) c_{n+2} + (n+1) c_n = 0$, thus the recursion relation is $c_{n+2} = \frac{-(n+1) c_n}{(n+2)(n+1)} = -\frac{c_n}{n+2}$, $n = 0, 1, 2, \dots$. Then the even coefficients are given by $c_2 = -\frac{c_0}{2}$, $c_4 = -\frac{c_2}{4} = \frac{c_0}{2 \cdot 4}$, $c_6 = -\frac{c_4}{6} = -\frac{c_0}{2 \cdot 4 \cdot 6}$, and in general, $c_{2n} = (-1)^n \frac{c_0}{2 \cdot 4 \cdot \dots \cdot 2n} = \frac{(-1)^n c_0}{2^n n!}$. The odd coefficients are $c_3 = -\frac{c_1}{3}$, $c_5 = -\frac{c_3}{5} = \frac{c_1}{3 \cdot 5}$, $c_7 = -\frac{c_5}{7} = -\frac{c_1}{3 \cdot 5 \cdot 7}$, and in general, $c_{2n+1} = (-1)^n \frac{c_1}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1)} = \frac{(-2)^n n! c_1}{(2n+1)!}$. The solution is $y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$.

6. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. Hence, the equation $y'' = y$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} - c_n] x^n = 0$. So the recursion relation is $c_{n+2} = \frac{c_n}{(n+2)(n+1)}$, $n = 0, 1, \dots$. Given c_0 and c_1 , $c_2 = \frac{c_0}{2 \cdot 1}$, $c_4 = \frac{c_2}{4 \cdot 3} = \frac{c_0}{4!}$, $c_6 = \frac{c_4}{6 \cdot 5} = \frac{c_0}{6!}$, $\dots$, $c_{2n} = \frac{c_0}{(2n)!}$ and $c_3 = \frac{c_1}{3 \cdot 2}$, $c_5 = \frac{c_3}{5 \cdot 4} = \frac{c_1}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{c_1}{5!}$, $c_7 = \frac{c_5}{7 \cdot 6} = \frac{c_1}{7!}$, $\dots$, $c_{2n+1} = \frac{c_1}{(2n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

The solution can be written as $y(x) = c_0 \cosh x + c_1 \sinh x$

$$\left[\text{or } y(x) = c_0 \frac{e^x + e^{-x}}{2} + c_1 \frac{e^x - e^{-x}}{2} = \frac{c_0 + c_1}{2} e^x + \frac{c_0 - c_1}{2} e^{-x} \right].$$

7. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, $xy' = \sum_{n=0}^{\infty} n c_n x^n$ and $(x^2 + 1)y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + [n(n-1) + n-1] c_n] x^n = 0$. The recursion relation is $c_{n+2} = -\frac{(n-1) c_n}{n+2}$, $n = 0, 1, 2, \dots$. Given c_0 and c_1 , $c_2 = \frac{c_0}{2}$, $c_4 = -\frac{c_2}{4} = -\frac{c_0}{2^2 \cdot 2!}$, $c_6 = -\frac{3c_4}{6} = (-1)^2 \frac{3c_0}{2^3 \cdot 3!}$, $\dots$, $c_{2n} = (-1)^{n-1} \frac{1 \cdot 3 \cdot \dots \cdot (2n-3) c_0}{2^n n!} = (-1)^{n-1} \frac{(2n-3)! c_0}{2^n 2^{n-2} n! (n-2)!} = (-1)^{n-1} \frac{(2n-3)! c_0}{2^{2n-2} n! (n-2)!}$ for $n = 2, 3, \dots$, $c_3 = \frac{0 \cdot c_1}{3} = 0 \Rightarrow c_{2n+1} = 0$ for $n = 1, 2, \dots$. Thus the solution is $y(x) = c_0 + c_1 x + c_0 \frac{x^2}{2} + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (2n-3)!}{2^{2n-2} n! (n-2)!} x^{2n}$.

8. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$ and

$$-xy(x) = -\sum_{n=0}^{\infty} c_n x^{n+1} = -\sum_{n=1}^{\infty} c_{n-1} x^n. \text{ The equation } y'' = xy \text{ becomes}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0 \text{ or } 2c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - c_{n-1}] x^n = 0. \text{ Equating}$$

coefficients gives $c_2 = 0$ and $c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \dots$. Since $c_2 = 0$,

$$c_{3n+2} = 0 \text{ for } n = 0, 1, 2, \dots. \text{ Given } c_0, c_3 = \frac{c_0}{3 \cdot 2}, c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \dots,$$

$$c_{3n} = \frac{c_0}{3n(3n-1)(3n-3)(3n-4) \cdots 6 \cdot 5 \cdot 3 \cdot 2}. \text{ Given } c_1, c_4 = \frac{c_1}{4 \cdot 3}, c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}, \dots,$$

$$c_{3n+1} = \frac{c_1}{(3n+1)3n(3n-2)(3n-3) \cdots 7 \cdot 6 \cdot 4 \cdot 3}. \text{ The solution can be written as}$$

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(3n-2)(3n-5) \cdots 7 \cdot 4 \cdot 1}{(3n)!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(3n-1)(3n-4) \cdots 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}$$

9. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum_{n=1}^{\infty} n c_n x^{n-1} = -\sum_{n=1}^{\infty} n c_n x^n = -\sum_{n=0}^{\infty} n c_n x^n$,

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n, \text{ and the equation } y'' - xy' - y = 0 \text{ becomes}$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - n c_n - c_n] x^n = 0. \text{ Thus, the recursion relation is}$$

$$c_{n+2} = \frac{n c_n + c_n}{(n+2)(n+1)} = \frac{c_n(n+1)}{(n+2)(n+1)} = \frac{c_n}{n+2} \text{ for } n = 0, 1, 2, \dots. \text{ One of the given conditions is}$$

$$y(0) = 1. \text{ But } y(0) = \sum_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + \cdots = c_0, \text{ so } c_0 = 1. \text{ Hence, } c_2 = \frac{c_0}{2} = \frac{1}{2}, c_4 = \frac{c_2}{4} = \frac{1}{2 \cdot 4},$$

$$c_6 = \frac{c_4}{6} = \frac{1}{2 \cdot 4 \cdot 6}, \dots, c_{2n} = \frac{1}{2^n n!}. \text{ The other given condition is } y'(0) = 0. \text{ But}$$

$$y'(0) = \sum_{n=1}^{\infty} n c_n(0)^{n-1} = c_1 + 0 + 0 + \cdots = c_1, \text{ so } c_1 = 0. \text{ By the recursion relation, } c_3 = \frac{c_1}{3} = 0, c_5 = 0, \dots,$$

$c_{2n+1} = 0$ for $n = 0, 1, 2, \dots$. Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}$$

10. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2}$ and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-2}^{\infty} (n+4)(n+3)c_{n+4} x^{n+2}$$

$$= 2c_2 + 6c_3 x + \sum_{n=0}^{\infty} (n+4)(n+3)c_{n+4} x^{n+2}$$

Thus, the equation $y'' + x^2y = 0$ becomes $2c_2 + 6c_3x + \sum_{n=0}^{\infty} [(n+4)(n+3)c_{n+4} + c_n]x^{n+2} = 0$. So

$$c_2 = c_3 = 0 \text{ and the recursion relation is } c_{n+4} = -\frac{c_n}{(n+4)(n+3)}, n = 0, 1, 2, \dots$$

But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion relation, $c_{4n+1} = c_{4n+2} = c_{4n+3} = 0$ for $n = 0, 1, 2, \dots$

Also, $c_0 = y(0) = 1$, so

$$c_4 = -\frac{c_0}{4 \cdot 3} = -\frac{1}{4 \cdot 3}, c_8 = -\frac{c_4}{8 \cdot 7} = \frac{(-1)^2}{8 \cdot 7 \cdot 4 \cdot 3}, \dots, c_{4n} = \frac{(-1)^n}{4n(4n-1)(4n-4)(4n-5) \cdots 4 \cdot 3}.$$

Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=0}^{\infty} c_{4n} x^{4n} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5) \cdots 4 \cdot 3}$$

11. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $xy = x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$,

$$x^2 y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n+1},$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} \quad [\text{replace } n \text{ with } n+3]$$

$$= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1},$$

and the equation $y'' + x^2 y' + xy = 0$ becomes $2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) c_{n+3} + n c_n + c_n] x^{n+1} = 0$.

So $c_2 = 0$ and the recursion relation is $c_{n+3} = \frac{-n c_n - c_n}{(n+3)(n+2)} = -\frac{(n+1) c_n}{(n+3)(n+2)}, n = 0, 1, 2, \dots$

But $c_0 = y(0) = 0 = c_2$ and by the recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \dots$

Also, $c_1 = y'(0) = 1$, so

$$c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{2}{4 \cdot 3}, c_7 = -\frac{5c_4}{7 \cdot 6} = (-1)^2 \frac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \frac{2^2 5^2}{7!}, \dots,$$

$$c_{3n+1} = (-1)^n \frac{2^2 5^2 \cdots (3n-1)^2}{(3n+1)!}. \text{ Thus, the solution is}$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[(-1)^n \frac{2^2 5^2 \cdots (3n-1)^2 x^{3n+1}}{(3n+1)!} \right]$$

12. (a) Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^{n+2}$,

$$xy'(x) = \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=-1}^{\infty} (n+2) c_{n+2} x^{n+2} = c_1 x + \sum_{n=0}^{\infty} (n+2) c_{n+2} x^{n+2}, \text{ and the equation}$$

$$x^2 y'' + xy' + x^2 y = 0 \text{ becomes } c_1 x + \sum_{n=0}^{\infty} \{[(n+2)(n+1) + (n+2)] c_{n+2} + c_n\} x^{n+2} = 0. \text{ So } c_1 = 0$$

and the recursion relation is $c_{n+2} = -\frac{c_n}{(n+2)^2}, n = 0, 1, 2, \dots$. But $c_1 = y'(0) = 0$ so $c_{2n+1} = 0$ for

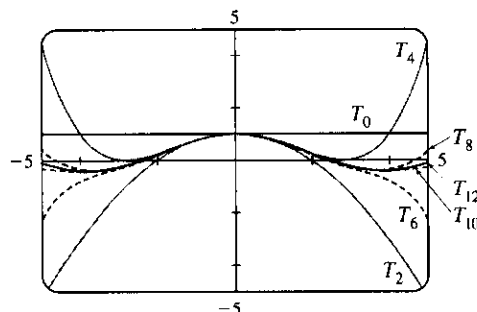
$n = 0, 1, 2, \dots$. Also, $c_0 = y(0) = 1$, so $c_2 = -\frac{1}{2^2}$, $c_4 = -\frac{c_2}{4^2} = (-1)^2 \frac{1}{4^2 2^2} = (-1)^2 \frac{1}{2^4 (2!)^2}$,

$c_6 = -\frac{c_4}{6^2} = (-1)^3 \frac{1}{2^6 (3!)^2}$, $\dots$, $c_{2n} = (-1)^n \frac{1}{2^{2n} (n!)^2}$. The solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} (n!)^2}$$

(b) The Taylor polynomials T_0 to T_{12} are shown in the graph.

Because T_{10} and T_{12} are close together throughout the interval $[-5, 5]$, it is reasonable to assume that T_{12} is a good approximation to the Bessel function on that interval.



18 Review

ET 17

CONCEPT CHECK

- (a) $ay'' + by' + cy = 0$ where a , b , and c are constants.

(b) $ar^2 + br + c = 0$

(c) If the auxiliary equation has two distinct real roots r_1 and r_2 , the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If the roots are real and equal, the solution is $y = c_1 e^{rx} + c_2 x e^{rx}$ where r is the common root. If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is $y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$.
- (a) An initial-value problem consists of finding a solution y of a second-order differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are constants.

(b) A boundary-value problem consists of finding a solution y of a second-order differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.
- (a) $ay'' + by' + cy = G(x)$ where a , b , and c are constants and G is a continuous function.

(b) The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution y_c of the complementary equation and y_p is any particular solution of the original differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

(c) See Examples 1–5 and the associated discussion in Section 18.2 [ET 17.2].

(d) See the discussion on pages 1188–1190 [ET 1152–1154].
- Second-order linear differential equations can be used to describe the motion of a vibrating spring or to analyze an electric circuit; see the discussion in Section 18.3 [ET 17.3].
- See Example 1 and the preceding discussion in Section 18.4 [ET 17.4].

- ## EXERCISES

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11. The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k_2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.
12. The auxiliary equation is $r^2 - 6r + 25 = 0$ and the general solution is $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $2 = y(0) = c_1$ and $1 = y'(0) = 3c_1 + 4c_2$. Thus the solution is $y(x) = e^{3x}(2 \cos 4x - \frac{5}{4} \sin 4x)$.
13. The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1 e^x + c_2 e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}(e^{4x} - e^x)$.
14. $y_c(x) = c_1 \cos(x/3) + c_2 \sin(x/3)$. For $9y'' + y = 3x$, try $y_{p1}(x) = Ax + B$. Then $y_{p1}(x) = 3x$. For $9y'' + y = e^{-x}$, try $y_{p2}(x) = Ae^{-x}$. Then $9Ae^{-x} + Ae^{-x} = e^{-x}$ or $y_{p2}(x) = \frac{1}{10}e^{-x}$. Thus the general solution is $y(x) = c_1 \cos(x/3) + c_2 \sin(x/3) + 3x + \frac{1}{10}e^{-x}$. But $1 = y(0) = c_1 + \frac{1}{10}$ and $2 = y'(0) = \frac{1}{3}c_2 + 3 - \frac{1}{10}$, so $c_1 = \frac{9}{10}$ and $c_2 = -\frac{27}{10}$. Hence the solution is $y(x) = \frac{1}{10}[9 \cos(x/3) - 27 \sin(x/3)] + 3x + \frac{1}{10}e^{-x}$.
15. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \dots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \dots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\frac{1}{3}$, $c_5 = \frac{(-1)^2}{3 \cdot 5}$, $c_7 = \frac{(-1)^3}{3 \cdot 5 \cdot 7} = \frac{(-1)^3 2^3 3!}{7!}$, $\dots$, $c_{2n+1} = \frac{(-1)^n 2^n n!}{(2n+1)!}$ for $n = 0, 1, 2, \dots$. Thus the solution to the initial-value problem is $y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}$.
16. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (n+2)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = \frac{c_n}{n+1}$ for $n = 0, 1, 2, \dots$. Given c_0 and c_1 , we have $c_2 = \frac{c_0}{1}$, $c_4 = \frac{c_2}{3} = \frac{c_0}{1 \cdot 3}$, $c_6 = \frac{c_4}{5} = \frac{c_0}{1 \cdot 3 \cdot 5}$, $\dots$, $c_{2n} = \frac{c_0}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} = c_0 \frac{2^{n-1}(n-1)!}{(2n-1)!}$. Similarly $c_3 = \frac{c_1}{2}$, $c_5 = \frac{c_3}{4} = \frac{c_1}{2 \cdot 4}$, $c_7 = \frac{c_5}{6} = \frac{c_1}{2 \cdot 4 \cdot 6}$, $\dots$, $c_{2n+1} = \frac{c_1}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} = \frac{c_1}{2^n n!}$. Thus the general solution is $y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_0 \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)! x^{2n}}{(2n-1)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!}$. But $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!} = x \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x^2)^n}{n!} = x e^{x^2/2}$, so $y(x) = c_1 x e^{x^2/2} + c_0 + c_0 \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)! x^{2n}}{(2n-1)!}$.
17. Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A$. Thus the general solution is $Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}$. But $0.01 = Q'(0) = c_1 + 0.03$ and $0 = Q''(0) = -10c_1 + 10c_2$, so $c_1 = -0.02 = c_2$. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

18. By Hooke's Law the spring constant is $k = 64$ and the initial-value problem is $2x'' + 16x' + 64x = 0$, $x(0) = 0$, $x'(0) = 2.4$. Thus the general solution is $x(t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t)$. But $0 = x(0) = c_1$ and $2.4 = x'(0) = -4c_1 + 4c_2 \Rightarrow c_1 = 0, c_2 = 0.6$. Thus the position of the mass is given by $x(t) = 0.6e^{-4t} \sin 4t$.

19. (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as

follows: $\rho = \frac{\text{mass of earth}}{\text{volume of earth}} = \frac{M}{\frac{4}{3}\pi R^3}$. If V_r is the volume of the portion of the earth which lies within a

distance r of the center, then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \frac{Mr^3}{R^3}$. Thus $F_r = -\frac{GM_r m}{r^2} = -\frac{GMm}{R^3}r$.

- (b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of

Motion, $m \frac{d^2 y}{dt^2} = F_y = -\frac{GMm}{R^3}y$, so $y''(t) = -k^2 y(t)$ where $k^2 = \frac{GM}{R^3}$. At the surface,

$-mg = F_R = -\frac{GMm}{R^2}$, so $g = \frac{GM}{R^2}$. Therefore $k^2 = \frac{g}{R}$.

- (c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 18.1 [ET 17.1], not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 18.1 [ET 17.1], the general solution of our differential equation for t is

$y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R \cos kt$ and $y'(t) = -kR \sin kt$. This is simple harmonic motion (see Section 18.3 [ET 17.3]) with amplitude R , frequency k , and phase angle 0. The period is $T = 2\pi/k$.

$R \approx 3960$ mi $= 3960 \cdot 5280$ ft and $g = 32$ ft/s², so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3}$ s⁻¹ and

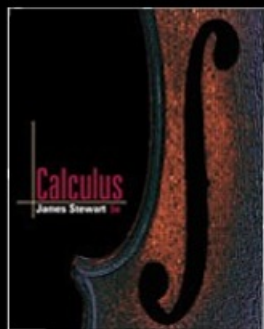
$T = 2\pi/k \approx 5079$ s ≈ 85 min.

- (d) $y(t) = 0 \Leftrightarrow \cos kt = 0 \Leftrightarrow kt = \frac{\pi}{2} + \pi n$ for some integer $n \Rightarrow$

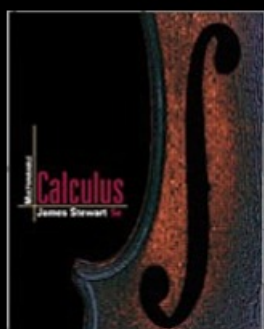
$y'(t) = -kR \sin(\frac{\pi}{2} + \pi n) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899$ mi/s $\approx 17,600$ mi/h.

Appendixes

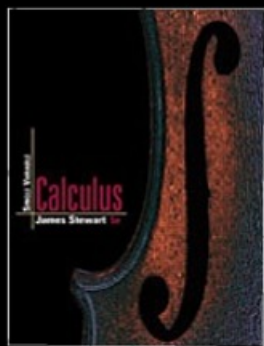
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Complete Solutions Manual
for
James Stewart's
Calculus – 5th Edition



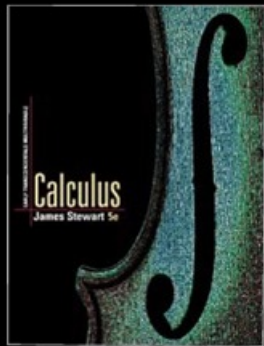
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James Stewart
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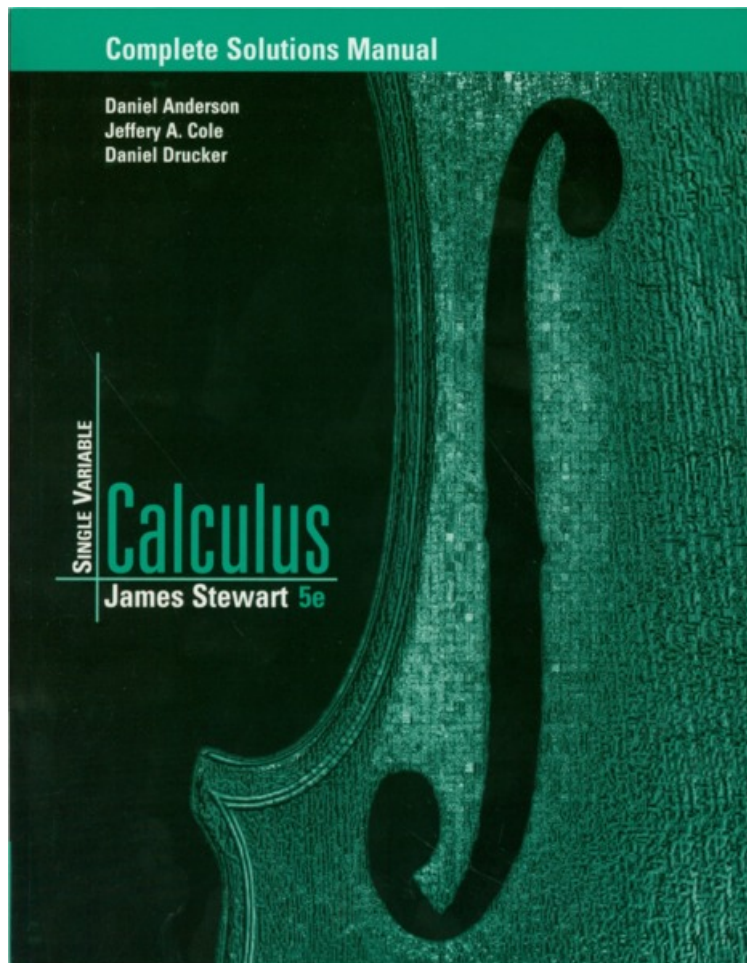
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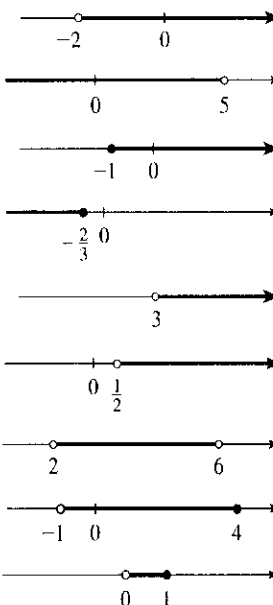
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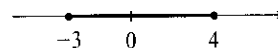
□ APPENDIXES

A Numbers, Inequalities, and Absolute Values

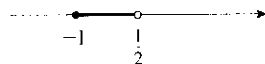
1. $|5 - 23| = |-18| = 18$
2. $|5| - |-23| = 5 - 23 = -18$
3. $|-π| = π$ because $π > 0$.
4. $|\pi - 2| = \pi - 2$ because $\pi - 2 > 0$.
5. $|\sqrt{5} - 5| = -(\sqrt{5} - 5) = 5 - \sqrt{5}$ because $\sqrt{5} - 5 < 0$.
6. $||-2| - |-3|| = |2 - 3| = |-1| = 1$
7. For $x < 2$, $x - 2 < 0$, so $|x - 2| = -(x - 2) = 2 - x$.
8. For $x > 2$, $x - 2 > 0$, so $|x - 2| = x - 2$.
9. $|x + 1| = \begin{cases} x + 1 & \text{for } x + 1 \geq 0 \Leftrightarrow x \geq -1 \\ -(x + 1) & \text{for } x + 1 < 0 \Leftrightarrow x < -1 \end{cases}$
10. $|2x - 1| = \begin{cases} 2x - 1 & \text{for } 2x - 1 \geq 0 \Leftrightarrow x \geq \frac{1}{2} \\ 1 - 2x & \text{for } 2x - 1 < 0 \Leftrightarrow x < \frac{1}{2} \end{cases}$
11. $|x^2 + 1| = x^2 + 1$ (since $x^2 + 1 \geq 0$ for all x).
12. Determine when $1 - 2x^2 < 0 \Leftrightarrow 1 < 2x^2 \Leftrightarrow x^2 > \frac{1}{2} \Leftrightarrow \sqrt{x^2} > \sqrt{\frac{1}{2}} \Leftrightarrow |x| > \sqrt{\frac{1}{2}} \Leftrightarrow x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$. Thus, $|1 - 2x^2| = \begin{cases} 1 - 2x^2 & \text{if } -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ 2x^2 - 1 & \text{if } x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}} \end{cases}$
13. $2x + 7 > 3 \Leftrightarrow 2x > -4 \Leftrightarrow x > -2$, so $x \in (-2, \infty)$.
14. $3x - 11 < 4 \Leftrightarrow 3x < 15 \Leftrightarrow x < 5$, so $x \in (-\infty, 5)$.
15. $1 - x \leq 2 \Leftrightarrow -x \leq 1 \Leftrightarrow x \geq -1$, so $x \in [-1, \infty)$.
16. $4 - 3x \geq 6 \Leftrightarrow -3x \geq 2 \Leftrightarrow x \leq -\frac{2}{3}$, so $x \in (-\infty, -\frac{2}{3}]$.
17. $2x + 1 < 5x - 8 \Leftrightarrow 9 < 3x \Leftrightarrow 3 < x$, so $x \in (3, \infty)$.
18. $1 + 5x > 5 - 3x \Leftrightarrow 8x > 4 \Leftrightarrow x > \frac{1}{2}$, so $x \in (\frac{1}{2}, \infty)$.
19. $-1 < 2x - 5 < 7 \Leftrightarrow 4 < 2x < 12 \Leftrightarrow 2 < x < 6$, so $x \in (2, 6)$.
20. $1 < 3x + 4 \leq 16 \Leftrightarrow -3 < 3x \leq 12 \Leftrightarrow -1 < x \leq 4$, so $x \in (-1, 4]$.
21. $0 \leq 1 - x < 1 \Leftrightarrow -1 \leq -x < 0 \Leftrightarrow 1 \geq x > 0$, so $x \in (0, 1]$.



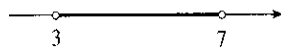
22. $-5 \leq 3 - 2x \leq 9 \Leftrightarrow -8 \leq -2x \leq 6 \Leftrightarrow 4 \geq x \geq -3$, so $x \in [-3, 4]$.



23. $4x < 2x + 1 \leq 3x + 2$. So $4x < 2x + 1 \Leftrightarrow 2x < 1 \Leftrightarrow x < \frac{1}{2}$, and $2x + 1 \leq 3x + 2 \Leftrightarrow -1 \leq x$. Thus, $x \in [-1, \frac{1}{2})$.



24. $2x - 3 < x + 4 < 3x - 2$. So $2x - 3 < x + 4 \Leftrightarrow x < 7$, and $x + 4 < 3x - 2 \Leftrightarrow 6 < 2x \Leftrightarrow 3 < x$, so $x \in (3, 7)$.



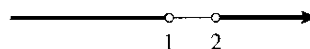
25. $(x - 1)(x - 2) > 0$. Case 1: (both factors are positive, so their product is positive)

$$x - 1 > 0 \Leftrightarrow x > 1, \text{ and } x - 2 > 0 \Leftrightarrow x > 2, \text{ so } x \in (2, \infty).$$

Case 2: (both factors are negative, so their product is positive)

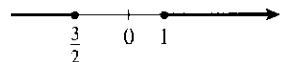
$$x - 1 < 0 \Leftrightarrow x < 1, \text{ and } x - 2 < 0 \Leftrightarrow x < 2, \text{ so } x \in (-\infty, 1).$$

Thus, the solution set is $(-\infty, 1) \cup (2, \infty)$.

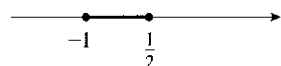


26. $(2x + 3)(x - 1) \geq 0$. Case 1: $2x + 3 \geq 0 \Leftrightarrow x \geq -\frac{3}{2}$, and $x - 1 \geq 0 \Leftrightarrow x \geq 1$, so $x \in [1, \infty)$.

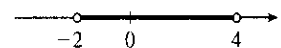
Case 2: $2x + 3 \leq 0 \Leftrightarrow x \leq -\frac{3}{2}$, and $x - 1 \leq 0 \Leftrightarrow x \leq 1$, so $x \in (-\infty, -\frac{3}{2}]$. Thus, the solution set is $(-\infty, -\frac{3}{2}] \cup [1, \infty)$.



27. $2x^2 + x \leq 1 \Leftrightarrow 2x^2 + x - 1 \leq 0 \Leftrightarrow (2x - 1)(x + 1) \leq 0$. Case 1: $2x - 1 \geq 0 \Leftrightarrow x \geq \frac{1}{2}$, and $x + 1 \leq 0 \Leftrightarrow x \leq -1$, which is an impossible combination. Case 2: $2x - 1 \leq 0 \Leftrightarrow x \leq \frac{1}{2}$, and $x + 1 \geq 0 \Leftrightarrow x \geq -1$, so $x \in [-1, \frac{1}{2}]$. Thus, the solution set is $[-1, \frac{1}{2}]$.



28. $x^2 < 2x + 8 \Leftrightarrow x^2 - 2x - 8 < 0 \Leftrightarrow (x - 4)(x + 2) < 0$. Case 1: $x > 4$ and $x < -2$, which is impossible. Case 2: $x < 4$ and $x > -2$. Thus, the solution set is $(-2, 4)$.

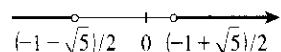


29. $x^2 + x + 1 > 0 \Leftrightarrow x^2 + x + \frac{1}{4} + \frac{3}{4} > 0 \Leftrightarrow (x + \frac{1}{2})^2 + \frac{3}{4} > 0$. But since $(x + \frac{1}{2})^2 \geq 0$ for every real x , the original inequality will be true for all real x as well. Thus, the solution set is $(-\infty, \infty)$.

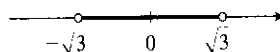


30. $x^2 + x > 1 \Leftrightarrow x^2 + x - 1 > 0$. Using the quadratic formula, we obtain

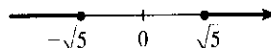
$$x^2 + x - 1 = \left(x - \frac{-1 - \sqrt{5}}{2}\right) \left(x - \frac{-1 + \sqrt{5}}{2}\right) > 0. \text{ Case 1: } x - \frac{-1 - \sqrt{5}}{2} > 0 \text{ and } x - \frac{-1 + \sqrt{5}}{2} > 0, \text{ so that } x > \frac{-1 + \sqrt{5}}{2}. \text{ Case 2: } x - \frac{-1 - \sqrt{5}}{2} < 0 \text{ and } x - \frac{-1 + \sqrt{5}}{2} < 0, \text{ so that } x < \frac{-1 - \sqrt{5}}{2}. \text{ Thus, the solution set is } (-\infty, \frac{-1 - \sqrt{5}}{2}) \cup (\frac{-1 + \sqrt{5}}{2}, \infty).$$



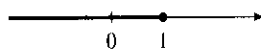
31. $x^2 < 3 \Leftrightarrow x^2 - 3 < 0 \Leftrightarrow (x - \sqrt{3})(x + \sqrt{3}) < 0$. *Case 1:* $x > \sqrt{3}$ and $x < -\sqrt{3}$, which is impossible.
Case 2: $x < \sqrt{3}$ and $x > -\sqrt{3}$. Thus, the solution set is $(-\sqrt{3}, \sqrt{3})$.
 Another method: $x^2 < 3 \Leftrightarrow |x| < \sqrt{3} \Leftrightarrow -\sqrt{3} < x < \sqrt{3}$.



32. $x^2 \geq 5 \Leftrightarrow x^2 - 5 \geq 0 \Leftrightarrow (x - \sqrt{5})(x + \sqrt{5}) \geq 0$. *Case 1:* $x \geq \sqrt{5}$ and $x \geq -\sqrt{5}$, so $x \in [\sqrt{5}, \infty)$.
Case 2: $x \leq \sqrt{5}$ and $x \leq -\sqrt{5}$, so $x \in (-\infty, -\sqrt{5}]$. Thus, the solution set is $(-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$.
 Another method: $x^2 \geq 5 \Leftrightarrow |x| \geq \sqrt{5} \Leftrightarrow x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$.



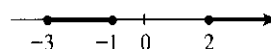
33. $x^3 - x^2 \leq 0 \Leftrightarrow x^2(x - 1) \leq 0$. Since $x^2 \geq 0$ for all x , the inequality is satisfied when $x - 1 \leq 0 \Leftrightarrow x \leq 1$. Thus, the solution set is $(-\infty, 1]$.



34. $(x + 1)(x - 2)(x + 3) = 0 \Leftrightarrow x = -1, 2$, or -3 . Constructing a table:

Interval	$x + 1$	$x - 2$	$x + 3$	$(x + 1)(x - 2)(x + 3)$
$x < -3$	—	—	—	—
$-3 < x < -1$	—	—	+	+
$-1 < x < 2$	+	—	+	—
$x > 2$	+	+	+	+

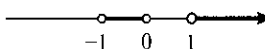
Thus, $(x + 1)(x - 2)(x + 3) \geq 0$ on $[-3, -1]$ and $[2, \infty)$, and the solution set is $[-3, -1] \cup [2, \infty)$.



35. $x^3 > x \Leftrightarrow x^3 - x > 0 \Leftrightarrow x(x^2 - 1) > 0 \Leftrightarrow x(x - 1)(x + 1) > 0$. Constructing a table:

Interval	x	$x - 1$	$x + 1$	$x(x - 1)(x + 1)$
$x < -1$	—	—	—	—
$-1 < x < 0$	—	—	+	+
$0 < x < 1$	+	—	+	—
$x > 1$	+	+	+	+

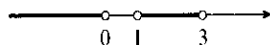
Since $x^3 > x$ when the last column is positive, the solution set is $(-1, 0) \cup (1, \infty)$.



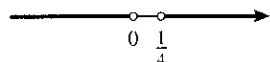
6. $x^3 + 3x < 4x^2 \Leftrightarrow x^3 - 4x^2 + 3x < 0 \Leftrightarrow x(x^2 - 4x + 3) < 0 \Leftrightarrow x(x-1)(x-3) < 0$.

Interval	x	$x-1$	$x-3$	$x(x-1)(x-3)$
$x < 0$	-	-	-	-
$0 < x < 1$	+	-	-	+
$1 < x < 3$	+	+	-	-
$x > 3$	+	+	+	+

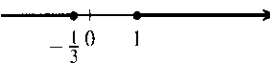
Thus, the solution set is $(-\infty, 0) \cup (1, 3)$.



17. $1/x < 4$. This is clearly true for $x < 0$. So suppose $x > 0$. then $1/x < 4 \Leftrightarrow 1 < 4x \Leftrightarrow \frac{1}{4} < x$. Thus, the solution set is $(-\infty, 0) \cup (\frac{1}{4}, \infty)$.



18. $-3 < 1/x \leq 1$. We solve the two inequalities separately and take the intersection of the solution sets. First, $-3 < 1/x$ is clearly true for $x > 0$. So suppose $x < 0$. Then $-3 < 1/x \Leftrightarrow -3x > 1 \Leftrightarrow x < -\frac{1}{3}$, so for this inequality, the solution set is $(-\infty, -\frac{1}{3}) \cup (0, \infty)$. Now $1/x \leq 1$ is clearly true if $x < 0$. So suppose $x > 0$. Then $1/x \leq 1 \Leftrightarrow 1 \leq x$, and the solution set here is $(-\infty, 0) \cup [1, \infty)$. Taking the intersection of the two solution sets gives the final solution set: $(-\infty, -\frac{1}{3}) \cup [1, \infty)$.



19. $C = \frac{5}{9}(F - 32) \Rightarrow F = \frac{9}{5}C + 32$. So $50 \leq F \leq 95 \Rightarrow 50 \leq \frac{9}{5}C + 32 \leq 95 \Rightarrow 18 \leq \frac{9}{5}C \leq 63 \Rightarrow 10 \leq C \leq 35$. So the interval is $[10, 35]$.

10. Since $20 \leq C \leq 30$ and $C = \frac{5}{9}(F - 32)$, we have $20 \leq \frac{5}{9}(F - 32) \leq 30 \Rightarrow 36 \leq F - 32 \leq 54 \Rightarrow 68 \leq F \leq 86$. So the interval is $[68, 86]$.

11. (a) Let T represent the temperature in degrees Celsius and h the height in km. $T = 20$ when $h = 0$ and T decreases by 10°C for every km (1°C for each 100-m rise). Thus, $T = 20 - 10h$ when $0 \leq h \leq 12$.

(b) From part (a), $T = 20 - 10h \Rightarrow 10h = 20 - T \Rightarrow h = 2 - T/10$. So $0 \leq h \leq 5 \Rightarrow 0 \leq 2 - T/10 \leq 5 \Rightarrow -2 \leq -T/10 \leq 3 \Rightarrow -20 \leq -T \leq 30 \Rightarrow 20 \geq T \geq -30 \Rightarrow -30 \leq T \leq 20$. Thus, the range of temperatures (in $^\circ\text{C}$) to be expected is $[-30, 20]$.

12. The ball will be at least 32 ft above the ground if $h \geq 32 \Leftrightarrow 128 + 16t - 16t^2 \geq 32 \Leftrightarrow 16t^2 - 16t - 96 \leq 0 \Leftrightarrow 16(t-3)(t+2) \leq 0$. $t = 3$ and $t = -2$ are endpoints of the interval we're looking for, and constructing a table gives $-2 \leq t \leq 3$. But $t \geq 0$, so the ball will be at least 32 ft above the ground in the time interval $[0, 3]$.

13. $|2x| = 3 \Leftrightarrow$ either $2x = 3$ or $2x = -3 \Leftrightarrow x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

14. $|3x + 5| = 1 \Leftrightarrow$ either $3x + 5 = 1$ or -1 . In the first case, $3x = -4 \Leftrightarrow x = -\frac{4}{3}$, and in the second case, $3x = -6 \Leftrightarrow x = -2$. So the solutions are -2 and $-\frac{4}{3}$.

15. $|x + 3| = |2x + 1| \Leftrightarrow$ either $x + 3 = 2x + 1$ or $x + 3 = -(2x + 1)$. In the first case, $x = 2$, and in the second case, $x + 3 = -2x - 1 \Leftrightarrow 3x = -4 \Leftrightarrow x = -\frac{4}{3}$. So the solutions are $-\frac{4}{3}$ and 2 .

46. $\left| \frac{2x-1}{x+1} \right| = 3 \Leftrightarrow$ either $\frac{2x-1}{x+1} = 3$ or $\frac{2x-1}{x+1} = -3$. In the first case, $2x-1 = 3x+3 \Leftrightarrow x = -4$, and in the second case, $2x-1 = -3x-3 \Leftrightarrow x = -\frac{2}{5}$.
47. By Property 5 of absolute values, $|x| < 3 \Leftrightarrow -3 < x < 3$, so $x \in (-3, 3)$.
48. By Properties 4 and 6 of absolute values, $|x| \geq 3 \Leftrightarrow x \leq -3$ or $x \geq 3$, so $x \in (-\infty, -3] \cup [3, \infty)$.
49. $|x-4| < 1 \Leftrightarrow -1 < x-4 < 1 \Leftrightarrow 3 < x < 5$, so $x \in (3, 5)$.
50. $|x-6| < 0.1 \Leftrightarrow -0.1 < x-6 < 0.1 \Leftrightarrow 5.9 < x < 6.1$, so $x \in (5.9, 6.1)$.
51. $|x+5| \geq 2 \Leftrightarrow x+5 \geq 2$ or $x+5 \leq -2 \Leftrightarrow x \geq -3$ or $x \leq -7$, so $x \in (-\infty, -7] \cup [-3, \infty)$.
52. $|x+1| \geq 3 \Leftrightarrow x+1 \geq 3$ or $x+1 \leq -3 \Leftrightarrow x \geq 2$ or $x \leq -4$, so $x \in (-\infty, -4] \cup [2, \infty)$.
53. $|2x-3| \leq 0.4 \Leftrightarrow -0.4 \leq 2x-3 \leq 0.4 \Leftrightarrow 2.6 \leq 2x \leq 3.4 \Leftrightarrow 1.3 \leq x \leq 1.7$, so $x \in [1.3, 1.7]$.
54. $|5x-2| < 6 \Leftrightarrow -6 < 5x-2 < 6 \Leftrightarrow -4 < 5x < 8 \Leftrightarrow -\frac{4}{5} < x < \frac{8}{5}$, so $x \in (-\frac{4}{5}, \frac{8}{5})$.
55. $1 \leq |x| \leq 4$. So either $1 \leq x \leq 4$ or $1 \leq -x \leq 4 \Leftrightarrow -1 \geq x \geq -4$. Thus, $x \in [-4, -1] \cup [1, 4]$.
56. $0 < |x-5| < \frac{1}{2}$. Clearly $0 < |x-5|$ for $x \neq 5$. Now $|x-5| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < x-5 < \frac{1}{2} \Leftrightarrow 4.5 < x < 5.5$. So the solution set is $(4.5, 5) \cup (5, 5.5)$.
57. $a(bx-c) \geq bc \Leftrightarrow bx-c \geq \frac{bc}{a} \Leftrightarrow bx \geq \frac{bc}{a} + c = \frac{bc+ac}{a} \Leftrightarrow x \geq \frac{bc+ac}{ab}$
58. $a \leq bx+c < 2a \Leftrightarrow a-c \leq bx < 2a-c \Leftrightarrow \frac{a-c}{b} \leq x < \frac{2a-c}{b}$ (since $b > 0$)
59. $ax+b < c \Leftrightarrow ax < c-b \Leftrightarrow x > \frac{c-b}{a}$ (since $a < 0$)
60. $\frac{ax+b}{c} \leq b \Leftrightarrow ax+b \geq bc$ (since $c < 0$) $\Leftrightarrow ax \geq bc-b \Leftrightarrow x \leq \frac{b(c-1)}{a}$ (since $a < 0$)
61. $|(x+y)-5| = |(x-2)+(y-3)| \leq |x-2| + |y-3| < 0.01 + 0.04 = 0.05$
62. Use the Triangle Inequality: $|x+3| < \frac{1}{2} \Rightarrow$
 $|4x+13| = |4(x+3)+1| \leq |4(x+3)| + |1| = 4|x+3| + 1 < 4(\frac{1}{2}) + 1 = 3$
Another method: $|x+3| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x+3 < \frac{1}{2} \Rightarrow -2 < 4x+12 < 2 \Rightarrow -1 < 4x+13 < 3 \Rightarrow$
 $|4x+13| < 3$
63. If $a < b$ then $a+a < a+b$ and $a+b < b+b$. So $2a < a+b < 2b$. Dividing by 2, we get $a < \frac{1}{2}(a+b) < b$.
64. If $0 < a < b$, then $\frac{1}{ab} > 0$. So $a < b \Rightarrow \frac{1}{ab} \cdot a < \frac{1}{ab} \cdot b \Leftrightarrow \frac{1}{b} < \frac{1}{a}$.
65. $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2} \sqrt{b^2} = |a||b|$
66. $\left| \frac{a}{b} \right| |b| = \left| \frac{a}{b} \cdot b \right| = |a|$ (using the result of Exercise 65). Dividing the equation through by $|b|$ gives $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$.
67. If $0 < a < b$, then $a \cdot a < a \cdot b$ and $a \cdot b < b \cdot b$ [using Rule 3 of Inequalities]. So $a^2 < ab < b^2$ and hence $a^2 < b^2$.
68. Following the hint, the Triangle Inequality becomes $|(x-y)+y| \leq |x-y| + |y| \Leftrightarrow |x| \leq |x-y| + |y| \Leftrightarrow$
 $|x-y| \geq |x| - |y|$.

69. Observe that the sum, difference and product of two integers is always an integer. Let the rational numbers be represented by $r = m/n$ and $s = p/q$ (where m, n, p and q are integers with $n \neq 0, q \neq 0$). Now
- $$r + s = \frac{m}{n} + \frac{p}{q} = \frac{mq + pn}{nq},$$
- but $mq + pn$ and nq are both integers, so $\frac{mq + pn}{nq} = r + s$ is a rational number by definition. Similarly, $r - s = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}$ is a rational number. Finally, $r \cdot s = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$ but mp and nq are both integers, so $\frac{mp}{nq} = r \cdot s$ is a rational number by definition.
70. (a) No. Consider the case of $\sqrt{2}$ and $-\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} + (-\sqrt{2}) = 0$ and 0, being an integer, is not irrational.
- (b) No. Consider the case of $\sqrt{2}$ and $\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} \cdot \sqrt{2} = 2$ is not irrational.

B Coordinate Geometry and Lines

- From the Distance Formula with $x_1 = 1, x_2 = 4, y_1 = 1, y_2 = 5$, we find the distance from $(1, 1)$ to $(4, 5)$ to be $\sqrt{(4-1)^2 + (5-1)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.
- The distance from $(1, -3)$ to $(5, 7)$ is $\sqrt{(5-1)^2 + [7-(-3)]^2} = \sqrt{4^2 + 10^2} = \sqrt{116} = 2\sqrt{29}$.
- $\sqrt{(-1-6)^2 + [3-(-2)]^2} = \sqrt{(-7)^2 + 5^2} = \sqrt{74}$
- $\sqrt{(-1-1)^2 + [-3-(-6)]^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$
- $\sqrt{(4-2)^2 + (-7-5)^2} = \sqrt{2^2 + (-12)^2} = \sqrt{148} = 2\sqrt{37}$
- $\sqrt{(b-a)^2 + (a-b)^2} = \sqrt{(a-b)^2 + (a-b)^2} = \sqrt{2(a-b)^2} = \sqrt{2}|a-b|$
- From (2), the slope is $\frac{11-5}{4-1} = \frac{6}{3} = 2$.
- $m = \frac{-3-6}{4-(-1)} = -\frac{9}{5}$
- With $P(-3, 3)$ and $Q(-1, -6)$, the slope m of the line through P and Q is $m = \frac{-6-3}{-1-(-3)} = -\frac{9}{2}$.
- $m = \frac{0-(-4)}{6-(-1)} = \frac{4}{7}$
- Since $|AC| = \sqrt{(-4-0)^2 + (3-2)^2} = \sqrt{(-4)^2 + 1^2} = \sqrt{17}$ and $|BC| = \sqrt{[-4-(-3)]^2 + [3-(-1)]^2} = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$, the triangle has two sides of equal length, and so is isosceles.
- (a) $|AB| = \sqrt{(11-6)^2 + [-3-(-7)]^2} = \sqrt{5^2 + 4^2} = \sqrt{41}$,
 $|AC| = \sqrt{(2-6)^2 + [-2-(-7)]^2} = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$, and
 $|BC| = \sqrt{(2-11)^2 + [-2-(-3)]^2} = \sqrt{(-9)^2 + 1^2} = \sqrt{82}$, so
 $|AB|^2 + |AC|^2 = 41 + 41 = 82 = |BC|^2$, and so $\triangle ABC$ is a right triangle.

(b) $m_{AB} = \frac{-3 - (-7)}{11 - 6} = \frac{4}{5}$ and $m_{AC} = \frac{-2 - (-7)}{2 - 6} = -\frac{5}{4}$. Thus $m_{AB} \cdot m_{AC} = -1$ and so AB is perpendicular to AC and $\triangle ABC$ must be a right triangle.

(c) Taking lengths from part (a), the base is $\sqrt{41}$ and the height is $\sqrt{41}$. Thus the area is $\frac{1}{2}bh = \frac{1}{2}\sqrt{41}\sqrt{41} = \frac{41}{2}$.

13. Using $A(-2, 9)$, $B(4, 6)$, $C(1, 0)$, and $D(-5, 3)$, we have

$$|AB| = \sqrt{[4 - (-2)]^2 + (6 - 9)^2} = \sqrt{6^2 + (-3)^2} = \sqrt{45} = \sqrt{9} \sqrt{5} = 3\sqrt{5},$$

$$|BC| = \sqrt{(1 - 4)^2 + (0 - 6)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45} = \sqrt{9} \sqrt{5} = 3\sqrt{5},$$

$$|CD| = \sqrt{(-5 - 1)^2 + (3 - 0)^2} = \sqrt{(-6)^2 + 3^2} = \sqrt{45} = \sqrt{9} \sqrt{5} = 3\sqrt{5}, \text{ and}$$

$$|DA| = \sqrt{[-2 - (-5)]^2 + (9 - 3)^2} = \sqrt{3^2 + 6^2} = \sqrt{45} = \sqrt{9} \sqrt{5} = 3\sqrt{5}. \text{ So all sides are of equal length and}$$

we have a rhombus. Moreover, $m_{AB} = \frac{6 - 9}{4 - (-2)} = -\frac{1}{2}$, $m_{BC} = \frac{0 - 6}{1 - 4} = 2$, $m_{CD} = \frac{3 - 0}{-5 - 1} = -\frac{1}{2}$, and

$m_{DA} = \frac{9 - 3}{-2 - (-5)} = 2$, so the sides are perpendicular. Thus, A , B , C , and D are vertices of a square.

14. (a) Using $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$, we have

$$|AB| = \sqrt{[3 - (-1)]^2 + (11 - 3)^2} = \sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5},$$

$$|BC| = \sqrt{(5 - 3)^2 + (15 - 11)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}, \text{ and}$$

$$|AC| = \sqrt{[5 - (-1)]^2 + (15 - 3)^2} = \sqrt{6^2 + 12^2} = \sqrt{180} = 6\sqrt{5}. \text{ Thus, } |AC| = |AB| + |BC|.$$

(b) $m_{AB} = \frac{11 - 3}{3 - (-1)} = \frac{8}{4} = 2$ and $m_{AC} = \frac{15 - 3}{5 - (-1)} = \frac{12}{6} = 2$. Since the segments AB and AC have the same slope, A , B and C must be collinear.

15. The slope of the line segment AB is $\frac{4 - 1}{7 - 1} = \frac{1}{2}$, the slope of CD is $\frac{7 - 10}{-1 - 5} = \frac{1}{2}$, the slope of BC is

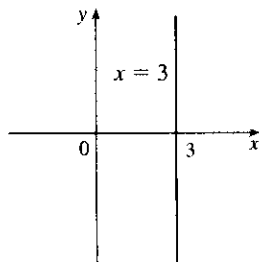
$$\frac{10 - 4}{5 - 7} = -3, \text{ and the slope of } DA \text{ is } \frac{1 - 7}{1 - (-1)} = -3. \text{ So } AB \text{ is parallel to } CD \text{ and } BC \text{ is parallel to } DA.$$

Hence $ABCD$ is a parallelogram.

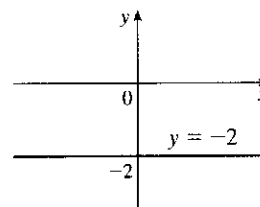
16. The slopes of the four sides are $m_{AB} = \frac{3 - 1}{11 - 1} = \frac{1}{5}$, $m_{BC} = \frac{8 - 3}{10 - 11} = -5$, $m_{CD} = \frac{6 - 8}{0 - 10} = \frac{1}{5}$, and

$m_{DA} = \frac{1 - 6}{1 - 0} = -5$. Hence $AB \parallel CD$, $BC \parallel DA$, $AB \perp BC$, $BC \perp CD$, $CD \perp DA$, and $DA \perp AB$, and so $ABCD$ is a rectangle.

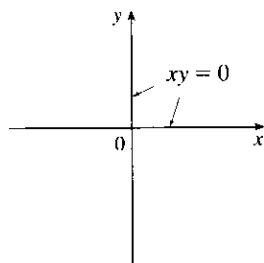
17. $x = 3$



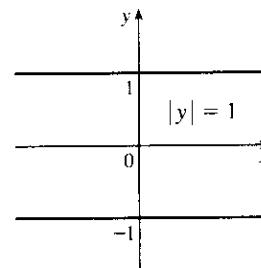
18. $y = -2$



19. $xy = 0 \Leftrightarrow x = 0$ or $y = 0$. The graph consists of the coordinate axes.

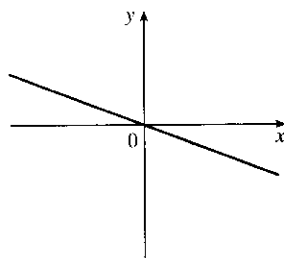


20. $|y| = 1 \Leftrightarrow y = 1$ or $y = -1$

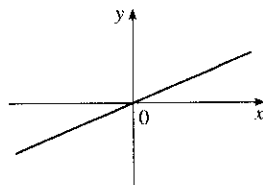


21. By the point-slope form of the equation of a line, an equation of the line through $(2, -3)$ with slope 6 is $y - (-3) = 6(x - 2)$ or $y = 6x - 15$.
22. $y - 4 = -3[x - (-1)]$ or $y = -3x + 1$
23. $y - 7 = \frac{2}{3}(x - 1)$ or $y = \frac{2}{3}x + \frac{19}{3}$
24. $y - (-5) = -\frac{7}{2}[x - (-3)]$ or $y = -\frac{7}{2}x - \frac{31}{2}$
25. The slope of the line through $(2, 1)$ and $(1, 6)$ is $m = \frac{6 - 1}{1 - 2} = -5$, so an equation of the line is $y - 1 = -5(x - 2)$ or $y = -5x + 11$.
26. For $(-1, -2)$ and $(4, 3)$, $m = \frac{3 - (-2)}{4 - (-1)} = 1$. So $y - 3 = 1(x - 4)$ or $y = x - 1$.
27. By the slope-intercept form of the equation of a line, an equation of the line is $y = 3x - 2$.
28. By the slope-intercept form of the equation of a line, an equation of the line is $y = \frac{2}{5}x + 4$.
29. Since the line passes through $(1, 0)$ and $(0, -3)$, its slope is $m = \frac{-3 - 0}{0 - 1} = 3$, so an equation is $y = 3x - 3$.
Another method: From Exercise 61, $\frac{x}{1} + \frac{y}{-3} = 1 \Rightarrow -3x + y = -3 \Rightarrow y = 3x - 3$.
30. For $(-8, 0)$ and $(0, 6)$, $m = \frac{6 - 0}{0 - (-8)} = \frac{3}{4}$. So an equation is $y = \frac{3}{4}x + 6$.
Another method: From Exercise 61, $\frac{x}{-8} + \frac{y}{6} = 1 \Rightarrow -3x + 4y = 24 \Rightarrow y = \frac{3}{4}x + 6$.
31. Since $m = 0$, $y - 5 = 0(x - 4)$ or $y = 5$.
32. Since m is undefined, we have the vertical line $x = 4$.
33. Putting the line $x + 2y = 6$ into its slope-intercept form gives us $y = -\frac{1}{2}x + 3$, so we see that this line has slope $-\frac{1}{2}$. Thus, we want the line of slope $-\frac{1}{2}$ that passes through the point $(1, -6)$: $y - (-6) = -\frac{1}{2}(x - 1) \Leftrightarrow y = -\frac{1}{2}x - \frac{11}{2}$.
34. $2x + 3y + 4 = 0 \Leftrightarrow y = -\frac{2}{3}x - \frac{4}{3}$, so $m = -\frac{2}{3}$ and the required line is $y = -\frac{2}{3}x + 6$.
35. $2x + 5y + 8 = 0 \Leftrightarrow y = -\frac{2}{5}x - \frac{8}{5}$. Since this line has slope $-\frac{2}{5}$, a line perpendicular to it would have slope $\frac{5}{2}$, so the required line is $y - (-2) = \frac{5}{2}[x - (-1)] \Leftrightarrow y = \frac{5}{2}x + \frac{1}{2}$.
36. $4x - 8y = 1 \Leftrightarrow y = \frac{1}{2}x - \frac{1}{8}$. Since this line has slope $\frac{1}{2}$, a line perpendicular to it would have slope -2 , so the required line is $y - (-\frac{2}{3}) = -2(x - \frac{1}{2}) \Leftrightarrow y = -2x + \frac{1}{3}$.

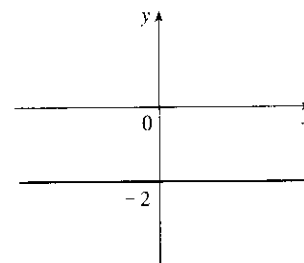
37. $x + 3y = 0 \Leftrightarrow y = -\frac{1}{3}x$,
so the slope is $-\frac{1}{3}$ and the
 y -intercept is 0.



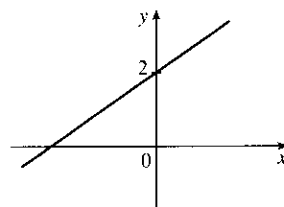
38. $2x - 5y = 0 \Leftrightarrow y = \frac{2}{5}x$, so
the slope is $\frac{2}{5}$ and the y -intercept
is 0.



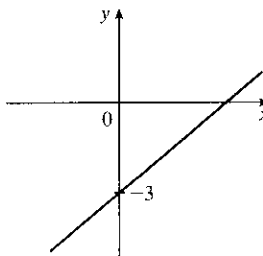
39. $y = -2$ is a horizontal line with
slope 0 and y -intercept -2 .



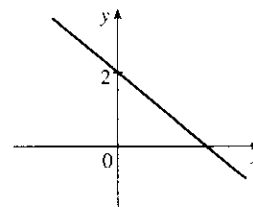
40. $2x - 3y + 6 = 0 \Leftrightarrow$
 $y = \frac{2}{3}x + 2$, so the slope is $\frac{2}{3}$
and the y -intercept is 2.



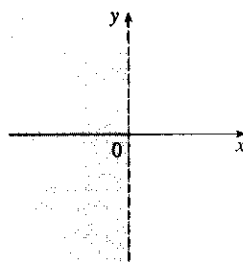
41. $3x - 4y = 12 \Leftrightarrow$
 $y = \frac{3}{4}x - 3$, so the slope is $\frac{3}{4}$
and the y -intercept is -3 .



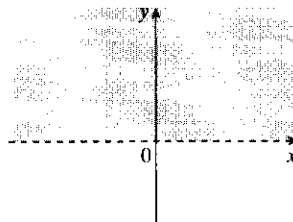
42. $4x + 5y = 10 \Leftrightarrow$
 $y = -\frac{4}{5}x + 2$, so the slope is
 $-\frac{4}{5}$ and the y -intercept is 2.



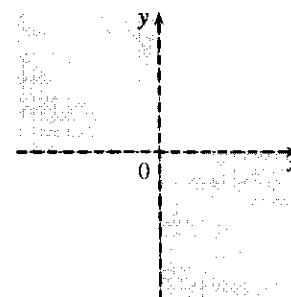
43. $\{(x, y) \mid x < 0\}$



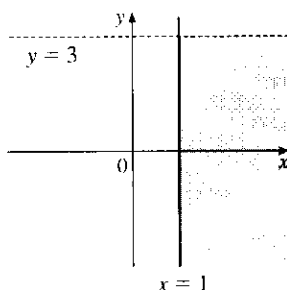
44. $\{(x, y) \mid y > 0\}$



45. $\{(x, y) \mid xy < 0\} =$
 $\{(x, y) \mid x < 0 \text{ and } y > 0\}$
 $\cup \{(x, y) \mid x > 0 \text{ and } y < 0\}$

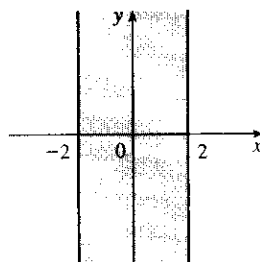


46. $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$

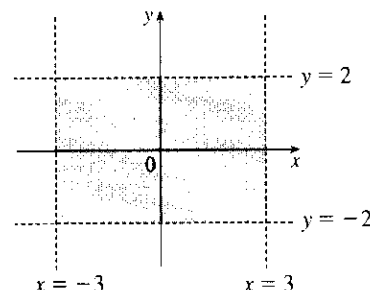


47. $\{(x, y) \mid |x| \leq 2\} =$

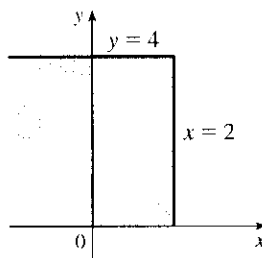
$\{(x, y) \mid -2 \leq x \leq 2\}$



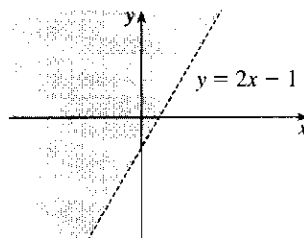
48. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$



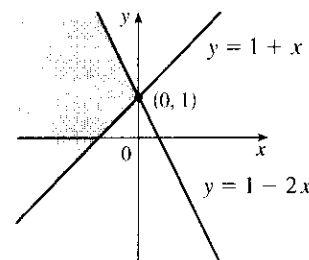
49. $\{(x, y) \mid 0 \leq y \leq 4, x \leq 2\}$



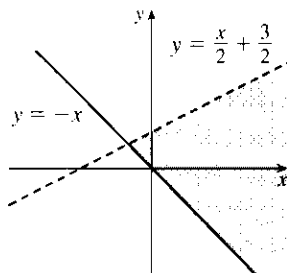
50. $\{(x, y) \mid y > 2x - 1\}$



51. $\{(x, y) \mid 1 + x \leq y \leq 1 - 2x\}$



52. $\{(x, y) \mid -x \leq y < \frac{x+3}{2}\}$

53. Let $P(0, y)$ be a point on the y -axis. The distance from P to $(5, -5)$ is $\sqrt{(5-0)^2 + (-5-y)^2} = \sqrt{5^2 + (y+5)^2}$. The distance from P to $(1, 1)$ is $\sqrt{(1-0)^2 + (1-y)^2} = \sqrt{1^2 + (y-1)^2}$. We wantthese distances to be equal: $\sqrt{5^2 + (y+5)^2} = \sqrt{1^2 + (y-1)^2} \Leftrightarrow$

$$5^2 + (y+5)^2 = 1^2 + (y-1)^2 \Leftrightarrow$$

$$25 + (y^2 + 10y + 25) = 1 + (y^2 - 2y + 1) \Leftrightarrow 12y = -48 \Leftrightarrow$$

 $y = -4$. So the desired point is $(0, -4)$.54. Let M be the point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Then

$$|MP_1|^2 = \left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \left(y_1 - \frac{y_1+y_2}{2}\right)^2 = \left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2 \text{ and}$$

$$|MP_2|^2 = \left(x_2 - \frac{x_1+x_2}{2}\right)^2 + \left(y_2 - \frac{y_1+y_2}{2}\right)^2 = \left(\frac{x_2-x_1}{2}\right)^2 + \left(\frac{y_2-y_1}{2}\right)^2. \text{ Hence, } |MP_1| = |MP_2|; \text{ that is, } M \text{ is equidistant from } P_1 \text{ and } P_2.$$

55. (a) Using the midpoint formula from Exercise 54 with $(1, 3)$ and $(7, 15)$, we get $\left(\frac{1+7}{2}, \frac{3+15}{2}\right) = (4, 9)$.(b) Using the same formula, we get $\left(\frac{-1+8}{2}, \frac{6-12}{2}\right) = \left(\frac{7}{2}, -3\right)$.

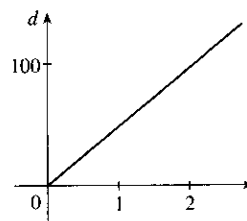
56. The midpoint M_1 of AB is $(\frac{1+3}{2}, \frac{0+6}{2}) = (2, 3)$, the midpoint M_2 of BC is $(\frac{3+8}{2}, \frac{6+2}{2}) = (\frac{11}{2}, 4)$, and the midpoint M_3 of CA is $(\frac{8+1}{2}, \frac{2+0}{2}) = (\frac{9}{2}, 1)$. The lengths of the medians are $|AM_2| = \sqrt{(\frac{11}{2} - 1)^2 + (4 - 0)^2} = \sqrt{(\frac{9}{2})^2 + 4^2} = \sqrt{\frac{145}{4}} = \frac{\sqrt{145}}{2}$,
 $|BM_3| = \sqrt{(\frac{9}{2} - 3)^2 + (1 - 6)^2} = \sqrt{(\frac{3}{2})^2 + (-5)^2} = \sqrt{\frac{109}{4}} = \frac{\sqrt{109}}{2}$, and
 $|CM_1| = \sqrt{(2 - 8)^2 + (3 - 2)^2} = \sqrt{(-6)^2 + 1^2} = \sqrt{37}$.
57. $2x - y = 4 \Leftrightarrow y = 2x - 4 \Rightarrow m_1 = 2$ and $6x - 2y = 10 \Leftrightarrow 2y = 6x - 10 \Leftrightarrow y = 3x - 5 \Rightarrow m_2 = 3$. Since $m_1 \neq m_2$, the two lines are not parallel. To find the point of intersection: $2x - 4 = 3x - 5 \Leftrightarrow x = 1 \Rightarrow y = -2$. Thus, the point of intersection is $(1, -2)$.
58. $3x - 5y + 19 = 0 \Leftrightarrow 5y = 3x + 19 \Leftrightarrow y = \frac{3}{5}x + \frac{19}{5} \Rightarrow m_1 = \frac{3}{5}$ and $10x + 6y - 50 = 0 \Leftrightarrow 6y = -10x + 50 \Leftrightarrow y = -\frac{5}{3}x + \frac{25}{3} \Rightarrow m_2 = -\frac{5}{3}$. Since $m_1 m_2 = \frac{3}{5}(-\frac{5}{3}) = -1$, the two lines are perpendicular. To find the point of intersection: $\frac{3}{5}x + \frac{19}{5} = -\frac{5}{3}x + \frac{25}{3} \Leftrightarrow 9x + 57 = -25x + 125 \Leftrightarrow 34x = 68 \Leftrightarrow x = 2 \Rightarrow y = \frac{3}{5} \cdot 2 + \frac{19}{5} = \frac{25}{5} = 5$. Thus, the point of intersection is $(2, 5)$.
59. With $A(1, 4)$ and $B(7, -2)$, the slope of segment AB is $\frac{-2-4}{7-1} = -1$, so its perpendicular bisector has slope 1. The midpoint of AB is $(\frac{1+7}{2}, \frac{4+(-2)}{2}) = (4, 1)$, so an equation of the perpendicular bisector is $y - 1 = 1(x - 4)$ or $y = x - 3$.
60. (a) Side PQ has slope $\frac{4-0}{3-1} = 2$, so its equation is $y - 0 = 2(x - 1) \Leftrightarrow y = 2x - 2$. Side QR has slope $\frac{6-4}{-1-3} = -\frac{1}{2}$, so its equation is $y - 4 = -\frac{1}{2}(x - 3) \Leftrightarrow y = -\frac{1}{2}x + \frac{11}{2}$. Side RP has slope $\frac{0-6}{1-(-1)} = -3$, so its equation is $y - 0 = -3(x - 1) \Leftrightarrow y = -3x + 3$.
- (b) M_1 (the midpoint of PQ) has coordinates $(\frac{1+3}{2}, \frac{0+4}{2}) = (2, 2)$. M_2 (the midpoint of QR) has coordinates $(\frac{3-1}{2}, \frac{4+6}{2}) = (1, 5)$. M_3 (the midpoint of RP) has coordinates $(\frac{1-1}{2}, \frac{0+6}{2}) = (0, 3)$. RM_1 has slope $\frac{2-6}{2-(-1)} = -\frac{4}{3}$ and hence equation $y - 2 = -\frac{4}{3}(x - 2) \Leftrightarrow y = -\frac{4}{3}x + \frac{14}{3}$. PM_2 is a vertical line with equation $x = 1$. QM_3 has slope $\frac{3-4}{0-3} = \frac{1}{3}$ and hence equation $y - 3 = \frac{1}{3}(x - 0) \Leftrightarrow y = \frac{1}{3}x + 3$. PM_2 and RM_1 intersect where $x = 1$ and $y = -\frac{4}{3}(1) + \frac{14}{3} = \frac{10}{3}$, or at $(1, \frac{10}{3})$. PM_2 and QM_3 intersect where $x = 1$ and $y = \frac{1}{3}(1) + 3 = \frac{10}{3}$, or at $(1, \frac{10}{3})$, so this is the point where all three medians intersect.
61. (a) Since the x -intercept is a , the point $(a, 0)$ is on the line, and similarly since the y -intercept is b , $(0, b)$ is on the line. Hence, the slope of the line is $m = \frac{b-0}{0-a} = -\frac{b}{a}$. Substituting into $y = mx + b$ gives $y = -\frac{b}{a}x + b \Leftrightarrow \frac{b}{a}x + y = b \Leftrightarrow \frac{x}{a} + \frac{y}{b} = 1$.
- (b) Letting $a = 6$ and $b = -8$ gives $\frac{x}{6} + \frac{y}{-8} = 1 \Leftrightarrow -8x + 6y = -48$ [multiply by -48] $\Leftrightarrow 6y = 8x - 48 \Leftrightarrow 3y = 4x - 24 \Leftrightarrow y = \frac{4}{3}x - 8$.

62. (a) Let
- d
- = distance traveled (in miles) and
- t
- = time elapsed (in hours). (b)

At $t = 0$, $d = 0$ and at $t = 50 \text{ minutes} = 50 \cdot \frac{1}{60} = \frac{5}{6} \text{ h}$, $d = 40$.

Thus, we have two points: $(0, 0)$ and $(\frac{5}{6}, 40)$, so $m = \frac{40 - 0}{5/6 - 0} = 48$

and $d = 48t$.

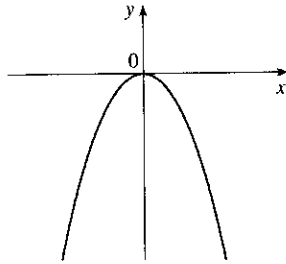


- (c) The slope is 48 and represents the car's speed in mi/h.

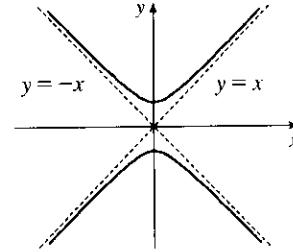
C Graphs of Second-Degree Equations

1. An equation of the circle with center $(3, -1)$ and radius 5 is $(x - 3)^2 + (y + 1)^2 = 5^2 = 25$.
2. From (1), the equation is $(x + 2)^2 + (y + 8)^2 = 100$.
3. The equation has the form $x^2 + y^2 = r^2$. Since $(4, 7)$ lies on the circle, we have $4^2 + 7^2 = r^2 \Rightarrow r^2 = 65$. So the required equation is $x^2 + y^2 = 65$.
4. The equation has the form $(x + 1)^2 + (y - 5)^2 = r^2$. Since $(-4, -6)$ lies on the circle, we have $r^2 = (-4 + 1)^2 + (-6 - 5)^2 = 130$. So an equation is $(x + 1)^2 + (y - 5)^2 = 130$.
5. $x^2 + y^2 - 4x + 10y + 13 = 0 \Leftrightarrow x^2 - 4x + y^2 + 10y = -13 \Leftrightarrow (x^2 - 4x + 4) + (y^2 + 10y + 25) = -13 + 4 + 25 = 16 \Leftrightarrow (x - 2)^2 + (y + 5)^2 = 4^2$. Thus, we have a circle with center $(2, -5)$ and radius 4.
6. $x^2 + y^2 + 6y + 2 = 0 \Leftrightarrow x^2 + (y^2 + 6y + 9) = -2 + 9 \Leftrightarrow x^2 + (y + 3)^2 = 7$. Thus, we have a circle with center $(0, -3)$ and radius $\sqrt{7}$.
7. $x^2 + y^2 + x = 0 \Leftrightarrow (x^2 + x + \frac{1}{4}) + y^2 = \frac{1}{4} \Leftrightarrow (x + \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$. Thus, we have a circle with center $(-\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.
8. $16x^2 + 16y^2 + 8x + 32y + 1 = 0 \Leftrightarrow 16(x^2 + \frac{1}{2}x + \frac{1}{16}) + 16(y^2 + 2y + 1) = -1 + 1 + 16 \Leftrightarrow 16(x + \frac{1}{4})^2 + 16(y + 1)^2 = 16 \Leftrightarrow (x + \frac{1}{4})^2 + (y + 1)^2 = 1$. Thus, we have a circle with center $(-\frac{1}{4}, -1)$ and radius 1.
9. $2x^2 + 2y^2 - x + y = 1 \Leftrightarrow 2(x^2 - \frac{1}{2}x + \frac{1}{16}) + 2(y^2 + \frac{1}{2}y + \frac{1}{16}) = 1 + \frac{1}{8} + \frac{1}{8} \Leftrightarrow 2(x - \frac{1}{4})^2 + 2(y + \frac{1}{4})^2 = \frac{5}{4} \Leftrightarrow (x - \frac{1}{4})^2 + (y + \frac{1}{4})^2 = \frac{5}{8}$. Thus, we have a circle with center $(\frac{1}{4}, -\frac{1}{4})$ and radius $\frac{\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{10}}{4}$.
10. $x^2 + y^2 + ax + by + c = 0 \Leftrightarrow (x^2 + ax + \frac{1}{4}a^2) + (y^2 + by + \frac{1}{4}b^2) = -c + \frac{1}{4}a^2 + \frac{1}{4}b^2 \Leftrightarrow (x + \frac{1}{2}a)^2 + (y + \frac{1}{2}b)^2 = \frac{1}{4}(a^2 + b^2 - 4c)$. For this to represent a nondegenerate circle, $\frac{1}{4}(a^2 + b^2 - 4c) > 0$ or $a^2 + b^2 > 4c$. If this condition is satisfied, the circle has center $(-\frac{1}{2}a, -\frac{1}{2}b)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2 - 4c}$.

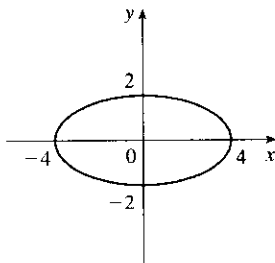
11. $y = -x^2$. Parabola



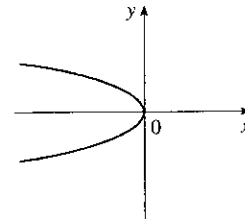
12. $y^2 - x^2 = 1$. Hyperbola



13. $x^2 + 4y^2 = 16 \Leftrightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$. Ellipse

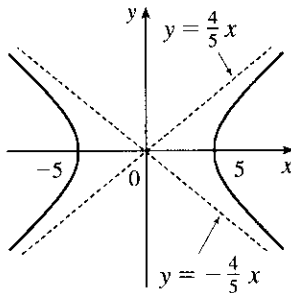


14. $x = -2y^2$. Parabola

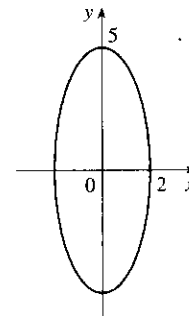


15. $16x^2 - 25y^2 = 400 \Leftrightarrow \frac{x^2}{25} - \frac{y^2}{16} = 1$.

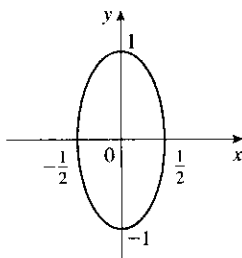
Hyperbola



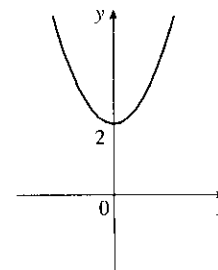
16. $25x^2 + 4y^2 = 100 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{25} = 1$. Ellipse



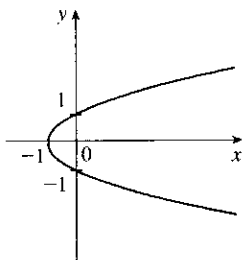
17. $4x^2 + y^2 = 1 \Leftrightarrow \frac{x^2}{1/4} + y^2 = 1$. Ellipse



18. $y = x^2 + 2$. Parabola with vertex at (0, 2)

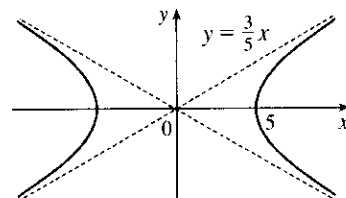


19. $x = y^2 - 1$. Parabola with vertex at $(-1, 0)$

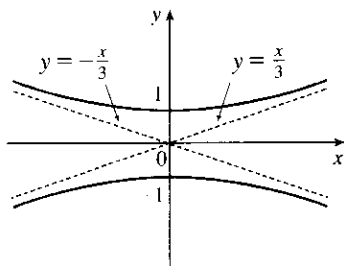


20. $9x^2 - 25y^2 = 225 \Leftrightarrow \frac{x^2}{25} - \frac{y^2}{9} = 1$.

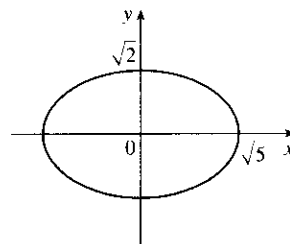
Hyperbola



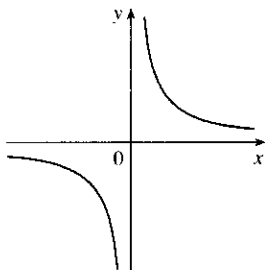
21. $9y^2 - x^2 = 9 \Leftrightarrow y^2 - \frac{x^2}{9} = 1$. Hyperbola



22. $2x^2 + 5y^2 = 10 \Leftrightarrow \frac{x^2}{5} + \frac{y^2}{2} = 1$. Ellipse

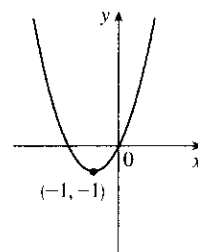


23. $xy = 4$. Hyperbola

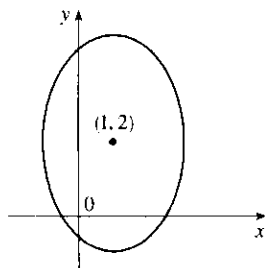


24. $y = x^2 + 2x = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1$.

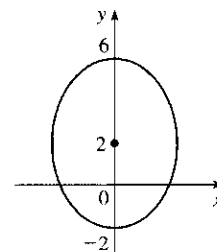
Parabola with vertex at $(-1, -1)$



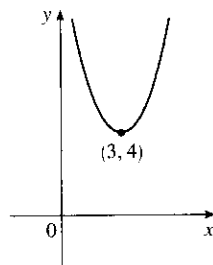
25. $9(x - 1)^2 + 4(y - 2)^2 = 36 \Leftrightarrow \frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{9} = 1$. Ellipse centered at $(1, 2)$



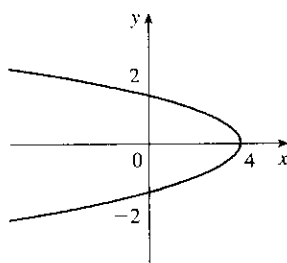
26. $16x^2 + 9y^2 - 36y = 108 \Leftrightarrow 16x^2 + 9(y^2 - 4y + 4) = 108 + 36 = 144 \Leftrightarrow \frac{x^2}{9} + \frac{(y - 2)^2}{16} = 1$. Ellipse centered at $(0, 2)$



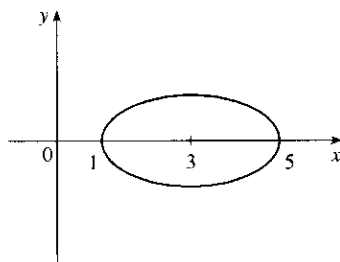
27. $y = x^2 - 6x + 13 = (x^2 - 6x + 9) + 4 = (x - 3)^2 + 4$. Parabola with vertex at $(3, 4)$



29. $x = -y^2 + 4$. Parabola with vertex at $(4, 0)$

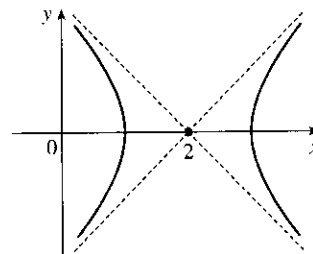


31. $x^2 + 4y^2 - 6x + 5 = 0 \Leftrightarrow (x^2 - 6x + 9) + 4y^2 = -5 + 9 = 4 \Leftrightarrow \frac{(x - 3)^2}{4} + y^2 = 1$. Ellipse centered at $(3, 0)$

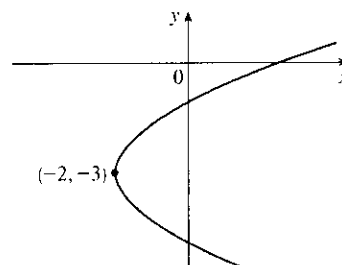


33. $y = 3x$ and $y = x^2$ intersect where $3x = x^2 \Leftrightarrow 0 = x^2 - 3x = x(x - 3)$, that is, at $(0, 0)$ and $(3, 9)$.

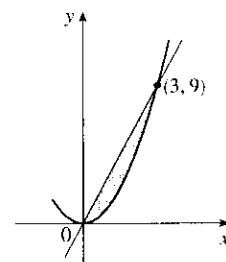
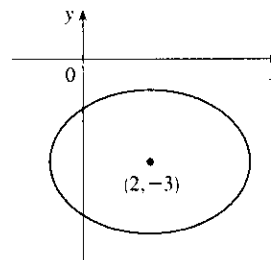
28. $x^2 - y^2 - 4x + 3 = 0 \Leftrightarrow (x^2 - 4x + 4) - y^2 = -3 + 4 = 1 \Leftrightarrow (x - 2)^2 - y^2 = 1$. Hyperbola centered at $(2, 0)$



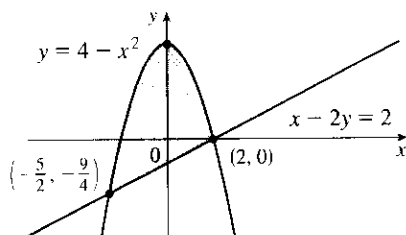
30. $y^2 - 2x + 6y + 5 = 0 \Leftrightarrow y^2 + 6y + 9 = 2x + 4 \Leftrightarrow (y + 3)^2 = 2(x + 2)$. Parabola with vertex $(-2, -3)$



32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0 \Leftrightarrow 4(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = -61 + 16 + 81 = 36 \Leftrightarrow \frac{(x - 2)^2}{9} + \frac{(y + 3)^2}{4} = 1$. Ellipse centered at $(2, -3)$



34.



$y = 4 - x^2$, $x - 2y = 2$. Substitute y from the first equation into the second: $x - 2(4 - x^2) = 2 \Leftrightarrow 2x^2 + x - 10 = 0$
 $\Leftrightarrow (2x + 5)(x - 2) = 0 \Leftrightarrow x = -\frac{5}{2}$ or 2 . So the points of intersection are $(-\frac{5}{2}, -\frac{9}{4})$ and $(2, 0)$.

35. The parabola must have an equation of the form $y = a(x - 1)^2 - 1$. Substituting $x = 3$ and $y = 3$ into the equation gives $3 = a(3 - 1)^2 - 1$, so $a = 1$, and the equation is $y = (x - 1)^2 - 1 = x^2 - 2x$. Note that using the other point $(-1, 3)$ would have given the same value for a , and hence the same equation.

36. The ellipse has an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Substituting $x = 1$ and $y = -\frac{10\sqrt{2}}{3}$ gives

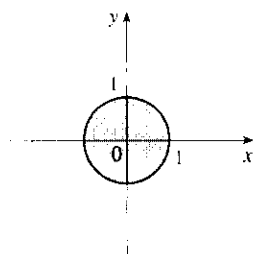
$$\frac{1^2}{a^2} + \frac{(-10\sqrt{2}/3)^2}{b^2} = \frac{1}{a^2} + \frac{200}{9b^2} = 1. \text{ Substituting } x = -2 \text{ and } y = \frac{5\sqrt{5}}{3} \text{ gives}$$

$$\frac{(-2)^2}{a^2} + \frac{(5\sqrt{5}/3)^2}{b^2} = \frac{4}{a^2} + \frac{125}{9b^2} = 1. \text{ From the first equation, } \frac{1}{a^2} = 1 - \frac{200}{9b^2}. \text{ Putting this into the second}$$

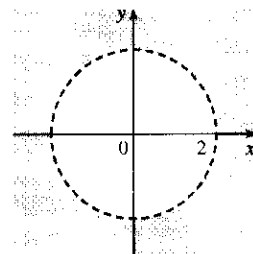
$$\text{equation gives } 4\left(1 - \frac{200}{9b^2}\right) + \frac{125}{9b^2} = 1 \Leftrightarrow 3 = \frac{675}{9b^2} \Leftrightarrow b^2 = \frac{675}{27} = 25, \text{ so } b = 5. \text{ Hence}$$

$$\frac{1}{a^2} = 1 - \frac{200}{9(5)^2} = \frac{1}{9} \text{ and so } a = 3. \text{ The equation of the ellipse is } \frac{x^2}{9} + \frac{y^2}{25} = 1.$$

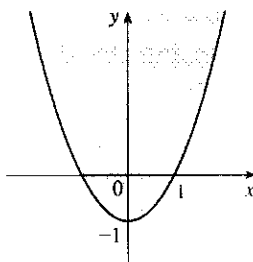
37. $\{(x, y) \mid x^2 + y^2 \leq 1\}$



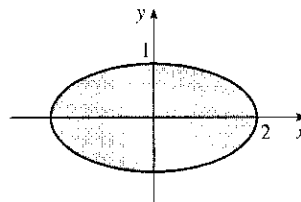
38. $\{(x, y) \mid x^2 + y^2 > 4\}$



39. $\{(x, y) \mid y \geq x^2 - 1\}$



40. $\{(x, y) \mid x^2 + 4y^2 \leq 4\}$



D Trigonometry

1. $210^\circ = 210\left(\frac{\pi}{180}\right) = \frac{7\pi}{6} \text{ rad}$

2. $300^\circ = 300\left(\frac{\pi}{180}\right) = \frac{5\pi}{3} \text{ rad}$

3. $9^\circ = 9\left(\frac{\pi}{180}\right) = \frac{\pi}{20} \text{ rad}$

4. $-315^\circ = -315\left(\frac{\pi}{180}\right) = -\frac{7\pi}{4} \text{ rad}$

5. $900^\circ = 900\left(\frac{\pi}{180}\right) = 5\pi \text{ rad}$

6. $36^\circ = 36\left(\frac{\pi}{180}\right) = \frac{\pi}{5} \text{ rad}$

7. $4\pi \text{ rad} = 4\pi\left(\frac{180}{\pi}\right) = 720^\circ$

8. $-\frac{7\pi}{2} \text{ rad} = -\frac{7\pi}{2}\left(\frac{180}{\pi}\right) = -630^\circ$

9. $\frac{5\pi}{12} \text{ rad} = \frac{5\pi}{12}\left(\frac{180}{\pi}\right) = 75^\circ$

10. $\frac{8\pi}{3} \text{ rad} = \frac{8\pi}{3}\left(\frac{180}{\pi}\right) = 480^\circ$

11. $-\frac{3\pi}{8} \text{ rad} = -\frac{3\pi}{8}\left(\frac{180}{\pi}\right) = -67.5^\circ$

12. $5 \text{ rad} = 5\left(\frac{180}{\pi}\right) = \left(\frac{900}{\pi}\right)^\circ$

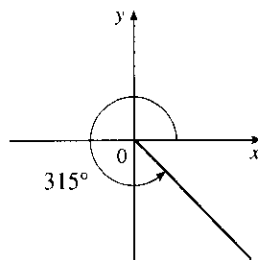
13. Using Formula 3, $a = r\theta = 36 \cdot \frac{\pi}{12} = 3\pi \text{ cm}$.

14. Using Formula 3, $a = r\theta = 10 \cdot 72\left(\frac{\pi}{180}\right) = 4\pi \text{ cm}$.

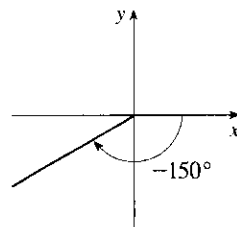
15. Using Formula 3, $\theta = a/r = \frac{1}{1.5} = \frac{2}{3} \text{ rad} = \frac{2}{3}\left(\frac{180}{\pi}\right) = \left(\frac{120}{\pi}\right)^\circ \approx 38.2^\circ$.

16. $a = r\theta \Rightarrow r = \frac{a}{\theta} = \frac{6}{3\pi/4} = \frac{8}{\pi} \text{ cm}$

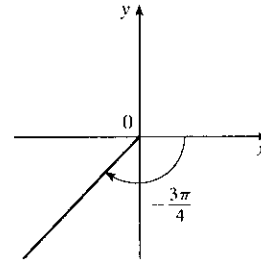
17.



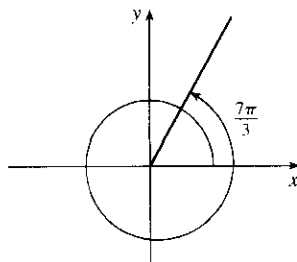
18.



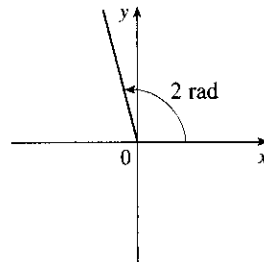
19.



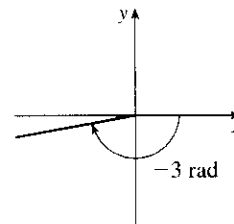
20.



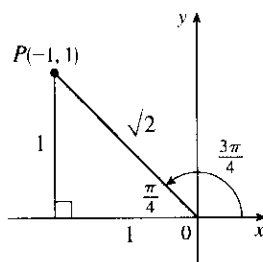
21.



22.



23.

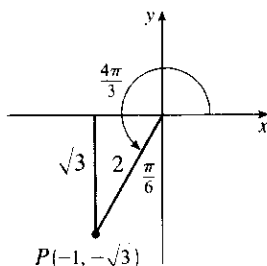


From the diagram we see that a point on the terminal side is $P(-1, 1)$.

Therefore, taking $x = -1$, $y = 1$, $r = \sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$,

$\tan \frac{3\pi}{4} = -1$, $\csc \frac{3\pi}{4} = \sqrt{2}$, $\sec \frac{3\pi}{4} = -\sqrt{2}$, and $\cot \frac{3\pi}{4} = -1$.

24.

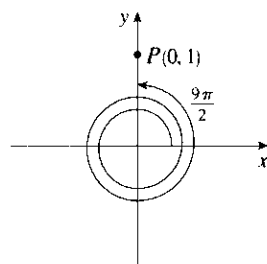


From the diagram and Figure 8, we see that a point on the terminal side is $P(-1, -\sqrt{3})$.

Therefore, taking $x = -1$, $y = -\sqrt{3}$, $r = 2$ in the definitions of the trigonometric ratios, we have $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$,

$\cos \frac{4\pi}{3} = -\frac{1}{2}$, $\tan \frac{4\pi}{3} = \sqrt{3}$, $\csc \frac{4\pi}{3} = -\frac{2}{\sqrt{3}}$, $\sec \frac{4\pi}{3} = -2$, and $\cot \frac{4\pi}{3} = \frac{1}{\sqrt{3}}$.

25.



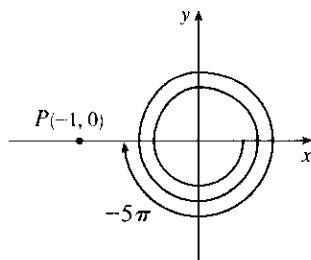
From the diagram we see that a point on the terminal line is $P(0, 1)$.

Therefore taking $x = 0$, $y = 1$, $r = 1$ in the definitions of the

trigonometric ratios, we have $\sin \frac{9\pi}{2} = 1$, $\cos \frac{9\pi}{2} = 0$, $\tan \frac{9\pi}{2} = y/x$ is

undefined since $x = 0$, $\csc \frac{9\pi}{2} = 1$, $\sec \frac{9\pi}{2} = r/x$ is undefined since $x = 0$, and $\cot \frac{9\pi}{2} = 0$.

26.



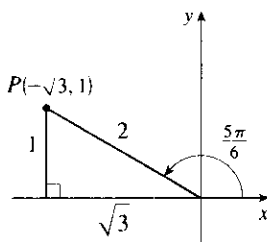
From the diagram, we see that a point on the terminal line is $P(-1, 0)$.

Therefore taking $x = -1$, $y = 0$, $r = 1$ in the definitions of the

trigonometric ratios we have $\sin(-5\pi) = 0$, $\cos(-5\pi) = -1$,

$\tan(-5\pi) = 0$, $\csc(-5\pi)$ is undefined, $\sec(-5\pi) = -1$, and $\cot(-5\pi)$ is undefined.

27.



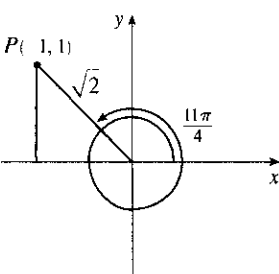
Using Figure 8 we see that a point on the terminal line is $P(-\sqrt{3}, 1)$.

Therefore taking $x = -\sqrt{3}$, $y = 1$, $r = 2$ in the definitions of the

trigonometric ratios, we have $\sin \frac{5\pi}{6} = \frac{1}{2}$, $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$,

$\tan \frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$, $\csc \frac{5\pi}{6} = 2$, $\sec \frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$, and $\cot \frac{5\pi}{6} = -\sqrt{3}$.

28.



From the diagram, we see that a point on the terminal line is $P(-1, 1)$.

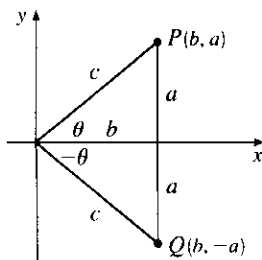
Therefore taking $x = -1$, $y = 1$, $r = \sqrt{2}$ in the definitions of the

trigonometric ratios we have $\sin \frac{11\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos \frac{11\pi}{4} = -\frac{1}{\sqrt{2}}$,

$\tan \frac{11\pi}{4} = -1$, $\csc \frac{11\pi}{4} = \sqrt{2}$, $\sec \frac{11\pi}{4} = -\sqrt{2}$, and $\cot \frac{11\pi}{4} = -1$.

29. $\sin \theta = y/r = \frac{3}{5} \Rightarrow y = 3, r = 5$, and $x = \sqrt{r^2 - y^2} = 4$ (since $0 < \theta < \frac{\pi}{2}$). Therefore taking $x = 4, y = 3, r = 5$ in the definitions of the trigonometric ratios, we have $\cos \theta = \frac{4}{5}, \tan \theta = \frac{3}{4}, \csc \theta = \frac{5}{3}, \sec \theta = \frac{5}{4}$, and $\cot \theta = \frac{4}{3}$.
30. Since $0 < \alpha < \frac{\pi}{2}, \alpha$ is in the first quadrant where x and y are both positive. Therefore, $\tan \alpha = y/x = \frac{2}{1} \Rightarrow y = 2, x = 1$, and $r = \sqrt{x^2 + y^2} = \sqrt{5}$. Taking $x = 1, y = 2, r = \sqrt{5}$ in the definitions of the trigonometric ratios, we have $\sin \alpha = \frac{2}{\sqrt{5}}, \cos \alpha = \frac{1}{\sqrt{5}}, \csc \alpha = \frac{\sqrt{5}}{2}, \sec \alpha = \sqrt{5}$, and $\cot \alpha = \frac{1}{2}$.
31. $\frac{\pi}{2} < \phi < \pi \Rightarrow \phi$ is in the second quadrant, where x is negative and y is positive. Therefore $\sec \phi = r/x = -1.5 = -\frac{3}{2} \Rightarrow r = 3, x = -2$, and $y = \sqrt{r^2 - x^2} = \sqrt{5}$. Taking $x = -2, y = \sqrt{5}, r = 3$ in the definitions of the trigonometric ratios, we have $\sin \phi = \frac{\sqrt{5}}{3}, \cos \phi = -\frac{2}{3}, \tan \phi = -\frac{\sqrt{5}}{2}, \csc \phi = \frac{3}{\sqrt{5}}$, and $\cot \phi = -\frac{2}{\sqrt{5}}$.
32. Since $\pi < x < \frac{3\pi}{2}, x$ is in the third quadrant where x and y are both negative. Therefore $\cos x = x/r = -\frac{1}{3} \Rightarrow x = -1, r = 3$, and $y = -\sqrt{r^2 - x^2} = -\sqrt{8} = -2\sqrt{2}$. Taking $x = -1, r = 3, y = -2\sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin x = -\frac{2\sqrt{2}}{3}, \tan x = 2\sqrt{2}, \csc x = -\frac{3}{2\sqrt{2}}, \sec x = -3$, and $\cot x = \frac{1}{2\sqrt{2}}$.
33. $\pi < \beta < 2\pi$ means that β is in the third or fourth quadrant where y is negative. Also since $\cot \beta = x/y = 3$ which is positive, x must also be negative. Therefore $\cot \beta = x/y = \frac{3}{1} \Rightarrow x = -3, y = -1$, and $r = \sqrt{x^2 + y^2} = \sqrt{10}$. Taking $x = -3, y = -1$ and $r = \sqrt{10}$ in the definitions of the trigonometric ratios, we have $\sin \beta = -\frac{1}{\sqrt{10}}, \cos \beta = -\frac{3}{\sqrt{10}}, \tan \beta = \frac{1}{3}, \csc \beta = -\sqrt{10}$, and $\sec \beta = -\frac{\sqrt{10}}{3}$.
34. Since $\frac{3\pi}{2} < \theta < 2\pi, \theta$ is in the fourth quadrant where x is positive and y is negative. Therefore $\csc \theta = r/y = -\frac{4}{3} \Rightarrow r = 4, y = -3$, and $x = \sqrt{r^2 - y^2} = \sqrt{7}$. Taking $x = \sqrt{7}, y = -3$, and $r = 4$ in the definitions of the trigonometric ratios, we have $\sin \theta = -\frac{3}{4}, \cos \theta = \frac{\sqrt{7}}{4}, \tan \theta = -\frac{3}{\sqrt{7}}, \sec \theta = \frac{4}{\sqrt{7}}$, and $\cot \theta = -\frac{\sqrt{7}}{3}$.
35. $\sin 35^\circ = \frac{x}{10} \Rightarrow x = 10 \sin 35^\circ \approx 5.73576 \text{ cm}$
36. $\cos 40^\circ = \frac{x}{25} \Rightarrow x = 25 \cos 40^\circ \approx 19.15111 \text{ cm}$
37. $\tan \frac{2\pi}{5} = \frac{x}{8} \Rightarrow x = 8 \tan \frac{2\pi}{5} \approx 24.62147 \text{ cm}$
38. $\cos \frac{3\pi}{8} = \frac{22}{x} \Rightarrow x = \frac{22}{\cos \frac{3\pi}{8}} \approx 57.48877 \text{ cm}$

39.

(a) From the diagram we see that $\sin \theta = \frac{y}{r} = \frac{a}{c}$, and

$$\sin(-\theta) = \frac{-a}{c} = -\frac{a}{c} = -\sin \theta.$$

(b) Again from the diagram we see that $\cos \theta = \frac{x}{r} = \frac{b}{c} = \cos(-\theta)$.

40. (a) Using (12a) and (12b), we have

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

(b) From (10a) and (10b), we have $\tan(-\theta) = -\tan \theta$, so (14a) implies that

$$\tan(x-y) = \tan(x+(-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

41. (a) Using (12a) and (13a), we have

$$\begin{aligned} \frac{1}{2} [\sin(x+y) + \sin(x-y)] &= \frac{1}{2} [\sin x \cos y + \cos x \sin y + \sin x \cos y - \cos x \sin y] \\ &= \frac{1}{2} (2 \sin x \cos y) = \sin x \cos y \end{aligned}$$

(b) This time, using (12b) and (13b), we have

$$\begin{aligned} \frac{1}{2} [\cos(x+y) + \cos(x-y)] &= \frac{1}{2} [\cos x \cos y - \sin x \sin y + \cos x \cos y + \sin x \sin y] \\ &= \frac{1}{2} (2 \cos x \cos y) = \cos x \cos y \end{aligned}$$

(c) Again using (12b) and (13b), we have

$$\begin{aligned} \frac{1}{2} [\cos(x-y) - \cos(x+y)] &= \frac{1}{2} [\cos x \cos y + \sin x \sin y - \cos x \cos y + \sin x \sin y] \\ &= \frac{1}{2} (2 \sin x \sin y) = \sin x \sin y \end{aligned}$$

42. Using (13b), $\cos\left(\frac{\pi}{2} - x\right) = \cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x = 0 \cdot \cos x + 1 \cdot \sin x = \sin x$.

43. Using (12a), we have $\sin\left(\frac{\pi}{2} + x\right) = \sin \frac{\pi}{2} \cos x + \cos \frac{\pi}{2} \sin x = 1 \cdot \cos x + 0 \cdot \sin x = \cos x$.

44. Using (13a), we have $\sin(\pi - x) = \sin \pi \cos x - \cos \pi \sin x = 0 \cdot \cos x - (-1) \sin x = \sin x$.

45. Using (6), we have $\sin \theta \cot \theta = \sin \theta \cdot \frac{\cos \theta}{\sin \theta} = \cos \theta$.

46. $(\sin x + \cos x)^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x = (\sin^2 x + \cos^2 x) + \sin 2x$ [by (15a)]
 $= 1 + \sin 2x$ [by (7)]

47. $\sec y - \cos y = \frac{1}{\cos y} - \cos y$ [by (6)] $= \frac{1 - \cos^2 y}{\cos y} = \frac{\sin^2 y}{\cos y}$ [by (7)] $= \frac{\sin y}{\cos y} \sin y = \tan y \sin y$ [by (6)]

48. $\tan^2 \alpha - \sin^2 \alpha = \frac{\sin^2 \alpha}{\cos^2 \alpha} - \sin^2 \alpha = \frac{\sin^2 \alpha - \sin^2 \alpha \cos^2 \alpha}{\cos^2 \alpha} = \frac{\sin^2 \alpha (1 - \cos^2 \alpha)}{\cos^2 \alpha} = \tan^2 \alpha \sin^2 \alpha$ [by (6), (7)]

49. $\cot^2 \theta + \sec^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta} + \frac{1}{\cos^2 \theta}$ [by (6)] $= \frac{\cos^2 \theta \cos^2 \theta + \sin^2 \theta}{\sin^2 \theta \cos^2 \theta}$
 $= \frac{(1 - \sin^2 \theta)(1 - \sin^2 \theta) + \sin^2 \theta}{\sin^2 \theta \cos^2 \theta}$ [by (7)] $= \frac{1 - \sin^2 \theta + \sin^4 \theta}{\sin^2 \theta \cos^2 \theta}$
 $= \frac{\cos^2 \theta + \sin^4 \theta}{\sin^2 \theta \cos^2 \theta}$ [by (7)] $= \frac{1}{\sin^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \csc^2 \theta + \tan^2 \theta$ [by (6)]

50. $2 \csc 2t = \frac{2}{\sin 2t} = \frac{2}{2 \sin t \cos t}$ [by (15a)] $= \frac{1}{\sin t \cos t} = \sec t \csc t$

51. Using (14a), we have $\tan 2\theta = \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \frac{2 \tan \theta}{1 - \tan^2 \theta}$.

52. $\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = \frac{1 + \sin \theta + 1 - \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} = \frac{2}{1 - \sin^2 \theta} = \frac{2}{\cos^2 \theta}$ [by (7)] $= 2 \sec^2 \theta$

53. Using (15a) and (16a),

$$\begin{aligned} \sin x \sin 2x + \cos x \cos 2x &= \sin x (2 \sin x \cos x) + \cos x (2 \cos^2 x - 1) = 2 \sin^2 x \cos x + 2 \cos^3 x - \cos x \\ &= 2(1 - \cos^2 x) \cos x + 2 \cos^3 x - \cos x \quad [\text{by (7)}] \\ &= 2 \cos x - 2 \cos^3 x + 2 \cos^3 x - \cos x = \cos x \end{aligned}$$

Or: $\sin x \sin 2x + \cos x \cos 2x = \cos(2x - x)$ [by 13(b)] $= \cos x$

54. Working backward, we start with equations (12a) and (13a):

$$\begin{aligned} \sin(x + y) \sin(x - y) &= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y) \\ &= \sin^2 x \cos^2 y - \sin x \cos y \cos x \sin y + \cos x \sin y \sin x \cos y - \cos^2 x \sin^2 y \\ &= \sin^2 x (1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y \quad [\text{by (7)}] \\ &= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y = \sin^2 x - \sin^2 y \end{aligned}$$

55. $\frac{\sin \phi}{1 - \cos \phi} = \frac{\sin \phi}{1 - \cos \phi} \cdot \frac{1 + \cos \phi}{1 + \cos \phi} = \frac{\sin \phi (1 + \cos \phi)}{1 - \cos^2 \phi} = \frac{\sin \phi (1 + \cos \phi)}{\sin^2 \phi}$ [by (7)]

$$= \frac{1 + \cos \phi}{\sin \phi} = \frac{1}{\sin \phi} + \frac{\cos \phi}{\sin \phi} = \csc \phi + \cot \phi \quad [\text{by (6)}]$$

56. $\tan x + \tan y = \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y} = \frac{\sin(x + y)}{\cos x \cos y}$ [by (12a)]

57. Using (12a),

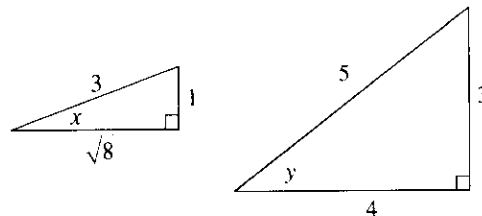
$$\begin{aligned} \sin 3\theta + \sin \theta &= \sin(2\theta + \theta) + \sin \theta = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta + \sin \theta \\ &= \sin 2\theta \cos \theta + (2 \cos^2 \theta - 1) \sin \theta + \sin \theta \quad [\text{by (16a)}] \\ &= \sin 2\theta \cos \theta + 2 \cos^2 \theta \sin \theta - \sin \theta + \sin \theta = \sin 2\theta \cos \theta + \sin 2\theta \cos \theta \quad [\text{by (15a)}] \\ &= 2 \sin 2\theta \cos \theta \end{aligned}$$

58. We use (12b) with $x = 2\theta$, $y = \theta$ to get

$$\begin{aligned} \cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\ &= (2 \cos^2 \theta - 1) \cos \theta - 2 \sin^2 \theta \cos \theta \quad [\text{by (16a) and (15a)}] \\ &= (2 \cos^2 \theta - 1) \cos \theta - 2(1 - \cos^2 \theta) \cos \theta \quad [\text{by (7)}] \\ &= 2 \cos^3 \theta - \cos \theta - 2 \cos \theta + 2 \cos^3 \theta = 4 \cos^3 \theta - 3 \cos \theta \end{aligned}$$

59. Since $\sin x = \frac{1}{3}$ we can label the opposite side as having length 1, the hypotenuse as having length 3, and use the Pythagorean Theorem to get that the adjacent side has length $\sqrt{8}$. Then, from the diagram, $\cos x = \frac{\sqrt{8}}{3}$. Similarly we have that $\sin y = \frac{3}{5}$. Now use (12a):

$$\sin(x+y) = \sin x \cos y + \cos x \sin y = \frac{1}{3} \cdot \frac{4}{5} + \frac{\sqrt{8}}{3} \cdot \frac{3}{5} = \frac{4}{15} + \frac{3\sqrt{8}}{15} = \frac{4+6\sqrt{2}}{15}.$$



60. Use (12b) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59 to get

$$\cos(x+y) = \cos x \cos y - \sin x \sin y = \frac{\sqrt{8}}{3} \cdot \frac{4}{5} - \frac{1}{3} \cdot \frac{3}{5} = \frac{8\sqrt{2}-3}{15}$$

61. Using (13b) and the values for $\cos x$ and $\sin y$ obtained in Exercise 59, we have

$$\cos(x-y) = \cos x \cos y + \sin x \sin y = \frac{\sqrt{8}}{3} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{3}{5} = \frac{8\sqrt{2}+3}{15}$$

62. Using (13a) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59, we get

$$\sin(x-y) = \sin x \cos y - \cos x \sin y = \frac{1}{3} \cdot \frac{4}{5} - \frac{\sqrt{8}}{3} \cdot \frac{3}{5} = \frac{4-6\sqrt{2}}{15}$$

63. Using (15a) and the values for $\sin y$ and $\cos y$ obtained in Exercise 59, we have

$$\sin 2y = 2 \sin y \cos y = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$$

64. Using (16a) with $\cos y = \frac{4}{5}$, we have $\cos 2y = 2 \cos^2 y - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$.

65. $2 \cos x - 1 = 0 \Leftrightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$ for $x \in [0, 2\pi]$.

66. $3 \cot^2 x = 1 \Leftrightarrow 3 = 1/\cot^2 x \Leftrightarrow \tan^2 x = 3 \Leftrightarrow \tan x = \pm\sqrt{3} \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ and } \frac{5\pi}{3}$.

67. $2 \sin^2 x = 1 \Leftrightarrow \sin^2 x = \frac{1}{2} \Leftrightarrow \sin x = \pm\frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

68. $|\tan x| = 1 \Leftrightarrow \tan x = -1 \text{ or } \tan x = 1 \Leftrightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \text{ or } x = \frac{\pi}{4}, \frac{5\pi}{4}$.

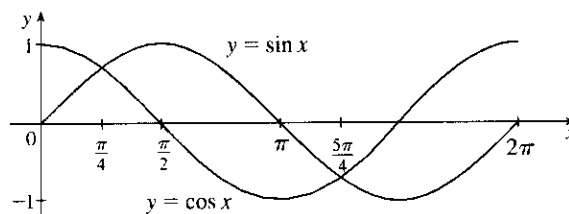
69. Using (15a), we have $\sin 2x = \cos x \Leftrightarrow 2 \sin x \cos x - \cos x = 0 \Leftrightarrow \cos x(2 \sin x - 1) = 0 \Leftrightarrow \cos x = 0 \text{ or } 2 \sin x - 1 = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$. Therefore, the solutions are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

70. By (15a), $2 \cos x + \sin 2x = 0 \Leftrightarrow 2 \cos x + 2 \sin x \cos x = 0 \Leftrightarrow 2 \cos x(1 + \sin x) = 0 \Leftrightarrow \cos x = 0 \text{ or } 1 + \sin x = 0 \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } \sin x = -1 \Rightarrow x = \frac{3\pi}{2}$. So the solutions are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

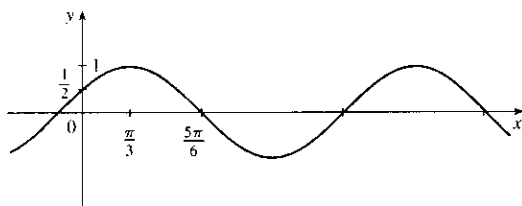
71. $\sin x = \tan x \Leftrightarrow \sin x - \tan x = 0 \Leftrightarrow \sin x - \frac{\sin x}{\cos x} = 0 \Leftrightarrow \sin x \left(1 - \frac{1}{\cos x}\right) = 0 \Leftrightarrow \sin x = 0 \text{ or } 1 - \frac{1}{\cos x} = 0 \Rightarrow x = 0, \pi, 2\pi \text{ or } 1 = \frac{1}{\cos x} \Rightarrow \cos x = 1 \Rightarrow x = 0, 2\pi$. Therefore the solutions are $x = 0, \pi, 2\pi$.

72. By (16a), $2 + \cos 2x = 3 \cos x \Leftrightarrow 2 + 2 \cos^2 x - 1 = 3 \cos x \Leftrightarrow 2 \cos^2 x - 3 \cos x + 1 = 0 \Leftrightarrow (2 \cos x - 1)(\cos x - 1) = 0 \Leftrightarrow \cos x = 1 \text{ or } \cos x = \frac{1}{2} \Rightarrow x = 0, 2\pi \text{ or } x = \frac{\pi}{3}, \frac{5\pi}{3}$.

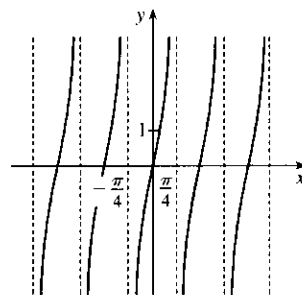
73. We know that $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, and from Figure 13(a), we see that $\sin x \leq \frac{1}{2} \Rightarrow 0 \leq x \leq \frac{\pi}{6}$ or $\frac{5\pi}{6} \leq x \leq 2\pi$ for $x \in [0, 2\pi]$.
74. $2 \cos x + 1 > 0 \Rightarrow 2 \cos x > -1 \Rightarrow \cos x > -\frac{1}{2}$. $\cos x = -\frac{1}{2}$ when $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ and from Figure 13(b), we see that $\cos x > -\frac{1}{2}$ when $0 \leq x < \frac{2\pi}{3}, \frac{4\pi}{3} < x \leq 2\pi$.
75. $\tan x = -1$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$, and $\tan x = 1$ when $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. From Figure 14 we see that $-1 < \tan x < 1 \Rightarrow 0 \leq x < \frac{\pi}{4}, \frac{3\pi}{4} < x < \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4} < x \leq 2\pi$.
76. We know that $\sin x = \cos x$ when $x = \frac{\pi}{4}, \frac{5\pi}{4}$, and from the diagram we see that $\sin x > \cos x$ when $\frac{\pi}{4} < x < \frac{5\pi}{4}$.



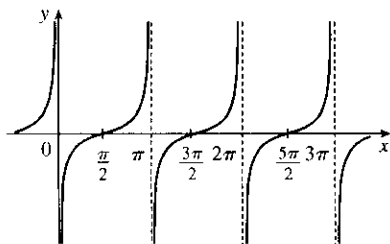
77. $y = \cos(x - \frac{\pi}{3})$. We start with the graph of $y = \cos x$ and shift it $\frac{\pi}{3}$ units to the right.



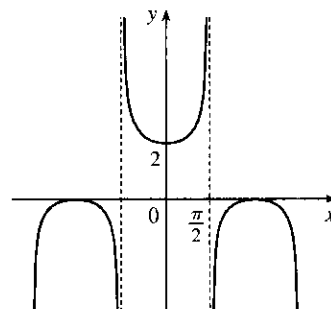
78. $y = \tan 2x$. Start with the graph of $y = \tan x$ with period π and compress it to a period of $\frac{\pi}{2}$.



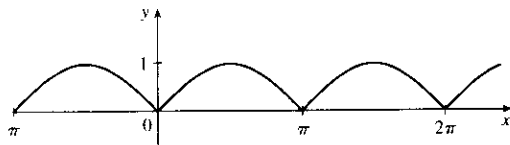
79. $y = \frac{1}{3} \tan(x - \frac{\pi}{2})$. We start with the graph of $y = \tan x$, shift it $\frac{\pi}{2}$ units to the right and compress it to $\frac{1}{3}$ of its original vertical size.



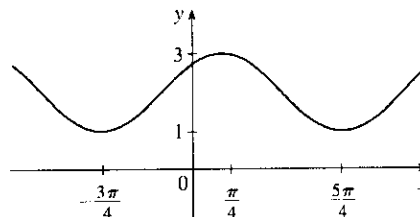
80. $y = 1 + \sec x$. Start with the graph of $y = \sec x$ and raise it by one unit.



81. $y = |\sin x|$. We start with the graph of $y = \sin x$ and reflect the parts below the x -axis about the x -axis.



82. $y = 2 + \sin(x + \frac{\pi}{4})$. Start with the graph of $y = \sin x$, and shift it $\frac{\pi}{4}$ units to the left and 2 units up.



83. From the figure in the text, we see that $x = b \cos \theta$, $y = b \sin \theta$, and from the distance formula we have that the distance c from (x, y) to $(a, 0)$ is $c = \sqrt{(x - a)^2 + (y - 0)^2} \Rightarrow$

$$\begin{aligned} c^2 &= (b \cos \theta - a)^2 + (b \sin \theta)^2 = b^2 \cos^2 \theta - 2ab \cos \theta + a^2 + b^2 \sin^2 \theta \\ &= a^2 + b^2 (\cos^2 \theta + \sin^2 \theta) - 2ab \cos \theta = a^2 + b^2 - 2ab \cos \theta \quad [\text{by (7)}] \end{aligned}$$

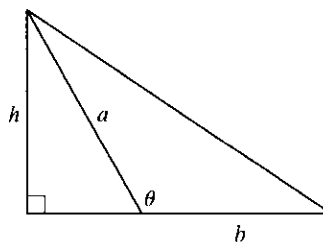
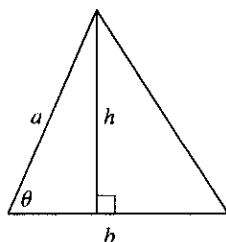
84. $|AB|^2 = |AC|^2 + |BC|^2 - 2|AC||BC| \cos \angle C = (820)^2 + (910)^2 - 2(820)(910) \cos 103^\circ$
 $\approx 1,836,217 \Rightarrow |AB| \approx 1355 \text{ m}$

85. Using the Law of Cosines, we have $c^2 = 1^2 + 1^2 - 2(1)(1) \cos(\alpha - \beta) = 2[1 - \cos(\alpha - \beta)]$. Now, using the distance formula, $c^2 = |AB|^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$. Equating these two expressions for c^2 , we get $2[1 - \cos(\alpha - \beta)] = \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \Rightarrow$
 $1 - \cos(\alpha - \beta) = 1 - \cos \alpha \cos \beta - \sin \alpha \sin \beta \Rightarrow \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$.

86. $\cos(x + y) = \cos(x - (-y)) = \cos x \cos(-y) + \sin x \sin(-y)$
 $= \cos x \cos y - \sin x \sin y \quad [\text{using Equations 10a and 10b}]$

87. In Exercise 86 we used the subtraction formula for cosine to prove the addition formula for cosine. Using that formula with $x = \frac{\pi}{2} - \alpha$, $y = \beta$, we get $\cos[(\frac{\pi}{2} - \alpha) + \beta] = \cos(\frac{\pi}{2} - \alpha) \cos \beta - \sin(\frac{\pi}{2} - \alpha) \sin \beta \Rightarrow$
 $\cos[\frac{\pi}{2} - (\alpha - \beta)] = \cos(\frac{\pi}{2} - \alpha) \cos \beta - \sin(\frac{\pi}{2} - \alpha) \sin \beta$. Now we use the identities given in the problem, $\cos(\frac{\pi}{2} - \theta) = \sin \theta$ and $\sin(\frac{\pi}{2} - \theta) = \cos \theta$, to get $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.

88. If $0 < \theta < \frac{\pi}{2}$, we have the case depicted in the first diagram. In this case, we see that the height of the triangle is $h = a \sin \theta$. If $\frac{\pi}{2} \leq \theta < \pi$, we have the case depicted in the second diagram. In this case, the height of the triangle is $h = a \sin(\pi - \theta) = a \sin \theta$ (by the identity proved in Exercise 78). So in either case, the area of the triangle is $\frac{1}{2}bh = \frac{1}{2}ab \sin \theta$.



89. Using the formula from part (a), the area of the triangle is $\frac{1}{2}(10)(3) \sin 107^\circ \approx 14.34457 \text{ cm}^2$.

E Sigma Notation

$$1. \sum_{i=1}^5 \sqrt{i} = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$$

$$3. \sum_{i=4}^6 3^i = 3^4 + 3^5 + 3^6$$

$$5. \sum_{k=0}^4 \frac{2k-1}{2k+1} = -1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$$

$$7. \sum_{i=1}^n i^{10} = 1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$$

$$9. \sum_{j=0}^{n-1} (-1)^j = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$$

$$11. 1 + 2 + 3 + 4 + \cdots + 10 = \sum_{i=1}^{10} i$$

$$13. \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20} = \sum_{i=1}^{19} \frac{i}{i+1}$$

$$15. 2 + 4 + 6 + 8 + \cdots + 2n = \sum_{i=1}^n 2i$$

$$17. 1 + 2 + 4 + 8 + 16 + 32 = \sum_{i=0}^5 2^i$$

$$19. x + x^2 + x^3 + \cdots + x^n = \sum_{i=1}^n x^i$$

$$21. \sum_{i=4}^8 (3i-2) = [3(4)-2] + [3(5)-2] + [3(6)-2] + [3(7)-2] + [3(8)-2] = 10 + 13 + 16 + 19 + 22 = 80$$

$$22. \sum_{i=3}^6 i(i+2) = 3 \cdot 5 + 4 \cdot 6 + 5 \cdot 7 + 6 \cdot 8 = 15 + 24 + 35 + 48 = 122$$

$$23. \sum_{j=1}^6 3^{j+1} = 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 = 9 + 27 + 81 + 243 + 729 + 2187 = 3276$$

(For a more general method, see Exercise 47.)

$$24. \sum_{k=0}^8 \cos k\pi = \cos 0 + \cos \pi + \cos 2\pi + \cos 3\pi + \cos 4\pi + \cos 5\pi + \cos 6\pi + \cos 7\pi + \cos 8\pi$$

$$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 1$$

$$25. \sum_{n=1}^{20} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 0$$

$$2. \sum_{i=1}^6 \frac{1}{i+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

$$4. \sum_{i=4}^6 i^3 = 4^3 + 5^3 + 6^3$$

$$6. \sum_{k=5}^8 x^k = x^5 + x^6 + x^7 + x^8$$

$$8. \sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$$

$$10. \sum_{i=1}^n f(x_i) \Delta x_i = f(x_1) \Delta x_1 + f(x_2) \Delta x_2$$

$$+ f(x_3) \Delta x_3 + \cdots + f(x_n) \Delta x_n$$

$$12. \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} = \sum_{i=3}^7 \sqrt{i}$$

$$14. \frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27} = \sum_{i=3}^{23} \frac{i}{i+4}$$

$$16. 1 + 3 + 5 + 7 + \cdots + (2n-1) = \sum_{i=1}^n (2i-1)$$

$$18. \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} = \sum_{i=1}^6 \frac{1}{i^2}$$

$$20. 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{i=0}^n (-1)^i x^i$$

$$26. \sum_{i=1}^{100} 4 = \underbrace{4 + 4 + 4 + \cdots + 4}_{(100 \text{ summands})} = 100 \cdot 4 = 400$$

$$27. \sum_{i=0}^4 (2^i + i^2) = (1 + 0) + (2 + 1) + (4 + 4) + (8 + 9) + (16 + 16) = 61$$

$$28. \sum_{i=-2}^4 2^{3-i} = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} = 63.5$$

$$29. \sum_{i=1}^n 2i = 2 \sum_{i=1}^n i = 2 \cdot \frac{n(n+1)}{2} \text{ [by Theorem 3(c)]} = n(n+1)$$

$$30. \sum_{i=1}^n (2 - 5i) = \sum_{i=1}^n 2 - \sum_{i=1}^n 5i = 2n - 5 \sum_{i=1}^n i = 2n - \frac{5n(n+1)}{2} = \frac{4n}{2} - \frac{5n^2 + 5n}{2} = -\frac{n(5n+1)}{2}$$

$$\begin{aligned} 31. \sum_{i=1}^n (i^2 + 3i + 4) &= \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + \sum_{i=1}^n 4 = \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 4n \\ &= \frac{1}{6} [(2n^3 + 3n^2 + n) + (9n^2 + 9n) + 24n] = \frac{1}{6} (2n^3 + 12n^2 + 34n) \\ &= \frac{1}{3} n(n^2 + 6n + 17) \end{aligned}$$

$$\begin{aligned} 32. \sum_{i=1}^n (3 + 2i)^2 &= \sum_{i=1}^n (9 + 12i + 4i^2) = \sum_{i=1}^n 9 + 12 \sum_{i=1}^n i + 4 \sum_{i=1}^n i^2 \\ &= 9n + 6n(n+1) + \frac{2n(n+1)(2n+1)}{3} = \frac{27n + 18n^2 + 18n + 4n^3 + 6n^2 + 2n}{3} \\ &= \frac{1}{3} (4n^3 + 24n^2 + 47n) = \frac{1}{3} n(4n^2 + 24n + 47) \end{aligned}$$

$$\begin{aligned} 33. \sum_{i=1}^n (i+1)(i+2) &= \sum_{i=1}^n (i^2 + 3i + 2) = \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + \sum_{i=1}^n 2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n = \frac{n(n+1)}{6} [(2n+1) + 9] + 2n \\ &= \frac{n(n+1)}{3} (n+5) + 2n = \frac{n}{3} [(n+1)(n+5) + 6] = \frac{n}{3} (n^2 + 6n + 11) \end{aligned}$$

$$\begin{aligned} 34. \sum_{i=1}^n i(i+1)(i+2) &= \sum_{i=1}^n (i^3 + 3i^2 + 2i) = \sum_{i=1}^n i^3 + 3 \sum_{i=1}^n i^2 + 2 \sum_{i=1}^n i \\ &= \left[\frac{n(n+1)}{2} \right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} \\ &= n(n+1) \left[\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1 \right] = \frac{n(n+1)}{4} (n^2 + n + 4n + 2 + 4) \\ &= \frac{n(n+1)}{4} (n^2 + 5n + 6) = \frac{n(n+1)(n+2)(n+3)}{4} \end{aligned}$$

$$\begin{aligned} 35. \sum_{i=1}^n (i^3 - i - 2) &= \sum_{i=1}^n i^3 - \sum_{i=1}^n i - \sum_{i=1}^n 2 = \left[\frac{n(n+1)}{2} \right]^2 - \frac{n(n+1)}{2} - 2n \\ &= \frac{1}{4} n(n+1) [n(n+1) - 2] - 2n = \frac{1}{4} n(n+1)(n+2)(n-1) - 2n \\ &= \frac{1}{4} n [(n+1)(n-1)(n+2) - 8] = \frac{1}{4} n [(n^2 - 1)(n+2) - 8] = \frac{1}{4} n (n^3 + 2n^2 - n - 10) \end{aligned}$$

36. By Theorem 3(c) we have that $\sum_{i=1}^n i = \frac{n(n+1)}{2} = 78 \Leftrightarrow n(n+1) = 156 \Leftrightarrow n^2 + n - 156 = 0 \Leftrightarrow (n+13)(n-12) = 0 \Leftrightarrow n = 12 \text{ or } -13$. But $n = -13$ produces a negative answer for the sum, so $n = 12$.

37. By Theorem 2(a) and Example 3, $\sum_{i=1}^n c = c \sum_{i=1}^n 1 = cn$.

38. Let S_n be the statement that $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$.

1. S_1 is true because $1^3 = \left(\frac{1 \cdot 2}{2} \right)^2$.

2. Assume S_k is true. Then $\sum_{i=1}^k i^3 = \left[\frac{k(k+1)}{2} \right]^2$, so

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \left[\frac{k(k+1)}{2} \right]^2 + (k+1)^3 = \frac{(k+1)^2}{4} [k^2 + 4(k+1)] = \frac{(k+1)^2}{4} (k+2)^2 \\ &= \left(\frac{(k+1)[(k+1)+1]}{2} \right)^2 \end{aligned}$$

showing that S_{k+1} is true.

Therefore, S_n is true for all n by mathematical induction.

39. $\sum_{i=1}^n [(i+1)^4 - i^4] = (2^4 - 1^4) + (3^4 - 2^4) + (4^4 - 3^4) + \cdots + [(n+1)^4 - n^4]$
 $= (n+1)^4 - 1^4 = n^4 + 4n^3 + 6n^2 + 4n$

On the other hand,

$$\begin{aligned} \sum_{i=1}^n [(i+1)^4 - i^4] &= \sum_{i=1}^n (4i^3 + 6i^2 + 4i + 1) = 4 \sum_{i=1}^n i^3 + 6 \sum_{i=1}^n i^2 + 4 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\ &= 4S + n(n+1)(2n+1) + 2n(n+1) + n \left[\text{where } S = \sum_{i=1}^n i^3 \right] \\ &= 4S + 2n^3 + 3n^2 + n + 2n^2 + 2n + n = 4S + 2n^3 + 5n^2 + 4n \end{aligned}$$

Thus, $n^4 + 4n^3 + 6n^2 + 4n = 4S + 2n^3 + 5n^2 + 4n$, from which it follows that

$$4S = n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1) = n^2(n+1)^2 \text{ and } S = \left[\frac{n(n+1)}{2} \right]^2.$$

40. The area of G_i is

$$\begin{aligned} \left(\sum_{k=1}^i k \right)^2 - \left(\sum_{k=1}^{i-1} k \right)^2 &= \left[\frac{i(i+1)}{2} \right]^2 - \left[\frac{(i-1)i}{2} \right]^2 = \frac{i^2}{4} [(i+1)^2 - (i-1)^2] \\ &= \frac{i^2}{4} [(i^2 + 2i + 1) - (i^2 - 2i + 1)] = \frac{i^2}{4} (4i) = i^3 \end{aligned}$$

Thus, the area of $ABCD$ is $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$.

$$41. (a) \sum_{i=1}^n [i^4 - (i-1)^4] = (1^4 - 0^4) + (2^4 - 1^4) + (3^4 - 2^4) + \cdots + [n^4 - (n-1)^4] = n^4 - 0 = n^4$$

$$(b) \sum_{i=1}^{100} (5^i - 5^{i-1}) = (5^1 - 5^0) + (5^2 - 5^1) + (5^3 - 5^2) + \cdots + (5^{100} - 5^{99}) = 5^{100} - 5^0 = 5^{100} - 1$$

$$(c) \sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \cdots + \left(\frac{1}{99} - \frac{1}{100} \right) = \frac{1}{3} - \frac{1}{100} = \frac{97}{300}$$

$$(d) \sum_{i=1}^n (a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) = a_n - a_0$$

42. Summing the inequalities $-|a_i| \leq a_i \leq |a_i|$ for $i = 1, 2, \dots, n$, we get $-\sum_{i=1}^n |a_i| \leq \sum_{i=1}^n a_i \leq \sum_{i=1}^n |a_i|$. Since

$$|x| \leq c \Leftrightarrow -c \leq x \leq c, \text{ we have } \left| \sum_{i=1}^n a_i \right| \leq \sum_{i=1}^n |a_i|. \text{ Another method: Use mathematical induction.}$$

$$43. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left(\frac{i}{n} \right)^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^n i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \lim_{n \rightarrow \infty} \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \\ = \frac{1}{6} (1)(2) = \frac{1}{3}$$

$$44. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left[\left(\frac{i}{n} \right)^3 + 1 \right] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{i^3}{n^4} + \frac{1}{n} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n^4} \sum_{i=1}^n i^3 + \frac{1}{n} \sum_{i=1}^n 1 \right] \\ = \lim_{n \rightarrow \infty} \left[\frac{1}{n^4} \left(\frac{n(n+1)}{2} \right)^2 + \frac{1}{n} (n) \right] = \lim_{n \rightarrow \infty} \frac{1}{4} \left(1 + \frac{1}{n} \right)^2 + 1 = \frac{1}{4} + 1 = \frac{5}{4}$$

$$45. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left[\left(\frac{2i}{n} \right)^3 + 5 \left(\frac{2i}{n} \right) \right] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{16}{n^4} i^3 + \frac{20}{n^2} i \right] = \lim_{n \rightarrow \infty} \left[\frac{16}{n^4} \sum_{i=1}^n i^3 + \frac{20}{n^2} \sum_{i=1}^n i \right] \\ = \lim_{n \rightarrow \infty} \left[\frac{16}{n^4} \frac{n^2(n+1)^2}{4} + \frac{20}{n^2} \frac{n(n+1)}{2} \right] = \lim_{n \rightarrow \infty} \left[\frac{4(n+1)^2}{n^2} + \frac{10n(n+1)}{n^2} \right] \\ = \lim_{n \rightarrow \infty} \left[4 \left(1 + \frac{1}{n} \right)^2 + 10 \left(1 + \frac{1}{n} \right) \right] = 4 \cdot 1 + 10 \cdot 1 = 14$$

$$46. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[\left(1 + \frac{3i}{n} \right)^3 - 2 \left(1 + \frac{3i}{n} \right) \right] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[1 + \frac{9i}{n} + \frac{27i^2}{n^2} + \frac{27i^3}{n^3} - 2 - \frac{6i}{n} \right] \\ = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{81}{n^4} i^3 + \frac{81}{n^3} i^2 + \frac{9}{n^2} i - \frac{3}{n} \right] \\ = \lim_{n \rightarrow \infty} \left[\frac{81}{n^4} \frac{n^2(n+1)^2}{4} + \frac{81}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{9}{n^2} \frac{n(n+1)}{2} - \frac{3}{n} n \right] \\ = \lim_{n \rightarrow \infty} \left[\frac{81}{4} \left(1 + \frac{1}{n} \right)^2 + \frac{27}{2} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + \frac{9}{2} \left(1 + \frac{1}{n} \right) - 3 \right] = \frac{81}{4} + \frac{54}{2} + \frac{9}{2} - 3 = \frac{195}{4}$$

47. Let $S = \sum_{i=1}^n ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplying both sides by r gives us

$rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$. Subtracting the first equation from the second, we find

$$(r-1)S = ar^n - a = a(r^n - 1), \text{ so } S = \frac{a(r^n - 1)}{r - 1} \text{ (since } r \neq 1).$$

$$48. \sum_{i=1}^n \frac{3}{2^{i-1}} = 3 \sum_{i=1}^n \left(\frac{1}{2}\right)^{i-1} = \frac{3 \left[\left(\frac{1}{2}\right)^n - 1\right]}{\frac{1}{2} - 1} \text{ [using Exercise 47 with } a = 3 \text{ and } r = \frac{1}{2}] = 6 \left[1 - \left(\frac{1}{2}\right)^n\right]$$

$$49. \sum_{i=1}^n (2i + 2^i) = 2 \sum_{i=1}^n i + \sum_{i=1}^n 2 \cdot 2^{i-1} = 2 \frac{n(n+1)}{2} + \frac{2(2^n - 1)}{2 - 1} = 2^{n+1} + n^2 + n - 2.$$

For the first sum we have used Theorem 3(c), and for the second, Exercise 47 with $a = r = 2$.

$$\begin{aligned} 50. \sum_{i=1}^m \left[\sum_{j=1}^n (i+j) \right] &= \sum_{i=1}^m \left[\sum_{j=1}^n i + \sum_{j=1}^n j \right] \text{ [Theorem 2(b)]} = \sum_{i=1}^m \left[ni + \frac{n(n+1)}{2} \right] \text{ [Theorem 3(b) and (c)]} \\ &= \sum_{i=1}^m ni + \sum_{i=1}^m \frac{n(n+1)}{2} = \frac{nm(m+1)}{2} + \frac{nm(n+1)}{2} = \frac{nm}{2} (m+n+2) \end{aligned}$$

G Complex Numbers

$$1. (5 - 6i) + (3 + 2i) = (5 + 3) + (-6 + 2)i = 8 + (-4)i = 8 - 4i$$

$$2. (4 - \frac{1}{2}i) - (9 + \frac{5}{2}i) = (4 - 9) + (-\frac{1}{2} - \frac{5}{2})i = -5 + (-3)i = -5 - 3i$$

$$\begin{aligned} 3. (2 + 5i)(4 - i) &= 2(4) + 2(-i) + (5i)(4) + (5i)(-i) = 8 - 2i + 20i - 5i^2 \\ &= 8 + 18i - 5(-1) = 8 + 18i + 5 = 13 + 18i \end{aligned}$$

$$4. (1 - 2i)(8 - 3i) = 8 - 3i - 16i + 6(-1) = 2 - 19i$$

$$5. \overline{12 + 7i} = 12 - 7i$$

$$6. 2i(\frac{1}{2} - i) = i - 2(-1) = 2 + i \Rightarrow \overline{2i(\frac{1}{2} - i)} = \overline{2 + i} = 2 - i$$

$$7. \frac{1 + 4i}{3 + 2i} = \frac{1 + 4i}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i} = \frac{3 - 2i + 12i - 8(-1)}{3^2 + 2^2} = \frac{11 + 10i}{13} = \frac{11}{13} + \frac{10}{13}i$$

$$8. \frac{3 + 2i}{1 - 4i} = \frac{3 + 2i}{1 - 4i} \cdot \frac{1 + 4i}{1 + 4i} = \frac{3 + 12i + 2i + 8(-1)}{1^2 + 4^2} = \frac{-5 + 14i}{17} = -\frac{5}{17} + \frac{14}{17}i$$

$$9. \frac{1}{1 + i} = \frac{1}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{1 - i}{1 - (-1)} = \frac{1 - i}{2} = \frac{1}{2} - \frac{1}{2}i$$

$$10. \frac{3}{4 - 3i} = \frac{3}{4 - 3i} \cdot \frac{4 + 3i}{4 + 3i} = \frac{12 + 9i}{16 - 9(-1)} = \frac{12 + 9i}{25} = \frac{12}{25} + \frac{9}{25}i$$

$$11. i^3 = i^2 \cdot i = (-1)i = -i$$

$$12. i^{100} = (i^2)^{50} = (-1)^{50} = 1$$

$$13. \sqrt{-25} = \sqrt{25}i = 5i$$

$$14. \sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{3 \cdot 12}i^2 = \sqrt{36}(-1) = -6$$

15. $\overline{12 - 5i} = 12 + 5i$ and $|12 - 5i| = \sqrt{12^2 + (-5)^2} = \sqrt{144 + 25} = \sqrt{169} = 13$

16. $\overline{-1 + 2\sqrt{2}i} = -1 - 2\sqrt{2}i$ and $|-1 + 2\sqrt{2}i| = \sqrt{(-1)^2 + (2\sqrt{2})^2} = \sqrt{1 + 8} = \sqrt{9} = 3$

17. $\overline{-4i} = 0 - 4i = 0 + 4i = 4i$ and $|-4i| = \sqrt{0^2 + (-4)^2} = \sqrt{16} = 4$

18. Let $z = a + bi$, $w = c + di$.

$$\begin{aligned} \text{(a)} \quad \overline{z + w} &= \overline{(a + bi) + (c + di)} = \overline{(a + c) + (b + d)i} \\ &= (a + c) - (b + d)i = (a - bi) + (c - di) = \bar{z} + \bar{w} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \overline{zw} &= \overline{(a + bi)(c + di)} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i. \\ \text{On the other hand, } \bar{z}\bar{w} &= (a - bi)(c - di) = (ac - bd) - (ad + bc)i = \overline{zw}. \end{aligned}$$

(c) Use mathematical induction and part (b): Let S_n be the statement that $\overline{z^n} = \bar{z}^n$.

S_1 is true because $\overline{z^1} = \bar{z} = \bar{z}^1$. Assume S_k is true, that is $\overline{z^k} = \bar{z}^k$. Then

$$\overline{z^{k+1}} = \overline{z^{1+k}} = \overline{z z^k} = \overline{z} \overline{z^k} \quad [\text{part (b) with } w = z^k] = \bar{z}^1 \bar{z}^k = \bar{z}^{1+k} = \bar{z}^{k+1}, \text{ which shows that } S_{k+1} \text{ is true.}$$

Therefore, by mathematical induction, $\overline{z^n} = \bar{z}^n$ for every positive integer n .

Another proof: Use part (b) with $w = z$, and mathematical induction.

19. $4x^2 + 9 = 0 \Leftrightarrow 4x^2 = -9 \Leftrightarrow x^2 = -\frac{9}{4} \Leftrightarrow x = \pm\sqrt{-\frac{9}{4}} = \pm\sqrt{\frac{9}{4}}i = \pm\frac{3}{2}i.$

20. $x^4 = 1 \Leftrightarrow x^4 - 1 = 0 \Leftrightarrow (x^2 - 1)(x^2 + 1) = 0 \Leftrightarrow x^2 - 1 = 0 \text{ or } x^2 + 1 = 0 \Leftrightarrow$
 $x = \pm 1 \text{ or } x = \pm i.$

21. $x^2 + 2x + 5 = 0 \Leftrightarrow x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

22. $2x^2 - 2x + 1 = 0 \Leftrightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{2 \pm \sqrt{-4}}{4} = \frac{2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i$

23. By the quadratic formula, $z^2 + z + 2 = 0 \Leftrightarrow z = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{-7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i.$

24. $z^2 + \frac{1}{2}z + \frac{1}{4} = 0 \Leftrightarrow 4z^2 + 2z + 1 = 0 \Leftrightarrow$
 $z = \frac{-2 \pm \sqrt{2^2 - 4(4)(1)}}{2(4)} = \frac{-2 \pm \sqrt{-12}}{8} = \frac{-2 \pm 2\sqrt{3}i}{8} = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i.$

25. For $z = -3 + 3i$, $r = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$ and $\tan \theta = \frac{3}{-3} = -1 \Rightarrow \theta = \frac{3\pi}{4}$ (since z lies in the second quadrant). Therefore, $-3 + 3i = 3\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$.

26. For $z = 1 - \sqrt{3}i$, $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ and $\tan \theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \Rightarrow \theta = \frac{5\pi}{3}$ (since z lies in the fourth quadrant). Therefore, $1 - \sqrt{3}i = 2(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})$.

27. For $z = 3 + 4i$, $r = \sqrt{3^2 + 4^2} = 5$ and $\tan \theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}(\frac{4}{3})$ (since z lies in the first quadrant).
 Therefore, $3 + 4i = 5[\cos(\tan^{-1} \frac{4}{3}) + i \sin(\tan^{-1} \frac{4}{3})]$.

28. For $z = 8i$, $r = \sqrt{0^2 + 8^2} = 8$ and $\tan \theta = \frac{8}{0}$ is undefined, so $\theta = \frac{\pi}{2}$ (since z lies on the positive imaginary axis).
 Therefore, $8i = 8(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$.

29. For $z = \sqrt{3} + i$, $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ and $\tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow z = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$.

For $w = 1 + \sqrt{3}i$, $r = 2$ and $\tan \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow w = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$.

Therefore, $zw = 2 \cdot 2[\cos(\frac{\pi}{6} + \frac{\pi}{3}) + i \sin(\frac{\pi}{6} + \frac{\pi}{3})] = 4(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$,

$z/w = \frac{2}{2}[\cos(\frac{\pi}{6} - \frac{\pi}{3}) + i \sin(\frac{\pi}{6} - \frac{\pi}{3})] = \cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6})$, and $1 = 1 + 0i = 1(\cos 0 + i \sin 0) \Rightarrow$

$1/z = \frac{1}{2}[\cos(0 - \frac{\pi}{6}) + i \sin(0 - \frac{\pi}{6})] = \frac{1}{2}[\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6})]$. For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{2}(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6})$.

30. For $z = 4\sqrt{3} - 4i$, $r = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{64} = 8$ and $\tan \theta = \frac{-4}{4\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{11\pi}{6} \Rightarrow$

$z = 8(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6})$. For $w = 8i$, $r = \sqrt{0^2 + 8^2} = 8$ and $\tan \theta = \frac{8}{0}$ is undefined, so $\theta = \frac{\pi}{2} \Rightarrow$

$w = 8(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$. Therefore, $zw = 8 \cdot 8[\cos(\frac{11\pi}{6} + \frac{\pi}{2}) + i \sin(\frac{11\pi}{6} + \frac{\pi}{2})] = 64(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$,

$z/w = \frac{8}{8}[\cos(\frac{11\pi}{6} - \frac{\pi}{2}) + i \sin(\frac{11\pi}{6} - \frac{\pi}{2})] = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$, and

$1 = 1 + 0i = 1(\cos 0 + i \sin 0) \Rightarrow 1/z = \frac{1}{8}[\cos(0 - \frac{11\pi}{6}) + i \sin(0 - \frac{11\pi}{6})] = \frac{1}{8}[\cos(\frac{\pi}{6}) + i \sin(\frac{\pi}{6})]$.

For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{8}(\cos \frac{11\pi}{6} - i \sin \frac{11\pi}{6})$.

31. For $z = 2\sqrt{3} - 2i$, $r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$ and $\tan \theta = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$\Rightarrow \theta = -\frac{\pi}{6} \Rightarrow z = 4[\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6})]$. For $w = -1 + i$, $r = \sqrt{2}$,

$\tan \theta = \frac{1}{-1} = -1 \Rightarrow \theta = \frac{3\pi}{4} \Rightarrow w = \sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$. Therefore,

$zw = 4\sqrt{2}[\cos(-\frac{\pi}{6} + \frac{3\pi}{4}) + i \sin(-\frac{\pi}{6} + \frac{3\pi}{4})] = 4\sqrt{2}(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12})$,

$z/w = \frac{4}{\sqrt{2}}[\cos(-\frac{\pi}{6} - \frac{3\pi}{4}) + i \sin(-\frac{\pi}{6} - \frac{3\pi}{4})] = \frac{4}{\sqrt{2}}[\cos(-\frac{11\pi}{12}) + i \sin(-\frac{11\pi}{12})]$

$= 2\sqrt{2}(\cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12})$, and

$1/z = \frac{1}{4}[\cos(-\frac{\pi}{6}) - i \sin(-\frac{\pi}{6})] = \frac{1}{4}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$.

32. For $z = 4(\sqrt{3} + i) = 4\sqrt{3} + 4i$, $r = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{64} = 8$ and $\tan \theta = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow$

$z = 8(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$. For $w = -3 - 3i$, $r = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$ and

$\tan \theta = \frac{-3}{-3} = 1 \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow w = 3\sqrt{2}(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4})$. Therefore,

$zw = 8 \cdot 3\sqrt{2}[\cos(\frac{\pi}{6} + \frac{5\pi}{4}) + i \sin(\frac{\pi}{6} + \frac{5\pi}{4})] = 24\sqrt{2}(\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12})$,

$z/w = \frac{8}{3\sqrt{2}}[\cos(\frac{\pi}{6} - \frac{5\pi}{4}) + i \sin(\frac{\pi}{6} - \frac{5\pi}{4})] = \frac{4\sqrt{2}}{3}[\cos(-\frac{13\pi}{12}) + i \sin(-\frac{13\pi}{12})]$, and

$1/z = \frac{1}{8}(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6})$.

33. For $z = 1 + i$, $r = \sqrt{2}$ and $\tan \theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$. So by

De Moivre's Theorem,

$$(1 + i)^{20} = \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^{20} = \left(2^{1/2} \right)^{20} \left(\cos \frac{20 \cdot \pi}{4} + i \sin \frac{20 \cdot \pi}{4} \right)$$

$$= 2^{10}(\cos 5\pi + i \sin 5\pi) = 2^{10}[-1 + i(0)] = -2^{10} = -1024$$

34. For $z = 1 - \sqrt{3}i$, $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ and $\tan \theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \Rightarrow \theta = \frac{5\pi}{3} \Rightarrow$
 $z = 2 \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right)$. So by De Moivre's Theorem,

$$\begin{aligned} (1 - \sqrt{3}i)^5 &= [2(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})]^5 = 2^5 (\cos \frac{5 \cdot 5\pi}{3} + i \sin \frac{5 \cdot 5\pi}{3}) \\ &= 2^5 (\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) = 32 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = 16 + 16\sqrt{3}i \end{aligned}$$

35. For $z = 2\sqrt{3} + 2i$, $r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4$ and $\tan \theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow$
 $z = 4 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$. So by De Moivre's Theorem,

$$(2\sqrt{3} + 2i)^5 = [4(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})]^5 = 4^5 (\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}) = 1024 \left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right] = -512\sqrt{3} + 512i$$

36. For $z = 1 - i$, $r = \sqrt{2}$ and $\tan \theta = \frac{-1}{1} = -1 \Rightarrow \theta = \frac{7\pi}{4} \Rightarrow z = \sqrt{2} \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) \Rightarrow$

$$\begin{aligned} (1 - i)^8 &= [\sqrt{2}(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4})]^8 = 2^4 (\cos \frac{8 \cdot 7\pi}{4} + i \sin \frac{8 \cdot 7\pi}{4}) \\ &= 16(\cos 14\pi + i \sin 14\pi) = 16(1 + 0i) = 16 \end{aligned}$$

37. $1 = 1 + 0i = 1(\cos 0 + i \sin 0)$. Using Equation 3 with $r = 1$, $n = 8$, and $\theta = 0$, we have

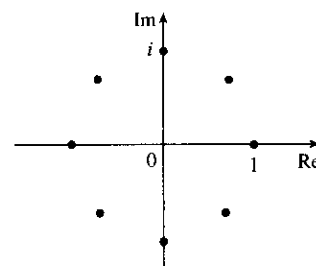
$$w_k = 1^{1/8} \left[\cos \left(\frac{0 + 2k\pi}{8} \right) + i \sin \left(\frac{0 + 2k\pi}{8} \right) \right] = \cos \frac{k\pi}{4} + i \sin \frac{k\pi}{4}, \text{ where } k = 0, 1, 2, \dots, 7.$$

$$w_0 = 1(\cos 0 + i \sin 0) = 1, w_1 = 1(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$$

$$w_2 = 1(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = i, w_3 = 1(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$$

$$w_4 = 1(\cos \pi + i \sin \pi) = -1, w_5 = 1(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

$$w_6 = 1(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) = -i, w_7 = 1(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$



38. $32 = 32 + 0i = 32(\cos 0 + i \sin 0)$. Using Equation 3 with $r = 32$, $n = 5$, and $\theta = 0$, we have

$$w_k = 32^{1/5} \left[\cos \left(\frac{0 + 2k\pi}{5} \right) + i \sin \left(\frac{0 + 2k\pi}{5} \right) \right] = 2(\cos \frac{2}{5}\pi k + i \sin \frac{2}{5}\pi k), \text{ where } k = 0, 1, 2, 3, 4.$$

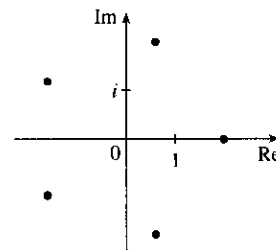
$$w_0 = 2(\cos 0 + i \sin 0) = 2$$

$$w_1 = 2(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})$$

$$w_2 = 2(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5})$$

$$w_3 = 2(\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5})$$

$$w_4 = 2(\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5})$$



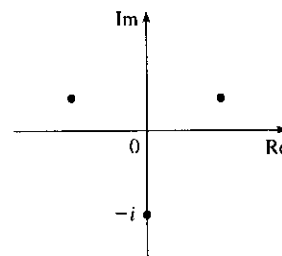
39. $i = 0 + i = 1(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$. Using Equation 3 with $r = 1$, $n = 3$, and $\theta = \frac{\pi}{2}$, we have

$$w_k = 1^{1/3} \left[\cos \left(\frac{\frac{\pi}{2} + 2k\pi}{3} \right) + i \sin \left(\frac{\frac{\pi}{2} + 2k\pi}{3} \right) \right], \text{ where } k = 0, 1, 2.$$

$$w_0 = \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_1 = \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_2 = \left(\cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} \right) = -i$$



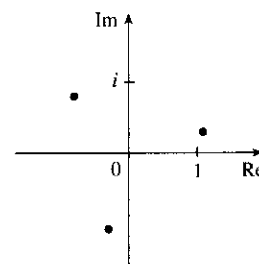
40. $1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$. Using Equation 3 with $r = \sqrt{2}$, $n = 3$, and $\theta = \frac{\pi}{4}$, we have

$$w_k = (\sqrt{2})^{1/3} \left[\cos \left(\frac{\frac{\pi}{4} + 2k\pi}{3} \right) + i \sin \left(\frac{\frac{\pi}{4} + 2k\pi}{3} \right) \right], \text{ where } k = 0, 1, 2.$$

$$w_0 = 2^{1/6} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$w_1 = 2^{1/6} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = 2^{1/6} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = -2^{-1/3} + 2^{-1/3}i$$

$$w_2 = 2^{1/6} \left(\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right)$$



41. Using Euler's formula (6) with $y = \frac{\pi}{2}$, we have $e^{i\pi/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + 1i = i$.

42. Using Euler's formula (6) with $y = 2\pi$, we have $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$.

43. Using Euler's formula (6) with $y = \frac{\pi}{3}$, we have $e^{i\pi/3} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

44. Using Euler's formula (6) with $y = -\pi$, we have $e^{-i\pi} = \cos(-\pi) + i \sin(-\pi) = -1$.

45. Using Equation 7 with $x = 2$ and $y = \pi$, we have $e^{2+i\pi} = e^2 e^{i\pi} = e^2(\cos \pi + i \sin \pi) = e^2(-1 + 0) = -e^2$.

46. Using Equation 7 with $x = \pi$ and $y = 1$, we have $e^{\pi+i} = e^\pi \cdot e^{1i} = e^\pi(\cos 1 + i \sin 1) = e^\pi \cos 1 + (e^\pi \sin 1)i$.

47. $F(x) = e^{rx} = e^{(a+bi)x} = e^{ax+bx i} = e^{ax}(\cos bx + i \sin bx) = e^{ax} \cos bx + i(e^{ax} \sin bx) \Rightarrow$

$$\begin{aligned} F'(x) &= (e^{ax} \cos bx)' + i(e^{ax} \sin bx)' = (ae^{ax} \cos bx - be^{ax} \sin bx) + i(ae^{ax} \sin bx + be^{ax} \cos bx) \\ &= a[e^{ax}(\cos bx + i \sin bx)] + b[e^{ax}(-\sin bx + i \cos bx)] = ae^{rx} + b[e^{ax}(i^2 \sin bx + i \cos bx)] \\ &= ae^{rx} + bi[e^{ax}(\cos bx + i \sin bx)] = ae^{rx} + bie^{rx} = (a + bi)e^{rx} = re^{rx} \end{aligned}$$

48. (a) From Exercise 47, $F(x) = e^{(1+i)x} \Rightarrow F'(x) = (1+i)e^{(1+i)x}$. So

$$\int e^{(1+i)x} dx = \frac{1}{1+i} \int F'(x) dx = \frac{1}{1+i} F(x) + C = \frac{1-i}{2} F(x) + C = \frac{1-i}{2} e^{(1+i)x} + C$$

$$(b) \int e^{(1+i)x} dx = \int e^x e^{ix} dx = \int e^x (\cos x + i \sin x) dx = \int e^x \cos x dx + i \int e^x \sin x dx \quad (1).$$

Also,

$$\begin{aligned} \frac{1-i}{2} e^{(1+i)x} &= \frac{1}{2} e^{(1+i)x} - \frac{1}{2} i e^{(1+i)x} = \frac{1}{2} e^{x+ix} - \frac{1}{2} i e^{x+ix} \\ &= \frac{1}{2} e^x (\cos x + i \sin x) - \frac{1}{2} i e^x (\cos x + i \sin x) \\ &= \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + \frac{1}{2} i e^x \sin x - \frac{1}{2} i e^x \cos x \\ &= \frac{1}{2} e^x (\cos x + \sin x) + i \left[\frac{1}{2} e^x (\sin x - \cos x) \right] \quad (2) \end{aligned}$$

Equating the real and imaginary parts in (1) and (2), we see that $\int e^x \cos x dx = \frac{1}{2} e^x (\cos x + \sin x) + C$ and

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$